# Characterizations of Ternary Semirings by their Soft Intersectional Ideals

By

**Asim Zamir**

**Department of Mathematics and Statistics**
**Faculty of Basic and Applied Sciences**
**International Islamic University, Islamabad, Pakistan.**

**2016**

1. Semirings (Mathematics)
2. Logic, symbolic and mathematical

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*A Dissertation*
*Submitted in the Partial Fulfillment of the*
*Requirements for the Degree of*
***MASTER OF SCIENCE***
***In***
***MATHEMATICS***

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# Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF SCIENCE in MATHEMATICS***

We accept this dissertation as conforming to the required standard

1 ____________________
Dr Munir Ahmed
(External Examiner)

2 ____________________
Dr Nayyar Mehmood
(Internal Examiner)

3 ____________________
Dr Tahir Mahmood
(Supervisor)

4 ____________________
Prof Dr M Arshad Zia
(Chairman)

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad,
Pakistan.
2016

# DECLARATION

I hereby, declare, that this thesis neither as a whole nor as apart thereof has been copied out from any source It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning

**Asim Zamir**
MS Mathematics
Reg No 174-FBAS/MSMA/S-14
Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad, Pakistan

*Dedicated*

*to*

*my*

*loving Parents*

*And My Friends*

*And respectful teachers.*

# *Acknowledgements*

All praises to almighty "**ALLAH**" The creator of the universe, who blessed me with the knowledge and enabled me to complete this dissertation All respects to **Holy prophet MUHAMMAD (P.B.U.H)**, who is the last messenger, whose life is a perfect model for the whole humanity

I express my deep sense of gratitude to my supervisor **Dr. Tahir Mahmood** (assistant professor IIU, Islamabad) for his thought provoking untiring and patient guidance during the course of this work Indeed, I could not complete my thesis without his inspiring suggestions, encouragement, active participation and guidance at every stage of my research work

I pay my thanks to whole faculty of my department I also feel much pleasure in acknowledging nice company of my friends in university

I also thanks especially to **Usman Tariq** and my research fellows **Mohsin Ali Khan** and **Azhar Rauf Khan** who support and encourage me directly or indirectly during my research work

My deepest sense of indebtedness goes to my beloved parents who always pray for me Words are not adequate to express the love and support of my parents for their constant encouragement and moral support during my whole educational life and particularly in my research work.

## Structure of the Thesis

The thesis is organized chapter wise as follows:

**Chapter 1:**

This chapter is introductory and sets up the background for the problems taken in the thesis. It overviews ideals in Semirings, k-ideals in Semirings, ideals in Ternary Semirings, k-ideals in Ternary Semirings, Soft Sets, Soft-Union-Intersection Sum, Soft-Union-Intersection Product, Soft Intersectional k-Ideals in Semirings and related results are discussed.

**Chapter 2:**

This chapter contains the discussion of the [X,Y] Soft Intersection k-subsemirings, [X,Y] Soft Intersectional k-ideals. In the last section of the chapter, we investigate some important results of k-regular semirings in terms of [X,Y] Soft Intersectional k-ideals.

**Chapter 3:**

In this chapter, we introduce Soft Intersectional k-ternary subsemirings, Soft Intersectional k-ternary Ideals. In the last section of the chapter, we introduce [X,Y] Soft Intersectional k-ternary subsemirings, [X,Y] Soft Intersection k-ternary Ideals, k-regular ternary semirings and most concerning results are investigated.

# Contents

# Preface

In our daily life, we have to face the uncertain situations but if we measure the parameters of uncertainty scientifically then we can take decisions comparatively in a better way There exist different types and different handling methods for uncertainties According to scientific method, it is a big challenge for us to deal with uncertainties The uses of automata theory, rough theory, fuzzy theory and probability theory in the domain of economics, engineering, information science and medical science are described

The process of developing the new techniques to deal uncertainties is rapidly growing by different researches In this regard Molodtsov [23] defined the soft set theory to deal particular type of uncertainties The algebraic structures by using soft sets is extensively using to elaborate the practical implication of uncertainties in a wide range of scientific fields including mentioned above That's why many researchers taking keen interest in various fields of algebra and broading its limits by using soft set theory Some initial and basic operations on soft sets defined by P K Maji [22], M I Ali [2] and Sezgin and Atagun [24] played the important role in the further progress in this field See [1,4]

In 1934 Vandiver [26] first introduced the concept of semiring Infact semiring is the generalization of ring Dutta and Kar [12] made research in 2003 about the concept of ternary semiring, regular ternary semiring and k-regular ternary semiring and their major properties The research about the quasi-ideals and bi-ideals of ternary semi-group is made by Dixit and Dewan [5] S Kar [16] made research about the properties of quasi-ideal and bi-ideal in ternary semiring M K Dubey [6] defines k-quasi ideals and k- bi ideals in ternary semirings and investigated some related properties

Soft ideals in Soft semiring were defined by F Feng et al [14] Further, X Ma and J Zhan [19] define (M,N)-soft union ideals in (M,N)-soft union subsemirings T Mahmood and U Tariq [20] further discussed (X,Y)-soft intersection subsemirings in terms of (X,Y)-soft intersection bi ideals and (X,Y)-soft intersection quasi ideals After that soft intersectional ternary semirings in terms of soft intersectional ternary ideals was discussed by T Mahmood et. al [21]

In our work we discussed soft intersectional sets in ternary semirings by using ternary k-ideals We define soft union-intersection sum and soft union-intersection product We introduce the concept of soft intersectional ternary k-subsemirings, soft intersectional ternary k-ideals, soft intersectional ternary k-quasi ideals and soft intersectional ternary k-bi ideals of ternary semiring Furthermore, we characterize k-regular ternary semirings by using soft intersectional ternary k-bi ideals and soft intersectional ternary k-quasi ideals Finally, we discuss [X,Y] soft intersectional ternary k-subsemirings, [X,Y] soft intersectional ternary k-ideals in ternary semirings

# Chapter 1

## 1. Preliminaries

In this chapter, we will discuss the concerned definitions with examples and some important results that are used in next chapters For undefined terms and notions we refer to [4, 6,13, 16, 21, 23, 26]

### 1.1. Semirings

#### 1.1.1. Definition

A non-empty set $R$ together with two binary operations " + " and " " is said to be semiring if it satisfies the following conditions

(i) $(R,+)$ is a semigroup

(ii) $(R,\ )$ is a semigroup

(iii) " " distributes over " + " from both sides

Then we can write $(R,+,\ )$ or simply $R$ is a semiring

#### 1.1.2. Example

$N_0$ (the set of all non negative integers) under the binary operations of ordinary addition and ordinary multiplication is a semiring

**1.1.3. Definition**

$R$ is called commutative if " " is commutative in $R$.

**1.1.4. Definition**

An element $0 \in R$, satisfying $0\ l = l\ 0 = 0$ and $0 + l = l + 0 = l\ \forall\ l \in R$. is called zero (or obsorbing element) of $R$

**1.1.5. Definition**

An element $1 \in R$, satisfying the condition $1.l = l.1 = l\ \forall\ l \in R$ is called identity of the semiring $R$

**1.1.6. Definition**

A subset $A \neq \phi$ of a semiring $R$ is called a subsemiring of $R$ if $A$ itself is a semiring under the operations inherited from $R$

**1.1.7. Theorem**

A subset $A \neq \phi$ of a semiring $R$ is called a subsemiring of $R$ if we have $l + m \in A$ and $l.m \in A, \forall\ l, m \in A$.

**1.1.8. Remark**

From now to onward, else or otherwise stated, for $l, m \in R$, instead of writing $l.m$ we will write $lm$

**1.1.9. Examples**

1 All rings are semirings with subrings as subsemirings

2 The set $R$ of all $n \times n$ matrices with entries from non-negative real numbers is a semiring with usual binary operations of addition " + " and multiplication " " of matrices

3 The set of whole numbers as well as the set of non-negative rational numbers are commutative semirings under usual addition and multiplication of real numbers The set of whole numbers is a subsemiring of the set of non-negative rational numbers

**1.1.10. Definition**

For $A, B \subseteq R$. We defne

$A + B = \{l + m \ l \in A, m \in B\}$ and

$$AB = \left\{ \sum_{finite} l_i m_i \ l_i \in A, m_i \in B \right\}$$

**1.1.11. Definition**

An element $l \in R$ is called multiplicatively idempotent if $ll = l^2 = l$.

**1.1.12. Definition**

$R$ is said to be multiplivatively idempotent if each element of $R$ is multiplivatively idempotent

**1.1.13. Definition [15]**

If $\phi \neq I \subseteq R$ satisfying the $I + I \subseteq I$ and $RI \subseteq I$ $(IR \subseteq I)$, then $I$ is called left (right) ideal of $R$ If $I$ is both left and right ideal of $R$, then it is said to be ideal (or two-sided ideal) of $R$

**1.1.14. Definitin**

Let $R$ be a semiring and $\phi \neq B \subseteq R$ Then $B$ is called bi-ideal of $R$ if $B$ is subsemiring of $R$ and satisfying $BRB \subseteq B$

**1.1.15. Example**

Let R $= \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\}$ be a semiring under the addition and multiplication of matrices

Then $B_1 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\}$ and $B_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right\}$ are the bi-ideals of R

**1.1.16. Definition**

Let $R$ be a semiring If $\phi \neq Q \subseteq R$ satisfying $Q + Q \subseteq Q$ and $RQ \widehat{\cap} QR \subseteq Q$, then $Q$ is called quasi-ideal of $R$, where $\widehat{\cap}$ denotes the intersection of sets

**1.1.17. Example**

Let $R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z^0\right\}$ be a semiring under the addition and multiplication of matrices

Then $Q_1 = \left\{\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \mid u \in Z^0\right\}$ and $Q_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} \mid v \in Z^0 \right\}$ are the quasi-ideals of $R$

**1.1.18. Definition**

A semiring $R$ is called Von-Neumann regular if we for any $a \in R$ there exist $x \in R$ such that $axa = a$ or we have $a \in aRa$ for all $a \in R$

**1.1.19. Definition [25]**

A semiring $R$ is called left (right) weakly regular if we have $a \in RaRa$ $(a \in aRaR)$ for all $a \in R$

**1.1.20. Remarks**

Clearly, if $R$ is commutative, then $R$ is (right or left) weakly regular if and only if $R$ is Von-Neumann regular

**1.1.21. Theorem [25]**

A semiring $R$ is Von-Neumann regular if and only if for any right ideal $A$ and left ideal $B$ of $R$, $A \widehat{\cap} B = AB$

## 1.2. Ternary Semirings

**1.2.1. Definition [12]**

A non-empty set $S$ together with a binary operation of addition " + " and ternary multiplication" " respectively, is said to be a ternary semiring if $(S, +)$ is a commutative

semigroup satisfying the following conditions

(i) $(abc)de = a\,(bcd)e = ab(cde)$

(ii) $(a + b)cd = acd + bcd$

(iii) $a(b + c)d = abd + acd$

(iv) $ab(c + d) = abc + abd \;\; \forall a, b, c, d, e \in S$

Ternary semiring is represented by $T_{sr}$ We can see that any semiring can be reduced to a $T_{sr}$ However, a $T_{sr}$ does not necessarily reduce to a semiring by this example We consider $Z^-$ the set of all negative integers under usual addition and multiplication, we see that $Z^-$ is an additive semigroup which is closed under the ternary multiplication but is not closed under the binary multiplication Moreover, $Z^-$ is a ternary semiring but it is not a semiring under usual addition and multiplication

### 1.2.2. Remarks

From now to onward, else or otherwise stated, $S$ will denoted ternary semiring

### 1.2.3. Definition [13]

An additive semigroup $U$ of $S$ is called a ternary subsemiring of $S$ if for all $u_1, u_2, u_3 \in U$ then $u_1 u_2 u_3 \in U$

### 1.2.4 Definition [12]

A $T_{sr}$ S is said to be commutative if

$$abc = bac = bca \qquad \forall\, a, b, c \in S$$

### 1.2.5. Definition

Let $S$ be a $T_{sr}$ If there exist an element $0 \in S$ such that

(i) $0 + x = x$

(ii) $0xy = x0y = xy0 = 0$

$\forall\, x, y \in S$, then "0" is called the zero element of the ternary semiring S

### 1.2.6. Example

In a $T_{sr}$ $Z$, the set of all integers "0" is called the zero of $Z$ because for any $a, b \in Z$

(i) $0 + a = a$

(ii) $0ab = a0b = ab0 = 0$

**1.2.7. Definition [12]**

$T_{sr}$ with 1 means $11x = 1x1 = x11 = x \; \forall \, x \in S$.

**1.2.8. Definition** [12]

An element $l \in S$ is called multiplicatively idempotent If $lll = l^3 = l$

**1.2.9. Definition** [12]

An element $l \in S$ is called additively idempotent if $l + l = l$

**1.2.10. Definition** [13]

An additive subsemigroup $I$ of $S$ is called left (right, lateral) ideals of S

if $s_1s_2\iota \in I \; (\iota s_1s_2 \in I, s_1\iota s_2 \in I) \; \forall \, s_1, s_2 \in S$ and $\iota \in I$ If $I$ is a left, right and lateral ideal of $S$, then $I$ is called an ideal of $S$

**1.2.11. Definition [16]**

An additive subsemigroup $Q$ of a $T_{sr}$ $S$ is called a quasi-ideal of $S$ if

$QSS \,\widehat{\cap}\, (SQS + SSQSS) \,\widehat{\cap}\, SSQ \subseteq Q$

**1.2.12. Definition [16]**

Let $S$ be a $T_{sr}$ and $\phi \neq B \subseteq S$ Then $B$ is called a bi-ideal of $S$ if $B$ is subsemigroup of $S$ and satisfying $BSBSB \subseteq B$.

**1.2.13. Definition [12]**

A $T_{sr}$ is called regular if for any $a \in S, \exists\, x \in S$ such that such that $a = axa$ or $a \in aSa\ \forall a \in S$

**1.2.14. Theorem [12]**

A $T_{sr}$ is regular if for any right ideal $A$ and left ideal $B$, and lateral Ideal $C, A \widehat{\cap} B \widehat{\cap} C = ABC$

**1.2.15. Definition [13]**

$S$ is called weakly left ( right , lateral) regular if we have

$$l \in S(lS)^3 (l \in (lS)^3 S, l \in SSlSlSlSS), \forall\, l \in S$$

**1.2.16. Definition [6]**

For $\phi \neq K \subseteq S$, the k-closure of $K$ is denoted and defined as $\overline{K} = \{u \in S | u + g = h \text{ for some } g, h \in K\}$.

**1.2.17. Definition [6]**

A ternary subsemiring (left ideal, right ideal, lateral ideal, ideal, quasi ideal, bi-ideal) $K$ of $S$ is called ternary k-subsemiring ( k-left ideal, k-right ideal, k-lateral ideal, k-ideal, k-quasi ideal, k-bi ideal) of $S$, respectively, if $u + g = h$ implies $u \in S$ and $g, h, \in K$, it will be represented by $\mathbb{K}_{t-SS}(\mathbb{K}_{t-L_l}, \mathbb{K}_{t-R_l}, \mathbb{K}_{t-La_l}\ \mathbb{K}_{t-I}, \mathbb{K}_{t-Q_l}, \mathbb{K}_{t-B_l})$ respectively

**1.2.18. Definition [6]**

A $T_{sr}$ is said to be k-regular if for each $u \in S$ there exist $g, h \in S$ such that $u + ugu = uhu$ It will be represented by $\mathbb{K}_{t-r}$

**1.2.19. Theorem [6]**

A $T_{sr}$ is k-regular if and only if $E \mathbin{\widehat{\cap}} F \mathbin{\widehat{\cap}} G = EFG$, for any E, F and G as $\mathbb{K}_{t-R_i}$, $\mathbb{K}_{t-La_i}$ and $\mathbb{K}_{t-L_i}$ of S respectively

**1.2.20. Theorem [10]**

The following conditons in a $T_{sr}$ are equivalent

(i) $S$ is $\mathbb{K}_{t-r}$

(ii) For every $\mathbb{K}_{t-B_i}$ $B$ of $S$, $B = \overline{BSBSB}$

(iii)For every $\mathbb{K}_{t-Q_i}$ $Q$ of $S$, $Q = \overline{QSQSQ}$

**Proof:** Straightforward

## 1.3. Soft Intersectional k-Ideals in Semirings

**1.3.1. Definition [23]**

If Z be the initial universal set, E is a set of parameters and $A, B, C, \quad \subseteq E =$

$R$

An ordered set $\hat{\lambda}_A = \{\left(u, \hat{\lambda}_A(u)\right)\ u \in E, \hat{\lambda}_A(u) \in P(Z)\}$ over Z, is known as a soft set, where $\hat{\lambda}_A : E \to P(Z)$ such that $\hat{\lambda}_A(u) = \Phi$ if $u \notin A$, $\hat{\lambda}_A$ is called approximate function

Note that, the collection of all soft sets over Z will be denoted by CS (Z)

**1.3.2. Definition [3]**

The upper inclusion set of $\hat{\lambda}_A$ for $\alpha \neq \phi$ and $A \subseteq R$ is denoted and defined as

$$U(\hat{\lambda}_A, \alpha) = \{x \in A \mid \hat{\lambda}_A(u) \supseteq \alpha\}$$

**1.3.3. Definition [19]**

For $\hat{\lambda}_A, \hat{\lambda}_B \in CS(Z)$

(i) $\hat{\lambda}_A \subseteq \hat{\lambda}_B$ if $\hat{\lambda}_A(u) \subseteq \hat{\lambda}_B(u), \forall\, u \in R$

(ii) $\hat{\lambda}_A \cup \hat{\lambda}_B = \hat{\lambda}_{A\cup B}$ where $\hat{\lambda}_{A\cup B}(u) = \hat{\lambda}_A(u) \cup \hat{\lambda}_B(u), \forall\, u \in R$

(iii) $\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B = \hat{\lambda}_{A\hat{\cap}B}$ where $\hat{\lambda}_{A\hat{\cap}B}(u) = \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_B(u), \forall\, u \in R$

**1.3.4. Definition [20]**

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS(Z)$ Then soft union-intersection sum is defined by

$$(\hat{\lambda}_A \oplus \hat{\lambda}_B)(u) = \begin{cases} \bigcup_{u+(x_1+y_1)=(x_2+y_2)} \{\hat{\lambda}_A(x_1) \mathbin{\hat{\cap}} \hat{\lambda}_A(x_2) \mathbin{\hat{\cap}} \hat{\lambda}_B(y_1) \mathbin{\hat{\cap}} \hat{\lambda}_B(y_2)\} \\ \Phi \quad \text{if } u \text{ cannot be expressed as } u + (x_1 + y_1) = (x_2 + y_2) \end{cases}$$

### 1.3.5. Definition [20]

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS(Z)$ Then soft union-intersection product is denoted and defined as

$$(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B)(u) = \begin{cases} \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ (\bigcap_{i=1}^{m} \hat{\lambda}_A(g_i) \hat{\cap} \hat{\lambda}_B(h_i)) \hat{\cap} (\bigcap_{j=1}^{n} \hat{\lambda}_A(g_j) \hat{\cap} \hat{\lambda}_B(h_j)) \right\} & \\ \Phi & \text{if } u \text{ cannot be expressed as } u + \sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j \end{cases}$$

$$\forall u \in R$$

### 1.3.6. Definition [20]

Let $R$ be a semiring and $\phi \neq G \subseteq R$ Then characteristic soft set is denoted and defined by

$$\mathcal{C}_G(x) = \begin{cases} Z & \text{if } x \in G \\ \phi & \text{if } x \in R \backslash G \end{cases}$$

The soft set $\mathcal{C}_R \in CS(Z)$ is called identity soft set and is denoted by $\mathbb{C}$

### 1.3.7. Lemma [19]

Let $R$ be a semiring and $G, H \subseteq R$ Then we have

(i) $G \subseteq H \Leftrightarrow \mathcal{C}_G \subseteq \mathcal{C}_H$

(ii) $\mathcal{C}_G \hat{\cap} \mathcal{C}_H = \mathcal{C}_{G \hat{\cap} H}$

(iii) $\mathcal{C}_G \hat{\odot} \mathcal{C}_H = \mathcal{C}_{GH}$

### 1.3.8. Definition [20]

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional k-subsemiring of $R$ if it satisfies

(i) $\hat{\lambda}_A(u+v) \supseteq \hat{\lambda}_A(u) \,\widehat{\cap}\, \hat{\lambda}_A(v)$ $\forall\, u,v \in \mathrm{R}$

(ii) $\hat{\lambda}_A(uv) \supseteq \hat{\lambda}_A(u) \,\widehat{\cap}\, \hat{\lambda}_A(v)$ $\forall\, u,v \in R$

(iii) If $u+g=h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \,\widehat{\cap}\, \hat{\lambda}_A(h)$

Soft intersectional k-subsemiring is represented by $\hat{\lambda}_{A_{k-SS}}^{\;si}$

### 1.3.9. Definition [20]

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional k-left ideal ( k-interior ideal, k-right ideal ) If it satisfies

(i) $\hat{\lambda}_A(u+v) \supseteq \hat{\lambda}_A(u) \,\widehat{\cap}\, \hat{\lambda}_A(v)$ $\forall\, u,v \in R$

(ii) $\hat{\lambda}_A(uv) \supseteq \hat{\lambda}_A(v)\ \{\hat{\lambda}_A(uv) \supseteq \hat{\lambda}_A(u), \hat{\lambda}_A(uvw) \supseteq \hat{\lambda}_A(v)\}$ $\forall\, u,v \in R$

(iii) If $u+g=h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \,\widehat{\cap}\, \hat{\lambda}_A(h)$

Soft intersectional k-left ideal ( k-interior ideal, k-right ideal ) is represented by $\hat{\lambda}_{A_{k-L_i}}^{\;si}$ ($\hat{\lambda}_{A_{k-Ia_i}}^{\;si}$ and $\hat{\lambda}_{A_{k-R_i}}^{\;si}$) respectively

A soft set $\hat{\lambda}_A \in CS(Z)$ is called soft intersectional k-ideal ($\hat{\lambda}_{A_{k-I}}^{\;si}$) if it is $\hat{\lambda}_{A_{k-L_i}}^{\;si}$ and $\hat{\lambda}_{A_{k-R_i}}^{\;si}$

### 1.3.10. Definition [20]

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional k-quasi ideal of $R$ if it satisfies

(i) $\hat{\lambda}_A(u+v) \supseteq \hat{\lambda}_A(u) \,\widehat{\cap}\, \hat{\lambda}_A(v)$ $\forall\, u,v \in R$

(ii) $\hat{\lambda}_A \supseteq (\hat{\lambda}_A \,\widehat{\odot}\, \mathbb{C}) \cap (\mathbb{C} \,\widehat{\odot}\, \hat{\lambda}_A)$

(iii) If $u+g=h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \,\widehat{\cap}\, \hat{\lambda}_A(h)$

Soft intersectional k-quasi ideal is represented by $\hat{\lambda}_{A_{k-Q_i}}^{si}$

**1.3.11. Definition [20]**

A $\hat{\lambda}_{A_{k-SS}}^{si} \in CS(Z)$ is called soft intersectional k-bi ideal of $R$ if it satisfies

$\hat{\lambda}_A(uvw) \supseteq \hat{\lambda}_A(u) \cap \hat{\lambda}_A(w) \quad \forall\, u, v, w \in R$

Soft intersectional k-bi ideal is represented by $\hat{\lambda}_{A_{k-B_i}}^{si}$.

It is obvious that $\hat{\lambda}_A(0) \supseteq \hat{\lambda}_A(u), \quad \forall\, u \in R$.

# Chapter 2

## 2. Generalized k-ideals in Semirings using Soft Intersectional Sets

In this chapter, we review the paper of T Mahmood and U Tariq [20] contains the study of [X,Y] soft intersectional k-ideals in semiring with investigation of the related results

### 2.1. [X,Y] soft intersectional k-ideals

Here we discuss [X,Y] soft intersectional k-subsemiring, [X,Y] soft intersectional k-ideals, [X,Y] soft intersectional k-quasi ideals, [X,Y] soft intersectional k-bi ideals, [X,Y] soft intersectional k-interior ideals and investigate some related results In our next discussion we use $\phi \subseteq X \subset Y \subseteq Z$

#### 2.1.1. Definition

$\hat{\lambda}_A \in CS(Z)$ is called $[X,Y]$ Soft intersectional k-subsemiring of $R$ if

(i) $\hat{\lambda}_A(u+v) \cup X \supseteq \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(v) \,\hat{\cap}\, Y \qquad \forall\, u,v \in R$

(ii) $\hat{\lambda}_A(uv) \cup X \supseteq \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(v) \,\hat{\cap}\, Y \qquad \forall\, u,v \in R$

(iii) If $u+g=h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \,\hat{\cap}\, \hat{\lambda}_A(h) \,\hat{\cap}\, Y$

$[X,Y]$ Soft intersectional k-subsemiring is represented by $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{SI}}$

### 2.1.2. Definition

$\hat{\lambda}_A \in CS(Z)$ is called $[X,Y]$ Soft intersectional k- left ideal (k-right ideal) if

(i) $\hat{\lambda}_A(u+v) \cup X \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y$ $\quad \forall\, u, v \in R$

(ii) $\hat{\lambda}_A(uv) \cup X \supseteq \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y$ $(\hat{\lambda}_A(uv) \cup X \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} Y)$

(iii) If $u+g=h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h) \mathbin{\hat{\cap}} Y$

$[X,Y]$ soft intersectional k- left ideal (k-right ideal) is represented by $\hat{\lambda}_{A_{k-L_i}}^{[X,Y]^{si}}$ $(\hat{\lambda}_{A_{k-R_i}}^{[X,Y]^{si}})$

A soft set $\hat{\lambda}_A \in CS(Z)$ is called $\hat{\lambda}_{A_{k-I}}^{[X,Y]^{si}}$ if it is $\hat{\lambda}_{A_{k-L_i}}^{[X,Y]^{si}}$ as well as $\hat{\lambda}_{A_{k-R_i}}^{[X,Y]^{si}}$ of $R$

### 2.1.3. Definition

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional k-quasi ideal of $R$ if it satisfies

(iv) $\hat{\lambda}_A(u+v) \cup X \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y$ $\quad \forall\, u, v \in R$

(v) $\hat{\lambda}_A(u) \cup X \supseteq (\hat{\lambda}_A \mathbin{\hat{\odot}} \mathbb{C})(u) \cap (\mathbb{C} \mathbin{\hat{\odot}} \hat{\lambda}_A)(u) \mathbin{\hat{\cap}} Y$

(vi) If $u+g=h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h) \mathbin{\hat{\cap}} Y$

Soft intersectional k-quasi ideal is represented by $\hat{\lambda}_{A_{k-Q_i}}^{[X,Y]^{si}}$

### 2.1.4. Definition

A $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{si}} \in CS(Z)$ is called soft intersectional k-bi ideal of $R$ if it satisfies

$$\hat{\lambda}_A(uvw) \cup X \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(w) \mathbin{\hat{\cap}} Y. \qquad \forall\, u, v, w \in R$$

Soft intersectional k-bi ideal is denoted by $\hat{\lambda}_{A_{k-B_i}}^{[X,Y]^{si}}$

### 2.1.5. Definition

A $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{si}} \in CS(Z)$ is called soft intersectional k-interior ideal of S if it satisfies

$$\hat{\lambda}_A(uvw) \cup X \supseteq \hat{\lambda}_A(v) \,\hat{\cap}\, Y. \qquad \forall\, u, v, w \in R$$

Soft intersectional k-interior ideal is denoted by $\hat{\lambda}_{A_{k-Ia_i}}^{[X,Y]^{si}}$

### 2.1.6. Definition

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS\,(Z)$ Then $\hat{\lambda}_A \subseteq \hat{\lambda}_B \Leftrightarrow (\hat{\lambda}_A(u) \,\hat{\cap}\, Y) \cup X \subseteq (\hat{\lambda}_B\,(u) \,\hat{\cap}\, Y) \cup X$

$\forall\, u \in R$

### 2.1.7. Definition

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS\,(Z)$ Then $\hat{\lambda}_A = \hat{\lambda}_B \Leftrightarrow \hat{\lambda}_A \subseteq \hat{\lambda}_B \text{ and } \hat{\lambda}_B \subseteq \hat{\lambda}_A$

Obviously, $\hat{\lambda}_A(0) \cup X \supseteq \hat{\lambda}_A(u) \,\hat{\cap}\, Y \text{ and } (\hat{\lambda}_A(0) \,\hat{\cap}\, Y) \cup X \supseteq (\hat{\lambda}_A(u) \,\hat{\cap}\, Y) \cup X$

$\forall\, u \in R$

### 2.1.8. Theorem

Let $\hat{\lambda}_A \in CS(Z)$. Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{si}}$ iff $\hat{\lambda}_A$ satisfies

(i) If $u + g = h \Rightarrow \hat{\lambda}_A\,(u) \cup X \supseteq \hat{\lambda}_A(g) \,\hat{\cap}\, \hat{\lambda}_A(h) \,\hat{\cap}\, Y$

(ii) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(iii) $\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A \subseteq \hat{\lambda}_A$

**Proof**

Suppose $\hat{\lambda}_A \in CS(Z)$ be an $\hat{\lambda}_{A_{k-SS}}^{[X\,Y]^{SI}}$ Let $u \in R$ Then

$$((\hat{\lambda}_A \oplus \hat{\lambda}_A)(u) \cap Y) \cup X$$

$$= \bigcup_{u+(g_1+h_1)=(g_2+h_2)} \{(\hat{\lambda}_A(g_1) \widehat{\cap} \hat{\lambda}_A(g_2) \widehat{\cap} \hat{\lambda}_A(h_1) \widehat{\cap} \hat{\lambda}_A(h_2)) \widehat{\cap} Y\} \cup X$$

$$= \bigcup_{u+(g_1+h_1)=(g_2+h_2)} \left\{ \begin{matrix} (\hat{\lambda}_A(g_1) \widehat{\cap} \hat{\lambda}_A(h_1) \widehat{\cap} Y) \widehat{\cap} \\ ((\hat{\lambda}_A(g_2) \widehat{\cap} \hat{\lambda}_A(h_2) \widehat{\cap} Y)) \widehat{\cap} Y \end{matrix} \right\} \cup X$$

$$\subseteq \bigcup_{u+(g_1+h_1)=(g_2+h_2)} \left\{ \begin{matrix} ((\hat{\lambda}_A(g_1+h_1) \cup X) \widehat{\cap} \\ (\hat{\lambda}_A(g_2+h_2) \cup X) \widehat{\cap} Y \end{matrix} \right\} \cup X$$

$$= \bigcup_{u+(g_1+h_1)=(g_2+h_2)} \{((\hat{\lambda}_A(g_1+h_1)) \cap (\hat{\lambda}_A(g_2+h_2) \cap Y) \widehat{\cap} Y\} \cup X$$

$$\subseteq \bigcup_{u+(g_1+h_1)=(g_2+h_2)} (\hat{\lambda}_A(u) \widehat{\cap} Y) \cup X$$

$$= (\hat{\lambda}_A(u) \widehat{\cap} Y) \cup X$$

It follows that $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

Now

$$\left((\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A)(u) \widehat{\cap} Y\right) \cup X$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ (\bigcap_{i=1}^{n} \hat{\lambda}_A(g_i) \widehat{\cap} \hat{\lambda}_A(h_i) \widehat{\cap} (\bigcap_{j=1}^{n} \hat{\lambda}_A(g_j) \widehat{\cap} \hat{\lambda}_A(h_j)) \widehat{\cap} Y \right\} \cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i=\sum_{j=1}^{n} g_j h_j} \left\{ (\bigcap_{i=1}^{n} \hat{\lambda}_A(g_i h_i) \,\hat{\cap}\, \bigcap_{j=1}^{n} \hat{\lambda}_A(g_j h_j)) \,\hat{\cap}\, Y \right\} \cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i=\sum_{j=1}^{n} g_j h_j} \left\{ (\bigcap_{i=1}^{n} \bigcap_{j=1}^{n} \hat{\lambda}_A(u) \,\hat{\cap}\, Y \right\} \cup X$$

$$\subseteq (\hat{\lambda}_A(u) \,\hat{\cap}\, Y) \cup X$$

Conversely

$$(\hat{\lambda}_A(u+v) \,\hat{\cap}\, Y) \cup X$$

$$\supseteq (\hat{\lambda}_A(u+v) \,\hat{\cap}\, Y) \cup X$$

$$= ((\hat{\lambda}_A \oplus \hat{\lambda}_A)(u+v) \,\hat{\cap}\, Y) \cup X$$

$$= \bigcup_{(u+(g_1+h_1)=(g_2+h_2)} \{(\hat{\lambda}_A(g_1) \,\hat{\cap}\, \hat{\lambda}_A(g_2) \,\hat{\cap}\, \hat{\lambda}_A(h_1) \,\hat{\cap}\, \hat{\lambda}_A(h_2)) \,\hat{\cap}\, Y\} \cup X$$

$$\supseteq ((\hat{\lambda}_A(0) \,\hat{\cap}\, \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(v)) \,\hat{\cap}\, Y) \cup X$$

$$= (((\hat{\lambda}_A(0) \cup X) \,\hat{\cap}\, (\hat{\lambda}_A(u) \,\hat{\cap}\, Y) \,\hat{\cap}\, (\hat{\lambda}_A(v) \,\hat{\cap}\, Y)) \,\hat{\cap}\, Y) \cup X$$

$$\supseteq (\hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(v) \,\hat{\cap}\, Y) \cup X$$

Hence $\hat{\lambda}_A(u+v) \cup X \supseteq \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(v) \,\hat{\cap}\, Y$ holds

$\hat{\lambda}_A(uvw) \cup X \supseteq \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(v) \,\hat{\cap}\, \hat{\lambda}_A(w) \,\hat{\cap}\, Y$ is analogous Thus $\hat{\lambda}_A$ is $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{st}}$

### 2.1.9. Theorem

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-L_i}}^{[X,Y]^{st}}$ $(\hat{\lambda}_{A_{k-R_i}}^{[X,Y]^{st}})$ iff $\hat{\lambda}_A$ satisfies

(i) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(ii) $\mathbb{C}\hat{\odot}\hat{\lambda}_A \subseteq \hat{\lambda}_A \left(\hat{\lambda}_A\hat{\odot}\mathbb{C} \subseteq \hat{\lambda}_A\right)$

(iii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h) \mathbin{\hat{\cap}} Y$

**Proof:** Straightforward

### 2.1.10. Theorem

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-B_i}}^{[X,Y]^{st}}$ iff $\hat{\lambda}_A$ satisfies

(i) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(ii) $\hat{\lambda}_A\hat{\odot}\hat{\lambda}_A \subseteq \hat{\lambda}_A$

(iii) $\hat{\lambda}_A\hat{\odot}\mathbb{C}\hat{\odot}\hat{\lambda}_A \subseteq \hat{\lambda}_A$

(iv) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h) \mathbin{\hat{\cap}} Y$

**Proof:** Straightforward

### 2.1.11. Theorem

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-Q_i}}^{[X,Y]^{st}}$ iff $\hat{\lambda}_A$ satisfies

(i) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(ii) $\left(\hat{\lambda}_A\hat{\odot}\mathbb{C}\right) \mathbin{\hat{\cap}} \left(\mathbb{C}\hat{\odot}\hat{\lambda}_A\right) \subseteq \hat{\lambda}_A$

(iii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h) \mathbin{\hat{\cap}} Y$

**Proof:** Straightforward

**2.1.12. Theorem**

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-Ia_t}}^{[X,Y]^{st}}$ iff $\hat{\lambda}_A$ satisfies

(i) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(ii) $\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(iii) $\mathbb{C} \widehat{\odot} \hat{\lambda}_A \widehat{\odot} \mathbb{C} \subseteq \hat{\lambda}_A$

(iv) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \mathbin{\widehat{\cap}} \hat{\lambda}_A(h) \mathbin{\widehat{\cap}} Y$

**Proof:** Straightforward

**2.1.13 Theorem**

Let $\phi \neq A \subseteq R$ Then A is $\mathbb{K}_{SS}$ ( $\mathbb{K}_{L_t}$, $\mathbb{K}_{Ia_t}$, $\mathbb{K}_{R_t}$, $\mathbb{K}_I$, $\mathbb{K}_{Q_t}$, $\mathbb{K}_{B_t}$) $\Leftrightarrow$ $C_A$ is $C_{A_{k-SS}}^{[X,Y]^{st}}$

$(C_{A_{k-L_t}}^{[X,Y]^{st}}, C_{A_{k-Ia_t}}^{[X,Y]^{st}}, C_{A_{k-R_t}}^{[X,Y]^{st}}, C_{A_{k-I}}^{[X,Y]^{st}}, C_{A_{k-Q_t}}^{[X,Y]^{st}}, C_{A_{k-B_t}}^{[X,Y]^{st}})$ of R

**Proof:** Straightforward

**2.1.14. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{k-I}^{[X,Y]^{st}}$ if and only if each nonempty subset

$$U(\hat{\lambda}_A, \alpha) = \{u \in S \mid \hat{\lambda}_A(u) \supseteq \alpha \cap Y\}$$

is $\mathbb{K}_I$ of $R$ for each $\alpha \subseteq U$ under the condition $\alpha \supseteq X$

**Proof:**

Let $\hat{\lambda}_A \in CS(Z)$ be an $\hat{\lambda}_{A_{k-I}}^{[X,Y]^{st}}$ such that $\hat{\lambda}_A(u) \supseteq X$ for every $u \in R$ and $u, v \in U(\hat{\lambda}_A; \alpha)$. Then

$$\hat{\lambda}_A(u+v) = \hat{\lambda}_A(u+v) \cup X$$

$$\supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y$$

$$\supseteq \alpha \mathbin{\hat{\cap}} Y$$

which implies $u + v \in U(\hat{\lambda}_A, \alpha)$

Next, we let $u \in R$ and $v \in U(\hat{\lambda}_A, \alpha)$ Then

$$\hat{\lambda}_A(uv) = \hat{\lambda}_A(uv) \cup X$$

$$\supseteq \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y$$

$$\supseteq \alpha \mathbin{\hat{\cap}} Y$$

$\Rightarrow uv \in U(\hat{\lambda}_A, \alpha)$ Similarly, we get $vu \in U(\hat{\lambda}_A, \alpha)$ for $v \in R$ *and* $u \in U(\hat{\lambda}_A, \alpha)$

Now, let $u \in S$ *and* $g, h \in U(\hat{\lambda}_A, \alpha)$ such that $u + g = h$ Then

$$\hat{\lambda}_A(u) = \hat{\lambda}_A(u) \cup X$$

$$\supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h) \mathbin{\hat{\cap}} Y$$

$$\supseteq \alpha \mathbin{\hat{\cap}} Y$$

$\Rightarrow u \in U(\hat{\lambda}_A, \alpha)$ Therefore, $U(\hat{\lambda}_A, \alpha)$ is an $\mathbb{K}_I$ of R

Conversely

Let each nonempty subset $U(\hat{\lambda}_A, \alpha)$ be an $\mathbb{K}_I$ of $R$ Then, for $u, v \in R$ there are

$\alpha_1, \alpha_2 \subseteq U$ such that $\alpha_1 \supseteq X, \alpha_2 \supseteq X$ with $\hat{\lambda}_A(u) = \alpha_1$ *and* $\hat{\lambda}(v) = \alpha_2$ Thus, $\hat{\lambda}_A(u) \supseteq \alpha \supseteq \alpha \mathbin{\hat{\cap}} Y$ and

$\hat{\lambda}_A(v) \supseteq \alpha \supseteq \alpha \widehat{\cap} Y$ for $\alpha = \alpha_1 \widehat{\cap} \alpha_2 \supseteq X$ Hence $u, v \in U(\hat{\lambda}_A, \alpha)$ Next $u + v \in U(\hat{\lambda}_A, \alpha)$ for u, $v \in U(\hat{\lambda}_A, \alpha)$.

since, $U(\hat{\lambda}, \alpha)$ is an $\mathbb{K}_I$ of $R$ Then

$$\hat{\lambda}_A(u+v) \cup X = \hat{\lambda}_A(u+v)$$

$$\supseteq \alpha \widehat{\cap} Y$$

$$= \alpha_1 \widehat{\cap} \alpha_2 \widehat{\cap} Y$$

$$= \hat{\lambda}_A(u) \widehat{\cap} \hat{\lambda}_A(v) \widehat{\cap} Y.$$

The verification is complete

Also, we have $uv \in U(\hat{\lambda}_A, \alpha)$ for $u \in R$ and $v \in U(\hat{\lambda}_A, \alpha)$ Then

$$\hat{\lambda}_A(uv) \cup X = \hat{\lambda}_A(uv)$$

$$\supseteq \alpha \widehat{\cap} Y$$

$$= \alpha_1 \widehat{\cap} \alpha_2 \widehat{\cap} Y$$

$$= \hat{\lambda}_A(u) \widehat{\cap} \hat{\lambda}_A(v) \widehat{\cap} \hat{\lambda}_A(w) \widehat{\cap} Y$$

$$\supseteq \hat{\lambda}_A(u) \widehat{\cap} \alpha \widehat{\cap} Y$$

$$\supseteq \hat{\lambda}_A(u) \widehat{\cap} Y.$$

Similarly, we get $\hat{\lambda}_A(vuw) \cup X \supseteq \hat{\lambda}_A(v) \widehat{\cap} Y$ and $\hat{\lambda}_A(wvu) \cup X \supseteq \hat{\lambda}_A(w) \widehat{\cap} Y$ The verification is complete

Now we let $\hat{\lambda}_A(g) = \alpha_1, \hat{\lambda}_A(h) = \alpha_2$ $and$ $u + g = h$ Then $\hat{\lambda}_A(g) \supseteq \alpha_1 \widehat{\cap} \alpha_2 and$

$\hat{\lambda}_A(h) \supseteq \alpha_1 \widehat{\cap} \alpha_2$ obviously So $g, h \in U(\hat{\lambda}_A, \alpha_1 \widehat{\cap} \alpha_2)$ Since $U(\hat{\lambda}_A, \alpha_1 \widehat{\cap} \alpha_2)$ is $\mathbb{K}_{t-I}$, then

$u \in U(\hat{\lambda}_A; \alpha_1 \widehat{\cap} \alpha_2)$ Thus

$$\begin{aligned} \hat{\lambda}_A(u) \cup X &= \hat{\lambda}_A(u) \\ &\supseteq \alpha_1 \widehat{\cap} \alpha_2 \widehat{\cap} Y \\ &= \hat{\lambda}_A(g) \widehat{\cap} \hat{\lambda}_A(h) \widehat{\cap} Y \end{aligned}$$

Hence $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-I}}^{[X,Y]^{st}}$

**2.1.15. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{st}} \left(\hat{\lambda}_{A_{k-B_i}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-Q_i}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-I_i}}^{[X,Y]^{st}}\right)$ if and only if each nonempty subset $U(\hat{\lambda}_A, \alpha) = \{u \in S \mid \hat{\lambda}_A(u) \supseteq \alpha \cap Y\}$ is $\mathbb{K}_S(\mathbb{K}_B, \mathbb{K}_Q, \mathbb{K}_I)$ of R

**Proof** Similar to previous Lemma

**2.1.16. Lemma**

A soft set $\hat{\lambda}_A \in CS(Z)$ is $\hat{\lambda}_{A_{tk-L_i}}^{[X,Y]^{st}}$ $(\hat{\lambda}_{A_{tk-R_i}}^{[X,Y]^{st}})$ of R $\Leftrightarrow$ $\hat{\lambda}_A$ satisfies

(i) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(ii) $\mathbb{C}\widehat{\odot}\hat{\lambda}_A \subseteq \hat{\lambda}_A$ $(\hat{\lambda}_A\widehat{\odot}\mathbb{C} \subseteq \hat{\lambda}_A)$

(iii)If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \widehat{\cap} \hat{\lambda}_A(h) \widehat{\cap} Y$.

**Proof:** Straightforward

**2.1.17. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{st}}$ ( $\hat{\lambda}_{A_{k-I}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-B_t}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-Q_t}}^{[X,Y]^{st}}$ ) if and only if

$U(\hat{\lambda}_A, \alpha) = \{x \in A \mid \hat{\lambda}_A(u) \supseteq \alpha \widehat{\cap} Y\}$ is $\mathbb{K}_{SS}$ ($\mathbb{K}_I, \mathbb{K}_B, \mathbb{K}_Q$) of $R$

**Proof:** Straightforward

**2.1.18. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{st}}$ ( $\hat{\lambda}_{A_{k-I}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-B_t}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-Q_t}}^{[X,Y]^{st}}$ ) if and only if

$\hat{\lambda}_A \widehat{\cap} Y$ is an $\hat{\lambda}_{A_{k-SS}}^{[X,Y]^{st}}$ ( $\hat{\lambda}_{A_{k-I}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-B_t}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{k-Q_t}}^{[X,Y]^{st}}$ ) of $R$

**Proof:** Straightforward

**2.1.19. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$. Then every $\hat{\lambda}_{A_{k-L_t}}^{[X,Y]^{st}}$ ( $\hat{\lambda}_{A_{k-R_t}}^{[X,Y]^{st}}$ ) is an $\hat{\lambda}_{A_{k-Q_t}}^{[X,Y]^{st}}$.

**Proof** Proof is straightforward

**2.1.20. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$. Then every $\hat{\lambda}_{A_{k-Q_t}}^{[X,Y]^{st}}$ is an $\hat{\lambda}_{A_{k-B_t}}^{[X,Y]^{st}}$

**Proof:** Proof is straightforward

**2.1.21. Defination**

Let $\hat{\lambda}_A \in CS(Z)$ Then

(i) $\hat{\lambda}_A \mathbin{\widehat{\cap}} \hat{\lambda}_B = ((\hat{\lambda}_A \cap \hat{\lambda}_B) \mathbin{\widehat{\cap}} Y) \cup X$

(ii) $\hat{\lambda}_A \cup \hat{\lambda}_B = ((\hat{\lambda}_A \cup \hat{\lambda}_B) \mathbin{\widehat{\cap}} Y) \cup X$

(iii) $\hat{\lambda}_A \oplus \hat{\lambda}_B = ((\hat{\lambda}_A \oplus \hat{\lambda}_B) \mathbin{\widehat{\cap}} Y) \cup X$

(iv) $\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B = ((\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B) \mathbin{\widehat{\cap}} Y) \cup X$

**2.1.22. Lemma**

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS(Z)$ are $\hat{\lambda}_{A\,k-R_i}^{[X,Y]^{st}}$, $\hat{\lambda}_{B\,k-L_i}^{[X,Y]^{st}}$ respectively Then

$\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B \subseteq \hat{\lambda}_A \mathbin{\widehat{\cap}} \hat{\lambda}_B$

**Proof:**

Let $\in S$ If $(\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B)(u) = \phi$ or $(\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B)(u) = X$ Then $\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B \subseteq \hat{\lambda}_A \mathbin{\widehat{\cap}} \hat{\lambda}_B$

Otherwise $(\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B)$(u)

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \mathbin{\widehat{\cap}} (\hat{\lambda}_B(h_i) \right) \mathbin{\widehat{\cap}} \\ \left( \bigcap_{j=1}^{n} \hat{\lambda}_A(g_j) \mathbin{\widehat{\cap}} \hat{\lambda}_B(h_j) \right) \end{array} \right) \mathbin{\widehat{\cap}} Y \right\} \cup X$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} ((\hat{\lambda}_A(g_i) \mathbin{\widehat{\cap}} Y) \mathbin{\widehat{\cap}} (\hat{\lambda}_B(h_i) \mathbin{\widehat{\cap}} Y) \right) \mathbin{\widehat{\cap}} \\ \left( \bigcap_{j=1}^{n} ((\hat{\lambda}_A(g_j) \mathbin{\widehat{\cap}} Y)) \mathbin{\widehat{\cap}} (\hat{\lambda}_B(h_j) \mathbin{\widehat{\cap}} Y) \right) \end{array} \right) \mathbin{\widehat{\cap}} Y \right\} \cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} ((\hat{\lambda}_A(g_i h_i) \cup X) \mathbin{\widehat{\cap}} (\hat{\lambda}_B(g_i h_i) \cup X) \mathbin{\widehat{\cap}} \right) \mathbin{\widehat{\cap}} \\ \left( \bigcap_{j=1}^{n} ((\hat{\lambda}_A(g_i h_i) \cup X)) \mathbin{\widehat{\cap}} (\hat{\lambda}_B(g_i h_i) \cup X) \right) \end{array} \right) \mathbin{\widehat{\cap}} Y \right\} \cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i=\sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{pmatrix} \bigcap_{i=1}^{m}((\hat{\lambda}_A(g_i h_i) \widehat{\cap} Y) \widehat{\cap} (\hat{\lambda}_B(g_i h_i) \widehat{\cap} Y)) \end{pmatrix} \widehat{\cap} \\ \begin{pmatrix} \bigcap_{j=1}^{n}((\hat{\lambda}_A(g_i h_i) \widehat{\cap} Y)) \widehat{\cap} (\hat{\lambda}_B(g_i h_i) \widehat{\cap} Y) \end{pmatrix} \right) \widehat{\cap} Y \right\} \cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i=\sum_{j=1}^{n} g_j h_j} \left\{ \begin{pmatrix} \hat{\lambda}_A\left(\sum_{i=1}^{m} g_i h_i\right) \widehat{\cap} \hat{\lambda}_B\left(\sum_{i=1}^{m} g_i h_i\right) \widehat{\cap} \\ \hat{\lambda}_A\left(\sum_{j=1}^{n} g_j h_j\right) \widehat{\cap} \hat{\lambda}_B\left(\sum_{j=1}^{n} g_j h_j\right) \end{pmatrix} \widehat{\cap} Y \right\} \cup X$$

$\subseteq \quad (\hat{\lambda}_A \widehat{\cap} \hat{\lambda}_B)(u)$

Thus, $\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B \subseteq \hat{\lambda}_A \widehat{\cap} \hat{\lambda}_B$

## 2.2. k-Regular Semirings

In this section, we disuss k-regular semirings ($\mathbb{K}_r$) in terms of $\hat{\lambda}_{A_{k-I}}^{[X,Y]^{st}}$, $\hat{\lambda}_{A_{k-Q_i}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{A_{k-B_i}}^{[X,Y]^{st}}$ of $R$.

### 2.2.1. Theorem

A semiring R is $\mathbb{K}_r$ iff $\hat{\lambda}_A \widehat{\cap} \hat{\lambda}_B = \hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B$, for any $\hat{\lambda}_A$ and $\hat{\lambda}_B$ as $\hat{\lambda}_{A_{k-R_i}}^{st}$ and $\hat{\lambda}_{B_{k-L_i}}^{st}$ of R respectively

**Proof:** Let we suppose $R$ is $\mathbb{K}_r$ and $u \in R$ Then there exist $g, h \in R$ such that

$u + ugu = uhu$

Now $(\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B)(u)$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \cap (\hat{\lambda}_B(h_i) \right) \cap \\ (\bigcap_{j=1}^{n} \left( \hat{\lambda}_A(g_j) \right) \cap (\hat{\lambda}_B(h_j)) \end{array} \right) \cap Y \right\} \cup X$$

$$\supseteq (\hat{\lambda}_A(ug) \cap \hat{\lambda}_A(uh) \cap \hat{\lambda}_B(u)) \cap Y) \cup X$$

$$\supseteq ((\hat{\lambda}_A(u) \cap \hat{\lambda}_B(u) \cap Y) \cup X$$

$$= (\hat{\lambda}_A \cap \hat{\lambda}_B)(u)$$

Thus, $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \supseteq \hat{\lambda}_A \cap \hat{\lambda}_B$ and by Lemma 3 12 $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \subseteq \hat{\lambda}_A \cap \hat{\lambda}_B$

Hence, $\hat{\lambda}_A \cap \hat{\lambda}_B = \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$.

Conversely.

Let R and L be $\mathbb{K}_{R_t}$, and $\mathbb{K}_{L_t}$ Then by Theorem, 2 1 13 $\mathcal{C}_R$ and $\mathcal{C}_L$ are $\mathcal{C}_{R_{k-R_t}}^{st}$ and

$\mathcal{C}_{L_{k-L_t}}^{st}$ respectively Then by hypothesis

$\mathcal{C}_R \hat{\odot} \mathcal{C}_L = \mathcal{C}_R \cap \mathcal{C}_L$

$\Rightarrow (\mathcal{C}_{R\hat{\odot}L} \cap Y) \cup X = (\mathcal{C}_{R\cap L} \cap Y) \cup X$

$\Rightarrow R \hat{\odot} L = R \cap L$

$\Rightarrow S$ is $\mathbb{K}_r$

**2.2.2. Theorem**

For a semiring $R$, following are equivalent

(i) $R$ is $\mathbb{K}_r$

(ii) $\hat{\lambda}_A \subseteq \hat{\lambda}_A \hat{\odot} \mathbb{C} \hat{\odot} \hat{\lambda}_A$ for every $\hat{\lambda}_{A_{k-R_t}}^{st}$ of $R$

(iii) $\hat{\lambda}_A \subseteq \hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A$ for every $\hat{\lambda}_{A_{k-Q_i}}^{si}$ of $R$

***Proof*** $(i) \Rightarrow (ii)$

Let $\hat{\lambda}_A$ be a $\hat{\lambda}_{k-Q_i}^{si}$ of $R$ Then for any $u \in R$ there exist $g, h \in R$ such that $u + ugu = uhu$

Now

$$(\hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A)(u)$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{matrix} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A \widehat{\odot} \mathbb{C})(g_i)) \mathbin{\widehat{\cap}} (\bigcap_{i=1}^{m} \hat{\lambda}_A(h_i) \right) \mathbin{\widehat{\cap}} \\ (\bigcap_{j=1}^{n} (\hat{\lambda}_A \widehat{\odot} \mathbb{C})(g_j) \mathbin{\widehat{\cap}} (\bigcap_{j=1}^{n} \hat{\lambda}_A(h_j)) \end{matrix} \right) \mathbin{\widehat{\cap}} Y \right\} \cup X$$

$$\supseteq \left\{ \left( \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A \widehat{\odot} \mathbb{C})(ug) \right) \mathbin{\widehat{\cap}} \left( \bigcap_{j=1}^{n} (\hat{\lambda}_A \widehat{\odot} \mathbb{C})(uh) \right) \mathbin{\widehat{\cap}} \hat{\lambda}_A(u) \right) \mathbin{\widehat{\cap}} Y \right\} \cup X$$

$$= \left\{ \left( \begin{matrix} \bigcap_{i=1}^{n} \left( \bigcup_{ug+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \begin{matrix} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \right) \mathbin{\widehat{\cap}} \\ (\bigcap_{j=1}^{n} \left( \hat{\lambda}_A(h_j) \right) \end{matrix} \right\} \right) \mathbin{\widehat{\cap}} \\ \bigcap_{j=1}^{n} \left( \bigcup_{uh+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \begin{matrix} \left( \bigcap_{i=1}^{n} (\hat{\lambda}_A(g_i) \right) \mathbin{\widehat{\cap}} \\ (\bigcap_{j=1}^{n} \left( \hat{\lambda}_A(h_j) \right) \end{matrix} \right\} \mathbin{\widehat{\cap}} \hat{\lambda}_A(u) \right) \end{matrix} \right) \mathbin{\widehat{\cap}} Y \right\} \cup X$$

$$\supseteq \left\{ \left( \begin{matrix} \left( \widehat{\lambda_A}(ugu) \mathbin{\widehat{\cap}} \hat{\lambda}_A(uhu) \right) \mathbin{\widehat{\cap}} \\ \{\hat{\lambda}_A(ugu) \mathbin{\widehat{\cap}} \hat{\lambda}_A(uhu)\} \mathbin{\widehat{\cap}} \hat{\lambda}_A(u) \end{matrix} \right) \mathbin{\widehat{\cap}} Y \right\} \cup X \quad \text{because} \begin{bmatrix} ug + ugug = uhug \text{ and} \\ uh + uguh = uhuh \end{bmatrix}$$

$$\supseteq \{\hat{\lambda}_A(u) \hat{\cap} Y\} \cup X$$

$(ii) \Longrightarrow (iii)$ is straightforward

$(iii) \Rightarrow (i)$ Let $Q$ be any $\mathbb{K}_{Q_i}$ of $R$ Then by Theorem 2 1 13 $C_Q$ is an $\hat{\lambda}^{si}_{ik-Q_i}$ of $R$

Now by using the given condition, we can write

$$(C_Q \hat{\cap} Y) \cup X \subseteq C_Q \hat{\odot} \mathbb{C} \hat{\odot} C_Q = C_{\overline{Q\mathbb{C}Q}}$$

$$\Rightarrow Q \subseteq \overline{Q\mathbb{C}Q}$$

Also, we know that $\overline{Q\mathbb{C}Q} \subseteq \overline{\mathbb{C}Q} \hat{\cap} \overline{Q\mathbb{C}} = Q$ Thus $Q = \overline{Q\mathbb{C}Q}$ Therefore by Theorem 2 4, R is $\mathbb{K}_r$

### 2.2.3. Theorem

For a semiring $R$, following are equivalent

(i) $R$ is $\mathbb{K}_r$

(ii) $\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_A$ for every $\hat{\lambda}^{si}_{A_{k-B_i}}$, $\hat{\lambda}^{si}_{B_{k-I}}$.

(iii) $\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_A$ for every $\hat{\lambda}^{si}_{A_{k-Q_i}}$, $\hat{\lambda}^{si}_{B_{k-I}}$.

**Proof:** By using previous Theorem 2 2 2, one can easily prove it

### 2.2.4. Theorem

For a semiring $R$, the following statements are equivalent

(i) $R$ is $\mathbb{K}_r$

(ii) $\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$ for every $\hat{\lambda}^{[X,Y]^{si}}_{A_{k-Q_i}}$ and $\hat{\lambda}^{[X,Y]^{si}}_{B_{k-L_i}}$

(iii) $\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$ for every $\hat{\lambda}^{[X,Y]^{si}}_{A_{k-B_i}}$ and $\hat{\lambda}^{[X,Y]^{si}}_{B_{k-L_i}}$

(iv) $\hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$ for every $\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{B_{k-Q_l}}^{[X,Y]^{st}}$

(v) $\hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$ for every $\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{B_{k-B_l}}^{[X,Y]^{st}}$

(vi) $\hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B \,\hat{\cap}\, \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$ for every $\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$, $\hat{\lambda}_{B_{k-Q_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{C_{k-L_l}}^{[X,Y]^{st}}$

(vii) $\hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B \,\hat{\cap}\, \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$ for every $\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$, $\hat{\lambda}_{B_{k-B_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{C_{k-L_l}}^{[X,Y]^{st}}$

**Proof.** $(i) \implies (iii)$ Let $\hat{\lambda}_A, \hat{\lambda}_B$ be any $\hat{\lambda}_{A_{k-B_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{B_{k-L_l}}^{[X,Y]^{st}}$ respectively Since $R$ is $\mathbb{K}_r$, then for any $u \in S$ there exist $v, w \in R$ such that $u + uvu = uwu$ Now,

$$(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B)(u)$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \,\hat{\cap}\, \hat{\lambda}_B(h_i) \right) \hat{\cap} \\ (\bigcap_{j=1}^{n} (\hat{\lambda}_A(g_j) \,\hat{\cap}\, \hat{\lambda}_B(h_j)) \end{array} \right) \hat{\cap}\, Y \right\} \cup X$$

$$\supseteq \{(\hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_B(vu) \,\hat{\cap}\, \hat{\lambda}_B(wu)) \,\hat{\cap}\, Y\} \cup X$$

$$\supseteq \{\hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_B(u) \,\hat{\cap}\, Y\} \cup X$$

$$\supseteq (\hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B)(u)$$

Hence $\hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$

$(iii) \implies (ii)$ is easy to prove by using Lemma 2 1 20

$(ii) \implies (i)$ Let $\hat{\lambda}_A, \hat{\lambda}_B$ be any $\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{B_{kt-L_l}}^{[X,Y]^{st}}$ By Lemma 2 1 19

$\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$ is an $\hat{\lambda}_{B_{k-Q_l}}^{[X,Y]^{st}}$ Then by hypothesis, we have $\hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$ Also

by Lemma 2 1 22, we have $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \subseteq \hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B$ Hence $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B = \hat{\lambda}_A \,\hat{\cap}\, \hat{\lambda}_B$

Thus $R$ is $\mathbb{K}_r$

$(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ are straightforward

$(i) \Longrightarrow (vii)$ Let $\hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C$ be any $\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$, $\hat{\lambda}_{B_{k-B_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{C_{k-L_l}}^{[X,Y]^{st}}$ respectively Since $R$ is $\mathbb{K}_r$, then for any $u \in S$ there exist $v, w \in S$ such that $u + uvu = uwu$ Now

$$(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u)$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{pmatrix} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B)(g_i)) \mathbin{\tilde{\cap}} (\bigcap_{i=1}^{m} \hat{\lambda}_C(h_i) \right) \mathbin{\tilde{\cap}} \\ (\bigcap_{j=1}^{n} (\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B)(g_j) \mathbin{\tilde{\cap}} (\bigcap_{j=1}^{n} \hat{\lambda}_C(h_j)) \end{pmatrix} \mathbin{\tilde{\cap}} Y \right\} \cup X$$

$$\supseteq \left\{ \left( \left( \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B)(u) \right) \mathbin{\tilde{\cap}} \left( \hat{\lambda}_C(vu) \right) \mathbin{\tilde{\cap}} \hat{\lambda}_C(wu) \right) \mathbin{\tilde{\cap}} Y \right\} \cup X$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i = \sum_{j=1}^{n} g_j h_j} \left\{ \left( \begin{pmatrix} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \mathbin{\tilde{\cap}} \hat{\lambda}_B(h_i) \right) \mathbin{\tilde{\cap}} \\ (\bigcap_{j=1}^{n} (\hat{\lambda}_A(g_j) \mathbin{\tilde{\cap}} \hat{\lambda}_B(h_j)) \end{pmatrix} \mathbin{\tilde{\cap}} \hat{\lambda}_C(u) \right) \mathbin{\tilde{\cap}} Y \right\} \cup X$$

$$\supseteq \{(\hat{\lambda}_i(uv) \mathbin{\tilde{\cap}} \hat{\lambda}_B(uw) \mathbin{\tilde{\cap}} \hat{\lambda}_C(u)) \mathbin{\tilde{\cap}} Y\} \cup X$$

$$\supseteq \{(\hat{\lambda}_A(u) \mathbin{\tilde{\cap}} \hat{\lambda}_B(u) \mathbin{\tilde{\cap}} \hat{\lambda}_C(u)) \mathbin{\tilde{\cap}} Y\} \cup X$$

$$\supseteq (\hat{\lambda}_A \mathbin{\tilde{\cap}} \hat{\lambda}_B \mathbin{\tilde{\cap}} \hat{\lambda}_C)(u)$$

Thus $\hat{\lambda}_A \mathbin{\tilde{\cap}} \hat{\lambda}_B \mathbin{\tilde{\cap}} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$

$(vii) \Longrightarrow (vi)$ Straightforward

$(vi) \Longrightarrow (i)$ Let $\hat{\lambda}_A, \hat{\lambda}_B$ be any $\hat{\lambda}_{A_{k-R_l}}^{[X,Y]^{st}}$ and $\hat{\lambda}_{B_{k-L_l}}^{[X,Y]^{st}}$ respectively Then

$$\hat{\lambda}_A \mathbin{\tilde{\cap}} \hat{\lambda}_B = \hat{\lambda}_A \mathbin{\tilde{\cap}} \mathbb{C} \mathbin{\tilde{\cap}} \hat{\lambda}_B$$

$$\subseteq \hat{\lambda}_A \hat{\odot} \mathbb{C} \hat{\odot} \hat{\lambda}_B$$

$$\subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B$$

But $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \subseteq \hat{\lambda}_A \mathbin{\tilde{\cap}} \hat{\lambda}_B$ is always hold for any $\hat{\lambda}_{A_{k-R_l}}^{st}$ and $\hat{\lambda}_{B_{k-L_l}}^{st}$. Hence $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B = \hat{\lambda}_A \mathbin{\tilde{\cap}} \hat{\lambda}_B$ Then by Theorem 2 2 1, $R$ is $\mathbb{K}_r$

# Chapter 3

## 3. Generalized Ternary *k*-Ideals in Ternary Semirings using Soft Intersectional Sets

In this chapter, we present the certain algebraic structure of Soft Intersectional ternary k-subsemirings, Soft Intersectional ternary k-ideals in terms of soft union intersection sum and soft union intersection product Next, we define [X,Y] soft Soft Intersectional ternary k-subsemirings, [X,Y] Soft Intersectional ternary k-ideals and investigate some important properties

### 3.1. Soft Intersectional Ternary k-Ideals

#### 3.1.1. Definition

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS(Z)$ Then soft union-intersection sum is defined by

$$(\hat{\lambda}_A \oplus \hat{\lambda}_B)(u) = \begin{cases} \bigcup_{u+(x_1+y_1)=(x_2+y_2)} \{\hat{\lambda}_A(x_1) \hat{\cap} \hat{\lambda}_A(x_2) \hat{\cap} \hat{\lambda}_B(y_1) \hat{\cap} \hat{\lambda}_B(y_2)\} \\ \Phi \quad \text{if } u \text{ cannot be expressed as } u + (x_1 + y_1) = (x_2 + y_2) \end{cases}$$

$$\forall u \in S$$

#### 3.1.2. Definition

Let $\hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C \in CS(Z)$ Then soft union-intersection product is denoted and defined as

$$(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u) = \begin{cases} \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \cap \hat{\lambda}_B(h_i) \cap \hat{\lambda}_C(k_i)) \right) \cap \left( \bigcap_{j=1}^{n} (\hat{\lambda}_A(g_j) \cap \hat{\lambda}_B(h_j) \cap \hat{\lambda}_C(k_j)) \right) \right\} \\ \Phi \qquad \text{if } u \text{ cannot be expressed as } u + \sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j \end{cases}$$

$\forall\, u \in S$

### 3.1.3. Definition

Let $S$ be a ternary semiring ($T_{ST}$) and $\phi \neq G \subseteq S$ Then characteristic soft set is denoted and defined by

$$\mathcal{C}_G(x) = \begin{cases} U & \text{if } x \in G \\ \phi & \text{if } x \in S \setminus G \end{cases}$$

The soft set $\mathcal{C}_S \in CS(Z)$ is called identity soft set and is denoted by $\mathbb{C}$

### 3.1.4. Lemma

Let S be a $\mathbb{K}_{t-ss}$ and $F, G, H \subseteq S$. Then we have

(i) $G \subseteq H \Leftrightarrow \mathcal{C}_G \subseteq \mathcal{C}_H$

(ii) $\mathcal{C}_G \hat{\cap} \mathcal{C}_H = \mathcal{C}_{G \hat{\cap} H}$

(iii) $\mathcal{C}_F \hat{\odot} \mathcal{C}_G \hat{\odot} \mathcal{C}_H = \mathcal{C}_{FGH}$

**Proof:** Straightforward

### 3.1.5. Definition

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional ternary k-subsemiring of $S$ if it satisfies

(i) $\hat{\lambda}_A(u+v) \supseteq \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v) \qquad \forall\, u, v \in S$

(ii) $\hat{\lambda}_A(uvw) \supseteq \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v) \hat{\cap} \hat{\lambda}_A(w) \qquad \forall\, u, v, w \in S$

(iii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \hat{\cap} \hat{\lambda}_A(h)$

Soft intersectional ternary k-subsemiring is represented by $\hat{\lambda}_{A_{tk-SS}}^{st}$

### 3.1.6. Example

Let $U = \{0, -\acute{e}, -\acute{f}, -\acute{g}, -\acute{h}\}$ be a $T_{sr}$ with the following defined operations

| + | o | -é | -f́ | -ǵ | -h́ |
|---|---|---|---|---|---|
| o | o | -é | -f́ | -ǵ | -h́ |
| -é | -é | -f́ | -ǵ | -h́ | o |
| -f́ | -f́ | -g | -h́ | o | -é |
| -ǵ | -ǵ | -h́ | o | -é | -f́ |
| -h́ | -h́ | o | -é | -f́ | -ǵ |

| | o | -é | -f́ | -ǵ | -h́ |
|---|---|---|---|---|---|
| o | o | o | o | o | o |
| -é | o | é | f́ | ǵ | h́ |
| -f́ | o | f́ | h́ | é | ǵ |
| -c | o | ǵ | é | h́ | f́ |
| -d | o | h́ | g | f́ | é |

| | o | é | f́ | ǵ | h́ |
|---|---|---|---|---|---|
| o | o | o | o | o | o |
| -é | o | -é | -f́ | -ǵ | -h́ |
| -f́ | o | -f́ | -h́ | -é | -ǵ |
| -ǵ | o | -ǵ | -e | -h́ | -f́ |
| -h́ | o | -h́ | -g | -f́ | -é |

Define $\hat{\lambda}_A(o) = \{o, -\acute{f}, -\acute{h}\}$ and $\hat{\lambda}_A(-\acute{e}) = \hat{\lambda}_A(-\acute{f}) = \hat{\lambda}_A(-\acute{g}) = \hat{\lambda}_A(-\acute{h}) = \{-\acute{f}, -\acute{h}\}$

Then $\hat{\lambda}_A$ is $\hat{\lambda}_{A_{tk-SS}}^{st}$

**3.1.7. Definition**

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional ternary k-left ideal ( ternary k-lateral ideal ternary k-right ideal ) If it satisfies

(i) $\hat{\lambda}_A(u+v) \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v)$ $\forall\, u, v \in S$

(ii) $\hat{\lambda}_A(uvw) \supseteq \hat{\lambda}_A(w)$ $(\hat{\lambda}_A(uvw) \supseteq \hat{\lambda}_A(u), \hat{\lambda}_A(uvw) \supseteq \hat{\lambda}_A(v))$ $\forall\, u, v, w \in S$

(iii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h)$

Soft intersectional ternary k-left ideal ( ternary k-lateral ideal, ternary k-right ideal ) will be denoted by $\hat{\lambda}_{A_{tk-L_i}}^{si}$ ($\hat{\lambda}_{A_{tk-La_i}}^{si}$ and $\hat{\lambda}_{A_{tk-R_i}}^{si}$) respectively

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional ternary k-ideal ($\hat{\lambda}_{A_{tk-I}}^{si}$) if it is $\hat{\lambda}_{A_{tk-L_i}}^{si}$, $\hat{\lambda}_{A_{tk-La_i}}^{si}$ and $\hat{\lambda}_{A_{tk-R_i}}^{si}$

**3.1.8. Definition**

$\hat{\lambda}_A \in CS(Z)$ is called soft intersectional ternary k-quasi ideal of S if it satisfies

(i) $\hat{\lambda}_A(u+v) \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v)$ $\forall\, u, v \in S$

(ii) $\hat{\lambda}_A \supseteq (\hat{\lambda}_A \mathbin{\hat{\odot}} \mathbb{C} \mathbin{\hat{\odot}} \mathbb{C}) \mathbin{\hat{\cap}} ((\mathbb{C} \mathbin{\hat{\odot}} \hat{\lambda}_A \mathbin{\hat{\odot}} \mathbb{C}) \oplus (\mathbb{C} \mathbin{\hat{\odot}} \mathbb{C} \mathbin{\hat{\odot}} \hat{\lambda}_A \mathbin{\hat{\odot}} \mathbb{C} \mathbin{\hat{\odot}} \mathbb{C})) \mathbin{\hat{\cap}} (\mathbb{C} \mathbin{\hat{\odot}} \mathbb{C} \mathbin{\hat{\odot}} \hat{\lambda}_A)$

(iii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h)$

Soft intersectional ternary k-quasi ideal is represented by $\hat{\lambda}_{A_{tk-Q_i}}^{si}$

**3.1.9. Definition**

A $\hat{\lambda}_{A_{tk-SS}}^{st} \in CS(Z)$ is called soft intersectional ternary k-bi ideal of S if it satisfies

$\hat{\lambda}_A(abcde) \supseteq \hat{\lambda}_A(a) \hat{\cap} \hat{\lambda}_A(c) \hat{\cap} \hat{\lambda}_A(e) \qquad \forall\, a,b,c,d,e \in S$

Soft intersectional ternary k-bi ideal is represented by $\hat{\lambda}_{A_{tk-B_i}}^{st}$

It is obvious that $\hat{\lambda}_A(0) \supseteq \hat{\lambda}_A(u), \quad \forall\, u \in S$

**3.1.10. Lemma**

$\hat{\lambda}_A$ is $\hat{\lambda}_{A_{tk-SS}}^{st}$ of S $\Leftrightarrow$ it satisfies $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$, $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \subseteq \hat{\lambda}_A$

and If $u + g = h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \hat{\cap} \hat{\lambda}_A(h)$

**Proof** Suppose $\hat{\lambda}_A$ is $\hat{\lambda}_{A_{tk-SS}}^{st}$ of S, then

$(\hat{\lambda}_A \oplus \hat{\lambda}_A)(u)$

$$= \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1) \hat{\cap} \hat{\lambda}_A(p_2) \hat{\cap} \hat{\lambda}_A(q_1) \hat{\cap} \hat{\lambda}_A(q_2)\}$$

$$= \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{\hat{\lambda}_A(p_1) \hat{\cap} \hat{\lambda}_A(q_1) \hat{\cap} \hat{\lambda}_A(p_2) \hat{\cap} \hat{\lambda}_A(q_2)\}$$

$$\subseteq \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1 + q_1) \hat{\cap} \hat{\lambda}_A(p_2 + q_2)\}$$

$$\subseteq \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(u)\}$$

$$= \hat{\lambda}_A(u)$$

It follows that $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

Now

$(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \hat{\odot} \hat{\lambda}_A)(u)$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ \begin{array}{c} \left(\bigcap_{i=1}^{n}(\hat{\lambda}_A(p_i) \hat{\cap} \hat{\lambda}_A(q_i) \hat{\cap} \hat{\lambda}_A(r_i)\right) \hat{\cap} \\ (\bigcap_{j=1}^{n}(\hat{\lambda}_A(p_j) \hat{\cap} \hat{\lambda}_A(q_j) \hat{\cap} \hat{\lambda}_A(r_j)) \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ (\bigcap_{i=1}^{n} \hat{\lambda}_A(p_i q_i r_i)) \hat{\cap} (\bigcap_{j=1}^{n} \hat{\lambda}_A(p_j q_j r_j) \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ (\bigcap_{i=1}^{n} \bigcap_{j=1}^{n} \hat{\lambda}_A(u) \right\}$$

$$\subseteq \hat{\lambda}_A(u)$$

So $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \subseteq \hat{\lambda}_A$

Conversely.

let us assume $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$, $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \subseteq \hat{\lambda}_A$ and If $u + g = h$

$\Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \hat{\cap} \hat{\lambda}_A(h)$

Then $\forall\, u, v, w \in$ S

$$\hat{\lambda}_A(u + v) \supseteq (\hat{\lambda}_A \oplus \hat{\lambda}_A)(u + v)$$

$$= \bigcup_{(u+v)+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1) \hat{\cap} \hat{\lambda}_A(p_2) \hat{\cap} \hat{\lambda}_A(q_1) \hat{\cap} \hat{\lambda}_A(q_2)\}$$

$$\supseteq \hat{\lambda}_A(0) \hat{\cap} \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v)$$

$$\supseteq \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v)$$

$$\hat{\lambda}_A(uvw) \supseteq (\hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \hat{\odot} \hat{\lambda}_A)(uvw)$$

$$= \bigcup_{uvw+\sum_{i=1}^{m} p_i q_i r_i = \sum_{j=1}^{n} p_j q_j r_j} \left\{ \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(p_i) \cap \hat{\lambda}_A(q_i) \cap \hat{\lambda}_A(r_i) \right) \cap \\ (\bigcap_{j=1}^{n} (\hat{\lambda}_A(p_j) \cap \hat{\lambda}_A(q_j) \cap \hat{\lambda}_A(r_j) \end{array} \right\}$$

$$\supseteq \hat{\lambda}_A(u) \cap \hat{\lambda}_A(v) \cap \hat{\lambda}_A(w)$$

Thus, $\hat{\lambda}_A(u+v) \supseteq \hat{\lambda}_A(u) \cap \hat{\lambda}_A(v)$. $\hat{\lambda}_A(uvw) \supseteq \hat{\lambda}_A(u) \cap \hat{\lambda}_A(v) \cap \hat{\lambda}_A(w)$ and

If $u + g = h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \cap \hat{\lambda}_A(h) \quad \forall\, u, v, w \in S$

Hence $\hat{\lambda}_A$ is $\hat{\lambda}_{A_{tk-SS}}^{si}$ of S

**3.1.11. Lemma**

$\hat{\lambda}_A \in CS(Z)$ is $\hat{\lambda}_{A_{tk-L_i}}^{si}$ $(\hat{\lambda}_{A_{tk-R_i}}^{si}, \hat{\lambda}_{A_{tk-La_i}}^{si})$ of S $\Leftrightarrow \hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$,

If $u + g = h \Rightarrow \hat{\lambda}_A(u) \supseteq \hat{\lambda}_A(g) \cap \hat{\lambda}_A(h)$

and $\mathbb{C} \hat{\odot} \mathbb{C} \hat{\odot} \hat{\lambda}_A \subseteq \hat{\lambda}_A$ $(\hat{\lambda}_A \hat{\odot} \mathbb{C} \hat{\odot} \mathbb{C} \subseteq \hat{\lambda}_A, \mathbb{C} \hat{\odot} \hat{\lambda}_A \hat{\odot} \mathbb{C} \subseteq \hat{\lambda}_A)$

**Proof** Suppose $\hat{\lambda}_A \in CS(Z)$ is $\hat{\lambda}_{A_{tk-L_i}}^{si}$ of $S$, then

$(\hat{\lambda}_A \oplus \hat{\lambda}_A)(u)$

$= \cup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1) \cap \hat{\lambda}_A(p_2) \cap \hat{\lambda}_A(q_1) \cap \hat{\lambda}_A(q_2)\}$

$$= \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{\hat{\lambda}_A(p_1) \cap \hat{\lambda}_A(q_1) \cap \hat{\lambda}_A(p_2) \cap \hat{\lambda}_A(q_2)\}$$

$$\subseteq \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1 + q_1) \cap \hat{\lambda}_A(p_2 + q_2)\}$$

$$\subseteq \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(u)\}$$

$$= \hat{\lambda}_A(u)$$

It follows that $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

Now

$$(\mathbb{C}\hat{\odot}\mathbb{C}\hat{\odot}\,\hat{\lambda}_A)(u)$$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ (\bigcap_{i=1}^{m} (\mathbb{C}(p_i) \hat{\cap} \mathbb{C}(q_i) \hat{\cap}\ \hat{\lambda}_A(r_i)) \hat{\cap} (\bigcap_{j=1}^{n} (\mathbb{C}(p_j) \hat{\cap} \mathbb{C}(q_j) \hat{\cap}\ \hat{\lambda}_A(r_j)) \right\}$$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ (\bigcap_{i=1}^{m} (Z \hat{\cap} Z \hat{\cap}\ \hat{\lambda}_A(r_i)) \hat{\cap} (\bigcap_{j=1}^{n} (Z \hat{\cap} Z \hat{\cap}\ \hat{\lambda}_A(r_j)) \right\}$$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ (\bigcap_{i=1}^{m} \hat{\lambda}_A(r_i)) \hat{\cap} (\bigcap_{j=1}^{n} \hat{\lambda}_A(r_j)) \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ (\bigcap_{i=1}^{m} \hat{\lambda}_A(p_i q_i r_i)) \hat{\cap} (\bigcap_{j=1}^{n} \hat{\lambda}_A(p_j q_j r_j)) \right\}$$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ \hat{\lambda}_A\left(\sum_{i=1}^{m} p_i q_i r_i\right) \hat{\cap}\ \hat{\lambda}_A\left(\sum_{i=1}^{m} p_j q_j r_j\right) \right\}$$

$$\subseteq\ \hat{\lambda}_A(u)$$

Conversely

$$\hat{\lambda}_A(u+v)\ \supseteq (\hat{\lambda}_A \oplus \hat{\lambda}_A)(u+v)$$

$$= \bigcup_{(u+v)+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1) \hat{\cap} \hat{\lambda}_A(p_2) \hat{\cap} \hat{\lambda}_A(q_1) \hat{\cap}\ \hat{\lambda}_A(q_2)\}$$

$$\supseteq\ \hat{\lambda}_A(0) \hat{\cap}\ \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v)$$

$$\supseteq\ \hat{\lambda}_A(u) \hat{\cap}\ \hat{\lambda}_A(v)$$

$$\hat{\lambda}_A(uvw) \supseteq\ (\mathbb{C}\hat{\odot}\mathbb{C}\hat{\odot}\,\hat{\lambda}_A)(uvw)$$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i = \sum_{j=1}^{n} p_j q_j r_j} \left\{ \left( \bigcap_{i=1}^{m} (\mathbb{C}(p_i) \mathbin{\widehat{\cap}} \mathbb{C}(q_i) \mathbin{\widehat{\cap}} \hat{\lambda}_A(r_i)) \mathbin{\widehat{\cap}} \left( \bigcap_{j=1}^{n} (\mathbb{C}(p_j) \mathbin{\widehat{\cap}} \mathbb{C}(q_j) \mathbin{\widehat{\cap}} \hat{\lambda}_A(r_j) \right) \right\}$$

$$\supseteq \mathbb{C}(u) \mathbin{\widehat{\cap}} \mathbb{C}(v) \mathbin{\widehat{\cap}} \hat{\lambda}_A(w)$$

$$\supseteq \hat{\lambda}_A(w)$$

**3.1.12. Theorem**

Let $\phi \neq A \subseteq S$ Then $A$ is $\mathbb{K}_{t-ss}$ ( $\mathbb{K}_{t-L_i}$, $\mathbb{K}_{t-La_i}$, $\mathbb{K}_{t-R_i}$, $\mathbb{K}_{t-Q_i}$, $\mathbb{K}_{t-B_i}$) $\Leftrightarrow$ $\mathcal{C}_A$ is $\hat{\lambda}^{si}_{A_{tk-SS}}$ ($\hat{\lambda}^{si}_{A_{tk-L_i}}$, $\hat{\lambda}^{si}_{A_{tk-La_i}}$, $\hat{\lambda}^{si}_{tk-R_i}$, $\hat{\lambda}^{si}_{A_{tk-Q_i}}$, $\hat{\lambda}^{si}_{A_{tk-B_i}}$) of S

**Proof** Suppose $A$ is $\mathbb{K}_{t-ss} \in CS(Z)$ and $u, v, w \in S$

Case (1) For $u, v, w \in A$, we have $u + v, uvw \in A$ Then

$\mathcal{C}_A(u + v) = Z = Z \mathbin{\widehat{\cap}} Z = \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \mathcal{C}_A(v)$ and

$\mathcal{C}_A(uvw) = Z = Z \mathbin{\widehat{\cap}} Z \mathbin{\widehat{\cap}} Z = \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \mathcal{C}_A(v) \mathbin{\widehat{\cap}} \mathcal{C}_A(w)$

As $A$ is $\mathbb{K}_{t-ss}$ of S, so $u + v = w$ implies $u \in A$ for some $u \in S$ and $v, w \in A$

$\mathcal{C}_A(u) \supseteq \mathcal{C}_A(v) \mathbin{\widehat{\cap}} \mathcal{C}_A(w)$

Case (2) For at least one, say $v \notin W$, we have $\mathcal{C}_A(v) = \phi$. Then

$\mathcal{C}_A(u + v) \supseteq \phi = \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \phi = \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \mathcal{C}_A(v)$ and

$\mathcal{C}_A(uvw) \supseteq \phi = \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \phi \mathbin{\widehat{\cap}} \mathcal{C}_A(w) = \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \mathcal{C}_A(v) \mathbin{\widehat{\cap}} \mathcal{C}_A(w)$

As $v \notin W$ so $u + v \notin W$ and we cannot write $u + v \neq w$ Then $\mathcal{C}_A(u) \not\supseteq \mathcal{C}_A(v) \mathbin{\widehat{\cap}} \mathcal{C}_A(w)$ but this contradicts our supposition

By combining both cases, we have

$\mathcal{C}_A(u + v) \supseteq \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \mathcal{C}_A(v)$, $\mathcal{C}_A(uvw) \supseteq \mathcal{C}_A(u) \mathbin{\widehat{\cap}} \mathcal{C}_A(v) \mathbin{\widehat{\cap}} \mathcal{C}_A(w)$ and

if $u + v = w$, then $\mathcal{C}_A(u) \supseteq \mathcal{C}_A(v) \mathbin{\widehat{\cap}} \mathcal{C}_A(w)$

Conversely,

assume that $\mathcal{C}_A$ is $\hat{\lambda}_{A_{tk-SS}}^{st}$ and $u, v, w \in A$ Then

$$\mathcal{C}_A(u+v) \supseteq \mathcal{C}_A(u) \hat{\cap} \mathcal{C}_A(v) = Z \hat{\cap} Z = Z \text{ and}$$

$$\mathcal{C}_A(uvw) \supseteq \mathcal{C}_A(u) \hat{\cap} \mathcal{C}_A(v) \hat{\cap} \mathcal{C}_A(w) = Z \hat{\cap} Z \hat{\cap} Z = Z$$

Thus $u+v, uvw \in A, \forall\, u, v, w \in A$.

Hence A is $\mathbb{K}_{t-ss}$ of $S$

**3.1.13. Lemma**

Let $\hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C \in CS(Z)$ be $\hat{\lambda}_{A_{tk-R_i}}^{st}$, $\hat{\lambda}_{B_{tk-La_i}}^{st}$ and $\hat{\lambda}_{C_{tk-L_i}}^{st}$ respectively Then

$\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

**Proof:**

Let $u \in S$ If $(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u) = \phi$ or $(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u) = X$ Then $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

Otherwise $(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u)$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i = \sum_{j=1}^{n} p_j q_j r_j} \left\{ \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(p_i) \hat{\cap} \hat{\lambda}_B(q_i) \hat{\cap} \hat{\lambda}_C(r_i) \right) \hat{\cap} \\ ( \bigcap_{j=1}^{n} (\hat{\lambda}_A(p_j) \hat{\cap} \hat{\lambda}_B(q_j) \hat{\cap} \hat{\lambda}_C(r_j) \end{array} \right\}$$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i = \sum_{j=1}^{n} p_j q_j r_j} \left\{ \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(p_i) \hat{\cap} \hat{\lambda}_B(q_i) \hat{\cap} \hat{\lambda}_C(r_i) \right) \hat{\cap} \\ \bigcap_{j=1}^{n} \left(\hat{\lambda}_A(p_j)\right) \hat{\cap} \hat{\lambda}_B(q_j) \hat{\cap} \hat{\lambda}_C(r_j) \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i = \sum_{j=1}^{n} p_j q_j r_j} \left\{ \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(p_i q_i r_i) \hat{\cap} \hat{\lambda}_B(p_i q_i r_i) \hat{\cap} \hat{\lambda}_C(p_i q_i r_i) \right) \hat{\cap} \\ \bigcap_{j=1}^{n} \left(\hat{\lambda}_A(p_j q_j r_j)\right) \hat{\cap} \hat{\lambda}_B(p_j q_j r_j) \hat{\cap} \hat{\lambda}_C(p_j q_j r_j) \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ \begin{array}{l} \hat{\lambda}_A\left(\sum_{i=1}^{m} p_i q_i r_i\right) \mathbin{\hat{\cap}} \hat{\lambda}_B\left(\sum_{i=1}^{m} p_i q_i r_i\right) \mathbin{\hat{\cap}} \hat{\lambda}_C\left(\sum_{i=1}^{m} p_i q_i r_i\right) \mathbin{\hat{\cap}} \\ \hat{\lambda}_A\left(\sum_{j=1}^{n} p_j q_j r_j\right) \mathbin{\hat{\cap}} \hat{\lambda}_B\left(\sum_{j=1}^{n} p_j q_j r_j\right) \mathbin{\hat{\cap}} \hat{\lambda}_C\left(\sum_{j=1}^{n} p_j q_j r_j\right) \end{array} \right\}$$

$\subseteq \quad (\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C)(\mathrm{u})$

Thus, $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C$

**3.1.14. Theorem**

For $S$, the statements given below are equivalent

(i) $S$ is $\mathbb{K}_{t-R}$

(ii) $M \mathbin{\hat{\cap}} N \mathbin{\hat{\cap}} O = MNO$, for any $M$, $N$ and $O$ as $\mathbb{K}_{t-R_i}$, $\mathbb{K}_{t-La_i}$ and $\mathbb{K}_{t-L_i}$ of S respectively

(iii) $\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C = \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$, for any $\hat{\lambda}_A$, $\hat{\lambda}_B$ and $\hat{\lambda}_C$ as $\hat{\lambda}^{si}_{A_{tk-R_i}}$, $\hat{\lambda}^{si}_{B_{tk-La_i}}$ and $\hat{\lambda}^{si}_{C_{tk-L_i}}$ of S respectively

**Proof:** $(i) \Leftrightarrow (ii)$ is followed by Lemma 3.1.4

$(i) \Rightarrow (iii)$

$(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u)$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i=\sum_{j=1}^{n} g_j h_j k_j} \left\{ \left(\bigcap_{i=1}^{n} (\hat{\lambda}_A(g_i) \mathbin{\hat{\cap}} \hat{\lambda}_B(h_i) \mathbin{\hat{\cap}} \hat{\lambda}_C(k_i)\right) \mathbin{\hat{\cap}} \left(\bigcap_{j=1}^{n} (\hat{\lambda}_A(g_j) \mathbin{\hat{\cap}} \hat{\lambda}_B(h_j) \mathbin{\hat{\cap}} \hat{\lambda}_C(k_j)\right) \right\}$$

$\supseteq \hat{\lambda}_A(uab) \mathbin{\hat{\cap}} \hat{\lambda}_B(aub) \mathbin{\hat{\cap}} \hat{\lambda}_C(abu)$

$\supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_B(u) \mathbin{\hat{\cap}} \hat{\lambda}_C(u)$

$= (\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C)(u)$

Thus, $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \supseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$ and by Lemma 3.1.13 $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

Hence $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C = \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

$(iii) \Rightarrow (i)$

Let R, E and L be $\mathbb{K}_{t-R_i}$, $\mathbb{K}_{t-La_i}$ and $\mathbb{K}_{t-L_i}$. Then by Theorem 3.1.12 $\mathcal{C}_R, \mathcal{C}_E$ and $\mathcal{C}_L$ are $\hat{\lambda}^{si}_{tk-R_i}$, $\hat{\lambda}^{si}_{tk-La_i}$ and $\hat{\lambda}^{si}_{tk-L_i}$ of S respectively. Then by hypothesis

$\mathcal{C}_R \hat{\odot} \mathcal{C}_E \hat{\odot} \mathcal{C}_L = \mathcal{C}_R \hat{\cap} \mathcal{C}_E \hat{\cap} \mathcal{C}_L$

$\Rightarrow \mathcal{C}_{R \hat{\odot} E \hat{\odot} L} = \mathcal{C}_{R \cap E \cap L}$

$\Rightarrow R \hat{\odot} E \hat{\odot} L = R \hat{\cap} E \hat{\cap} L$

$\Rightarrow S$ is $\mathbb{K}_{t-R}$

**3.1.15. Theorem**

For $S$ with 1, the following are equivalent

(i) $S$ is ternary k-right weakly regular

(ii) All $\mathbb{K}_{t-R_i}$ of S are idempotent

(iii) $M \hat{\cap} N \hat{\cap} O = MNO$, for any $M$, $N$ and $O$ as $\mathbb{K}_{t-R_i}$, $\mathbb{K}_{t-La_i}$ and $\mathbb{K}_{t-L_i}$ of S respectively

(iv) All $\hat{\lambda}^{si}_{tk-R_i}$ of S are fully idempotent

(v) $\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C = \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$, for any $\hat{\lambda}_A$, $\hat{\lambda}_B$ and $\hat{\lambda}_C$ as $\hat{\lambda}_{A}{}^{si}_{tk-R_i}$, $\hat{\lambda}_{B}{}^{si}_{tk-La_i}$ and $\hat{\lambda}_{C}{}^{si}_{tk-L_i}$ of S respectively

If $S$ satisfies the commutative law then (i) $\Rightarrow$(iv) are equivalent to

(vi) $S$ is Von-Neumann regular

**Proof:** $(i) \Rightarrow (iv)$

Let $\hat{\lambda}_A$ be a $\hat{\lambda}^{si}_{tk-R_l}$ of S and $w \in S$ Then

$$\hat{\lambda}^3(u) = (\hat{\lambda}_A \hat{\odot} \hat{\lambda}_A \hat{\odot} \hat{\lambda}_A)(u)$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{c} \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \,\hat{\cap}\, \hat{\lambda}_A(h_i) \,\hat{\cap}\, \hat{\lambda}_A(k_i) \right) \hat{\cap} \\ ( \bigcap_{j=1}^{n} (\hat{\lambda}_A(g_j) \,\hat{\cap}\, \hat{\lambda}_A(h_j) \,\hat{\cap}\, \hat{\lambda}_A(k_j)) \end{array} \right\}$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{c} \left( \bigcap_{i=1}^{m} \{\hat{\lambda}_A(g_i h_i k_i)\} \,\hat{\cap}\, \left(\hat{\lambda}_A(h_i) \,\hat{\cap}\, \hat{\lambda}_A(k_i)\right) \right) \hat{\cap} \\ ( \bigcap_{j=1}^{n} (\{\hat{\lambda}_A(g_j h_j k_j)\} \,\hat{\cap}\, (\hat{\lambda}_A(h_j) \,\hat{\cap}\, \hat{\lambda}_A(k_j))) \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{c} [\hat{\lambda}_A(\sum_{i=1}^{m}(g_i h_i k_i)) \,\hat{\cap}\, ( \bigcap_{i=1}^{m} \hat{\lambda}_A(h_i) \,\hat{\cap}\, \hat{\lambda}_A(k_i))] \,\hat{\cap} \\ [\hat{\lambda}_A(\sum_{j=1}^{n}(g_j h_j k_j)) \,\hat{\cap}\, ( \bigcap_{j=1}^{n} \hat{\lambda}_A(h_j) \,\hat{\cap}\, \hat{\lambda}_A(k_j))] \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{c} [\hat{\lambda}_A(\sum_{i=1}^{m}(g_i h_i k_i)) \,\hat{\cap}\, \hat{\lambda}_A(\sum_{j=1}^{n}(g_j h_j k_j))] \,\hat{\cap} \\ ( \bigcap_{i=1}^{m} \hat{\lambda}_A(h_i) \,\hat{\cap}\, \hat{\lambda}_A(k_i)) \,\hat{\cap}\, ( \bigcap_{j=1}^{n} \hat{\lambda}_A(h_j) \,\hat{\cap}\, \hat{\lambda}_A(k_j)) \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \hat{\lambda}_A(u) \,\hat{\cap}\, ( \bigcap_{i=1}^{m} \hat{\lambda}_A(h_i) \,\hat{\cap}\, \hat{\lambda}_A(k_i)) \,\hat{\cap}\, ( \bigcap_{j=1}^{n} \hat{\lambda}_A(h_j) \,\hat{\cap}\, \hat{\lambda}_A(k_j)) \right\}$$

$$\subseteq \hat{\lambda}_A(u)$$

Now

$$\hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(u)$$

$$\subseteq \{\hat{\lambda}_A(ua_i u) \,\hat{\cap}\, \hat{\lambda}_A(ub_i u) \,\hat{\cap}\, \hat{\lambda}_A(uc_i d_i) \,\hat{\cap}\, \hat{\lambda}_A(ua_j u) \,\hat{\cap}\, \hat{\lambda}_A(ub_j u) \,\hat{\cap}\, \hat{\lambda}_A(uc_j d_j)\}$$

$$\subseteq \{ \bigcap_{i=1}^{m} \{\hat{\lambda}_A(ua_i u) \,\hat{\cap}\, \hat{\lambda}_A(ub_i u) \,\hat{\cap}\, \hat{\lambda}_A(uc_i d_i)\} \,\hat{\cap}\, \bigcap_{j=1}^{n} \{\hat{\lambda}_A(ua_j u) \,\hat{\cap}\, \hat{\lambda}_A(ub_j u) \,\hat{\cap}\, \hat{\lambda}_A(uc_j d_j)\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} ua_i u(ub_i u)uc_i d_i=\sum_{j=1}^{n} ua_j u(ub_j u)uc_j d_j} \left\{ \begin{array}{c} \bigcap_{i=1}^{m}\{\hat{\lambda}_A(ua_i u) \Cap \hat{\lambda}_A(ub_i u) \Cap \hat{\lambda}_A(uc_i d_i)\} \Cap \\ \bigcap_{j=1}^{n}\{\hat{\lambda}_A(ua_j u) \Cap \hat{\lambda}_A(ub_j u) \Cap \hat{\lambda}_A(uc_j d_j)\} \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} x_i y_i z_i=\sum_{j=1}^{n} x_j y_j z_j} \left\{ \begin{array}{c} \bigcap_{i=1}^{m}\{\hat{\lambda}_A(x_i) \Cap \hat{\lambda}_A(y_i) \Cap \hat{\lambda}_A(z_i)\} \Cap \\ \bigcap_{j=1}^{n}\{\hat{\lambda}_A(x_j) \Cap \hat{\lambda}_A(y_j) \Cap \hat{\lambda}_A(z_j)\} \end{array} \right\}$$

$\subseteq (\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A)(u)$

$\subseteq (\hat{\lambda}_A{}^3)(u)$

Hence $\hat{\lambda}_A{}^3 = \hat{\lambda}_A$

$(iv) \Rightarrow (i)$

Let $w \in S$ and $A = wSS$ be a $\mathbb{K}_{t-R_l}$ of S generated by w Then $w \in A$ and the characteristic function $\mathcal{C}_A$ of $A$ is $\mathcal{C}_{A_{lk-R_l}}^{sl}$ of S and by hypothesis

$\mathcal{C}_A = \mathcal{C}_A \widehat{\odot} \mathcal{C}_A \widehat{\odot} \mathcal{C}_A = \mathcal{C}_{A^3}$

$\Rightarrow A = A^3$

$\Rightarrow w \in A^3 = (wSS)^3$

$\Rightarrow w \in wSwSwSS$

Thus S is $\mathbb{K}_{trw-r}$

$(i) \Rightarrow (v)$

$(\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_B \widehat{\odot} \hat{\lambda}_C)(u)$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i=\sum_{j=1}^{n} g_j h_j k_j} \left\{ (\bigcap_{i=1}^{n}(\hat{\lambda}_A(g_i) \Cap \hat{\lambda}_B(h_i) \Cap \hat{\lambda}_C(k_i)) \Cap (\bigcap_{j=1}^{n}(\hat{\lambda}_A(g_j) \Cap \hat{\lambda}_B(h_j) \Cap \hat{\lambda}_C(k_j)) \right\}$$

$\supseteq \hat{\lambda}_A(uab) \Cap \hat{\lambda}_B(aub) \Cap \hat{\lambda}_C(abu)$

$\supseteq \hat{\lambda}_A(u) \Cap \hat{\lambda}_B(u) \Cap \hat{\lambda}_C(u)$

$= (\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C)(u)$

Thus, $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \supseteq \hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C$

Since S is $\mathbb{K}_{trw-r}$ so for $w \in S$ can be written as $u = \sum_{i=1}^{n} ua_i ub_i uc_i d_i$ where

$a_i, c_i, b_i, d_i \in S$

Now $(\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C)(u) = \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_B(u) \mathbin{\hat{\cap}} \hat{\lambda}_C(u)$

$\subseteq \hat{\lambda}_A(ua_i u) \mathbin{\hat{\cap}} \hat{\lambda}_B(ub_i u) \mathbin{\hat{\cap}} \hat{\lambda}_C(uc_i d_i)$

Thus $(\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C)(u)$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} ua_i u(ub_i u)uc_i d_i = \sum_{j=1}^{n} ua_j u(ub_j u)uc_j d_j} \left\{ \begin{array}{l} \bigcap_{i=1}^{m} \{\hat{\lambda}_A(ua_i u) \mathbin{\hat{\cap}} \hat{\lambda}_B(ub_i u) \mathbin{\hat{\cap}} \hat{\lambda}_C(uc_i d_i)\} \mathbin{\hat{\cap}} \\ \bigcap_{j=1}^{n} \{\hat{\lambda}_A(ua_j u) \mathbin{\hat{\cap}} \hat{\lambda}_B(ub_j u) \mathbin{\hat{\cap}} \hat{\lambda}_C(uc_j d_j)\} \end{array} \right\}$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} x_i y_i z_i = \sum_{j=1}^{n} x_j y_j z_j} \left\{ \begin{array}{l} \bigcap_{i=1}^{m} \{\hat{\lambda}_A(x_i) \mathbin{\hat{\cap}} \hat{\lambda}_B(y_i) \mathbin{\hat{\cap}} \hat{\lambda}_C(z_i)\} \mathbin{\hat{\cap}} \\ \bigcap_{j=1}^{n} \{\hat{\lambda}_A(x_j) \mathbin{\hat{\cap}} \hat{\lambda}_B(y_j) \mathbin{\hat{\cap}} \hat{\lambda}_C(z_j)\} \end{array} \right\}$$

$= (\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u)$

$$\Rightarrow \hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$$

$$\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C = \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$$

$(v) \Longrightarrow (iii)$ Let E, F and G be $\mathbb{K}_{t-R_i}$, $\mathbb{K}_{t-La_i}$ and $\mathbb{K}_{t-L_i}$ Then by Theorem 3 1 12.

$\mathcal{C}_E, \mathcal{C}_F$ and $\mathcal{C}_G$ are $\hat{\lambda}^{si}_{tk-R_i}$, $\hat{\lambda}^{si}_{tk-La_i}$ and $\hat{\lambda}^{si}_{tk-L_i}$ of S respectively Then by hypothesis

$\mathcal{C}_M \hat{\odot} \mathcal{C}_N \hat{\odot} \mathcal{C}_O = \mathcal{C}_M \mathbin{\hat{\cap}} \mathcal{C}_N \mathbin{\hat{\cap}} \mathcal{C}_O$

$\Rightarrow \mathcal{C}_{M \hat{\odot} N \hat{\odot} O} = \mathcal{C}_{E = M \hat{\cap} N \hat{\cap} O}$

$\Rightarrow M \hat{\odot} N \hat{\odot} O = M \mathbin{\hat{\cap}} N \mathbin{\hat{\cap}} O$

Similarly $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ and $(i) \Longrightarrow (vi)$ are straightforward

**3.1.16. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$ Then every $\hat{\lambda}^{si}_{B_{tk-L_i}}(\hat{\lambda}^{si}_{A_{tk-R_i}}, \hat{\lambda}^{si}_{A_{tk-La_i}})$ is an $\hat{\lambda}^{si}_{A_{tk-Q_i}}$

**Proof** Proof is straightforward

**3.1.17. Remark**

Now we show that the converse of the Lemma 3 1 16 is not true in general

**3.1.18. Example**

Let $S = \left\{\begin{pmatrix} p & q \\ r & s \end{pmatrix}\ p, q, r, s \in Z^- \cup \{0\}\right\}$ and $A = \left\{\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\ p \in Z^- \cup \{0\}\right\}$ Then $S$ is a ternary semiring under usual operations, and $A$ is a $\mathbb{K}_{t-Q_i}$ of $S$ but $A$ is not $\mathbb{K}_{t-R_i}$, $\mathbb{K}_{t-La_i}$ and $\mathbb{K}_{t-L_i}$ of $S$ Then by Theorem 3 1 12 $\mathcal{C}_A$ is an $\mathcal{C}^{si}_{A_{tk-Q_i}}$ but not $\mathcal{C}^{si}_{A_{tk-L_i}}\left(\mathcal{C}^{si}_{A_{tk-La_i}}, \mathcal{C}^{si}_{A_{tk-R_i}}\right)$

**3.1.19. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$. Then every $\hat{\lambda}^{si}_{A_{tk-Q_i}}$ is an $\hat{\lambda}^{si}_{A_{tk-B_i}}$

**Proof** Proof is straightforward

**3.1.20. Theorem**

For a $T_{sr}$, following are equivalent

(i) S is $\mathbb{K}_{t-r}$

(ii) $\hat{\lambda}_A \subseteq \hat{\lambda}_A \widetilde{\odot}\ \mathbb{C} \widetilde{\odot}\ \hat{\lambda}_A \widetilde{\odot}\ \mathbb{C} \widetilde{\odot}\ \hat{\lambda}_A$ for every $\hat{\lambda}_{A_{tk-Q_i}}^{si}$ of S

(iii) $\hat{\lambda}_A \subseteq \hat{\lambda}_A \widetilde{\odot}\ \mathbb{C} \widetilde{\odot}\ \hat{\lambda}_A \widetilde{\odot}\ \mathbb{C} \widetilde{\odot}\ \hat{\lambda}_A$ for every $\hat{\lambda}_{A_{tk-B_i}}^{si}$ of S

**Proof:** $(i) \Rightarrow (iii)$

Let $\hat{\lambda}_A$ be a $\hat{\lambda}_{A_{tk-B_i}}^{si}$ of $S$ Then for any $u \in S$ there exist $g_1, g_2 \in S$ such that

$u + ug_1u = ug_2u$

Now

$$(\hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A)(u)$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{l} (\bigcap_{i=1}^{n} (\hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A)(g_i) \cap \mathbb{C}(h_i) \cap \hat{\lambda}_A(k_i) \cap \\ (\bigcap_{j=1}^{n} (\hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A)(g_j) \cap \mathbb{C}(h_j) \cap \hat{\lambda}_A(k_j)) \end{array} \right\}$$

$$\supseteq \bigcap_{i=1}^{n} (\hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A)(umu) \cap \bigcap_{j=1}^{n} (\hat{\lambda}_A \widehat{\odot} \mathbb{C} \widehat{\odot} \hat{\lambda}_A)(unu) \cap \hat{\lambda}_A(u)$$

$$= \left( \begin{array}{l} \bigcap_{i=1}^{n} \left( \bigcup_{ul_1u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{l} \left( \bigcap_{i=1}^{n} (\hat{\lambda}_A(g_i) \widehat{\cap} \hat{\lambda}_A(k_i) \right) \widehat{\cap} \\ (\bigcap_{j=1}^{n} \left( \hat{\lambda}_A(g_j) \widehat{\cap} \hat{\lambda}_A(k_j) \right) \end{array} \right\} \right) \cap \\ \bigcap_{j=1}^{n} \left( \bigcup_{um_1u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{l} \left( \bigcap_{i=1}^{n} (\hat{\lambda}_A(g_i) \widehat{\cap} \hat{\lambda}_A(k_i) \right) \widehat{\cap} \\ (\bigcap_{j=1}^{n} \left( \hat{\lambda}_A(g_j) \widehat{\cap} \hat{\lambda}_A(k_j) \right) \end{array} \right\} \right) \cap \hat{\lambda}_A(u) \end{array} \right)$$

$$\supseteq \left( (\hat{\lambda}_A(ul_1ul_2u) \widehat{\cap} \hat{\lambda}_A(ul_3ul_4u) \right) \widehat{\cap} \hat{\lambda}_A(um_1um_2u) \widehat{\cap} \hat{\lambda}_A(um_3um_4u) \cap \hat{\lambda}_A(u)$$

$$\supseteq \hat{\lambda}_A(u)$$

$(iii) \Longrightarrow (ii)$ is straightforward by using Lemma 3 1 19

$(ii) \Longrightarrow (i)$ Let Q be any $\mathbb{K}_{t-Q_i}$ of S Then by Theorem 3 1 12 $C_Q$ is an ${C_Q}^{si}_{tk-Q_i}$ of S

Now by using the given condition, we can write

$$C_Q \subseteq C_Q \hat{\odot} \mathbb{C} \hat{\odot} C_Q \hat{\odot} \mathbb{C} \hat{\odot} C_Q = C_{\overline{Q\mathbb{C}Q\mathbb{C}Q}}$$

$$\Rightarrow Q \subseteq \overline{Q\mathbb{C}Q\mathbb{C}Q}$$

Also, we know that $\overline{Q\mathbb{C}Q\mathbb{C}Q} \subseteq \overline{\mathbb{C}QQ} \mathbin{\hat{\cap}} \overline{Q\mathbb{C}Q} \mathbin{\hat{\cap}} \overline{QQ\mathbb{C}} = Q$ Therefore by Theorem 2 2 4 S is $\mathbb{K}_{t-r}$

**3.1.21. Theorem**

For a $T_{sr}$ S, the following statements are equivalent

(i) S is $\mathbb{K}_{t-r}$

(ii) $\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$ for every ${\hat{\lambda}_A}^{si}_{tk-Q_i}$, ${\hat{\lambda}_B}^{si}_{tk-La_i}$ and ${\hat{\lambda}_C}^{si}_{kt-L_i}$

(iii) $\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$ for every ${\hat{\lambda}_A}^{si}_{tk-B_i}$, ${\hat{\lambda}_B}^{si}_{tk-La_i}$ and ${\hat{\lambda}_C}^{si}_{kt-L_i}$

(iv) $\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$ for every ${\hat{\lambda}_A}^{si}_{tk-R_i}$, ${\hat{\lambda}_B}^{si}_{tk-La_i}$ and ${\hat{\lambda}_C}^{si}_{tk-Q_i}$

(v) $\hat{\lambda}_A \mathbin{\hat{\cap}} \hat{\lambda}_B \mathbin{\hat{\cap}} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$ for every ${\hat{\lambda}_A}^{si}_{tk-R_i}$, ${\hat{\lambda}_B}^{si}_{tk-La_i}$ and ${\hat{\lambda}_C}^{si}_{tk-B_i}$

**Proof** $(i) \Longrightarrow (iii)$ Let $\hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C$ be any ${\hat{\lambda}_A}^{si}_{tk-B_i}$, ${\hat{\lambda}_B}^{si}_{tk-La_i}$ and ${\hat{\lambda}_C}^{si}_{tk-L_i}$ respectively Since S is $\mathbb{K}_{t-r}$, then for any $u \in S$ there exist $v, w \in S$ such that $u + ugu = uhu$. Now,

$$(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u)$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i = \sum_{j=1}^{n} g_j h_j k_j} \left\{ \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(g_i) \mathbin{\hat{\cap}} \hat{\lambda}_B(h_i) \mathbin{\hat{\cap}} \hat{\lambda}_C(k_i) \right) \mathbin{\hat{\cap}} \left( \bigcap_{j=1}^{n} (\hat{\lambda}_A(g_j) \mathbin{\hat{\cap}} \hat{\lambda}_B(h_j) \mathbin{\hat{\cap}} \hat{\lambda}_C(k_j)) \right) \right\}$$

$$\supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_B(ugu) \mathbin{\hat{\cap}} \hat{\lambda}_B(uhu) \mathbin{\hat{\cap}} \hat{\lambda}_C(gku) \mathbin{\hat{\cap}} \hat{\lambda}_C(hku)$$

$$\supseteq \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_B(u) \hat{\cap} \hat{\lambda}_C(u)$$

$$\supseteq (\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C)(u)$$

$(iii) \Longrightarrow (ii)$ is straightforward

$(ii) \Longrightarrow (i)$ Let $\hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C$ be any $\hat{\lambda}_{A_{kt-R_i}}^{si}, \hat{\lambda}_{B_{kt-La_i}}^{si}$ and $\hat{\lambda}_{C_{kt-L_i}}^{si}$ By Lemma 3 1 16 $\hat{\lambda}_{A_{tk-R_i}}^{si}$ is an $\hat{\lambda}_{A_{tk-Q_i}}^{si}$ Then by hypothesis, we have $\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C$ Also by Lemma 3 1 13, we have

$\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$. Hence $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C = \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

Thus S is k-ternary regular

$(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ are straightforward

## 3.2. $[X, Y]$ Soft Intersectional k- Ternary Ideals

Here we discuss [X,Y] soft intersectional ternary k-subsemiring, [X,Y] soft intersectional ternary k-ideals and investigate some related results

In our next discussion we use $\phi \subseteq X \subset Y \subseteq Z$

### 3.2.1. Definition

$\hat{\lambda}_A \in CS(Z)$ is known as $[X, Y]$ *Soft intersectional ternary k − subsemiring* of $S$ if

(i) $\hat{\lambda}_A(u+v) \cup X \supseteq \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v) \hat{\cap} Y \quad \forall u, v \in S$

(ii) $\hat{\lambda}_A(uvw) \cup X \supseteq$

$\hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v) \hat{\cap} \hat{\lambda}_A(w) \hat{\cap} Y \quad \forall u, v, w \in S$

(iii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \hat{\cap} \hat{\lambda}_A(h) \hat{\cap} Y$

$[X, Y]$ Soft intersectional ternary k-subsemiring is represented by $\hat{\lambda}_{A_{tk-SS}}^{[X,Y]^{st}}$

### 3.2.2. Example

Let $S = \{o, \acute{e}, \acute{f}\}$ with defined "+" and " " as follows

| + | $o$ | $\acute{e}$ | $\acute{f}$ |
|---|---|---|---|
| $o$ | $o$ | $\acute{e}$ | $\acute{f}$ |
| $\acute{e}$ | $\acute{e}$ | $o$ | $\acute{f}$ |
| $\acute{f}$ | $\acute{f}$ | $\acute{f}$ | $o$ |

| | $o$ | $\acute{e}$ | $\acute{f}$ |
|---|---|---|---|
| $o$ | $o$ | $o$ | $o$ |
| $\acute{e}$ | $o$ | $o$ | $o$ |
| $\acute{f}$ | $o$ | $o$ | $\acute{f}$ |

| . | $o$ | $\acute{e}$ | $\acute{f}$ |
|---|---|---|---|
| $o$ | $o$ | $o$ | $o$ |
| $\acute{f}$ | $o$ | $o$ | $\acute{f}$ |

Define a soft set $\hat{\lambda}_A$ of S over $Z = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ such that $\hat{\lambda}_A(0) = \{1, 2, 3, 4\}$, $\hat{\lambda}_A(\acute{e}) = \{0, 1, 2, 3\}$ and $\hat{\lambda}_A(\acute{f}) = \{1, 2, 3\}$ If $X = \{0, 1, 2\}$ and $Y = \{0, 1, 2, 3\}$, then one can easily check that $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{tk-SS}}^{[X,Y]^{st}}$ but it is not $\hat{\lambda}_{A_{tk-SS}}^{st}$, since $\hat{\lambda}_A(0) \not\supseteq \hat{\lambda}_A(u)$

### 3.2.3. Definition

$\hat{\lambda}_A \in CS(Z)$ is known as $[X, Y]$ Soft intersectional ternary k- left ideal ( ternary k-right ideal, ternary k- lateral ideal) if

(i) $\hat{\lambda}_A(u + v) \cup X \supseteq \hat{\lambda}_A(u) \,\hat{\cap}\, \hat{\lambda}_A(v) \,\hat{\cap}\, Y$ $\qquad \forall\, u, v \in S$

(ii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \,\hat{\cap}\, \hat{\lambda}_A(h) \,\hat{\cap}\, Y$

(iii) $\hat{\lambda}_A(uvw) \cup X \supseteq \hat{\lambda}_A(w) \widehat{\cap} Y$ { $\hat{\lambda}_A(uvw) \cup X \supseteq \hat{\lambda}_A(u) \widehat{\cap} Y$, $\hat{\lambda}_A(uvw) \cup X \supseteq \hat{\lambda}_A(v) \widehat{\cap} Y$} and will be denoted by $\hat{\lambda}_{A_{tk-L_i}}^{[X,Y]^{SI}}$ ($\hat{\lambda}_{A_{tk-R_i}}^{[X,Y]^{SI}}$, $\hat{\lambda}_{A_{tk-La_i}}^{[X,Y]^{SI}}$).

A soft set $\hat{\lambda}_A \in CS(Z)$ is called $[X,Y]$ Soft intersectional ternary k-ideal ($\hat{\lambda}_{A_{tk-I}}^{[X,Y]^{SI}}$) if it is $\hat{\lambda}_{A_{tk-L_i}}^{[X,Y]^{SI}}$, $\hat{\lambda}_{A_{tk-R_i}}^{[X,Y]^{SI}}$ and $\hat{\lambda}_{A_{tk-La_i}}^{[X,Y]^{SI}}$

### 3.2.4. Definition

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS(Z)$ Then

$$\hat{\lambda}_A \subseteq \hat{\lambda}_B \Leftrightarrow \left(\hat{\lambda}_A(u) \widehat{\cap} Y\right) \cup X \subseteq \left(\hat{\lambda}_B(u) \widehat{\cap} Y\right) \cup X, \qquad \forall u \in S$$

### 3.2.5. Definition

Let $\hat{\lambda}_A, \hat{\lambda}_B \in CS(Z)$ Then $\hat{\lambda}_A = \hat{\lambda}_B \Leftrightarrow \hat{\lambda}_A \subseteq \hat{\lambda}_B$ *and* $\hat{\lambda}_B \subseteq \hat{\lambda}_A$

Obviously, $\hat{\lambda}_A(0) \cup X \supseteq \hat{\lambda}_A(u) \widehat{\cap} Y$ *and* $(\hat{\lambda}_A(0) \widehat{\cap} Y) \cup X \supseteq (\hat{\lambda}_A(u) \widehat{\cap} Y) \cup X \quad \forall u \in S$.

### 3.2.6. Theorem

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{tk-SS}}^{[X,Y]^{SI}}$ iff $\hat{\lambda}_A$ satisfies

(i) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \widehat{\cap} \hat{\lambda}_A(h) \widehat{\cap} Y$

(ii) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(iii) $\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A \subseteq \hat{\lambda}_A$

**Proof:**

Suppose $\hat{\lambda}_A \in CS(Z)$be an $\hat{\lambda}_{A_{tk-SS}}^{[X,Y]^{SI}}$ Let $u \in S$ Then

$$((\hat{\lambda}_A \oplus \hat{\lambda}_A)(u) \cap Y) \cup X$$

$$= \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1) \widehat{\cap} \hat{\lambda}_A(p_2) \widehat{\cap} \hat{\lambda}_A(q_1) \widehat{\cap} \hat{\lambda}_A(q_2)) \widehat{\cap} Y\} \cup X$$

$$= \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{(\hat{\lambda}_A(p_1) \widehat{\cap} \hat{\lambda}_A(q_1) \widehat{\cap} Y) \widehat{\cap} (\hat{\lambda}_A(p_2) \widehat{\cap} \hat{\lambda}_A(q_2) \widehat{\cap} Y)) \widehat{\cap} Y\} \cup X$$

$$\subseteq \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{((\hat{\lambda}_A(p_1+q_1) \cup X) \widehat{\cap} (\hat{\lambda}_A(p_2+q_2) \cup X) \widehat{\cap} Y\} \cup X$$

$$= \bigcup_{u+(p_1+q_1)=(p_2+q_2)} \{((\hat{\lambda}_A(p_1+q_1)) \cap (\hat{\lambda}_A(p_2+q_2) \cap Y) \widehat{\cap} Y\} \cup X$$

$$\subseteq \bigcup_{(u+(p_1+q_1)=(p_2+q_2)} (\hat{\lambda}_A(u) \widehat{\cap} Y) \cup X$$

$$= (\hat{\lambda}_A(u) \widehat{\cap} Y) \cup X$$

It follows that $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

Now

$$\left((\hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A \widehat{\odot} \hat{\lambda}_A)(u) \widehat{\cap} Y\right) \cup X$$

$$= \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i=\sum_{j=1}^{n} g_j h_j k_j} \left\{ \begin{array}{l} \left(\bigcap_{i=1}^{n}(\hat{\lambda}_A(g_i) \widehat{\cap} \hat{\lambda}_A(h_i) \widehat{\cap} \hat{\lambda}_A(k_i)\right) \widehat{\cap} \\ (\bigcap_{j=1}^{n}(\hat{\lambda}_A(g_j) \widehat{\cap} \hat{\lambda}_A(h_j) \widehat{\cap} \hat{\lambda}_A(k_j)) \widehat{\cap} Y \end{array} \right\} \cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i=\sum_{j=1}^{n} g_j h_j k_j} \left\{ (\bigcap_{i=1}^{n} \hat{\lambda}_A(g_i h_i k_i)) \mathbin{\widehat{\cap}} (\bigcap_{j=1}^{n} \hat{\lambda}_A(g_j h_j k_j) \mathbin{\widehat{\cap}} Y \right\}$$

$$\cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} g_i h_i k_i=\sum_{j=1}^{n} g_j h_j k_j} \left\{ (\bigcap_{i=1}^{n} \bigcap_{j=1}^{n} \hat{\lambda}_A(u) \mathbin{\widehat{\cap}} Y \right\} \cup X$$

$$\subseteq (\hat{\lambda}_A (u) \mathbin{\widehat{\cap}} Y) \cup X$$

Conversely

$$(\hat{\lambda}_A(u+v) \mathbin{\widehat{\cap}} Y) \cup X$$

$$\supseteq (\hat{\lambda}_A(u+v) \mathbin{\widehat{\cap}} Y) \cup X$$

$$= ((\hat{\lambda}_A \oplus \hat{\lambda}_A)(u+v) \mathbin{\widehat{\cap}} Y) \cup X$$

$$= \bigcup_{u+(g_1+h_1)=(g_2+h_2)} \{(\hat{\lambda}_A(g_1) \mathbin{\widehat{\cap}} \hat{\lambda}_A(g_2) \mathbin{\widehat{\cap}} \hat{\lambda}_A(h_1) \mathbin{\widehat{\cap}} \hat{\lambda}_A(h_2)) \mathbin{\widehat{\cap}} Y\} \cup X$$

$$\supseteq (( \hat{\lambda}_A(0) \mathbin{\widehat{\cap}} \hat{\lambda}_A(u) \mathbin{\widehat{\cap}} \hat{\lambda}_A(v)) \mathbin{\widehat{\cap}} Y) \cup X$$

$$= (((\hat{\lambda}_A(0) \cup X) \mathbin{\widehat{\cap}} (\hat{\lambda}_A(u) \mathbin{\widehat{\cap}} Y) \mathbin{\widehat{\cap}} (\hat{\lambda}_A(v) \mathbin{\widehat{\cap}} Y)) \mathbin{\widehat{\cap}} Y) \cup X$$

$$\supseteq (\hat{\lambda}_A(u) \mathbin{\widehat{\cap}} \hat{\lambda}_A(v) \mathbin{\widehat{\cap}} Y) \cup X$$

Hence $\hat{\lambda}_A(u+v) \cup X \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y$ holds

$\hat{\lambda}_A(uvw) \cup X \supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v) \mathbin{\hat{\cap}} \hat{\lambda}_A(w) \mathbin{\hat{\cap}} Y$ is analogous Thus $\hat{\lambda}_A$ is $\hat{\lambda}_{A_{tk-SS}}^{[X,Y]^{st}}$

### 3.2.7. Theorem

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{tk-L_i}}^{[X,Y]^{st}}$ $(\hat{\lambda}_{A_{tk-R_i}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{tk-La_i}}^{[X,Y]^{st}})$ iff $\hat{\lambda}$ satisfies

(i) $\hat{\lambda}_A \oplus \hat{\lambda}_A \subseteq \hat{\lambda}_A$

(ii) If $u + g = h \Rightarrow \hat{\lambda}_A(u) \cup X \supseteq \hat{\lambda}_A(g) \mathbin{\hat{\cap}} \hat{\lambda}_A(h) \mathbin{\hat{\cap}} Y$

(iii) $\mathbb{C} \mathbin{\hat{\odot}} \hat{\lambda}_A \mathbin{\hat{\odot}} \hat{\lambda}_A \subseteq \hat{\lambda}_A$ $(\hat{\lambda}_A \mathbin{\hat{\odot}} \hat{\lambda}_A \mathbin{\hat{\odot}} \mathbb{C} \subseteq \hat{\lambda}_A, \hat{\lambda}_A \mathbin{\hat{\odot}} \mathbb{C} \mathbin{\hat{\odot}} \mathbb{C} \subseteq \hat{\lambda}_A)$

**Proof** Omitted (same as above Theorem 3 2 6 )

### 3.2.8. Theorem

Let $\phi \neq A \subseteq S$ Then A is $\mathbb{K}_{t-SS}$ $(\mathbb{K}_{t-L_i}, \mathbb{K}_{t-La_i}, \mathbb{K}_{t-R_i}, \mathbb{K}_{t-Q_i}, \mathbb{K}_{t-B_i})$

of $S \Leftrightarrow \mathcal{C}_A$ is $\hat{\lambda}_{A_{tk-SS}}^{[X,Y]^{st}}$ $(\hat{\lambda}_{A_{tk-L_i}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{tk-La_i}}^{[X,Y]^{st}}, \hat{\lambda}_{tk-R_i}^{[X,Y]^{st}}, \hat{\lambda}_{A_{tk-Q_i}}^{[X,Y]^{st}}, \hat{\lambda}_{A_{tk-B_i}}^{[X,Y]^{st}})$ of S

**Proof:** Straightforward

### 3.2.9. Example

Let $S = \{o, 1, \acute{e}, \acute{f}, \acute{g}\}$ be a $T_{sr}$ with the following defined addition and multiplication

| + | o | 1 | é | $\acute{f}$ | $\acute{g}$ |
|---|---|---|---|---|---|
| o | o | 1 | é | $\acute{f}$ | $\acute{g}$ |
| 1 | 1 | $\acute{f}$ | 1 | é | 1 |
| é | é | 1 | é | $\acute{f}$ | é |
| $\acute{f}$ | $\acute{f}$ | é | $\acute{f}$ | 1 | $\acute{f}$ |
| $\acute{g}$ | $\acute{g}$ | 1 | é | $\acute{f}$ | $\acute{g}$ |

&

| . | o | 1 | é | $\acute{f}$ | $\acute{g}$ |
|---|---|---|---|---|---|
| o | o | o | o | o | o |
| 1 | o | 1 | é | $\acute{f}$ | $\acute{g}$ |
| é | o | é | e | é | $\acute{g}$ |
| $\acute{f}$ | o | $\acute{f}$ | é | 1 | $\acute{g}$ |
| $\acute{g}$ | o | $\acute{g}$ | $\acute{g}$ | $\acute{g}$ | o |

Then, $A = \{o, \acute{g}\}$ is a $\mathbb{K}_{t-l}$ of $S$ One can easily show that $\mathcal{C}_A$ is an $\mathcal{C}_{A_{tk-l}}^{[X,Y]^{st}}$

**3.2.10. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{tk-l}}^{[X,Y]^{st}}$ if and only if each nonempty subset

$$U(\hat{\lambda}_A, \alpha) = \{u \in S \mid \hat{\lambda}_A(u) \supseteq \alpha \cap Y\}$$

is $\mathbb{K}_{t-l}$ of S for each $\alpha \subseteq U$ under the condition $\alpha \supseteq X$

**Proof** Let $\hat{\lambda}_A \in CS(Z)$ be an $\hat{\lambda}_{A_{tk-l}}^{[X,Y]^{st}}$ such that $\hat{\lambda}_A(u) \supseteq X$ for every $u \in S$ and $u, v \in U(\hat{\lambda}_A, \alpha)$ Then

$$\begin{aligned} \hat{\lambda}_A(u+v) &= \hat{\lambda}_A(u+v) \cup X \\ &\supseteq \hat{\lambda}_A(u) \mathbin{\hat{\cap}} \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y \\ &\supseteq \alpha \mathbin{\hat{\cap}} Y, \end{aligned}$$

which implies $u + v \in U(\hat{\lambda}_A, \alpha)$.

Next, we let $u \in S$ and $v, w \in U(\hat{\lambda}_A; \alpha)$ Then

$$\begin{aligned} \hat{\lambda}_A(uvw) &= \hat{\lambda}_A(uvw) \cup X \\ &\supseteq \hat{\lambda}_A(v) \mathbin{\hat{\cap}} Y \\ &\supseteq \alpha \mathbin{\hat{\cap}} Y \end{aligned}$$

$\Rightarrow vuw \in U(\hat{\lambda}_A, \alpha)$ Similarly, we get $wuv \in U(\hat{\lambda}_A, \alpha)$ for $v \in S$ *and* $u \in U(\hat{\lambda}_A, \alpha)$

Now, let $u \in S$ *and* $g, h \in U(\hat{\lambda}_A, \alpha)$ such that $u + g = h$ Then

$$\hat{\lambda}_A(u) = \hat{\lambda}_A(u) \cup X$$

$\supseteq \hat{\lambda}_A(g) \hat{\cap} \hat{\lambda}_A(h) \hat{\cap} Y$

$\supseteq \alpha \hat{\cap} Y$

$\Rightarrow u \in U(\hat{\lambda}_A, \alpha)$ Therefore, $U(\hat{\lambda}_A, \alpha)$ is an $\mathbb{K}_{t-I}$ of S

Conversely

Let each nonempty subset $U(\hat{\lambda}_A, \alpha)$ be an $\mathbb{K}_{t-I}$ of S Then, for $u, v \in S$ there are

$\alpha_1, \alpha_2 \subseteq U$ such that $\alpha_1 \supseteq X, \alpha_2 \supseteq X$ with $\hat{\lambda}_A(u) = \alpha_1$ *and* $\hat{\lambda}_A(v) = \alpha_2$ Thus, $\hat{\lambda}_A(u) \supseteq \alpha \supseteq \alpha \hat{\cap} Y$ and

$\hat{\lambda}_A(v) \supseteq \alpha \supseteq \alpha \hat{\cap} Y$ for $\alpha = \alpha_1 \hat{\cap} \alpha_2 \supseteq X$ Hence $u, v \in U(\hat{\lambda}_A, \alpha)$ Next $u + v \in U(\hat{\lambda}_A, \alpha)$ for u, $v \in U(\hat{\lambda}_A, \alpha)$,

since, $U(\hat{\lambda}_A, \alpha)$ is an $\mathbb{K}_{t-I}$ of S Then

$\hat{\lambda}_A(u + v) = \hat{\lambda}_A(u + v) \cup X$

$\supseteq \alpha \hat{\cap} Y$

$= \alpha_1 \hat{\cap} \alpha_2 \hat{\cap} Y$

$= \hat{\lambda}_A(u) \hat{\cap} \hat{\lambda}_A(v) \hat{\cap} Y$

The verification is complete

Also, we have $uv \in U(\hat{\lambda}_A, \alpha)$ for $u \in S$ and $v \in U(\hat{\lambda}_A, \alpha)$ Then

$\hat{\lambda}_A(uvw) \cup X = \hat{\lambda}_A(uvw)$

$\supseteq \alpha \hat{\cap} Y$

$= \alpha_1 \widehat{\cap} \alpha_2 \widehat{\cap} Y$

$= \hat{\lambda}_A(u) \widehat{\cap} \hat{\lambda}_A(v) \widehat{\cap} \hat{\lambda}_A(w) \widehat{\cap} Y$

$\supseteq \hat{\lambda}_A(u) \widehat{\cap} \alpha \widehat{\cap} Y$

$\supseteq \hat{\lambda}_A(u) \widehat{\cap} Y$

Similarly, we get $\hat{\lambda}_A(vuw) \cup X \supseteq \hat{\lambda}_A(v) \widehat{\cap} Y$ and $\hat{\lambda}_A(wvu) \cup X \supseteq \hat{\lambda}_A(w) \widehat{\cap} Y$ The verification is complete

Now we let $\hat{\lambda}_A(g) = \alpha_1, \hat{\lambda}_A(h) = \alpha_2$ *and* $u + g = h$ Then $\hat{\lambda}_A(g) \supseteq \alpha_1 \widehat{\cap} \alpha_2$ *and*

$\hat{\lambda}_A(h) \supseteq \alpha_1 \widehat{\cap} \alpha_2$ obviously So $g, h \in U(\hat{\lambda}, \alpha_1 \widehat{\cap} \alpha_2)$ Since $U(\hat{\lambda}_A, \alpha_1 \widehat{\cap} \alpha_2)$ is $\mathbb{K}_{t-I}$I,

then

$u \in U(\hat{\lambda}_A, \alpha_1 \widehat{\cap} \alpha_2)$ Thus

$\hat{\lambda}_A(u) \cup X = \hat{\lambda}_A(u)$

$\supseteq \alpha_1 \widehat{\cap} \alpha_2 \widehat{\cap} Y$

$= \hat{\lambda}_A(g) \widehat{\cap} \hat{\lambda}_A(h) \widehat{\cap} Y$

Hence $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{tk-I}}^{[X,Y]^{st}}$

**3.2.11. Lemma**

Let $\hat{\lambda}_A \in CS(Z)$ Then $\hat{\lambda}_A$ is an $\hat{\lambda}_{A_{tk-SS}}^{[X,Y]^{st}}$ ( $\hat{\lambda}_{A_{tk-I}}^{[X,Y]^{st}}$ ) if and only if

$U(\hat{\lambda}_A; \alpha) = \{x \in A \mid \hat{\lambda}_A(u) \supseteq \alpha \widehat{\cap} Y\}$ is $\mathbb{K}_{t-SS}$ ($\mathbb{K}_{t-I}$) of S

**Proof** - Similar to previous Lemma 3 2 10

**3.2.12. Defination**

Let $\hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C \in CS(Z)$ Then

(v) $\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B = ((\hat{\lambda}_A \cap \hat{\lambda}_B) \hat{\cap} Y) \cup X$

(vi) $\hat{\lambda}_A \cup \hat{\lambda}_B = ((\hat{\lambda}_A \cup \hat{\lambda}_B) \hat{\cap} Y) \cup X$

(vii) $\hat{\lambda}_A \oplus \hat{\lambda}_B = ((\hat{\lambda}_A \oplus \hat{\lambda}_B) \hat{\cap} Y) \cup X$

(viii) $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C = ((\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C) \hat{\cap} Y) \cup X$

**3.2.13. Lemma**

Let $\hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C \in CS(Z)$ are $\hat{\lambda}_{A_{tk-R_i}}^{[X,Y]^{si}}, \hat{\lambda}_{B_{tk-La_i}}^{[X,Y]^{si}}, \hat{\lambda}_{C_{tk-L_i}}^{[X,Y]^{si}}$ respectively Then

$\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

**Proof:**

Let $u \in S$ If $(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u) = \phi$ or $(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)(u) = X$ Then

$\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

Otherwise $(\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C)$(u)

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i = \sum_{j=1}^{n} p_j q_j r_j} \left\{ \begin{array}{c} \left( \left( \bigcap_{i=1}^{m} (\hat{\lambda}_A(p_i) \hat{\cap} \hat{\lambda}_B(q_i) \hat{\cap} \hat{\lambda}_C(r_i) \right) \hat{\cap} \right. \\ \left. \left( \bigcap_{j=1}^{n} (\hat{\lambda}_A(p_j) \hat{\cap} \hat{\lambda}_B(q_j) \hat{\cap} \hat{\lambda}_C(r_j)) \right) \right) \end{array} \right\}$$

$$= \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i = \sum_{j=1}^{n} p_j q_j r_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} ((\hat{\lambda}_A(p_i) \hat{\cap} Y) \hat{\cap} (\hat{\lambda}_B(q_i) \hat{\cap} Y) \hat{\cap} (\hat{\lambda}_C(r_i) \hat{\cap} Y) \right) \hat{\cap} \\ \left( \bigcap_{j=1}^{n} ((\hat{\lambda}_A(p_j) \hat{\cap} Y)) \hat{\cap} (\hat{\lambda}_B(q_j) \hat{\cap} Y) \hat{\cap} (\hat{\lambda}_C(r_j) \hat{\cap} Y) \right) \end{array} \right) \hat{\cap} Y \right\} \cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} \begin{array}{c} ((\hat{\lambda}_A(p_i q_i r_i) \cup X) \hat{\cap} (\hat{\lambda}_B(p_i q_i r_i) \cup X) \hat{\cap} \\ (\hat{\lambda}_C(p_i q_i r_i) \cup X) \end{array} \right) \hat{\cap} \\ (\bigcap_{j=1}^{n} \begin{array}{c} \left( \left( \hat{\lambda}_A \left( p_j q_j r_j \right) \cup X \right) \right) \hat{\cap} \left( \hat{\lambda}_B \left( p_j q_j r_j \right) \cup X \right) \hat{\cap} \\ (\hat{\lambda}_C(p_j q_j r_j) \cup X) \end{array} \end{array} \right) \hat{\cap} Y \right\}$$

$$\cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ \left( \begin{array}{c} \left( \bigcap_{i=1}^{m} \begin{array}{c} ((\hat{\lambda}_A(p_i q_i r_i) \hat{\cap} Y) \hat{\cap} (\hat{\lambda}_B(p_i q_i r_i) \hat{\cap} Y) \hat{\cap} \\ (\hat{\lambda}_C(p_i q_i r_i) \hat{\cap} Y) \end{array} \right) \hat{\cap} \\ (\bigcap_{j=1}^{n} \begin{array}{c} \left( \left( \hat{\lambda}_A \left( p_j q_j r_j \right) \hat{\cap} Y \right) \right) \hat{\cap} \left( \hat{\lambda}_B \left( p_j q_j r_j \right) \hat{\cap} Y \right) \hat{\cap} \\ (\hat{\lambda}_C(p_j q_j r_j) \hat{\cap} Y) \end{array} \end{array} \right) \hat{\cap} Y \right\}$$

$$\cup X$$

$$\subseteq \bigcup_{u+\sum_{i=1}^{m} p_i q_i r_i=\sum_{j=1}^{n} p_j q_j r_j} \left\{ \left( \begin{array}{c} \hat{\lambda}_A \left( \sum_{i=1}^{m} p_i q_i r_i \right) \hat{\cap} \hat{\lambda}_B \left( \sum_{i=1}^{m} p_i q_i r_i \right) \hat{\cap} \hat{\lambda}_C \left( \sum_{i=1}^{m} p_i q_i r_i \right) \hat{\cap} \\ \hat{\lambda}_A \left( \sum_{j=1}^{n} p_j q_j r_j \right) \hat{\cap} \hat{\lambda}_B \left( \sum_{j=1}^{n} p_j q_j r_j \right) \hat{\cap} \hat{\lambda}_C \left( \sum_{j=1}^{n} p_j q_j r_j \right) \end{array} \right) \hat{\cap} Y \right\}$$

$$\cup X$$

$$\subseteq \quad (\hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C)(u)$$

Thus, $\hat{\lambda}_A \hat{\odot} \hat{\lambda}_B \hat{\odot} \hat{\lambda}_C \subseteq \hat{\lambda}_A \hat{\cap} \hat{\lambda}_B \hat{\cap} \hat{\lambda}_C$

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