Some Simple Flows of Phan-Thien-Tanner Fluids

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2013

1 - Flow - fluid mechanics

2 - Fluid mechanics

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Reg. # 49.FBAS/MSMA/F10

A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
IN
MATHEMATICS

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DECLARATION

I, hereby declare that this thesis neither as a whole nor as a part there of has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my supervisor. No portion of the work presented in this thesis has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTRER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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In the end and most importantly, I would like to tribute my parents, brothers and sisters whose constant, unending and countless prayers buoyed me up.

Anwar Ahmed

Dedicated

To

Hazrat e Aqdas Shah Saeed Ahmed Raipuri (R.A)

(1926-2012)

My achievements and success owe to his ideology, encouragement and affection.

My Allah rests his soul in Heaven.

Preface

Flows of non-Newtonian fluids have been extensively studied in recent past because of their importance in technology and industry. In particular these flows can be found in journal bearings, commercial viscometers, swirl nozzles, chemical and mechanical mixing equipment and electric motors. The relationship between stress and strain for non-Newtonian fluids is mostly represented by nonlinear differential constitutive equations. The commonly used non-Newtonian models are due to Maxwell [1], Green and Tobolsky [2], Oldroyd [3], Giesekus [4-7], White and Metzner [8] and Phan-Thein & Tanner [9, 10]. Phan-Thein & Tanner (PTT) model is fairly simple quasi-linear viscoelastic model constitutive equation, which was derived using network theory by Phan-Thien and Tanner. This model incorporates not only shear-thinning, shear viscosity and normal-stress differences but also an elongational parameter ϵ and so reproduces many of the characteristics of the rheology of polymer solutions and other non-Newtonian liquids. The elongational parameter imposes an upper limit on the elongational viscosity, which is inversely proportional to ϵ. When ϵ goes to zero the PTT constitutive equation reduces to the Johnson-Segalman model without the presence of the solvent viscosity, while the simplified form of the PTT model is equivalent to the upper convected Maxwell model. The PTT model is being employed increasingly to predict the flow and heat transfer of viscoelastic fluids. Recent papers include those of Oliveira and Pinho [11], Alves et al. [12], Mirzazadeh et al. [13] and Hashemabadi et al. [14]. The analytical solutions for elementary flows of non-Newtonian fluids using the nonlinear differential constitutive equations are of considerable importance because of the fact that they offer a simple way of checking the ability of the chosen non-Newtonian model to represent some specific behavior and are also useful to prescribe inlet and outlet boundary conditions in numerical flow simulations and to validate numerical prediction. Some recent studies on the analytical solutions of different non-Newtonian constitutive equations can be found in refs. [15-26]. Keeping the above facts in mind we in this dissertation, present the exact analytical solutions of PTT constitutive equation for two types of elementary flows i.e., I) fully developed pipe and channel flow II) tangential flow in a concentric annulus. The dissertation is organized as follows.

Chapter 1 includes some basic definitions, equations and concepts regarding polymeric fluids. Fully developed channel and pipe flows of PTT fluid are analyzed in Chapter 2. In chapter 3 purely tangential flow of PTT fluid in a concentric annulus is presented. A comprehensive bibliography is included at the end of the dissertation.

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Chapter 1

Preliminaries

In this chapter we will present some basic definitions and flow equations which are used in the subsequent chapters. The part of this chapter is based on the material from internet and [21].

1.1 Fluid mechanics

It is the branch of science that deals with nature and properties of the fluid both in motion and rest.

1.2 Flow

A substance or material goes under deformation in the presence of different forces. If the deformation exceeds continuously without limit then this phenomenon is known as flow.

1.3 Fluid

A fluid is a substance or material that deforms continuously under the action of applied shear stress.

1.4 Velocity field

In dealing with fluid motion, we shall necessarily be concerned with the description of velocity field. In general at given instant the velocity field **u** is function of space coordinates x, y, z and time t. The velocity at any point in flow field might vary from one instant to another. The complete representation of velocity field is given

$$\mathbf{u} = \mathbf{u}(x, y, z, t), \tag{1.1}$$

or

$$\mathbf{u} = u(x, y, z, t)\hat{\mathbf{i}} + v(x, y, z, t)\hat{\mathbf{j}} + w(x, y, z, t)\hat{\mathbf{k}}. \tag{1.2}$$

1.5 Classification of fluids

There are two main types of fluids

1.5.1 Ideal fluids

A fluid for which viscosity is zero is termed as ideal fluid. An ideal fluid is fictitious and does not exist in nature however many fluids under certain engineering applications show negligible viscosity effects and can be treated as ideal fluids.

1.5.2 Real fluids

All fluids for which the viscosity is not zero are known as real fluids. These are further divided into two main classes.

Newtonian fluids

All fluids which satisfy the Newton's law of viscosity are called Newtonian fluids. The Newton's law of viscosity is stated as "shear stress is directly and linearly proportional to rate of deformation". For a steady one dimensional flow between two parallel walls driven by the motion of upper wall

$$\mathbf{u} = u(y)\mathbf{i} \tag{1.3a}$$

and therefore Newton's law of viscosity takes the form

$$\tau_{yx} \propto \frac{d\mathbf{u}}{dy}, \tag{1.3b}$$

or

$$\tau_{yx} = \mu \left(\frac{d\mathbf{u}}{dy} \right), \tag{1.4}$$

where τ_{yx} is the shear stress acting in the plane normal to y-axis and in the direction parallel to x-axis and μ is constant of proportionality, known as absolute or dynamic viscosity. Water, air and gasoline are examples of Newtonian fluids.

Non-Newtonian fluids

All fluids which do not obey Newton's law of viscosity are called non-Newtonian fluids. These types of fluid obey the power law model in which shear stress is directly but nonlinearly proportional to the rate of deformation. Mathematically,

$$\tau_{yx} \propto \left(\frac{du}{dy}\right)^{n}, n \neq 1, \tag{1.5}$$

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^{n}. \tag{1.6}$$

where n is called the flow behavior index and k is the consistency index. Examples of non-Newtonian fluids are shampoo, gel, blood etc.

1.6 Kinematic viscosity

It is the ratio of the viscosity to density of the fluid, and it is given as

$$v = \frac{\mu}{\rho}. \tag{1.6}$$

1.7 Body force

The forces which do not require any physical contact with boundary and distributed over the whole volume of the fluid are known as body forces. Gravitational and electromagnetic forces are categorized as body forces. These are in fact long rang forces.

1.8 Pressure

Pressure is the surface force that acts normal to the area under consideration. The force per unit area is called pressure. Let A is the surface area of fluid and F is magnitude of force acting normal to surface, then pressure P due to the force on unit area of this surface is defined as

$$P = \frac{F}{A}. \tag{1.7}$$

1.9 Governing equations

1.9.1 Equation of continuity

The mathematical form of law of conservation of mass for fluid is known as equation of continuity. It has the following form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \mathbf{u}) = 0, \tag{1.8}$$

and for incompressible fluids it reduces to

$$\boldsymbol{\nabla}.\mathbf{u} = 0. \tag{1.9}$$

1.9.2 Equation of motion

The motion of fluid is governed by law of conservation of momentum. The application of this law to an arbitrary control volume in flowing fluid yield the following equation commonly known an equation of motion.

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \operatorname{div}\mathbf{T} + \rho\mathbf{b}. \tag{1.10}$$

In above equation $\mathbf{T}$ is Cauchy stress tensor and $\mathbf{b}$ is body force per unit mass.

1.10 Flow phenomena in polymeric liquids

1.10.1 The chemical nature of polymeric liquids

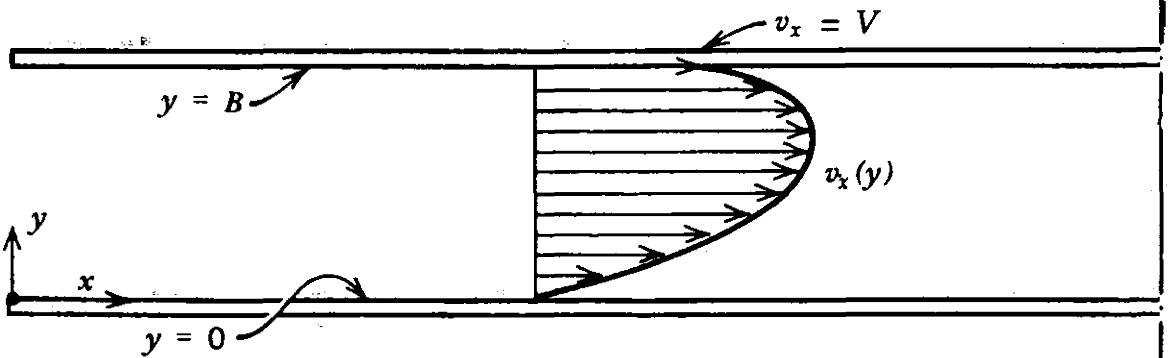
A macromolecule (or polymer) is a large molecule composed of many small simple chemical units, generally called structural units. In some polymers each structural unit is connected to precisely two other structural units, and the resulting chain structure is called a linear macromolecule. In other polymers most structural units are connected to two other units, although some structural units connect three or more units, and we talk of branched molecules. Where the chains terminate, special units called end groups are found. For the sake of completeness we mention also that in some macromolecular materials all structural units are interconnected resulting in a three-dimensional cross-linked or network structure rather than in separate molecules.

It is sometimes useful to distinguish between synthetic and natural (biological) macromolecules. Many synthetic polymers are built from a single structural unit, and the polymer is then referred to as a homopolymer. Typical examples of synthetic homopolymers are polyethylene, polystyrene, and polyvinylchloride. In contrast copolymers are built from two or more different structural units. According to the manner in which the structural units combine, copolymers are further classified as random copol-copolymers, block copolymers, or graft copolymers. The motivation for producing copolymers is to obtain materials with a wider range of mechanical properties than is possible with the homopolymers alone.

1.10.2 Non-Newtonian viscosity

Probably the single most important characteristic of polymeric liquids is the fact that they have a "shear-rate dependent" or "non-Newtonian" viscosity. Still different behavior is shown by fluids that will not flow unless acted on by at least some critical shear stress, called the yield stress. We call these viscoplastic fluids. Certain types of paints, greases, and pastes are examples of viscoplastic fluids.

1.10.3 Normal stress effects



A number of important effects in the flow of polymeric liquids may be attributed to the fact that polymeric liquids exhibit normal stress differences in "shear flows". For polymeric fluids the first normal stress difference is practically always negative and numerically much larger than the second normal stress difference. This means that to a first approximation polymeric fluids exhibit in addition to the shear stresses an extra tension along the streamlines, that is, in the (1) direction. It was shown by Weissenberg that the simple notion of an extra tension along streamlines may be used to obtain qualitative explanations of a large number of experiments. In terms of chemical structure, the extra tension along the streamlines in polymeric fluids arises from the stretching and alignment of the polymer molecules along the streamlines. The thermal motions make the polymer molecules act as small "rubber bands" wanting to snap back, and it is in this way that the extra tension arises. The second normal stress difference has been found experimentally to be positive, but usually much smaller than the magnitude of the first normal stress difference. This means that in a shear flow the fluid exhibits a small extra tension in the (2) direction. A simple structural explanation for this extra tension is lacking, and the simplest kinetic theories of polymeric fluids are not capable of describing this effect; more elaborate theories are successful, however. We emphasize that the second normal stress difference is quite small, and it is normally observable only in situations where the first normal stress difference, for geometrical reasons, has no effect.

1.10.4 Rod-climbing

In this experiment we insert rotating rods into two beakers, one containing a Newtonian liquid and the other a polymer solution. We see that the Newtonian liquid near the rotating rod is pushed outward by the centrifugal force, and a dip in the liquid surface near the center of the beaker results. This is typical of the flow near the rotating shaft of a stirrer. The contrasting behavior of the polymer solution is striking. The polymer solution moves in the opposite direction, toward the center of the beaker and climbs up the rod. Moreover, for comparable rotational speeds, the response of the polymer solution can be far more dramatic than that of the Newtonian liquid.

1.11 Dimensionless groups in non-newtonion fluid mechanics

In Newtonian fluid mechanics the Reynolds number appears as a dimensionless group that may be interpreted roughly as a ratio of the magnitude of inertial forces to that of viscous forces. In any given flow situation other dimensionless numbers may arise (for example, geometric ratios), but the Reynolds number is generally the most important dimensionless group. Dimensionless groups are particularly useful for scaling arguments and also for cataloging the flow regimes. For example, in tangential annular flow with the inner cylinder rotating, one can make visual observations and then determine the ranges of Reynolds numbers and radius ratios for which one has laminar flow, Taylor vortices, undulating vortices, and turbulence. For viscoelastic fluids the key dimensionless group is the Deborah number, introduced by Reiner. This number may be interpreted as the ratio of the magnitude of the elastic forces to that of the viscous forces. It is defined as the ratio of a characteristic time (or "time scale") of the fluid, λ, to a characteristic time (or "time scale") of the flow system, t_{flow}

$$De = \frac{\lambda}{t_{flow}}. \tag{1.11}$$

The characteristic time of the fluid is often taken to be the largest time constant describing the slowest molecular motions, or else some mean time constant determined by linear viscoelasticity. Sometimes the characteristic time is chosen to be a time constant in a constitutive equation. The characteristic time for the flow is usually taken to be the time interval during which a typical fluid element experiences a significant sequence of kinematic events, sometimes it is taken to be the duration of an experiment or experimental observation. If the flow following a material particle is steady, then the characteristic time is taken to be the reciprocal of a characteristic strain rate. A second dimensionless group, the Weissenberg number, that is sometimes used in polymer fluid dynamics involves a ratio of λ to this second characteristic time. The Weissenberg number We is defined by:

$$We = \lambda\kappa \tag{1.15}$$

where κ is a characteristic strain rate in the flow. For many problems, however, there is only one characteristic time that can be identified, and for these problems we choose to use the Deborah number as the dimensionless group. Two limiting values of the Deborah number can be identified with classical mechanics. If the Deborah number is small, then thermal motions keep the polymer molecules more or less in their equilibrium configurations, and the polymeric fluid shows only minor qualitative differences from a Newtonian fluid. We say that Newtonian fluid behavior is obtained in the limit $De \to 0$. Conversely, if the Deborah number is large, polymer molecules that are distorted by the flow will not have time to relax during the time scale of the process or experiment. In the limit $De \to \infty$ the experiment happens so fast that the polymer molecules have no time to change configuration, and the fluid behaves more or less as a Hookean elastic solid.

Chapter 2

Exact solution for fully developed flow of Phan-Thien-Tanner fluids

In this chapter analytical solutions are obtained for velocity field, the components of stress and the function of viscosity for Phan-Thien–Tanner (PTT) fluids in fully developed channel and pipe flow. Here analysis is performed for the PTT equation invovling both the linearized and the exponential forms. It is observed that shear stress at the wall in PTT fluid is very small according to the value of Newtonian or upper-convected Maxwell fluids. This chapter is based on the contents of ref. [11].

2.1 The fluid model

The constitutive equation for Phan-Thien & Tanner (PTT) fluid [9] is written in the general form as

$$f(tr(\boldsymbol{\tau}))\boldsymbol{\tau} + \lambda \overset{\nabla}{\boldsymbol{\tau}} = 2\eta \mathbf{D}, \tag{2.1}$$

where λ is the relaxation time, $\boldsymbol{\tau}$ is the extra-stress, η is a constant viscosity coefficient, D is thedeformation-rate tensors and $\overset{\nabla}{\boldsymbol{\tau}}$ denotes Oldroyd's upper-convected derivative given by

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathbf{Du}/\mathbf{D}t - \boldsymbol{\tau}.\nabla\mathbf{u} - \nabla\mathbf{u}^T.\boldsymbol{\tau}. \tag{2.2}$$

The linearized PTT model assumes the following form for $f(tr(\boldsymbol{\tau}))$ i.e.,

$$f(tr(\boldsymbol{\tau})) = 1 + \frac{\in \lambda}{\eta} tr(\boldsymbol{\tau}), \tag{2.3}$$

while for exponential PTT model

$$f(tr(\boldsymbol{\tau})) = \exp\left(\frac{\in \lambda}{\eta} tr(\boldsymbol{\tau})\right). \tag{2.4}$$

The parameter $\in$ appearing in (2.3) and (2.4) is related to the extended behaviour of the fluid model. Also it should be noted that the linearized form (2.3) can be deduced from exponential form (2.4) and when $\in$ vanishes if the trace of the stress tensor is small and that both the linear and exponenetial forms reduce to upper-convected Maxwell (UCM) model.

2.2 Analytical solution

The analysis presented here holds for axisymmetric pipe flows and channel flows in two-dimension. However, for simplifications we present it here with the suitable modifications required for pipe flow presented through specific parameters for the planar case. Let u and v denote the velocity component in streamwise and cross-stream radial directions respectively. The flow is defined such that y is representing either a radial or a transversal coordinate in (x, y)-plane. The $y = H$ is the wall and symmetry line is located at $y = 0$, H is half of the pipe radius or the channel width. Here an index with comma is presenting a partial derivative and a superscript j identifies the flow case with $j = 0$ for channel and $j = 1$ for pipe flows. The velocity field depends on the coordinate y in developed flows. The no-slip condition at the wall and equation of continuity gives $v = 0$. Where stress tensor τ is given by

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \tag{2.5}$$

with its trace defined as

$$tr(\tau) = \tau_{xx} + \tau_{yy}, \tag{2.6}$$

Let us define

$$\tau_{kk=}\tau_{xx+}\tau_{yy}. \tag{2.7}$$

Since

$$\bar{\mathbf{D}} = \nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T, \tag{2.8}$$

and

$$\mathbf{u} = [u(y), 0, 0], \tag{2.9}$$

therefore

$$\nabla\mathbf{u} = \begin{bmatrix} 0 & 0 \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix} \qquad \nabla\mathbf{u}^T = \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ 0 & 0 \end{bmatrix} \tag{2.10}$$

For steady flow $\partial\mathbf{u}/\partial t = 0$ and further $(\mathbf{u}.\nabla)\mathbf{u} = 0$, so

$$\overset{\nabla}{\tau} = -\tau.\nabla\mathbf{u} - \nabla\mathbf{u}^T.\tau. \tag{2.11}$$

Using the above results we get

$$\overset{\nabla}{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} -2\tau_{xy}\frac{\partial u}{\partial y} & -\tau_{yy}\frac{\partial u}{\partial y} \\ -\tau_{yy}\frac{\partial u}{\partial y} & 0 \end{bmatrix}. \tag{2.12}$$

In view of (2.7), (2.8) and (2.12), Eq. (2.1) can be expressed as

$$f(\tau_{kk}) \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} = \begin{bmatrix} 2\lambda\tau_{xy}\frac{\partial u}{\partial y} & \lambda\tau_{yy}\frac{\partial u}{\partial y} \\ \lambda\tau_{yy}\frac{\partial u}{\partial y} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 2\eta\frac{\partial u}{\partial y} \\ 2\eta\frac{\partial u}{\partial y} & 0 \end{bmatrix} \tag{2.13}$$

From above equation we can write

$$f(\tau_{kk})\tau_{xx} = 2\lambda u_{,y}\tau_{xy}, \tag{2.14}$$

$$f(\tau_{kk})\tau_{yy} = 0, \tag{2.15}$$

$$f(\tau_{kk})\tau_{xy} = \eta u_{,y} + \lambda\tau_{yy}u_{,y}. \tag{2.16}$$

Eq. (2.15) yields $\tau_{yy} = 0$, since $f(\tau_{kk}) \neq 0$. Therefore $\tau_{kk} = \tau_{xx}$. The longitudinal momentum equation for the flow under consideration read

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}. \tag{2.17}$$

This gives

$$p_{,x} = \frac{\partial \tau_{xy}}{\partial y}. \tag{2.18}$$

Integration of Eq. (2.18) and utilizatilizing the conditions at the boundry for $y = 0$, we have $\tau_{xy} = 0$, which yields

$$\tau_{xy} = p_{,x}y. \tag{2.19}$$

The above equation can be put into the form

$$\tau_{xy} = p_{,x}\frac{y}{2^j}, \tag{2.20}$$

where for pipe flow

$$j = 1$$

and for the channel flow

$$j = 0$$

Multiplication of Eq. (2.14) by τ_{xy} and Eq. (2.16) by τ_{xx} and by subtraction, we get

$$\tau_{xx} = \frac{2\lambda}{\eta}p_{,x}^2\frac{y^2}{2^{2j}}. \tag{2.21}$$

We observe that Eq. (2.21) is in accordance with the boundary condition required at centerline, i.e., for $y = 0$ we have $\tau_{xx} = 0$. From Eq. (2.16), Eq. (2.20) and Eq. (2.21) we can write

$$u_{,y} = \frac{1}{\eta}f(\tau_{xx})\tau_{xy}. \tag{2.22}$$

With the help of Eq. (2.21) we obtain

$$u_{,y} = \frac{1}{\eta}f\left(\frac{2\lambda}{\eta}p_{,x}^2\frac{y^2}{2^{2j}}\right)p_{,x}\frac{y}{2^j}, \tag{2.23}$$

or

$$\frac{du}{dy} = f\left(\frac{2\lambda}{\eta}p_{,x}^2\frac{y^2}{2^{2j}}\right)\frac{p_{,x}}{\eta}\frac{y}{2^j}. \tag{2.24}$$

In order to get the velocity profile as the function f is specified, we see that the above equation is an explicit form of differential equation in y .

2.3 Linear PTT model

For PTT model in linear form, the funtion f is expressed by Eq.(2.3) i.e.,

$$f(tr(\boldsymbol{\tau})) = 1 + \frac{\in \lambda}{\eta} tr(\boldsymbol{\tau}).$$

Thus in Eq. (2.24) the right hand side becomes

$$f\left(\frac{2\lambda}{\eta} p_{,x}^2 \frac{y^2}{2^{2j}}\right) = 1 + \frac{\in \lambda}{\eta}\left(\frac{2\lambda}{\eta} p_{,x}^2 \frac{y^2}{2^{2j}}\right). \tag{2.25}$$

This implies that

$$\frac{du}{dy} = \left(1 + \frac{\in \lambda}{\eta}\left(\frac{2\lambda}{\eta} p_{,x}^2 \frac{y^2}{2^{2j}}\right)\right) p_{,x} \frac{y}{2^j}. \tag{2.26}$$

Integration of above equation gives

$$u(y) = p_{,x} \frac{y^2}{2^j} + \frac{2 \in \lambda^2}{4\eta^2} p_{,x}^3 \frac{y^4}{2^{2j+j}} + C. \tag{2.27}$$

Using the conditions at the wall i.e.,

$$u(y) = 0 \text{ at } y = H,$$

we can easily find the value of C as

$$C = -p_{,x} \frac{H^2}{2^j} - \frac{2 \in \lambda^2}{4\eta^2} p_{,x}^3 \frac{H^4}{2^{2j+j}}.$$

Finally

$$u(y) = \frac{-p_{,x}}{2^{j+1}\eta}(H^2 - y^2)\left(1 + \frac{\in \lambda^2 p_{,x}^2}{2^{2j}\eta^2}(H^2 + y^2)\right). \tag{2.28}$$

Scaling the gaps with respect to H , the velocities with cross-sectional average velocities $\bar{u}$ and pressure or stresses with $\eta\bar{u}/H$, we get the non-dimensional velocity profile as

$$\frac{u(y)}{\bar{u}} = \frac{-p_{,x} H^2}{2^{j+1}\eta\bar{u}}\left[1 - \left(\frac{y}{H}\right)^2\right]\left[1 + \frac{\in \lambda^2 p_{,x}^2 H^2}{2^{2j}\eta^2}(1 + \left(\frac{y}{H}\right)^2)\right]. \tag{2.29}$$

Define

$$\frac{-p_{,x}H^2}{2^{j+1}\kappa\eta} = \bar{u}_N. \tag{2.30}$$

Squaring both sides gives

$$\frac{p_{,x}^2 H^2}{2^{2j}\eta^2} = \frac{4\kappa^2\bar{u}^2}{H^2}\left(\frac{\bar{u}_N}{\bar{u}}\right)^2. \tag{2.31}$$

Using the above values in Eq. (2.28) we find

$$\frac{u(y)}{\bar{u}} = \kappa\frac{\bar{u_N}}{\bar{u}}\left(1-\left(\frac{y}{H}\right)^2\right)\left(1+4\kappa^2\in De^2\left(\frac{\bar{u_N}}{\bar{u}}\right)^2\left(1+\left(\frac{y}{H}\right)^2\right)\right), \tag{2.32}$$

where κ take the value 1.5 for plane flow and the value 2 for axisymmetric flow. In Eq. (2.32) the parameter $\bar{u_N}$ is defined by Eq. (2.30) is representing the cross-sectional average velocity for the upper-convected Maxwell fluid or the Newtonian cases. Note also that $\bar{u_N}/\bar{u}$ is nothing other than a pressure gradient in non-dimensional. The $De = \lambda\bar{u}/H$ is the Deborah number in dimensionless group, it is a measure of the level of elasticity in the fluid, and is basicaly the velocity $\bar{u}$ which is the average velocity. For $\in= 0$ or $De = 0$, the Eq. (2.32) takes the form to the known profile in parabolic, with a maximum velocity present at the centreline given as $(u_o)_N = \kappa\bar{u_N}$. The second term present in the brackets of Eq. (2.32) represents a corrective term, infact to the parabolic profile, and is attached to the PTT model.

The pressure gradient -$p_{,x}$ is usually not known but it can be related to the cross-sectional average velocity via the definition of flow rate. According to Bird et al. [21]. we define

$$\bar{u} \equiv \frac{1}{H^{j+1}}\int_0^H 2^j y^j u(y)dy, \tag{2.33}$$

or

$$\bar{u} \equiv \frac{1}{H^{j+1}} \int_0^H 2^j y^j \left[\frac{-p_{,x}}{2^{j+1}\eta}(H^2 - y^2) \left(1 + \frac{\in \lambda^2 p_{,x}^2}{2^{2j}\eta^2}(H^2 + y^2) \right) \right] dy. \tag{2.34}$$

Integration yields

$$\bar{u} = \frac{-p_{,x}H^2}{\eta(j+1)(j+3)} \left(1 + \frac{\in \lambda^2 p_{,x}^2 H^2}{2^{(2j-1)}\eta^2} \frac{(j+3)}{(j+5)} \right). \tag{2.35}$$

For the inverse problem solution, the evaluation of the pressure difference for a given flux, it is advantageous to work with the normalized velocity profile (2.29) which needs to be integrated to yield the following non-dimensional cubic Eq. for $\bar{u_N}/\bar{u}$ i.e.,

$$\frac{(j+1)(j+3)}{2^{j+1}\kappa} = \frac{\bar{u_N}}{\bar{u}} \left[1 + b \left(\frac{\bar{u_N}}{\bar{u}} \right)^2 \right]. \tag{2.36}$$

The left hand side of Eq. (2.36) equal one both for channel and pipe flow and therefore we get

$$1 = \frac{\bar{u_N}}{\bar{u}} \left(1 + b \left(\frac{\bar{u_N}}{\bar{u}} \right)^2 \right), \tag{2.37}$$

with

$$b = \frac{8(3+j)\kappa^2}{(5+j)} \in De^2. \tag{2.38}$$

Eq. (2.37) shows that $\bar{u_N} < \bar{u}$.

This implies that the flow rate for a PTT fluid is higher then for a Newtonian fluid for an identical pressure gradint. This is because of shear thinning behaviour of PTT fluid. Now we apply Cardano-Tartaglia formula on Eq. s (2.37) as follows:

Let $z = \bar{u_N}/\bar{u}$ and $m = n = 1/b$, then Eq. (2.37) reads

$$z^3 + mz = n. \tag{2.39}$$

Using the transformations

$$z = t - u, \ m = 3tu, \ n = t^3 - u^3,$$

we can write

$$n = t^3 - \left[\frac{m}{3t}\right]^3, \tag{2.40}$$

which gives

$$\left(t^3\right)^2 - n\left(t^3\right) - \frac{m^3}{27} = 0. \tag{2.41}$$

The above equation is a quadratic equation. in t^3 and can be easily solved to give

$$t^3 = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}. \tag{2.42}$$

Using the positive root for the real solution, we get

$$t = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}. \tag{2.43}$$

For this value of t , we can easily find the required value of u i.e.,

$$u = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}. \tag{2.44}$$

Having in hand the values of t and u we find the following value of z

$$z = \sqrt[3]{\frac{1}{2b} + \frac{\alpha^{1/2}}{(108b^3)^{1/2}}} - \sqrt[3]{-\frac{1}{2b} + \frac{\alpha^{1/2}}{(108b^3)^{1/2}}}, \tag{2.45}$$

or

$$z = \frac{1}{(2\beta b)^{1/3}}\left[\delta^{1/3} - (-\beta + \alpha^{1/2})^{1/3}\right], \tag{2.46}$$

where

$$\alpha = 3^3 b + 4; \quad \beta = 3^{3/2} b^{1/2}; \quad \delta = \alpha^{1/2} + \beta.$$

Returning to the original variable we have

$$\frac{\bar{u_N}}{\bar{u}} = \frac{(423)^{1/6}(\delta^{2/3} - 2^{2/3})}{6b^{1/2}\delta^{1/3}} \tag{2.47}$$

Eq. (2.47) gives the explicit relation for the pressure gradient as a function of the cross-sectional average velocity $\bar{u}$, the main results of the analysis for the linearized PTT fluid are therefore the velocity profile (2.28) and (2.29), the flux Eq. (2.35) and the unknown driving pressure gradient at given flow rate obtained from (2.47). The maximum velocity at the centreline ($y = 0$) is also useful and is given by

$$\frac{u_0}{\bar{u}} = \kappa \frac{1 + 4\kappa^2 \in De^2 (\bar{u_N}/\bar{u})^2}{1 + b(\bar{u_N}/\bar{u})^2}, \tag{2.48}$$

showing that it is smaller than in the Newtonian case. Expressions for the normalized stress components are readily obtained after scaling with the wall shear stress for the Newtonian (or UCM) fluid. From Eq. (2.21), the non-dimensional normal stress is

$$\frac{\tau_{xx}}{2\kappa\eta\bar{u}/H} = 4\kappa De \left(\frac{\bar{u_N}}{\bar{u}}\right)^2 \left(\frac{y}{H}\right)^2 \tag{2.49}$$

and the normalized shear stress component is calculated from (2.20):

$$\frac{\tau_{xy}}{2\kappa\eta\bar{u}/H} = -\left(\frac{\bar{u_N}}{\bar{u}}\right)\left(\frac{y}{H}\right). \tag{2.50}$$

The rate of normalized shear strain from Eq. (2.24) takes the form

$$\frac{\dot{\gamma}(y)}{2\kappa\bar{u}/H} = -\frac{\bar{u_N}}{\bar{u}}\left(\frac{y}{H}\right)\left(1 + 8\kappa^2 \in De^2 \left(\frac{\bar{u_N}}{\bar{u}}\right)^2 \left(\frac{y}{H}\right)^2\right). \tag{2.51}$$

and the viscosity profile is

$$\mu(\dot{\gamma}) \equiv \frac{\tau_{xy}}{\dot{\gamma}} \Longrightarrow \frac{\mu(\dot{\gamma})}{\eta} = \left(1 + 8\kappa^2 \in De^2 \left(\frac{\bar{u_N}}{\bar{u}}\right)^2 \left(\frac{y}{H}\right)^2\right)^{-1}. \tag{2.52}$$

The values of these quantities at the wall are useful to define non-dimensional quantities and are obtained after setting $y = H$;

$$\frac{\mu_w}{\eta} = \left(1 + 8\kappa^2 \in De^2 \left(\frac{\bar{u_N}}{\bar{u}}\right)^2\right)^{-1}, \quad \frac{(\tau_{xy})_w}{2\kappa\eta\bar{u}/H} = \frac{\bar{u_N}}{\bar{u}} \text{ and } \frac{(\tau_{xx})_w}{2\kappa\eta\bar{u}/H} = 4\kappa De \left(\frac{\bar{u_N}}{\bar{u}}\right)^2 \tag{2.53}$$

Where $(\tau_{xy})_w$ is defined positive. These values are smaller than the corresponding values for the UCM fluid, a point to be taken into account when comparing non-dimensional values.

2.4 Exponential PTT model

Observe that normal and shear stress profiles are independent of the function $f(\mathbf{tr}(\boldsymbol{\tau}))$, therefore they are still given by Eq. s (2.49) and (2.50), respectively. However, the ratio $\bar{u_N}/\ \bar{u}$ in those expressions is different since it depends on the new velocity distribution. This distribution is obtained in a similar way by inserting the new function (Eq. (2.4)) into Eq. (2.24) followed by integration, to yield the dimensional and the corresponding non-dimensional forms of the velocity profile, respectively. Now in view of Eq. (2.4), Eq. (2.24) takes the form

$$\frac{du}{dy} = \exp\left(\frac{\in\lambda}{\eta}\frac{2\lambda}{\eta}\frac{p_{,x}^2 y^2}{2^{2j}}\right)\frac{p_{,x}y}{\eta 2^j}. \tag{2.54}$$

Integrating above Eq. we get

$$u(y) = \frac{\exp(\in\lambda^2 H^2 p_{,x}^2/(2^{(2j-1)}\eta^2))}{-p_{,x}\in\lambda^2/(2^{j-2}\eta)}\left(1-\exp\left(-\frac{\in\lambda^2 p_{,x}^2}{\eta^2 2^{2j-1}}(H^2-y^2)\right)\right), \tag{2.55}$$

and in order to non-dimensionalize the above Eq.(2.55), we use the same scalings as done before for linear PTT model and get

$$\frac{u(y)}{\bar{u}} = \kappa\frac{\bar{u_N}}{\bar{u}}\frac{\exp(b(\bar{u_N}/\bar{u})^2)}{b(\bar{u_N}/\bar{u})^2}(1-\exp(-b(\bar{u_N}/\bar{u})^2(1-(y/H)^2))). \tag{2.56}$$

where $b \equiv 8\kappa^2 \in De^2$. For the axisymmetric case it is also valid, after effecting the possible changes. Appropriately, in the small limit of $b(\bar{u_N}/\bar{u})^2$, Eq. (2.56) tends to the parabolic profile, as it should. For exponential PTT model, the expression of cross-sectional average velocity is no longer same for both axisymmetric and planar case. For channel, we obtain

$$\bar{u} \equiv \frac{1}{H}\int_0^H u(y)dy, \tag{2.57}$$

and then

$$\bar{u} = \frac{-\eta}{4 \in \lambda^2 p_{,x}}\left(\exp\left(\frac{2 \in \lambda^2 p_{,x}^2 H^2}{\eta^2}\right) - \frac{\eta}{2\lambda p_{,x} H}\frac{\pi^{1/2}\operatorname{erf}(i(\lambda p_{,x} H/\eta)\sqrt{2 \in})}{i\sqrt{2 \in}}\right). \tag{2.58}$$

and also for the pipe flow

$$\bar{u} \equiv \frac{1}{H^2}\int_0^H 2yu(y)dy, \tag{2.59}$$

and therefore

$$\bar{u} = \frac{-\eta}{2 \in \lambda^2 p_{,x}}\exp\left(\frac{\in \lambda^2 p_{,x}^2 H^2}{2\eta^2}\right)\left(1 + \frac{\exp(- \in \lambda^2 H^2 p_{,x}^2/(2\eta^2)) - 1}{\in \lambda^2 H^2 p_{,x}^2/(2\eta^2)}\right). \tag{2.60}$$

The normalized form of shear-rate and profiles of viscosity are easily evaluated from (2.56) and (2.50):

$$\frac{\dot{\gamma}(y)}{2\kappa\bar{u}/H} = -\frac{\bar{u}_N}{\bar{u}}\left(\frac{y}{H}\right)\exp\left(8\kappa^2 \in De^2\left(\frac{\bar{u}_N}{\bar{u}}\right)^2\left(\frac{y}{H}\right)^2\right). \tag{2.61}$$

and

$$\frac{\mu(\dot{\gamma})}{\eta} = \exp\left(-8\kappa^2 \in De^2\left(\frac{\bar{u}_N}{\bar{u}}\right)^2\left(\frac{y}{H}\right)^2\right). \tag{2.62}$$

Similar to the linear PTT model, we investigate the dimensionless velocity profile across the channel or pipe to give the parameter $\bar{u}_N/\bar{u}$ for the problem of obtaining the pressure difference for given rate of flow and for planar case ($\kappa = 1.5$) we obtain the following equation:

$$1 = \frac{3}{2}\frac{\bar{u}_N}{\bar{u}}\frac{\exp(b(\bar{u}_N/\bar{u})^2)}{b(\bar{u}_N/\bar{u})^2}\left(1 + \frac{i\pi^{1/2}\exp(-b(\bar{u}_N/\bar{u})^2)\operatorname{erf}(ib^{1/2}\bar{u}_N/\bar{u})}{2b^{1/2}(\bar{u}_N/\bar{u})}\right). \tag{2.63}$$

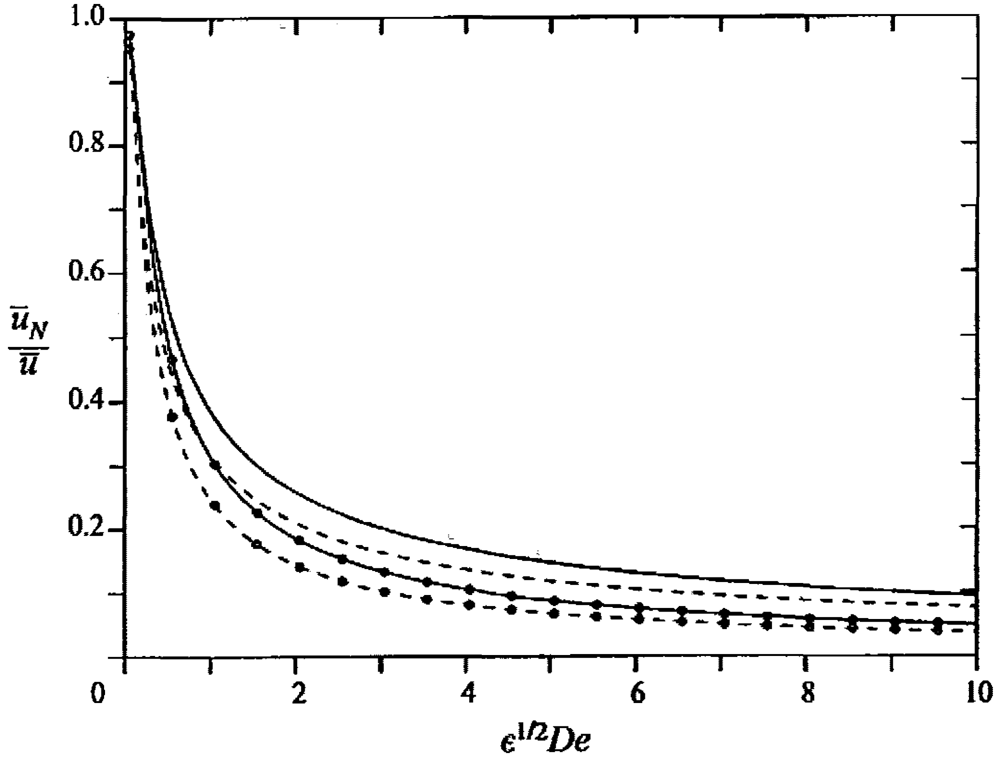


Figure 1: Variation of the average velocity ratio $\bar{u_N}/\bar{u}$ with $\in^{1/2}$ De (solid line: plane flow; dashed line: axisymmetric flow; no symbol:linear PTT; symbols: exponential PTT).

$\in^{1/2}$ *De*	Channel	Pipe	$\in^{1/2}$ *De*	Channel	Pipe
0.05	0.9746	0.9526	1.0	0.3134	0.2471
0.1	0.9132	0.8545	2.0	0.1858	0.1439
0.2	0.7695	0.6732	3.0	0.1342	0.1033
0.3	0.6515	0.5492	4.0	0.1059	0.08114
0.4	0.5628	0.4642	5.0	0.08778	0.06713
0.5	0.4953	0.4028	6.0	0.07519	0.05740
0.6	0.4427	0.3565	7.0	0.06589	0.05024
0.7	0.4007	0.3203	8.0	0.05873	0.04474
0.8	0.3663	0.2911	9.0	0.05303	0.04036
0.9	0.3376	0.2672	10.0	0.04839	0.03680

Table 1: Numerical solution of Eqs. (2.63) and (2.64) and for the axisymmetric case ($\kappa = 2$) we find

$$1 = 2\frac{\bar{u_N}}{\bar{u}}\frac{\exp(b(\bar{u_N}/\bar{u})^2)}{b(\bar{u_N}/\bar{u})^2}\left(1 - \frac{\exp(-b(\bar{u_N}/\bar{u})^2)}{b(\bar{u_N}/\bar{u})^2}\right). \tag{2.64}$$

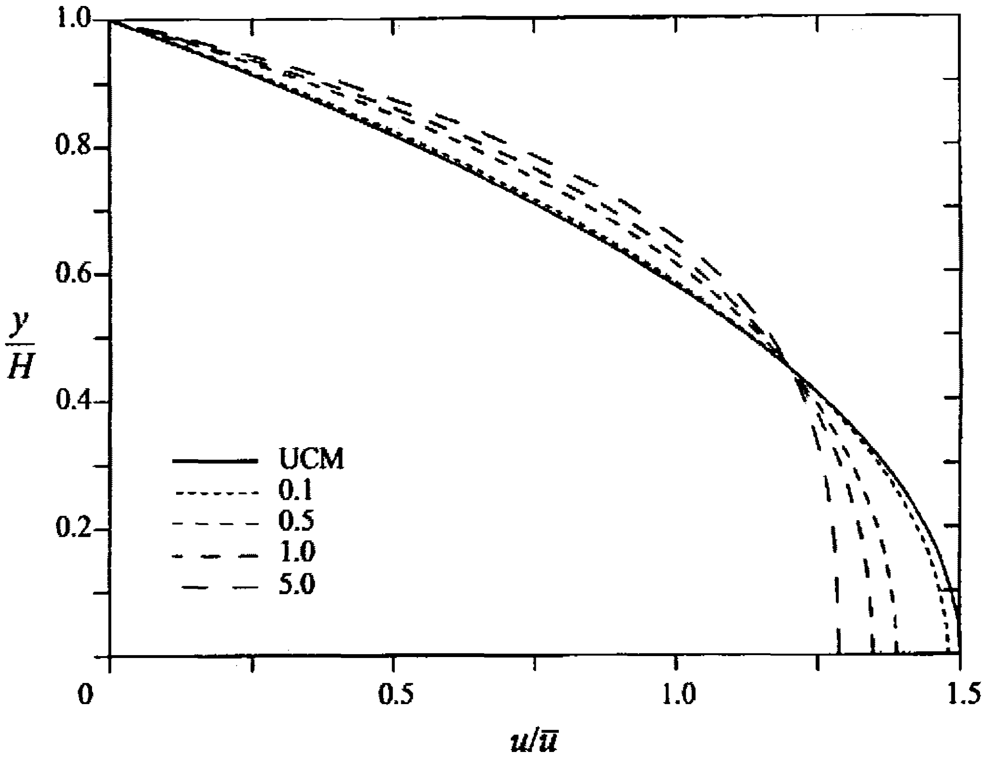


Figure 2: Velocity profile of the linear PTT fluid in channel flow as a functon of the dimensionless group $\epsilon^{1/2}$ *De* (solid line: parabolic profile; dashed lines: $\epsilon^{1/2}$ $De =$

0.1, 0.5, 1.0, 5.0).

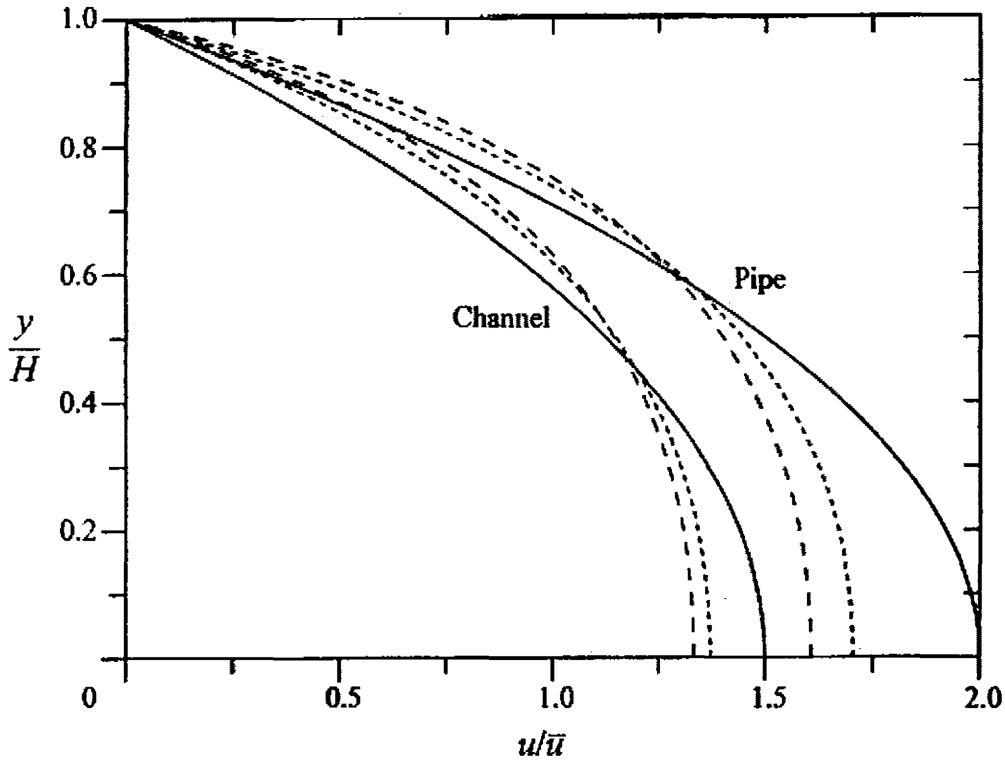


Figure 3: Comparisons of the velocity profiles for the linear and the exponential PTT model in channel and pipe flow ($De = 2$ and $\epsilon = 0.1$) (Solid lines: parabolic profiles;

dashed lines: linear and exponential PTT).

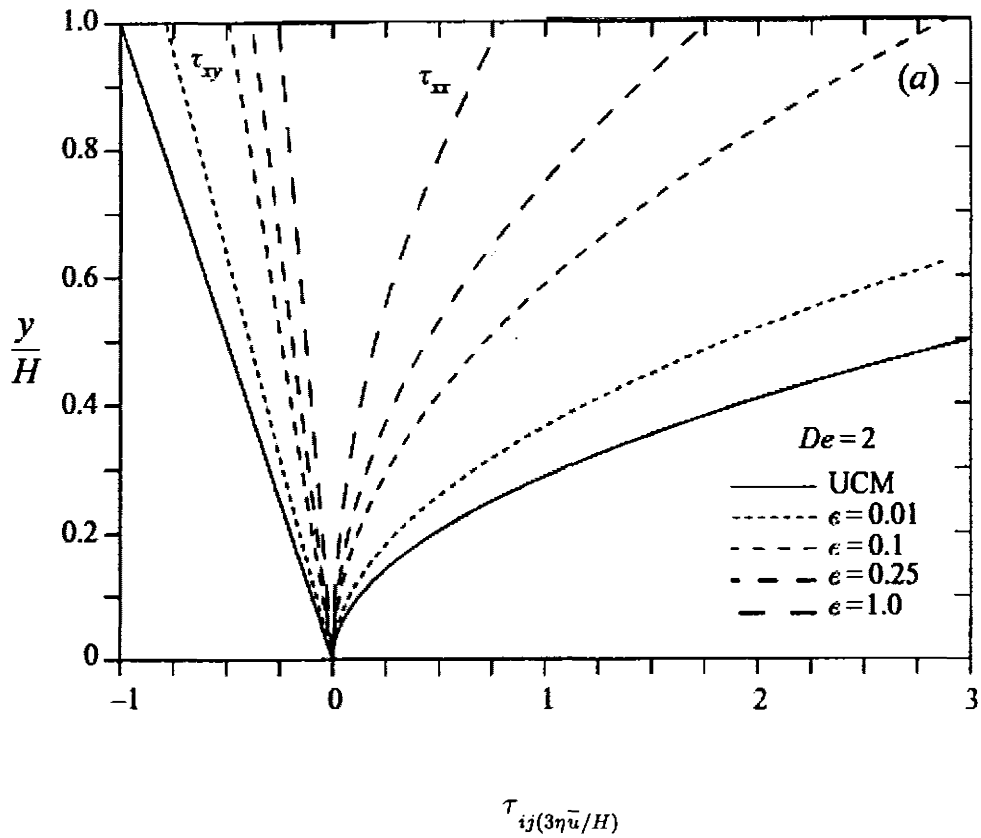


Figure 4(a): Profiles of normalized shear and normal stress components for varying

$\in$ and constant $De = 2$ (Solid lines: UCM fluid; dashed lines: $\in = 0.01, 0.1, 0.25, 1.0$).

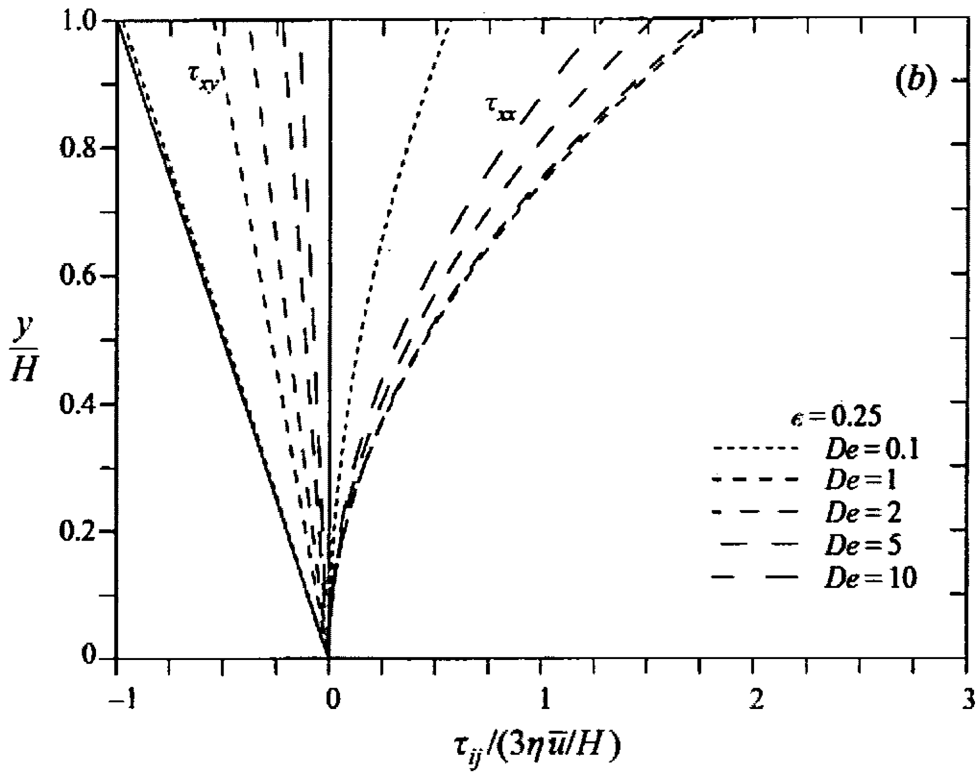


Figure 4(b): Profiles of normalized shear and normal strss components for varying De at constant $\in = 0.25$ (solid lines: newtonion fluids; dashed lines: $De = 0.1, 1, 2, 5, 10$).

Contrary to the previous case, these nonlinear equations are not amenable to an analytical solution and therefore the use of numerical methods is inevitable. We have solved the above equations with a built in routine "FindRoot" in **Mathematica,** the solution of which is given in Fig.1. A few such values are also tabulated in Table 1 from where other values can be extracted by interpolation.

2.5 Discussion

It is clear from Figure 1 that, for identical pressure gradients, the PTT fluid can carry a larger flow rate than the Newtonian or UCM fluids, especially for $\in^{1/2}$ De larger than 2. This effect is due to an increased shear-thinning behaviour with the parameter $\in^{1/2}$ De and is more intense with the exponential form of the PTT model. The shear-thinning behaviour is also observed in the flatter velocity profiles pertaining to the plane flow of the linear PTT fluid in Figure 2. As $\in^{1/2}$ De increases the velocity profiles flatten in the centre in a similar way to those of shear-thinning power-law fluids. The exponential form of the PTT model leads to velocity distributions (Eqs. (2.55), (2.56), (2.63)and (2.64)) that are similar to those in Figure 2 except for the increased shear-thinning behaviour as a consequence of the corresponding higher values of the function f, as seen in Figure 3. The distributions of the normalized τ_{xy} and τ_{xx} across the channel width are shown together for the linear PTT model in Figure 4(a), for varying $\in$ at constant De, and in Figure 4(b) for varying De at constant $\in$. The trends in Figure 4(a) are expected since $\in \rightarrow 0$ brings the PTT model close to the UCM model and so the stresses should increase in magnitude. In the latter graph, however, the trend is not monotonic and for high elasticity (high De) the normal stresses are seen to decrease, an unexpected outcome. Inspection of the relevant equations shows that both stress components depend only on the dimensionless group $\in^{1/2}$ De, but the normal stress also depends separately on De alone. For high De, Eq. (2.49) shows $\tau_{xx} \propto De^{-1/3}$ hence justifying the decrease of τ_{xx} with elasticity seen in figure 4(b). This peculiar effect can be removed if the stresses are made non-dimensional with their own value of shear stress at the wall. Then, the variation of $\tau_{xy}/(\tau_{xy})_w$ will coincide with that for the UCM or the Newtonian models, and the normal stress will be given by

$$\frac{\tau_{xx}}{(\tau_{xy})_w} = 4\kappa De(\bar{u_N}/\bar{u})(y/H)^2. \tag{2.65}$$

which, for high De, tends to $\approx (De/\in)^{1/3}$ at the wall, because $\bar{u_N}/\bar{u} \approx 1/b^{1/3} \approx 1/(\in De^2)^{1/3}$. Hence, the above non-dimensional normal stress now increases monotonically with De.

Chapter 3

Analytical solution for flow of PTT-viscoelastic fluid in a concentric annulus

The aim of this chapter is to present the analytical solution for purely tangential flow in steady state form of a viscoelastic fluid obeying the Phan-Thein-Tanner (PTT) constitutive equation in a concentric annulus with relative rotation of the inner and outer cylinders. The obtained solutions are valid for both linear and exponential PTT fluid model. This chapter is based on a recent paper by Mirzazadeh et al. [13].

3.1 Governing Equations

For the flow in a concentric annulus, we have

$$\mathbf{u} = [V_r, V_\theta, V_z],$$

$$V_\theta = V_\theta(r) \quad , \quad V_z = V_r = 0, \tag{3.1}$$

where V_θ , V_z and V_r are the tangential, axial and radial components of velocity. Let us define ratio of inner cylinder radius (R_i) to the outer cylinder radius (R_0) as κ, then the radial gap width δ is equal to $R_0(1-\kappa)$. Let Ω_i and Ω_0 denote the angular velocities of the inner and outer cylinders respectively. For the velocity field given by Eq. (3.1) the radial and tangential momentum equations reduces to:

$$-\rho\frac{V_\theta^2}{r} = \frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} - \frac{\tau_{\theta\theta}}{r} - \frac{\partial P}{\partial r}, \tag{3.2}$$

$$\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} = 0, \tag{3.3}$$

where τ_{rr} , $\tau_{\theta\theta}$ and $\tau_{r\theta}$ are the components of the stress tensors and r, z and θ refers to the radial, axial and tangential directions respectively. In this chapter we consider a relation of PTT constitutive equation that is general than the relation used in chapter 2 and is given by

$$f(tr(\boldsymbol{\tau}))\boldsymbol{\tau} + \lambda\overset{\nabla}{\boldsymbol{\tau}} + \frac{\xi}{2}\lambda(\mathbf{D}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{D}) = \eta\mathbf{D}, \tag{3.4}$$

where η is the viscosity coefficient of the model, ξ a parameter related to the no slip between the molecular network and the continum medium. For $\xi = 0$ we recover the model of PTT fluid used in chapter 2. For steady tangential annular flow Eq. (3.4) gives

$$f(tr(\boldsymbol{\tau}))\tau_{\theta\theta} = \lambda(2-\xi)\dot{\gamma}\tau_{r\theta}, \tag{3.5}$$

$$f(tr(\boldsymbol{\tau}))\tau_{rr} = -\lambda\xi\dot{\gamma}\tau_{r\theta}, \tag{3.6}$$

$$f(tr_\tau)\tau_{r\theta} = \eta\dot{\gamma} + \lambda\left(1-\frac{\xi}{2}\right)\dot{\gamma}\tau_{rr} - \frac{\lambda\xi}{2}\dot{\gamma}\tau_{\theta\theta}, \tag{3.7}$$

where $\dot{\gamma} = rd(V_\theta/\gamma)/d\gamma$. The boundary conditions for this problem arise from no-slip at the walls and are given by:

$$r = R_i \Rightarrow V_\theta = R_i\Omega_i, \tag{3.8}$$

$$r = R_0 \Rightarrow V_\theta = R_0\Omega_0. \tag{3.9}$$

3.2 Analytical solution for the complete PTT model

With the help of Eqs. (3.5) and (3.6), the following relation between the normal stresses can be established:

$$\frac{\tau_{rr}}{\tau_{\theta\theta}} = -\frac{\xi}{2-\xi}, \tag{3.10}$$

while the trace of the stress tensor is

$$\tau_{rr} + \tau_{\theta\theta} = tr(\boldsymbol{\tau}). \tag{3.11}$$

$$f(tr(\boldsymbol{\tau})) = 1 + \frac{\in\lambda}{\eta} tr(\boldsymbol{\tau}) \tag{3.12}$$

Using Eq. (3.11) we can write from Eq. (3.12)

$$f(\tau_{rr} + \tau_{\theta\theta}) = 1 + \frac{\in\lambda}{\eta}(\tau_{rr} + \tau_{\theta\theta}). \tag{3.13}$$

From Eq. (3.11) we have

$$\tau_{rr} = -\frac{\xi}{2-\xi}\tau_{\theta\theta}. \tag{3.14}$$

Using Eq. (3.14) in Eq. (3.13) we get the following form of f

$$f = 1 + \frac{2\in\lambda(1-\xi)}{\eta(2-\xi)}\tau_{\theta\theta}. \tag{3.15}$$

By dividing Eq. (3.7) by Eq. (3.5), we find that

$$\frac{f(\tau_{rr}+\tau_{\theta\theta})\tau_{r\theta}}{f(\tau_{rr}+\tau_{\theta\theta})\tau_{\theta\theta}}=\frac{\eta\dot{\gamma}+\lambda\left(1-\frac{\xi}{2}\right)\dot{\gamma}\tau_{rr}-\frac{\lambda\xi}{2}\dot{\gamma}\tau_{\theta\theta}}{\lambda(2-\xi)\dot{\gamma}\tau_{r\theta}}. \tag{3.16}$$

In view of Eq. (3.13) the above equation yields

$$\lambda\xi\tau_{\theta\theta}^2-\eta\tau_{\theta\theta}+\lambda(2-\xi)\tau_{r\theta}^2=0, \tag{3.17}$$

which is a quadratic equation in $\tau_{r\theta}$. The solution of above equation is

$$\tau_{\theta\theta}=\frac{\eta}{2\lambda\xi}\left[1\pm\sqrt{1-\frac{4\lambda^2\xi(2-\xi)\tau_{r\theta}^2}{\eta^2}}\right] \tag{3.18}$$

To find the appropriate sign in Eq. (3.18) we employ the simplified PTT model for which the value of ξ is zero and therefore Eq. (3.17) reduces to:

$$\tau_{\theta\theta}=\frac{2\lambda}{\eta}\tau_{r\theta}^2. \tag{3.19}$$

If the positive sign were taken in Eq. (3.18) and ξ is allowed to approach zero, $\tau_{\theta\theta}$ would approach infinity, which contradicts Eq. (3.19) and this root must therefore be discarded. If the negative sign in Eq. (3.18) is used instead, and l'Hopital's law is applied, we find:

$$\lim_{\xi\to 0}\tau_{\theta\theta}=\frac{2\lambda}{\eta}\tau_{r\theta}^2, \tag{3.20}$$

which is in agreement with Eq. (3.19). We conclude, therefore, that the correct solution is:

$$\tau_{\theta\theta}=\frac{\eta}{2\lambda\xi}\left[1-\sqrt{1-\frac{4\lambda^2\xi(2-\xi)\tau_{r\theta}^2}{\eta^2}}\right]. \tag{3.21}$$

For more detail we refer the reader to ref. [12].

To obtain the shear rate $\dot{\gamma}$, we substitute $\tau_{\theta\theta}$ from Eq. (3.21) into Eq. (3.5) and solve the resulting equation for $\dot{\gamma}$. These steps are outlined below and required value for $\dot{\gamma}$ is given in Eq. (3.22).

$$f(tr(\boldsymbol{\tau}))\tau_{\theta\theta} = \lambda(2-\xi)\dot{\gamma}\tau_{r\theta}$$

$$f(tr(\boldsymbol{\tau}))\frac{\eta}{2\lambda\xi}\left[1-\sqrt{1-\frac{4\lambda^2\xi(2-\xi)\tau_{r\theta}^2}{\eta^2}}\right] = \lambda(2-\xi)\dot{\gamma}\tau_{r\theta}$$

$$\dot{\gamma} = \frac{f(tr(\boldsymbol{\tau}))\frac{\eta}{2\lambda\xi}\left[1-\sqrt{1-\frac{4\lambda^2\xi(2-\xi)\tau_{r\theta}^2}{\eta^2}}\right]}{\lambda(2-\xi)\tau_{r\theta}}$$

$$\dot{\gamma} = \frac{\left[1+\frac{\epsilon\lambda}{\eta}\left(-\frac{\xi}{2-\xi}\tau_{\theta\theta}+\tau_{\theta\theta}\right)\right]\frac{\eta}{2\lambda\xi}\left[1-\sqrt{1-\frac{4\lambda^2\xi(2-\xi)\tau_{r\theta}^2}{\eta^2}}\right]}{\lambda(2-\xi)\tau_{r\theta}}$$

$$\dot{\gamma} = \frac{\left[1+\frac{\epsilon(1-\xi)}{(2-\xi)}\left\{1-\sqrt{1-\frac{4\lambda^2\xi(2-\xi)\tau_{r\theta}^2}{\eta^2}}\right\}\right]\times\left(\frac{\eta}{2\lambda\xi}\right)\left[1-\sqrt{1-\frac{4\lambda^2\xi(2-\xi)\tau_{r\theta}^2}{\eta^2}}\right]}{\lambda(2-\xi)\tau_{r\theta}} \tag{3.22}$$

We now introduce the following dimensionless variables:

$$\overset{*}{r} = r/R_o \ , \ \overset{*}{V_\theta} = V_\theta/V_c \ , \ \overset{*}{\tau}_{r\theta}=\tau_{r\theta}/\eta V_c/\delta \text{ and } We = \lambda V_c/\delta \tag{3.22a}$$

The introduction of these dimensionless variable leads to the following dimensionless form of Eq. (3.22).

$$\overset{*}{r}\frac{d}{d\overset{*}{r}}\left(\frac{\overset{*}{V_\theta}}{\overset{*}{r}}\right) = \frac{\left[1+\frac{1}{\chi}\left\{1-\sqrt{1-4\overset{*}{We}^2\chi\overset{*}{\tau}_{r\theta}^2}\right\}\right]\times\left[1-\sqrt{1-4\overset{*}{We}^2\chi\overset{*}{\tau}_{r\theta}^2}\right]}{2(1-\kappa)\overset{*}{We}^2\chi\overset{*}{\tau}_{r\theta}}, \tag{3.23}$$

where χ is a parameter which combines $\in$ and ξ and is defined as $\frac{\xi(2-\xi)}{\in(1-\xi)}$ and $\overset{*}{We}$ is the modified Weissenberg number defined as $\overset{*}{We} = We\sqrt{\in(1-\xi)}$. Integration of Eq. (3.3) after non-dimensionalisation leads to:

$$\frac{\overset{*}{\tau}_{r\theta}}{\overset{*}{\tau}_{wi}} = \frac{\kappa^2}{\overset{*}{r}^2} \tag{3.24}$$

where $\overset{*}{\tau}_{wi}$ is the dimensionless wall shear stress on the inner cylinder. Substitution of $\overset{*}{\tau}_{r\theta}$ from Eq. (3.24) into Eq. (3.23) followed by integration leads to the dimensionless form of the velocity profile:

$$\frac{\overset{*}{V_\theta}}{\overset{*}{r}} = \frac{\left(1+\frac{2}{\chi}\right)}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi} \times \left[\begin{array}{c} \overset{*}{r}^2 - \sqrt{\overset{*}{r}^4 - n^2} \\ +nArctg\left(\frac{\sqrt{\overset{*}{r}^4-n^2}}{n}\right) \end{array}\right] + \frac{n^2}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi^2}\left(\frac{1}{\overset{*}{r}^2}\right) + C, \tag{3.25}$$

where

$$n = 2\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}\sqrt{\chi} \tag{3.26}$$

The boundary conditions can be put into non-dimensional form as follows:

$$r = R_i \Rightarrow V_\theta = R_i\Omega_i$$

gives

$$\overset{*}{r} = \kappa \Rightarrow \overset{*}{V_\theta} = \frac{2\kappa}{(1+\kappa)\left|\beta-1\right|}. \tag{3.27}$$

and

$$r = R_o \Rightarrow V_\theta = R_o\Omega_o$$

yields

$$\overset{*}{r} = 1 \Rightarrow \overset{*}{V_\theta} = \frac{2\beta}{(1+\kappa)\left|\beta-1\right|}. \tag{3.28}$$

where

$$\beta = \frac{\Omega_o}{\Omega_i}. \tag{3.29}$$

The application of boundry condition (3.27) to Eq. (3.25) gives

$$\left[\begin{array}{c} -\frac{2\kappa}{\kappa(1+\kappa)|\beta-1|} + \\ \frac{\left(1+\frac{2}{\chi}\right)}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi} \end{array}\right] \times \left[\begin{array}{c} \kappa^2 - \sqrt{\kappa^4 - n^2} + \\ nArctg\left(\frac{\sqrt{\kappa^4-n^2}}{n}\right) \end{array}\right] + \frac{n^2}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi^2}\left(\frac{1}{\kappa^2}\right) = -C \tag{3.30}$$

Similarly Eq. (3.25) in view of boundry condition (3.28) gives

$$-\frac{2\beta}{(1+\kappa)\left|\beta-1\right|}+\frac{\left(1+\frac{2}{\chi}\right)}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^{2}\chi}\times\begin{bmatrix}1-\sqrt{1-n^2}+\\ nArctg\left(\frac{\sqrt{1-n^2}}{n}\right)\end{bmatrix}+\frac{n^2}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^{2}\chi^2}=-C \tag{3.31}$$

Comparison of Eq. (3.30) and Eq. (3.31) yield the following equation for the unknown shear stress $\overset{*}{\tau}_{wi}$ at the inner cylinder

$$\frac{\left(1+\frac{2}{\chi}\right)}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^{2}\chi}\times\left\{\begin{matrix}(1-\kappa^2)+\sqrt{\kappa^4-n^2}-\\ \sqrt{1-n^2}+\\ n\begin{bmatrix}Arctg\left(\frac{\sqrt{1-n^2}}{n}\right)-\\ Arctg\left(\frac{\sqrt{\kappa^4-n^2}}{n}\right)\end{bmatrix}\end{matrix}\right\}-\begin{bmatrix}\frac{n^2}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^{2}\chi^2}\left(\frac{1-\kappa^2}{\kappa^2}\right)\\ +\frac{2(\beta-1)}{(1+\kappa)\left|\beta-1\right|}\end{bmatrix}=0 \tag{3.32}$$

Eq. (3.32) is strongly non-linear but can be solved numerically for the dimensionless wall shear stress $\overset{*}{\tau}_{wi}$ on the inner cylinder. Once $\overset{*}{\tau}_{wi}$ is known, the constant C in Eq. (3.25) can be obtained by applying either of the two boundary conditions (Eqs. (3.27) or (3.28).

3.3 Exact solution for the simplified PTT model

For simplified PTT model ξ is zero and $f(tr_\tau)=1+\frac{\epsilon\lambda}{\eta}tr_\tau$ therefore, we get following scalar equations from the above constitutive relation i.e.,

$$f(tr(\tau))\tau_{\theta\theta}=2\lambda\dot{\gamma}\tau_{r\theta}, \tag{3.33}$$

$$f(tr(\tau))\tau_{rr}=0, \tag{3.34}$$

$$f(tr(\tau))\tau_{r\theta}=\dot{\gamma}\eta. \tag{3.35}$$

Moreover, we have

$$\tau_{\theta\theta} = \frac{2\lambda}{\eta}\tau_{r\theta}^2 \tag{3.36}$$

and then

$$f(\tau_{\theta\theta}) = 1 + \frac{2 \in \lambda^2}{\eta^2}\tau_{r\theta}^2. \tag{3.37}$$

Now putting the values from Eq. (3.36) and Eq. (3.37) in Eq. (3.33)

$$1 + \frac{2 \in \lambda^2}{\eta^2}\tau_{r\theta}^2 \frac{2\lambda}{\eta}\tau_{r\theta}^2 = 2\lambda\dot{\gamma}\tau_{r\theta} \tag{3.38}$$

This gives

$$\dot{\gamma} = \left[1 + \frac{2 \in \lambda^2}{\eta^2}\tau_{r\theta}^2\right]\frac{\tau_{r\theta}}{\eta}, \tag{3.39}$$

or

$$r\frac{d}{dr}\left(\frac{V_\theta}{r}\right) = \left[1 + \frac{2 \in \lambda^2}{\eta^2}\tau_{r\theta}^2\right]\frac{\tau_{r\theta}}{\eta}. \tag{3.40}$$

With the help of dimensionless variables defined in Eq. (3.22a), we get

$$\overset{*}{r}\frac{d}{d\overset{*}{r}}\left(\frac{\overset{*}{V_\theta}}{\overset{*}{r}}\right) = \frac{\left(1 + 2 \in We^2 \overset{*}{\tau}_{r\theta}{}^2\right)\overset{*}{\tau}_{r\theta}}{1-\kappa}. \tag{3.41}$$

Now from Eq.(3.24)

$$\overset{*}{\tau}_{r\theta}{}^2 = \frac{\kappa^4}{r^4}\overset{*}{\tau}_{wi}{}^2. \tag{3.42}$$

Substitution of from above equation in Eq. (3.41) results in

$$\frac{d}{d\overset{*}{r}}\left(\frac{\overset{*}{V_\theta}}{\overset{*}{r}}\right) = \frac{\kappa^2\overset{*}{\tau}_{wi}}{1-\kappa}\left[-\frac{1}{2\overset{*}{r}{}^2} - \frac{\in We^2\kappa^4\overset{*}{\tau}_{wi}{}^2}{6\overset{*}{r}{}^6}\right]. \tag{3.43}$$

Integration yields

$$\frac{\overset{*}{V_\theta}}{\overset{*}{r}} = -\frac{\kappa^2\overset{*}{\tau}_{wi}}{(1-\kappa)\overset{*}{r}{}^2}\left[\frac{1}{2} + \frac{\in We^2\kappa^4\overset{*}{\tau}_{wi}{}^2}{3\overset{*}{r}{}^4}\right] + C. \tag{3.44}$$

In view of the boundry condition

$$\overset{*}{r} = \kappa \Rightarrow \overset{*}{V}_\theta = \frac{2\kappa}{(1+\kappa)\,|\beta - 1|},$$

Eq. (3.44) becomes

$$\frac{2\kappa}{\kappa(1+\kappa)\,|\beta-1|} + \frac{\kappa^2 \overset{*}{\tau}_{wi}}{(1-\kappa)\kappa^2}\frac{1}{2} + \frac{\in We^2\kappa^6 \overset{*}{\tau}_{wi}{}^3}{3(1-\kappa)\kappa^6} = C. \tag{3.45}$$

Similarly for the boundry condition

$$\overset{*}{r} = 1 \Rightarrow \overset{*}{V}_\theta = \frac{2\beta}{(1+\kappa)\,|\beta - 1|},$$

we have from Eq. (3.44) the following relation

$$\frac{2\beta}{(1+\kappa)\,|\beta-1|} + \frac{\kappa^2 \overset{*}{\tau}_{wi}}{(1-\kappa)}\frac{1}{2} + \frac{\in We^2\kappa^6 \overset{*}{\tau}_{wi}{}^3}{3(1-\kappa)} = C. \tag{3.46}$$

Equating the left hand sides of Eq. (3.45) and Eq. (3.46), we obtain

$$-\frac{6(1-\kappa)(\beta-1)}{\in We^2(1-\kappa^6)(1+\kappa)\,|\beta-1|} + \frac{3(1-\kappa^2)}{2\in We^2(1-\kappa^6)}\overset{*}{\tau}_{wi} + \overset{*}{\tau}_{wi}{}^3 = 0, \tag{3.47}$$

or

$$\overset{*}{\tau}_{wi}{}^3 + p\overset{*}{\tau}_{wi} + q = 0. \tag{3.48}$$

where the constants p and q in Eq. (3.48) are given by

$$p = \frac{3(1-\kappa^2)}{2\in We^2(1-\kappa^6)}. \tag{3.49}$$

$$q = -\frac{6(1-\kappa)(\beta-1)}{\in We^2(1-\kappa^6)(1+\kappa)\,|\beta-1|}. \tag{3.50}$$

The real solution of Eq. (3.48) can be expressed as:

$$\overset{*}{\tau}_{wi} = \frac{1}{6}\sqrt[3]{-108q + 12\sqrt{12p^3 + 81q^2}} - \frac{2p}{\sqrt[3]{-108q + 12\sqrt{12p^3 + 81q^2}}} \tag{3.51}$$

By introducing boundary conditions from Eqs. (3.26) or (3.28) into Eq. (3.44) and using $\overset{*}{\tau}_{wi}$ from Eq. (3.51), the second constant C is easily obtained, clearly the numerical value of C in this case (for SPTT) is different from the numerical value of that for PTT. For the limiting case ($\in We^2 \to 0$) the previous equation reduces to the well-known solution for a Newtonian fluid. In engineering calculations the torque friction factor f defined as $(\tau_w/\rho V_c^2/2)$ is a parameter of interest. Usually, the product of f and the rotational Reynolds number Re, which is defined as $(\rho V_c \delta/\eta)$ is often required.

Now we obtain the pressure variation across the annular gap. By substitution of τ_{rr} from Eq. (3.10) into Eq. (3.2), we arrive at:

$$\frac{\partial P}{\partial r} = \rho \frac{V_\theta^2}{r} - \frac{\xi}{2-\xi} \frac{1}{r} \frac{\partial (r \tau_{\theta\theta})}{\partial r} - \frac{\tau_{\theta\theta}}{r}. \tag{3.52}$$

Note that in Eq. (3.54) right hand side is a function of the radial coordinate only so that integration of this equation leads to an equation for the pressure distribution across the annular gap. Let us define the pressure distribution for simplified PTT constitutive equation. For this we take ξ equal to zero in Eq. (3.54) and non-dimensionalizing the resulting equation to get

$$\frac{\partial \overset{*}{P}}{\partial \overset{*}{r}} = \mathrm{Re} \frac{\overset{*}{V}_\theta^{\,2}}{\overset{*}{r}} - 2We \frac{\overset{*}{\tau}_{r\theta}^{\,2}}{\overset{*}{r}}, \tag{3.53}$$

where the non-dimensional pressure $\overset{*}{P}$ is defined as $(P/\eta V_c/\delta)$. By introducing $\overset{*}{\tau}_{r\theta}$ from Eq. (3.24) and $\overset{*}{V}_\theta$ from Eq. (3.44) into Eq. (3.55) and then integrating we arrive at:

$$\int \frac{\partial \overset{*}{P}}{\partial \overset{*}{r}} d\overset{*}{r} = \mathrm{Re} \int \frac{\overset{*}{V}_\theta^{\,2}}{\overset{*}{r}} d\overset{*}{r} - 2We \int \frac{\overset{*}{\tau}_{r\theta}^{\,2}}{\overset{*}{r}} d\overset{*}{r}, \tag{3.54}$$

$$\overset{*}{P} - \overset{*}{P_i} = \Psi(\overset{*}{r}) - \Psi(\kappa), \tag{3.55}$$

where $\overset{*}{P_i}$ is the dimensionless pressure on the inner cylinder and $\Psi(\overset{*}{r})$ is as follows:

$$\Psi(\overset{*}{r}) = \Psi_1(\overset{*}{r}) + \Psi_2(\overset{*}{r}), \tag{3.56}$$

$$\Psi_1(\overset{*}{r}) = \frac{\mathrm{Re}}{2}\left[C^2\overset{*}{r}^2 - \frac{2\kappa^2\overset{*}{\tau}_{wi}^2 C}{1-\kappa}\ln\overset{*}{r} - \frac{\kappa^4\overset{*}{\tau}_{wi}^2}{4(1-\kappa)^2}\frac{1}{\overset{*}{r}^2}\right], \tag{3.57}$$

$$\Psi_2(\overset{*}{r}) = \in We^2\left[\mathrm{Re}\left\{\frac{\kappa^6\overset{*}{\tau}_{wi}^3 C}{6(1-\kappa)}\frac{1}{\overset{*}{r}^4} - \frac{\kappa^8\overset{*}{\tau}_{wi}^2}{18(1-\kappa)}\frac{1}{\overset{*}{r}^6} - \frac{\kappa^{12}\in We^2\overset{*}{\tau}_{wi}^6}{90(1-\kappa)^2}\frac{1}{\overset{*}{r}^{10}}\right\} + We\frac{\kappa^4\overset{*}{\tau}_{wi}^2}{2}\frac{1}{\overset{*}{r}^4}\right]. \tag{3.58}$$

It should be pointed out for a Newtonian fluid ($\in We^2 \to 0$) and then the second term on the right hand side of Eq. (3.58), ($\Psi_2(\overset{*}{r})$) must be equal to zero.

3.4 Results and Discussion

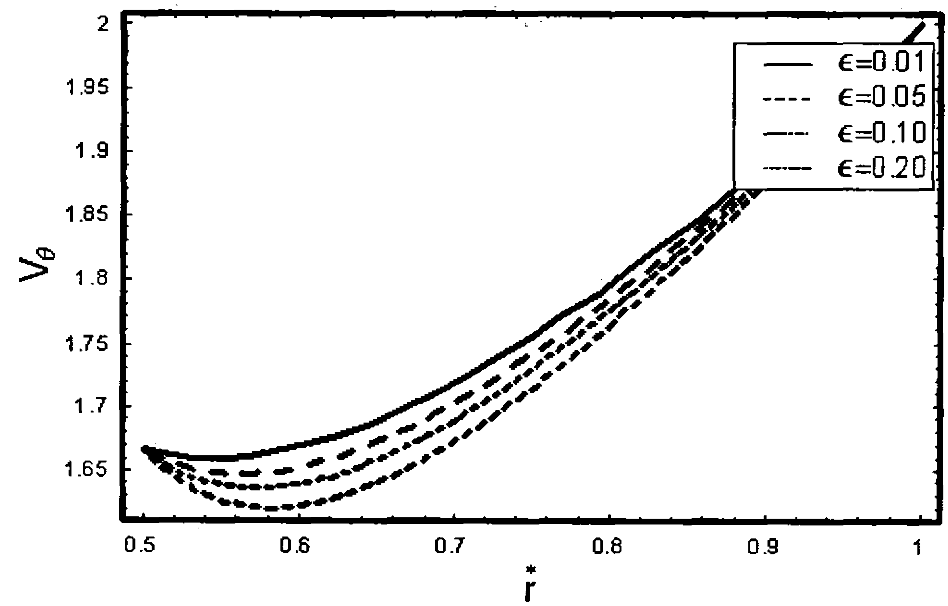


Figure 3.1: Variation of velocity $\overset{*}{V_\theta}$ for SPTT fluid with respect to $\overset{*}{r}$ for different

values of $\in$. The other parameters are $We = 5$, κ=0.5.

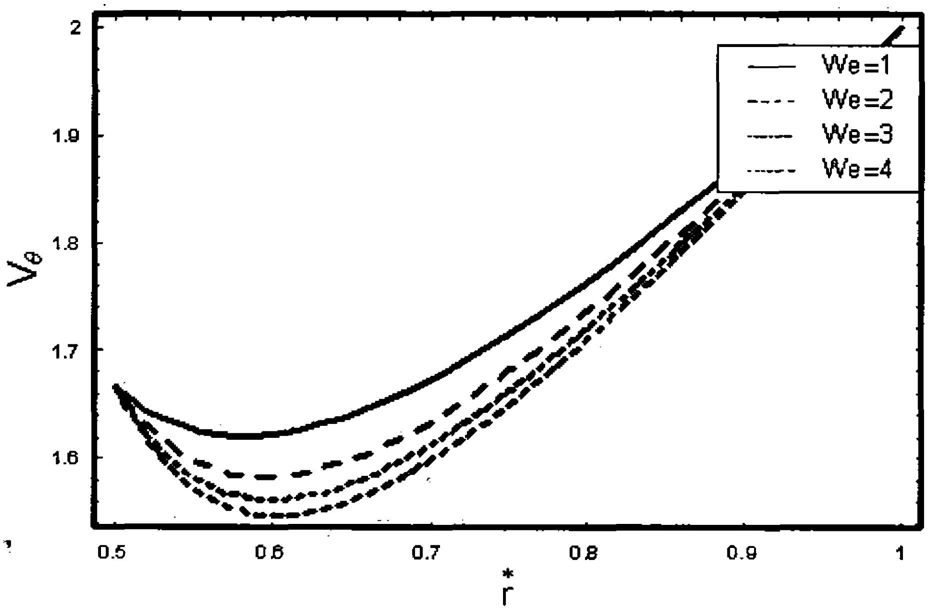


Figure 3.2: Variation of velocity $\overset{*}{V_\theta}$ for SPTT fluid with respect to $\overset{*}{r}$ for different values of We. The other parameters are $\in = .01$, κ=0.5.

In this section we start by showing the behaviour of velocity component $\overset{*}{V_\theta}$ for SPTT fluid for different values of We and $\in$. Figures 3.1 and 3.2 are plotted to serve the purpose. These figures reveals that the velocity component $\overset{*}{V_\theta}$ decreases by increasing We and $\in$. Moreover, it is also observed that $\overset{*}{V_\theta}$ attains a minimum values in the annular gap. The radial location of this minimun velocity is of interest in many situations [15] and therefore at the end we proceed to find the radial location of the minimum velocity both for PTT and simplified PTT constitutive equations.

The tangential velocity distribution for PTT constitutive equation is given as follows:

$$\overset{*}{V}_\theta = \frac{\left(1+\frac{2}{\chi}\right)}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi} \times \left[\begin{array}{c} \overset{*3}{r} - \overset{*}{r}\sqrt{\overset{*4}{r} - n^2} + \\ \overset{*}{r} n Arctg\left(\frac{\sqrt{\overset{*4}{r}-n^2}}{n}\right) \end{array}\right] + \frac{n^2}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi^2}\left(\frac{1}{\overset{*}{r}}\right) + C\overset{*}{r}. \tag{3.59}$$

Differentiating it with respect to $\overset{*}{r}$ we get

$$\frac{d\overset{*}{V}_\theta}{d\overset{*}{r}} = \frac{\left(1+\frac{2}{\chi}\right)}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi} \times \left[\begin{array}{c} 3\overset{*2}{r} - \sqrt{\overset{*4}{r} - n^2} - \frac{2\overset{*4}{r}}{\sqrt{\overset{*4}{r}-n^2}} + \\ n Arctg\left(\frac{\sqrt{\overset{*4}{r}-n^2}}{n}\right) + \frac{2n^2}{\sqrt{\overset{*4}{r}-n^2}} \end{array}\right] - \frac{n^2}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi^2}\left(\frac{1}{\overset{*2}{r}}\right) + C \tag{3.60}$$

Equating $d\overset{*}{V}_\theta/d\overset{*}{r}$ to zero yield the following equation

$$\frac{\left(1+\frac{2}{\chi}\right)}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi} \times \left[\begin{array}{c} 3\overset{*2}{r}_{\min} - \sqrt{\overset{*4}{r}_{\min} - n^2} + \\ n Arctg\left(\frac{\sqrt{\overset{*4}{r}_{\min}-n^2}}{n}\right) \end{array}\right] + \frac{n^2}{4(1-\kappa)\kappa^2\overset{*}{\tau}_{wi}\overset{*}{We}^2\chi^2}\left(\frac{1}{\overset{*2}{r}}\right) + C = 0. \tag{3.61}$$

Eq. (3.61) is a strongly non-linear and therefore has to be solved numerically to find the value of $\overset{*}{r}_{\min}$ as a function of the Weissenberg number, radius ratio χ, and the angular velocity ratio (β). Now we repeat the above procedure to find the values of $\overset{*}{r}_{\min}$ for simplified PTT model. For simplified PTT model $\overset{*}{V}_\theta$ is given by

$$\overset{*}{V}_\theta = -\frac{\kappa^2\overset{*}{\tau}_{wi}}{(1-\kappa)\overset{*}{r}}\frac{1}{2} + \frac{\in We^2\kappa^6\overset{*}{\tau}_{wi}^3}{3(1-\kappa)\overset{*5}{r}} + C\overset{*}{r}. \tag{3.62}$$

Differentiating it with respect to $\overset{*}{r}$ we have

$$\frac{d\overset{*}{V}_\theta}{d\overset{*}{r}} = \frac{\kappa^2\overset{*}{\tau}_{wi}}{2(1-\kappa)\overset{*2}{r}} + \frac{5 \in We^2\kappa^6\overset{*}{\tau}_{wi}^3}{3(1-\kappa)\overset{*6}{r}} + C. \tag{3.63}$$

Now setting

$$\frac{d\overset{*}{V}_\theta}{d\overset{*}{r}} = 0, \tag{3.64}$$

gives

$$\frac{\kappa^2 \overset{*}{\tau}_{wi}}{2(1-\kappa)\overset{*}{r}^2} + \frac{5 \in We^2 \kappa^6 \overset{*}{\tau}_{wi}^{\,3}}{3(1-\kappa)\overset{*}{r}^6} + C = 0, \tag{3.65}$$

or

$$\frac{\kappa^2 \overset{*}{\tau}_{wi} \overset{*}{r}^6}{2(1-\kappa)\overset{*}{r}^2} + \frac{5 \in We^2 \kappa^6 \overset{*}{\tau}_{wi}^{\,3}}{3(1-\kappa)} + C\overset{*}{r}^6 = 0, \tag{3.66}$$

or

$$\left(\overset{*}{r}^2\right)^3 + a(\overset{*}{r}^2)^2 + b = 0, \tag{3.67}$$

The real solution of the above cubic equation is

$$\overset{*}{r}_{\min} = \frac{\sqrt{6d(d^2 + 4a^2 - 2ad)}}{6d}, \tag{3.68}$$

where

$$d = \sqrt[3]{(-108b - 8a^3 + 12\sqrt{(81b^2 + 12ba^3)})}, \tag{3.69}$$

$$a = \frac{\kappa^2 \overset{*}{\tau}_{wi}}{2(1-\kappa)C}, \tag{3.70}$$

$$b = \frac{5 \in We^2 \kappa^6 \overset{*}{\tau}_{wi}}{3C}. \tag{3.71}$$

Here again, we recover the well-known result for the limiting case of a Newtonian fluid (i.e., $\in We^2 = 0$).

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