Mixed convection in the stagnation point flow due to a moving extensible surface

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE **MASTER OF SCIENCE IN MATHEMATICS**

We accept this dissertation as conforming to the required standard.

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Dedicated To

My sweet and beloved
Behna
and to my Parents

Acknowledgement

"The mind is its own place, and in itself

Can make a Heaven of Hell, a Hell of Heav'n"

All praise be to ALLAH, the ALMIGHTY, WHO bestowed upon me with one of HIS chief blessings for mankind – the thinking mind.

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Date: October 19, 2011 ***Muhammad Usman***

DECLARTION

I hereby declare that this thesis, neither as a whole nor as a part thereof, has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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Preface

The study of flow field caused by a stretching surface is relevant to several practical applications in engineering process as aerodynamic extrusion of plastic sheets, melting-spinning, paper production, glass and polymer industry and many others. It must however be pointed out that the problem of boundary layer flow adjacent to continuous moving sheet is physically different from that of the classical Blasius flow past a stationary flat plate and these two problems can never be mathematically transformed from one to another. Starting from the pioneering work of Skiadis [1, 2], who analyzed the boundary layer on continuous semi infinite sheets and cylindrical rods moving steadily in an otherwise quiescent environment. An extensive effort has been made to gain information regarding the stretching flow problem in various situations. Such situation includes consideration of heat transfer, mass transfer, porous medium, non-Newtonian fluid, magnetohydrodynamic fluids etc. Some very recent attempts have been made in the investigation [3-13]. Crane [14] presented a similarity solution in closed analytical form for steady two dimensional incompressible boundary layer flow over a surface which is being stretched in its own plane with velocity varying linearly with the distance from a fixed point. Banks [15] considered a surface that is stretched with a non-linear velocity. Magyari and Keller [16, 17] and Magyari et al. [18] presented a closed form solution for the case when mass transfer (suction or injection) is proportional to a power of the distance x from the fixed point. Kukin [19] has considered the boundary layer flow due to a moving sheet that obeys a more general stretching law. After that Ishak et al. [20] discussed the boundary layer flow due to a moving extensible surface. Motivated by all these facts, the aim of this dissertation is to investigate the effect of mixed convection MHD stagnation point flow due to a moving extensible surface. The arrangement of this dissertation is as follows.

Chapter 1 introduces some basic definitions and concepts [21] related to this dissertation. The study of chapter 2 is based on the work of Ishak et.al. [20], all the graphs and tables are reproduced with an excellent accuracy. In chapter 3, the work of Ishak et. al. [20] is generalized in the direction of mixed convection stagnation point flow. The governing equations are solved by using Keller Box method [22] and shooting method [23]. The influence of mixed convection and stagnation point parameter are analyzed trough graphs.

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Chapter 1

Preliminaries

In this chapter, some basic definitions [21] related to this dissertation, equations of motion of fluid in "moving extensible surface" are discussed for the better understanding of the reader, which will be used in the next chapters.

1.1 Definitions

1.2 Fluid mechanics

It is that branch of science in which we study the fluid at rest and motion.

1.2.1 Velocity field

Both by the experiment and theory, the properties of the fluid as a function of position and time, in a given flow, is regarded as solution of the problem. The velocity field $V(x, y, z, t)$ has three components u, v, w as

$$\mathbf{V}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) = u(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t})\mathbf{i} + v(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t})\mathbf{j} + w(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t})\mathbf{k}\ , \tag{1.1}$$

all of which are scaler fields itself.

1.2.2 Fluid

Any substance that has flowing ability is called fluid.

1.2.3 Flow

The process that deforms a substance continuously because of the force applied on it, is called flowing.

1.3 Thermodynamic properties of a fluid

In flow analysis the following are three basic companions of velocity:

1.3.1 Pressure

The stress on static fluid at a point is called pressure. Flow is often driven because of change in pressure, especially in ducts. Generally, pressure can be defined as a magnitude of force per unit area, which is expressed as

$$P = \frac{F}{A}. \tag{1.2}$$

1.3.2 Density

Amount of mass per unit volume is known as density of any substance. In liquids, density is usually assumed as constant however, it is highly changeable in gases. Mathematically, it can be written as

$$\rho = \lim_{\delta v \longrightarrow 0} \frac{\delta m}{\delta v}, \tag{1.3}$$

where δm is mass of fluid within unit volume δv.

1.3.3 Temperature

Thermal energy possessed by the particles of a substance is regarded as temperature. The internal level of energy is directly related to temperature. When there are strong differences in temperature at different positions of the fluid flow, chances of heat transfer occur.

1.3.4 Heat transfer

Heat transfer is in fact flow of heat from one body to another. It always move from warmer to cooler substance. There are three kinds of heat transfer as

1.3.5 Conduction

Whenever heat is transferred through direct contact of two bodies at different temperatures, then this phenomenon of heat transfer is called conduction.

1.3.6 Convection

It is the transport of heat energy through fluid. It is the process in which heat is transferred from solid boundary to moving fluid. It has further two kinds namely

- Forced or assisted convection.
- Natural or free convection.

Forced convection

When the flow of fluid is under the action of applied force (e.g. pump, fan, mixer) and if heat transfer occur due to these applied forces, then it is called forced convection. Heating and cooling of parts of body due to circulation of blood and aerodynamic heating are good examples of forced convection.

Natural convection

If the process of heat transfer occur with out any applied forces,it is natural convection. Generally, one can say that, as density of the fluid varies due to the difference in temperature at different position, this change of density experienced a buoyant force which is responsible for natural convection. Upward movement of air due to burning fire is an example of natural convection.

1.3.7 Radiation

Transfer of heat through electromagnetic waves is known as radiation.

1.3.8 Viscosity

The resistance in relative motion of fluid is called viscosity. It can be defined as

$$\mu = \frac{Shear\ stress}{rate\ of\ shear\ strain}. \tag{1.4}$$

1.3.9 Kinematic viscosity

Kinematic viscosity is the quotient of viscosity to density. Mathematically, it can be represented as

$$\nu = \frac{\mu}{\rho}. \tag{1.5}$$

1.3.10 Thermal conductivity

The relation between flow of heat per unit area with temperature differences is known as thermal conductivity. It is denoted by k, and mathematically, it can be expressed as

$$\mathbf{q} = -k\nabla T. \tag{1.6}$$

1.4 Types of fluid

1.4.1 Ideal fluid

The ideal fluids are the fluids with no viscosity. Though such fluids do not exist, in nature, we can assume the fluids to be indeed only for the simplification of the problem computationally.

1.4.2 Real fluid

The real fluids are those fluids which have some viscosity. Such fluids can be further classified into two kinds as

Newtonian fluid

A fluid in which applied shear stress is proportional to the velocity gradients is called Newtonian fluid. In Newtonian fluid viscosity is independent of rate of shear or velocity gradients (which is Newton law of viscosity). It can be expressed as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.7}$$

where τ_{yx} is the shear stress, du/dy is velocity gradients perpendicular to the direction of shear stress and μ is the viscosity of the fluid here.

1.4.3 Non-Newtonian fluid

A fluid which do not follow above relation is called non-Newtonian fluid. In non-Newtonian fluid, viscosity is function of rate of strain tensor D. Some of the many examples are polymer solution, colloidal suspensions, paper pulp in water, latex paint, blood plasma, syrup and shampoo, etc. A mathematical relation can be established for this type of fluid which is given by

$$\tau_{yx} = \mu \left(\frac{du}{dy}\right)^n, \; n \neq 1 \tag{1.8}$$

where n is consistency index and μ represent behavior.

1.5 Types of flows

1.5.1 Uniform flow

A flow of uniform velocity is known as uniform flow.

1.5.2 Non-uniform flow

A flow in which velocity is not same at each point is called non-uniform flow.

1.5.3 Laminar flow

The flow of a fluid that approximately follow some shape of a streamlined surface without intersecting its own path, such flows are called laminar flow.

1.5.4 Turbulent flow

When the paths of particles cross each other and have no definite path is called turbulent flow.

1.5.5 Compressible flow

A flow with variable density is regarded as compressible flow. All the gases are generally considered to have compressible flow.

1.5.6 Incompressible flow

A flow with constant density is called incompressible flow. All the liquid flows are generally considered to have incompressible flow.

1.5.7 Steady flow

If the velocity field of a fluid is not a function of time, then the flow is called steady flow.

1.5.8 Unsteady flow

If the velocity field of a fluid also depend on time, then the flow is called unsteady flow.

1.6 Types of forces

1.6.1 Inertial force

In inertial frame of reference the product of mass and acceleration of the body is called inertial force.

1.6.2 Body force

A force whose magnitude is proportional to the volume of the fluid element is called body force. Gravitational and magnetic forces are the examples of body forces.

1.6.3 Surface force

Force acting on the boundaries of a medium through direct contact is called surface force.

1.7 Basic equations

1.7.1 Equation of continuity

Let us choose a control volume having sides of lengths dx, dy, dz and velocity of control volume as $\mathbf{V} = (u, v, w)$ and ρ, as density. Since mass of the fluid in control volume is $\rho dxdydz$ and the rate of its increase in time is

$$\frac{\partial \rho}{\partial t} dxdydz. \tag{1.9}$$

It is quite clear from the above equation that when density is constant then above term will be vanished. The net rate of mass flux coming out of control volume is given by

$$\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dxdydz. \tag{1.10}$$

After adding the Eq. (1.9) and Eq. (1.10), we get

$$\frac{\partial \rho}{\partial t} dxdydz + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dxdydz = 0, \tag{1.11}$$

since,

$$\boldsymbol{\nabla}.\rho \mathbf{V} = \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dxdydz, \tag{1.12}$$

so Eq. (1.11) becomes

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.\rho \mathbf{V} = 0, \tag{1.13}$$

which is well known mass-conservation equation (continuity equation). As for incompressible flow, the the density is independent on space as well as time, so above relation takes the form

$$\boldsymbol{\nabla}.\mathbf{V} = 0. \tag{1.14}$$

But for compressible, steady flow, the continuity Eq. (1.13) reduces to the form

$$\boldsymbol{\nabla}.\rho V = 0. \tag{1.15}$$

1.7.2 Momentum equation

For an infinitesimal element, the basic differential momentum equation in vector form is

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + \nabla.\boldsymbol{T} + \rho \boldsymbol{b}, \tag{1.16}$$

where p is pressure, $\boldsymbol{b}$ is body force, $\boldsymbol{T}$ is a Cauchy stress tenser defined as

$$\boldsymbol{T} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}, \tag{1.17}$$

where τ_{xx}, τ_{yy} and τ_{zz} are normal stresses and the remaining components in $\boldsymbol{T}$ are known as shear stresses.

1.7.3 Energy equation

The energy equation is described as

$$\rho c_p \frac{de}{dt} = \mathrm{T.L} - \nabla.\mathrm{q}. \tag{1.18}$$

In Cartesian coordinates, it is given as

$$\rho c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu\phi, \tag{1.19}$$

where ϕ is viscous dissipation function.

1.8 Numerical scheme

Every physical problem in science involving change with respect to space or time is formulated mathematically in terms of partial differential equations or set of such equations. Most of the problems can not be solved analytically, usually we go for estimated solutions by means of numerical method of such problems. Only the big advantage of numerical scheme is that it can solve the non−linear problems with an ease. Finite difference and finite element methods are used commonly to approximate the solution of non-linear differential equations numerically.

1.8.1 Keller Box method

Keller (1970) introduced the box method which is also known as Keller-Box method. It is widely used to obtained the solutions of boundary layer problems [22]. This method generally can be explained in the following steps as follows

Step 1: Higher order differential equations are reduced to the system of first order differential equations.

Step 2: First order derivatives are replaced by their central difference approximations and the function with its average value. We obtained set of algebraic equations to be solved.

Step 3 : In this step, the set of algebraic equations are non−linear, linearize by using Newton's method.

Step 4: After completing third step, we have system of linear algebraic equations, which can be put in the form of tridiagonal matrix and solved by block-elemination method [22].

Chapter 2

MHD boundary-layer flow due to a moving extensible surface

This chapter deals with the MHD boundary layer flow due to a moving extensible surface. The surface occupies the negative x-axis and is continuously moving in the positive x-direction. The governing system of partial differential equations is first transformed into system of ordinary differential equations and then transformed equations are solved using implicit finite difference scheme, namely the Keller-Box method [22]. The important features of the flow and heat transfer characteristics for different variations in the governing parameters are analyzed and discussed through graphs. This chapter is review of the paper by Ishak et al. [20].

2.1 Mathematical formulation

Let us consider the steady, two dimensional flow of an incompressible viscous electrically conducting fluid. The fluid is flowing due to stretching sheet, which is placed in an ambient fluid of uniform temperature T_∞. We consider that a variable magnetic field $\mathbf{B}(x)$ is applied normal to the sheet in the absence of electric field and temperature of the sheet is $T_w(x)$, wher $T_w(x) > T_\infty$. It is assumed that the induced magnetic field is neglected due to small magnetic Reynolds number assumptions. Under these assumptions, the MHD equations for the steady flow and the heat transfer in the boundary layer over the stretching sheet are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2(x)}{\rho}u, \tag{2.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{2.3}$$

where u and v velocity component along x and $y-$axis respectively, y-axis is measured in the direction normal to the surface, T is the fluid temperature, ρ and α are the density and thermal diffusivity of the fluid respectively. The flowing fluid is confined to $y > 0$. The appropriate boundary conditions are

$$\left.\begin{array}{c} u(x,0)=u_s(x),\ v(x,0)=0,\ T(x,0)=T_w(x) \\ u\to 0,\ T\to T_\infty \text{ as } y\to\infty \end{array}\right\}, \tag{2.4}$$

where

$$T_w(x)=T_\infty+T_0\left(\frac{x_0}{|x|}\right)^m \text{ and } u_s(x)=u_0\left(\frac{x_0}{|x|}\right)^n. \tag{2.5}$$

Here, $u_s(x)$ is the velocity of the extensible sheet with which it is moving from the negative $x-$axis to the positive $x-$direction, $n > 0$, x_0 and u_0 are the characteristic length and velocity respectively. T_0 being the characteristic temperature. The boundary conditions must be completed with the condition

$$u\to 0,\ T\to T_\infty \text{ as } x\to -\infty. \tag{2.6}$$

To obtain similarity solution , we assume that the magnetic field $\mathbf{B}(x)$ is of the form

$$\mathbf{B}(x)=\left(\frac{x_0}{|x|}\right)^{\frac{n+1}{2}}\mathbf{B}_0, \tag{2.7}$$

where B_0 is the constant magnetic field. Introducing similarity variables of the form

$$\left.\begin{array}{c} \eta=y\left(\frac{u_s}{2\nu|x|}\right)^{1/2}\ \ \psi=(2\nu u_s\,|\,x\,|)^{1/2}f(\eta) \\ \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty} \end{array}\right\}, \tag{2.8}$$

Where ψ is stream function which is defined as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. After introducing similarity variables, Eqs.(2.2) and (2.3) take the form as

$$f''' + (n+1)ff'' - 2nf'^2 - 2Mf' = 0, \tag{2.9}$$

$$\frac{1}{Pr}\theta'' + (n-1)f\theta' - 2mf'\theta = 0. \tag{2.10}$$

Where $M = \sigma B_0^2/(\rho u_0/x_0)$ the magnetic parameter. Pr the Prandtle number and primes denote differentiation with respect to η. The boundary conditions Eq. (2.4) become

$$\begin{aligned} f(0) &= 0,\ f'(0) = 1,\ \theta(0) = 1, \\ f'(\eta) &\to 0,\ \theta(\eta) \to 0 \text{ as } \eta \to \infty. \end{aligned} \tag{2.11}$$

The skin friction coefficient C_f and local Nusselt number Nu_x are given by

$$\left.\begin{aligned} C_f = \frac{\tau_w}{\rho u_s^2/2} = \sqrt{\frac{2}{Re}}f''(0) \\ Nu_x = \frac{xq_w}{k(T_w - T_\infty)} = \sqrt{\frac{Re}{2}}\theta'(0) \end{aligned}\right\}, \tag{2.12}$$

where τ_w is shear stress at wall, q_w is heat transfer from the surface and k being dynamic viscosity and $Re_x = u_s\,|x|\,/v$ is the local Reynold number.

2.2 Numerical solution

Eqs. (2.9) and (2.10) subject to boundary conditions Eq. (2.11) are solved using implicit finite difference scheme commonly known as Keller Box method, described in the book by Cebeci and Bradshaw [22]. The solution is obtained by using few steps discussed in chapter 1.

2.2.1 Perturbation solution for small M

In order to solve differential Eqs. (2.9) and Eq. (2.10) subject to boundary conditions, Eq. (2.11) for small values of magnetic parameter M, we assume the solution [7] of the form

$$\left.\begin{aligned} f(\eta) = \sum f_i(\eta)M^i, \\ \theta(\eta) = \sum \theta_i(\eta)M^i. \end{aligned}\right\} \tag{2.13}$$

where f_i and θ_i are the perturbations in f and θ respectively. We obtain the following set of equations

$$f_0''' + (n-1)f_0 f_0'' - 2n f_0'^2 = 0, \tag{2.14}$$

$$\frac{1}{Pr}\theta_0'' + (n-1)f_0\theta_0' - 2m f_0'\theta_0 = 0, \tag{2.15}$$

and for $i \geq 1$

$$f_i''' + (n-1)\sum_{j=0}^{i} f_j f_{i-j}'' - 2n\sum_{j=0}^{i} f_j' f_{i-j}' - 2f_{i-1}' = 0, \tag{2.16}$$

$$\frac{1}{Pr}\theta_i'' + (n-1)\sum_{j=0}^{i} f_j\theta_{i-j}' - 2m\sum_{j=0}^{i} f_j'\theta_{i-j} = 0, \tag{2.17}$$

subject to boundary equations

$$\begin{aligned} &f_i(0) = 0,\ f_i'(0) = \delta_{i0},\ \theta_i(0) = \delta_{i0}, \\ &f_i'(\eta) \to 0,\ \theta_i(\eta) \to 0,\ \text{as } \eta \to \infty, \end{aligned} \tag{2.18}$$

for $i \geq 0$, where δ_{ij} is the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{2.19}$$

The perturbation form of skin–friction and local Nusselt number is

$$\left.\begin{aligned} \tfrac{1}{2}C_f Re_x^{\frac{1}{2}} = \tfrac{1}{\sqrt{2}}f''(0) \approx \tfrac{1}{\sqrt{2}}\sum_{i=0}^{4} f_i''(0)M^i \\ Nu_x/Re_x^{1/2} = -\tfrac{1}{\sqrt{2}}\sum_{i=0}^{4}\theta_i'(0)M^i \end{aligned}\right\}, \tag{2.20}$$

for $M << 1$.

2.2.2 Asymptotic solution for large M

In order to find the solution which is valid for large value of $M(M >> 1)$. We can define

$$F(z) = M^{\frac{1}{2}} f(\eta), \ \phi(z) = \theta(\eta), \ z = M^{1/2}\eta. \tag{2.21}$$

Now our basic equations Eqs. (2.9) and (2.10) take the form

$$F''' - 2F' + \beta\left[(n-1)FF'' - 2nF'^2\right] = 0, \tag{2.22}$$

$$\frac{1}{Pr}\phi'' + \beta\left[(n-1)F\phi' - 2mF'\phi\right] = 0, \tag{2.23}$$

subject to boundary conditions

$$\begin{gathered} F(0) = 0, \ F'(0) = 1, \ \phi(0) = 1, \\ F'(\infty) \to 0, \ \phi(\infty) \to 0. \end{gathered} \tag{2.24}$$

Where $\beta = \frac{1}{M} << 1$ and now prime denote differentiation w.r.t. z. As $\beta << 1$ so we can perturb Eqs. (2.22) and Eq. (2.23) with parameter ε as

$$F(z) = \sum_{i=0}^{\infty} F_i(z)\beta^i, \ \phi(z) = \sum_{i=0}^{\infty} \phi_i(z)\beta^i. \tag{2.25}$$

By putting Eq. (2.25) into (2.22) and (2.23) we get the following differential equations

$$\left.\begin{gathered} F_0''' - 2F_0' = 0, \ \phi_0'' = 0, \\ F_0(0) = 0, F_0'(0) = 1, \ \phi_0'(0) = 0, \\ F_0'(\infty) \to 0, \ \phi_0(\infty) \to 0 \end{gathered}\right\}. \tag{2.26}$$

Clearly $\phi_0(z)$ does not satisfy the corresponding boundary conditions, so for very large M, the problem of thermal field is singular. Further, we have

$$\begin{gathered} F_i''' - 2F_i' + (n-1)\sum_{j=0}^{i-1} F_j F_{i-j-1}'' - 2n\sum_{j=0}^{i-1} F_j' F_{i-j-1}' = 0, \\ F_i(0) = 0, \ F_i'(0) = 0, \ F_i'(\infty) \to 0, \end{gathered} \tag{2.27}$$

for $i \geq 1$. The skin-friction coefficient Eq. (2.12) is given by

$$\frac{1}{2}ReC_{fx}^{1/2} = \frac{1}{\sqrt{2}}f''(0) \approx \sqrt{M}\sum_{j=0}^{4} F_i''(0)M^{-i}, \tag{2.28}$$

for $M >> 1$.

2.3 Results and discussion

Implicit finite difference scheme (Keller-Box method) [22] is translated into matlab program and solve the Eqs. (2.9) and (2.10) subject to the boundary conditions Eq. (2.11) for various values of the velocity index n, magnetic parameter M and for convenience Pr and m are kept fixed at $Pr = 7$ and $m = 10$. We observe different cases for n.

2.3.1 Case I: For $0 < n < 1$

The values of $f''(0)$ are given in Table 1 for some values of n and $M = 0, 1, 2$. We reproduced all the values given in Table 1 by Ishak et al. [20]. It is observed that in the absence of magnetic field ($M = 0$), calculated value by the present method are highly accurate with the Kuiken [19], in which he used series method to do this for all values of n considered. Moreover, for $M \neq 0$, values of $f''(0)$ are also given for different n in Table 1. Figures 2.1 and 2.2 are presented to show the effects of different values of n on the velocity and temperature profiles respectively. It is also ensured through the graph that boundary conditions are satisfied which further supports the validity of the present code. It is seen from Fig. 2.1 for $M = 1$, that velocity is decreasing as n increasing between 0 to 1. In other words, boundary layer thickness decreases with increase in n and the values of skin-friction coefficient $|f''(0)|$ increases absolutely with increase in n. This observation is in agreement with the values of $f''(0)$ given in Table 1 for $M = 1$. Qualitatively, the same feature is observed for the effect on n on the temperature profile, but the effect of n on velocity profile are comparatively more prominent than on the temperature profiles. All the values in Table 1 are negative due to reason that the sheet exerts a drag force on the fluid.

	$M = 0$			
n	Kuiken [19]	Present values	$M = 1$	$M = 2$
0.2	−0.38191349	−0.3819	−1.4191	−2.0017
0.4	−0.63898882	−0.6390	−1.5310	−2.0830
0.6	−0.83961231	−0.8396	−1.6352	−2.1612
0.8	−1.00779210	−1.0078	−1.7331	−2.2365

Table1: Values of $f''(0)$ when $0 < n < 1$.

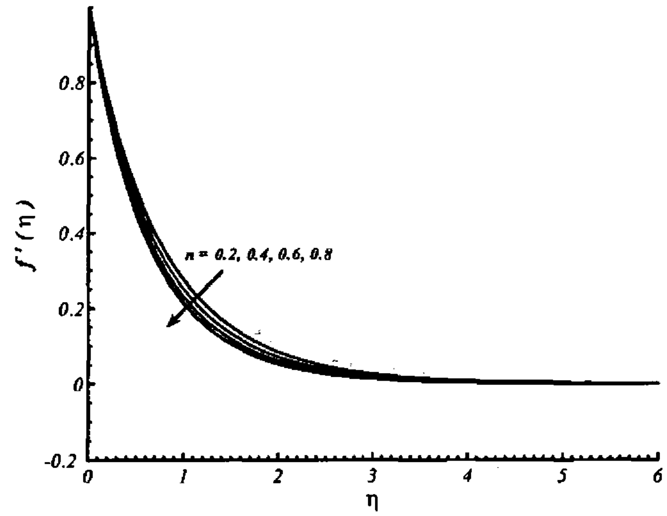

Fig. 2.1 : Effects on velocity profiles for different values of n $(0 < n < 1)$, when $M = 1$.

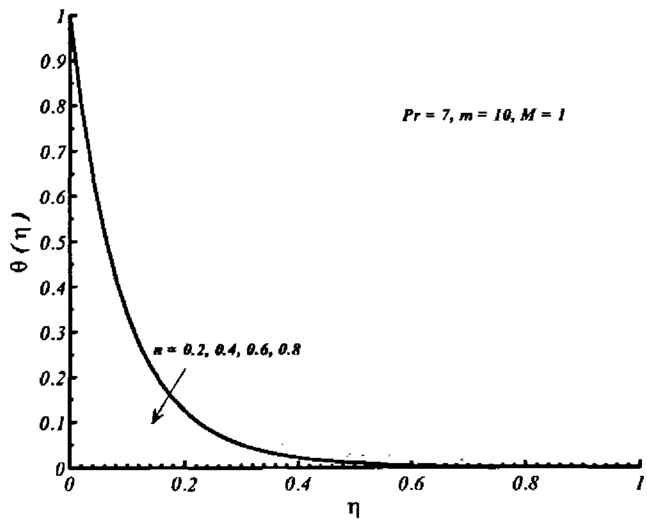

Fig. 2.2 : Effects on temperature profiles for different values of n $(0 < n < 1)$, when $Pr = 7, m = 10$ and $M = 1$.

2.3.2 Case II: For $n > 1$

Table 2 presents the values of $f''(0)\ n^{-\frac{1}{2}}$ for different values of n as considered in Kukin [19] for $M = 0$. Again, in this Table , an excellent agreement is found with those reported in [19]. In addition, two solutions are found for large values of n, which is quite new and not reported in [19]. The velocity and temperature profiles which support dual solution for different values of n are drawn in Figures 2.3 and 2.4 respectively. It is observed that velocity profile decreasing with increase in large values of n. The reversed flow $(f'(\eta) < 0)$ occur in the second solution, as satisfy the boundary conditions, so it should not be neglected. Fig. 2.4 shows a monotonically decreasing manner of the temperature profile as η increases. Again, the effects of dual solutions are not prominent for temperature profile as shown in Fig. 2.4. Moreover boundary layer thickness decrease with the increase in n, which in turn increases the temperature gradient at the surface, hence increases the heat transfer rate here. The velocity and temperature profiles for $n = 2$, $m = 10$ and $Pr = 7$ for different values of magnetic parameter M are shown in Fig. 2.5 and Fig. 2.6 respectively. It is seen that the magnetic field can be introduced to lower the boundary layer thickness at surface, in turn, it increases the velocity gradients at the surface, but temperature gradients decreases with increase in M. Thus, the effects of magnetic parameter on temperature profiles are less pronounce than on the velocity profiles. This effect becomes less important for large values of n, for example when $n = 1000$, as shown in Fig. 2.7. Second solution is also calculated for this value of n too and it produces the reversed of flow as mentioned earlier.

		Present results	
n	Kuiken [19]	First solution	Second solution
1.5	−1.19485513	−1.194849192	
2	−1.21601870	−1.216011581	
2.5	−1.22896403	−1.228955721	
3	−1.23767321	−1.237663702	
4	−1.24863030	−1.248618389	−1.254882713
5	−1.25523518	−1.255220869	−1.262182634
10	−1.26849732	−1.268471007	−1.276872999
100	−1.28047587	−1.280233693	−1.290277821
1000	−1.28167528	−1.281626246	−1.2915129821

Table 2: Values of $f''(0)n^{-\frac{1}{2}}$ for different values of n when $M = 0$.

The values of the skin friction coefficient $f''(0)$ and local Nusselt number $-\theta'(0)$ are also obtained by using series expansion Eq. (2.21) for small values of M and the series Eq. (2.29) for large values of M. Beside using the applied numerical scheme given by Eq. (2.13). For $P_r = 7$, $m = 10$ and $n = 10$, Eq. (2.21) becomes

$$\left.\begin{aligned} \tfrac{1}{2}C_f \mathrm{Re}_x^{1/2} &\approx (-4.0113 - 0.2552M + 0.0096M^2 - 0.0011M^3 + 0.0002M^4) \\ Nu_x/\mathrm{Re}_x^{1/2} &\approx (-12.2194 + 0.0673M - 0.0025M^2 + 0.0003M^3 - 0.0001M^4) \end{aligned}\right\}. \tag{2.29}$$

while for $n = 10$, Eq. (2.29) becomes

$$\begin{aligned}\tfrac{1}{2}C_f\,\mathrm{Re}_x^{1/2} &\approx \sqrt{M}(-1.4142 - 5.7688M^{-1}\\ &+12.2727M^{-2} + 30.9766M^{-3} - 1.4606M^{-4}).\end{aligned} \tag{2.30}$$

These values of $f''(0)$ and $-\theta'(0)$ are presented in Table 3 and 4 for different values of M. It is observed that there is an excellent agreement between the numerical solution of the equation and perturbation solution for small and large values of M as shown in Table. 3 and 4 for $f''(0)$ and $-\theta'(0)$ respectively. Fig. 2.8 is shown to compare the values of skin friction coefficient $f''(0)$ by numerical scheme and perturbation scheme for small and large values of M.

M	Numerical, Eq. (2.12)	Small M, Eq. (2.20)	Large M, Eq. (2.28)
0	−4.0113	−4.0113	
0.1	−4.0368	−4.0368	
0.2	−4.0620	−4.0620	
0.3	−4.0871	−4.0871	
0.4	−4.1120	−4.1119	
0.5	−4.1367	−4.1366	
1	−4.2580		
2	−4.4898		
4	−4.9188		
10	−6.0206		
100	−14.7078		−14.7064
1000	−44.9036		−44.9034
10000	−141.4790		−141.4790

Table 3: Values of $f''(0)$ when $n = 10$ (for the first solution).

M	Numerical, Eq.(2.12)	Small$M$, Eq. (2.20)
0	12.2194	12.2194
0.1	12.2127	12.2127
0.2	12.2060	12.2060
0.3	12.1993	12.1994
0.4	12.1927	12.1929
0.5	12.1862	12.1864
1	12.1540	
2	12.0923	
4	11.9776	
10	11.6791	
100	9.2117	
1000	4.2512	
10000	–	

Table 4 : Values of $-\theta'(0)$ when $Pr = 7$, $m = 10$ and $n = 10$ (for the first solution).

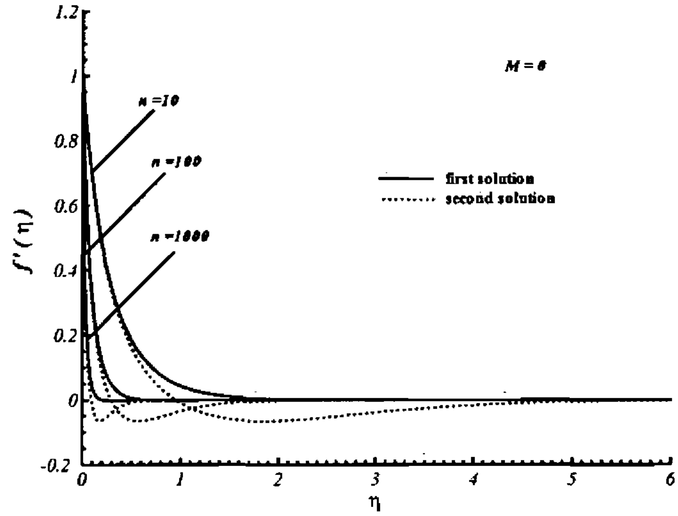

Fig. 2.3 : Effects on velocity profiles for
$n = 10, 100, 1000$ when $M = 0$.

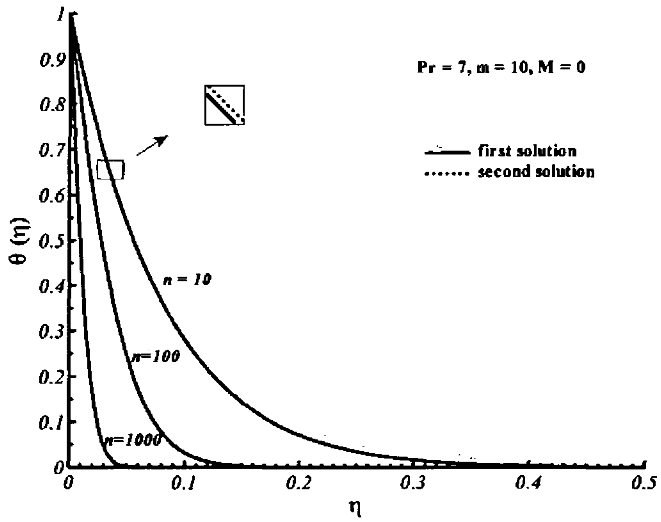

Fig. 2.4 : Effects on variation of temperature profiles for
$n = 10, 100$ and 1000 when $Pr = 7,\ m = 10$ and $M = 0$.

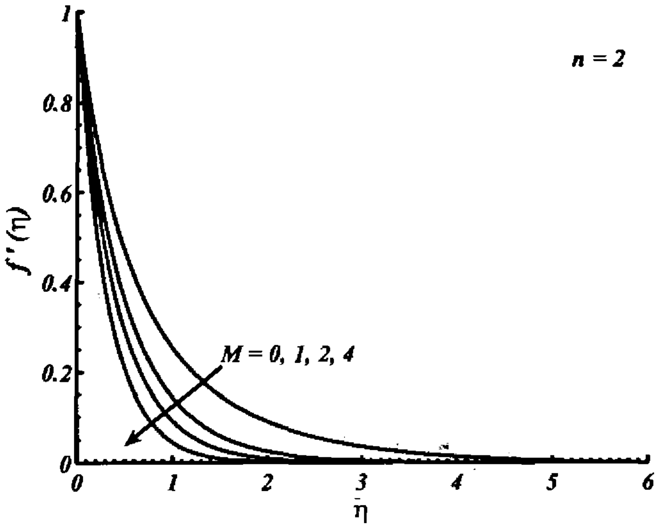

Fig. 2.5 : Effects on velocity profiles for $M = 0, 1, 2$ and 4

when $n = 2$.

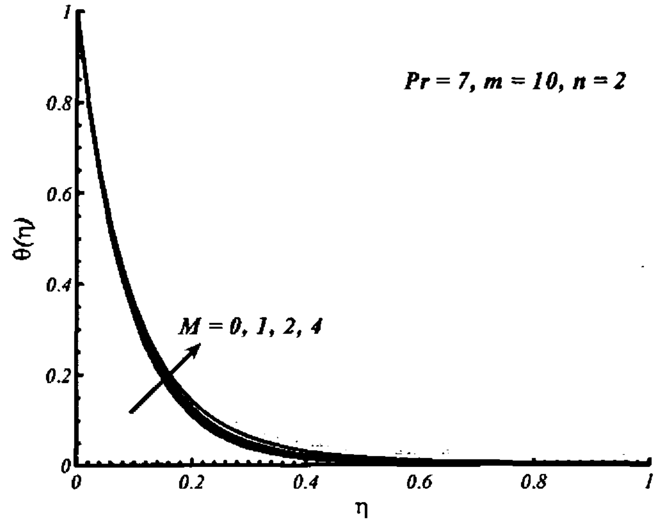

Fig. 2.6 : Effects on temperature profiles for $M = 0, 1, 2$ and

4 when $m = 10$, $Pr = 7$ and $n = 2$.

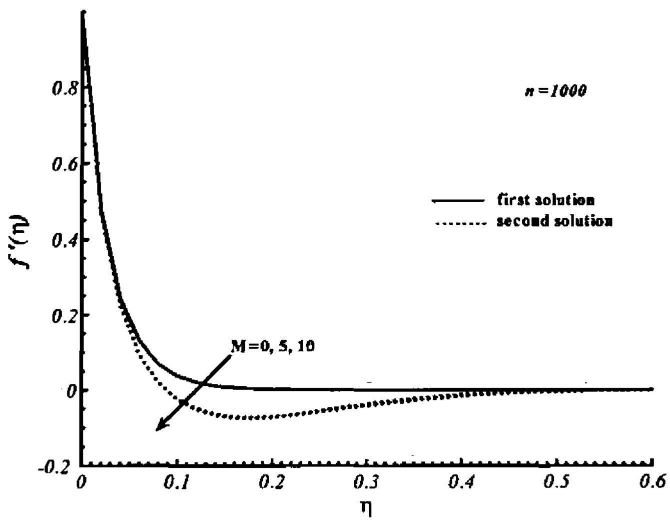

Fig. 2.7 : Effects on velocity profiles for $M = 0, 5$ and 10 when $n = 1000$.

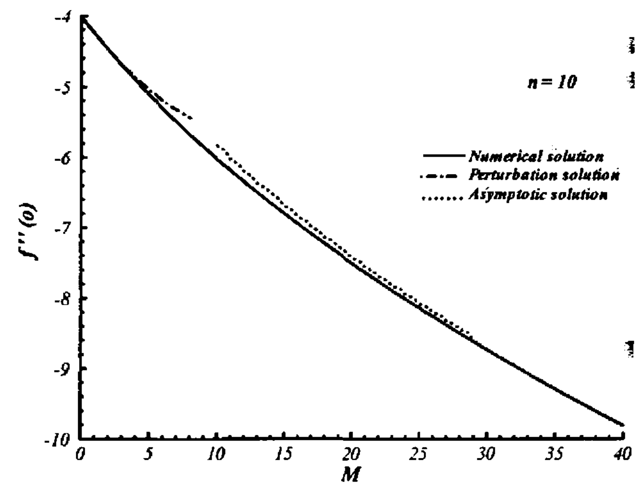

Fig. 2.8 : Variation of $f''(0)$ with M when $n = 10$

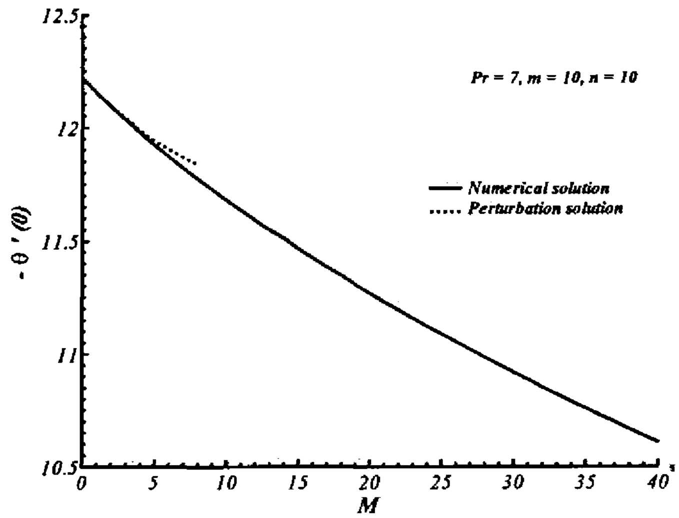

Fig. 2.9 : Variation of $-\theta'(0)$ with M when $Pr = 7, m = 10$ and $n = 10$.

2.3.3 Case III: For $n = 1$

The velocity and temperature profiles for different values of M are presented in Figs. 2.10 and 2.11. It is seen that the velocity boundary layer thickness decreases and thermal boundary layer thickness increases with the increase in magnetic parameter M. Thus, the skin friction coefficient increases but the rate of heat transfer at the surface decreases as M increases. These results are consistent with the results both for case $0 < n < 1$ and $n > 1$. For $n = 1$, the momentum Eq. (2.9) admits the first integral

$$f''^2 = 2mf'^2 + \frac{4}{3}f'^3$$

$$f''(0) = -\sqrt{2m + \frac{4}{3}}. \tag{2.31}$$

Table 5 presents some results for $f''(0)$ obtained by numerical scheme and using relation Eq. (2.31) for some values of M when $n = 1$. It is observed that the agreement is excellent.

M	Eq. (2.9)	Eq. (2.31)
0	-1.1547	-1.1547
1	-1.8257	-1.8257
10	-4.6188	-4.6188
100	-14.1892	-14.1892
1000	-44.7363	-44.7363
10000	-141.4261	-141.4261

Table 5: Values of $f''(0)$ for $n = 1$.

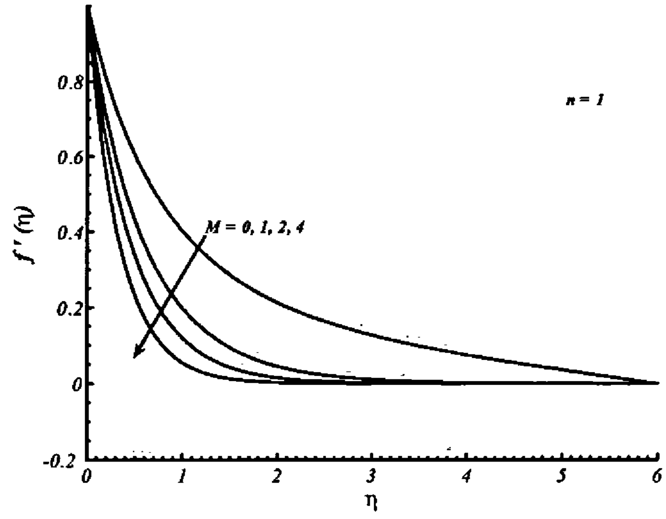

Fig. 2.10 : Effects on velocity profiles for different values of M when $n = 1$.

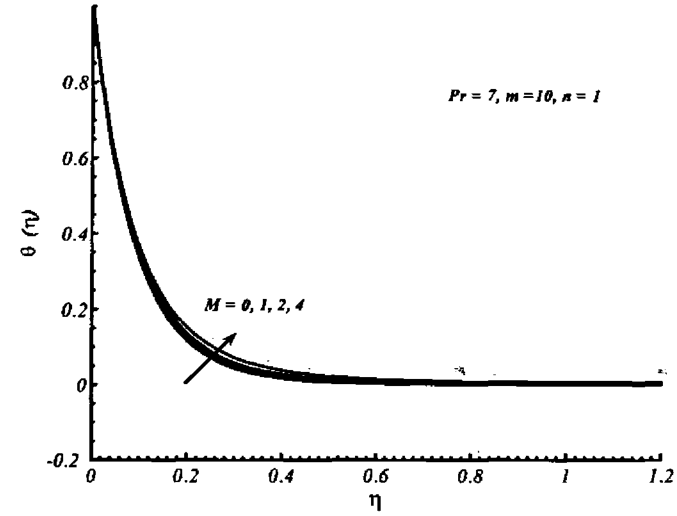

Fig. 2.11 : Effects on temperature profiles for different values of M when$Pr = 7, m = 10$ and $n = 1$.

2.3.4 Case IV: For $n = -1$:

In this case Eq.(2.9) has the exact analytical solution as

$$f(\eta) = \frac{1 - e^{-\sqrt{2(m-1)}\eta}}{\sqrt{2(m-1)}}. \tag{2.32}$$

Chapter 3

Mixed convection in the stagnation point flow due to a moving extensible surface

In this chapter, we investigate the steady, two-dimensional mixed convection MHD stagnation point flow of an incompressible fluid over a moving extensible surface. Moving extensible surface obeys a more general law of stretching is considered. The sheet occupies the negative x-axis and is moving continuously in the positive x-direction. The governing system of nonlinear partial differential equations are transformed into the ordinary differential equations by similarity transformation. In order to solve the ODE's numerically, we use a finite difference scheme Keller box and shooting method with Runge-Kutta fourth order scheme. The effects of various parameters on the flow and heat transfer is analyzed and discussed through graphs in detail. This chapter is an extension of the work done by Ishak et al. [20], discussed in chapter 2.

3.1 Mathematical formulation

Consider a moving extensible sheet occupies the negative x-axis and is continuously moving in the positive x-direction which is placed in an ambient fluid of uniform temperature T_∞. We consider the steady two dimensional stagnation-point flow of an incompressible, electrically conducting viscous fluid caused by moving extensible sheet. The velocity, the temperature of the moving sheet are the same which is already discussed in previous chapter. Under these assumptions together with the Bouissinesq approximation and boundary layer approximation, the equations for steady two dimensional MHD stagnation point flow over the stretching surface are Eq. (2.1), Eq. (2.2) reduce to the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+U\frac{dU}{dx}-\sigma\frac{B^2(x)}{\rho}u+g\beta(T-T_\infty), \tag{3.1}$$

where the equation of heat transfer in the boundary layer over the stretching surface is same as Eq. (2.3). Where y-axis is taken as normal to the sheet and fluid is occupying the space $y>0$, g is the gravitational acceleration and β is the thermal expansion coefficient. Let us assume the appropriate boundary conditions as

$$\left.\begin{array}{c} u=u_s(x),\ v=0,\ T=T_w(x)\ at\ y=0,\\ u=u_e(x),\ v=0,\ T\rightarrow T_\infty \text{ as } y\rightarrow\infty,\\ u\rightarrow 0,\ T\rightarrow T_\infty \text{ as } x\rightarrow-\infty \end{array}\right\}. \tag{3.2}$$

Where

$$u_e(x)=\left(\frac{x_0}{|x|}\right)^n u_\infty,\ u_s(x),\ T_w(x),\ B(x)\ . \tag{3.3}$$

are clearly defined in previous chapter. Here u_0 is characteristic velocity, $u_e(x)$ is the velocity at the edge of the boundary layer of thickness δ and u_∞ is ambient fluid velocity.

After introducing the similarity transformation

$$\eta=y\sqrt{\frac{u_e}{2\,|x|\,\nu}},\ \psi=\sqrt{2\nu\,|x|\,u_e}f(\eta),\ \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}. \tag{3.4}$$

After introducing Eq. (3.4).we get following non-linear ordinary differential equations

$$f''' + (n-1)ff'' + 2n(\epsilon^2 - f'^2) + 2M(\epsilon - f') + \lambda\theta = 0, \tag{3.5}$$

However, the heat transfer equation is same as in Eq. (2.10). Boundary conditions reduce to the form

$$\left.\begin{array}{c} f(0)=0,\ f'(0)=1,\ f'(\infty)=\epsilon, \\ \theta(0)=1,\ \theta(\infty)=0 \end{array}\right\}. \tag{3.6}$$

Where $\varepsilon = \frac{u_\infty}{u_0}$ is stagnation point parameter.

3.2 Numerical solution

Eqs. (3.5) and (2.10) subject to boundary conditions Eq. (3.6) are solved numerically by using Keller box method [22] and shooting method [23] for different values of parameters, λ (mixed convection parameter) and ε.

3.3 Results and discussion

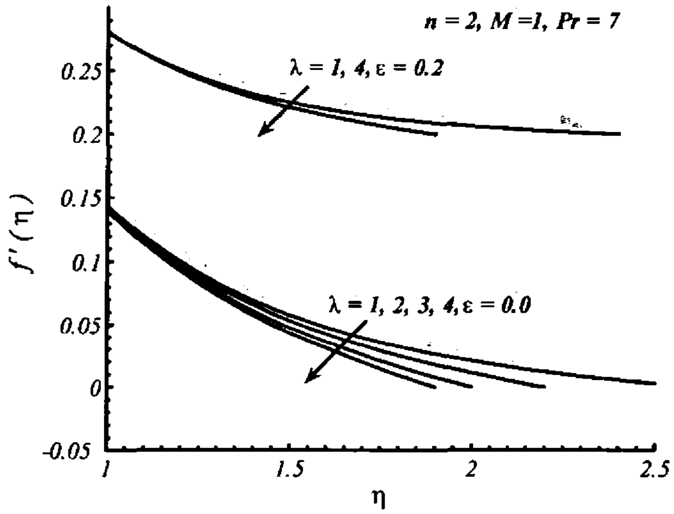

Fig. 3.1 : Velocity profiles for different values of λ.

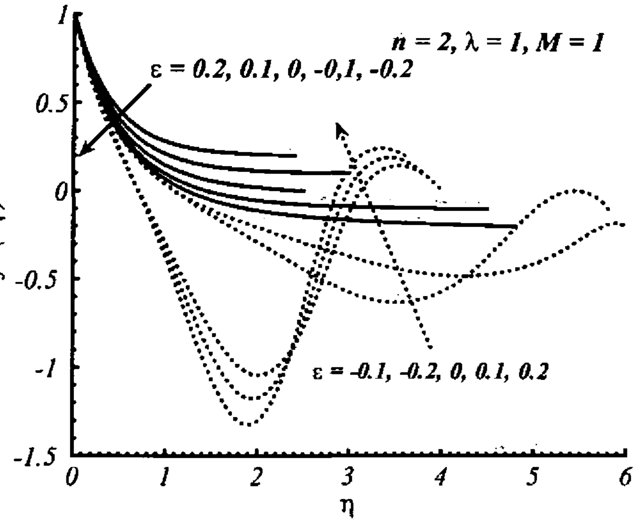

Fig. 3.2 : Velocity profiles for different values of ϵ.

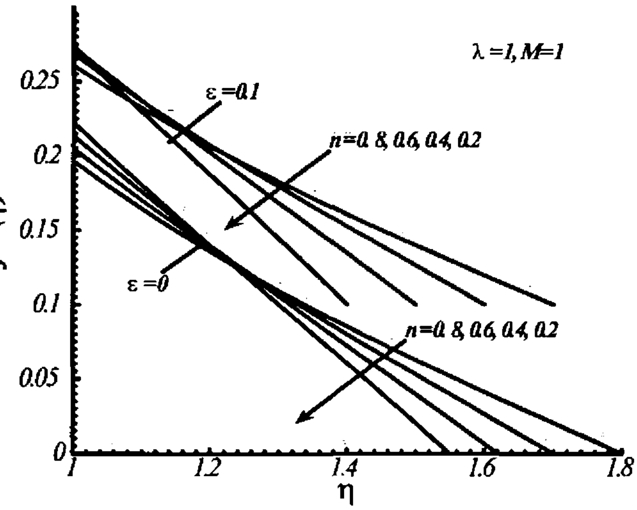

Fig. 3.3 : Velocity profiles for different values of n when

$\varepsilon = 0$ and $\varepsilon = 0.1$.

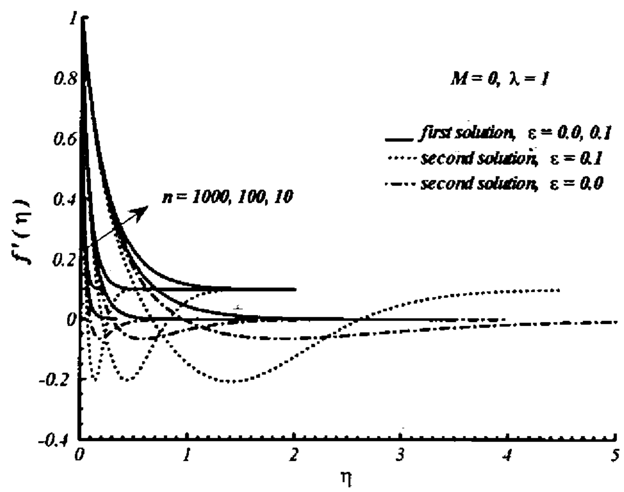

Fig. 3.4 : Velocity profiles for different values of n

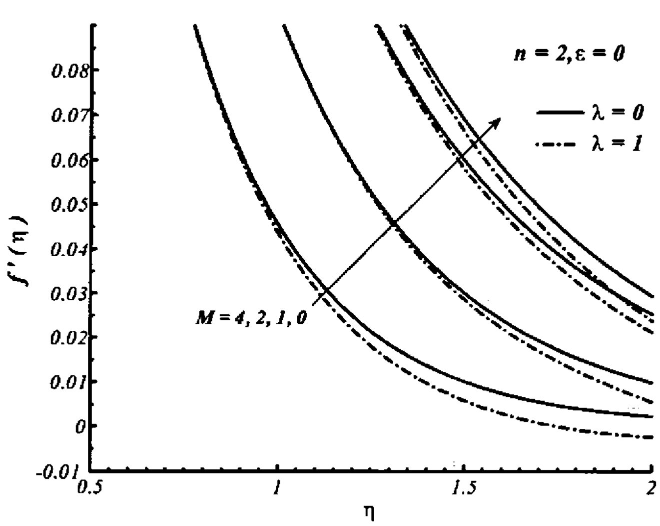

Fig. 3.5 : Velocity profiles for different values of magnetic parameter M when $\lambda = 0$ and $\lambda = 1$.

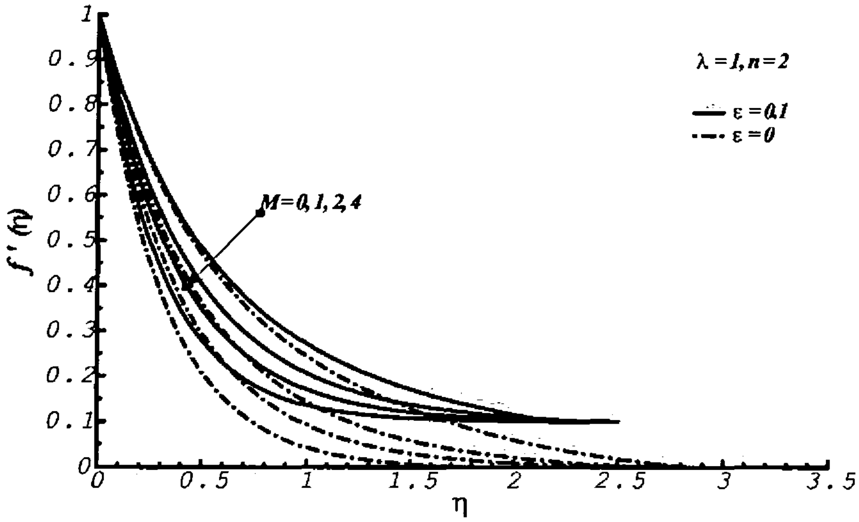

Fig. 3.6 : Velocity profiles for different values of M when $\varepsilon = 0$ and $\varepsilon = 0.1$.

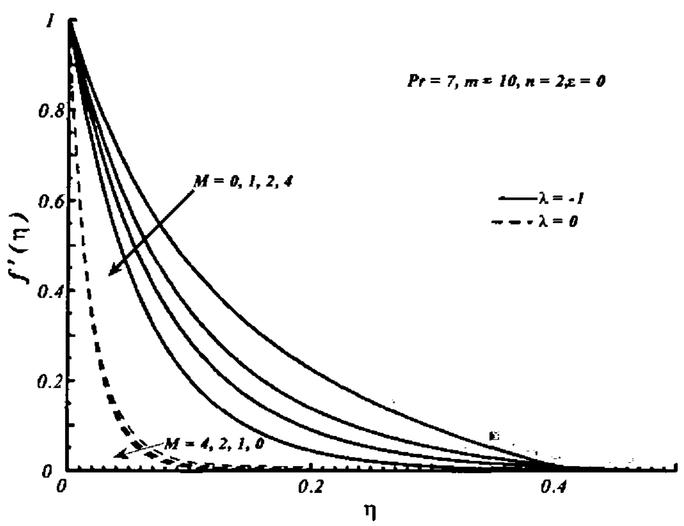

Fig. 3.7 : Velocity profiles for different values of M when $\lambda = 0$ and $\lambda = -1$.

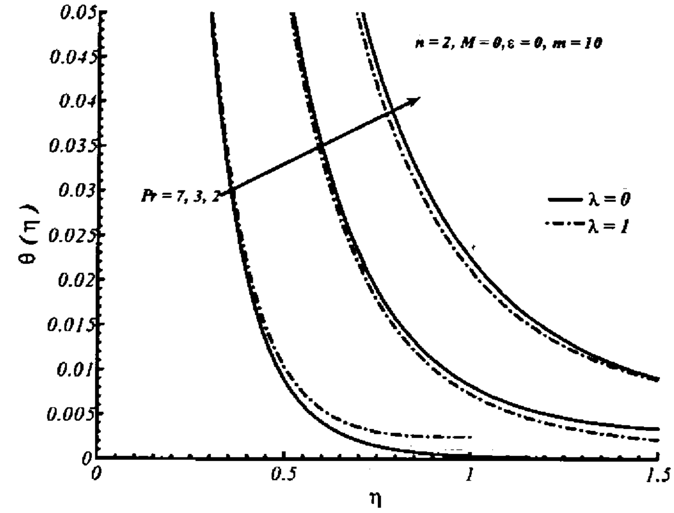

Fig. 3.8 : Temperature profiles for different values of Pr when $\lambda = 0$ and $\lambda = 1$.

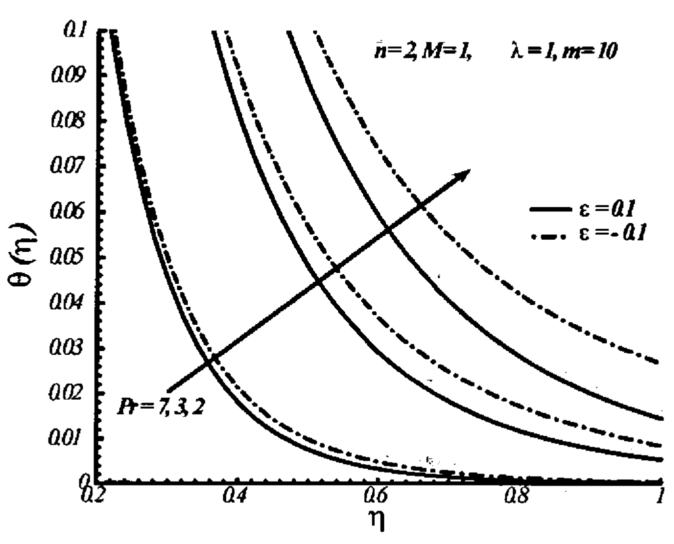

Fig. 3.9 : Temperature profiles for different values of Prandtle number Pr when $\varepsilon = 0.1$ and $\varepsilon = -0.1$.

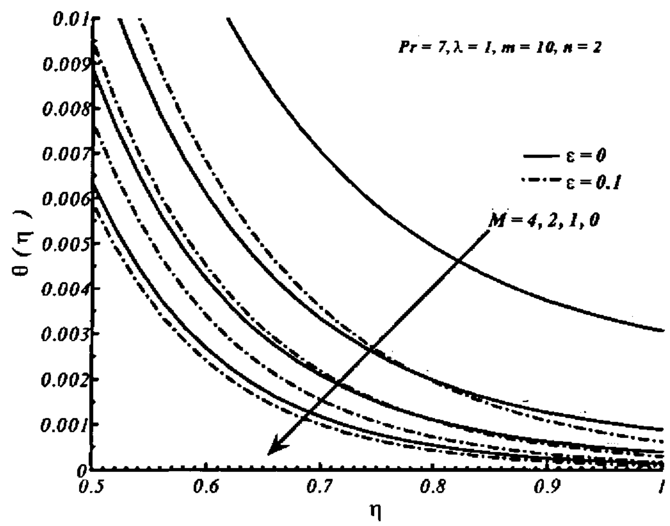

Fig. 3.10 : Temperature profiles for different values of M, when $\varepsilon = 0$ and $\varepsilon = 0.1$.

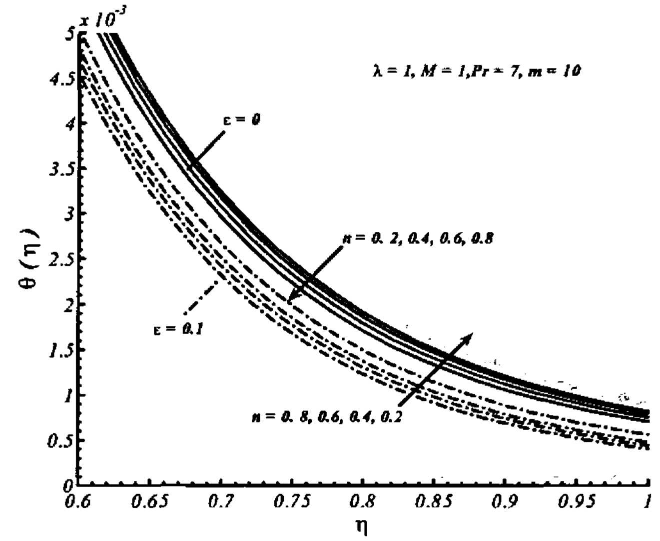

Fig. 3.11 : Temperature profiles for different values of n, when $\varepsilon = 0$ and $\varepsilon = 0.1$.

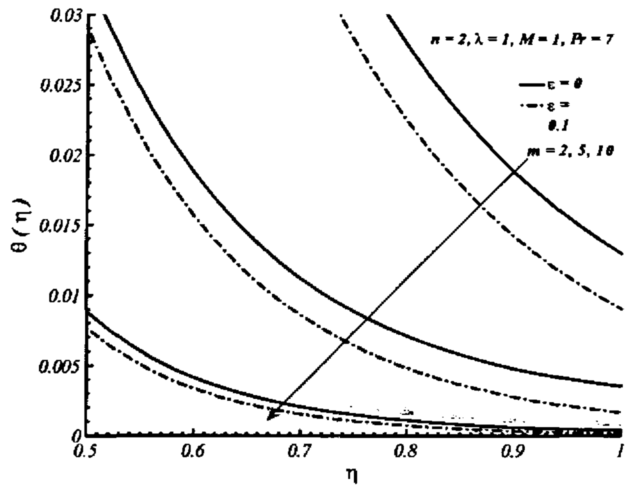

Fig. 3.12 : Temperature profiles for different values of m, when $\varepsilon = 0$ and $\varepsilon = 0.1$.

To investigate the effects of pertinent parameters M, n, λ, ε and Pr on the velocity and temperature profiles, Figures 3.1 to 3.12 are drawn. By increasing the mixed convection parameter λ, the boundary layer thickness decreases, whatever ε may be, as shown in Fig. 3.1. In turn velocity gradients at the surface increases by increasing the mixed convection parameter. It is important to mention here that the effects of different values on velocity profiles are not much pronounced, therefore , we draw graphs for η between 1 to 2.5 and range is minimized to 0.25, so that its effects can be considered for discussion. Further more, it is observed that dual solution exist for MHD mixed convection stagnation point flow over an extensible surface. Fig. 3.2 ensures the existence of dual solution for different values of $\epsilon = 0$ and $\epsilon \neq 0$. It is already discussed in previous chapter about the existence of the solution for $\lambda = 0$ and $\varepsilon = 0$. Fig.3.3 illustrate the effect of different values of n for $\varepsilon = 0$ and $\varepsilon = 0.1$. Again in the figure we reduced the domain for η and range for graph for clear observation. It is observed that boundary layer thickness reduces with the increase in the index n. The effects of large values of index n on the velocity profiles are shown in Fig. 3.4. Again, in this figure dual solution are drawn for $\varepsilon = 0$ and $\varepsilon = 0.1$. Solid lines and dotted lines represents the curves for different values of $n = 1000, 100$ and 10 for $\varepsilon = 0.1$. However, dash-dot-dashed lines and solid lines are for $\varepsilon = 0$

when $n = 1000, 100$ and 10. It is observed that either $\varepsilon = 0$ or $\varepsilon \neq 0$, boundary layer thickness decreases with increase in index n. By increasing magnetic parameter M, the boundary layer thickness decreases and the velocity gradients at the surface increases as shown in figure 3.5 for $\lambda = 0$ and $\lambda = 1$. It is further ensured that the boundary layer thickness further reduces, when mixed convection is introduced for any value of the magnetic parameter M in the Fig. 3.5. For non zero value of the stagnation parameter ε, it is seen that boundary layer thickness reduces due to increase in magnetic parameter M as shown in Fig. 3.6. For negative value of mixed convection parameter $\lambda = -1$, effects of magnetic parameter M on the velocity profile is shown in Fig. 3.7. For comparison only, $\lambda = 0$ is considered here. It is observed that as soon as values of mixed convection parameter λ goes from zero to negative, boundary layer thickness increases in this case and in turn velocity gradient at the surface decreases. The effects of these pertinent parameters on the temperature profiles are shown in the Figs. 3.8 to 3.12. Increase in the values of prandtle number Pr helps to decrease boundary layer thickness as shown in Fig 3.8. The effects of prandtle number Pr and $\varepsilon = 0.1$and $\varepsilon = -0.1$ on the temperature profiles are shown in Fig. 3.9. It is seen that due to negative value of ε, thermal boundary layer thickness increases for each value of Pr. Figure 3.10 illustrates the effects of magnetic parameter M on temperature profiles. It is interesting to note that by increasing M, thermal boundary layer thickness increases, as in this case, it is observed that stagnation parameter can control the thermal boundary layer, as shown in the Fig. 3.10. A very low pronounced effects of small n for $\varepsilon = 0$ and $\varepsilon = 0.1$ on temperature profiles are shown in the Fig. 3.11. By increasing the index n, temperature profile decreases both for $\epsilon = 0$ and $\epsilon \neq 0$. In last Fig. 3.12, it is shown that by increasing m for $\epsilon = 0$ and $\epsilon = 0.1$, thermal boundary layer thickness decreases and in turn heat transfer rate increase for $\epsilon = 0$ and $\epsilon = 0.1$.

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