بسم الله الرحمن الرحيم

Some Contribution in Lattice Ordered Soft Semigroups

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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DECLARATION

I hereby, declare, that this thesis neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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Dedicated

To

My Grandmother,

My

Loving parents,

My friends

And respectful teachers

Acknowledgements

All praises to almighty "**ALLAH**" the creator of the universe, who blessed me with the knowledge and enabled me to complete the dissertation. All respects to Holy Prophet **MUHAMMAD (S.A.W)**, who is the last messenger, whose life is a perfect model for the whole humanity.

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Muhammad Mubashar Rafique Abbasi

Structure of Thesis

Chapter 1

In this chapter we recall some basic definitions and notions. These definitions will help in later Chapters.

Chapter 2

In this chapter we review the research paper "On Soft Ideals over Semigroups". In this paper the concept of soft ideal over semigroup has been discussed.

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Chapter 3

In this chapter, the concept of lattice (anti-lattice) ordered soft semigroups and some properties of lattice (anti-lattice) ordered soft semigroups has been introduced. Also the concept of lattice (anti-lattice) ordered soft ideals (quasi-ideals, bi-ideals) and its properties has been defined.

Preface

It is known that many problems in different directions such as engineering, economics and medical are commonly not accurate. There are always many types of uncertainties involved in the data. The classical tools used to deal with all these uncertainties are useful only under definite domain. In dealing with uncertainties, lots of notions have been newly grown, which includes the theory of Fuzzy sets [15], theory of Intuitionistic Fuzzy sets, theory of Rough sets and so on. As a result of these theories many new techniques have been grown. Molodtsov introduced soft set theory in 1999 [13]. This theory has become an important instrument to handle the vagueness and ambiguity in different fields of life since more than a decade. Maji et al. [12] gave the operations on soft sets and he also introduced a method of tackling a decision making argument [11]. Later Ali et al. [2] improved the operations and the results given by Maji et al. [12]. Ali et al. [2] then gave further additional operations for theory of soft sets and proved De Morgan's laws by making use of these operations. F. Feng at al. [7] discussed soft sets combined with Fuzzy sets and Rough sets. Soft sets have vast importance due to their algebraic structures that occasionally have different behavior than that of original algebraic structures. Aktas and Cagman [1] presented algebraic structure on soft sets and also presented soft groups. Jun et al. [10] implemented the theory of soft sets to ordered semigroups. Y.B. Jun at al. [9] discussed applications of soft sets in ideal theory. F. Feng et al. [6] discussed soft semirings. Ali et al. [3] discussed soft semigroups, soft ideals, soft quasi-ideals and soft bi-ideals and gave new concepts over theory of classical semigroups. In general, when we talk about a soft ideal (quasi-ideal, bi-ideal), we actually mean that we are letting a collection of ideals (quasi-ideals, bi-ideals) over a semigroup. Thus, the concept of soft ideal (quasi-ideal, bi-ideal) is a more general concept than the concept of ideal (quasi-ideal, bi-ideal). Ordering of elements became an important and vital fact in theory of soft sets since recently it was defined by Ali et al. [5]. Ali et al. [5] introduced lattice ordered soft set and defined some basic operations in it.

In this thesis, the concept of lattice (anti-lattice) ordered soft semigroups and some properties of lattice (anti-lattice) ordered soft semigroups have been introduced. Also the concept of lattice (anti-lattice) ordered soft ideals (quasi-ideals, bi-ideals) and its properties has been defined.

Contents

Chapter 1

Preliminaries

In this chapter we recall some basic definitions and notions. These definitions will help in later chapters. For undefined terms and notions we refer to ([8], [10]).

1.1 Soft Set

In this section we recall soft sets, their basic operations and results.

1.1.1 Definition [13]

Let $\hat{D}$ be a universal set, E represents the set of parameters under consideration and $\mathcal{F} \subseteq E$. Then a pair $(\alpha, \mathcal{F})$ is called soft set over $\hat{D}$, where α is a mapping $\alpha : \mathcal{F} \to P(\hat{D})$.

1.1.2 Definition [12]

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft sets over a common universal set $\hat{D}$. Then $(\beta, \mathcal{L})$ is called soft subset of $(\alpha, \mathcal{F})$, if $\mathcal{L} \subseteq \mathcal{F}$ and $\beta(\sigma) \subseteq \alpha(\sigma)$, $\forall\ \sigma \in \mathcal{L}$.

1.1.3 Definition [3]

Let $\hat{D}$ be a universal set, E represents the set of parameters under consideration and $\mathcal{F} \subseteq E$. Then

1. A soft set $(\alpha, \mathcal{F})$ is called relative null soft set (with respect to the set $\mathcal{F}$) denoted by $N_{(\hat{D},\mathcal{F})}$ if $\alpha(\rho) = \phi$, $\forall\ \rho \in \mathcal{F}$.

2. A soft set $(\alpha, \mathcal{F})$ is called relative whole soft set (with respect to the set $\mathcal{F}$) denoted by $W_{(\hat{D},\mathcal{F})}$ if $\alpha(\rho) = \hat{D}$, $\forall\ \rho \in \mathcal{F}$.

3. A soft set $(\alpha, \mathcal{F})$ is called the empty soft set denoted by $\phi_{\hat{D}}$ if the parametric set $\mathcal{F}$ is empty. i.e $\mathcal{F} = \phi$.

4. A soft set $(\alpha, \mathcal{F})$ is called absolute soft set denoted by $\mathcal{F}_{(\hat{D},E)}$ if $\alpha(\rho) = \hat{D}$, $\forall$ $\rho \in E$.

1.1.4 Definition [2]

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft sets over a universal set $\hat{D}$. Then

1. Restricted intersection of $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is denoted and defined as $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \phi$ and $\gamma(\varsigma) = \alpha(\varsigma)\tilde{\cap}\beta(\varsigma)$, $\forall\ \varsigma \in \Re$.

2. Restricted union of $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is denoted and defined as $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \phi$ and $\gamma(\varsigma) = \alpha(\varsigma)\tilde{\cup}\beta(\varsigma)$, $\forall\ \varsigma \in \Re$.

3. Extended intersection of $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is denoted and defined as $(\alpha, \mathcal{F})\tilde{\cap}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L}$ and $\forall\ \varsigma \in \Re$.

$$\gamma(\varsigma) = \begin{cases} \alpha(\varsigma) & \text{if } \varsigma \in \mathcal{F} - \mathcal{L} \\ \beta(\varsigma) & \text{if } \varsigma \in \mathcal{L} - \mathcal{F} \\ \alpha(\varsigma)\tilde{\cap}\beta(\varsigma) & \text{if } \varsigma \in \mathcal{F}\tilde{\cap}\mathcal{L} \end{cases}$$

4. Extended union of $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is denoted and defined as $(\alpha, \mathcal{F})\tilde{\cup}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L}$ and $\forall\ \varsigma \in \Re$.

$$\gamma(\varsigma) = \begin{cases} \alpha(\varsigma) & \text{if } \varsigma \in \mathcal{F} \ominus \mathcal{L} \\ \beta(\varsigma) & \text{if } \varsigma \in \mathcal{L} \ominus \mathcal{F} \\ \alpha(\varsigma)\tilde{\cup}\beta(\varsigma) & \text{if } \varsigma \in \mathcal{F}\tilde{\cap}\mathcal{L} \end{cases}$$

1.1.5 Definition [12]

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft sets over a universal set $\hat{D}$. Then

1. Basic intersection of $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is denoted and defined as $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F} \otimes \mathcal{L}$ and $\gamma(\rho, \sigma) = \alpha(\rho)\tilde{\cap}\beta(\sigma)$, $\forall\ (\rho, \sigma) \in \Re$, where $\mathcal{F} \otimes \mathcal{L}$ is the cartesian product of $\mathcal{F}$ and $\mathcal{L}$.

2. Basic union of $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is denoted and defined as $(\alpha, \mathcal{F})\tilde{\cup}_B(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F} \otimes \mathcal{L}$ and $\gamma(\rho, \sigma) = \alpha(\rho)\tilde{\cup}\beta(\sigma)$, $\forall\ (\rho, \sigma) \in \Re$.

1.2 Lattices

1.2.1 Definition

A binary relation $\preccurlyeq$ defined on a non-empty set $\mathcal{F}$ is called a partial order on the set $\mathcal{F}$ if the following conditions hold :

1. $\rho_1 \preccurlyeq \rho_1$ (reflexivity)

2. $\rho_1 \preccurlyeq \rho_2$ and $\rho_2 \preccurlyeq \rho_1$ implies $\rho_1 = \rho_2$ (antisymmetry)

3. $\rho_1 \preccurlyeq \rho_2$ and $\rho_2 \preccurlyeq \rho_3$ implies $\rho_1 \preccurlyeq \rho_3$ (transitivity)

for all $\rho_1, \rho_2, \rho_3 \in \mathcal{F}$.

If, in addition, for every $\rho_1, \rho_2 \in \mathcal{F}$, either $\rho_1 \preccurlyeq \rho_2$ or $\rho_2 \preccurlyeq \rho_1$, then we say $\preccurlyeq$ is total order on $\mathcal{F}$. A non-empty set with a partial order on it is called a partially order set, or more briefly a poset. And if the relation is a total order then we speak it a totally order set or simply a chain.

1.2.2 Examples

1. Let $P(\mathcal{F})$ denotes the power set of $\mathcal{F}$, i.e., the set of all subsets of $\mathcal{F}$. Then " $\subseteq$ " is a partial order on $P(\mathcal{F})$.

2. Let $\mathcal{F}$ be the set of natural numbers and let $\preccurlyeq$ be the relation "divides". Then $\preccurlyeq$ is a partial order on $\mathcal{F}$.

3. Let $\mathcal{F}$ be tbe set of real numbers and let $\preccurlyeq$ be the usual ordering. Then $\preccurlyeq$ is a total order on $\mathcal{F}$.

1.2.3 Definition

It is well known that a restriction of a partial order is again a partial, so we consider the partial order $\preccurlyeq$ on $\mathcal{F} \otimes \mathcal{L}$ by defining, $(\rho_1, \sigma_1) \preccurlyeq_{\mathcal{F}\otimes\mathcal{L}} (\rho_2, \sigma_2)$ if and only if $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$ and $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$ and $\sigma_1, \sigma_2 \in \mathcal{L}$. For the rest of the thesis this order will be consider on $\mathcal{F} \otimes \mathcal{L}$.

1.2.4 Definition

Let $\mathcal{F}$ be a non-empty subset of an ordered set R. Then an element $M \in R$ is called an *upper bound* of $\mathcal{F}$ if $\rho \preccurlyeq M, \forall\, \rho \in \mathcal{F}$. Similarly, an element $m \in R$ is called a *lower bound* of $\mathcal{F}$ if $m \preccurlyeq \rho, \forall\, \rho \in \mathcal{F}$.

1.2.5 Definition

Suppose that $\mathcal{F} \subset R$. If $M \in R$ is an upper bound of $\mathcal{F}$ such that $M' \preccurlyeq M$ for every upper bound M' of $\mathcal{F}$, then M is called the *supremum* of $\mathcal{F}$, denoted as $M = sup\mathcal{F}$. If $m \in R$ is a lower bound of $\mathcal{F}$ such that $m' \preccurlyeq m$ for every lower bound m' of $\mathcal{F}$, then m is called the $infimum$ of $\mathcal{F}$, denoted as $m = inf\mathcal{F}$.

1.2.6 Definition

Let $\preccurlyeq$ be a partial order on $\mathcal{F}$. Then the pair $\mathcal{F} = (\mathcal{F}, \preccurlyeq)$ is a lattice if $\forall\, \rho_1, \rho_2 \in \mathcal{F}$ the set $\{\rho_1, \rho_2\}$ has a supremum and an infimum.

1.2.7 Theorem

Let $\mathcal{F}$ be a non-empty set. If $\wedge$ and $\vee$ are two binary operations on $\mathcal{F}$. Then $\mathcal{F}$ is a lattice if and only if for each $\rho_1, \rho_2, \rho_3 \in \mathcal{F}$ the following hold:

1. $\rho_1 \wedge \rho_2 = \rho_2 \wedge \rho_1$ and $\rho_1 \vee \rho_2 = \rho_2 \vee \rho_1$
2. $(\rho_1 \wedge \rho_2) \wedge \rho_3 = \rho_1 \wedge (\rho_2 \wedge \rho_3)$ and $(\rho_1 \vee \rho_2) \vee \rho_3 = \rho_1 \vee (\rho_2 \vee z)$
3. $\rho_1 \vee \rho_1 = \rho_1$ and $\rho_1 \wedge \rho_1 = \rho_1$
4. $\rho_1 \wedge (\rho_1 \vee \rho_2) = \rho_1$ and $\rho_1 \vee (\rho_1 \wedge \rho_2) = \rho_1$

1.2.8 Definition

If in a lattice $\mathcal{F}$ there are elements 0 and **1** such that $0 \preccurlyeq \rho$ and $\rho \preccurlyeq 1, \forall\, \rho \in \mathcal{F}$. Then $\mathcal{F}$ is called bounded lattice.

1.2.9 Examples

1. Let $\mathcal{F}$ be the set of propositions, let $\vee$ denotes the connective" *or* " and $\wedge$ denotes the connective "*and*". Then 1 to 4 are well-known properties of lattice from propositional logic.

2. Let $\mathcal{F}$ be the set of natural numbers and $\vee$ denotes the least common multiple and $\wedge$ denotes greatest common divisor. Then $\mathcal{F}$ is a lattice.

3. For any non-empty set $\mathcal{F}$, $(P(\mathcal{F}), \cap, \cup)$ is a bounded lattice.

1.2.10 Definition

If a lattice $\mathcal{F}$ has 0 and 1 and for each $\rho \in \mathcal{F}$ there exists an element ρ' such that $\rho \wedge \rho' = 0$ and $\rho \vee \rho' = 1$. Then $\mathcal{F}$ is complimented.

1.2.11 Definition

A distributive lattice $\mathcal{F}$ is a lattice which satisfies either of the distributive laws holds i.e,

1. $\rho_1 \vee (\rho_2 \wedge \rho_3) = (\rho_1 \vee \rho_2) \wedge (\rho_1 \vee \rho_3)$.

2. $\rho_1 \wedge (\rho_2 \vee \rho_3) = (\rho_1 \wedge \rho_2) \vee (\rho_1 \wedge \rho_3)$. $\forall\ \rho_1, \rho_2, \rho_3 \in \mathcal{F}$.

1.2.12 Definition

If distributive laws holds in lattice $\mathcal{F}$. Then $\mathcal{F}$ is called distributive lattice. A bounded Distributive Lattice which is also complimented is called a Boolean Algebra.

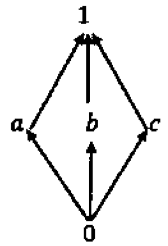
1.2.13 Definition

A lattice $\mathcal{F}$ is called Modular if and only if $\rho_1 \leq \rho_2 \Longrightarrow \rho_2 \wedge (\rho_1 \vee \rho_3) \leq \rho_1 \vee (\rho_2 \wedge \rho_3)$, $\forall\ \rho_1, \rho_2, \rho_3 \in \mathcal{F}$.

1.2.14 Example

1. Every totally ordered set is modular lattice.

2. The following lattice, known as M_5 is modular.



M_5
Fig−1.1

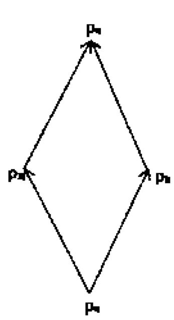
1.3 Lattice Ordered Soft Set

1.3.1 Definition [5]

A soft set $(\alpha, \mathcal{F})$ is called lattice (anti-lattice) ordered soft set if for the mapping $\alpha : \mathcal{F} \rightarrow P(\hat{D})$, $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)(\alpha(\rho_2) \subseteq \alpha(\rho_1))$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$.

1.3.2 Examples

Let $\hat{D} = \{d_1, d_2, d_3, d_4, d_5\}$ he a universal set and $\mathcal{F} = \{\rho_1, \rho_2, \rho_3, \rho_4\}$ be the set of parameters. Then order among the elements of $\mathcal{F}$ is shown in Fig-1.2.



Fig−1.2

$(\alpha, \mathcal{F})$

	d_1	d_2	d_3	d_4	d_5
ρ_1	0	0	0	0	0
ρ_2	1	0	1	0	1
ρ_3	0	1	0	1	0
ρ_4	1	1	1	1	1

Table−1.1

From Table-1.1 lattice of sets are $\mathcal{F}(\rho_1) \subseteq \mathcal{F}(\rho_2) \subseteq \mathcal{F}(\rho_4)$ and $\mathcal{F}(\rho_1) \subseteq \mathcal{F}(\rho_3) \subseteq \mathcal{F}(\rho_4)$. Then it is clear that $(\alpha, \mathcal{F})$ is lattice ordered soft set.

1.3.3 Example

Let $\hat{D} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ be the set of six big stores and $\mathcal{L} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where

σ_1; Large stores.

σ_2; Very large stores.

σ_3; Huge stores.

σ_4; Very huge stores.

Then clearly there is an order in the elements of parameters set $\mathcal{L}$. This order can be describe as $\sigma_1 \preccurlyeq \sigma_2 \preccurlyeq \sigma_3 \preccurlyeq \sigma_4$. The soft set $(\beta, \mathcal{L})$ represented as in Table-1.2.

$(\beta, \mathcal{L})$

	d_1	d_2	d_3	d_4	d_5	d_6
σ_1	1	1	1	1	1	1
σ_2	0	1	1	1	1	0
σ_3	0	1	1	0	1	0
σ_4	0	1	0	0	0	0

Table–1.2

From Table 1.2, we have $\beta(\sigma_1) \supseteq \beta(\sigma_2) \supseteq \beta(\sigma_3) \supseteq \beta(\sigma_4)$. It is clear that $(\beta, \mathcal{L})$ is an anti-lattice ordered soft set.

1.3.4 Theorem [5]

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two lattice (anti-lattice) ordered soft sets. Then the following statements hold:

1. Restricted intersection of two lattice (anti-lattice) ordered soft sets $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is lattice (anti-lattice) ordered soft set.

2. Restricted union of two lattice (anti-lattice) ordered soft sets $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is lattice (anti-lattice) ordered soft set.

3. Extended union of two lattice (anti-lattice) ordered soft sets $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is lattice (anti-lattice) ordered soft set, if either $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$ or $(\beta, \mathcal{L}) \subseteq (\alpha, \mathcal{F})$.

4. Basic intersection of two lattice (anti-lattice) ordered soft sets $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is lattice (anti-lattice) ordered soft set.

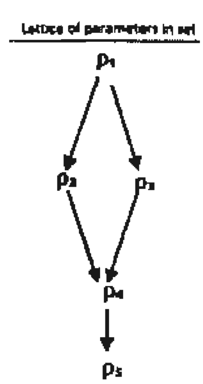
5. Basic union of two lattice (anti-lattice) ordered soft sets $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is lattice (anti-lattice) ordered soft set.

1.3.5 Remark [5]

In general the extended intersection of two lattice (anti-lattice) ordered soft sets $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ may not be a lattice (anti-lattice) ordered soft set.

1.3.6 Example [5]

Let $E = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ with lattice order as shown in Fig-1.3.



$Fig-1.3$

Lattice order soft set $(\alpha,\mathcal{F})$

	d_1	d_2	d_3	d_4	d_5
ρ_1	0	0	0	1	0
ρ_2	0	0	0	1	1
ρ_3	0	1	1	1	0
ρ_4	0	1	1	1	1

Table−1.3*A*

Lattice order soft set $(\beta,\mathcal{L})$

	d_1	d_2	d_3	d_4	d_5
ρ_1	1	0	0	0	0
ρ_2	1	0	0	1	0
ρ_4	1	0	0	1	1
ρ_5	1	1	0	1	1

Table−1.3*B*

Lattice order soft set $(\gamma,\Re)$

	d_1	d_2	d_3	d_4	d_5
ρ_1	0	0	0	0	0
ρ_2	0	0	0	1	0
ρ_3	0	1	1	1	0
ρ_4	0	0	0	1	1
ρ_5	1	1	0	1	1

Table−1.3*C*

Let $\mathcal{F} = \{\rho_1, \rho_2, \rho_3, \rho_4\}$, $\mathcal{L} = \{\rho_1, \rho_2, \rho_4, \rho_5\}$. Consider $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ as lattice order soft sets over a set $\hat{D} = \{d_1, d_2, d_3, d_4, d_5\}$ as shown in $Table-1.3A$ and $Table-1.3B$ respectively

Here $\alpha(\rho_1) \subseteq \alpha(\rho_2) \subseteq \alpha(\rho_4)$, $\alpha(\rho_1) \subseteq \alpha(\rho_3) \subseteq \alpha(\rho_4)$ and $\beta(\rho_1) \subseteq \beta(\rho_2) \subseteq \beta(\rho_4) \subseteq \beta(\rho_5)$. Then their extended intersection $(\alpha, \mathcal{F})\tilde{\cap}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L} = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ is given in $Table - 1.3C$

From $Table - 1.3C$ we have $\gamma(\rho_1) \subseteq \gamma(\rho_2) \subseteq \gamma(\rho_3)$ and $\gamma(\rho_1) \subseteq \gamma(\rho_2) \subseteq \gamma(\rho_4) \subseteq \gamma(\rho_5)$. As $\rho_3 \preccurlyeq \rho_5$ but $\gamma(\rho_3) \nsubseteq \gamma(\rho_5)$. So $(\gamma, \Re)$ is not a lattice order soft set.

1.4 Semigroup

1.4.1 Definition

Let S_g be a non-empty set and " $*$ " be a binary operation on S_g. Then $(S_g, *)$ is called a semigroup if this operation is associative, that is $a * (b * c) = (a * b) * c \quad \forall a, b, c \in S_g$.

A semigroup $(S_g, *)$ is called *commutative* if $a * b = b * a \quad \forall a, b \in S_g$.

1.4.2 Examples

1. $(\mathbf{N}, +)$ is a semigroup.

2. Let $S = \{a_1, a_2, a_3, ...\}$ such that " $*$ " is defined on S_g by $a_i * a_j = a_i$. Then $(S_g, *)$ is a semigroup.

3. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with the Table-1.4 given below is a semigroup under multiplication.

.	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	2	3	4	4	4	4	4
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	5	5	5	5	5	5	5	5
7	5	6	7	8	8	8	8	8
8	8	8	8	8	8	8	8	8

Table–1.4

1.4.3 Definition

Let $(S_g, *)$ be a semigroup. Then a non-empty subset H of S_g is said to be a subsemigroup of S_g if and only if $\forall\ a, b \in H$, we have $a * b \in H$.

1.4.4 Example

The set $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in Z \right\}$ is a subsemigroup of $M_{2\times 2}(Z)$.

Now we define the left, right and two sided ideals in semigroup and so we give some examples about them. Also, we define quasi-ideals and bi-ideals in semigroup.

Let $(S_g, .)$ be a semigroup, if $\phi \neq \mathcal{F}, \mathcal{L} \subseteq S_g$. Then $\mathcal{F}\mathcal{L} = \{\rho\sigma : \rho \in \mathcal{F}, \sigma \in \mathcal{L}\}$.

1.4.5 Definition

Let $(S_g, .)$ be a semigroup. A non-empty subset $\mathcal{F}$ of S_g is called a left (right) ideal of S_g, if $S_g\mathcal{F} \subseteq \mathcal{F}(\mathcal{F}S_g \subseteq \mathcal{F})$. If $\mathcal{F}$ is both a left ideal and a right ideal, then it is called an ideal (or a two sided ideal) of S_g.

1.4.6 Example

Let $Z_{14} = \{\bar{0}, \bar{1}, \bar{2}, ..., \bar{13}\}$ be the semigroup under multiplication modulo 14. Then $I = \{\bar{0}, \bar{7}\}$ and $J = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}, \bar{12}\}$ are two sided ideals of Z_{14}.

1.4.7 Definition

Let $(S_g, *)$ be a semigroup. A subset $\phi \neq Q \subseteq S_g$ is called a quasi-ideal of S_g if and only if Q is a subsemigroup of $(S_g, *)$ satisfying $S_gQ \cap QS_g \subseteq Q$.

1.4.8 Example

Let $S_g = \{1, 3, 5, 7, 9\}$ be a semigroup with the following Cayley Table-1.5

.	1	3	5	7	9
1	1	1	1	1	1
3	1	1	1	3	5
5	1	3	5	1	1
7	1	1	1	7	9
9	1	7	9	1	1

Table−1.5

$Q_1 = \{1\}$, $Q_2 = \{1, 3, 5\}$ $Q_3 = \{1, 3\}$, $Q_4 = \{1, 3, 5, 7, 9\}$ are quasi-ideals over a semigroup S_g.

1.4.9 Theorem

Let S_g be a semigroup. If S_g is commutative, then every quasi-ideal of S_g is a two sided ideal of S_g.

1.4.10 Definition

Let $(S_g, *)$ be a semigroup. A subset $\phi \neq B \subseteq S_g$ is called a bi-ideal of S_g if and only if B is a subsemigroup of $(S_g, *)$ satisfying $BS_gB \subseteq B$.

1.4.11 Example

Let $S_g = \{1, 2, 3, 4\}$ be a semigroup with the following cayley Table-1.6.

.	1	2	3	4
1	1	2	3	4
2	2	3	3	4
3	4	3	4	3
4	4	4	3	4

Table–1.6

$B_1 = \{3, 4\}$, $B_2 = \{2, 3, 4\}$, $B_3 = \{1, 2, 3, 4\}$ are bi-ideals of tbe semigroup S_g.

1.4.12 Remarks

1. Intersection and union of any collection of ideals is ideal.

2. Intersection and union of any collection of quasi-ideals is quasi-ideal.

3. Every quasi-ideal is bi-ideal.

Chapter 2

On Soft Ideals over Semigroups

In this chapter we review the reserch paper "On Soft Ideals over Semigroups"[3]. In this paper the concept of soft ideals over semigroup has been discussed.

2.1 Soft Semigroups

2.1.1 Definition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft sets over a semigroup S_g. The restricted product of $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is defined as $(\alpha, \mathcal{F})\hat{O}(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cap}\mathcal{L}$ and $\gamma(\varsigma) = \alpha(\varsigma)\beta(\varsigma)$, $\forall\ \varsigma \in \Re$.

2.1.2 Definition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft sets over a semigroup S_g. Then the operation " $*$ " is defined as $(\alpha, \mathcal{F}) * (\beta, \mathcal{L}) = (H, \mathcal{F} \otimes \mathcal{L})$, where $\gamma(\rho, \sigma) = \alpha(\rho) * \beta(\sigma)$, $\rho \in \mathcal{F}$, $\sigma \in \mathcal{L}$, further $\mathcal{F} \otimes \mathcal{L}$ is the cartesian product of $\mathcal{F}$ and $\mathcal{L}$. From now to onward simply write $(\alpha, \mathcal{F})(\beta, \mathcal{L})$ and $\alpha(\rho)\beta(\sigma)$ instead of $(\alpha, \mathcal{F}) * (\beta, \mathcal{L})$ and $\alpha(\rho) * \beta(\sigma)$

respectively.

2.1.3 Definition

A non null and non-empty soft set $(\alpha, \mathcal{F})$ over a semigroup S_g is said to be soft semigroup if $(\alpha, \mathcal{F})(\alpha, \mathcal{F}) \subseteq (\alpha, \mathcal{F})$.

or

If $(\alpha, \mathcal{F})$ is a soft set over S_g. Then $(\alpha, \mathcal{F})$ is a soft semigroup over S_g if and only if $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is a subsemigroup of S_g, whenever $\alpha(\rho) \neq \phi$.

2.1.4 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft semigroups over a semigroup S_g. Then $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L})$ also a soft semigroup over a semigroup S_g, whenever $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L})$ is non null and non empty.

2.1.5 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft semigroups over a semigroup S_g such that $\mathcal{F}\tilde{\cap}\mathcal{L} = \phi$. Then $(\alpha, \mathcal{F})\tilde{\cup}_E(\beta, \mathcal{L})$ also a soft semigroup over a semigroup S_g.

2.1.6 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft semigroups over a semigroup S_g. Then $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L})$ also a soft semigroup over a semigroup S_g. Whenever $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L})$ is non null.

2.1.7 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be any two soft semigroups over a commutative semigroup S_g. Then $(\alpha, \mathcal{F}) * (\beta, \mathcal{L})$ is also a soft semigroup over a semigroup S_g. Whenever $(\alpha, \mathcal{F}) * (\beta, \mathcal{L})$ is a non null.

If S_g is a non-commutative. Then $(\alpha, \mathcal{F}) * (\beta, \mathcal{L})$ is not necessarily a soft semigroup.

2.1.8 Example

Let the semigroup $S_g = \{1, b, c, d\} = \mathcal{F}$ with the Cayley Table-2.1

.	1	b	c	d
1	1	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	d

Table–2.1

we define a soft semigroups $(\alpha, \mathcal{F})$ and $(\beta, \{1, b\})$ over a semigroup S_g. As

$\alpha(1) = \{1\}$, $\alpha(b) = \{b\}$, $\alpha(c) = \{c\}$, $\alpha(d) = \{d\}$

and

$\beta(1) = \{1, b\}$, $\beta(b) = \{1, c\}$

Now $(\alpha, \mathcal{F}) * (\beta, \{1, b\}) = (\gamma, \mathcal{F} \times \{1, b\})$ and $\gamma(\rho, \sigma) = \alpha(\rho) * \beta(\sigma)$, where $\rho \in \mathcal{F}$, $\sigma \in \{1, b\}$. Now $\gamma(d, 1) = \{d\}\{1, b\} = \{d, c\}$. Which is not a subsemigroup of S. Therefore $(\gamma, \mathcal{F} \times \{1, b\})$ is not a soft semigroup over S_g.

2.2 Soft Ideals

In this section, we study some properties of soft ideals.

2.2.1 Definition

A pair $(\alpha, \mathcal{F})$ over a semigroup S_g is said to be soft left (right) ideal over the semigroup S_g, if $\mathcal{F}_{S_g}\hat{O}(\alpha, \mathcal{F}) \subseteq (\alpha, \mathcal{F})((\alpha, \mathcal{F})\hat{O}\mathcal{F}_{S_g} \subseteq (\alpha, \mathcal{F}))$, where $\mathcal{F}_{S_g}$ is an absolute soft set over S_g. A soft set over S_g is soft ideal if it is both soft left (right) ideal over S_g.

2.2.2 Definition

A non null and non-empty soft set $(\alpha, \mathcal{F})$ over a semigroup S_g is a soft ideal over S_g. if and only if $\alpha(\rho) \neq \phi$, $\forall\ \rho \in \mathcal{F}$ is an ideal of S_g.

2.2.3 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be any two soft ideals over a semigroup S_g. Then $(\alpha, \mathcal{F}) * (\beta, \mathcal{L})$ is also a soft ideal over S_g, whenever $(\alpha, \mathcal{F}) * (\beta, \mathcal{L})$ is non null.

2.2.4 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be any two soft semigroups (ideals) over S_{g_1} and S_{g_2} respectively. Then $(\alpha, \mathcal{F}) \times (\beta, \mathcal{L})$ is also a soft semigroup (ideal) over $S_{g_1} \times S_{g_2}$, whenever $(\alpha, \mathcal{F}) \times (\beta, \mathcal{L})$ is non null.

2.2.5 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be any two soft ideals over a semigroup S_g. Then $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L})$ is also a soft ideal over S_g, contained in both $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$, whenever $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L})$ is non null and non empty.

2.2.6 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft ideals over a semigroup S_g. Then $(\alpha, \mathcal{F})\tilde{\cup}_E(\beta, \mathcal{L})$ is also a soft ideal over S_g, containing both $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$.

2.2.7 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft ideals over a semigroup S_g. Then $(\alpha, \mathcal{F})\hat{O}(\beta, \mathcal{L})$ is a soft ideal contained in both $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$, whenever $(\alpha, \mathcal{F})\hat{O}(\beta, \mathcal{L})$ is non null and non empty.

2.2.8 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft ideals over a semigroup S_g. Then $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L})$ is a soft ideal over S_g whenever $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L})$ is non null.

2.2.9 Proposition

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft ideals over a semigroup S_g. Then $(\alpha, \mathcal{F})\bar{\cup}_B(\beta, \mathcal{L})$ is a soft ideal over S_g.

2.3 Soft Quasi-Ideals and Soft Bi-Ideals

The notion of quasi-ideal in a semigroup was firts introduced by Steinfeld in [14]. In fact, the concept of quasi-ideal play an important role in the characterization of different algebraic structures. In semigroups, it is interesting to note that the restricted intersection and the basic intersection of a soft left ideal and a soft right ideal over a semigroup S_g is neither a soft left ideal nor a soft right ideal over S_g. This interseting fact can be illustrated in the following example.

2.3.1 Example

Let $S_g = \{a, b, c, d, e\}$ be a semigroup with the following Cayley Table-2.2.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

Table−2.2

Let $(\mathbb{R}, \mathcal{F})$ and $(\mathbb{Z}, \mathcal{L})$ be soft sets over S_g, where $\mathcal{F} = \mathcal{L} = S_g$ and $\mathbb{R}$ and $\mathbb{Z}$ are defined as $\mathbb{R}(a) = \{a\}$, $\mathbb{R}(b) = \{a, b, c\} = \mathbb{R}(c)$, $\mathbb{R}(d) = \mathbb{R}(e) = \{a, d, e\}$, $\mathbb{Z}(a) = \{a\}$, $\mathbb{Z}(b) = \{a, b, d\}$, $\mathbb{Z}(c) = \{a, c, e\}$, $\mathbb{Z}(d) = \{a, b, d\}$, $\mathbb{Z}(e) = \{a, c, e\}$. Then $(\mathbb{R}, \mathcal{F})$ is a soft right ideal over S_g and $(\mathbb{Z}, \mathcal{L})$ be the soft left ideal over S_g.

Let $(Q, \Re) = (\mathbb{R}, \mathcal{F}) \tilde{\cap}_R (\mathbb{Z}, \mathcal{L})$, where $\Re = \mathcal{F} \cap \mathcal{L} = S_g$ and $Q(\varsigma) = \mathbb{R}(\varsigma) \cap \mathbb{Z}(\varsigma)$ $\forall$ $\varsigma \in \Re$. Then $Q(a) = \{a\}$, $Q(b) = \{a, b\}$, $Q(c) = \{a, c\}$, $Q(d) = \{a, d\}$, $Q(e) = \{a, e\}$. Thus it is clear that $(Q, \Re)$ is neither a soft right ideal nor a soft left ideal over S_g.

Similarly it can be shown that $(\mathbb{R}, \mathcal{F}) \tilde{\cap}_B (\mathbb{Z}, \mathcal{L})$ is neither a soft left ideal nor a soft right ideal over S_g.

2.3.2 Definition

A non null soft set $(\alpha, \mathcal{F})$ is said to be soft quasi ideal over a semigroup S_g if $(\alpha, \mathcal{F})\mathcal{F}_{S_g} \tilde{\cap}_R \mathcal{F}_{S_g}(\alpha, \mathcal{F}) \subseteq (\alpha, \mathcal{F})$. where $\mathcal{F}_{S_g}$ is an absolute soft set over a semigroup S_g.

2.3.3 Definition

A non null and non empty soft set $(\alpha, \mathcal{F})$ over a semigroup S_g is called a soft quasi ideal over S_g if and only if $\alpha(\rho)$ is a quasi ideal of S_g, whenever $\alpha(\rho) \neq \phi$, $\forall$ $\rho \in \mathcal{F}$.

Soft quasi-ideals over a semigroup S_g have the following properties.

1. Let $(\mathbb{R}, \mathcal{F})$ be a soft right ideal over S_g and $(\mathbb{Z}, \mathcal{L})$ be a soft left ideal over S_g. Then $(\mathbb{R}, \mathcal{F}) \tilde{\cap}_R (\mathbb{Z}, \mathcal{L})$ is a soft quasi-ideal over S_g, whenever $(\mathbb{R}, \mathcal{F}) \tilde{\cap}_R (\mathbb{Z}, \mathcal{L})$ is a non null and non empty.

2. Let $(\mathbb{R}, \mathcal{F})$ be a soft right ideal over S_g and $(\mathbb{Z}, \mathcal{L})$ be a soft left ideal over S_g. Then $(\mathbb{R}, \mathcal{F}) \tilde{\cap}_E (\mathbb{Z}, \mathcal{L})$ is a soft quasi-ideal over S_g.

3. Let $(\mathbb{R}, \mathcal{F})$ be a soft right ideal over S_g and $(\mathbb{Z}, \mathcal{L})$ be a soft left ideal over S_g Then $(\mathbb{R}, \mathcal{F}) \tilde{\cap}_B (\mathbb{Z}, \mathcal{L})$ is a soft quasi-ideal over S_g, whenever $(\mathbb{R}, \mathcal{F}) \tilde{\cap}_B (\mathbb{Z}, \mathcal{L})$ is a non null.

4. Let $(\mathbb{R}, \mathcal{F})$ be a soft right (left) ideal over S_g. Then $(\mathbb{R}, \mathcal{F})$ is a soft quasi ideal over S_g.

It is easy to see that if $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are two soft quasi-ideals over a semigroup S_g, then the following statements hold:

1. $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L})$ is a soft quasi-ideal over S_g, whenever $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L})$ is a non null and non empty.

2. $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L})$ is a soft quasi-ideal over S_g, whenever $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L})$ is a non null.

3. $(\alpha, \mathcal{F})\tilde{\cap}_E(\beta, \mathcal{L})$ is a soft quasi-ideal over S_g.

2.3.4 Definition

A soft set $(\alpha, \mathcal{F})$ over a semigroup S_g is said to be soft bi-ideal over a semigroup S_g if

1. $(\alpha, \mathcal{F})$ is a soft semigroup over a semigroup S_g.

2. $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is bi-ideal over a semigroup S_g.

2.3.5 Theorem

A soft set $(\alpha, \mathcal{F})$ over a semigroup S_g is said to be soft bi-ideal over S_g if and only if $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho) \neq \phi$ is a bi-ideal over S_g.

2.3.6 Theorem

Every soft quasi-ideal over a semogroup S_g is a soft bi-ideal over S_g.

2.3.7 Theorem

Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two soft quasi ideals over a semigroup S_g. Then $(\alpha, \mathcal{F}) * (\beta, \mathcal{L})$ is a soft bi-ideal over S_g, where " $*$ " is a binary operation defined on S_g.

Chapter 3

A Study in Lattice Ordered Soft Semigroups

In this chapter, the concept of lattice (anti-lattice) ordered soft semigroups and some properties of lattice (anti-lattice) ordered soft semigroups has been introduced. Also the concept of lattice (anti-lattice) ordered soft ideal (quasi-ideal, bi-ideal) and its properties has been defined, and related properties are discussed.

From now to onward we will give the notation to soft set as SS, semigroup as SG, subsemigroup as Ssg, soft semigroup as SSG, lattice (anti lattice) ordered soft set as L(anti-L)OSS, lattice (anti-lattice) ordered soft semigroup as L(anti-L)OSSG, lattice (anti-lattice) ordered soft subsemigroup as L(anti-L)OSSSG, lattice (anti-lattice) order quasi-idealistic soft semigroups as L(anti-L)OQISSG and lattice (anti-lattice) order bi-idealistic soft semigroups as L(anti-L)OBISSG.

3.1 Lattice Ordered Soft Semigroups

3.1.1 Definition

Let S_g be a SG and $(\alpha, \mathcal{F})$ be a non-empty SS over S_g. Then $(\alpha, \mathcal{F})$ is called L(anti-L)OSSG over S_g, if

1. $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is Ssg of S_g.

2. $\forall\ \rho_1, \rho_2 \in \mathcal{F}$, $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)(\alpha(\rho_2) \subseteq \alpha(\rho_1))$.

3.1.2 Example

Let $S_g = \{1, 2, 3, 4\}$ be a SG with the following Cayley Table-3.1 and with the ordered $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4$.

.	1	2	3	4
1	1	2	3	4
2	2	3	3	4
3	4	3	4	3
4	4	4	3	4

Table−3.1

Let $\mathcal{F} = \{1, 2, 3\}$ and define a mapping $\alpha : \mathcal{F} \to P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(2) = \{1, 4\}$, $\alpha(3) = \{1, 3, 4\}$. Then $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is Ssg of S_g and $\forall\ \rho_1, \rho_2 \in \mathcal{F}$ with $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$. Then $(\alpha, \mathcal{F})$ is LOSSG over S_g.

3.1.3 Example

Let $S_g = \{1, 2, 3, 4, 5\}$ be a SG with the Cayley Table-3.2 and having the order $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4 \preccurlyeq 5$.

.	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	3
3	1	2	3	1	1
4	1	1	1	4	5
5	1	4	5	1	1

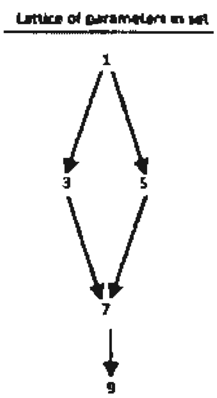
Let $\mathcal{F} = \{1,2,3\}$ and define a mapping $\alpha : \mathcal{F} \rightarrow P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(2) = \{1,2\}$, $\alpha(3) = \{1,2,3\}$. Then clearly $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is Ssg of S_g and for any $\rho_1, \rho_2 \in \mathcal{F}$, $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$. Then $(\alpha, \mathcal{F})$ is LOSSG over S_g.

3.1.4 Example

Let $S_g = \{1,3,5,7,9\}$ be a SG with the following Cayley Table-3.3 and with the order by the following Hasse diagram shown in Fig-3.1.

.	1	3	5	7	9
1	1	1	1	1	1
3	1	1	1	3	5
5	1	3	5	1	1
7	1	1	1	7	9
9	1	7	9	1	1

Table–3.3



Fig–3.1

Let $\mathcal{F} = \{1,3,5\}$ and define a mapping $\alpha : \mathcal{F} \rightarrow P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(3) = \{1,3\}$, $\alpha(5) = \{1,5\}$. Then clearly $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is Ssg of S_g and for any $\rho_1, \rho_2 \in \mathcal{F}$, with $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$. Then $(\alpha, \mathcal{F})$ is LOSSG over S_g.

On the same SG define another parametric set $\mathcal{L} = \{3,7,9\}$ and define a mapping $\beta : \mathcal{L} \rightarrow P(S_g)$ by $\beta(3) = \{1,3,5,7\}$, $\beta(7) = \{1,3,5\}$, $\beta(9) = \{1,3\}$. Then $\forall\ \sigma \in \mathcal{L}$,

$\beta(\sigma)$ is Ssg of S_g and for any $\sigma_1, \sigma_2 \in \mathcal{L}$, $\sigma_1 \preccurlyeq \sigma_2$ implies $\beta(\sigma_2) \subseteq \beta(\sigma_1)$. Then $(\beta, \mathcal{L})$ is anti-LOSSG over S_g.

3.1.5 Example

Let $S_g = \{1,2,3,4,5,6,7,8\}$ be a SG with the following Cayley Table-3.4 and with the order $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4 \preccurlyeq 5 \preccurlyeq 6 \preccurlyeq 7 \preccurlyeq 8$.

.	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	2	3	4
3	1	2	3	4	4	4	4	4
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	5	5	5	5	5	5	5	5
7	5	6	7	8	8	8	8	8
8	8	8	8	8	8	8	8	8

Table−3.4

Let $\mathcal{F} = \{1,3,4\}$ and define a mapping $\alpha : \mathcal{F} \to P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(3) = \{1,8\}$, $\alpha(4) = \{1,5,8\}$. Then $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is Ssg of S_g and $\forall\ \rho_1, \rho_2 \in \mathcal{F}$, with $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$. Then $(\alpha, \mathcal{F})$ is LOSSG over S_g. On a same SG define another parametric set $\mathcal{L} = \{3,4,6,7\}$ and define a mapping $\beta : \mathcal{L} \to P(S_g)$ by $\beta(3) = \{1,2,4,5\}$, $\beta(4) = \{1,4,5\}$, $\beta(6) = \{4,5\}$, $\beta(7) = \{4\}$. Then $\forall\ \sigma \in \mathcal{L}$, $\beta(\sigma)$ is Ssg of S_g and $\forall\ \sigma_1, \sigma_2 \in \mathcal{L}$, with $\sigma_1 \preccurlyeq \sigma_2$ implies $\beta(\sigma_2) \subseteq \beta(\sigma_1)$. Then $(\beta, \mathcal{L})$ is anti-LOSSG over S_g.

3.1.6 Definition

Let S_g be a SG and $(\alpha, \mathcal{F})$ be a LOSSG over S_g. Then support of $(\alpha, \mathcal{F})$ is denoted and define as $Supp(\alpha, \mathcal{F}) = \{\rho \in \mathcal{F}, \alpha(\rho) \neq \emptyset\}$.

3.1.7 Definition

Let S_g be a SG with $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be LOSSG over the same SG S_g. Then $(\alpha, \mathcal{F})$ is LOSSSG of $(\beta, \mathcal{L})$ if it holds,

1. $\mathcal{F} \subseteq \mathcal{L}$.

2. for $\rho \in Supp(\alpha, \mathcal{F})$, implies $\alpha(\rho)$ is Ssg of $\beta(\rho)$.

3.2 Basic Operations on Lattice (anti-lattice) Ordered Soft Semigroups

3.2.1 Theorem

Restricted intersection of two L(anti-L)OSSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OSSG if it is non null.

Proof. Let S_g be a SG, E be an ordered set of parameter with $\mathcal{F}, \mathcal{L} \subseteq E$, $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two LOSSG over S_g. Then by definition $(\alpha, \mathcal{F})\bar{\cap}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ with $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \emptyset$, $\forall\ \varsigma \in \Re$, $\gamma(\varsigma) = \alpha(\varsigma)\bar{\cap}\beta(\varsigma)$. Then result follows by the fact that the intersection of any number of Ssgs is Ssg provided it is non-empty. Now Let $\mathcal{F}\bar{\cap}\mathcal{L} \neq \emptyset$. As $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire the partial order from E. Hence for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$, implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$, implies $\beta(\sigma_1) \subseteq \beta(\sigma_2)\ \forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $\varsigma_1, \varsigma_2 \in \Re$, $\varsigma_1 \preccurlyeq \varsigma_2$ implies $\alpha(\varsigma_1) \subseteq \alpha(\varsigma_2)$ and $\beta(\varsigma_1) \subseteq \beta(\varsigma_2)$. Also for $\alpha(\varsigma_1)\bar{\cap}\beta(\varsigma_1) \subseteq \alpha(\varsigma_2)\bar{\cap}\beta(\varsigma_2)$, this implies $\gamma(\varsigma_1) \subseteq \gamma(\varsigma_2)$ for $\varsigma_1 \preccurlyeq_{\Re} \varsigma_2$. Thus $(\alpha, \mathcal{F})\bar{\cap}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ is a LOSSG over S_g. Similary the result can be shown for anti-LOSSG. ■

The example given below describes that the restricted union of two LOSSGs may or may not be a LOSSG.

3.2.2 Example

Let $S_g = \{1,2,3,4,5\}$ be a SG with Cayley Table-3.5 and with usual order $1 \preceq 2 \preceq 3 \preceq 4 \preceq 5$.

.	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	3
3	1	2	3	1	1
4	1	1	1	4	5
5	1	4	5	1	1

Table−3.5

Let $\mathcal{F} = \{1,2,4\}$, $\mathcal{L} = \{1,2,3,5\}$ be two parametric sets. Define a mapping $\alpha : \mathcal{F} \to P(S_g)$ and $\beta : \mathcal{L} \to P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(2) = \{1,2\}$, $\alpha(4) = \{1,2,3\}$ and $\beta(1) = \{1,5\}$, $\beta(2) = \{1,3,5\}$, $\beta(3) = \{1,3,4,5\}$, $\beta(5) = \{1,2,3,4,5\}$. Then clearly for all $\rho, \sigma \in \mathcal{F}, \mathcal{L}$ and $\rho \preccurlyeq \sigma$ implies $\alpha(\rho) \subseteq \alpha(\sigma)(\beta(\rho) \subseteq \beta(\sigma))$. So $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are two LOSSGs over a SG S_g. Then their restricted union $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} = \{1,2\}$ is given by $\gamma(1) = \{1,5\}$, $\gamma(2) = \{1,2,3,5\}$, As $1 \preccurlyeq 2$ and $\gamma(1) \subseteq \gamma(2)$, but $\gamma(2) = \{1,2,3,5\}$ is not a Ssg. So $(\gamma, \Re)$ is not a LOSSG.

3.2.3 Theorem

Restricted union of two L(anti-L)OSSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OSSG over S_g if $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$ or $(\beta, \mathcal{L}) \subseteq (\alpha, \mathcal{F})$.

Proof. Let $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$ with $\mathcal{F} \subseteq \mathcal{L}$. As by definition $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\gamma, \Re)$

with $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \phi$. Then for any $\varsigma \in \Re$, we encompass $\gamma(\varsigma) = \alpha(\varsigma)\tilde{\cup}\beta(\varsigma)$. Now as $\Re = \mathcal{F}\tilde{\cap}\mathcal{L}$ with $\mathcal{F} \subseteq \mathcal{L}$ subsequently we encompass $\Re = \mathcal{F}$. So this implies that $(\gamma, \Re) = (\alpha, \mathcal{F})$ implies $\gamma(\varsigma) = \alpha(\varsigma)$ implies $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\alpha, \mathcal{F})$, but $(\alpha, \mathcal{F})$ is LOSSG and $(\gamma, \Re) = (\alpha, \mathcal{F})$, implies $(\gamma, \Re) = (\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L})$ is also LOSSG. ∎

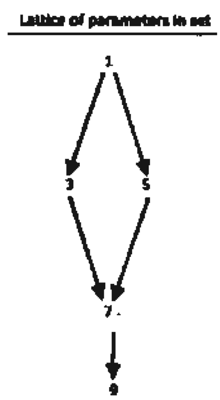
The example given below describes that the extended intersection and extended union of two LOSSGs may not be a LOSSG.

3.2.4 Example

Let $S_g = \{1, 3, 5, 7, 9\}$ be a SG with Cayley Table-3.6 and with order by the following Hasse diagram shown in Fig-3.2.

.	1	3	5	7	9
1	1	1	1	1	1
3	1	1	1	3	5
5	1	3	5	1	1
7	1	1	1	7	9
9	1	7	9	1	1

Table–3.6



Fig–3.2

Let $\mathcal{F} = \{1, 3, 7\}$, $\mathcal{L} = \{1, 3, 5, 9\}$ be two parametric sets. Define a mapping $\alpha : \mathcal{F} \to P(S_g)$ and $\beta : \mathcal{L} \to P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(3) = \{1, 3\}$, $\alpha(7) = \{1, 3, 5\}$ and $\beta(1) = \{1, 9\}$, $\beta(3) = \{1, 3, 5, 7, 9\}$, $\beta(5) = \{1, 5, 9\}$, $\beta(9) = \{1, 3, 5, 7, 9\}$. Then clearly for all $\rho, \sigma \in \mathcal{F}, \mathcal{L}$ and $\rho \preccurlyeq \sigma$ implies $\alpha(\rho) \subseteq \alpha(\sigma)(\beta(\rho) \subseteq \beta(\sigma))$. So $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are two LOSSGs over S_g.

Here $\alpha(1) \subseteq \alpha(3) \subseteq \alpha(7)$ and $\beta(1) \subseteq \beta(3) \subseteq \beta(9)$, $\beta(1) \subseteq \beta(5) \subseteq \beta(9)$. Then their extended intersection $(\alpha, \mathcal{F})\tilde{\cap}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L} = \{1, 3, 5, 7, 9\}$ is given by $\gamma(1) = \{1\}$, $\gamma(3) = \{1\}$, $\gamma(5) = \{1, 5, 9\}$, $\gamma(7) = \{1, 3, 5\}$,

$\gamma(9) = \{1,3,5,7,9\}$. As $5 \preccurlyeq 7$ but $\gamma(5) \nsubseteq \gamma(7)$. So $(\gamma, \Re)$ is not a LOSSG. Similarly extended union $(\alpha, \mathcal{F})\tilde{\cup}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L} = \{1,3,5,7,9\}$ is given by $\gamma(1) = \{1,9\}$, $\gamma(3) = \{1,3,5,7,9\}$, $\gamma(5) = \{1,5,9\}$, $\gamma(7) = \{1,3,5\}$, $\gamma(9) = \{1,3,5,7,9\}$. As $3 \preccurlyeq 7$ but $\gamma(3) \nsubseteq \gamma(7)$. So $(\gamma, \Re)$ is not a LOSSG.

3.2.5 Theorem

Extended union of two L(anti-L)OSSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OSSG if $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$ or $(\beta, \mathcal{L}) \subseteq (\alpha, \mathcal{F})$.

Proof. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two LOSSGs over S_g and $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$, where $\mathcal{F} \subseteq \mathcal{L}$ and $\alpha(\rho) \subseteq \beta(\rho)$, $\forall\ \rho \in \mathcal{F}$. Then by definition $(\alpha, \mathcal{F})\tilde{\cup}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L}$ as $\mathcal{F} \subseteq \mathcal{L}$ then $\Re = \mathcal{L}$, this implies that $\gamma(\varsigma) = \beta(\varsigma)\ \forall\ \varsigma \in \Re$. So $(\gamma, \Re) = (\beta, \mathcal{L})$. As $(\beta, \mathcal{L})$ is LOSSG implies $(\gamma, \Re)$. So $(\alpha, \mathcal{F})\tilde{\cup}_E(\beta, \mathcal{L}) = (\gamma, \Re)$ is LOSSG over SG S_g In the same way the result can be shown for anti-LOSSGs. ■

3.2.6 Theorem

Basic intersection of two L(anti-L)OSSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OSSG, if it is non null.

Proof. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two LOSSGs over S_g. Then $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F} \otimes \mathcal{L}$. Then for any $\rho \in \mathcal{F}$, $\sigma \in \mathcal{L}$ and for $(\rho, \sigma) \in \mathcal{F} \otimes \mathcal{L}$, we have $\gamma(\rho, \sigma) = \alpha(\rho)\tilde{\cap}\beta(\sigma)$, where $\alpha(\rho)$ and $\beta(\sigma)$ are Ssgs of S_g. As $\alpha(\rho)\tilde{\cap}\beta(\sigma) \neq \phi$. As intersection of any numbers of Ssgs of S_g is Ssg, so $(\gamma, \Re)$ is Ssg of S_g Since both $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire a partial order from E. Therfore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$ implies $\beta(\sigma_1) \subseteq \beta(\sigma_2)$, $\forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \Re$. Now $\preceq$ is the partial order on $\Re$ which is generated by partial order on $\mathcal{F}$ and $\mathcal{L}$. If $(\rho_1, \sigma_1) \preceq (\rho_2, \sigma_2)$, then

$\alpha(\rho_1) \subseteq \alpha(\rho_2)$ and $\beta(\sigma_1) \subseteq \beta(\sigma_2)$ implies $\alpha(\rho_1)\tilde{\cap}\beta(\sigma_1) \subseteq \alpha(\rho_2)\tilde{\cap}\beta(\sigma_2)$ this implies $\gamma(\rho_1,\sigma_1) \subseteq \gamma(\rho_2,\sigma_2)$ for $(\rho_1,\sigma_1) \preceq_{\Re} (\rho_2,\sigma_2)$. Thus $(\alpha,\mathcal{F})\tilde{\cap}_B(\beta,\mathcal{L}) = (\gamma,\Re)$ is a LOSSG. In the same way the result can be shown for anti-LOSSG. ■

The example given below describes that the basic union of two LOSSGs may or may not be a LOSSG.

3.2.7 Example

Let $S_g = \{1,2,3,4,5\}$ be a SG with Cayley Table-3.7 and with usual order $1 \preceq 2 \preceq 3 \preceq 4 \preceq 5$.

.	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	3
3	1	2	3	1	1
4	1	1	1	4	5
5	1	4	5	1	1

Table−3.7

Let $\mathcal{F} = \{1,2,4\}$, $\mathcal{L} = \{1,2,3,5\}$ be two parametric sets. Define a mapping $\alpha : \mathcal{F} \to P(S_g)$ and $\beta : \mathcal{L} \to P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(2) = \{1,2\}$, $\alpha(4) = \{1,2,3\}$ and $\beta(1) = \{1,5\}$, $\beta(2) = \{1,3,5\}$, $\beta(3) = \{1,3,4,5\}$, $\beta(5) = \{1,2,3,4,5\}$. Then clearly for all $\rho,\sigma \in \mathcal{F},\mathcal{L}$ and $\rho \preccurlyeq \sigma$ implies $\alpha(\rho) \subseteq \alpha(\sigma)(\beta(\rho) \subseteq \beta(\sigma))$. So $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ are two LOSSGs over a SG S_g. Then their basic union $(\alpha,\mathcal{F})\tilde{\cup}_B(\beta,\mathcal{L}) = (\gamma,\Re)$, where $\Re = \mathcal{F}\otimes\mathcal{L} = \{(1,1),(1,2),(1,3),(1,5),(2,1),(2,2),(2,3),(2,5),(4,1),(4,2),(4,3),(4,5)\}$ is given by $\gamma(1,1) = \{1,5\}$, $\gamma(1,2) = \{1,3,5\}$, $\gamma(1,3) = \{1,3,4,5\}$, $\gamma(1,5) = \{1,2,3,4,5\}$, $\gamma(2,1) = \{1,2,5\}$, $\gamma(2,2) = \{1,2,3,5\}$, $\gamma(2,3) = \{1,2,3,4,5\}$, $\gamma(2,5) = \{1,2,3,4,5\}$, $\gamma(4,1) = \{1,2,3,5\}$, $\gamma(4,2) = \{1,2,3,5\}$, $\gamma(4,3) = \{1,2,3,4,5\}$, $\gamma(4,5) =$

$\{1,2,3,4,5\}$. As $(1,1) \preccurlyeq (2,1)$ and $\gamma((1,1)) \subseteq \gamma(2,1)$, but $\gamma(2,1) = \{1,2,5\}$ is not a Ssg over S_g. So $(\gamma, \Re)$ is not a LOSSG.

3.2.8 Theorem

Basic union of two L(anti-L)OSSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OSSG if for all $(\rho, \sigma) \in \mathcal{F} \otimes \mathcal{L}$ either $\alpha(\rho) \subseteq \beta(\sigma)$ or $\beta(\sigma) \subseteq \alpha(\rho)$.

Proof. For any $(\rho, \sigma) \in \mathcal{F} \otimes \mathcal{L}$, we consider that $\alpha(\rho) \subsetneq \beta(\sigma)$. By definition $(\alpha, \mathcal{F})\tilde{\cup}_B(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F} \otimes \mathcal{L}$ and for any $(\rho, \sigma) \in \mathcal{F} \otimes \mathcal{L}$ we encompass $\gamma(\rho, \sigma) = \alpha(\rho)\tilde{\cup}\beta(\sigma)$. As $\alpha(\rho) \subseteq \beta(\sigma)$, so $\alpha(\rho)\tilde{\cup}\beta(\sigma) = \beta(\sigma)$ implies $(\gamma, \Re) = (\beta, \mathcal{L})$, but $(\beta, \mathcal{L})$ is a LOSSG over a SG S_g so it pursues that $(\alpha, \mathcal{F})\tilde{\cup}_B(\beta, \mathcal{L}) = (\gamma, \Re)$ is a LOSSG over S_g. In the same way the result can be shown for anti-LOSSG. ■

3.3 Properties of Lattice Ordered Soft Ideals

3.3.1 Definition

A SS $(\alpha, \mathcal{F})$ over a SG S_g is said to be L(anti-L)OISSG if it satisfies the following conditions.

1. $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is an ideal of $\mathcal{F}$.

2. $\forall\ \rho_1, \rho_2 \in \mathcal{F}$, $\rho_1 \preccurlyeq \rho_2 \Rightarrow \alpha(\rho_1) \subseteq \alpha(\rho_2)(\alpha(\rho_2) \subseteq \alpha(\rho_1))$.

3.3.2 Example

Let $S_g = \{1,2,3,4\}$ be a SG with the following Cayley Table-3.8 and with the order $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4$.

.	1	2	3	4
1	1	2	3	4
2	2	3	3	4
3	4	3	4	3
4	4	4	3	4

Table−3.8

Let $\mathcal{F} = \{1,2\}$ and $\mathcal{L} = \{1,2,3\}$ are two parametric sets. Define a mapping $\alpha : \mathcal{F} \rightarrow P(S_g)$ by $\alpha(1) = \{3,4\} = \alpha(2)$ and $\beta : \mathcal{L} \rightarrow P(S_g)$ by $\beta(1) = \{3,4\}$, $\beta(2) = \{1,2,3,4\} = \beta(3)$. Then clearly $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are LOISSGs over S_g.

3.3.3 Theorem

Restricted intersection of two L(anti-L)OISSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OISSG.

Proof. Let S_g be a SG, E be an ordered set of parameters with $\mathcal{F}, \mathcal{L} \subseteq E$. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two L(anti-L)OISSGs over S_g. By definition $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ with $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \emptyset$. Then for $\varsigma \in \Re$, $\gamma(\varsigma) = \alpha(\varsigma)\tilde{\cap}\beta(\varsigma)$. Then results follows by the fact that the intersection of any collection of ideals is ideal provided it is non-empty. Since $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire the partial ordered from E. Therefore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$, we have $\alpha(\rho_1) \subseteq \alpha(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$, we have $\beta(\sigma_1) \subseteq \beta(\sigma_2)\ \forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $\varsigma_1, \varsigma_2 \in \Re$, $\alpha(\varsigma_1) \subseteq \alpha(\varsigma_2)$ and $\beta(\varsigma_1) \subseteq \beta(\varsigma_2)$. Also for $\alpha(\varsigma_1)\tilde{\cap}\beta(\varsigma_1) \subseteq \alpha(\varsigma_2)\tilde{\cap}\beta(\varsigma_2)$ this implies $\gamma(\varsigma_1) \subseteq \gamma(\varsigma_2)$ for $\varsigma_1 \preccurlyeq_{\Re} \varsigma_2$. Thus $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \sigma) = (\gamma, \Re)$ is a LOISSG over S_g Similary the result can be shown for anti-LOISSGs. ∎

3.3.4 Theorem

Restricted union of two L(anti-L)OISSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OISSG.

Proof. Let S_g be a SG, E be an ordered set of parameter with $\mathcal{F}, \mathcal{L} \subseteq E$. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two LOISSGs over S_g. By definition $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ with $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \emptyset$ Then for $\varsigma \in \Re$, $\gamma(\varsigma) = \alpha(\varsigma)\tilde{\cup}\beta(\varsigma)$. Then results follows by the fact that the unoin of any number of ideals is ideal. Since $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire the partial ordered from E. Therefore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$, we have $\alpha(\rho_1) \subseteq \alpha(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$, we have $\beta(\sigma_1) \subseteq \beta(\sigma_2)\ \forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $\varsigma_1, \varsigma_2 \in \Re$, $\alpha(\varsigma_1) \subseteq \alpha(\varsigma_2)$ and $\beta(\varsigma_1) \subseteq \beta(\varsigma_2)$. Also for $\alpha(\varsigma_1)\tilde{\cup}\beta(\varsigma_1) \subseteq \alpha(\varsigma_2)\tilde{\cup}\beta(\varsigma_2)$ this implies $\gamma(\varsigma_1) \subseteq \gamma(\varsigma_2)$ for $\varsigma_1 \preccurlyeq_{\Re} \varsigma_2$. Thus $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ is a LOISSGs over S_g. Similary the result can be shown for anti-LOISSG over S_g. ■

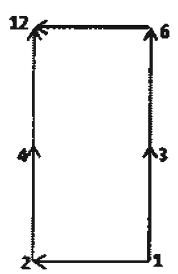
The example given below describes that the extended intersection and extended union of two L(anti-L)OISSGs may not be a L(anti-L)OISSGs.

3.3.5 Example

Consider $S_g = \{1, 2, 3, 4, 6, 12\}$ be a SG with the Cayley Table and lattice ordered shown in Table-3.9 and Fig- 3.3. respectively.

.	1	2	3	4	6	12
1	4	4	4	4	4	1
2	6	6	6	6	6	2
3	4	4	4	4	6	3
4	4	4	4	4	4	4
6	6	6	6	6	6	6
12	4	4	4	4	4	12

Table–3.9



Fig–3.3

Let $\mathcal{F} = \{1, 2, 12\}$, $\mathcal{L} = \{1, 3, 4\}$. be two parametric sets. Define a mapping $\alpha : \mathcal{F} \rightarrow P(S_g)$ and $\beta : \mathcal{L} \rightarrow P(S_g)$ by $\alpha(1) = \{4, 6\}$, $\alpha(2) = \{1, 4, 6\}$, $\alpha(12) =$

$\{1,3,4,6\}$ and $\beta(1)=\{4,6\}$, $\beta(3)=\{1,3,4,6,12\}$, $\beta(4)=\{1,4,6\}$. Then clearly for all $\rho,\sigma\in\mathcal{F},\mathcal{L}$ and $\rho\preccurlyeq\sigma$ implies $\alpha(\rho)\subseteq\alpha(\sigma)$ and $\beta(\rho)\subseteq\beta(\sigma)$. So $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ are two LOISSG over S_g. Then their extended intersection $(\alpha,\mathcal{F})\tilde{\cap}_E(\beta,\mathcal{L})=(\gamma,\Re)$, where $\Re=\mathcal{F}\tilde{\cup}\mathcal{L}=\{1,2,3,4,12\}$ is given by $\gamma(1)=\{4,6\}$, $\gamma(2)=\{1,4,6\}$, $\gamma(3)=\{1,3,4,6,12\}$, $\gamma(4)=\{1,4,6\}$, $\gamma(12)=\{1,3,4,6\}$. As $3\preccurlyeq 12$ but $\gamma(3)\nsubseteq\gamma(12)$. So $(\gamma,\Re)$ is not a LOISSG. Similarly extended union $(\alpha,\mathcal{F})\tilde{\cup}_E(\beta,\mathcal{L})=(\gamma,\Re)$, where $\Re=\mathcal{F}\tilde{\cup}\mathcal{L}=\{1,2,3,4,12\}$ is given by $\gamma(1)=\{4,6\}$, $\gamma(2)=\{1,4,6\}$, $\gamma(3)=\{1,3,4,6,12\}$, $\gamma(4)=\{1,4,6\}$, $\gamma(12)=\{1,3,4,6\}$. As $3\preccurlyeq 12$ but $\gamma(3)\nsubseteq\gamma(12)$. So $(\gamma,\Re)$ is not a LOISSG.

3.3.6 Theorem

Extended union of two L(anti-L)OISSGs $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ is L(anti-L)OISSG if $(\alpha,\mathcal{F})\subseteq(\beta,\mathcal{L})$ or $(\beta,\mathcal{L})\subseteq(\alpha,\mathcal{F})$.

Proof. Let $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ be two LOISSGs over S_g and $(\alpha,\mathcal{F})\subseteq(\beta,\mathcal{L})$, where $\mathcal{F}\subseteq\mathcal{L}$ and $\alpha(\rho)\subseteq\beta(\rho)$, $\forall\ \rho\in\mathcal{F}$. Let $(\alpha,\mathcal{F})\tilde{\cup}_E(\beta,\mathcal{L})=(\gamma,\Re)$, where $\Re=\mathcal{F}\tilde{\cup}\mathcal{L}$ as $\mathcal{F}\subseteq\mathcal{L}$ then $\Re=\mathcal{L}$ this implies that $\gamma(\varsigma)=\beta(\varsigma)\ \forall\ \varsigma\in\Re$. So $(\gamma,\Re)=(\beta,\mathcal{L})$ is LOISSG. Similary the result can be shown for anti-LOISSGs over S_g. ■

3.3.7 Theorem

Basic intersection of two L(anti-L)OISSGs $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ is L(anti-L)OISSG, if it is non null.

Proof. Let $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ be two L(anti-L)OISSGs over S_g. Then by definition $(\alpha,\mathcal{F})\tilde{\cap}_B(\beta,\mathcal{L})=(\gamma,\Re)$, where $\Re=\mathcal{F}\otimes\mathcal{L}$. Then for any $\rho\in\mathcal{F}$, $\sigma\in\mathcal{L}$ and for $(\rho,\sigma)\in\mathcal{F}\otimes\mathcal{L}$, we have $\gamma(\rho,\sigma)=\alpha(\rho)\tilde{\cap}\beta(\sigma)$, where $\alpha(\rho)$ and $\beta(\sigma)$ are BIs of S_g. As $\alpha(\rho)\tilde{\cap}\beta(\sigma)\neq\phi$. As intersection of any numbers of BIs of S_g is BI, so $(\gamma,\Re)$ is BI of

S_g. Since both $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire a partial order from E. Therfore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$ implies $\beta(\sigma_1) \subseteq \beta(\sigma_2)$, $\forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \Re$. Now $\preceq$ is the partial order on $\Re$ which is generated by partial order on $\mathcal{F}$ and $\mathcal{L}$. If $(\rho_1, \sigma_1) \preceq (\rho_2, \sigma_2)$ then $\alpha(\rho_1) \subseteq \alpha(\rho_2)$ and $\beta(\sigma_1) \subseteq \beta(\sigma_2)$ implies $\alpha(\rho_1)\tilde{\cap}\beta(\sigma_1) \subseteq \alpha(\rho_2)\tilde{\cap}\beta(\sigma_2)$ this implies $\gamma(\rho_1, \sigma_1) \subseteq \gamma(\rho_2, \sigma_2)$ for $(\rho_1, \sigma_1) \preceq_{\Re} (\rho_2, \sigma_2)$. Thus $(\alpha, \mathcal{F})\tilde{\cap}_B(\beta, \mathcal{L}) = (\gamma, \Re)$ is a LOISSG over S_g. In the same way the result can be shown for anti-LOISSGs over S_g. ∎

3.3.8 Theorem

Basic union of two L(anti-L)OISSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OISSG.

Proof. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two LOISSGs over S_g. Then $(\alpha, \mathcal{F})\tilde{\cup}_B(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F} \otimes \mathcal{L}$. Then for any $\rho \in \mathcal{F}$, $\sigma \in \mathcal{L}$ and for $(\rho, \sigma) \in \mathcal{F} \otimes \mathcal{L}$, we have $\gamma(\rho, \sigma) = \alpha(\rho)\tilde{\cup}\beta(\sigma)$, where $\alpha(\rho)$ and $\beta(\sigma)$ are BIs of S_g. As $\alpha(\rho)\tilde{\cup}\beta(\sigma) \neq \phi$. As union of any numbers of BIs of S_g is BI, so $(\gamma, \Re)$ is BI of S_g. Since both $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire a partial order from E. Therfore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)\ \forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$ implies $\beta(\sigma_1) \subseteq \beta(\sigma_2)\ \forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \Re$. Now $\preceq$ is the partial order on $\Re$ which is generated by partial order on $\mathcal{F}$ and $\mathcal{L}$. If $(\rho_1, \sigma_1) \preceq (\rho_2, \sigma_2)$ then $\alpha(\rho_1) \subseteq \alpha(\rho_2)$ and $\beta(\sigma_1) \subseteq \beta(\sigma_2)$ implies $\alpha(\rho_1)\tilde{\cup}\beta(\sigma_1) \subseteq \alpha(\rho_2)\tilde{\cup}\beta(\sigma_2)$ this implies $\gamma(\rho_1, \sigma_1) \subseteq \gamma(\rho_2, \sigma_2)$ for $(\rho_1, \sigma_1) \preceq_{\Re} (\rho_2, \sigma_2)$. Thus $(\alpha, \mathcal{F})\tilde{\cup}_B(\beta, \mathcal{L}) = (\gamma, \Re)$ is a LOISSG. In the same way the result can be shown for anti-LOISSGs. ∎

3.4 Properties of Lattice Ordered Quasi Idealistic Soft Semigroup

3.4.1 Definition

Let S_g be a SG with $\mathcal{F} \subseteq S_g$. A non null SS $(\alpha, \mathcal{F})$ over S_g is called L(anti-L)OQISSG over S_g if

1. $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is quasi ideal over S_g.

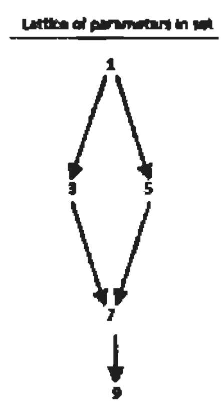
2. $\forall\ \rho_1, \rho_2 \in \mathcal{F}$, with $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)(\alpha(\rho_2) \subseteq \alpha(\rho_1))$.

3.4.2 Example

Let $S_g = \{1, 3, 5, 7, 9\}$ be a SG with the following Cayley Table-3.10 and with the order by the following Hasse diagram shown in Fig-3.4.

.	1	3	5	7	9
1	1	1	1	1	1
3	1	1	1	3	5
5	1	3	5	1	1
7	1	1	1	7	9
9	1	7	9	1	1

Table−3.10



Fig−3.4

Let $\mathcal{F} = \{1, 3\}$ and $\mathcal{L} = \{1, 3, 5\}$ be two parametric sets define a mapping $\alpha : \mathcal{F} \to P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(3) = \{1, 3, 5\}$ and $\beta : \mathcal{L} \to P(S_g)$ by $\beta(1) = \{1, 3\}$, $\beta(3) = \{1, 3, 5\}$, $\beta(5) = \{1, 3, 5, 7, 9\}$. Then $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are LOQISSGs over S_g.

3.4.3 Theorem

Restricted intersection of two L(anti-L)OQISSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OQISSG if it is non null.

Proof. Let S_g be a SG, E be an ordered set of parameter with $\mathcal{F}, \mathcal{L} \subseteq E$. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ be two LOQISSGs over S_g. By definition $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ with $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \emptyset$. Then for $\varsigma \in \Re$, $\gamma(\varsigma) = \alpha(\varsigma)\tilde{\cap}\beta(\varsigma)$. Then results follows by the fact that the intersection of any collection of QIs is QI provided it is non-empty. Since $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire the partial ordered from E. Therefore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$, we have $\alpha(\rho_1) \subseteq \alpha(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$, we have $\beta(\sigma_1) \subseteq \beta(\sigma_2)$, $\forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $\varsigma_1, \varsigma_2 \in \Re$, $\alpha(\varsigma_1) \subseteq \alpha(\varsigma_2)$ and $\beta(\varsigma_1) \subseteq \beta(\varsigma_2)$. Also for $\alpha(\varsigma_1)\tilde{\cap}\beta(\varsigma_1) \subseteq \alpha(\varsigma_2)\tilde{\cap}\beta(\varsigma_2)$ this implies $\gamma(\varsigma_1) \subseteq \gamma(\varsigma_2)$ for $\varsigma_1 \preccurlyeq_{\Re} \varsigma_2$. Thus $(\alpha, \mathcal{F})\tilde{\cap}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ is a LOQISSG. Similary the result can be shown for anti-LOQISSGs. ■

3.4.4 Theorem

Restricted union of two L(anti-L)OQISSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OQISSG if it is non null and either $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$ or $(\beta, \mathcal{L}) \subseteq (\alpha, \mathcal{F})$.

Proof. Let $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$ with $\mathcal{F} \subseteq \mathcal{L}$. By definition $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ with $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \emptyset$ and for any $\varsigma \in \Re$, we encompass $\gamma(\varsigma) = \alpha(\varsigma)\tilde{\cup}\beta(\varsigma)$. Now as $\Re = \mathcal{F}\tilde{\cap}\mathcal{L}$ with $\mathcal{F} \subseteq \mathcal{L}$, then we encompass $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} = \mathcal{F}$.So $\forall\varsigma \in \Re$ implies $\gamma(\varsigma) = \alpha(\varsigma)$ implies $(\gamma, \Re) = (\alpha, \mathcal{F})$ implies $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\alpha, \mathcal{F})$ but $(\alpha, \mathcal{F})$ is LOQISSG it pursues that $(\alpha, \mathcal{F})\tilde{\cup}_R(\beta, \mathcal{L}) = (\gamma, \Re)$ is LOQISSG. Similary the result can be shown for anti-LOQISSGs. ■

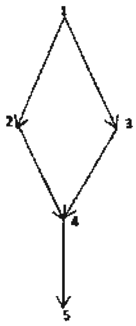
The example given below describes that the extended intersection and extended union of two L(anti-L)OQISSGs may not be a L(anti-L)OQISSG.

3.4.5 Example

Let $S_g = \{1,2,3,4,5\}$ be a SG having Cayley Table-3.11 and with lattice ordered as shown in Fig-3.5

.	1	2	3	4	5
1	1	2	3	4	5
2	2	2	2	2	2
3	3	2	3	3	2
4	4	2	4	4	2
5	5	5	5	5	5

Table–3.11



Fig–3.5

Let $\mathcal{F} = \{1,4\}$, $\mathcal{L} = \{1,2,3\}$. be two parametric sets. Define a mapping $\alpha : \mathcal{F} \to P(S_g)$ and $\beta : \mathcal{L} \to P(S_g)$ by $\alpha(1) = \{2\}$, $\alpha(4) = \{2,5\}$, and $\beta(1) = \{2,3\}$, $\beta(2) = \{2,3,5\}$, $\beta(3) = \{2,3\}$. Then for all $\rho, \sigma \in \mathcal{F}, \mathcal{L}$ and $\rho \preccurlyeq \sigma$ implies $\alpha(\rho) \subseteq \alpha(\sigma)$ and $\beta(\rho) \subseteq \beta(\sigma)$. So $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are two lattice ordered quasi-idealistic soft SG over S_g.

Here $\alpha(1) \subseteq \alpha(4)$ and $\beta(1) \subseteq \beta(2)$, $\beta(1) \subseteq \beta(3)$. Then their extended intersection $(\alpha, \mathcal{F})\tilde{\cap}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L} = \{1,2,3,4\}$ is given by $\gamma(1) = \{2\}$, $\gamma(2) = \{2,3,5\}$, $\gamma(3) = \{2,3\}$, $\gamma(4) = \{2,5\}$. As $2 \preccurlyeq 4$ but $\gamma(2) \nsubseteq \gamma(4)$. So $(\gamma, \Re)$ is not a LOQISSG. Similarly extended union $(\alpha, \mathcal{F})\tilde{\cup}_E(\beta, \mathcal{L}) = (\gamma, \Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L} = \{1,2,3,4\}$ is given by $\gamma(1) = \{2,3\}$, $\gamma(2) = \{2,3,5\}$, $\gamma(3) = \{2,3\}$, $\gamma(4) = \{2,5\}$. As $1 \preccurlyeq 4$ and $2 \preccurlyeq 4$ but $\gamma(1) \nsubseteq \gamma(4)$ and $\gamma(2) \nsubseteq \gamma(4)$. So $(\gamma, \Re)$ is not a LOQISSG.

3.4.6 Theorem

Extended union of two L(anti-L)OQISSGs $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ is L(anti-L)OQISSG if $(\alpha, \mathcal{F}) \subseteq (\beta, \mathcal{L})$ or $(\beta, \mathcal{L}) \subseteq (\alpha, \mathcal{F})$.

Proof. Let $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ be two LOQISSGs over S_g and $(\alpha,\mathcal{F}) \subseteq (\beta,\mathcal{L})$ where $\mathcal{F} \subseteq \mathcal{L}$ and $\alpha(\rho) \subseteq \beta(\rho)$, $\forall\ \rho \in \mathcal{F}$. Let $(\alpha,\mathcal{F})\tilde{\cup}_E(\beta,\mathcal{L}) = (\gamma,\Re)$, where $\Re = \mathcal{F}\tilde{\cup}\mathcal{L}$ as $\mathcal{F} \subseteq \mathcal{L}$ then $\Re = \mathcal{L}$ this implies that $\gamma(\varsigma) = \beta(\varsigma)$, $\forall\ \varsigma \in \Re$. So $(\gamma,\Re) = (\beta,\mathcal{L})$ is LOQISSG. Similary the result can be shown for anti-LOQISSGs. ■

3.4.7 Theorem

Basic intersection of two L(anti-L)OQISSGs is again L(anti-L)OQISSG if it is non null.

Proof. Let $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ be LOQISSGs over S_g. Then by definition $(\alpha,\mathcal{F})\tilde{\cap}_B(\beta,\mathcal{L}) = (\gamma,\Re)$, where $\Re = \mathcal{F} \otimes \mathcal{L}$. we have $\gamma(\rho,\sigma) = \alpha(\rho)\tilde{\cap}\beta(\sigma)$, where $\alpha(\rho)$ and $\beta(\sigma)$ are a QIs of S_g. As $\alpha(\rho)\tilde{\cap}\beta(\sigma) \neq \phi$. As intersection of any numbers of QIs of S_g is QI, so $(\gamma,\Re)$ is QI of S_g Since both $\mathcal{F},\mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire a partial order from E. Therfore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$, $\forall\ \rho_1,\rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$ implies $\beta(\sigma_1) \subseteq \beta(\sigma_2)$, $\forall\ \sigma_1,\sigma_2 \in \mathcal{L}$. Therefore for any $(\rho_1,\sigma_1),(\rho_2,\sigma_2) \in \Re$. Now $\preceq$ is the partial order on $\Re$ which is generated by partial order on $\mathcal{F}$ and $\mathcal{L}$. If $(\rho_1,\sigma_1) \preceq (\rho_2,\sigma_2)$ then $\alpha(\rho_1) \subseteq \alpha(\rho_2)$ and $\beta(\sigma_1) \subseteq \beta(\sigma_2)$ implies $\alpha(\rho_1)\tilde{\cap}\beta(\sigma_1) \subseteq \alpha(\rho_2)\tilde{\cap}\beta(\sigma_2)$ this implies $\gamma(\rho_1,\sigma_1) \subseteq \gamma(\rho_2,\sigma_2)$ for $(\rho_1,\sigma_1) \preceq_{\Re} (\rho_2,\sigma_2)$. Thus $(\alpha,\mathcal{F})\tilde{\cap}_B(\beta,\mathcal{L}) = (\gamma,\Re)$ is LOQISSG. In the same way the result can be shown for anti-LOQISSG. ■

3.4.8 Theorem

Basic union of two L(anti-L)OQISSGs $(\alpha,\mathcal{F})$ and $(\beta,\mathcal{L})$ is L(anti-L)OQISSG if for all $(\rho,\sigma) \in \mathcal{F} \otimes \mathcal{L}$ either $\alpha(\rho) \subseteq \beta(\sigma)$ or $\beta(\sigma) \subseteq \alpha(\rho)$.

Proof. For any $(\rho,\sigma) \in \mathcal{F} \otimes \mathcal{L}$, we consider that $\alpha(\rho) \subseteq \beta(\sigma)$. By definition $(\alpha,\mathcal{F})\tilde{\cup}_B(\beta,\mathcal{L}) = (\gamma,\Re)$, where $\Re = \mathcal{F} \otimes \mathcal{L}$ and for any $(\rho,\sigma) \in \mathcal{F} \otimes \mathcal{L}$ we encompass

$\gamma(\rho,\sigma) = \alpha(\rho)\tilde{\cup}\beta(\sigma)$. As $\alpha(\rho) \subseteq \beta(\sigma)$, so $\alpha(\rho)\tilde{\cup}\beta(\sigma) = \beta(\sigma)$ implies $(\gamma, \Re) = (\beta, \mathcal{L})$, but $(\beta, \mathcal{L})$ is a LOQISSG over a SG S_g so it pursues that $(\alpha, \mathcal{F})\tilde{\cup}_B(\beta, \mathcal{L}) = (\gamma, \Re)$ is a LOQISSG over S_g. In the same way the result can be shown for anti-LOQISSG. ■

3.4.9 Theorem

Let $(\alpha, \mathcal{F})$ be a lattice (anti-lattice) order left(right) idealistic soft semigroup over a SG S_g. Then $(\alpha, \mathcal{F})$ is L(anti-L)OQISSG over S_g.

Proof. Let S_g be a SG, E be an order set of parameter with $\mathcal{F} \subseteq E$. Let $(\alpha, \mathcal{F})$ be a lattice (anti-lattice) order left(right) idealistic soft semigroup over the SG S_g. It means $(\alpha, \mathcal{F})$ contains lattice order in it which means $\forall\ \rho_1, \rho_2 \in \mathcal{F}$ with $\rho_1 \preccurlyeq \rho_2$, implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)$. As $(\alpha, \mathcal{F})$ is left (right) idealistic soft SG over the SG S_g, so $\forall$ $\rho \in \mathcal{F}$, implies $\alpha(\rho)$ is left (right) ideal over S_g. Since every left (right) ideal over S_g is quasi-ideal over S_g. Then this left (right) idealistic soft SG over S_g becomes quasi-idealistic soft SG over S_g. Further $(\alpha, \mathcal{F})$ contains lattice ordered, so $(\alpha, \mathcal{F})$ becomes LOQISSG over S_g. Similary the result can be shown for anti-LOQISSGs over S_g. ■

3.4.10 Theorem

Let $(\mathbb{R}, \mathcal{F})$ be a L(anti-L)ORISSG over a SG S_g and $(\mathbb{Z}, \mathcal{L})$ be a L(anti-L)OLISSG over a SG S_g. Then $(\mathbb{R}, \mathcal{F})\tilde{\cap}_R(\mathbb{Z}, \mathcal{L})$ is a L(anti-L)OQISSG over a SG S_g.

Proof. Let S_g be the SG, E be an order set of parameter with $\mathcal{F}, \mathcal{L} \subseteq E$. Let $(\mathbb{R}, \mathcal{F})$ be a LORISSG over S_g and $(\mathbb{Z}, \mathcal{L})$ be a LOLISSG over S_g. By definition $(\mathbb{R}, \mathcal{F})\tilde{\cap}_R(\mathbb{Z}, \mathcal{L}) = (\gamma, \Re)$ with $\Re = \mathcal{F}\tilde{\cap}\mathcal{L} \neq \emptyset$ Then for $\varsigma \in \Re$, $\gamma(\varsigma) = \mathbb{R}(\varsigma)\tilde{\cap}\mathbb{Z}(\varsigma)$. where $\mathbb{R}(\varsigma)$ is RI over S_g and $\mathbb{Z}(\varsigma)$ is LI over S_g. So in all above cases $\gamma(\varsigma)$ becomes quasi ideal over S_g. Hence $(\gamma, \Re)$ becomes quasi-idealistic soft SG over S_g. Now show that this quasi-idealistic soft SG contains lattice order in it. As $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire the partial ordered from E. Therefore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$ implies $\mathbb{R}(\rho_1) \subseteq \mathbb{R}(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$ we have $\mathbb{Z}(\sigma_1) \subseteq \mathbb{Z}(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $\varsigma_1, \varsigma_2 \in \Re$, $\mathbb{R}(\varsigma_1) \subseteq \mathbb{R}(\varsigma_2)$ and $\mathbb{Z}(\varsigma_1) \subseteq \mathbb{Z}(\varsigma_2)$. Also for $\mathbb{R}(\varsigma_1)\tilde{\cap}\mathbb{Z}(\varsigma_1) \subseteq \mathbb{R}(\varsigma_2)\tilde{\cap}\mathbb{Z}(\varsigma_2)$ this implies $\gamma(\varsigma_1) \subseteq \gamma(\varsigma_2)$ for $\varsigma_1 \preccurlyeq_{\Re} \varsigma_2$. Thus $(\mathbb{R}, \mathcal{F})\tilde{\cap}_R(\mathbb{Z}, \mathcal{L}) = (\gamma, \Re)$ is a LOQISSG. Similary the result can be shown for anti-LOQISSGs. ■

3.4.11 Theorem

Let $(\mathbb{R}, \mathcal{F})$ be a L(anti-L)ORISSG over a SG S_g and $(\mathbb{Z}, \mathcal{L})$ be a L(anti-L)OLISSG over a SG S_g. Then $(\mathbb{R}, \mathcal{F})\tilde{\cap}_B(L, \mathcal{L})$ is a L(anti-L)OQISSG over a SG S_g.

Proof. Let S_g be the SG, E be an order set of parameter with $\mathcal{F}, \mathcal{L} \subseteq E$. Let $(\mathbb{R}, \mathcal{F})$ be a LORISSG over S_g and $(\mathbb{Z}, \mathcal{L})$ be a LOLISSG over S_g. By definition $(\mathbb{R}, \mathcal{F})\tilde{\cap}_B(\mathbb{Z}, \mathcal{L}) = (\gamma, \Re)$,where $\Re = \mathcal{F} \otimes \mathcal{L}$, we have $(\rho, \sigma) \in \Re$ implies $\gamma(\rho, \sigma) = \mathbb{R}(\rho)\tilde{\cap}\mathbb{Z}(\sigma)$. where $\mathbb{R}(\rho)$ is RI over S and $\mathbb{Z}(\sigma)$ is LI over S. So in all above cases $\gamma(\rho, \sigma)$ becomes quasi ideal over S_g. Hence $(\gamma, \Re)$ becomes quasi-idealistic soft SG over S_g. Now show that this quasi-idealistic soft SG contains lattice order in it. As $\mathcal{F}, \mathcal{L} \subseteq E$, so both $\mathcal{F}$ and $\mathcal{L}$ acquire the partial ordered from E. Therefore for any $\rho_1 \preccurlyeq_{\mathcal{F}} \rho_2$ implies $\mathbb{R}(\rho_1) \subseteq \mathbb{R}(\rho_2)$, $\forall\ \rho_1, \rho_2 \in \mathcal{F}$. Also for any $\sigma_1 \preccurlyeq_{\mathcal{L}} \sigma_2$ we have $\mathbb{Z}(\sigma_1) \subseteq \mathbb{Z}(\sigma_2)$, $\forall\ \sigma_1, \sigma_2 \in \mathcal{L}$. Therefore for any $\varsigma_1, \varsigma_2 \in \Re$, $\mathbb{R}(\varsigma_1) \subseteq \mathbb{R}(\varsigma_2)$ and $\mathbb{Z}(\varsigma_1) \subseteq \mathbb{Z}(\varsigma_2)$. Also for $\mathbb{R}(\varsigma_1)\tilde{\cap}\mathbb{Z}(\varsigma_1) \subseteq \mathbb{R}(\varsigma_2)\tilde{\cap}\mathbb{Z}(\varsigma_2)$ this implies $\gamma(\varsigma_1) \subseteq \gamma(\varsigma_2)$ for $\varsigma_1 \preccurlyeq_{\Re} \varsigma_2$. Thus $(\mathbb{R}, \mathcal{F})\tilde{\cap}_B(\mathbb{Z}, \mathcal{L}) = (\gamma, \Re)$ is a LOQISSGs. Similary the result can be shown for anti-LOQISSGs. ■

3.5 Properties of Lattice Ordered Bi-Idealistic Soft Semigroup

3.5.1 Definition

A non null SS $(\alpha, \mathcal{F})$ over a SG S_g is said to be L(anti-L)OBISSG over S_g if

1. $\forall\ \rho \in \mathcal{F}$, $\alpha(\rho)$ is bi-ideal over S_g.

2. $\forall\ \rho_1, \rho_2 \in \mathcal{F}$, with $\rho_1 \preccurlyeq \rho_2$ implies $\alpha(\rho_1) \subseteq \alpha(\rho_2)(\alpha(\rho_2) \subseteq \alpha(\rho_1))$.

3.5.2 Example

Let $S_g = \{1,2,3,4\}$ be a SG with the following cayley Table-3.12 and with ordered $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4$.

.	1	2	3	4
1	1	2	3	4
2	2	3	3	4
3	4	3	4	3
4	4	4	3	4

Table−3.12

Let $\mathcal{F} = \{1,4\}$ and $\mathcal{L} = \{1,2,4\}$ are two parametric sets. Define a mapping $\alpha : \mathcal{F} \to P(S_g)$ by $\alpha(1) = \{3,4\} = \alpha(4)$ and $\beta : \mathcal{L} \to P(S_g)$ by $\beta(1) = \{3,4\}$, $\beta(2) = \{2,3,4\}$, $\beta(4) = \{1,2,3,4\}$. Then $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are LOBISSG over S_g.

3.5.3 Example

Let $S_g = \{1,2,3,4,5,6,7,8\}$ be a SG with the following cayley Table-3.12 and with ordered $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4 \preccurlyeq 5 \preccurlyeq 6 \preccurlyeq 7 \preccurlyeq 8$.

.	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	2	3	4	4	4	4	4
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	5	5	5	5	5	5	5	5
7	5	6	7	8	8	8	8	8
8	8	8	8	8	8	8	8	8

Table–3.12

Let $\mathcal{F} = \{1,3\}$ and $\mathcal{L} = \{1,2,3\}$ are two parametric sets define a mapping $\alpha : \mathcal{F} \to P(S_g)$ by $\alpha(1) = \{1\}$, $\alpha(3) = \{1,8\}$ and $\beta : \mathcal{L} \to P(S_g)$ by $\beta(1) = \{1,5\}$, $\beta(2) = \{1,5,8\}$, $\beta(3) = \{1,5,6,8\}$. Then $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{L})$ are LOBISSGs over S_g.

3.5.4 Theorem

Every LOQISSG over a SG S_g is LOBISSG over a SG S_g.

Proof. Let S_g be a SG, E be an order set of parameter with $\mathcal{F} \subseteq E$. Let $(\alpha, \mathcal{F})$ be a LOQISSG over S_g. it means $(\alpha, \mathcal{F})$ is QISSG that contains latice order. Then $\alpha(\rho)$ is QI of S_g, $\forall\ \rho \in \mathcal{F}$. As every QI of S_g is BI of S_g, so $\alpha(\rho)$ is BI of S_g implies $\alpha(\rho)$ is BISSG over S_g. Further as $(\alpha, \mathcal{F})$ contains lattice ordered so $(\alpha, \mathcal{F})$ becomes LOBISSGs over S_g. ∎

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