Two-Dimensional Inertial Flow of Couple-Stress Fluid through a Permeable Slit

By

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A Thesis
Submitted in the Partial Fulfillment of the
Requirement for the Degree of
MASTER OF SCIENCE
In
MATHEMATICS

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<u>Certificate</u>

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IN MATHEMATICS

We accept this thesis as conforming to the required standard

1.____________________	2. ____________________
Dr. Rahila Naz	Dr. Ambreen Afsar Khan
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2024

<u>Thesis Certificate</u>

The thesis entitled ***"Two-Dimensional Inertial Flow of Couple-Stress Fluid through a Permeable Slit"*** submitted by ***Afifa Ahmed***, ***838-FBAS/MSMA/F22*** in partial fulfillment of MS Degree in Mathematics has been completed under my guidance and supervision. I am satisfied with the quality of her research work and allow her to submit this thesis for further process to graduate with Master of Science degree from the Department of Mathematics & Statistics, as per IIUI rules and regulations.

Date…………………………..   ____________________

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DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely based on my efforts under the supervision of my supervisor **Dr. Khadija Maqbool**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

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DEDICATION

First of all, I dedicate this thesis to Allah Almighty and the Holy Prophet (P.B.U.H). Secondly, I dedicate this thesis to my respected parents who pray for me all the time. I also dedicate it to my teachers who guided me at every stage, and then to all my family members, who helped and encouraged me.

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All praise and glory to **Allah Almighty** Who gave me the strength and courage to understand, learn, and complete this thesis.

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I would also like to thank my dear parents, **Mr. and Mrs. Fiaz Ahmed**, and my brothers and sisters, who have always supported me. Without their unwavering support, I would not have reached the pinnacle of success. They kept me motivated and understanding, and their encouragement and guidance since my childhood have enabled me to unfold my hidden abilities.

Sincere thanks to all my friends for their kindness and moral support during my study. Thanks for the friendship and memories.

Lastly, I offer my regards and blessings to all of those who supported me in any aspect during the completion of this thesis.

(Afifa Ahmed)

Contents

Preface

The research area of Newtonian and non-Newtonian fluid flow in a permeable channel has attained a lot of interest in the field of bioengineering. Few researchers like Espedal et al. [1], Mazumdar et al. [2], Waite et al. [3,4], and Hayat et al. [5] discussed the behavior of viscous flow in different geometries. Different mechanisms like membrane filtration, physiological flow through veins, transpiration cooling, biofluid flow through the proximal tube, and dialysis can be modeled by the viscous fluid flow through different conduits. Numerous researchers proposed the theoretical and experimental models of the filtration process including Berman et al. [6], who is a pioneer in the study of laminar flow of incompressible fluid through a permeable conduit also found velocity and pressure by the perturbation method with constant suction on the boundary. After Berman, Yuan et al. [7,8] extended their research for high and low seepage rates and found the analytical solutions for velocity with pressure by series solution method. Like wise Terril et al. [9] calculated the exact solution for the two-dimensional flow problems in a permeable tube. A study of laminar flow in permeable conduit including the pressure drop, axial and transverse velocity profile, and the flow rate has been presented by Jocelyne et al. [10]. Sandeep et al. [11] solved the problem of rectangular and cylindrical cross-sectional areas with porous boundaries and found the series solution of velocity and pressure field.

Further scientists have been exploring a wide range of flow problems involving non-Newtonian fluids, because these fluids have numerous practical applications in modern technology and industries. The theory of couple-stress fluid, proposed by Stokes [12,13], has gained the interest of many researchers in fluid mechanics over the past five decades. Couple-stress fluid is a type of non-Newtonian fluid that exhibits the rotational motion [14]. These fluids are essential in modeling various physical phenomena, such as fluid flow through porous media, lubrication, and biological fluid dynamics [15,16]. The fundamental equations that describe the behaviour of couple-stress fluid are inherently non-linear and more complex than the Navier-Stokes equations. Therefore, finding an exact solution of these equations is quite challenging. Different perturbation techniques have been frequently employed to obtain approximate solutions of these equations [17]. However, in recent research, we have employed the Langlois technique to obtained the solution of non-linear partial differential equations.

Moreover, reabsorption in fluid mechanics introduces additional complexity, especially when dealing with permeable boundaries where the fluid can be absorbed back into the medium through which it flows. However, constant reabsorption models play a vital role in medical applications such as drug delivery systems and renal physiology, where the precise control of fluid absorption rates is important [18,19]. These models are useful for predicting and optimizing the performance of systems that benefit from controlled reabsorption improving functionality and effectiveness. Siddiqui et al [20] discussed the analytical solution of the creeping flow of couple-stress fluid with constant reabsorption.

According to the best of our knowledge, the two-dimensional inertial flow of couple stress fluid through a permeable rectangular conduit has not been discussed earlier, therefore in this research work we are intended to discuss two-dimensional inertial flow of couple-stress fluid through a permeable slit with constant reabsorption. This thesis is organized into three chapters. Chapter one presents basic definitions and laws of fluid mechanics. Chapter two discusses the study of inertial flow of couple stress fluid through a rectangular conduit with no slip velocity at the channel wall. Chapter three includes the effect of inertial forces and lubricated walls on the two-dimensional flow of couple stress fluids through a porous slit with constant reabsorption.

Chapter 1

Preliminaries

This chapter includes the basic concepts and definitions of different types of flows, fluid properties, and laws of fluid mechanics.

1.1 Fluid

A fluid is a substance that deforms subject to shear stress. It's a state of matter that consists of liquid, gas, and plasma.

1.2 Fluid Mechanics

Fluid mechanics is a branch of engineering that explores the characteristics and behaviour of fluids, whether fluids are in motion or at rest.

1.3 Types of Fluid

Fluid mechanics, involves the examination of different types of fluids, each with their own unique characteristics and behaviours. There are mainly two types of fluids, such as Newtonian and non-Newtonian fluids.

1.3.1 Newtonian vs Non-Newtonian Fluid

Newtonian fluid follows the Newton's law of viscosity, that exhibits a direct relationship between shear stress and velocity gradient. While non-Newtonian fluid do not adhere this law.

Mathematically, for Newtonian fluid following relation holds:

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} \tag{1.1}$$

where τ_{xy} is the shear stress, and μ is constant viscosity.

However in case of non-Newtonian fluid it may be written as:

$$\tau_{xy} = \eta \frac{\partial u}{\partial y} \tag{1.2}$$

where $\eta = \left(\frac{\partial u}{\partial y}\right)^{n-1}$, and $n \neq 1$.

1.4 Types of Flow

Flows can be categorized based on various characteristics:

1.4.1 Laminar vs Turbulent Flow

Laminar flow refers to a type of fluid flow where all fluid particles follow a certain path and move smoothly without crossing each other. On the other hand, turbulent flow is characterized by fluid particles that do not follow a specific path and the fluid flows in an irregular pattern..

1.4.2 Steady vs Unsteady Flow

In steady flow, fluid properties such as velocity, pressure, and density do not depend on time during flow i.e. $S \neq S(t)$.While in case of unsteady flow fluid properties change with respect to time i.e. $S = S(t)$

Mathematically, steady flow can be defined as:

$$\frac{\partial S}{\partial t} = 0, \tag{1.3}$$

and unsteady flow can be written as:

$$\frac{\partial S}{\partial t} \neq 0. \tag{1.4}$$

where S may be the velocity, pressure or density.

1.4.3 Compressible vs Incompressible Flow

Compressible flow experiences a notable change in density with varying pressure, temperature, and space components. On contraroy incompressible flow have a density that does not change with respect to space, pressure and temperature.

1.4.4 Two-Dimensional Flow and Three-Dimensional Flow

Two-dimensional flow is a flow in which the flow properties are functions of only two spatial coordinates, whereas three-dimensional flow involves flow properties that are functions of three spatial coordinates.

Mathematically, two-dimensional flow in rectangular coordinates is written as

$$\boldsymbol{V} = (u_1(x, y, t), u_2(x, y, t), 0), \tag{1.5}$$

where the velocity component in third direction is zero, and properties do not vary along third direction.

While three-dimensional flow is written as

$$\boldsymbol{V} = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t)). \tag{1.6}$$

where u_1 is the velocity component in x-direction, u_2 is in y-direction, and u_3 is in z-direction and flow properties vary in all three directions.

1.5 Flow Properties

1.5.1 Density

Density is defined as mass per unit volume at a specific pressure and temperature.

Mathematically, it is defined as

$$\rho = \frac{m}{V}. \tag{1.7}$$

where V denotes the volume and m is mass of the fluid.

1.5.2 Pressure

Pressure is the magnitude of force per unit area.

Mathematically, it can be expressed as

$$P = \frac{|\mathbf{F}|}{A}. \tag{1.8}$$

where $|\mathbf{F}|$ denotes the magnitude of force, A is the area and pressure is denoted by P.

1.5.3 Shear Stress

A force per unit area that tends to create deformation in fluid flow is known as shear stress. It is denoted by $\boldsymbol{\tau}$ and mathematically, defined as

$$\boldsymbol{\tau} = \frac{\boldsymbol{F}}{A}. \tag{1.9}$$

1.5.4 Volumetric Flux

In fluid mechanics, the volumetric flux is the rate of volume flow across a unit area. It is denoted by Q, and mathematically defined as

$$Q = AS \tag{1.10}$$

where A represents the cross-sectional area, and S denotes average flow velocity throughout the fluid. It can also be written as

$$Q = \int \int_A S dA \tag{1.11}$$

1.6 Non-dimensional Parameters

1.6.1 Reynold's Number

The Reynold's number is a dimensionless quantity, that quantifies the ratio of inertial to the viscous forces. It helps to predict the flow patterns of different fluid flow.

Mathematically, it is defined as

$$\mathrm{Re} = \frac{\rho V L}{\mu}, \tag{1.12}$$

where ρ denotes density, μ is dynamic viscosity, V represents velocity of fluid and L is characteristics length.

The $\mathrm{Re} < 2300$ represents the laminar flow and $2300 < \mathrm{Re} < 4000$ indicates the transition from laminar to turbulent flow but $\mathrm{Re} > 4000$ predict the turbulent flow [21].

1.6.2 Couple-stress Parameter

Couple-stress parameter is the square root of ratio of viscosity coefficient and couple-stress coefficient.

Mathematically it is defined as

$$\alpha = \sqrt{\frac{\mu}{\mu_1}}. \tag{1.13}$$

where α is the couple-stress parameter, μ represents viscous coefficient (dynamic viscosity), and μ_1 is the couple-stress coefficient.

1.7 Permeable Slit

Slit is defined as a long, narrow cut or opening in some system and permeable slit means it has pores which allows fluid particles to get either in or out.

1.8 Slip vs No-slip Condition

No-slip boundary condition presumes that fluid layer in touch with the wall is similar to velocity of wall. There is no relative motion between wall and fluid due to no-slip velocity on the boundary. In case of slip condition, the relative movement between boundary and fluid layer occurs.

1.9 Basic Laws of Fluid Mechanics

1.9.1 Principle of Conservation of Mass

This law states that mass of the system remains constant with respect to all physical changes.

Mathematically, it can be defined as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho\, .\boldsymbol{V}) = 0, \tag{1.14}$$

where ρ is the fluid density, t is the time and $\boldsymbol{\nabla}$ is the divergence, $\boldsymbol{V}$ is the velocity vector. For incompressible flow, density is constant and above equation becomes

$$\boldsymbol{\nabla}.\boldsymbol{V} = 0. \tag{1.15}$$

1.9.2 Principle of Conservation of Momentum

The principle states that the total amount of momentum remains constant within some domain that is momentum is neither created nor destroyed, but it can only be changed by the action of forces as mentioned by the Newton's law of motion.

Mathematically, it can be defined as

$$\rho\left(\frac{\partial}{\partial t} + (\boldsymbol{V}.\boldsymbol{\nabla})\right)\boldsymbol{V} + \nabla p - \rho\boldsymbol{f} = \boldsymbol{\nabla\tau}\,. \tag{1.16}$$

where $\boldsymbol{V}$ is the velocity vector, t represents time, $\boldsymbol{\tau}$ is the Cauchy stress tensor, p denotes the hydrostatic pressure, and f denotes the body forces.

1.10 Methodology

The technique which is going to be used in this research is Recursive or Langlois technique, which was introduced by W.E.Langlois in 1963. This method will help us to make the non-linear system into linear system. After linearizing, we will use inverse method to convert the linear PDE's into set of ODE's. In this approach, one can linearize velocity profile, shear stress and pressure with the help of small dimensionless number ε. In order to obtain the 1st, 2nd and 3rd order solutions for velocity profile, shear stress and pressure, one can assume following series i.e.

$$u = \sum_{i=1}^{\infty}\varepsilon^i u^{(i)}, \quad v = \sum_{i=1}^{\infty}\varepsilon^i v^{(i)}, \quad p = p^{(0)} + \sum_{i=1}^{\infty}\varepsilon^i p^{(i)}. \tag{1.17}$$

Chapter 2

Effect of Constant Reabsorption on Inertial Flow Passed through a Permeable Slit

2.1 Introduction

In this chapter an incompressible, stead, two-dimensional, non-creeping flow of couple-stress fluid, through a permeable slit of small width has is discussed, and Cartesian coordinate system is taken into account. The fluid Reabsorption is assumed to be constant at permeable walls of the slit, and also no-slip boundary condition is used to solve the set of non-linear coupled partial differential equations. The Langlois's method is used to linearize the problem and to formulate the solutions for, shear stress, velocity profile, pressure difference, and stream function. The graphical results for velocity profile, pressure and wall shear stress show the impact of Reynold's number, reabsorption parameter and couple-stress parameter.

2.2 Mathematical Formulation

Consider two-dimensional inertial flow of couple-stress fluid through a permeable slit having dimension $L \times h \times W$ and $W < h < L$. A constant Reabsorption rate V_0 is uniformly distributed at the permeable walls of the slit and Cartesian coordinate system (x, y, z) is chosen for the slit. It is also assumed that $W << h$, therefore, the flow in z-direction is very small as compared to x and y-direction.

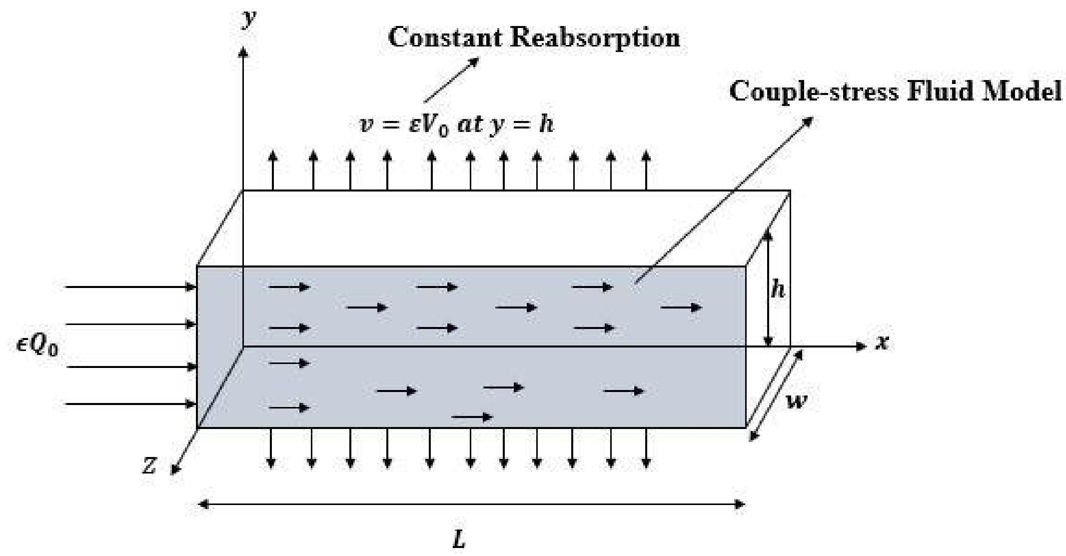


Fig. 2.1 : Schematic diagram of the problem

The above assumptions suggest the following velocity profile:

$$\boldsymbol{V} = (u(x, y), v(x, y)), \tag{2.1}$$

where $u(x, y)$ is velocity component in axial direction and $v(x, y)$ is the velocity component in transverse direction, respectively.

The governing equations for two-dimensional flow of couple-stress fluid through a slit having small width are as follows:

$$\boldsymbol{\nabla}.\ \boldsymbol{V} = 0, \tag{2.2}$$

$$\rho\left(\frac{\partial}{\partial t}+(\boldsymbol{V}.\boldsymbol{\nabla})\right)\boldsymbol{V}=-\nabla p+\mu\nabla^2\boldsymbol{V}-\mu_1\nabla^4\boldsymbol{V}, \tag{2.3}$$

where p is the hydrostatic pressure of the fluid, μ is the dynamic viscosity of the fluid and μ_1 is the material constant associated with couple-stress fluid.

The shear stress for the couple-stress fluid is given as follows:

$$\tau_{xy}=\mu\frac{\partial u}{\partial y}-\mu_1\frac{\partial^3 u}{\partial y^3}, \tag{2.4}$$

The boundary conditions according to the flow are as follows:

$$\frac{\partial u(x,y)}{\partial y}=0, \quad \frac{\partial^3 u(x,y)}{\partial y^3}=0, \quad v(x,y)=0, \qquad \text{at} \quad y=0, \tag{2.5}$$

$$u(x,y)=0, \quad \frac{\partial^2 u(x,y)}{\partial y^2}=0, \qquad v(x,y)=\varepsilon V_0, \qquad \text{at} \quad y=h,$$

$$\varepsilon Q_0=2W\int_0^h u(x,y)dy, \quad \text{at} \quad x=0.$$

For two-dimensional inertial flow, continuity and component form of momentum equations are given as follows:

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}, \tag{2.6}$$

$$\rho\left(\boldsymbol{V}.\boldsymbol{\nabla}\right)u=-\frac{\partial p}{\partial x}+\mu\nabla^2 u-\mu_1\nabla^4 u, \tag{2.7}$$

$$\rho\left(\boldsymbol{V}.\boldsymbol{\nabla}\right)v=-\frac{\partial p}{\partial y}+\mu\nabla^2 v-\mu_1\nabla^4 v. \tag{2.8}$$

2.2.1 Non-dimensional quantities

The following parameters are defined for non-dimensional analysis:

$$\begin{aligned} x' &= \frac{x}{L}, \quad y'=\frac{y}{h}, \quad u'=\frac{uh^2}{Q_0}, \quad v'=\frac{vhL}{Q_0}, V_0'=\frac{V_0hL}{Q_0}, \\ p' &= \frac{ph^4}{\mu L Q_0}, \quad \delta=\frac{h}{L}, \quad Re=\frac{\rho Q_0}{\mu h}, \quad \alpha=\sqrt{\frac{\mu}{\mu_1}} \end{aligned} \tag{2.9}$$

using above quantities in equations (2.6) – (2.8) and dropping primes one can write the following equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.10}$$

$$\delta Re\,(\boldsymbol{V}.\boldsymbol{\nabla})\,u=-\frac{\partial p}{\partial x}+\delta^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-N_1, \tag{2.11}$$

$$\delta^3 Re\,(\boldsymbol{V}.\boldsymbol{\nabla})\,v=-\frac{\partial p}{\partial y}+\delta^4\frac{\partial^2 v}{\partial x^2}+\delta^2\frac{\partial^2 v}{\partial y^2}-N_2, \tag{2.12}$$

where

$$N_1=\frac{1}{\alpha^2}\left(\delta^4\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+2\delta^2\frac{\partial^4 u}{\partial x^2\partial y^2}\right),$$

$$N_2=\frac{1}{\alpha^2}\left(\delta^6\frac{\partial^4 v}{\partial x^4}+\delta^2\frac{\partial^4 v}{\partial y^4}+2\delta^4\frac{\partial^4 v}{\partial x^2\partial y^2}\right).$$

The dimensionless form of boundary conditions will take the following form:

$$\frac{\partial u(x,y)}{\partial y}=0,\quad \frac{\partial^3 u(x,y)}{\partial y^3}=0,\quad v(x,y)=0,\qquad at\quad y=0, \tag{2.13}$$

$$u(x,y)=0,\quad \frac{\partial^2 u(x,y)}{\partial y^2}=0,\quad v(x,y)=\varepsilon V_0\qquad at\quad y=1,$$

$$\varepsilon=\frac{2W}{h}\int_0^1 u(x,y)dy,\qquad at\quad x=0.$$

2.3 Solution of the Problem

The set of equations (2.10) – (2.12) represents the non-linear partial differential equations in which three unknowns u, v,and p are present which will be determined by the recursive approach. It is already assumed that the slit is narrow and its width is very small as compared to its length, therefore the ratio of length to width (δ) is less than 1 and the terms which are in order of δ^2 will be ignored.

To find the solution of unknown quantities we will expand u, v, p, and τ_{xy} in the power series of ε (small dimensionless quantity)

$$u = \sum_{i=1}^{\infty} \varepsilon^i u^{(i)}, \qquad v = \sum_{i=1}^{\infty} \varepsilon^i v^{(i)}, \qquad p = p^{(0)} + \sum_{i=1}^{\infty} \varepsilon^i p^{(i)}, \quad \tau_{xy} = \sum_{i=1}^{\infty} \varepsilon^i \tau_{xy}^{(i)}. \tag{2.14}$$

where $p^{(0)}$ is a constant at inlet.

Now using the above mention series in equations (2.10) − (2.13) and then collecting powers of ε, one can get the following systems:

2.3.1 First Order System

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0, \tag{2.15}$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u^{(1)}}{\partial y^2} - \frac{1}{\alpha^2}\frac{\partial^4 u^{(1)}}{\partial y^4}, \tag{2.16}$$

$$0 = \frac{\partial p}{\partial y}, \tag{2.17}$$

$$\tau_{xy}^{(1)} = \mu\frac{\partial u^{(1)}}{\partial y} - \mu_1\frac{\partial^3 u^{(1)}}{\partial y^3}. \tag{2.18}$$

The corresponding boundary conditions for first order system are as follows:

$$\frac{\partial u^{(1)}}{\partial y} = 0, \quad \frac{\partial^3 u^{(1)}}{\partial y^3} = 0, \quad v^{(1)} = 0, \qquad \text{at} \quad y = 0, \tag{2.19}$$

$$u^{(1)} = 0, \quad \frac{\partial^2 u^{(1)}}{\partial y^2} = 0, \quad v^{(1)} = V_0, \qquad \text{at} \quad y = 1,$$

$$1 = \frac{2W}{h}\int_0^1 u^{(1)} dy, \quad \text{at} \quad x = 0.$$

To reduce the unknown quantities stream function is defined in following manner.

$$u^{(1)} = \frac{\partial \psi^{(1)}}{\partial y}, \qquad v^{(1)} = -\frac{\partial \psi^{(1)}}{\partial x}, \tag{2.20}$$

After replacing above relation in equations (2.15) − (2.18), one can write the following form:

$$0=-\frac{\partial p^{(1)}}{\partial x}+\frac{\partial^3\psi^{(1)}}{\partial y^3}-\frac{1}{\alpha^2}\frac{\partial^5\psi^{(1)}}{\partial y^5}, \tag{2.21}$$

$$0=\frac{\partial p^{(1)}}{\partial y}, \tag{2.22}$$

Upon using equation (2.22) into equation (2.21) following equation can be obtained.

$$0=\frac{\partial^6\psi^{(1)}}{\partial y^6}-\alpha^2\frac{\partial^4\psi^{(1)}}{\partial y^4}. \tag{2.23}$$

Boundary conditions in terms of stream function are mentioned as follows:

$$\frac{\partial^2\psi^{(1)}}{\partial y^2}=0, \qquad \frac{\partial\psi^{(1)}}{\partial x}=0, \qquad \text{at} \quad y=0, \tag{2.24}$$

$$\frac{\partial\psi^{(1)}}{\partial y}=0, \qquad \frac{\partial\psi^{(1)}}{\partial x}=-V_0, \qquad \text{at} \quad y=1,$$

$$\frac{\partial^3\psi^{(1)}}{\partial y^3}=0, \qquad \text{at} \quad y=1,$$

$$\frac{\partial^4\psi^{(1)}}{\partial y^4}=0, \qquad \text{at} \quad y=0,$$

$$\frac{h}{2W}=\psi^{(1)}(x,1),\ 0=\psi^{(1)}(x,0), \quad \text{at} \quad x=0.$$

To solve above BVP an Inverse method is used that suggests the following stream function:

$$\psi^{(1)}=V_0xR_1(y)+T_1(y), \tag{2.25}$$

After using above function in Eq.(2.23) and (2.24), one can get the following system of BVP's:

$$\frac{d^6R_1}{dy^6}-\alpha^2\frac{d^4R_1}{dy^4}=0, \tag{2.26}$$

The associated boundary conditions are:

$$R_1 = 0, \quad \frac{d^2 R_1}{dy^2} = 0, \quad \frac{d^4 R_1}{dy^4} = 0, \quad \text{at} \quad y = 0, \tag{2.27}$$

$$R_1 = -1, \quad \frac{dR_1}{dy} = 0, \quad \frac{d^3 R_1}{dy^3} = 0 \quad \text{at} \quad y = 1,$$

The second boundary value problem is given as follows:

$$\frac{d^6 T_1}{dy^6} - \alpha^2 \frac{d^4 T_1}{dy^4} = 0, \tag{2.28}$$

along with boundary conditions

$$T_1 = 0, \quad \frac{d^2 T_1}{dy^2} = 0, \quad \frac{d^4 T_1}{dy^4} = 0, \quad \text{at} \quad y = 0, \tag{2.29}$$

$$T_1 = \frac{h}{2W}, \quad \frac{dT_1}{dy} = 0, \quad \frac{d^3 T_1}{dy^3} = 0 \quad \text{at} \quad y = 1,$$

The general solutions of above BVP's are as follows:

$$R_1 = \sum_{i=1}^{4} c_i y^{i-1} + c_5 e^{\alpha y} + c_6 e^{-\alpha y}, \tag{2.30}$$

$$T_1 = \sum_{i=1}^{4} d_i y^{i-1} + d_5 e^{\alpha y} + d_6 e^{-\alpha y}, \tag{2.31}$$

After using their corresponding boundary conditions one can get the following solution.

$$R_1(y) = c_2 y + c_4 y^3 + 2c_5 \sinh(\alpha y), \tag{2.32}$$

$$T_1(y) = d_2 y + d_4 y^3 + 2d_5 \sinh(\alpha y). \tag{2.33}$$

where the values of unknown constant are given in appendix.

Using above expressions in Eq.(2.25) one can get the following stream function and velocity profile:

$$\psi^{(1)}(x,y) = (V_0xc_2 + d_2)y + (V_0xc_4 + d_4)y^3 + 2(V_0xc_5 + d_5)\sinh(\alpha y), \tag{2.34}$$

$$u^{(1)}(x,y) = (V_0xc_2 + d_2) + 3(V_0xc_4 + d_4)y^2 + 2\alpha(V_0xc_5 + d_5)\cosh(\alpha y), \tag{2.35}$$

$$v^{(1)}(x,y) = -V_0(c_2y + c_4y^3 + 2c_5\sinh(\alpha y), \tag{2.36}$$

With the help of above velocity profile and Eq.(2.21) pressure distribution can be formulated as follows:

$$p^{(1)}(x) = \frac{3\alpha^3(hx - V_0Wx^2)\cosh(\alpha h)}{2h\alpha W(-3 + h^2\alpha^2)\cosh(\alpha h) + 6W\sinh(\alpha h)}. \tag{2.37}$$

2.3.2 Second Order System

$$\frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} = 0, \tag{2.38}$$

$$\delta Re\left(\mathbf{V}^{(1)}.\boldsymbol{\nabla}\right)u^{(1)} = -\frac{\partial p^{(2)}}{\partial x} + \frac{\partial^2 u^{(2)}}{\partial y^2} - \frac{1}{\alpha^2}\frac{\partial^4 u^{(2)}}{\partial y^4}, \tag{2.39}$$

$$0 = \frac{\partial p^{(2)}}{\partial y}, \tag{2.40}$$

$$\tau_{xy}^{(2)} = \mu\frac{\partial u^{(2)}}{\partial y} - \mu_1\frac{\partial^3 u^{(2)}}{\partial y^3}, \tag{2.41}$$

and boundary conditions are given as follows:

$$\frac{\partial u^{(2)}}{\partial y} = 0, \quad \frac{\partial^3 u^{(2)}}{\partial y^3} = 0, \quad v^{(2)} = 0, \quad \text{at} \quad y = 0, \tag{2.42}$$

$$u^{(2)} = 0, \quad \frac{\partial^2 u^{(2)}}{\partial y^2} = 0, \quad v^{(2)} = 0, \quad \text{at} \quad y = 1,$$

$$0 = \int_0^1 u^{(2)}dy, \quad \text{at} \quad x = 0$$

After eliminating the pressure gradient from Eq. (2.39) – (2.40) and using stream function, following equation can be obtained:

$$\alpha^2 \delta Re \frac{\partial}{\partial y}\left(\mathbf{V}^{(2)}.\nabla\right) u^{(2)} = \frac{\partial^6 \psi^{(2)}}{\partial y^6} - \alpha^2 \frac{\partial^4 \psi^{(2)}}{\partial y^4}. \tag{2.43}$$

along with boundary conditions

$$\frac{\partial^2 \psi^{(2)}}{\partial y^2} = 0, \quad \frac{\partial \psi^{(2)}}{\partial x} = 0, \quad \text{at} \quad y = 0, \tag{2.44}$$

$$\frac{\partial \psi^{(2)}}{\partial y} = 0, \quad \frac{\partial \psi^{(2)}}{\partial x} = 0, \quad \text{at} \quad y = 1,$$

$$\frac{\partial^3 \psi^{(2)}}{\partial y^3} = 0, \quad \text{at} \quad y = 1,$$

$$\frac{\partial^4 \psi^{(2)}}{\partial y^4} = 0, \quad \text{at} \quad y = 0,$$

$$0 = \psi^{(2)}(x,1) = \psi^{(2)}(x,0), \quad \text{at} \quad x = 0$$

Now we will solve above BVP by assuming the following stream function.

$$\psi^{(2)}(x,y) = g(x)R_2(y) + T_2(y), \tag{2.45}$$

where $g(x) = x$.

Following same procedure adopted in first order system, one can get the following BVP's:

$$\frac{d^6 R_2}{dy^6} - \alpha^2 \frac{d^4 R_2}{dy^4} = f_1(y), \tag{2.46}$$

and boundary conditions are:

$$R_2 = 0, \quad \frac{d^2 R_2}{dy^2} = 0, \quad \frac{d^4 R_2}{dy^4} = 0, \quad \text{at} \quad y = 0, \tag{2.47}$$

$$R_2 = 0, \quad \frac{dR_2}{dy} = 0, \quad \frac{d^3 R_2}{dy^3} = 0, \quad \text{at} \quad y = 1.$$

Also,

$$\frac{d^6T_2}{dy^6} - \alpha^2\frac{d^4T_2}{dy^4} = f_2(y), \tag{2.48}$$

The related boundary conditions are:

$$T_2 = 0, \quad \frac{d^2T_2}{dy^2} = 0, \quad \frac{d^4T_2}{dy^4} = 0, \quad \text{at} \quad y = 0, \tag{2.49}$$

$$T_2 = 0, \quad \frac{dT_2}{dy} = 0, \quad \frac{d^3T_2}{dy^3} = 0, \quad \text{at} \quad y = 1,$$

where $f_1(y)$ and $f_2(y)$ are mentioned in appendix.

Solutions of above BVP's are obtained by DSolve command in MATHEMATICA. After getting the solution of stream function $\psi^{(2)}(x, y)$, one can find second order velocity components and second order pressure distribution.

2.3.3 Third Order System

$$\frac{\partial u^{(3)}}{\partial x} + \frac{\partial v^{(3)}}{\partial y} = 0, \tag{2.50}$$

$$M = -\frac{\partial p^{(3)}}{\partial x} + \frac{\partial^2 u^{(3)}}{\partial y^2} - \frac{1}{\alpha^2}\frac{\partial^4 u^{(3)}}{\partial y^4}, \tag{2.51}$$

$$0 = \frac{\partial p^{(3)}}{\partial y},$$

$$\tau_{xy}^{(3)} = \mu\frac{\partial u^{(3)}}{\partial y} - \mu_1\frac{\partial^3 u^{(3)}}{\partial y^3}, \tag{2.52}$$

where

$$M = \delta Re\left(u^{(1)}\frac{\partial u^{(2)}}{\partial x} + v^{(1)}\frac{\partial u^{(2)}}{\partial y}\right) + \left(u^{(2)}\frac{\partial u^{(1)}}{\partial x} + v^{(2)}\frac{\partial u^{(1)}}{\partial y}\right),$$

and boundary conditions are given as follows:

$$\frac{\partial^3 u^{(3)}}{\partial y^3} = 0, \quad \frac{\partial u^{(3)}}{\partial y} = 0, \quad v^{(3)} = 0, \quad \text{at} \quad y = 0, \tag{2.53}$$

$$\frac{\partial^2 u^{(3)}}{\partial y^2} = 0, \quad u^{(3)} = 0, \quad v^{(3)} = 0, \qquad \text{at} \quad y = 1,$$

$$0 = \int_0^1 u^{(3)} dy, \quad \text{at} \quad x = 0.$$

After eliminating pressure gradient and using stream function the following equation can be obtained:

$$-\alpha^2 \frac{\partial M}{\partial y} = \frac{\partial^6 \psi^{(3)}}{\partial y^6} - \alpha^2 \frac{\partial^4 \psi^{(3)}}{\partial y^4}, \tag{2.54}$$

and boundary conditions of stream function $\psi^{(3)}$ are as follows:

$$\frac{\partial^4 \psi^{(3)}}{\partial y^4} = 0, \quad \frac{\partial^2 \psi^{(3)}}{\partial y^2} = 0, \quad \frac{\partial \psi^{(3)}}{\partial x} = 0, \quad \text{at} \quad y = 0, \tag{2.55}$$

$$\frac{\partial^3 \psi^{(3)}}{\partial y^3} = 0, \quad \frac{\partial \psi^{(3)}}{\partial y} = 0, \quad \frac{\partial \psi^{(3)}}{\partial x} = 0, \qquad \text{at} \quad y = 1,$$

$$0 = \psi^{(3)}(x, 1) = \psi^{(3)}(x, 0), \quad \text{at} \quad x = 0$$

To write the simplified form of Eq.(2.51), one can get the following relation:

$$h(x) f_3(y) + f_4(y) = \frac{\partial^6 \psi^{(3)}}{\partial y^6} - \alpha^2 \frac{\partial^4 \psi^{(3)}}{\partial y^4}, \tag{2.56}$$

where

$$h(x) = x$$

and $f_3(y)$,and $f_4(y)$ are defined in appendix.

To find the solution by Inverse method, following choice of stream function $\psi^{(3)}(x, y)$ is used:

$$\psi^{(3)}(x, y) = h(x) R_3(y) + T_3(y), \tag{2.57}$$

Following same steps of 1st order system, one can obtain the following set of equations.

$$\frac{d^6 R_3}{dy^6} - \alpha^2 \frac{d^4 R_3}{dy^4} = f_3(y), \tag{2.58}$$

$$\frac{d^6T_3}{dy^6} - \alpha^2\frac{d^4T_3}{dy^4} = f_4(y), \tag{2.59}$$

and boundary conditions are given as follows:

$$R_3 = 0, \quad \frac{d^2R_3}{dy^2} = 0, \quad \frac{d^4R_3}{dy^4} = 0, \quad \text{at} \quad y = 0, \tag{2.60}$$

$$R_3 = 0, \quad \frac{dR_3}{dy} = 0, \quad \frac{d^3R_3}{dy^3} = 0, \quad \text{at} \quad y = 1,$$

$$T_3 = 0, \quad \frac{d^2T_3}{dy^2} = 0, \quad \frac{d^4T_3}{dy^4} = 0, \quad \text{at} \quad y = 0, \tag{2.61}$$

$$T_3 = 0, \quad \frac{dT_3}{dy} = 0, \quad \frac{d^3T_3}{dy^3} = 0, \quad \text{at} \quad y = 1,$$

Solution of above BVP's can be obtain by DSolve command in MATHEMATICA and solution of R_3, and T_3 will define third order velocity components, pressure, and shear stress. After combining, 1st, 2nd, and 3rd order solution , one can find expression of stream function, axial and transverse velocities, shear stress and pressure distribution in following manner:

$$\psi(x, y) = \psi^{(1)}(x, y) + \psi^{(2)}(x, y) + \psi^{(3)}(x, y), \tag{2.62}$$

$$u(x, y) = u^{(1)}(x, y) + u^{(2)}(x, y) + u^{(3)}(x, y), \tag{2.63}$$

$$v(x, y) = v^{(1)}(x, y) + v^{(2)}(x, y) + v^{(3)}(x, y), \tag{2.64}$$

$$\tau_{xy}(x, y) = \tau_{xy}^{(1)}(x, y) + \tau_{xy}^{(2)}(x, y) + \tau_{xy}^{(3)}(x, y), \tag{2.65}$$

$$p(x, y) = p^{(0)} + p^{(1)} + p^{(2)} + p^{(3)} \tag{2.66}$$

where $p^{(0)} = p(0, 0)$.

2.4 Results and Discussion

The graphical results are discussed in order to study influence of Reynold's number (Re), reabsorption parameter (V_0) and couple-stress parameter (α) on pressure difference, velocity profile at middle region ($x = 0.5$) of slit and shear stress at wall.

2.4.1 Effect of Reynold's Number (Re)

Fig. 2.2(a) illustrates the behaviour of horizontal velocity by rising values of Reynold's number at middle region of slit. It is observed that at the centre of rectangular cross section the flow is maximum due to pressure and away from this region the velocity of fluid is decaying towards walls due to friction, however at the boundaries it is zero due to no-slip velocity. In Fig. 2.2(b) graph depicts the behaviour of vertical velocity with respect to rising Reynold's number. It shows that vertical velocity accelerating towards walls of slit away from the center point, and no change has been observed at centerline. Fig. 2.2(c) indicates the effect of Reynold's number on pressure difference, which increases from entrance to exit point of slit by increasing values of Reynold's number. The effect of discrete values of Reynold's number on wall shear stress is depicted in Fig. 2.2(d). It is clear from this figure that wall shear stress rises with increase in values of Reynold's number and this rise is from inlet to outlet region.

2.4.2 Effect of Reabsorption Parameter (V_0)

Fig. 2.3(a) shows that the horizontal velocity of fluid rises with increasing values of reabsorption parameter V_0 at middle point ($x = 0.5$) of the slit. The vertical velocity of fluid is observed in Fig. 2.3(b) for separate values of reabsorption paramneter. It shows that at the center line of slit vertical velocity is not changing and away from center line the vertical velocity rises in forward and backward directions. Fig. 2.3.(c) shows effect of reabsorption parameter V_0 on pressure difference and it is noted that pressure difference rises from entrance to the point of the slit as reabsorption rate V_0 grows. The Fig. 2.3(d) indicates that wall shear stress is zero at entrance point and away from this point it rapidly increases towards walls of slit and it is observed that wall shear stress rises as reabsorption rate V_0 grows.

2.4.3 Effect of Couple-Stress Parameter (α)

Fig. 2.4(a) shows that the horizontal velocity rises for all increasing values of couple-stress parameter α at middle point of slit . Also, it is noted that the horizontal velocity rises at the centre, and decays towards walls of the slit. Fig. 2.4(b) indicates impact of couple-stress parameter on vertical velocity, which shows that for all increasing values of couple-stress parameter α, vertical velocity is not changing at center point and away from this region gradually accelerating to the walls of the slit. Fig. 2.4.(c) indicates impact of couple-stress parameter α on pressure difference and it is noticed that more pressure is required for the fluid flow when couple-stress parameter rises. Fig. 2.4(d) depicts that wall shear stress increases from entrance to exit point of slit for growing values of couple-stress parameter α.

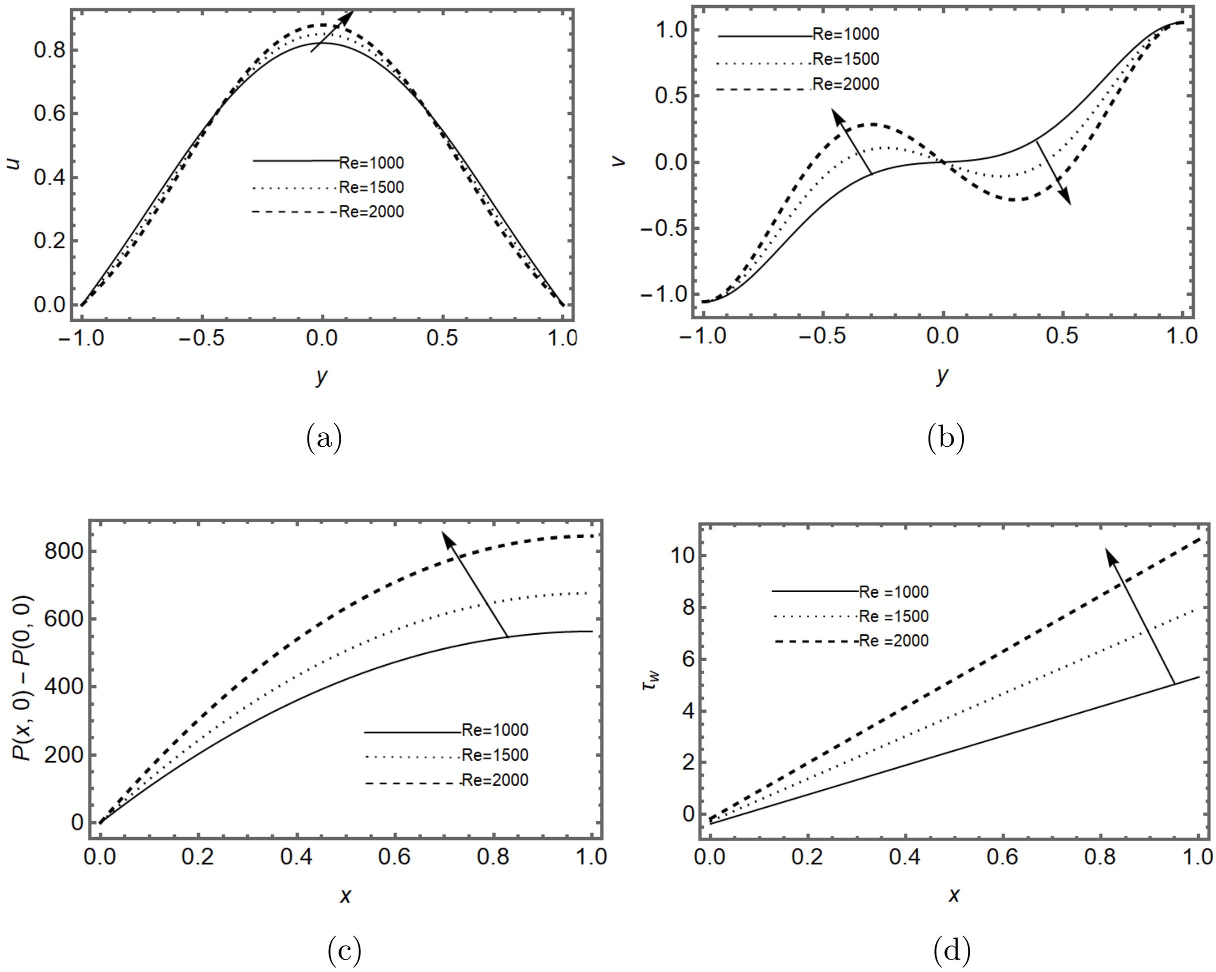
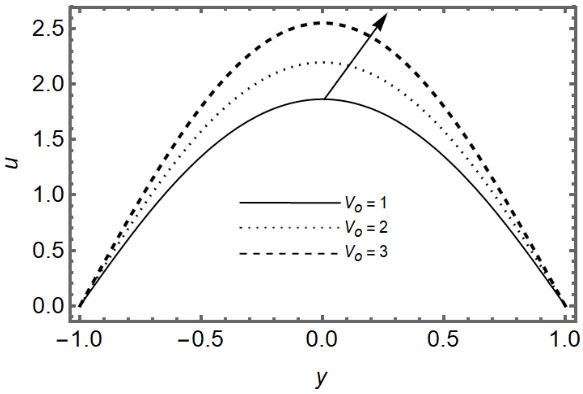
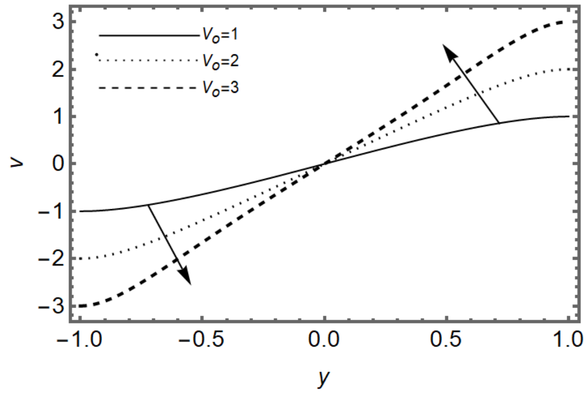


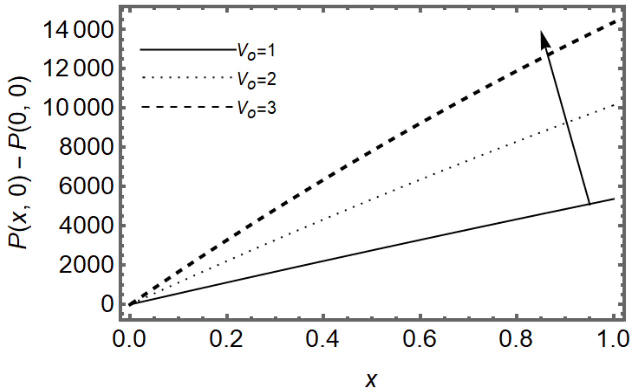
Fig.2.2 (a-d): Effect of Reynold's number on (a) horizontal and (b) vertical velocity at middle point $x = 0.5$, (c) pressure difference and (d) wall shear stress.



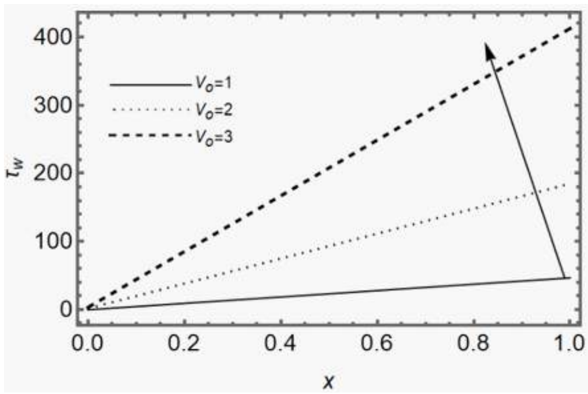
(a)



(b)

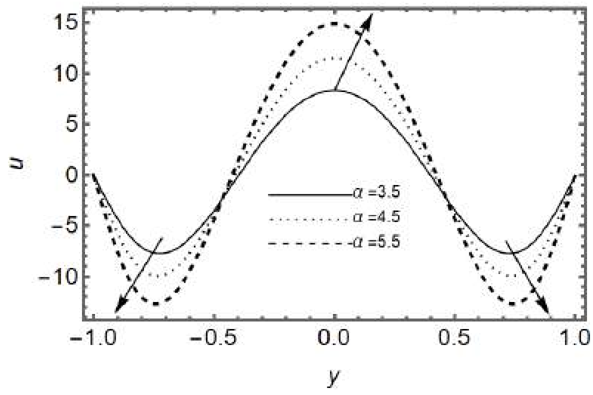


(c)

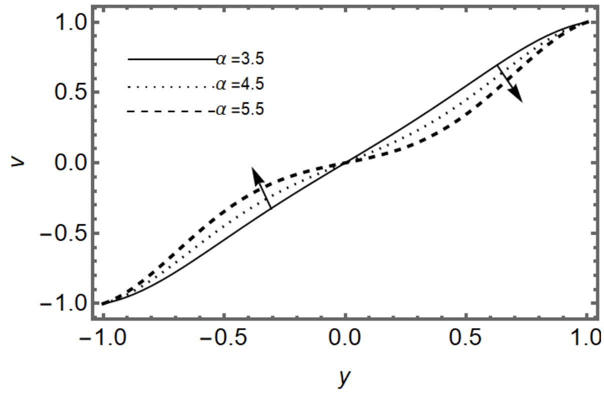


(d)

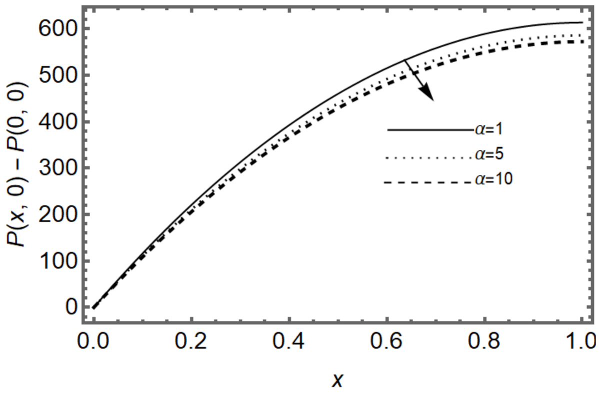
Fig.2.3 (a-d): Effect of reabsorption parameter V_0 on (a) horizontal and (b) vertical velocity at middle point $x = 0.5$, (c) pressure difference and (d) wall shear stress.



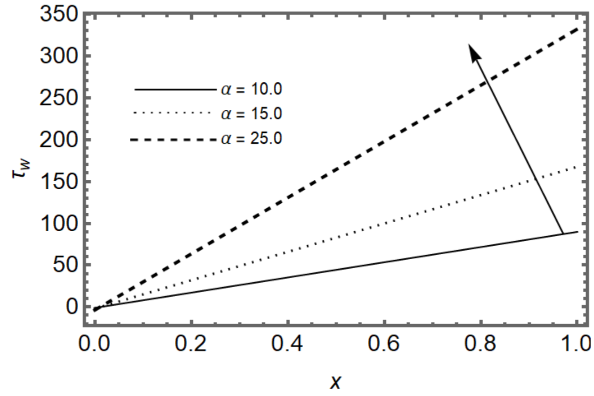
(a)



(b)



(c)



(d)

Fig.2.4 (a-d): Effect of couple-stress parameter α on (a) horizontal and (b) vertical velocity at middle point $x = 0.5$, (c) pressure difference and (d) wall shear stress.

Chapter 3

Slip Effect on Inertial Flow of Couple-Stress Fluid through a Permeable Slit

3.1 Introduction

In this chapter two-dimensional inertial (non-creeping) flow of couple-stress fluid through a permeable slit of small width is discussed. A slip boundary condition and a uniform reabsorption at permeable walls of slit is assumed to solve the set of non-linear partial differential equations. The resulting partial differential equations are solved with the help of Langlois technique and analytical results for velocity profile, pressure and shear stress are displayed through graphs.

3.2 Mathematical Formulation

Consider a steady two-dimensional inertial flow of couple stress fluid through a permeable slit having dimension $L \times h \times W$ and $W < h < L$. It is assumed that constant reabsorption rate V_0 is uniformly distributed at permeable walls of the slit and slip condition is used due to relative motion between fluid and the walls of the slit. Cartesian coordinate system (x, y, z) is chosen for the slit and it is assumed that $W << h$, therefore flow in z-direction is negligible.

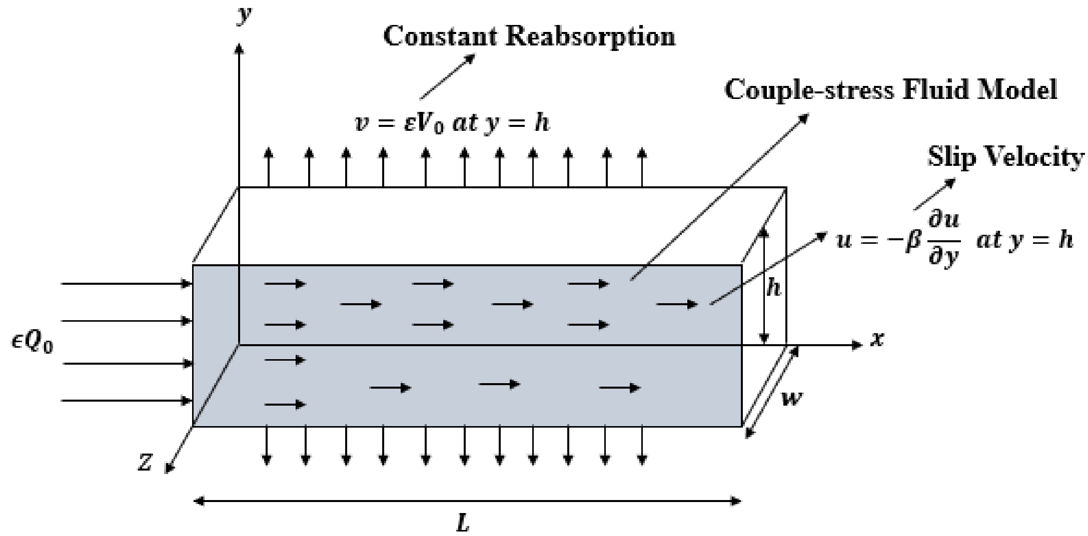


Fig. 3.1: Schematic diagram of the problem.

The two-dimensional and bidirectional flow suggests the following velocity profile:

$$\boldsymbol{V} = (u(x, y), v(x, y)), \tag{3.1}$$

where $u(x, y)$ and $v(x, y)$ represents the velocity components in x and y-directions, respectively.

Governing equations for couple stress fluid flow through a slit are described by the following expressions:

$$\boldsymbol{\nabla}.\boldsymbol{V} = 0, \tag{3.2}$$

$$\rho \left(\frac{\partial}{\partial t} + (\boldsymbol{V}.\boldsymbol{\nabla}) \right) \boldsymbol{V} = -\nabla p + \mu \nabla^2 \boldsymbol{V} - \mu_1 \nabla^4 \boldsymbol{V}, \tag{3.3}$$

where p is the hydrostatic pressure of fluid, μ is dynamic viscosity of fluid and μ_1 is material constant associated with couple-stress fluid.

The shear stress for couple stress fluid is defined as follows:

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} - \mu_1 \frac{\partial^3 u}{\partial y^3}, \tag{3.4}$$

and boundary conditions are given as follows:

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial^3 u}{\partial y^3} = 0, \quad v = 0, \quad \text{at} \quad y = 0,$$

$$u = -\beta \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad v = \varepsilon V_{0,} \quad \text{at} \quad y = h,$$

$$\varepsilon Q_0 = 2W \int_0^h u dy, \quad \text{at} \quad x = 0. \tag{3.5}$$

For two-dimensional inertial flow, the component form of continuity and momentum equations are given as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.6}$$

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u - \mu_1 \nabla^4 u, \tag{3.7}$$

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \nabla^2 v - \mu_1 \nabla^4 v. \tag{3.8}$$

3.2.1 Non-dimensional Parameter

For non-dimensional analysis following quantities are defined:

$$\begin{aligned} x' &= \frac{x}{L}, y' = \frac{y}{h}, u' = \frac{uh^2}{Q_0}, v' = \frac{vhL}{Q_0}, p' = \frac{ph^4}{\mu L Q_0}, \\ V_0' &= \frac{V_0 hL}{Q_0}, \delta = \frac{h}{L}, Re = \frac{\rho Q_0}{\mu h}, \alpha = \sqrt{\frac{\mu}{\mu_1}}, \beta' = \frac{\beta}{h}. \end{aligned} \tag{3.9}$$

using previous quantities in equations (3.6) – (3.8) and dropping primes, one can get following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.10}$$

$$\delta Re\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - N_3, \tag{3.11}$$

$$\delta^3 Re\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \delta^4\frac{\partial^2 v}{\partial x^2} + \delta^2\frac{\partial^2 v}{\partial y^2} - N_4, \tag{3.12}$$

where

$$N_3 = \frac{1}{\alpha^2}\left(\delta^4\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + 2\delta^2\frac{\partial^4 u}{\partial x^2 \partial y^2}\right),$$

$$N_4 = \frac{1}{\alpha^2}\left(\delta^6\frac{\partial^4 v}{\partial x^4} + \delta^2\frac{\partial^4 v}{\partial y^4} + 2\delta^4\frac{\partial^4 v}{\partial x^2 \partial y^2}\right),$$

and dimensionless form of boundary conditions are given as below:

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial^3 u}{\partial y^3} = 0, \quad v = 0, \quad \text{at} \quad y = 0,$$

$$u = -\beta\frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad v = \varepsilon V_0, \quad \text{at} \quad y = 1,$$

$$\varepsilon = \frac{2W}{h}\int_0^1 u dy, \quad \text{at} \quad x = 0. \tag{3.13}$$

3.3 Solution of the Problem

The set of equations (3.11 – 3.13) represent non-linear partial differential equations in which three unknowns u, v and p are involved. To reduce the complexity of problem it is assumed that width of slit is very small as compared to its length therefore, the ratio of length to width (δ) is less than 1 and we ignore terms of order δ^2. To find the solutions of u, v, p and τ_{xy} following form of series is used:

$$u = \sum_{i=1}^{\infty}\varepsilon^i u^{(i)}, \quad v = \sum_{i=1}^{\infty}\varepsilon^i v^{(i)}, \quad p = p^{(0)} + \sum_{i=1}^{\infty}\varepsilon^i p^{(i)}, \quad \tau_{xy} = \sum_{i=1}^{\infty}\varepsilon^i \tau_{xy}^{(i)}, \tag{3.14}$$

where $p^{(0)}$ is constant pressure.

To get the linear system of boundary value problem above series are used in Eqs.(3.10−3.13). After using above series and collecting powers of ε,one can get the following systems:

3.3.1 First Order System

$$\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(1)}}{\partial y}=0, \tag{3.15}$$

$$0=-\frac{\partial p^{(1)}}{\partial x}+\frac{\partial^2 u^{(1)}}{\partial y^2}-\frac{1}{\alpha^2}\frac{\partial^4 u^{(1)}}{\partial y^4}, \tag{3.16}$$

$$0=\frac{\partial p}{\partial y}, \tag{3.17}$$

$$\tau_{xy}^{(1)}=\mu\frac{\partial u^{(1)}}{\partial y}-\mu_1\frac{\partial^3 u^{(1)}}{\partial y^3}, \tag{3.18}$$

and boundary conditions for 1st order system are:

$$\frac{\partial u^{(1)}}{\partial y}=0,\quad \frac{\partial^3 u^{(1)}}{\partial y^3}=0,\quad v^{(1)}=0,\quad \text{at}\quad y=0, \tag{3.19}$$

$$u^{(1)}=-\beta\frac{\partial u^{(1)}}{\partial y},\quad \frac{\partial^2 u^{(1)}}{\partial y^2}=0,\quad v^{(1)}=V_0,\quad \text{at}\quad y=1, \tag{3.20}$$

$$1=\frac{2W}{h}\int_0^1 u^{(1)}dy,\quad \text{at}\quad x=0. \tag{3.21}$$

To reduce the numbers of unknown, we will introduce the following stream function $\psi^{(1)}(x,y)$:

$$u^{(1)}=\frac{\partial \psi^{(1)}}{\partial y},\quad v^{(1)}=-\frac{\partial \psi^{(1)}}{\partial x}. \tag{3.22}$$

After using above equation in first order system and eliminating pressure gradient from Eq.(3.16), one can write the following equation.

$$0=\frac{\partial^6 \psi^{(1)}}{\partial y^6}-\alpha^2\frac{\partial^4 \psi^{(1)}}{\partial y^4}, \tag{3.23}$$

and boundary conditions in the following form:

$$\frac{\partial^2\psi^{(1)}}{\partial y^2}=0,\quad \frac{\partial^4\psi^{(1)}}{\partial y^4}=0\quad \frac{\partial\psi^{(1)}}{\partial x}=0,\quad \text{at}\quad y=0, \tag{3.24}$$

$$\frac{\partial\psi^{(1)}}{\partial y}=-\beta\frac{\partial^2\psi^{(1)}}{\partial y^2},\quad \frac{\partial^3\psi^{(1)}}{\partial y^3}=0,\quad \frac{\partial\psi^{(1)}}{\partial x}=-V_0,\quad \text{at}\quad y=1, \tag{3.25}$$

$$\frac{h}{2W}=\psi^{(1)}(x,1)-\psi^{(1)}(x,0),\quad \text{at}\quad x=0. \tag{3.26}$$

To solve above BVP, we will use Inverse method that suggests the following assumption:

$$\psi^{(1)}=V_0xR_1(y)+T_1(y), \tag{3.27}$$

where $R_1(y)$ and $T_1(y)$ are unknown functions depending upon single variable y.

Using above solution in Eqs.(3.23 – 3.26) one can get the following system of ODE's :

$$\frac{d^6R_1}{dy^6}-\alpha^2\frac{d^4R_1}{dy^4}=0,\quad \frac{d^6T_1}{dy^6}-\alpha^2\frac{d^4R_1}{dy^4}=0, \tag{3.28}$$

and their corresponding boundary conditions are as follows:

$$R_1=0,\frac{d^2R_1}{dy^2}=0,\frac{d^4R_1}{dy^4}=0,\quad \text{at}\quad y=0, \tag{3.29}$$

$$R_1=-1,\frac{dR_1}{dy}=-\beta\frac{d^2R_1}{dy^2},\frac{d^3R_1}{dy^3}=0,\quad \text{at}\quad y=1,$$

$$T_1=0,\frac{d^2T_1}{dy^2}=0,\frac{d^4T_1}{dy^4}=0,\quad \text{at}\quad y=0, \tag{3.30}$$

$$T_1=\frac{h}{2W},\frac{dT_1}{dy}=-\beta\frac{d^2T_1}{dy^2},\frac{d^3T_1}{dy^3}=0,\quad \text{at}\quad y=1,$$

General solutions of above two BVP's are as follows:

$$R_1=\sum_{i=1}^{4}a_iy^{i-1}+a_5e^{\alpha y}+a_6e^{-\alpha y}, \tag{3.31}$$

$$T_1 = \sum_{i=1}^{4} b_i y^{i-1} + b_5 e^{\alpha y} + b_6 e^{-\alpha y}, \tag{3.32}$$

After using boundary conditions following form can be obtained:

$$R_1(y) = a_2 y + a_4 y^3 + 2a_5 \sinh(\alpha y), \tag{3.33}$$

$$T_1(y) = b_2 y + b_4 y^3 + 2b_5 \sinh(\alpha y), \tag{3.34}$$

Upon using $R_1(y)$ and $T_1(y)$ into Eq.(3.27) one can get following form:

$$\psi^{(1)}(x,y) = (V_0 x a_2 + b_2)y + (V_0 x a_4 + b_4)y^3 + 2(V_0 x a_5 + b_5)\sinh(\alpha y), \tag{3.35}$$

Substituting above equation into equation (3.22), one can find following expressions for velocity profiles:

$$u^{(1)}(x,y) = (V_0 x a_2 + b_2) + 3(V_0 x a_4 + b_4)y^2 + 2\alpha(V_0 x a_5 + b_5)\cosh(\alpha y), \tag{3.36}$$

$$v^{(1)}(x,y) = -V_0(a_2 y + a_4 y^3 + 2a_5 \sinh(\alpha y), \tag{3.37}$$

The solution of first order pressure distribution can be obtained with the help of first order velocity profile.

$$p^{(1)}(x) = -\frac{3\alpha^3(x - V_0 W x^2)\cosh(\alpha h)}{N_5}, \tag{3.38}$$

After using velocity profile into equation (3.18) one can write first order shear stress in the following form:

$$\tau_{xy}^{(1)} = \frac{3\alpha(1 - 2V_0 W x)(-y\mu\alpha^2\cosh(\alpha h) + (\mu - \alpha^2\mu_1)\sinh(\alpha y)}{N_5}, \tag{3.39}$$

where

$$N_5 = 2hW\alpha(-3 + h\alpha^2(3\beta + h))\cosh(\alpha h) - 6W(h\beta\alpha^2 - 1)\sinh(\alpha h).$$

3.3.2 Second Order System

$$\frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} = 0, \tag{3.40}$$

$$N_6 = -\frac{\partial p^{(2)}}{\partial x} + \frac{\partial^2 u^{(2)}}{\partial y^2} - \frac{1}{\alpha^2}\frac{\partial^4 u^{(2)}}{\partial y^4}, \tag{3.41}$$

$$0 = \frac{\partial p^{(2)}}{\partial y}, \tag{3.42}$$

$$\tau_{xy}^{(2)} = \mu\frac{\partial u^{(2)}}{\partial y} - \mu_1\frac{\partial^3 u^{(2)}}{\partial y^3}, \tag{3.43}$$

where

$$N_6 = \delta Re\left(u^{(1)}\frac{\partial}{\partial x} + v^{(1)}\frac{\partial}{\partial y}\right)u^{(1)},$$

and boundary conditions are as follows:

$$\frac{\partial u^{(2)}}{\partial y} = 0, \frac{\partial^3 u^{(2)}}{\partial y^3} = 0, v^{(2)} = 0, \quad \text{at} \quad y = 0, \tag{3.44}$$

$$u^{(2)} = -\beta\frac{\partial u^{(2)}}{\partial y}, \frac{\partial^2 u^{(2)}}{\partial y^2} = 0, v^{(2)} = 0, \quad \text{at} \quad y = 1,$$

$$0 = \int_0^1 u^{(2)} dy, \quad \text{at} \quad x = 0.$$

After eliminating pressure gradient and using stream function, Eq.(3.41) can be written in following form:

$$-\alpha^2\frac{\partial N_6}{\partial y} = \frac{\partial^6 \psi^{(2)}}{\partial y^6} - \alpha^2\frac{\partial^4 \psi^{(2)}}{\partial y^4}. \tag{3.45}$$

and boundary conditions in stream functions are as follows:

$$\frac{\partial^2 \psi^{(2)}}{\partial y^2} = 0, \frac{\partial^4 \psi^{(2)}}{\partial y^4} = 0, \frac{\partial \psi^{(2)}}{\partial x} = 0, \quad \text{at} \quad y = 0, \tag{3.46}$$

$$\frac{\partial \psi^{(2)}}{\partial y} = -\beta \frac{\partial^2 \psi^{(2)}}{\partial y^2}, \frac{\partial^3 \psi^{(2)}}{\partial y^3} = 0, \frac{\partial \psi^{(2)}}{\partial y} = 0, \quad \text{at} \quad y = 1,$$

$$0 = \psi^{(2)}(x,1) - \psi^{(2)}(x,0), \quad \text{at} \quad x = 0.$$

Inverse method suggests the following assumption:

$$\psi^{(2)}(x,y) = g(x)R_2(y) + T_2(y), \tag{3.47}$$

where R_2 and T_2 are unknown functions and $g(x) = x$.

After using above solution of stream function in Eq.$(3.45 - 3.46)$ one can write the following BVP's:

$$\frac{d^6 R_2}{dy^6} - \alpha^2 \frac{d^4 R_2}{dy^4} = g_1(y), \tag{3.48}$$

$$R_2 = 0, \frac{d^2 R_2}{dy^2} = 0, \frac{d^4 R_2}{dy^4} = 0, \quad \text{at} \quad y = 0,$$

$$R_2 = 0, \frac{dR_2}{dy} = 0, \frac{d^3 R_2}{dy^3} = 0, \quad \text{at} \quad y = 1,$$

and

$$\frac{d^6 T_2}{dy^6} - \alpha^2 \frac{d^4 T_2}{dy^4} = g_2(y), \tag{3.49}$$

$$T_2 = 0, \frac{d^2 T_2}{dy^2} = 0, \frac{d^4 T_2}{dy^4} = 0, \quad \text{at} \quad y = 0,$$

$$T_2 = 0, \frac{dT_2}{dy} = 0, \frac{d^3 T_2}{dy^3} = 0, \quad \text{at} \quad y = 1,$$

Solutions of above BVP's can be obtained by "DSolve" command in MATHEMATICA. The second order velocity profile and second order pressure can be obtained by Eq.(2.47)

3.3.3 Third Order System

$$\frac{\partial u^{(3)}}{\partial x} + \frac{\partial v^{(3)}}{\partial y} = 0, \tag{3.50}$$

$$N_7 = -\frac{\partial p^{(3)}}{\partial x} + \frac{\partial^2 u^{(3)}}{\partial y^2} - \frac{1}{\alpha^2} \frac{\partial^4 u^{(3)}}{\partial y^4}, \tag{3.51}$$

$$0 = \frac{\partial p^{(3)}}{\partial y}, \tag{3.52}$$

$$\tau_{xy}^{(3)} = \mu \frac{\partial u^{(3)}}{\partial y} - \mu_1 \frac{\partial^3 u^{(3)}}{\partial y^3}, \tag{3.53}$$

where

$$N_7 = \delta Re \left(u^{(1)} \frac{\partial u^{(2)}}{\partial x} + v^{(1)} \frac{\partial u^{(2)}}{\partial y} \right) + \left(u^{(2)} \frac{\partial u^{(1)}}{\partial x} + v^{(2)} \frac{\partial u^{(1)}}{\partial y} \right),$$

and associated boundary conditions for 3rd order system are as follows:

$$\frac{\partial u^{(3)}}{\partial y} = 0, \frac{\partial^3 u^{(3)}}{\partial y^3} = 0, v^{(3)} = 0, \quad \text{at} \quad y = 0, \tag{3.54}$$

$$u^{(3)} = -\beta \frac{\partial u^{(3)}}{\partial y}, \frac{\partial^2 u^{(3)}}{\partial y^2} = 0, v^{(3)} = 0, \quad \text{at} \quad y = 1,$$

$$0 = \int_0^1 u^{(3)} dy, \quad \text{at} \quad x = 0.$$

After eliminating pressure gradient and using stream function Eq.(3.51) can be written in the following form:

$$-\alpha^2 \frac{\partial}{\partial y}(N_7) = \frac{\partial^6 \psi^{(3)}}{\partial y^6} - \alpha^2 \frac{\partial^4 \psi^{(3)}}{\partial y^4}. \tag{3.55}$$

and boundary conditions in context of stream functions are as follows:

$$\frac{\partial^2 \psi^{(3)}}{\partial y^2} = 0, \frac{\partial^4 \psi^{(3)}}{\partial y^4} = 0, \frac{\partial \psi^{(3)}}{\partial x} = 0, \quad \text{at} \quad y = 0, \tag{3.56}$$

$$\frac{\partial \psi^{(3)}}{\partial y} = 0, \frac{\partial^3 \psi^{(3)}}{\partial y^3} = 0, \frac{\partial \psi^{(3)}}{\partial x} = 0, \quad \text{at} \quad y = 1,$$

$$0 = \psi^{(3)}(x, 1) - \psi^{(3)}(x, 0), \quad \text{at} \quad x = 0.$$

To solve above BVP we will assume following form of stream function:

$$\psi^{(3)}(x, y) = h(x) R_3(y) + T_3(y). \tag{3.57}$$

where $h(x) = x$.

After following all steps of first order solution, one can get the third order solution of stream function, that will help to find third order velocity, pressure and shear stress.

Now by combining first, second, and third order solution of stream function, velocity profile, pressure and shear stress when $\varepsilon \rightarrow 1$following expression can be obtained:

$$\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)}, \tag{3.58}$$

$$u = u^{(1)} + u^{(2)} + u^{(3)}, \tag{3.59}$$

$$v = v^{(1)} + v^{(2)} + v^{(3)}, \tag{3.60}$$

$$p = p_0 + p^{(1)} + p^{(2)} + p^{(3)}, \tag{3.61}$$

$$\tau_{xy} = \tau_{xy}^{(1)} + \tau_{xy}^{(2)} + \tau_{xy}^{(3)}.$$

3.4 Results and Discussion

This section displays the influence of Reynolds number Re, reabsorption parameter V_0, couple-stress parameter α and slip parameter β on pressure difference, shear stress, horizontal and vertical velocity components at middle position ($x = 0.5$) of slit.

3.4.1 Effect of Reynold's Number (Re)

Fig. 3.2(a) illustrates that horizontal velocity rises for different values of Reynold's number at center of slit and decays near walls of slit due to slip velocity. Fig. 3.2(b) depicts the behaviour of vertical velocity against increasing values of Reynold's number which shows that the vertical velocity is zero (not changing) at center point, and away from this regime it rises in forward and backward direction for different values of Reynold's number. Fig. 3.2(c) displays the impact of increasing values of Reynold's number on pressure difference and causing increase in pressure change during the fluid flow through a slit. The effect of Reynold's number on wall shear stress is observed in Fig. 3.2(d) and it shows that at the entrance of slit the wall shear stress is high for all values of Reynold's number.

3.4.2 Effect of Reabsorption Parameter (V_0)

Fig. 3.3(a) illustrates impact of reabsorption parameter (V_0) on horizontal velocity at middle point ($x = 0.5$) of slit. It shows that the horizontal velocity of fluid is non-zero at boundaries due to the presence of slip parameter and maximum at middle region of the slit due to pressure and inertial forces. The vertical velocity of fluid is observed in Fig. 3.3(b) for distinct values of V_0 and it shows that away from center point the vertical velocity of fluid is accelerating towards walls of slit, but it is zero at the center point due to permeability at walls. Fig. 3.3(c) indicates effect of reabsorption rate (V_0) on pressure difference, which shows that pressure difference rises as reabsorption rate becomes high inside the slit. Fig. 3.3(d) indicates that if reabsorption rates (V_0) become high then fluid flow requires high shear rate near the wall.

3.4.3 Effect of Couple-Stress Parameter (α)

Fig. 3.4(a) shows that horizontal velocity decays at middle region of slit for all increasing values of couple-stress parameter α, but horizontal velocity rises near the walls of slit. Fig. 3.4(b) shows effect of couple-stress parameter α on vertical velocity, and it is observed that away from center point the vertical velocity rises with the growing values of couple-stress parameter. The effect of couple-stress parameter on pressure distribution is observed in Fig. 3.4(c), and this figure shows that pressure difference rises in backward direction for all increasing values of couple-stress parameter. Fig. 3.4(d) indicates that the wall shear stress rises with increasing values of couple-stress parameter.

3.4.4 Effect of Slip Parameter (β)

The effect of slip parameter (β) on horizontal velocity is observed in Fig. 3.5(a), which shows that at center region of slit horizontal velocity rises in backward direction but rises in forward direction near the walls of slit. Fig. 3.5(b) displays the effect of slip parameter on vertical velocity, and it is noted that at the center of the slit vertical velocity is not changing but away from the center point it rises in forward and backward directions towards the walls of slit. Fig. 3.5(c) shows that the pressure difference rises in backward direction for all rising values of slip parameter. The wall shear stress decreases from entrance to exit point of slit , it is observed from Fig. 3.5(d).

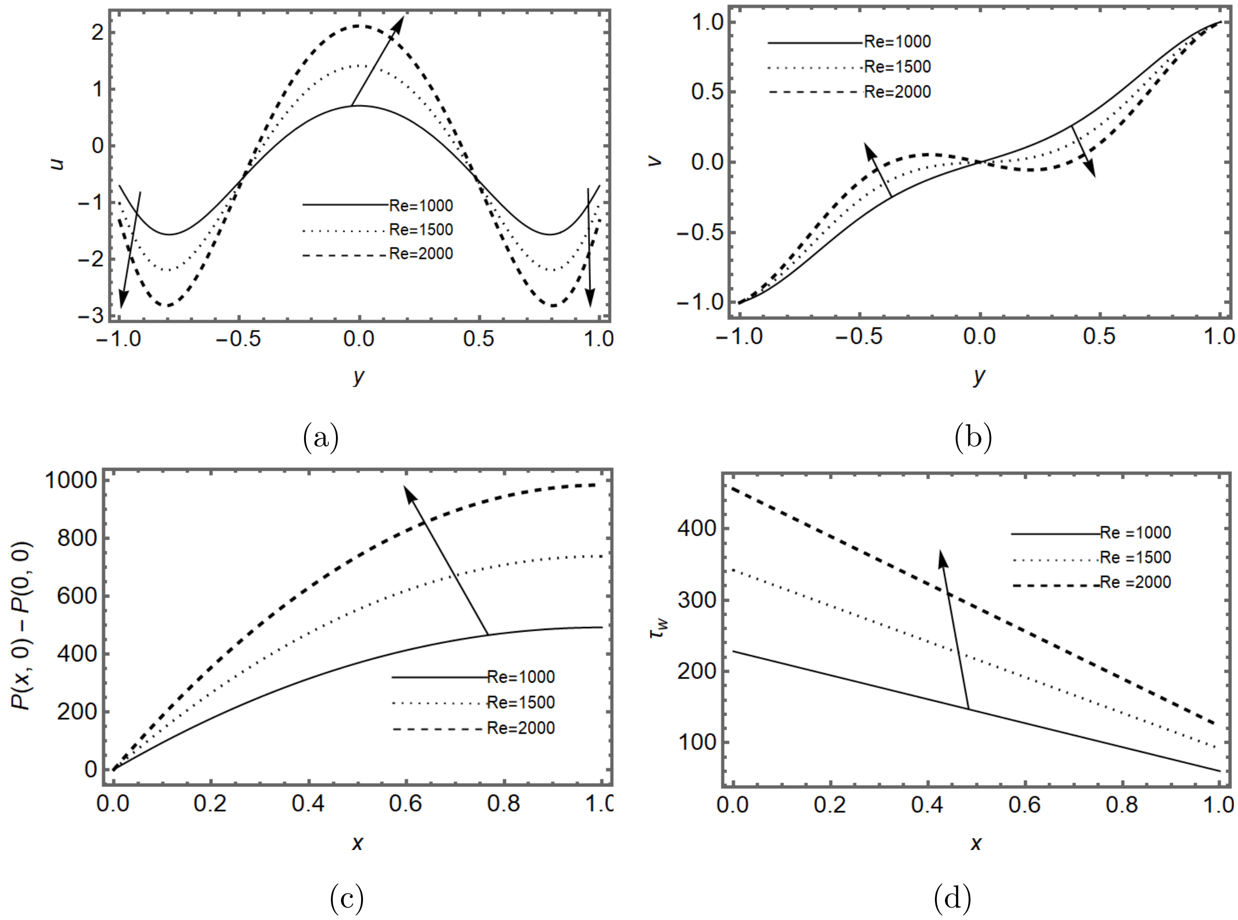


Fig.3.2 (a-d): Effect of Reynolds number on (a) horizontal and (b) vertical velocity at middle point $x = 0.5$, (c) pressure difference and (d) wall shear stress.

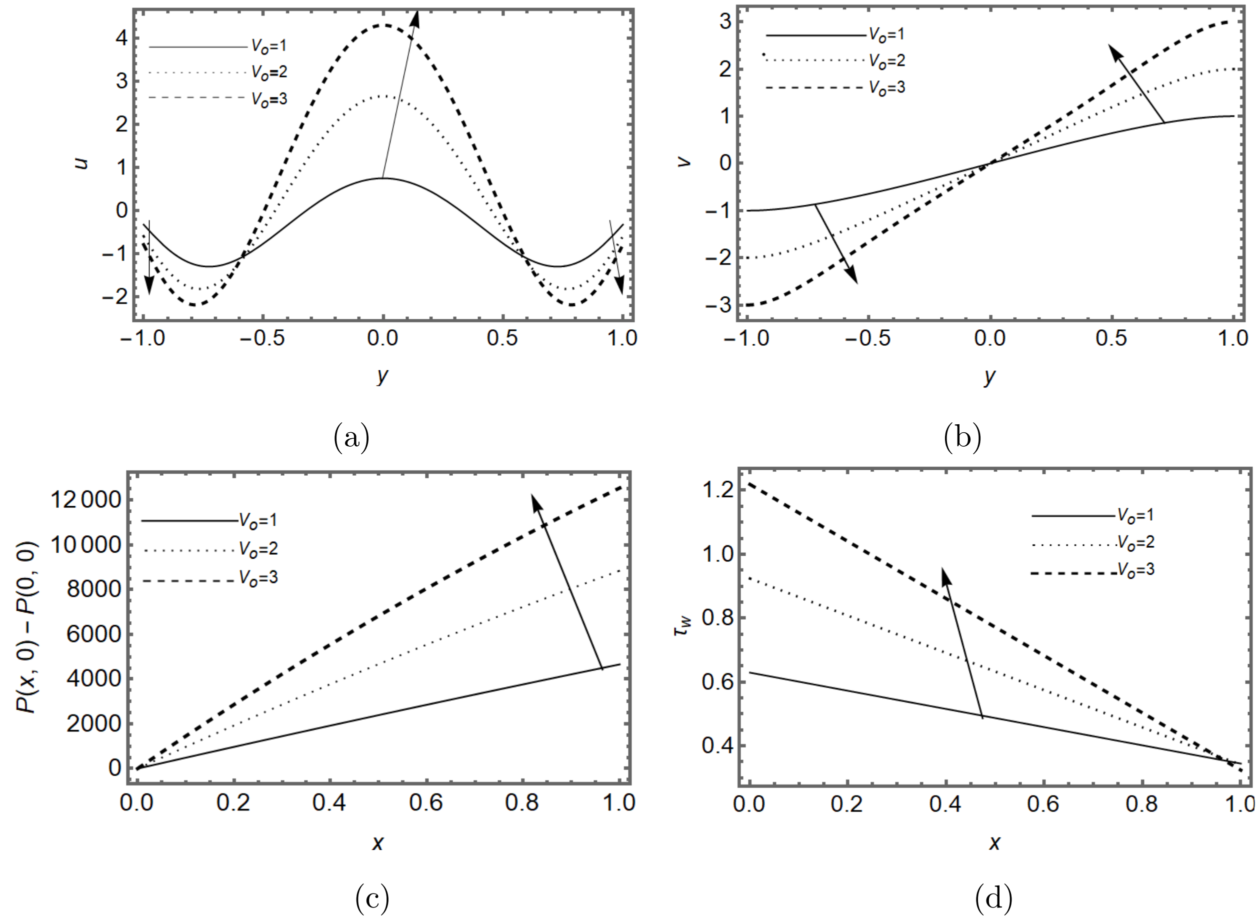


Fig.3.3 (a-d): Effect of reabsorption velocity V_0 on (a) horizontal and (b) vertical velocity, (c) pressure difference and (d) wall shear stress.

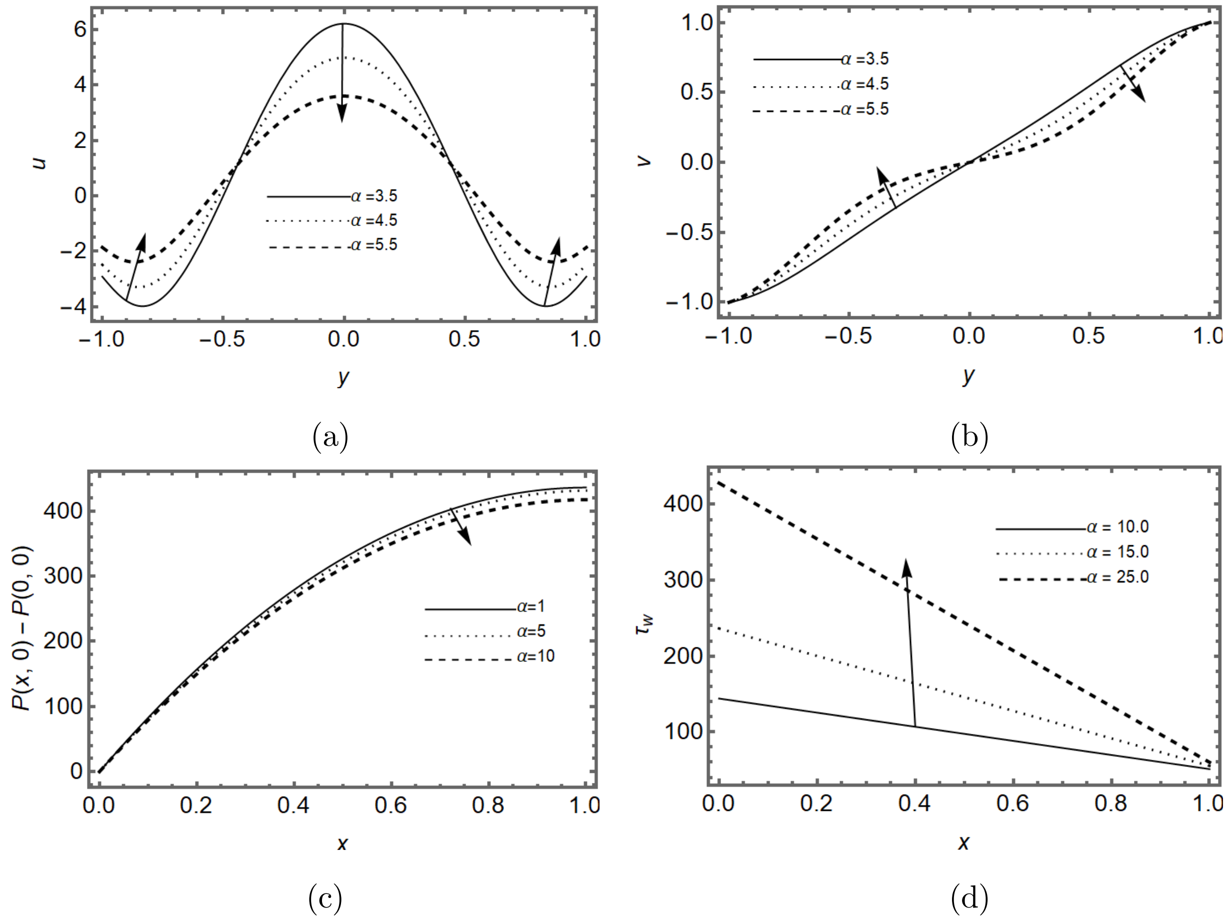


Fig.3.4 (a-d): Effect of couple stress parameter α on (a) horizontal and (b) vertical velocity, (c) pressure difference and (d) wall shear stress.

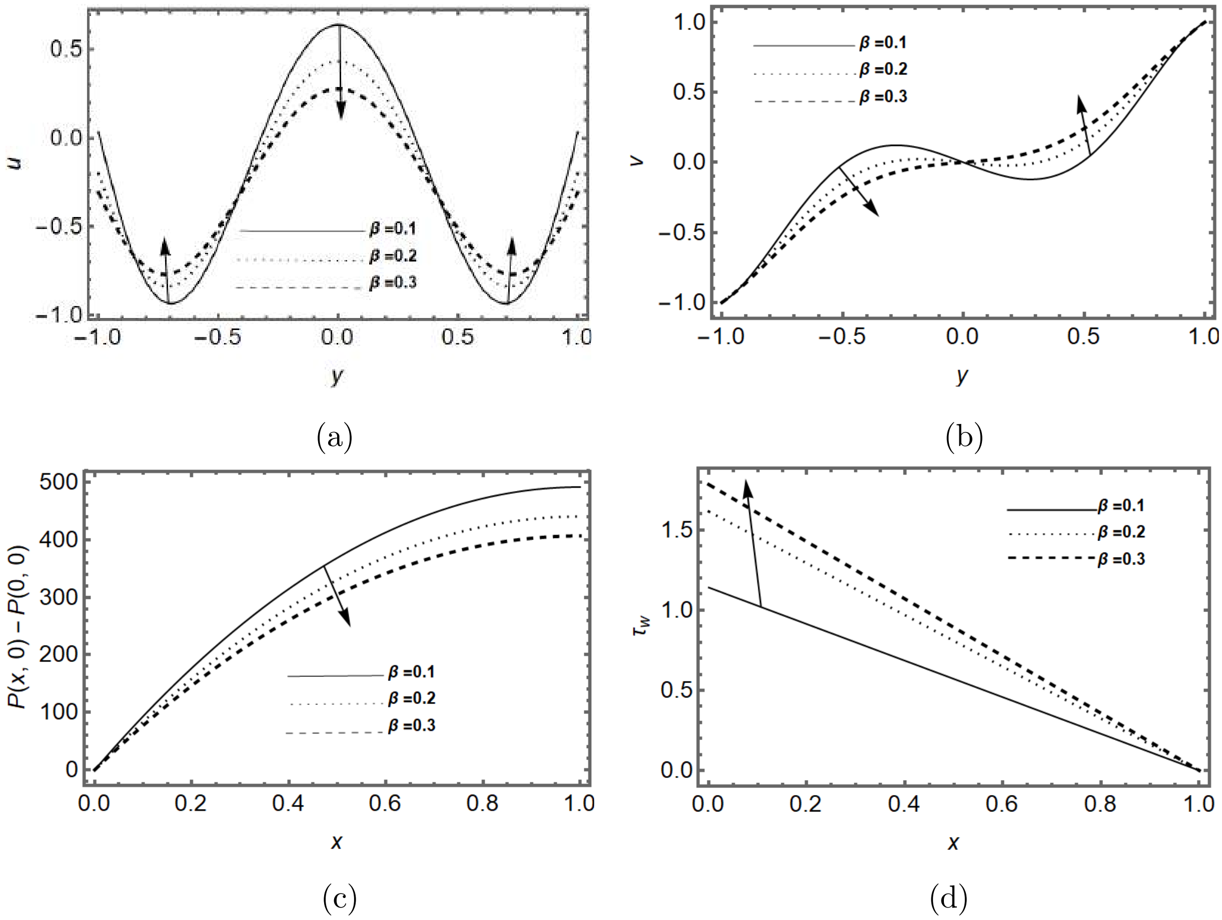


Fig.3.5 (a-d): Effect of slip parameter β on (a) horizontal and (b) vertical velocity, (c) pressure difference and (d) wall shear stress.

3.5 Conclusion

In present research two-dimensional inertial flow of couple-stress fluid through a permeable slit with constant reabsorption is observed. The mathematical models of couple-stress fluid are represented by the set of intricate non-linear partial differential equations and their solutions are obtained by Langlois technique using no-slip and slip boundary conditions. The analytical results of various flow characteristics like velocity profile, pressure difference and shear stress are calculated in this study, and graphical results for these flow characteristics are observed for different parameters. It is concluded from graphical results that the horizontal velocity rises with the extending values of Reynold's number (Re) , and reabsorption parameter (V_0), whereas for couple-stress parameter (α) horizontal velocity rises in forward direction with no-slip boundary condition and rises in backward direction for slip boundary condition. The transverse velocity of fluid is observed for Reynold's number, reabsorption parameter and couple-stress parameter, and it is observed that at center point of slit it is not changing but away from the center it is accelerating towards the wall of slit. Further, the graphical results also show that the pressure difference increases from entrance to exit point of slit for rising values of involving parameters. This research also concludes that wall shear stress rises with the rising values of all emerging parameters.

3.6 Appendix

$$a_2 = \cosh(\alpha h)(\alpha^3 h^2 + 2h\beta\alpha^3 - 2\alpha) - 2\beta\alpha^2 a_5 \sinh(\alpha h),\ a_4 = -\frac{\alpha^3 \cosh(\alpha h) a_5}{3},$$

$$a_5 = \frac{1}{2\alpha h \cosh(\alpha h) - 2\sinh(\alpha h)(1-\alpha^2\beta h) - \frac{2}{3}\alpha^3 h^2 \cosh(\alpha h)(h+3\beta)},$$

$$b_2 = \cosh(\alpha h)(\alpha^3 h^2 + 2h\beta\alpha^3 - 2\alpha) - 2\beta\alpha^2 b_5 \sinh(\alpha h),\ b_4 = -\frac{\alpha^3 \cosh(\alpha h) b_5}{3},$$

$$b_5 = \frac{1}{4W(\sinh(\alpha h)(1-\alpha^2\beta h) + \cosh(\alpha h)(\frac{1}{3}\alpha^3 h^3 - h - 2\beta\alpha^3 h^2)}.$$

$$c_1 = c_3 = a_1 = a_3 = b_1 = b_3 = 0.\ m_2(y) = \alpha^2 y^3 (1 + \cosh(2\alpha h))$$

$$\begin{aligned} c_2 &= \alpha \cosh(\alpha h)(\alpha^2 h^2 - 2) c_5, \qquad c_4 = -\frac{\alpha^3 \cosh(\alpha h) c_5}{3}, \\ c_5 &= \frac{3}{2(\alpha h \cosh(\alpha h)(3-\alpha^2 h^2) - 3\sinh(\alpha h))}, d_2 = \alpha \cosh(\alpha h)(\alpha^2 h^2 - 2) d_{5,} \end{aligned}$$

$$d_4 = -\frac{\alpha^3 \cosh(\alpha h) d_5}{3},\ d_5 = \frac{3}{4W(\alpha h \cosh(\alpha h)(\alpha^2 h^2 - 3) + 3\sinh(\alpha h))},$$

$$m_1 = -\frac{3\delta \operatorname{Re} \alpha^6 V_o}{4W(\alpha h(-3 + h\alpha^2(h+3\beta))\cosh(\alpha h) - 3(h\beta\alpha^2 - 1)\sinh(\alpha h))^2}$$

$$f_1(y) = \frac{3V_o^2 \alpha^5 \cosh(\alpha h)(2\alpha y^3 \cosh(\alpha h) + y(-3h^2 + y^2)\alpha \cosh(\alpha y) + 3(h-y)(h+y)\sinh(\alpha y))}{2(\alpha h(-3 + h^2\alpha^2)\cosh(\alpha h) + 3\sinh(\alpha h))^2}$$

$$f_2(y) = -\frac{3V_o^2 \alpha^5 \cosh(\alpha h)(2\alpha y^3 \cosh(\alpha h) + y(-3h^2 + y^2)\alpha \cosh(\alpha y) + 3(h-y)(h+y)\sinh(\alpha y))}{4W(\alpha h(-3 + h^2\alpha^2)\cosh(\alpha h) + 3\sinh(\alpha h))^2}$$

$$\begin{aligned} g_1(y) &= m_1(2V_o W(m_2 + 6\beta \sinh(\alpha h)(y\alpha \cosh(\alpha h) - \sinh(\alpha y) + \alpha \cosh(\alpha h) \\ &\quad (\alpha y(y^2 - 3h(h+2\beta))\cosh(\alpha y) + 3(h^2 - y^2 + 2h\beta)\sinh(\alpha y)))) \end{aligned}$$

$$\begin{aligned} g_2(y) &= m_1(-m_2 + 6\beta \sinh(\alpha h)(-y \cosh(\alpha y) + \sinh(\alpha y)) + \alpha \cosh(\alpha h) \\ &\quad (y(-y^2 + 3h(h+2\beta))\cosh(\alpha y) - 3(h^2 - y^2 + 2h\beta)\sinh(\alpha y)))) \end{aligned}$$

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