Lie Symmetry Analysis for the Flow of Nano-Fluids Over Stretching Surface

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A Dissertation
Submitted in the Partial Fulfilment of the
Requirements of the Degree of
MASTER of SCIENCES
IN
MATHEMATICS

Supervised by

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بسم الله الرحمن الرحيم

Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MS IN STATISTICS

We accept this dissertation as conforming to the required standard.

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2016

Dedication

I dedicate this dissertation to my family specially the most important person of my life, my beloved Baji for her endless efforts and encouragement they did for me. Her prayers and support are the huge asset of my life ever.

Usman Ali

Acknowledgement

Firstly, I am thankful to Almighty Allah. Who created us. Taught us everything we did not know. Provided us with balance, health, knowledge and intelligence to explore His world. I offer salutation upon the Hoy Prophet Hazrat Muhammad (PBUH), who has lightened the life of all mankind with his guidance. He is a source of knowledge and blessings for the entire creations. His teachings make us to ponder and to explore this world with direction of Islam.

Secondly, I express my profound gratitude to my respected supervisor **Dr. Ahmed Zeeshan** who helped me throughout my MS study with their cooperation guidance and guided me to complete my thesis within due course of time. This thesis would not have been successfully completed without their valuable intellectual tutelage, words of encouragement. Push for tenacity and opportunity to gain from their wealth of knowledge. I am also pay my regards to all my teachers who always directed me to right dimensions and made it possible for me to achieve high goals in my life.

I extended my gratitude to my family for encouragement and support, even in the gloomiest of times. My special thanks goes to **Mrs. Farhat Munir** (Baji), Naseem Akhtar and Abdul Rehman whose prayers and encouragement are the huge asset of my life ever.

I am also very thankful to my friends (7 Idiots) specially Aqib Majeed, Nasir Shahzad, Mudassir Maskeen, Khalil Ur Rehman, Muhammad Rizwan, Zubair Ali Mughal, Aqeela Qaiser and Roqaiya Qaiser who are always there to pray during my study in all respects.

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Declaration

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the supervision of my supervisor Dr. Ahmad Zeeshan. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

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Preface

Flow of nano-fluids has gain vital importance in recent years. Choi [1] was the first to use the terminology for the fluids which have nano-sized particles suspended in the base liquid. The flow of such fluids have high thermo-dynamical applications [2-5]. Many scientists discussed the flow behaviours of nano-fluid in diverse geometery [6-15]. Flow and heat transfer of an incompressible viscous fluid over a stretching sheet appear in several industrial process such as extrusion of polymers, the cooling of metallic plates, the aero-dynamical extrusion of plastic sheets etc. In the glass industry, blowing, floating or spinning of fibers are processes which involve the flow due to stretching surface [16-22]. The study of heat transfer and flow field is necessary for determining the quality of the final products of such processes. Sakiadas [2] presented the pioneering work in this field. He investigated the flow induced by a semi-infinite horizontally moving wall in an ambient fluid. The flow problems in such situations are complicated giving non-linear coupled partial differential equations. These equations are very hard to solve even with modern computational techniques. Lie group developed by Sophus Lie [23] gives a technique which reduces the independent variables of PDE and hence, reduce it to ODEs. Problems which can be solved numerically or analytically using many techniques. Many mathematicians have implemented those techniques successfully [24-30].

This thesis discusses the Lie group solution of the flow problem arises due to flow of nano-fluid over a stretching sheet. For this purpose three chapter are formulated.

In first chapter, we present some definitions and basic concepts relating the problem. Lie symmetry technique is also discussed.

In second chapter, MHD flow of the Newtonian fluid with variable viscosity over a stretching sheet with heat transfer is taken into account. The problem is solved using Lie-group method. Similarity transforms are generated and checked for consistency on boundary conditions and equations. The transform is then used to convert PDE to ODE. The ODEs are then solved using shooting method. The chapter has two section. In first section, the problem is modelled using continuity, Navier-Stoke's and energy equation. In second section, Lie symmetry is applied to solve the problem.

In third chapter, we investigated the flow of MHD nano-fluid over a stretching surface with slip effects in porous medium using Lie Symmetry analysis. The governing equations describing law of conservation of mass, momentum and energy are converted to system of

ODEs using similarity transform. Generators formed are checked for consistency. Equations are solved numerically and graphical results are displayed.

Contents

Chapter 1

1.1 Introduction

In this chapter we present some definitions and basic concepts relating the problem. Lie symmetry technique is also discussed.

1.2 Basics of Fluid

1.2.1 Fluid Mechanics

Fluid mechanics is the branch of applied mathematics in which we study the behaviour of fluids in the states of rest as well as in motion.

1.2.2 Pressure

Pressure is an effect which occurs when a force is applied on a surface per unit area. Mathematically,

$$P = \frac{F}{A}, \tag{1.1}$$

where P is pressure, F is the normal force and A is the area. The SI unit for pressure is Pascal (Pa), equal to one Newton per square meter $(Nm^{-2}$ or $kgm^{-1}s^{-2})$.

1.2.3 Density

Density is a measure of how much mass is contained in a given unit volume, i.e,

$$\rho = \frac{m}{V}, \tag{1.2}$$

where m is the mass and V is the volume.

1.2.4 Viscosity

Viscosity is a measure of the resistance of a fluid to deformation under shear stress. It is commonly perceived as "thickness" or resistance to pouring. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. Thus, methanol is "thin" having a low viscosity, while vegetable oil is "thick" having a high viscosity.

1.2.5 Fluid Mechanics

Fluid mechanics is the study of fluids which can be divided into fluid kinematics (the study of fluid motion) and fluid dynamics (the study of the effect of forces on fluid motion) which can further be divided into fluid statics (the study of fluids at rest) and fluid kinetics (the study of fluids in motion).

1.2.6 Fluid Kinematics

Kinematics is the branch of mechanics that deals with quantities involving space and time only. It is used to describe the motions of particles and objects, but does not take the forces that cause these motions into account.

1.3 Fluid Dynamics

Fluid dynamics is a sub-discipline of fluid mechanics that deals with fluid flow---the natural science of fluids (liquids and gases) in motion.
Fluid statics is the science of fluids at rest, and is a sub-field within fluid mechanics.

1.4 Classification of Fluid

1.4.1 Ideal Fluid

A non-existent, assumed fluid without either viscosity or compressibility is called an ideal fluid or perfect fluid. In nature this type of fluid does not exist. Furthermore, a gas subject to *Boyle's-Charle's law* is called a perfect or an ideal gas. It is the hypothetical form of fluid. However, the fluid with negligible viscosity may be considered as an ideal fluid.

1.4.2 Real Fluid

Real fluids are those in which fluid friction has significant effects on the fluid motion. In other words we cannot neglect the viscosity effects on the motion. Real fluids are further classified into two classes on the basis of *Newton's law of viscosity*. Shear stress is directly proportional to the rate of deformation. For one dimensional flow it can be written as

$$\tau_{yx} = \mu \frac{du}{dy}, \qquad (1.3)$$

where τ_{yx} is the shear stress and du/dy is the rate of deformation.

1.4.3 Newtonian Fluid

A Newtonian fluid (named after *Isaac Newton*) is a fluid whose stress versus strain (deformation) rate curve is linear and passes through the origin, *i.e*, Newtonian fluid obeys *Newton's law of viscosity*. Water, gasoline and mercury are some examples of Newtonian fluids.

1.4.4 Non-Newtonian Fluid

A non-Newtonian fluid is a fluid whose flow properties are not described by a single constant value of viscosity, i.e., it does not satisfy *Newton's law of viscosity*. For non-Newtonian fluids or

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n, \qquad n \neq 1$$

$$\tau_{yx} = \eta\left(\frac{du}{dy}\right), \tag{1.4}$$

where

$$\eta = k\left(\frac{du}{dy}\right)^{n-1}$$

is the apparent viscosity. Examples of non-Newtonian fluids are tooth paste, ketchup, gel, shampoo, blood, soaps etc. Here we discuss one of many types of non-Newtonian namely Nano-fluid.

1.5 Non-Newtonian Nano-fluid

Nano-fluid is a fluid containing Nano-meter-sized particles, called nanoparticles. These fluids are engineered colloidal suspensions of nanoparticles in a base fluid. The nanoparticles used in Nano-fluids are typically made of metals, oxides, carbides, or carbon nanotubes. Common base fluids include water and ethylene glycol.

1.6 Classification of Fluid Flow -- Based on Flow Pattern

While studying the motion of a rigid body we do not have to bother about the relative motion of the particles of the rigid body as they are very firmly fixed to each other and move as a whole. But for the study of the motion of fluids, things are not so simple because the fluid particles are attached with each other with very weak forces. There are various relative motions and a lot of possibilities for relative motion between the fluid particles.
To make things somewhat simple or for making the flow analysis feasible, fluid flow is visualized as a composition of fluid elements. These elements are defined by using certain similarities or patterns and mathematics is applied to them to study fluid flow comprehensively.

1.6.1 Rotational or Irrotational Flow

To classify any flow as rotational or irrotational the angular motion of the fluid elements is analysed. If the angle between the two intersecting lines of the boundary of the fluid element changes while moving in the flow, then the flow is a Rotational Flow. But if the fluid element rotates as a whole and there is no change in angles between the boundary lines then the flow cannot be rotational flow, so it is irrotational flow.

1.6.2 Laminar Flow

The flow of a fluid moving with a moderate speed has fluid layers moving past other layers as if some sheets are moving over other layers. Such flow of fluids is called Laminar Flow.
In Laminar Flow viscous shear stresses act between these layers of the fluid which defines the velocity distribution among these layers of flow. In laminar flows the shear stresses are defined by Newton's equation for shear stress.

1.6.3 Turbulent Flow

As the flow speed of the otherwise calm layers increases, these smoothly moving layers start moving randomly, and with further increase in flow velocity, the flow of fluid particles becomes completely random and no such laminar layers exist anymore. Shear stresses in the turbulent flow are more than those in laminar flow.

1.7 Classification of Fluid Flow--Based on Variation with Time and Space

When a fluid flows past a point or through a path different parameters associated with the flow of the fluid, certain parameters vary and others may remain constant.
The two basic parameters of any fluid flow are velocity of the fluid particle or element and the

pressure of the fluid at the point under consideration. The flow of fluids can be classified in different patterns based on the variation of the flow parameters with time and distance. The benefit of characterizing the fluid flow as certain patterns helps in analysing it under the appropriate solution paradigm.

1.7.1 Classification Based on Variation with Time

The classification of the fluid flow based on the variation of the fluid flow parameters with time characterizes the flow in two categories, steady and unsteady flow.

1.7.2 Steady and Unsteady Flow

If the flow parameters, such as velocity, pressure, density and discharge do not vary with time or are independent of time then the flow is steady. If the flow parameters vary with time then the flow is categorized as unsteady.

In real conditions it is very rare to have such flows with parameters exactly constant with time. The parameters usually vary with time but variation is within a small range such as the average of particular parameter is constant for certain duration of time.

1.7.3 Classification Based on Variation with Space

The other classification criterion for the fluid flow is based on the variation of the flow parameters with distance or space. It characterizes the flow as uniform or non-uniform.

1.8 Uniform or Non-Uniform Flow

The fluid flow is a uniform flow if the flow parameters remain constant with distance along the flow path. The fluid flow is non-uniform if the flow parameters vary and are different at different points on the flow path.

For a uniform flow, by its definition, the area of the cross section of the flow should remain constant. So a fitting example of the uniform flow is the flow of a liquid thorough a pipeline of constant diameter and contrary to this the flow through a pipeline of variable diameter would be necessarily non-uniform.

1.9 Flow Types

A steady flow can be uniform or non-uniform and similarly an unsteady flow can also be uniform or non-uniform. For a steady flow discharge is constant with time and for a uniform flow the area of cross section of the fluid flow is constant through the flow path.

1.9.1 Steady and Uniform Flow

Flow through a pipeline of constant diameter with a discharge constant with time.

1.9.2 Steady and Non-Uniform Flow

Fixed discharge flow through a tapering pipe. Water flow through a river with a constant discharge is also a good example of such flow as the span of river generally varies with distance and amount of water flow in river is constant.

1.9.3 Unsteady and Uniform Flow

A flow through pipeline of constant cross section with sudden changes in fluid discharge or pressure.

1.9.4 Unsteady and Non-Uniform Flow

Pressure surges in a flow through a pipe of variable cross section. A practical example can be the water flow in the network of canals during water release.

1.10 Fluid Flow Analysis -- Different Approaches

1.10.1 System Approach

A problem is half solved if it is defined properly. Like we use free body diagrams to solve the problems in mechanics, we define a system to solve problems of fluid flows.

A system is defined as a quantity of mass separated from surroundings by system boundaries across which no mass transfer occur. The boundaries of system can be moveable. Basic laws are applied to this system to solve fluid flow problems. This system approach is helpful in analysis of simple flows through channels or pipes where a fixed mass for analysis can be defined and tracked as it flows.

1.10.2 Control Volume Approach

For flows through complex shapes and machines like compressors or turbines it is difficult to define and track a particular mass. Thus, for analysis of flow we define a control volume and study the flow through this volume. It boundaries can coincide with the real physical boundaries of objects or can be imaginary boundaries defined for analysis. Control volume approach can be used to find flow velocities at different ends of the control volume and also can be used for force and motion analysis of the fluid flow.

1.10.3 Differential Approach

The analysis of fluid flow can be done by considering infinitesimal elements of system or control volume. This gives differential equations defining the flow and their solutions provide detailed picture of the flow.

1.10.4 Integral Approach

For overall analysis of the fluids finite elements of system or control volume are considered. It gives integral formulation, which is simple in analysis and gives overall picture of the fluid behaviour.

1.10.5 Lagrangian Approach

In Lagrangian approach fluid is considered to be formed of small fluid particles. The motion of these fluid particles is tracked and laws of particle mechanics are applied to them for analysis. With the increasing number of particles analysis becomes cumbersome.

1.10.6 Eulerian Approach

In Eulerian approach properties of fluid flow, such as, velocity, acceleration, pressure and density, are described as function of space and time. This provides a picture of the properties of flow at every point in space as it varies with time. This formulation of the flow field allows detailed mathematically analysis of any flow field.

These basic approaches are equally applicable to all fluid flow problems but Sometimes even in analysis of some simple fluid flow problems closed results cannot be obtained. In such problems numerical and experimental approaches are used.

1.11 Fluid Energy

A fluid possesses energy in various forms. When applied to a fluid, the first law of thermodynamics relates the change in the internal, kinetic, and potential energies of a mass of fluid to the work done on that fluid plus the heat added to the fluid. Changes in the energy content of a fluid are important in many applications. In some applications a fluid does work (e.g., turbines, windmills, waterwheels), in other applications work is done on the fluid (e.g., pumps, fans, compressors).

1.12 Internal Energy

The internal energy of a mass of fluid is a macroscopic measure of microscopic (molecular, atomic, and subatomic) energy content.

1.13 Kinetic Energy

The energy associated with fluid in motion is called kinetic energy, E_k. This energy is proportional to the mass of fluid in the system, and to the square of the fluid speed, V. For a mass of fluid the total kinetic energy is given by

$$E_k = \frac{1}{2}MV^2. \qquad (1.5)$$

The corresponding kinetic energy per unit mass is

$$e_k = \frac{1}{2}V^2, \qquad (1.6)$$

and a kinetic energy per unit volume is given by:

$$\rho e_k = \rho\frac{1}{2}V^2, \qquad (1.7)$$

1.14 Potential Energy

A change in the gravitational potential energy of a fluid occurs whenever the fluid moves with, or against, the force of gravity. Suppose we chose a coordinate system with the $z-$ axi vertical. Then the gravitational potential energy, E_G, of a small volume of fluid at height z, relative to the potential energy the volume of fluid has at the origin, is given by

$$E_G = Mgz. \qquad (1.8)$$

The potential energy per unit mass is given by

$$e_G = gz \qquad (1.9)$$

and the potential energy per unit volume is

$$\rho e_G = \rho g z. \qquad (1.10)$$

1.14 Heat Transfer

Heat transfer is a discipline of thermal engineering that concerns the transfer of thermal energy from one physical system to another. Heat transfer is classified into various mechanisms, such as heat conduction, convection, thermal radiation and phase-change transfer.

1.15 Conduction

Conduction is the transfer of energy through matter from particle to particle. It is the transfer and distribution of heat energy from atom to atom within a substance. For example, a spoon in a cup of hot soup becomes warmer because the heat from the soup is conducted along the spoon. Conduction is most effective in solids-but it can happen in fluids. Fun fact: Have you ever noticed that metals tend to feel cold? Believe it or not, they are not colder? They only feel colder because they conduct heat away from your hand. You perceive the heat that is leaving your hand as cold.

1.16 Convection

Convection is the transfer of heat by the actual movement of the warmed matter. Heat leaves the coffee cup as the currents of steam and air rise. Convection is the transfer of heat energy in a gas or liquid by movement of currents. (It can also happen is some solids, like sand.) The heat moves with the fluid. Consider this: convection is responsible for making macaroni rise and fall in a pot of heated water. The warmer portions of the water are less dense and therefore, they rise. Meanwhile, the cooler portions of the water fall because they are denser.

1.17 Thermal Radiation

Thermal radiation is electromagnetic radiation emitted from all matter due to its possessing thermal energy which is measured by the temperature of the matter.
Examples of thermal radiation are an incandescent light bulb emitting visible-light, infrared radiation emitted by a common household radiator or electric heater, as well as radiation from hot gas in outer space.

1.18 Diffusion

Diffusion is one of the fundamental processes by which material moves. It is thus important in biology and medicine, chemistry and geology, engineering and physics, and in just about every aspect of our lives. Diffusion is a consequence of the constant thermal motion of atoms, molecules, and particles, and results in material moving from areas of high to low concentration. Thus the end result of diffusion would be a constant concentration, throughout space, of each of the components in the environment.

1.19 Brownian Diffusion

Particles with a diameter smaller then $1\mu m$ exhibit irregular and random motion because their masses are small enough to render fluctuation by the bombardment of gas molecules. As a result of random motion particles as whole move toward to low concentration region from a high concentration region. This phenomenon in which similar to gas molecules, is referred to as Brownian diffusion of particles.

1.20 Thermophoresis

Thermophoresis, thermos-diffusion, or Soret effect, or Ludwig-Soret effect, is a phenomenon observed when a mixture of two or more types of motile particles (particles able to move) are subjected to the force of a temperature gradient and the different types of particles respond to it differently. The term "Sorét effect" (or Ludwig-Sorét effect) is normally intended to mean thermophoresis in liquids only. The word "thermophoresis" is most often intended to mean the behaviour in aerosols, not liquids, but the broader meaning is also common. The mechanisms of thermophoresis in liquid mixtures differ from those in gas mixtures, and are generally not as well understood.

1.21 Lie Symmetry

The invariance of the dependent and independent variables of the system under transformation will be discussed. Our main focus is special type of transformation, called Lie symmetry. A Lie symmetry depends on continuous parameters which map each solution of the equation to another solution of the same equation. Before we define a Lie Symmetry, we will state the more usual term of any kind of symmetry for differential equations.

1.21.1 Definition: Symmetry

The symmetry of a given differential equation means a transformation which maps one solution of the given differential equation to another.

Such a general definition of symmetry permits a huge variety of transformations under which the differential equations can be invariant.

Lie introduced one of such transformations in the form of point transformations.

Lie's method leads to group-invariant solutions and conservation laws applied to partial differential equations (PDEs). New solutions can be derived from known ones by exploiting the symmetries of PDEs. PDEs can be classified into equivalence classes. Furthermore, group-invariant solutions obtained via Lie's approach may provide insight into the physical models themselves and explicit solutions can serve as benchmarks in the design, accuracy testing, and comparison of numerical algorithms. Nowadays, the concept of symmetry plays a key role in the study and development of mathematics and physics. Indeed, the theory of Lie groups and Lie algebras is applied to diverse fields of mathematics including differential geometry, algebraic topology, bifurcation theory etc. Lie's original ideas greatly influenced the study of physically important systems of differential equations in classical and quantum mechanics, fluid dynamics, elasticity and many other applied areas. The application of Lie group methods to concrete physical systems involves tedious computations.

Consider the one parameter Lie group of infinitesimal transformations in (x,z,t,u,v,w,p,θ) given by

$$x^* = x + \varepsilon\xi^1(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.11)$$

$$z* = z + \varepsilon\xi^2(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.12)$$

$$t* = t + \varepsilon\xi^3(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.13)$$

$$u* = u + \varepsilon\mu^1(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.14)$$

$$v* = v + \varepsilon\mu^2(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.15)$$

$$w* = w + \varepsilon\mu^3(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.16)$$

$$p* = p + \varepsilon\mu^4(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.17)$$

$$\theta* = \theta + \varepsilon\mu^4(x,z,t,u,v,w,p,\theta) + O(\varepsilon^2) \quad (1.18)$$

where ε is the Lie group parameter.

The associated Lie algebra of infinitesimal symmetries is the set of the vector field of the form

$$X = \xi^1(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial x} + \xi^2(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial z}$$

$$+\xi^3(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial t} + \mu^1(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial u}$$

$$+\mu^2(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial v} + \mu^3(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial w}$$

$$+\mu^4(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial p} + \mu^5(x,z,t,u,v,w,p,\theta)\frac{\partial}{\partial \theta} \qquad (1.20)$$

The action of X is extended to all derivatives through the second prolongation

$$X^{(2)} = X + \mu_x^1\frac{\partial}{\partial u_x} + \mu_z^1\frac{\partial}{\partial u_z} + \mu_t^1\frac{\partial}{\partial u_t} + \mu_z^2\frac{\partial}{\partial v_z} + \mu_t^2\frac{\partial}{\partial v_t} + \mu_z^3\frac{\partial}{\partial w_z} + \mu_t^3\frac{\partial}{\partial w_t}$$

$$+\mu_z^4\frac{\partial}{\partial p_z} + \mu_z^5\frac{\partial}{\partial \theta_z} + \mu_t^5\frac{\partial}{\partial \theta_t} + \mu_{zz}^1\frac{\partial}{\partial u_{zz}} + \mu_{zz}^2\frac{\partial}{\partial v_{zz}} + \mu_{zz}^3\frac{\partial}{\partial w_{zz}} + \mu_{zz}^5\frac{\partial}{\partial \theta_{zz}} \qquad (1.21)$$

1.22 Uses and Applications of Lie Symmetry

Lie demonstrated that many techniques for finding solutions of differential equations can be unified and extended by considering symmetry groups. Today, we know several applications of Lie groups in the theory of differential equations. The most important ones are summarized below:

- Reduction of the order for ordinary differential equations,
- Mapping solutions to other solutions,
- Reduction of the number of independent variables of partial differential equations,
- Construction of invariant solutions,
- Construction of invariant solutions to boundary value problems,
- Construction of conservation laws,
- Detection of linearizing transformations of PDEs.

1.23 Nano-fluids

A nano-fluid is a fluid containing nano-meter sized particles, called nanoparticles. These fluids are engineered colloidal suspensions of nanoparticles in a base fluid. The nanoparticles used in Nano-fluids are typically made of metals, oxides, carbides, or carbon nanotubes. Common base fluids include water, ethylene glycol and oil. With a rich history of colloidal science and recent advances in particle synthesis methods, Nano-fluids have recently been engineered for a rapidly increasing number of applications. Nano-fluids are fundamentally characterized by the fact that Brownian agitation overcomes any settling motion due to gravity. Thus, a stable Nano-fluid is theoretically possible as long as particles stay small enough (usually <100nm). Maintaining this size, however, can be a challenge since particles frequently come into contact with each other - potentially leading to the formation of large particle agglomerates which can settle out of suspension. The term 'Nano-fluid' also indicates a mixture where the properties of both the nanoparticles and the base fluid contribute to the application. In this sense, a Nano-fluid is created when nanoparticles are controllably dispersed into a base fluid to enhance its properties.

Since Choi's seminal publication in 1995, the amount of published work dealing with Nano-fluids per year has increased rapidly - growing at an average of around 32% per year for the past five years.

1.24 Nano-fluid Synthesis Method

A nano-fluid can be synthesized by mixing a Nano-powder in a liquid. In fact the process is more involved. Carbon nanotube, metal oxide, carbide, nitride, and other Nano-powders can all be readily purchased from Nano-material manufacturers. Due to their chemical inertness, these Nano-powders can usually be handled outside glove boxes or other sealed containers during the preparation of the Nano-fluid. Sonication at high intensity and over extended periods of time is usually sufficient to break up the agglomerated powders and form a well-dispersed nanoparticle.

Chapter 2

Lie Group Analysis of Magnetoyhdrodynamic Flow of Newtonian Fluid with Temperature Dependent Viscosity Over a Stretching Surface

In this chapter, MHD flow of the Newtonian fluid with variable viscosity over a stretching sheet with heat transfer is taken into account. The problem is solved using Lie-group method. Similarity transforms are generated and checked for consistency on boundary conditions and equations. The transform is then used to convert PDE to ODE. The ODEs are then solved using shooting method. The chapter has two section. In first section, the problem is modelled using continuity, Navier-Stoke's and energy equation. In second section, Lie symmetry is applied to solve the problem. In this chapter, we have reviewed the article [1]

2.1 Mathematical Formulation

Consider a two dimensional boundary layer flow with temperature dependent viscosity over a stretching surface. The flow is supposed to be incompressible and no-slip condition is applied. The flow takes place for $y \geq 0$, where y-axis is taken perpendicular to the plate along $x-axis$. Also, the plate is maintained at temperature T_w, where ambient fluid temperature is T_∞. Using law of conservation of mass, moment and energy, the basic equations are

$$\nabla \cdot \overline{V} = 0 \tag{2.1}$$

$$\rho(\frac{\partial \overline{V}}{\partial t} + \overline{V} \cdot \nabla \overline{V}) = -\nabla p + \mu \nabla^2 \overline{V} + \overline{J} \times \overline{B} \tag{2.2}$$

$$\rho_c(\frac{\partial T}{\partial t} + \overline{V} \cdot \nabla T) = k \nabla^2 T \tag{2.3}$$

Where $\overline{V}$ is the velocity vector, in two-dimensional *i. e* $\overline{V} = (u, v, 0)$, T is the temperature, ρ is the density and μ is the viscosity of the fluid, k is thermal conductivity, $\overline{J}$ is the joul current and $\overline{B}$ is the uniform transverse magnetic field. If

$$\bar{\mu} = \mu^{*}(a + b(T_w - T)), \tag{2.4}$$

where μ^{*} is the reference viscosity. The flow equation becomes

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{2.5}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{\rho}\frac{\partial \bar{\mu}}{\partial \bar{T}}\frac{\partial \bar{T}}{\partial \bar{y}}\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\bar{\mu}}{\rho}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \sigma\frac{B_\circ^2}{\rho}\bar{u}, \tag{2.6}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{2.7}$$

along with the following boundary conditions [2]

$$\begin{aligned} &(i) \quad \bar{u} = c\bar{x},\ \bar{v} = 0,\ \bar{T} = T_w \ at\ \bar{y} = 0, \\ &(ii) \quad \bar{u} \to 0,\ \bar{T} \to T_\infty \ as\ \bar{y} \to y_\infty, \end{aligned} \tag{2.8}$$

here σ is the electrical conductivity of the fluid, $B_\circ$ is the strength of the uniform magnetic field and c is the constant. The variables in Eqs (2.5) - (2.8) hence dimensionless using the following transformation

$$x = \frac{c\bar{x}}{U_1},\ y = \sqrt{\frac{c}{v}}\bar{y},\ u = \frac{\bar{u}}{U_1},\ v = \frac{\bar{v}}{\sqrt{c\upsilon}},\ T = \frac{\bar{T} - T_\infty}{T_w - T_\infty}, \tag{2.9}$$

where U_1 is the characteristic velocity and $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity.

The equations (2.5)-(2.7) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.10}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-A\frac{\partial T}{\partial y}\frac{\partial u}{\partial y}+(a+A(1-T))\frac{\partial^2 u}{\partial y^2}-M^2u, \tag{2.11}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2}, \tag{2.12}$$

where $M^2=\frac{\sigma B_0^2}{\rho C}$ is the Hartmann number, $A=b(T_w-T_\infty)$ is the viscosity parameter and Pr= $\frac{\nu}{\alpha}$ is the Prandtle number. The corresponding boundary conditions (2.8) are

$$\begin{aligned} &u=x, v=0, T=1 \ at \ y=0,\\ &u\to 0, T\to 0 \ as \ y\to\infty. \end{aligned} \tag{2.13}$$

Using equation (2.10), the stream function can be defined as

$$u(x,y)=\frac{\partial\Psi}{\partial y},\quad v(x,y)=-\frac{\partial\Psi}{\partial x}. \tag{2.14}$$

Substituting (2.14) into (2.11)-(2.12), we get

$$\Psi_y\Psi_{xy}-\Psi_x\Psi_{yy}+AT_y\Psi_{yy}-(a+A(1-T))\Psi_{yyy}+M^2\Psi_y=0, \tag{2.15}$$

$$\Psi_yT_x-\Psi_xT_y-\frac{1}{\Pr}T_{yy}=0, \tag{2.16}$$

with boundary conditions

$$\begin{aligned} &\Psi_y=x,\ \Psi_x=0,\ T=1 \ at \ y=0,\\ &\Psi_y\to 0,\ T\to 0 \ as \ y\to\infty, \end{aligned} \tag{2.17}$$

where subscripts denotes partial derivatives.

2.2 Solution Procedure

Now we determine the similarity solutions using Lie-group method for which (2.15) and (2.16) and the boundary conditions (2.17) are invariant and then similarity variables can be found by using these symmetries.

Let us take one-parameter ε Lie group of infinitesimal transformation in $(x,y,\psi,T)\to(x^*,y^*,\psi^*,T^*)$

$$\begin{aligned} x^* &= x+\varepsilon\phi(x,y;\Psi,T)+O(\varepsilon^2),\\ y^* &= y+\varepsilon\zeta(x,y;\Psi,T)+O(\varepsilon^2),\\ \Psi^* &= \Psi+\varepsilon\eta(x,y;\Psi,T)+O(\varepsilon^2),\\ T^* &= T+\varepsilon F(x,y;\Psi,T)+O(\varepsilon^2). \end{aligned} \tag{2.18}$$

Where ε is small parameter.

A system of PDE's (2.15)-(2.16) satisfies a symmetry generated by the vector field

$$X=\phi\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial \Psi}+F\frac{\partial}{\partial T}. \tag{2.19}$$

If remains invariant by the transformation $(x,y;\Psi,T)\to(x^*,y^*;\Psi^*,T^*)$.

The solution $\Psi=\Psi(x,y)$ and $T=T(x,y)$ remains invariants under the symmetry (2.19) if

$$\varphi_\Psi = X(\Psi-\Psi(x,y))=0 \; when \; \Psi=\Psi(x,y), \tag{2.20}$$

$$\varphi_T = X(T-T(x,y))=0 \; when \; T=T(x,y). \tag{2.21}$$

Let us assume

$$\Delta_1=\Psi_y\Psi_{xy}-\Psi_x\Psi_{yy}+AT_y\Psi_{yy}-(a+A(1-T))\Psi_{yyy}+M^2\Psi_y, \tag{2.22}$$

$$\Delta_2=\Psi_yT_x-\Psi_xT_y-\frac{1}{\Pr}T_{yy}. \tag{2.23}$$

A vector X given by (2.19) is called Lie Point Symmetry vector field for (2.22) and (2.23) if

$$X^{[3]}(\Delta_j)|_{\Delta_j=0}, j=1,2, \tag{2.24}$$

where

$$\begin{aligned} X^{[3]} &= \phi\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial \psi}+F\frac{\partial}{\partial T}+\eta^{x}\frac{\partial}{\partial \Psi_x}+\eta^{y}\frac{\partial}{\partial \Psi_y}+F^{x}\frac{\partial}{\partial T_x}+F^{y}\frac{\partial}{\partial T_y}+\eta^{xy}\frac{\partial}{\partial \Psi_{xy}} \\ &+\eta^{yy}\frac{\partial}{\partial \Psi_{yy}}+F^{yy}\frac{\partial}{\partial T_{yy}}+\eta^{yyy}\frac{\partial}{\partial \Psi_{yyy}} \end{aligned} \tag{2.25}$$

is the third prolongation of X.

We need to differentiate (2.18) w.r.t each of the variables to get the prolongation of the given transformation. For this, we have the following total derivatives.

$$\begin{aligned} D_x &= \partial_x+\Psi_x\partial_\Psi+T_x\partial_T+\Psi_{xx}\partial_{\Psi_x}+T_{xx}\partial_{T_x}+\Psi_{xy}\partial_{\Psi_y}+\dots\ , \\ D_y &= \partial_y+\Psi_y\partial_\Psi+T_y\partial_T+\Psi_{yy}\partial_{\Psi_y}+T_{yy}\partial_{T_y}+\Psi_{xy}\partial_{\Psi_x}+\dots\ . \end{aligned} \tag{2.26}$$

Equations (2.24) produces the following system of linear PDEs.

$$AF\Psi_{yyy}-\Psi_{yy}\eta^{x}+(\Psi_{xy}+M^2)\eta^{y}+A\Psi_{yy}F^{y}+\Psi_y\eta^{xy}+(AT_y-\Psi_x)\eta^{yy}-(a+A(1-T))\eta^{yyy}=0, \tag{2.27}$$

$$-T_y\eta^{x}+T_x\eta^{y}+\Psi_yF^{x}-\Psi_xF^{y}-\frac{1}{\Pr}F^{yy}=0\ . \tag{2.28}$$

The components $\eta^{x},\eta^{y},F^{x},F^{y},\eta^{xy},\eta^{yy},F^{yy},\eta^{yyy}$ are to be found from the following

$$\eta^{s}=D_s\eta-\Psi_x D_s\phi-\Psi_y D_s\zeta,$$
$$F^{s}=D_sF-T_x D_s\phi-T_y D_s\zeta,$$
$$\eta^{js}=D_s\eta^{j}-\Psi_{jx}D_s\phi-\Psi_{jy}D_s\zeta,$$
$$F^{js}=D_sF^{j}-T_{jx}D_s\phi-T_{jy}D_s\zeta, \tag{2.29}$$

where s implies x and j implies y.

Substituting (2.29) into (2.27) and (2.28) and solving the obtained equations with the boundary conditions (2.17), we get

$$\phi=C_1x,\ \zeta=C_2,\ \eta=C_1\Psi+C_3,\ F=0\ . \tag{2.30}$$

so the non-linear equations (2.15) and (2.16) have three parameter Lie group of point symmetries which gives

$$X_1=x\frac{\partial}{\partial x}+\Psi\frac{\partial}{\partial\Psi},X_2=\frac{\partial}{\partial y},\ X_3=\frac{\partial}{\partial\Psi} \tag{2.31}$$

Here, X_1 comprises of scaling that generates one parameter group, where X_2 and X_3 are the translation. The entry in the i^{th} row and j^{th} column is elaborated as $[X_i,X_j]=X_iX_j-X_jX_i$.

The finite transformations corresponding to the symmetries X_1, X_2, and X_3 are respectively

Table 1: Solution of the invariant surface conditions (2.30)-(2.31)

Generators	Characteristics $\varphi = (\varphi_\psi, \varphi_T)$	Solution of the invariant surface conditions
X_1	$\varphi_\psi = \psi - x\psi_x$ $\varphi_T = -T_x$	$\Psi = xG(y)$ $T = T(y)$
X_2	$\varphi_\psi = -\psi_y$ $\varphi_T = -T_y$	$\Psi = \Psi(x)$ $T = T(x)$
X_3	$\varphi_\psi = 1$ $\varphi_T = 0$	N0 solution

$$X_1 : x^* = e^{\varepsilon 1}x, y^* = y, \Psi^* = e^{\varepsilon 2}\Psi, T^* = T,$$
$$X_2 : x^* = x, y^* = y + \varepsilon_2, \Psi^* = \Psi, T^* = T,$$
$$X_3 : x^* = x, y^* = y, \Psi^* = \Psi, T^* = T, \tag{2.32}$$

where ϵ_1, ϵ_2 are group parameters.

Table 1 shows the solution of the invariant surface conditions (2.20) and (2.21)

For X_1 the characteristic,

$$\Phi = (\Phi_\Psi, \Phi_T) \tag{2.33}$$

has the components

$$\Phi_\Psi = \Psi - x\Psi_x, \Phi_T = -T_x \ . \tag{2.34}$$

Thus, the solution of the invariants (2.20) and (2.21) are

$$\Psi = xG(y), T = T(y) \ . \tag{2.35}$$

Substitution from (2.35) into (2.15) and (2.16) yields

$$(\frac{dG}{dy})^2 - G\frac{d^2G}{dy^2} + A\frac{dT}{dy}\frac{d^2G}{dy^2} - (a + A(1-T))\frac{d^2G}{dy^2} + M^2\frac{dG}{dy} = 0, \quad (2.36)$$

$$\frac{d^2T}{dy^2} + \Pr G\frac{dT}{dy} = 0 . \quad (2.37)$$

The boundary conditions (2.17) will be

$$\begin{aligned} &\frac{dG}{dy} = 1, G = 0, T = 1, y = 0 , \\ &\frac{dG}{dy} \to 0, T \to 0 \ as\ y \to \infty . \end{aligned} \quad (2.38)$$

For X_2 the characteristic (2.33) has the component

$$\Phi_\Psi = -\Psi_y,\ \Phi_T = -T_y . \quad (2.39)$$

Thus, the solutions of the invariant surface conditions (2.20) and (2.21) are $\Psi = \Psi(x)$ and T=T(x), which contradicts the boundary conditions.

For X_3, the characteristic (2.33) has the component

$$\phi_\psi = 1,\ \phi_T = 0 . \quad (2.40)$$

Therefore, no solution invariant under X_3.

2.3 Numerical Scheme

The system of non-linear differentiation equations (2.36)-(2.37) with the boundary conditions (2.38) is solved numerically by using the Runge-Kutta scheme based on shooting technique.

We take a=1 in all calculations. From (2.14) and (2.35), we get

$$\frac{u}{x} = \frac{dG}{dy}, v = -G(y), T = T(y) . \qquad (2.41)$$

2.4 Results and Discussion

The impacts on horizontal velocity u/x for Pr=1.0 and varations of Hartmann number M are appeared in Fig (2.1). It is observed that the horizontal velocity decreases by increasing M. The variation of the horizontal velocity u/x with change in viscosity parameter A in Fig (2.2). Presently it can be seen that the horizontal velocity increments by increasing A. The variations in the vertical velocity for Pr=1.0, A=0 and the Hartmann number M. Fig (2.3) demonstrates that the vertical velocity decreases by increasing M. The impacts of the Pr=1.0 with M=0, and different values of the viscosity parameter A are discussed in fig 2.4. It is seen that the vertical velocity increases by increasing A. Fig (2.5) uncovers the impacts on the temperature profiles T for Pr =1.0, A=0 and the Hartmann number M. We see that, the temperature increases as M increments. The impacts of the temperature profiles T for Pr=1.0, M=0 and the viscosity parameter A. As found in Fig (2.6), the temperature decreases as A increases. Fig (2.7) depicts the impacts of the temperature profiles T for M=0 with A=0 and different values of Prandtl number Pr. It is seen that, the temperature T reduces with an increase in Pr.

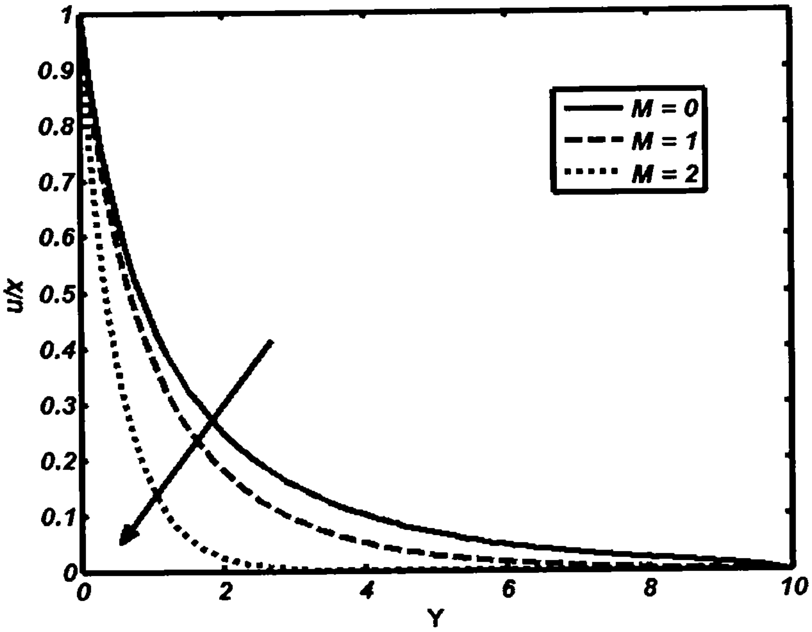

Fig 2.1: Horizontal velocity profiles for M with A=0 and Pr=1.0.

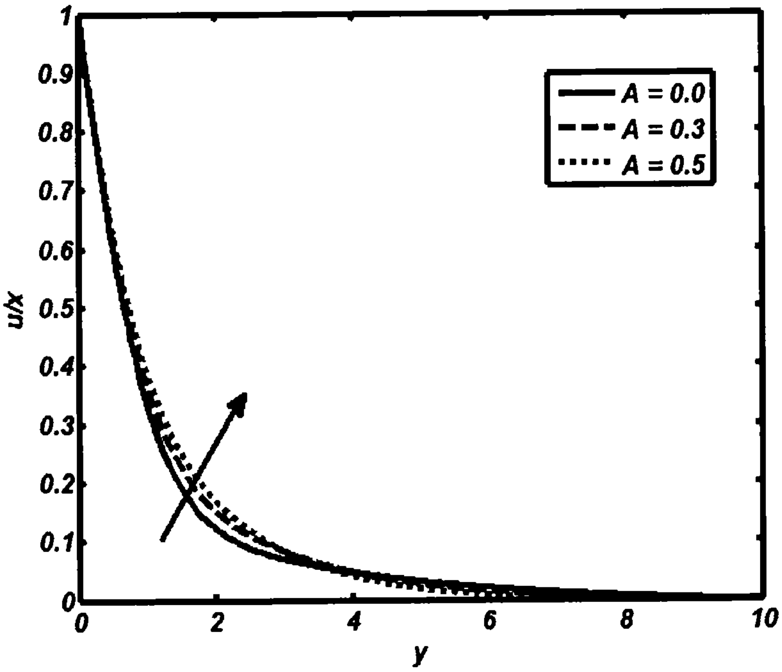

Fig 2.2: Horizontal velocity profiles for A with M=0 at Pr=1.0.

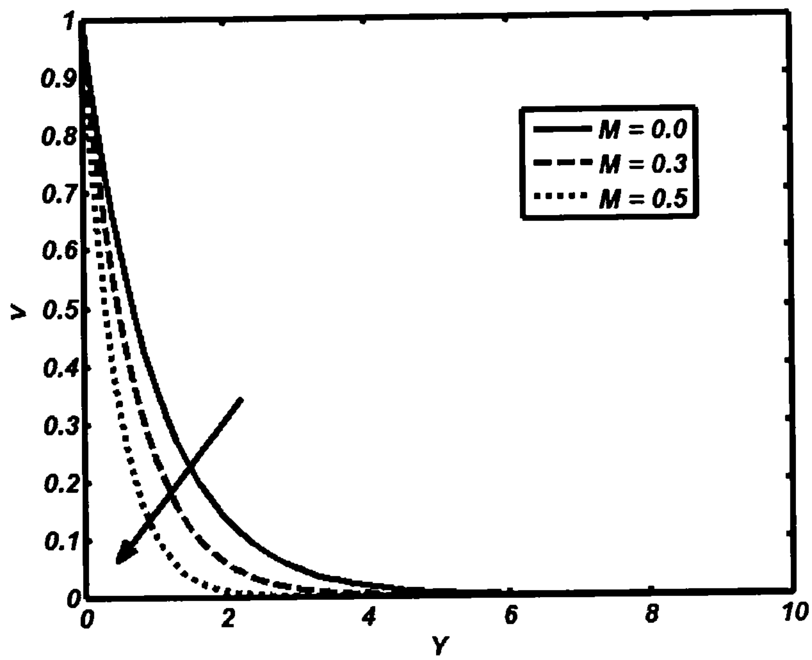

Fig 2.3: Vertical velocity profiles for M with A=0 at Pr=1.0.

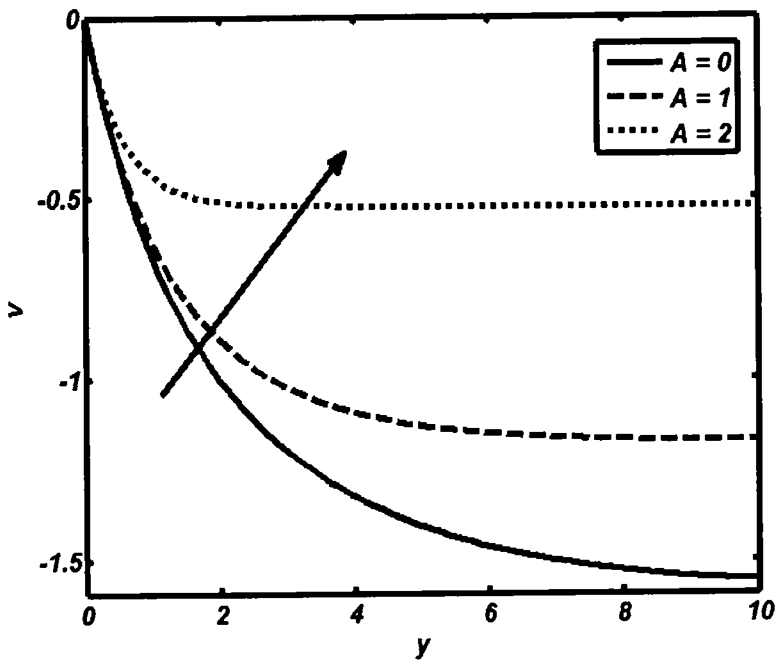

Fig 2.4: Vertical velocity profiles for A with M=0 and Pr=0.1.

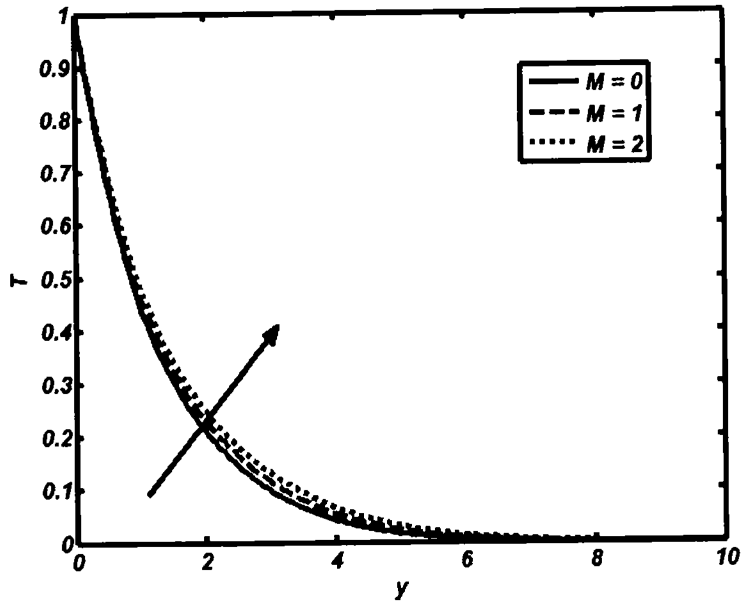

Fig 2.5: Temperature profile for M with Pr=1.0 and A=0.

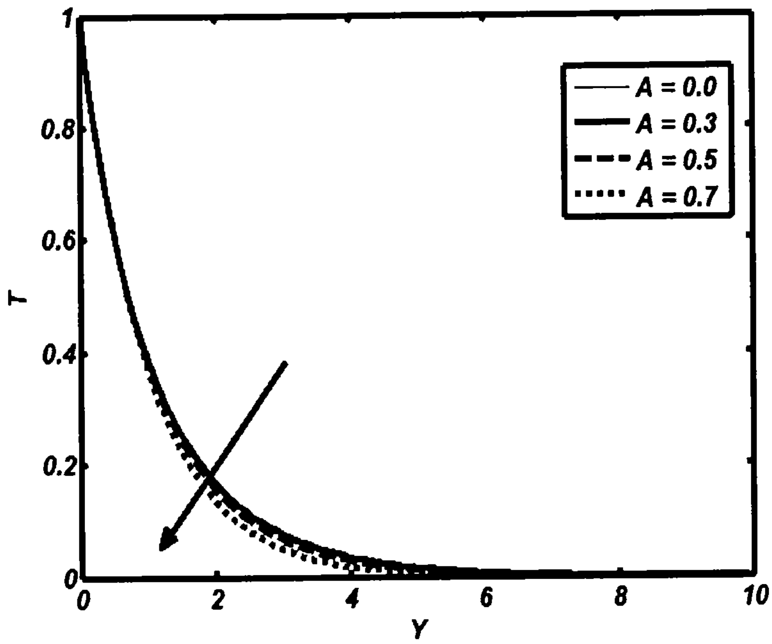

Fig 2.6: Temperature profile for A with Pr=1.0 and M=0.

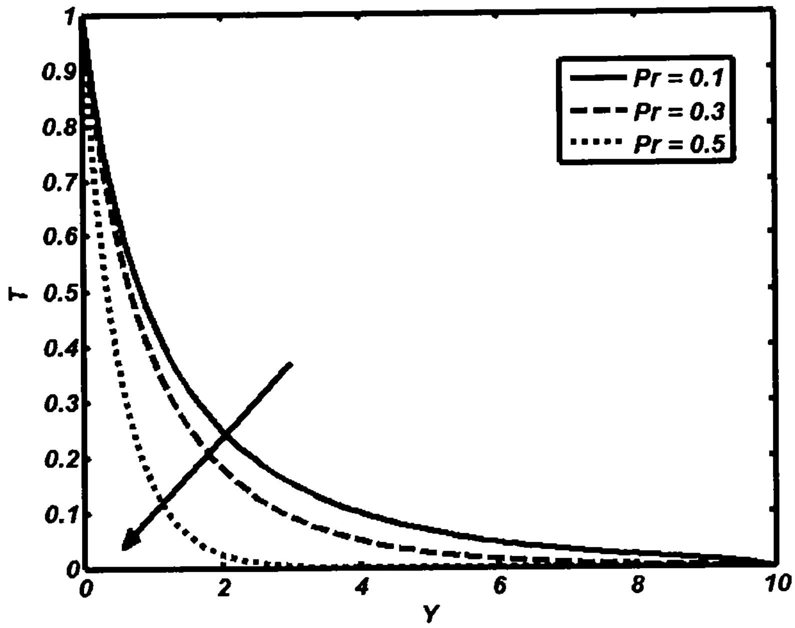

Figure 2.7: Temperature profile for Pr with A=1.0 and M=0.

2.5 Conclusion

The steady two-dimensional incompressible Magneto-hydrodynamic (MHD) boundary layer flow of variable viscosity over a heated stretching sheet including uniform transverse magnetic field has been investigated. The system of non-linear partial differential equations are solved using Lie-group method. The resulting ordinary differential equations are solved numerically using the shooting method with R-k scheme of 4^{th} order. The influence of the Hartman number M, the viscosity parameter A, and the Prandtl numbe Pr for horizontal velocity u/x, vertical velocity, temperature profiles T were examined. It has been observed that

- Horizontal and vertical velocity decreases and the temperature increases with the increase in the Hartmann number M.
- Horizontal and vertical velocity increases whereas the temperature decreases with the increase of the viscosity parameter A.
- Temperature decreases with increase in Prandtl number.

Chapter 3

Boundary Layer Flow of a Nano-fluid Over a Stretching Surface with Slip Coefficient

This chapter is formulated to investigate the flow of MHD nano-fluid over a stretching surface with slip effects in porous medium using Lie Symmetry analysis. The governing equations describing law of conservation of mass, momentum and energy are converted to system of ODEs using similarity transform. Generators formed are checked for consistency. Equations are solved numerically and graphical results are displayed.

3.1 Mathematical Formulation

The equations for conservation of mass, momentum, thermal energy and nanoparticle fraction for steady incompressible boundary layer flow are described as follows [4]

$$\nabla.\bar{V} = 0, \tag{3.1}$$

$$\rho_f(\frac{\partial \bar{V}}{\partial t} + \bar{V}.\nabla\bar{V}) = -\nabla p + \mu\nabla^2\bar{V} + \beta, \tag{3.2}$$

$$(\rho c)_f(\frac{\partial T}{\partial t} + \bar{V}.\nabla T) = k\nabla^2 T + (\rho c)_p.[D_B\nabla C.\nabla T + (\frac{D_T}{T_\infty})(\nabla T)^2], \tag{3.3}$$

$$(\frac{\partial C}{\partial t} + \bar{V}.\nabla C) = D_B\nabla^2 C + (\frac{D_T}{T_\infty})(\nabla T)^2], \tag{3.4}$$

where $\bar{V}$ is the velocity of the fluid, t is the time and ρ, μ, k, c are the density, dynamic viscosity, thermal conductivity and volumetric volume expansion coefficient of the Nano-fluid, (ρc) is the heat capacity and subscript 'p' is for particle and 'f' stands for base fluid. Further, the equations (3.3) and (3.4) consist of Brownian diffusion coefficient.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \qquad (3.5)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\gamma u^2-(\frac{\upsilon}{k})u-(\frac{\sigma B_\circ^2}{\rho})u\,, \qquad (3.6)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\nabla^2 T+\tau[D_b\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+(\frac{D_T}{T_\infty})(\frac{\partial T}{\partial y})^2]\,, \qquad (3.7)$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+(\frac{D_T}{T_\infty})(\frac{\partial^2 T}{\partial y^2})\,, \qquad (3.8)$$

Where, $\alpha=\frac{k}{(\rho c)_f}$ is the thermal diffusivity of the fluid, υ is the kinematic viscosity coefficient and $\tau=\frac{(\rho c)_p}{(\rho c)_f}$. The boundary conditions [3] of Equations. (3.5)-(3.8) are taken to be

$$\begin{aligned} &u=0,u=cx+\lambda\frac{\partial u}{\partial y},T=T_w,C=C_w \text{ at } y=0\,,\\ &u\to 0,T\to T_\infty,C\to C_\infty \; as\; y\to\infty\,. \end{aligned} \qquad (3.9)$$

The stream function Ψ are introduced as $u=\frac{\partial\Psi}{\partial y},v=-\frac{\partial\Psi}{\partial x}$, which satisfies Eq. (3.5). Eqs. (3.6)-(3.8) can then be written as

$$\Psi_y\Psi_{xy}-\Psi_x\Psi_{yy}-\Psi_{yyy}+\gamma(\Psi_y)^2+D\Psi_y+M^2\Psi_y=0\,, \qquad (3.10)$$

$$\Psi_y T_x-\Psi_x T_y-\frac{1}{\Pr}T_{yy}-Nt\Phi_y T_y-Nb(T_y)^2+M^2Ec(\Psi_y)^2=0\,, \qquad (3.11)$$

$$\Psi_y\Phi_x-\Psi_x\Phi_y-\frac{1}{Le}\Phi_{yy}-\frac{Nt}{Nb}T_{yy}=0\,. \qquad (3.12)$$

Also from equation (3.9), we have

$$\begin{aligned} &\Psi_y=0,\Psi_y=1+\lambda\Psi_{yy},T=T_w,C=C_w \; at\; y=0\,,\\ &\Psi_y\to 0,T\to T_\infty,C\to C_\infty \; as\; y\to\infty\,, \end{aligned} \qquad (3.13)$$

where subscripts denotes partial derivatives.

3.2 Solution of the Problem

Lie-group method are being used to find the similarity solutions for which (3.10)-(3.12) and the boundary conditions (3.13) are invariants, and then similarity variables can be found by using these symmetries.

Let us take one-parameter ϵ Lie group of infinitesimal transformations in $(x,y;\Psi,T,C)\rightarrow(x^{*},y^{*};\Psi^{*},T^{*},C^{*})$ as

$$\begin{aligned}
x^{*} &= x+\varepsilon\phi(x,y;\Psi,T,C)+O(\varepsilon^{2}),\\
y^{*} &= y+\varepsilon\zeta(x,y;\Psi,T,C)+O(\varepsilon^{2}),\\
\Psi^{*} &= \Psi+\varepsilon\eta(x,y;\Psi,T,C)+O(\varepsilon^{2}),\\
T^{*} &= T+\varepsilon F_{1}(x,y;\Psi,T,C)+O(\varepsilon^{2}),\\
C^{*} &= C+\varepsilon F_{2}(x,y;\Psi,T,C)+O(\varepsilon^{2}),
\end{aligned} \tag{3.14}$$

where ε is small parameter.

A system of PDEs (3.10)-(3.12) satisfies a symmetry generated by the vector field

$$X=\phi\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial\Psi}+F_{1}\frac{\partial}{\partial T}+F_{2}\frac{\partial}{\partial C}\ . \tag{3.15}$$

If it remains invariant by the transformation $(x,y;\Psi,T,C)\rightarrow(x^{*},y^{*};\Psi^{*},T^{*},C^{*})$.

The solution $\Psi=\Psi(x,y),T=T(x,y)$ and $C=C(x,y)$ are invariants under the symmetry (3.15) if equations (2.20) and (2.21) with $\varphi_{C}=X(C-C(x,y))=0$ when $C=C(x,y)$ satisfies.

Assume

$$\Delta_{3}=\Psi_{y}\Psi_{xy}-\Psi_{x}\Psi_{yy}-\Psi_{yyy}+\gamma(\Psi_{y})^{2}+D\Psi_{y}+M^{2}\Psi_{y}=0\ . \tag{3.16}$$

$$\Delta_{4}=\Psi_{y}T_{x}-\Psi_{x}T_{y}-\frac{1}{\Pr}T_{yy}-Nt\Phi_{y}T_{y}-Nb(T_{y})^{2}+M^{2}Ec(\Psi_{y})^{2}=0\ , \tag{3.17}$$

$$\Delta_{5}=\Psi_{y}C_{x}-\Psi_{x}C_{y}-\frac{1}{Le}C_{yy}-\frac{Nt}{Nb}T_{yy}=0\ . \tag{3.18}$$

A vector X in (3.15) is called Lie Point Symmetry vector field for (3.16)-(3.18) if

$$X^{[3]}(\Delta_j)|_{\Delta_j=0}, j=3,4,5\,, \tag{3.19}$$

where

$$X^{[3]}=\phi\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial\psi}+F_1\frac{\partial}{\partial T}+F_2\frac{\partial}{\partial C}+\eta^x\frac{\partial}{\partial\psi_x}+\eta^y\frac{\partial}{\partial\psi_y}+F_1^x\frac{\partial}{\partial T_x}+F_1^y\frac{\partial}{\partial T_y}$$
$$+F_2^x\frac{\partial}{\partial C_x}+F_2^y\frac{\partial}{\partial C_y}+\eta^{xy}\frac{\partial}{\partial\psi_{xy}}+\eta^{yy}\frac{\partial}{\partial\psi_{yy}}+F_1^{yy}\frac{\partial}{\partial T_{yy}}+F_2^{yy}\frac{\partial}{\partial C_{yy}}+\eta^{yyy}\frac{\partial}{\partial\psi_{yyy}}\,, \tag{3.20}$$

Is the third prolongation of X.

We need to differentiate (3.14) w.r.t each of the variables to get the prolongation of the given transformation. For this, we define the total derivatives as

$$\begin{aligned}D_x&=\partial_x+\Psi_x\partial_\Psi+T_x\partial_T+C_x\partial_C+\Psi_{xx}\partial_{\Psi_x}+T_{xx}\partial_{T_x}+C_{xx}\partial_{C_x}+\Psi_{xy}\partial_{\Psi_y}\ldots\,,\\ D_y&=\partial_y+\Psi_y\partial_\Psi+T_y\partial_T+C_y\partial_C+\Psi_{yy}\partial_{\Psi_y}+T_{yy}\partial_{T_y}+C_{yy}\partial_{C_y}+\Psi_{xy}\partial_{\Psi_x}\ldots\end{aligned} \tag{3.21}$$

Equations (3.16)-(3.18) produce the following system of linear PDE's

$$-\Psi_{yy}\eta^x+(\Psi_{xy}+2\gamma\psi_y+M^2+D)\eta^y+\Psi_y\eta^{xy}-\Psi_x\eta^{yy}-\eta^{yyy}=0\,, \tag{3.22}$$

$$-T_y\eta^x+(T_x+2M^2Ec\Psi_y)\eta^y-(\Psi_x+NtC_y+2NbT_y)F_1^x-NtT_yF_2^y-\frac{1}{\mathrm{Pr}}F_1^{yy}=0\,, \tag{3.23}$$

$$-C_y\eta^x+C_x\eta^y+F_2^x\Psi_y-F_2^y\psi_x-\frac{Nt}{Nb}F_1^{yy}-\frac{1}{Le}F_2^{yy}=0\,. \tag{3.24}$$

The components $\eta^x,\eta^y,F_1^x,F_1^y,F_2^x,F_2^y,\eta^{xy},\eta^{yy},F_1^{yy},F_2^{yy},\eta^{yyy}$ are to be find as follows.

$$\eta^s = D_s\eta - \Psi_x D_s\phi - \Psi_y D_s\zeta \ ,$$
$$F_1^s = D_s F_1 - T_x D_s\phi - T_y D_s\zeta \ ,$$
$$F_2^s = D_s F_2 - C_x D_s\phi - C_y D_s\zeta \ ,$$
$$\eta^{js} = D_s\eta^j - \Psi_{jx} D_s\phi - \Psi_{jy} D_s\zeta \ ,$$
$$F_1^{js} = D_s F_1^j - T_{jx} D_s\phi - T_{jy} D_s\zeta \ ,$$
$$F_2^{js} = D_s F_2^j - C_{jx} D_s\phi - C_{jy} D_s\zeta \ ,$$

(3.25)

where s [1]implies x and j implies y.

So the nonlinear equations (3.6)-(3.8) have the three-parameter Lie group of point symmetries generated by

$$X_4 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \ , \qquad (3.26)$$

$$X_5 = \frac{\partial}{\partial y} + \frac{\partial}{\partial \psi} \ , \qquad (3.27)$$

$$X_6 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial T} \ , \qquad (3.28)$$

$$X_7 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial C} \ . \qquad (3.29)$$

The finite transformation corresponding to the symmetries X_4, X_5, X_6, X_7 are respectively

Table 1: The invariant surface conditions

Generators	Characteristics $\varphi = (\varphi_\psi, \varphi_T, \varphi_C)$	Solution of the invariant surface conditions
X_4	$\varphi_\psi = -x\psi_y$ $\varphi_T = -T_x - xT_y$ $\varphi_C = -C_x - xC_y$	$\Psi = xG(y)$ $T = T(y)$ $C=C(y)$
X_5	$\varphi_\psi = 1-\psi_y$ $\varphi_T = -T_y$ $\varphi_C = -C_y$	$\Psi = \Psi(x)$ $T = T(x)$ $C=C(x)$
X_6	$\varphi_\psi = -\psi_y$ $\varphi_T = 1-T_y$ $\varphi_C = -C_y$	N0 solution
X_7	$\varphi_\psi = -\psi_y$ $\varphi_T = -T_y$ $\varphi_C = 1-C_y$	N0 solution

$$r = x+\varepsilon, s = x\varepsilon + \frac{1}{2}\varepsilon^2, f(r,s) = \psi, g(r,s) = T, h(r,s) = C \ , \tag{3.30}$$

$$r = x, s = y + x\varepsilon, f(r,s) = \varepsilon + \psi, g(r,s) = T, h(r,s) = C \ , \tag{3.31}$$

$$r = x, s = y + x\varepsilon, f(r,s) = \psi, g(r,s) = \varepsilon + T, h(r,s) = C \ , \tag{3.32}$$

$$r = x, s = y + x\varepsilon, f(r,s) = \psi, g(r,s) = T, h(r,s) = \varepsilon + C \ . \tag{3.33}$$

Further, by using the steps taken in previous chapter, Eqs. (3.10)-(3.12) are hence transformed to

$$f''' + ff'' - M^2 f' - Df' - F(f')^2 = 0 , \tag{3.34}$$

$$\frac{1}{\Pr}\theta'' + f\theta' + Nb\Phi'\theta' + Nt(\theta')^2 + M^2(f')^2 = 0 , \tag{3.35}$$

$$\Phi'' + Lef\Phi' + \frac{Nt}{Nb}\theta'' = 0 . \tag{3.36}$$

Subject to the boundary conditions

$$\begin{aligned} &f(0) = 0, f'(0) = 1 + \lambda f''(0),\ \theta(0) = 1,\ \Phi(0) = 1 , \\ &f'(\eta) = 0,\ \theta(\eta) = 0 \text{ as } \eta \to \infty . \end{aligned} \tag{3.37}$$

Also , the four parameters are expressed as follows

$$\begin{aligned} &\Pr = \frac{\nu}{\alpha}, Le = \frac{\nu}{D_B} , \\ &Nb = \frac{(\rho c)_p D_B(\Phi_w - \Phi_\infty)}{(\rho c)_f.\nu} , \\ &Nt = \frac{(\rho c)_p D_T(T_w - T_\infty)}{(\rho c)_f.T_\infty.\nu} . \end{aligned} \tag{3.38}$$

Here, Le and Pr denotes Lewis number and Prandtl number whereas Nt and Nb are thermophoresis parameter and Brownian diffusion parameter respectively. We notice that when Nb and Nt are zero, Eqs. (3.22) and (3.23) involve just two dependent variables, namely $f(\eta)$ and $\theta(\eta)$.

Quantities of particle interest are the skin-friction coefficient C_f, the local Nusselt number Nu_x and the local sherwood number Sh_x that are given as

$$C_f = \frac{\tau_w}{\rho U^2}, Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)} , \tag{3.39}$$

where τ_w, q_w, q_m are the shear stress, heat flux and mass flux at the surface. Here, we use variables given in Eqs. (3.17), we get

$$(2\text{Re}_x)^{1/2} = f''(0), (2\text{Re}_x)^{-1/2} Nu_x = -\theta'(0), \ (\text{Re}_x/2)^{-1/2} Sh_x = -\Phi'(0) \ , \tag{3.40}$$

where $\text{Re}_x = Ux/v$ is the local Reynolds number. In the present context $(\text{Re}_x)^{-1/2}Nu_x$ and $(\text{Re}_x)^{-1/2}Sh_x$ are referred to as the reduced Nusselt number and reduced Sherwood number denoted by Nur and Shr, which are represented by $-\theta'(0)$ and $-\Phi'(0)$ respectively.

3.3 Results and Discussion

Fig. (3.1) shows the velocity profile $f'(\eta)$ for different values of Hartmann number M. It is noted that an increase in Hartmann number causes an increase in the velocity $f'(\eta)$. The velocity profile $f'(\eta)$ has been taken for various values of slip parameter λ. It can be seen that increase in the slip parameter λ causes decrease in the velocity profile $f'(\eta)$, as shown in Fig. (3.2).

The effects of velocity are taken against Brownian diffusion coefficient D_B. Fig. (3.3) displays that the velocity $f'(\eta)$ decreases with increase in Brownian diffusion coefficient D_B.

The influence of the change of slip parameter λ for the temperature profile is studied. Fig. (3.4) indicates that the temperature $\theta(\eta)$ increases with increase in slip parameter λ. Fig. (3.5) shows the influence of the temperature $\theta(\eta)$ for Hartmann number. The increase in Hartmann number causes increases in temperature $\theta(\eta)$.The Brownian diffusion coefficient D_B demonstrates the variation in the temperature profile $\theta(\eta)$. Hence, Fig. (3.6) indicates that the temperature $\theta(\eta)$ decreases with increase in Prandtl number. As it is noticed that the concentration increases due to the increase in Brownian diffusion coefficient D_B in Fig. (3.7). Fig. (3.8) reveals variation of the concentration $\phi(\eta)$. The concentration increases with an increase in slip parameter λ.

Fig. (3.9) illustrates the variation of the concentration $\phi(\eta)$. As, increase in Prandtl number causes an increase in concentration. Table 1 shows the solution of the invariant surface conditions (2.20) and (2.21).

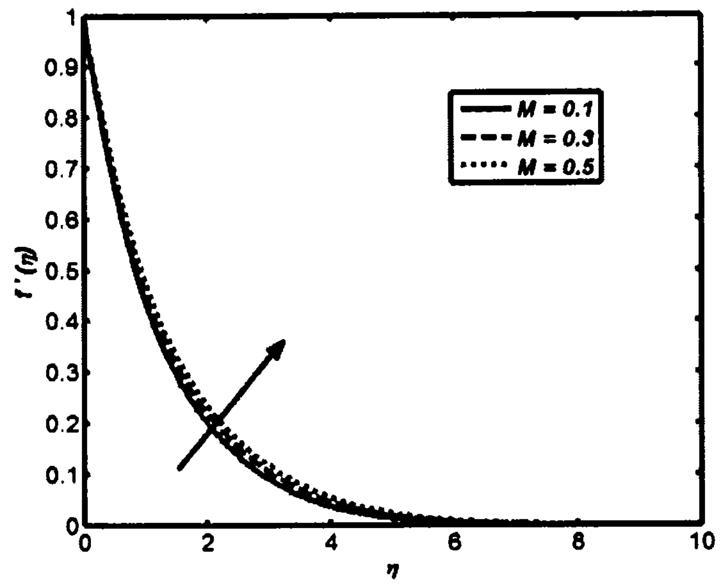

Fig 3.1: Velocity profile for M=0.1, 0.3, 0.5 .

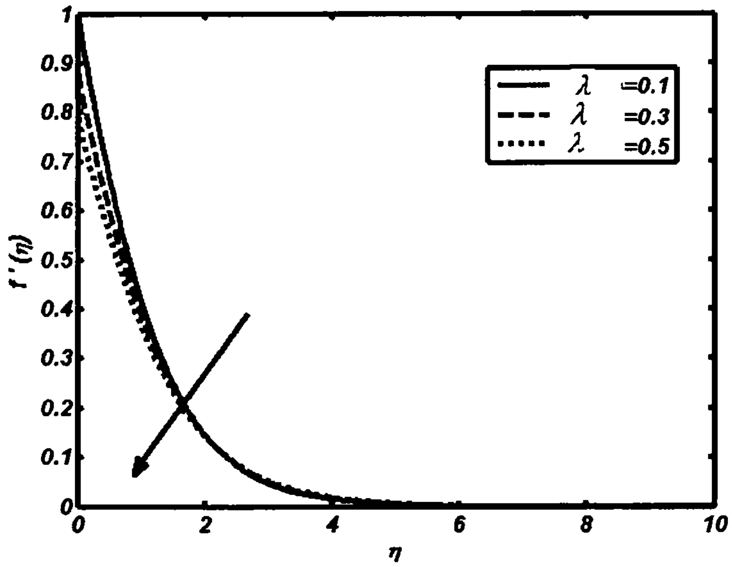

Fig 3.2: Velocity profile for λ=0.1, 0.3, 0.5 .

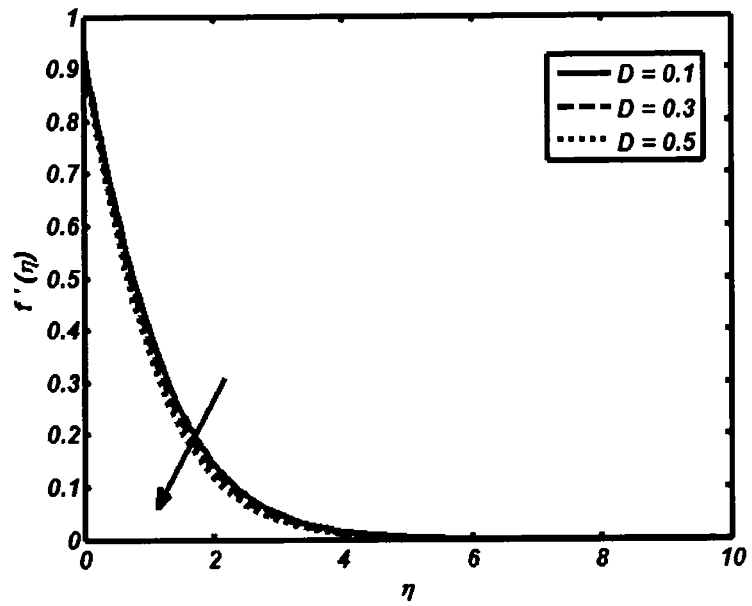

Fig 3.3: Velocity profile for D=0.1, 0.3, 0.5 .

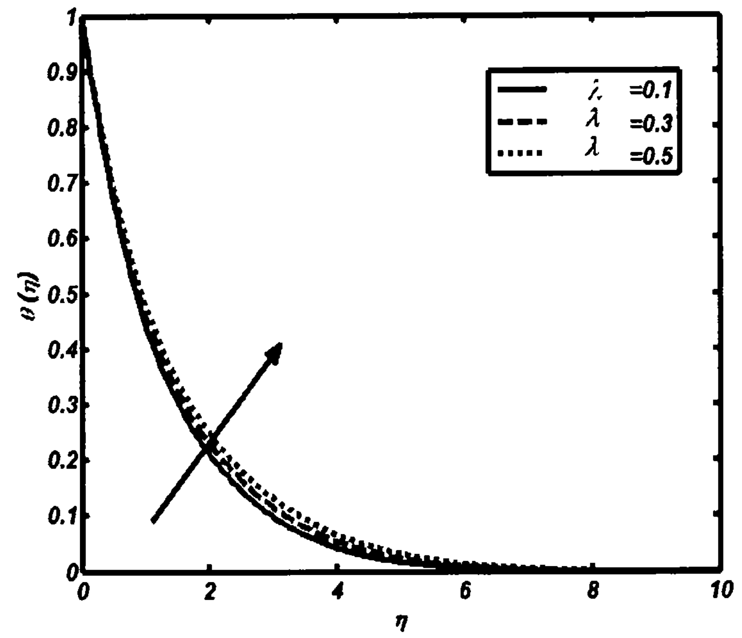

Fig 3.4: Temperature profile for λ =0.1, 0.3, 0.5 .

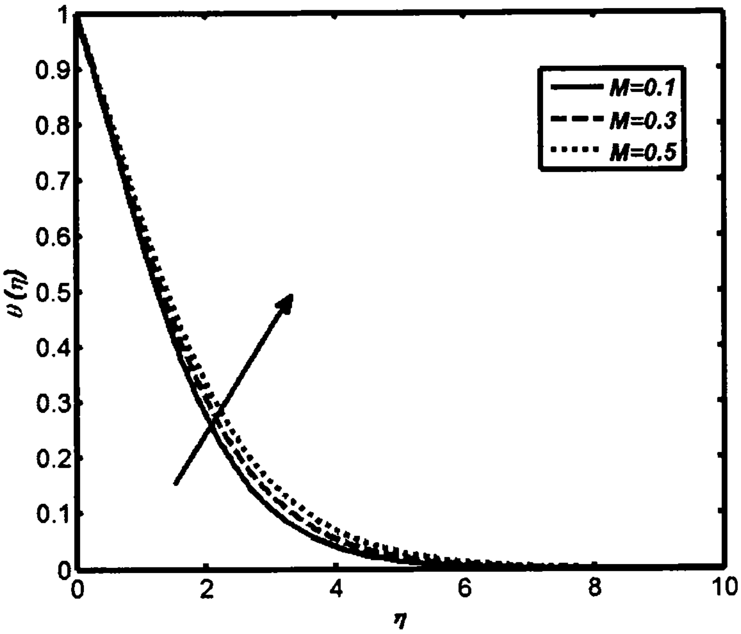

Fig 3.5: Temperature profile for M=0.1, 0.3, 0.5 .

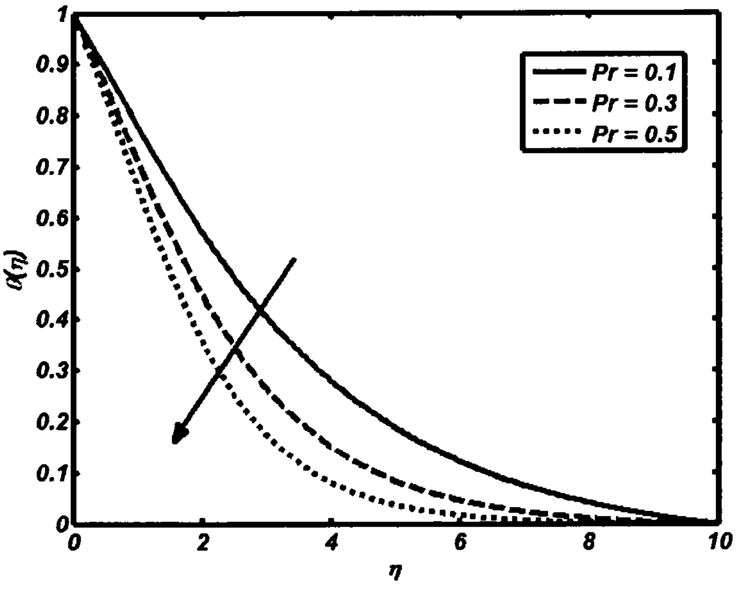

Fig 3.6: Temperature profile for Pr=0.1, 0.3, 0.5.

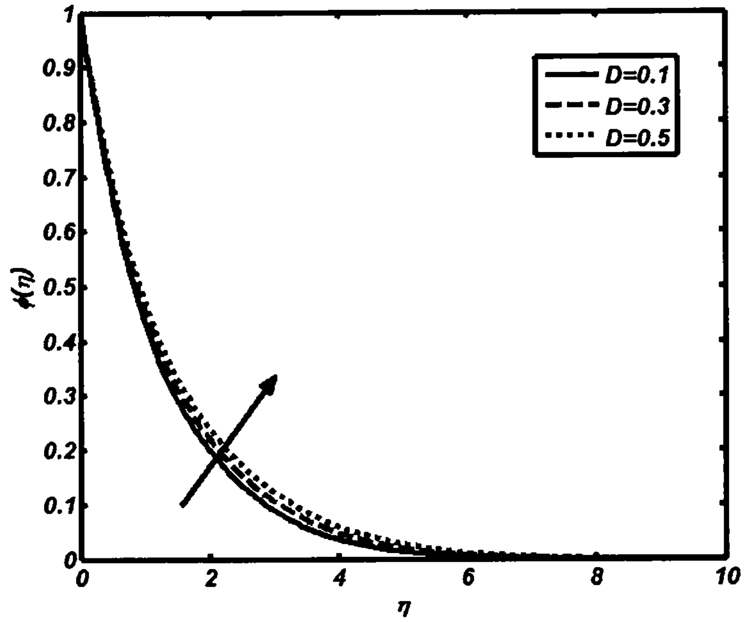

Fig 3.7: Concentration profile for D=0.1, 0.3 , 0.5 .

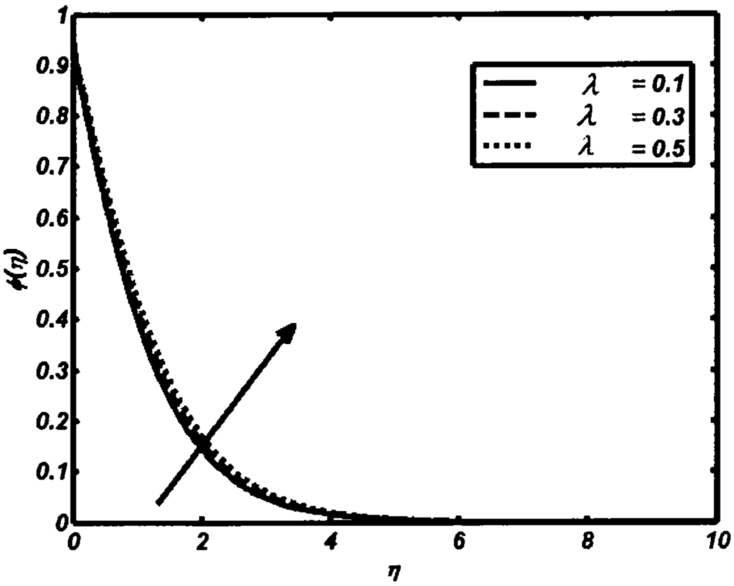

Fig 3.8: Concentration profile for λ =0.1, 0.3, 0.5.

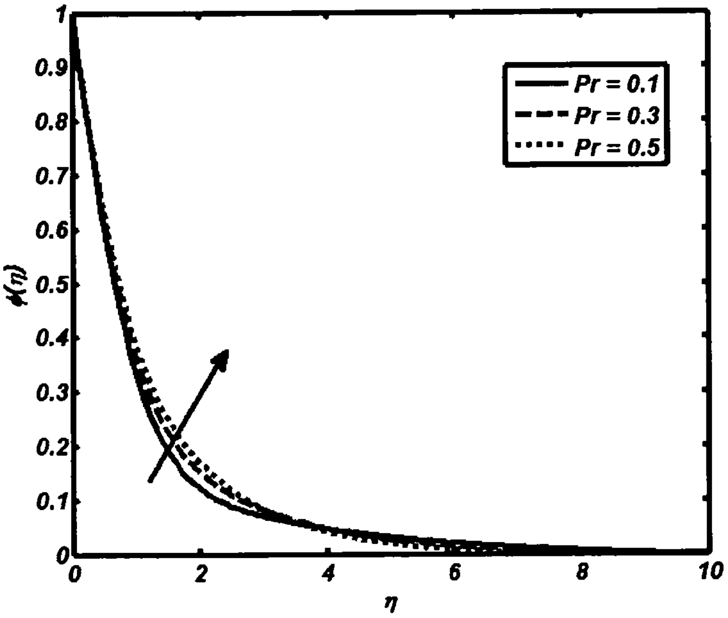

Fig 3.9: Concentration profile for Pr=0.1, 0.3, 0.5 .

3.4 Conclusion

In this chapter, we have concentrated on the flow of MHD nanofluid over a stretching surface along with slip effects in a permeable medium. The equations are modelled and utilized Lie symmetry method to tackle the problem so formed. The system of non-linear partial differential equations have been transformed into ordinary differential equations using Lie group method. The resulting equations are solved numerically using the Runge-Kutta scheme. Graphical results display the effects of different emerging parameters on temperature concentration and velocity. It has been observed that

- Velocity, temperature, and the concentration increases when increase in the value of Hartamann number M.
- Increase in the value of slip parameter λ causes increase in temperature and concentration as well, whereas velocity decreases with the increase in the value of slip parameter λ.
- Concentration increases with the increase in the value of the Brownian diffusion coefficient D_B.

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