Effects of Partial Slip on the Flow of Third Grade Nanofluid with Variable Viscosity: OHAM Solution

By

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Supervised by

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Department of Mathematics and Statistics
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International Islamic University, Islamabad
Pakistan
2011

1. Partial differentiation

2. Total derivatives

بسم الله الرحمن الرحيم

In the name of almighty ALLAH,
the most beneficent, the most merciful

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A Thesis
Submitted in the Partial Fulfillment of the Requirements for the Degree of
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In
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Certificate

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE **MASTRER OF SCIENCE** IN **MATHEMATICS**

We accept this thesis as conforming to the required standard.

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2011

Declaration

I, hereby declare that this dissertation, neither as a whole nor as a part thereof, has been copied out from any source. It is further that I have prepared this dissertation entirely on the basses of my personal effort made under the sincere guidance of my supervisor. No portion of the work, presented in this dissertation, has been submitted in support of any application of any degree or qualification of this or any other university or institute of learning.

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Dedicated to

Holy Prophet Hazrat Muhammad (PBUH)

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Preface

The flows of Non-Newtonian fluids are encountered in many industrial and technology aplications. In various industrial sectors including power generation, chemical production, air condition, transporation and microelectronics, the conventional heat transfer fluid such as air, water, mineral oil and ethylene glycol play an important role. These fluids are incompetent for high flux application such as superconducting magnets, superfast computing, novel supersonic jet aircraft and high power microwave tube due to their low thermal conductivity. Recent advances in nanotechnology and nanoscience have introduced a new type of fluid termed nanofluid [1]. This nanofluid is firstly introduced by Choi [2]. Nanofluids are dilute liquid suspension of nanoparticales with at least one their principals dimensions smaller than 100m [3]. Nanoparticales of various materials have been used for nanofluid production. These include copper, aluminium, copper oxide, alumina and titanic.

From previous investigations, nanofluids have been found to possess enhanced thermo physical properties such as thermal conductivity, thermal diffusivity, viscosity and convective heat transfer coefficients compared to those of base fluids like oil or water. The current review does concentrate on this relatively new class of fluids and not on colloids which are nanofluids because the latter have been used for a long time. Review of experimental studies clearly showed a lack of consistency in the reported results of different research groups regarding thermal properties [4, 5]. The effects of several important factors such as particle size and shapes, clustering of particles, temperature of the fluid, and dissociation of surfactant on the effective thermal conductivity of nanofluids have not been studied adequately. It is important to do more research so as to ascertain the effects of these factors on the thermal conductivity of wide range of nanofluids.

In chapter one, some basic definitions of fluids, fundamentals of fluid flow and basic idea of HAM and OHAM are presented.

Chapter two comprises the study of influence of variable viscosity and viscous dissipation on non-Newtonian flow which is review of Ellahi et al. [6]. This chapter concerns with the effect of constant and variable viscosity on velocity and temperature distributions for a third grade fluid in a pipe. Non-linear governing questions solved by HAM [7-11].

In chapter three, we study about effects of partial slip on the flow of third grade nanofluid with variable viscosity. We consider flow of third grade nanofluid between coaxial cylinders with constant and variable viscosity, to drive the solution of governing nonlinear boundary value problem, we have used one of the most modern perturbation methods, Optimal Homotopic Asymptotic Method (OHAM) [12-13] to find the solution of non-linear problem. The effects of heat transfer analysis on nanoparticles in the presence of nonlinear partial slip are also studied. The physical features of the pertinent parameters are presented in graphical forms.

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Chapter 1

Some Basic Descriptions

1.1 Introduction

In this chapter, some basic definitions and concepts of various types of fluids and fundamentals of fluid flow are discussed. The basic idea of Homotopy Analysis Method (HAM), Optimal Homotopic Asymptotic Method (OHAM) and advantages of Optimal Homotopic Asymptotic Method (OHAM) are also explained.

1.2 Basics of Fluid

1.2.1 Fluid

Fluid is a substance that continually deforms (flows) under an applied shear stress, no matter how small. Fluids are a subset of the phases of matter and include liquids, gases, plasmas and, to some extent, plastic solids.

1.2.2 Pressure

Pressure is an effect which occurs when a force is applied on a surface per unit area.

Mathematically,

$$P = \frac{F}{A}, \tag{1.1}$$

where P is pressure, F is the normal force and A is the area. The SI unit for pressure is Pascal

(Pa), equal to one Newton per square meter (Nm^{-2} or $kgm^{-1}s^{-2}$).

1.2.3 Density

Density is a measure of how much mass is contained in a given unit volume, i.e,

$$\rho = \frac{m}{V}, \tag{1.2}$$

where m is the mass and V is the volume.

1.2.4 Viscosity

Viscosity is a measure of the resistance of a fluid to deformation under shear stress. It is commonly perceived as "thickness", or resistance to pouring. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. Thus, methanol is "thin", having a low viscosity, while vegetable oil is "thick" having a high viscosity.

1.2.5 Eckert Number

The Eckert number is a dimensionless number used in fluid dynamics. It expresses the relationship between a flow's kinetic energy and enthalpy, and is used to characterize dissipation.

$$E_c = \frac{V^2}{c_p \Delta T} = \frac{\text{Kinetic Energy}}{\text{Enthalpy}} \tag{1.3}$$

1.2.6 Prandtl Number

The Prandtl number Pr is a dimensionless number; the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity.

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\text{Viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{c_p \mu}{k} \tag{1.4}$$

1.2.7 Fluid Mechanics

Fluid mechanics is the study of fluids which can be divided into fluid kinematics (the study of fluid motion) and fluid dynamics (the study of the effect of forces on fluid motion) which can further be divided into fluid statics (the study of fluids at rest) and fluid kinetics (the study of fluids in motion).

Fluid Kinematics

Kinematics is the branch of mechanics that deals with quantities involving space and time only. It is used to describe the motions of particles and objects, but does not take the forces that cause these motions into account.

Fluid Dynamics

Fluid dynamics is a sub-discipline of fluid mechanics that deals with fluid flow—the natural science of fluids (liquids and gases) in motion.

Fluid Statics

Fluid statics is the science of fluids at rest, and is a sub-field within fluid mechanics.

1.3 Classification of Fluid

1.3.1 Ideal Fluid

A non-existent, assumed fluid without either viscosity or compressibility is called an ideal fluid or perfect fluid. In nature this type of fluid does not exist. Furthermore, a gas subject to *Boyle's-Charle's law* is called a perfect or an ideal gas. It is the hypothetical form of fluids. However, the fluid with negligible viscosity may be considered as an ideal fluid.

1.3.2 Real Fluid

Real fluids are those in which fluid friction has significant effects on the fluid motion. In otherwords we can not neglect the viscosity effects on the motion. Real fluids are further classified into two classes on the basis of *Newton's law of viscosity*. "Shear stress is directly proportional to the rate of deformation". For one dimensional flow it can be written as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.5}$$

where τ_{yx} is the shear stress and du/dy is the rate of deformation.

1.3.3 Newtonian Fluid

A Newtonian fluid (named after *Isaac Newton*) is a fluid whose stress versus strain (deformation) rate curve is linear and passes through the origin, i.e., Newtonian fluid obeys *Newton's law of viscosity*. Water, gasoline and mercury are some examples of Newtonian fluids.

1.3.4 Non-Newtonian Fluid

A non-Newtonian fluid is a fluid whose flow properties are not described by a single constant value of viscosity, i.e., it does not satisfy *Newton's law of viscosity*. For non-Newtonian fluids

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n, \qquad n \neq 1 \tag{1.6}$$

or

$$\tau_{yx} = \eta\left(\frac{du}{dy}\right), \tag{1.7}$$

where

$$\eta = k\left(\frac{du}{dy}\right)^{n-1} \tag{1.8}$$

is the apparent viscosity. Examples of non-Newtonian fluids are tooth paste, ketchup, gel, shampoo, blood, soaps etc.

1.3.5 Nanofluid

Nanofluid is a fluid containing nanometer-sized particles, called nanoparticles. These fluids are engineered colloidal suspensions of nanoparticles in a base fluid. The nanoparticles used in nanofluids are typically made of metals, oxides, carbides, or carbon nanotubes. Common base fluids include water and ethylene glycol.

1.4 Classification of Fluid Flow–Based on Variation with Time and Space

When a fluid flows past a point or through a path different parameters associated with the flow of the fluid, certain parameters vary and others may remain constant.

The two basic parameters of any fluid flow are velocity of the fluid particle or element and the pressure of the fluid at the point under consideration. The flow of fluids can be classified in different patterns based on the variation of the flow parameters with time and distance.

1.4.1 Classification Based on Variation with Time

The classification of the fluid flow based on the variation of the fluid flow parameters with time characterizes the flow in two categories, steady and unsteady flow.

Steady and Unsteady flow

If the flow parameters, such as velocity, pressure, density and discharge do not vary with time or are independent of time then the flow is steady. If the flow parameters vary with time then the flow is categorized as unsteady.

In real conditions it is very rare to have such flows with parameters exactly constant with time. The parameters usually vary with time but variation is within a small range such as the average of particular parameter is constant for certain duration of time.

1.4.2 Classification Based on Variation with Space

The other classification criterion for the fluid flow is based on the variation of the flow parameters with distance or space. It characterizes the flow as uniform or non-uniform.

Uniform or Non-Uniform flow

The fluid flow is a uniform flow if the flow parameters remain constant with distance along the flow path. The fluid flow is non-uniform if the flow parameters vary and are different at different points on the flow path.

For a uniform flow, by its definition, the area of the cross section of the flow should remain constant. So a fitting example of the uniform flow is the flow of a liquid thorough a pipeline of constant diameter and contrary to this the flow through a pipeline of variable diameter would be necessarily non-uniform.

1.5 Flow Types

A steady flow can be uniform or non-uniform and similarly an unsteady flow can also be uniform or non-uniform. For a steady flow discharge is constant with time and for a uniform flow the area of cross section of the fluid flow is constant through the flow path.

1.5.1 Steady and Uniform Flow

Flow through a pipeline of constant diameter with a discharge constant with time.

1.5.2 Steady and Non-Uniform Flow

Fixed discharge flow through a tapering pipe. Water flow through a river with a constant discharge is also a good example of such flow as the span of river generally varies with distance and amount of water flow in river is constant.

1.5.3 Unsteady and Uniform Flow

A flow through pipeline of constant cross section with sudden changes in fluid discharge or pressure.

1.5.4 Unsteady and Non-Uniform Flow

Pressure surges in a flow through a pipe of variable cross section. A practical example can be the water flow in the network of canals during water release.

1.6 Fluid Energy

A fluid possesses energy in various forms. When applied to a fluid, the first law of thermodynamics relates the change in the internal, kinetic, and potential energies of a mass of fluid to the work done on that fluid plus the heat added to the fluid. Changes in the energy content of a fluid are important in many applications. In some applications a fluid does work (e.g., turbines, windmills, waterwheels), in other applications work is done on the fluid (e.g., pumps, fans, compressors).

1.6.1 Internal Energy

The internal energy of a mass of fluid is a macroscopic measure of microscopic (molecular, atomic, and subatomic) energy content.

1.6.2 Kinetic Energy

The energy associated with fluid in motion is called kinetic energy, E_k. This energy is proportional to the mass of fluid in the system and to the square of the fluid speed V. For a mass of fluid M, the total kinetic energy is given by

$$E_k = \frac{1}{2}MV^2. \tag{1.9}$$

The corresponding kinetic energy per unit mass is

$$e_k = \frac{1}{2}V^2 \tag{1.10}$$

and kinetic energy per unit volume is given by

$$\rho e_k = \rho\frac{1}{2}V^2. \tag{1.11}$$

1.6.3 Potential Energy

A change in the gravitational potential energy of a fluid occurs whenever the fluid moves with, or against, the force of gravity. Suppose we chose a coördinate system with the $z-$ $axis$ vertical. Then the gravitational potential energy, E_G, of a small volume of fluid at height z, relative to the potential energy the volume of fluid has at the origin, is given by

$$E_G = Mgz. \tag{1.12}$$

The potential energy per unit mass is given by

$$e_G = gz \tag{1.13}$$

and the potential energy per unit volume is

$$\rho e_G = \rho gz. \tag{1.14}$$

1.7 Heat Transfer

Heat transfer is a discipline of thermal engineering that concerns the transfer of thermal energy from one physical system to another. Heat transfer is classified into various mechanisms, such as heat conduction, convection, thermal radiation and phase-change transfer.

1.7.1 Conduction

Conduction is the transfer of energy through matter from particle to particle. It is the transfer and distribution of heat energy from atom to atom within a substance. For example, a spoon in a cup of hot soup becomes warmer because the heat from the soup is conducted along the spoon. Conduction is most effective in solids-but it can happen in fluids. Fun fact: Have you ever noticed that metals tend to feel cold? Believe it or not, they are not colder! They only feel colder because they conduct heat away from your hand. You perceive the heat that is leaving your hand as cold.

1.7.2 Convection

Convection is the transfer of heat by the actual movement of the warmed matter. Heat leaves the coffee cup as the currents of steam and air rise. Convection is the transfer of heat energy in a gas or liquid by movement of currents (it can also happen is some solids, like sand). The heat moves with the fluid. Consider this: convection is responsible for making macaroni rise and fall in a pot of heated water. The warmer portions of the water are less dense and therefore, they rise. Meanwhile, the cooler portions of the water fall because they are denser.

1.7.3 Thermal Radiation

Thermal radiation is electromagnetic radiation emitted from all matter due to its possessing thermal energy which is measured by the temperature of the matter. Examples of thermal radiation are an incandescent light bulb emitting visible-light, infrared radiation emitted by a common household radiator or electric heater, as well as radiation from hot gas in outer space.

1.8 Diffusion

Diffusion is one of the fundamental processes by which material moves. It is thus important in biology and medicine, chemistry and geology, engineering and physics, and in just about every aspect of our lives. Diffusion is a consequence of the constant thermal motion of atoms, molecules, and particles, and results in material moving from areas of high to low concentration. Thus the end result of diffusion would be a constant concentration, throughout space, of each of the components in the environment.

1.9 Brownian Diffusion

Particles with a diameter smaller then $1\mu m$ exhibit irregular and random motion because their masses are small enough to render fluctuation by the bombardment of gas molecules. As a result of random motion particles as whole move toward to low concentration region from a high contrition region from high concentration region. This phenomenon in which similar to gas molecules, is referred to as Brownian diffusion of particles.

1.10 Thermophoresis

Thermophoresis, thermodiffusion, or Soret effect, or Ludwig-Soret effect, is a phenomenon observed when a mixture of two or more types of motile particles (particles able to move) are subjected to the force of a temperature gradient and the different types of particles respond to it differently. The term "Sorêt effect" (or Ludwig-Sorêt effect) is normally intended to mean thermophoresis in liquids only. The word "thermophoresis" is most often intended to mean the behavior in aerosols, not liquids, but the broader meaning is also common. The mechanisms of thermophoresis in liquid mixtures differ from those in gas mixtures, and are generally not as well understood.

1.11 Fluid Flow Analysis – Different Approaches

1.11.1 System Approach

A problem is half solved if it is defined properly. Like we use free body diagrams to solve the problems in mechanics, we define a system to solve problems of fluid flows.

A system is defined as a quantity of mass separated from surroundings by system boundaries across which no mass transfer occur. The boundaries of system can be moveable. Basic laws are applied to this system to solve fluid flow problems. This system approach is helpful in analysis of simple flows through channels or pipes where a fixed mass for analysis can be defined and tracked as it flows.

1.11.2 Control Volume Approach

For flows through complex shapes and machines like compressors or turbines it is difficult to define and track a particular mass. Thus, for analysis of flow we define a control volume and study the flow through this volume. It boundaries can coincide with the real physical boundaries of objects or can be imaginary boundaries defined for analysis. Control volume approach can be used to find flow velocities at different ends of the control volume and also can be used for force and motion analysis of the fluid flow.

1.11.3 Differential Approach

The analysis of fluid flow can be done by considering infinitesimal elements of system or control volume. This gives differential equations defining the flow and their solutions provide detailed picture of the flow.

1.11.4 Integral Approach

For overall analysis of the fluids finite elements of system or control volume are considered. It gives integral formulation, which is simple in analysis and gives overall picture of the fluid behavior.

1.11.5 Lagrangian Approach

In Lagrangian approach fluid is considered to be formed of small fluid particles. The motion of these fluid particles is tracked and laws of particle mechanics are applied to them for analysis. With the increasing number of particles analysis becomes cumbersome.

1.11.6 Eulerian Approach

In Eulerian approach properties of fluid flow, such as, velocity, acceleration, pressure and density, are described as function of space and time. This provides a picture of the properties of flow at every point in space as it varies with time. This formulation of the flow field allows detailed mathematically analysis of any flow field.

These basic approaches are equally applicable to all fluid flow problems but Sometimes even in analysis of some simple fluid flow problems closed results cannot be obtained. In such problems numerical and experimental approaches are used.

1.12 Basic Idea of HAM

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[u(r)] = 0, \tag{1.15}$$

where N is a nonlinear operator, r denotes the independent variable, $u(r)$ is an unknown function. By means of generalizing the traditional homotopy method, Liao constructs the so-called zero-order deformation equation

$$(1-p)L[u^*(r;p) - u_0(r)] = p\hbar\left\{N[u^*(r;p)]\right\}, \tag{1.16}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary function, L is an auxiliary linear operator, $u_0(r)$ is an initial guess of $u(r)$ and $u^*(r;p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary objects such as $\hbar$ and L in

HAM. Obviously, when $p = 0$ and $p = 1$, both

$$u^*(r;0) = u_0(r), \qquad u^*(r;1) = u(r) \tag{1.17}$$

hold. Thus as p increases from 0 to 1, the solution $u^*(r;p)$ varies from the initial guess $u_0(r)$ to the solution $u(r)$. Expanding $u^*(r;p)$ in Taylor series with respect to p, one has

$$u^*(r;p) = u_0(r) + \sum_{m=1}^{\infty} u_m(r)p^m, \tag{1.18}$$

where

$$u_m(r) = \frac{1}{m!}\frac{\partial u^*(r;p)}{\partial pm}|p{=}0. \tag{1.19}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then the series Eq. (1.18) converges at $p = 1$ and one can get

$$u^*(r;1) = u_0(r) + \sum_{m=1}^{\infty} u_m(r), \tag{1.20}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao. If $\hbar = -1$, Eq. (1.16) becomes

$$(1-p)L[u^*(r;p) - u_0(r)] + p\{N[u^*(r;p)]\} = 0, \tag{1.21}$$

which is used mostly in the HPM. In view of Eq. (1.19), the governing equations can be deduced from the zero-order deformation Eq. (1.16). We define the vectors

$$u_i = \{u_0(r), u_0(r), \ldots, u_0(r)\}. \tag{1.22}$$

Differentiating Eq. (1.16) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m - \chi_m u_{m-1}] = \hbar Rm(u_{m-1}), \tag{1.23}$$

where

$$
\begin{aligned}
Rm(u_{m-1}) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}\{N[u^*(r;p)]\}}{\partial p^{m-1}}|p=0, \\
\chi_m &= \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}
\end{aligned}
\tag{1.24}
$$

It should be emphasized that $u_m(m \geq 1)$ are governed by the linear Eq. (1.23) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation softwares such as Maple, Matlab and Mathematica etc.

1.13 Basic Idea of OHAM

We apply the OHAM to find the solutions of following differential equation:

$$
\left.\begin{array}{c} L\left[u\left(r\right)\right] + g\left(r\right) + N\left[u\left(r\right)\right] = 0 \\ \\ B\left(u, \frac{du(r)}{dr}\right) = 0 \end{array}\right\},
\tag{1.25}
$$

where $u\left(r\right)$ in unknown function, g is a known function and B is boundary operator.

By means of OHAM we first construct a family of equations

$$
\left.\begin{array}{c} \left(1-p\right)\left[L\left(\varphi\left(r,p\right)\right) + g\left(r\right)\right] = H\left(p\right)\left[\begin{array}{c} L\left(\varphi_1\left(r,p\right)\right) \\ +g\left(r\right) + N\left(\varphi\left(r,p\right)\right) \end{array}\right] \\ \\ B_1\left(\varphi\left(r,p\right), \frac{d\varphi(r,p)}{dr}\right) = 0 \end{array}\right\},
\tag{1.26}
$$

where $r \in R$ and $0 \leq p \leq 1$ is an embedding parameter, $H\left(p\right)$ is a non-zero auxiliary function for $p \neq 0$ and $H\left(0\right) = 0$, $\varphi_i\left(r,p\right)$ $(i = 1-3)$ are unknown functions. Obviously when $p = 0$ and $p = 1$, we have

$$
\varphi_1\left(r,0\right) = u_0\left(r\right), \ \varphi_1\left(r,1\right) = u\left(r\right).
\tag{1.27}
$$

Thus as p increases from 0 to 1, the solution $\varphi\left(r,p\right)$ varies from $\varphi\left(r,0\right)$ to $\varphi\left(r,1\right)$. Now we choose the auxiliary function $H(p)$ in the form

$$H(p) = pK_1 + p^2K_2 + ... \qquad (1.28)$$

where $K_1, K_2, ...$ are constants.

Now we consider, the solution of Eq. (1.26) in form as

$$\varphi(r,p,K_j) = u_0(r) + \sum_{k\geq 1} u_k(r,K_j)\, p^k, \quad j = 1,2,...... \qquad (1.29)$$

Using Eq. (1.29) in Eq. (1.26) and equating the like terms of p we obtain the governing equations of *zeroth*, *first* and $k-$ order equations as follows

Zeroth Order Equation

$$L(u_0(r)) + g(r) = 0, \qquad B\left(u_0, \frac{du_0}{r}\right) = 0. \qquad (1.30)$$

First Order Equation

$$L(u_1(r)) = K_1 N_0(u_0(r), \qquad B\left(u_1, \frac{du_1}{r}\right) = 0. \qquad (1.31)$$

k-th order equation are determined as

$$\left.\begin{array}{c} L(u_k(r) - u_{k-1}(r)) = K_k N_0(u_0(r)) \\ +\displaystyle\sum_{j=1}^{k-1} K_j \left[\begin{array}{c} L(u_{k-j}(r)) \\ +N_{(k-j)}(u_0(r), u_1(r), ...u_{k-j}(r)) \end{array}\right] \\ \\ B\left(u_k, \frac{du_k}{dr}\right) = 0; \qquad k = 2,3,4.......... \end{array}\right\}. \qquad (1.32)$$

In general, the solution of Eq. (1.26) can be determined approximately in the form

$$u^{(m)}(r,p,K_j) = u_0(r) + \sum_{k=1}^{m} u_k(r,K_j). \qquad (1.33)$$

To find the value of K_j, we substitute Eq. (1.33) into Eq. (1.26) and as a result we get the following residual respectively

$$R(r,K_j) = L_1\left(u^{(m)}(r)\right) + g_1(r) + N_1\left(u^{(m)}(r), \theta^{(m)}(r), \phi^{(m)}(r)\right). \tag{1.34}$$

If $R(r,K_j) = 0$, $i = 1,2,3$ then $u^{(m)}(r,K_j)$, happens to be the exact solution. Generally such case will not arise for nonlinear problems, but we can minimize the functional by

$$J(K_j) = \int_a^b R^2(r,K_j)\,dr, \tag{1.35}$$

where a and b are two values, depending on the given problem for locating the desired K_j $(j = 1,2,3,......m)$ and finally the unknown constants K_j can be optimally identified from the conditions

$$\frac{\partial J}{\partial K_j} = 0, \tag{1.36}$$

with these constants known, the approximate solution of order m is well determined now. The constants K_j can be determined in another forms.Suppose if $k_i \in (a,b)$ $(i = 1,2,3,......m)$ then substituting k_i into Eq. (1.28), we obtain the following equation

$$R(k_1,K_j) = R(k_2,K_j) = ... = R(k_m,K_j) = 0. \tag{1.37}$$

1.14 Advantages of OHAM

All these traditional methods can not provide any guarantee for the convergence of approximation series. In 1992 Liao [14] took the lead to apply homotopy, a basic concept in topology to get analytic approximation of nonlinear problem and introduced a non auxiliary parameter to control to convergence region $[15-17]$. Recently, developed a new method namely "Optical Homtopic Asymptotic Method " in the frame of HAM to fin the solution of non-linear problem. A very interesting news is that, OHAM is valid for small and large parameter but also minimizes the residual error which explain its validity and great potential to solve the non-linear problem.

Chapter 2

The Influence of Variable Viscosity and Viscous Dissipation on the Non-Newtonian Flow: An Analytical Solution

2.1 Introduction

In this chapter, we review the work of Ellahi et al. [6]. The governing equations are formulated mathematically. The non-linear governing equations are solved analytically by Homotopy Analysis Method (HAM). In results, impact of different material parameters in the concerned equations is deploy by graphically.

2.2 Mathematical Formulation

Consider the steady, an incompressible, third grade fluid in a pipe. The $z-$axis is taken along the axis of the flow. The velocity field in cylindrical coordinates is given by

$$\mathbf{V} = [0, 0, v(r)]. \tag{2.1}$$

By definition of incompressible fluids, the continuity Eq. (1.18) is

$$\nabla \cdot \mathbf{V} = 0. \tag{2.2}$$

Energy equation is

$$\rho c_p \frac{D\theta}{Dt} = \mathbf{T} \cdot \mathbf{L} + k\nabla^2\theta. \tag{2.3}$$

For third grade fluid stress tenser is defined by

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \tag{2.4}$$

where p_1 is hydrostatic pressure, $\mathbf{I}$ is the identity tensor and $\alpha_i(i = 1, 2)$ and $\beta_j(j = 1, 2)$ are material constants. The *Rivlin-Ericksen* tensors are defined by the following general relations

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^t, \tag{2.5}$$

$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^t\mathbf{A}_{n-1}, \quad n > 1. \tag{2.6}$$

Thermodynamical limitations [18] comprise

$$\mu \geq 0, \ \alpha_1 \geq 0, \ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \ \beta_1 = \beta_2 = 0, \ \beta_3 \geq 0. \tag{2.7}$$

In view of Eq. (2.7), Eq. (2.4), we have

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{2.8}$$

Using the velocity field given in Eq. (2.1), we obtain

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{dv}{dr} & 0 & 0 \end{bmatrix}, \qquad \mathbf{L}^t = \begin{bmatrix} 0 & 0 & \frac{dv}{dr} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.9}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^t = \begin{bmatrix} 0 & 0 & \frac{dv}{dr} \\ 0 & 0 & 0 \\ \frac{dv}{dr} & 0 & 0 \end{bmatrix}, \tag{2.10}$$

$$\mathbf{A}_1\mathbf{L} + \mathbf{L}^t\mathbf{A}_1 = \begin{bmatrix} 2\left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.11}$$

For steady flow

$$\frac{\partial \mathbf{A}_1}{\partial t} = 0, \tag{2.12}$$

so

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^t\mathbf{A}_1 = \begin{bmatrix} 2\left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.13}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{dv}{dr}\right)^2 \end{bmatrix}, \tag{2.14}$$

$$tr(\mathbf{A}_1^2)\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 2\left(\frac{dv}{dr}\right)^3 \\ 0 & 0 & 0 \\ 2\left(\frac{dv}{dr}\right)^3 & 0 & 0 \end{bmatrix}, \tag{2.15}$$

$$\left.\begin{array}{c} \tau_{rr} = -p_1 + (2\alpha_1 + \alpha_2)\left(\frac{dv}{dr}\right)^2, \ \tau_{r\theta} = 0 = \tau_{\theta r}, \\ \tau_{rz} = \mu\frac{dv}{dr} + 2\beta_3\left(\frac{dv}{dr}\right)^3 = \tau_{zr}, \tau_{\theta\theta} = -p_1, \\ \tau_{\theta z} = 0 = \tau_{z\theta}, \ \tau_{zz} = -p_1 + \alpha_2\left(\frac{dv}{dr}\right)^2. \end{array}\right\}. \tag{2.16}$$

In the absence of body forces and using cylindrical coordinates (for the flow in a pipe), the momentum Eq. (1.19) will be in simplified form as

$$\frac{1}{r}\frac{d}{dr}\left[r\mu\left(\frac{dv}{dr}\right)\right] + \frac{2\beta_3}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right] = \frac{\partial \hat{p}}{\partial z}, \tag{2.17}$$

subject to the boundary conditions

$$v(R) = 0, \qquad \frac{dv}{dr}(0) = 0, \tag{2.18}$$

where

$$\hat{p} = p_1 - \alpha_2 \left(\frac{dv}{dr}\right)^2. \tag{2.19}$$

is the modified pressure. Now using the definition of product of two tensors, we have

$$\mathbf{T.L} = \tau_{zr}\frac{dv}{dr}, \tag{2.20}$$

$$\mathbf{T.L} = \mu\left(\frac{dv}{dr}\right)^2 + 2\beta_3\left(\frac{dv}{dr}\right)^4 \tag{2.21}$$

$$\nabla^2\theta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right). \tag{2.22}$$

The energy Eq. (2.3) becomes

$$\mu\left(\frac{dv}{dr}\right)^2 + 2\beta_3\left(\frac{dv}{dr}\right)^4 + k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right] = 0. \tag{2.23}$$

The relating boundary conditions are

$$\theta(R) = 0, \qquad \frac{d\theta}{dr}(0) = 0. \tag{2.24}$$

Using non-dimensionalization criteria, we set

$$v = \frac{\overline{v}}{v_0},\ r = \frac{\overline{r}}{R},\ \mu = \frac{\overline{\mu}}{\mu_0},\ \theta = \frac{\overline{\theta} - \theta_0}{\theta_1 - \theta_0} \tag{2.25}$$

$$c_1 = \frac{\partial\hat{p}}{\partial z},\ c = \frac{c_1 R^2}{v_0\mu_0},\ \Lambda = \frac{2\beta_3 v_0^2}{\mu_0 R^2},\ \Gamma = \frac{\mu_0 v_0^2}{k(\theta_1 - \theta_0)} \tag{2.26}$$

and the boundary value problems consisting of Eqs. (2.17), (2.18), (2.23) and (2.24) become

$$\frac{1}{r}\frac{d}{dr}\left[r\mu\left(\frac{dv}{dr}\right)\right] + \frac{\Lambda}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right] = c, \tag{2.27}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right] = 0, \tag{2.28}$$

$$v(1) = \theta(1) = 0, \qquad \frac{dv}{dr}(0) = \frac{d\theta}{dr}(0) = 0, \tag{2.29}$$

in which R, v_0, μ_0, θ_0, $\overline{\theta}$ and θ_1 are the radius, reference velocity, reference viscosity, reference temperature, pipe and fluid temperatures, respectively. Also, c_1 is the axial pressure drop, Λ is third grade parameter and Γ is related to the Prandtl and Eckert numbers. For simplicity we have omitted the bar symbols.

2.3 Solution of the Problem

We use homotopy analysis method (HAM) to solve the problem under consideration.

Case I: For the constant viscosity

When we take $\mu = 1$, the governing Eqs. (2.27) and (2.28) in simplified form reduce to

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + 3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 = c \tag{2.30}$$

and

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2 + \Lambda\Gamma\left(\frac{dv}{dr}\right)^4 = 0, \tag{2.31}$$

respectively. We use the method of higher order differential mapping [19], to choose the linear operator $\mathcal{L}$, i.e.,

$$\mathcal{L} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}. \tag{2.32}$$

The above operator satisfies the following relation

$$\mathcal{L}[C_1 + C_2 \ln r] = 0. \tag{2.33}$$

Here C_1 and C_2 are the arbitrary constants. Integrating the linear part of Eq. (2.30), we get

$$v_0(r) = \frac{1}{4}c(r^2 - 1), \tag{2.34}$$

as the initial approximation of velocity v, which satisfies the linear operator $\mathcal{L}$ and boundary conditions too.

Zeroth order deformation equation

For non-zero auxiliary parameter $\hbar$ and an embedding parameter $p \in [0,1]$, the zeroth order deformation equation in HAM is given by the following relation

$$(1-p)\mathcal{L}[v^*(r,p)-v_0(r)] = p\hbar \left[\begin{array}{c} \frac{d^2v^*}{dr^2} + \frac{1}{r}\frac{dv^*}{dr} + 3\Lambda\left(\frac{dv^*}{dr}\right)^2 \frac{d^2v^*}{dr^2} \\ +\frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 - c \end{array} \right], \tag{2.35}$$

subject to the following boundary conditions

$$v^*(1,p) = 0, \qquad \frac{dv^*}{dr}(0,p) = 0. \tag{2.36}$$

mth order deformation equation

If we differentiate $m-$times the zeroth order deformation Eqs. (2.35) and (2.36) with respect to p, dividing by $m!$ and finally taking $p=0$, we have the mth order deformation equation, of the following form

$$\mathcal{L}\left[v_m - \chi_m v_{m-1}\right] = \hbar R_m(r), \tag{2.37}$$

where

$$R_m(r) = v''_{m-1} + \frac{1}{r}v'_{m-1} + \Lambda\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{j=0}^{k} v'_{k-j}\left(\frac{1}{r}v'_j + 3v''_j\right) - c(1-\chi_m). \tag{2.38}$$

Corresponding boundary conditions take the following form

$$v'_m(0) = v_m(1) = 0, \tag{2.39}$$

where prime denotes the differentiation with respect to r. From Eq. (2.35) by setting $p=0$, it can be shown that

$$v^*(r,p) = v_0(r). \tag{2.40}$$

By the definition of homotopy, as p varies from 0 to 1, $v^*(r,p)$ varies from initial guess $v_0(r)$ to the exact solution $v(r)$, that is for properly chosen $\hbar$, we get

$$v^*(r,p) = v(r) \qquad \text{for } p = 1. \tag{2.41}$$

Then employing the Taylor's theorem, we can write

$$v^*(r,p) = v_0(r) + \sum_{m=1}^{\infty} v_m(r)p^m, \tag{2.42}$$

where

$$v_m(r) = \frac{1}{m!}\frac{\partial^m v^*(r,p)}{\partial p^m}\bigg|_{p=0}. \tag{2.43}$$

Now using Eq. (2.41) in Eq. (2.42), we get

$$v(r) = v_0(r) + \sum_{m=1}^{\infty} v_m(r). \tag{2.44}$$

Differentiate Eq. (2.35) with respect to p and set $p = 0$, then after solving the resulting equation we obtain the following

$$v_1(r) = \frac{1}{32}\hbar\Lambda c^3(r^4 - 1). \tag{2.45}$$

Again differentiating Eq. (2.35) with respect to p, putting $p = 0$ and using the similar procedure, we get

$$v_2(r) = \frac{1}{32}\hbar\Lambda c^3(r^4 - 1)(\hbar + 1) + \frac{1}{64}\hbar^2\Lambda^2 c^5\left(r^6 - 1\right). \tag{2.46}$$

Now from Taylor series, we have the three terms solution as

$$v(r) = v_0(r) + v_1(r) + v_2(r). \tag{2.47}$$

Finally, inserting Eqs. (2.34), (2.46) and (2.47) in Eq. (2.48), we get the expression for velocity as follows

$$v(r) = \frac{1}{4}c(r^2 - 1) + \frac{1}{32}\hbar\Lambda c^3(r^4 - 1)(\hbar + 2) + \frac{1}{64}\hbar^2\Lambda^2 c^5(r^6 - 1). \tag{2.48}$$

Now using Eqs. (2.31) and (2.49), with boundary conditions Eq. (2.29), we can find θ by using

Cauchy-Euler equation and computer software, 'Mathematica'. The result is given below

$$\left.\begin{array}{c}\theta(r)=A_1(r^4-1)+A_2(r^6-1)+A_3(r^8-1)+A_4(r^{10}-1)\\ +A_5(r^{12}-1)+A_6(r^{14}-1)+A_7(r^{16}-1)+A_8(r^{18}-1)\\ +A_9(r^{20}-1)+A_{10}(r^{22}-1)\end{array}\right\}. \tag{2.49}$$

The calculated values of coefficients $A_i(i=1,2,...10)$ are given in Appendix A.

Case II: For the variable viscosity

Let us now assume that the viscosity is space dependent and choose $\mu=r$. From Eq. (2.26), we have

$$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right)+\frac{\Lambda}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right]=c, \tag{2.51}$$

$$r\frac{d^2v}{dr^2}+2\frac{dv}{dr}+3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3=c, \tag{2.52}$$

$$\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}+\frac{3\Lambda}{r}\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}+\frac{\Lambda}{r^2}\left(\frac{dv}{dr}\right)^3=\frac{c}{r}, \tag{2.53}$$

with boundary conditions Eq. (2.29). Similarly Eq. (2.28) simplifies to

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\left(\frac{d\theta}{dr}\right)+\Gamma r\left(\frac{dv}{dr}\right)^2+\Gamma\Lambda\left(\frac{dv}{dr}\right)^4=0, \tag{2.54}$$

which corresponds to Eq. (2.26). The linear operator in this case will be

$$\mathcal{L}_1=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}, \tag{2.55}$$

provided that

$$\mathcal{L}_1\left[C_3+\frac{C_4}{r}\right]=0, \tag{2.56}$$

where C_3 and C_4 are constants of integration. Thus the initial approximation for the velocity v is

$$v_0(r)=\frac{1}{6}c(r^2-1). \tag{2.57}$$

With the use of Eq. (2.53), one can define the zeroth order deformation equation for v as

$$(1-p)\mathcal{L}_1[v^*(r,p)-v_0(r)] = p\hbar \left[\begin{array}{c} \frac{d^2v^*}{dr^2} + \frac{2}{r}\frac{dv^*}{dr} + \frac{3\Lambda}{r}\left(\frac{dv^*}{dr}\right)^2 \frac{d^2v^*}{dr^2} \\ +\frac{\Lambda}{r^2}\left(\frac{dv^*}{dr}\right)^3 - \frac{c}{r} \end{array} \right]. \tag{2.58}$$

Likewise the mth order deformation problem can be written as

$$\mathcal{L}\left[v_m - \chi_m v_{m-1}\right] = \hbar R_m(r), \tag{2.59}$$

where

$$R_m(r) = v''_{m-1} + \frac{2}{r}v'_{m-1} + \Lambda \sum_{k=0}^{m-1} v'_{m-1-k} \sum_{j=0}^{k} v'_{k-j}\left(\frac{1}{r^2}v'_j + 3v''_j\right) - c(1-\chi_m). \tag{2.60}$$

and boundary conditions will be same as in Eq. (2.36). The expression for θ can also be defined in the same manner. The mth order deformation equation can be obtained by using similar procedure like that of given in case I. Following the same procedure, we find three terms series solution of v as follows

$$\left. \begin{array}{c} v(r) = \frac{1}{6}c(r^2-1) + \frac{1}{6}\hbar c(r^2-1)(\hbar+2) + \frac{2}{81}\hbar\Lambda c^3(2\hbar+1)(r^3-1) \\ -\frac{1}{2}\hbar c(\hbar+2)(r-1) + \frac{1}{324}\hbar^2\Lambda^2c^5(r^4-1) - \frac{1}{12}\hbar^2\Lambda c^3(r^2-1) \end{array} \right\}, \tag{1.61}$$

For finding the solution of temperature θ, we use 'Mathematica' to solve the Cauchy-Euler equation. Then we obtain

$$\left. \begin{array}{c} \theta(r) = A_{11}(r^2-1) + A_{12}(r^3-1) + A_{13}(r^4-1) + A_{14}(r^5-1) + A_{15}(r^6-1) + \\ A_{16}(r^7-1) + A_{17}(r^8-1) + A_{18}(r^9-1) + A_{19}(r^{10}-1) + A_{20}(r^{11}-1) + \\ A_{21}(r^{12}-1) + A_{22}(r^{13}-1) + A_{23}(r^{14}-1) \end{array} \right\}, \tag{1.62}$$

where the coefficients $A_j(j = 11, 12, ...23)$ are given in Appendix A.

2.4 Graphs

In this section, we will discuss the results of velocity and temperature profiles for both constant and variable viscosity with the help of graphs.

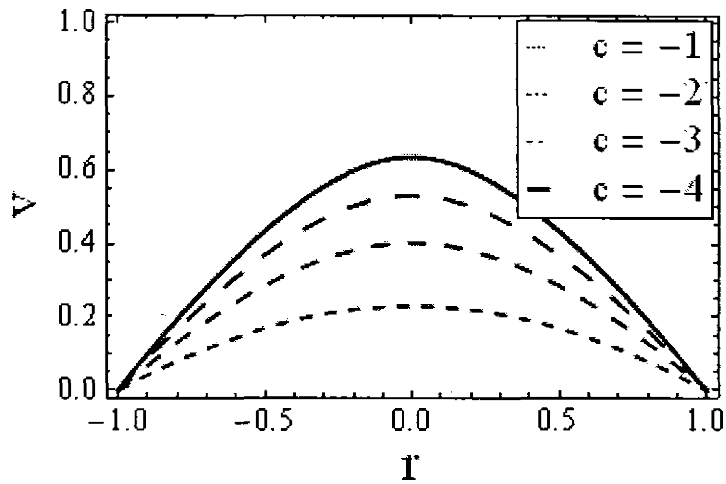

Fig. 2.1 :Influence of c on velocity when $\Lambda = 1$ and $\Gamma = 1$.

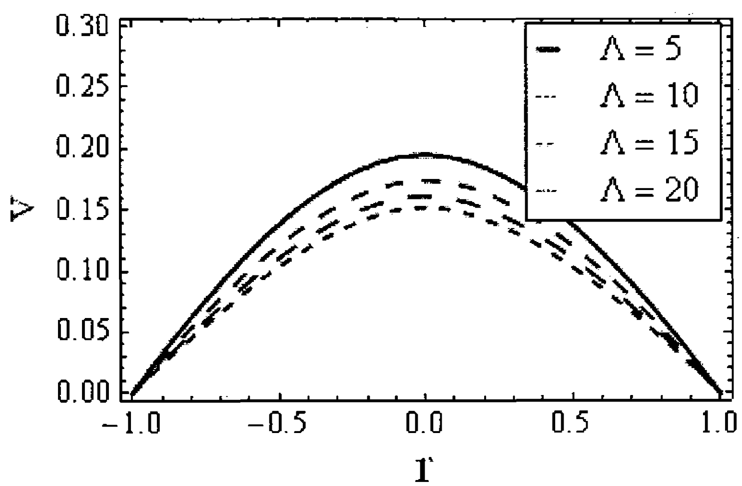

Fig. 2.2 :Influence of Λ on velocity when $c = -1$ and $\Gamma = 1$.

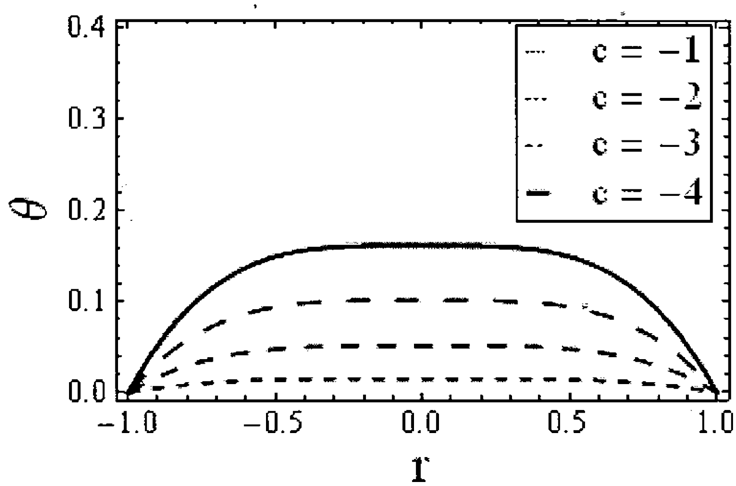

Fig. 2.3 :Influence of c on temperature when $\Gamma = 1$ and $\Lambda = 1$.

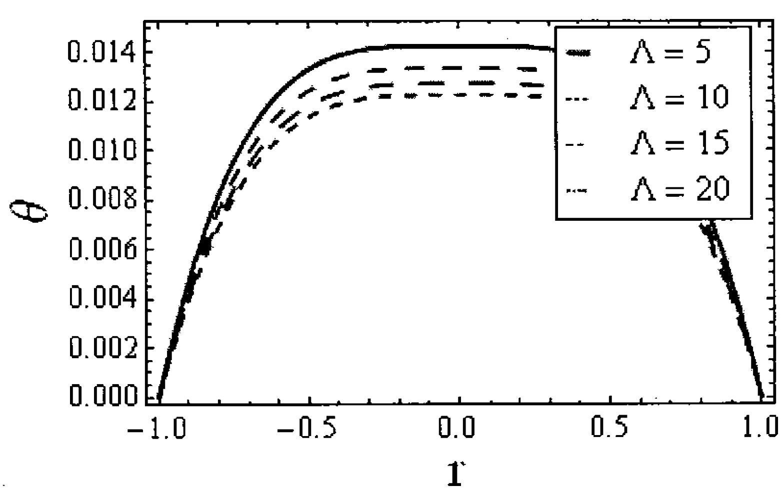

Fig. 2.4 :Influence of Λ on temperature when $c = -1$and $\Gamma = 1$.

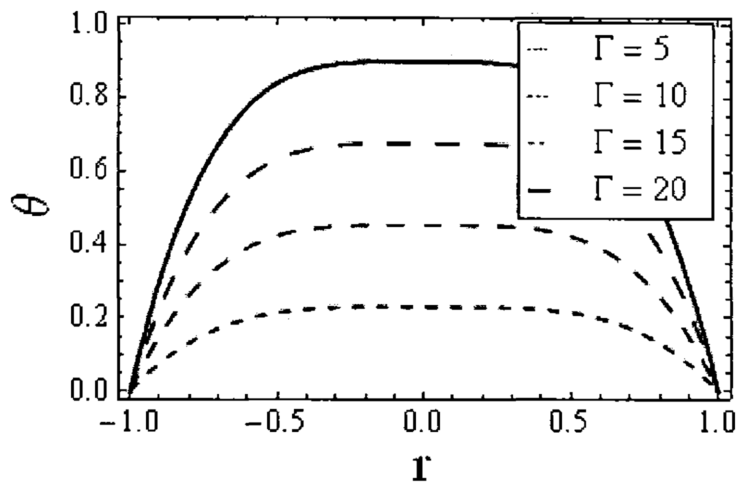

Fig. 2.5 :Influence of Γ on temperature when $c = -1$ and $\Lambda = 1$.

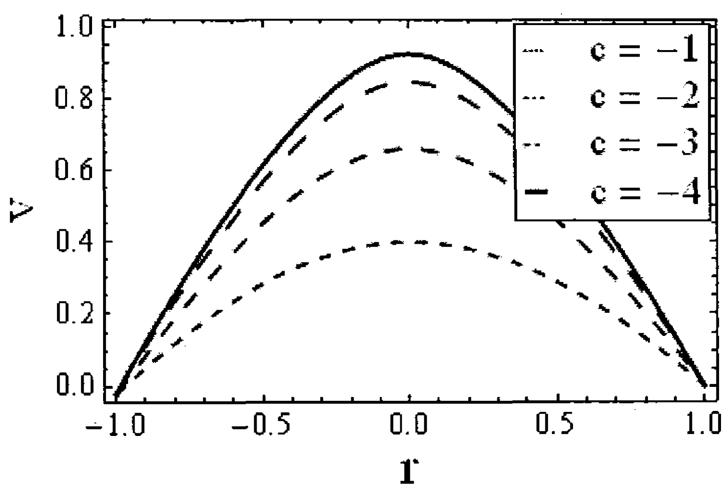

Fig. 2.6 :Influence of c on velocity when $\Lambda = 1$ and $\Gamma = 1$.

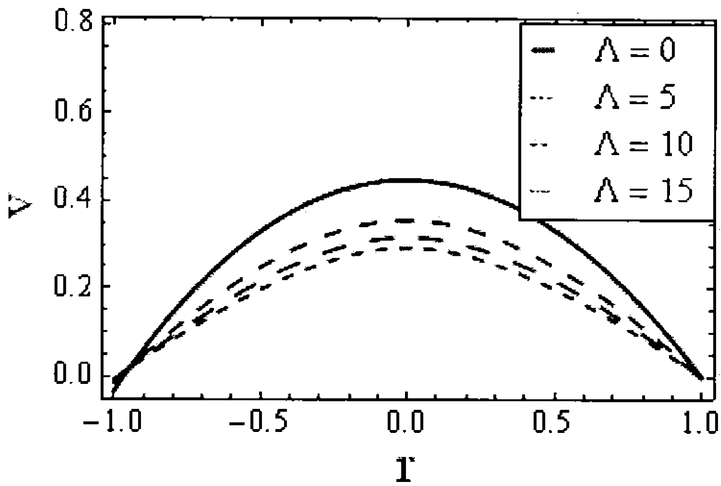

Fig. 2.7 :Influence of Λ on velocity when c = −1 and Λ = 1.

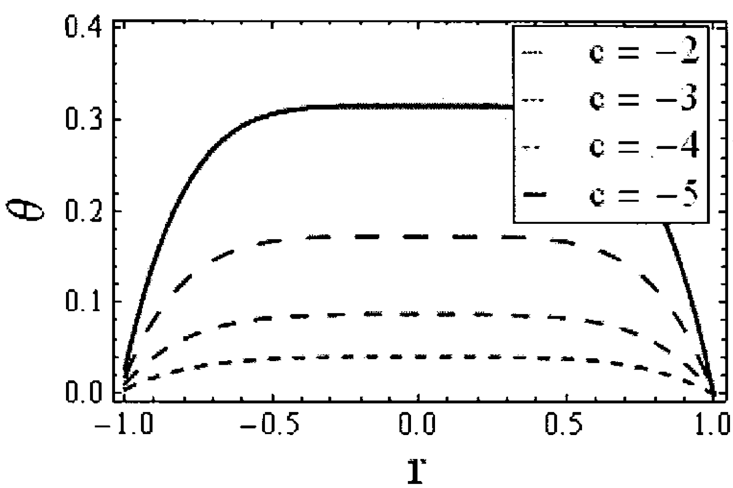

Fig. 2.8 :Influence of c on temperature when Γ = 1 and Λ = 1.

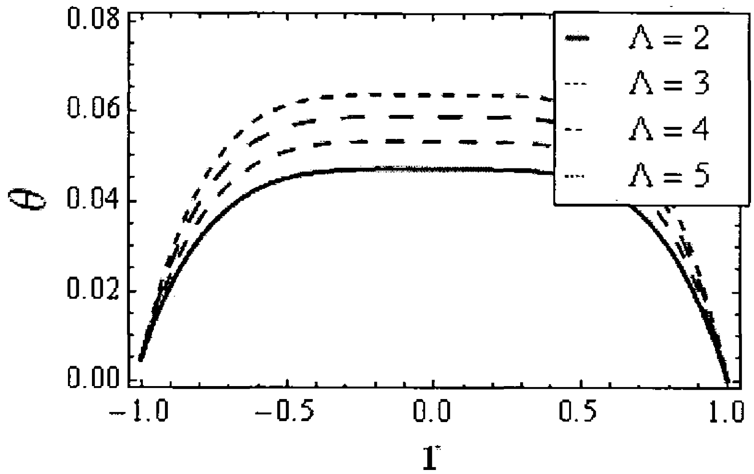

Fig. 2.9 :Influence of Λ on temperature when $\Gamma = 1$ and $c = -2$.

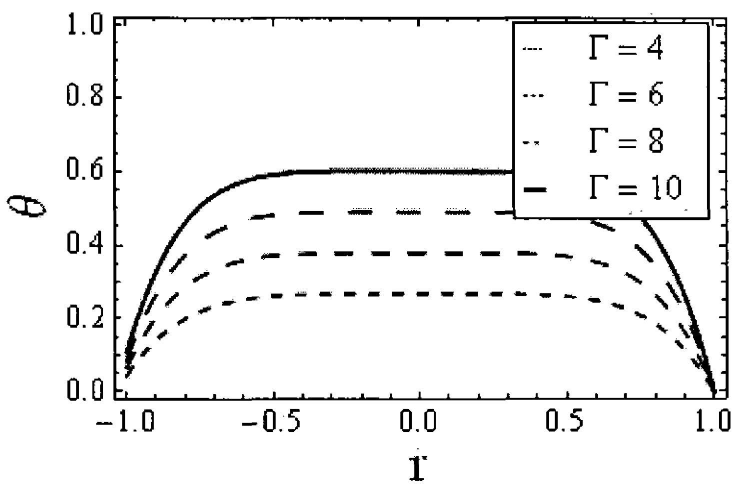

Fig. 2.10 :Influence of F on temperature when $c = -3$ and $\Lambda = 1.5$.

2.5 Results and Discussion

As mentioned above, the solution for the velocity and temperature distributions are plotted against the pipe radius. Figs. 2.1 to 2.5 show the variation of velocity and temperature profiles for constant viscosity case and for space dependent viscosity, Figs. 2.6 to 2.10 are presented. In these figures, the variation of the velocity v and temperature θ with the emerging parameters Λ, c and F is revealed.

In Fig. 2.1, the effect of pressure gradient c is depicted (when $\hbar$ is approximately equal to -0.05). It is clear that the velocity approaches its maximums at the center of the pipe and varies inversely with c. Also, the effect of c on θ (in Fig. 2.3) is similar to that of velocity. The effect of third grade parameter Λ on the velocity and temperature distributions are shown in Figs. 2.2 and 2.4 respectively. As expected, an increase in Λ results in a decrease in both velocity and temperature. However, the temperature profile is more flatter than the velocity profile for same values of Λ. Fig. 2.5 illustrates the effect of the parameter Γ on temperature distribution θ. It is concluded that θ increases with the increase of F and hence the thermal boundary layer thickness decreases.

So far, we disclosed the results of the velocity and temperature for constant viscosity model. Now we turn our consideration to the discussion of above mentioned parameters for space dependent viscosity. Figs. 2.6 to 2.10 represent the influence of all dealing parameters (c, Λ and Γ) on both, velocity and temperature solutions when viscosity is depending upon space. From these figures, it is observed that the impact of c, Λ and Γ on v and θ (when $\hbar$ is nearly equal to -0.01) is similar to that of constant viscosity case.

Chapter 3

Effects of Partial Slip on the Flow of Third Grade Nanofluid with Variable Viscosity

3.1 Introduction

Consider the steady, an incompressible third grade nano-fluid in coaxial cylinder. The following four field equations embody the conservation of total mass, momentum, thermal energy and nanoparticles, respectively. The field variables are the velocity $\mathbf{V}$, the temperature θ, and the nanoparticle volume fraction ϕ.

$$\boldsymbol{\nabla}.\mathbf{V} = 0, \tag{3.1}$$

$$\rho_f\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}\mathbf{V}\right) = \operatorname{div}\mathbf{T} + \left[\phi\rho_p + (1-\phi)\left\{\rho_f\left[1-\beta_T\left(\theta-\theta_w\right)\right]\right\}\right]g, \tag{3.2}$$

$$(\rho c)_f\left(\frac{\partial \theta}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}\theta\right) = k\boldsymbol{\nabla}^2\theta + (\rho c)_p\left[D_b\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\theta + \frac{D_T}{\theta_w}\nabla\theta.\nabla\theta\right], \tag{3.3}$$

$$\left(\frac{\partial \phi}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}\phi\right) = D_b\boldsymbol{\nabla}^2\phi + \frac{D_T}{\theta_w}\boldsymbol{\nabla}^2\theta. \tag{3.4}$$

Where

$$\mathbf{V} = \mathbf{V}\left[0, 0, u\left(r\right)\right].$$

Here ρ_f is the density of the base fluid and μ, k, β_T, are the viscosity,thermal conductivity, volumetric thermal expansion coefficient and volumetric solutal expansion coefficient of the nanofluid, while ρ_p is the density of the nanoparticles. The gravitational acceleration is denoted by g. The coefficients that appear in Eqs. (3.3) and (3.4) are the Brownian diffusion coefficient D_b, the thermophoretic diffusion coefficient D_T.

For third grade fluid stress tenser is defined by

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \tag{3.5}$$

Thermodynamical limitations comprise

$$\mu \geq 0,\ \alpha_1 \geq 0,\ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},\ \beta_1 = \beta_2 = 0,\ \beta_3 \geq 0. \tag{3.6}$$

we have

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{3.7}$$

The boundary conditions are taken to be

$$\begin{aligned} u(R_1) &= u_0 + \gamma\left[\frac{du}{dr}(R_1) + \frac{2\beta_3}{\mu}\left(\frac{du}{dr}(R_1)\right)^3\right], \\ u(R_2) &= \gamma\left[\frac{du}{dr}(R_2) + \frac{2\beta_3}{\mu}\left(\frac{du}{dr}(R_2)\right)^3\right], \\ \theta(R_1) &= \theta_m,\ \theta(R_2) = \theta_w\ ;\ \phi(R_1) = \phi_m,\ \phi(R_2) = \phi_w. \end{aligned} \tag{3.8}$$

Using non-dimensionalization criteria, we get the nonlinear governing equations of the form

$$\begin{aligned} &\frac{d\mu}{dr}\frac{du}{dr} + \frac{\mu}{r}\frac{du}{dr} + \mu\frac{d^2u}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 + 3\Lambda\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2} \\ &= c - G_r\theta - B_r\phi, \end{aligned} \tag{3.9}$$

$$\alpha\left(\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr}\right) + N_b\frac{d\theta}{dr}\frac{d\phi}{dr} + N_t\left(\frac{d\theta}{dr}\right)^2 = 0, \tag{3.10}$$

$$N_b\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}\right)+N_t\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=0, \tag{3.11}$$

subject to boundary conditions

$$\begin{aligned} u(1) &= 1+\gamma\left[\frac{du}{dr}(1)+\frac{\Lambda}{\mu}\left(\frac{du}{dr}(1)\right)^3\right], \\ u(k) &= \gamma\left[\frac{du}{dr}(k)+\frac{\Lambda}{\mu}\left(\frac{du}{dr}(k)\right)^3\right], \\ \theta(1) &= 1,\ \theta(k)=0;\ \phi(1)=1,\ \phi(k)=0. \end{aligned} \tag{3.12}$$

The non-dimensional quantities are defined by the following relations

$$\left.\begin{array}{c} \overline{u}=\frac{u}{u_0},\ \overline{r}=\frac{r}{R},\ \overline{\mu}=\frac{\mu}{\mu_0},\ \overline{\theta}=\frac{\theta-\theta_0}{\theta_1-\theta_0},\ c_1=\frac{\partial\hat{p}}{\partial z},\ c=\frac{c_1R^2}{v_0\mu_0}, \\ \overline{\phi}=\dfrac{\phi-\phi_w}{\phi_m-\phi_w},\ N_b=D_b\left(\phi_m-\phi_w\right),\ N_t=\frac{D_T(\theta_m-\theta_w)}{\theta_w}, \\ \Lambda=\frac{2\beta_3v_0^2}{\mu_0R^2},\ G_r=\frac{(\theta_m-\theta_w)\rho_{fw}R^2(1-\phi_w)g}{\mu_0u_0},\ B_r=\frac{(\rho_p-\rho_w)R^2(\phi_m-\phi_w)g}{\mu_0u_0} \end{array}\right\} \tag{3.13}$$

Here G_r, B_r, N_t and N_b are thermophoresis diffusion constant, Brownian diffusion constant, thermophoresis parameter and Brownian diffusion coefficient respectively.

3.2 Solution of the Problem

In this section, we discuss two models of viscosity namely; constant and variable viscosity for third grade nanofluid in absent of pressure garden. By using Optimal Homotopic Asymptotic Method (OHAM), we find the series solutions of the nonlinear governing equations

Case I: For constant viscosity model, we choose

$$\mu=1, \tag{3.14}$$

$$\mathcal{L}_i\left(\varphi_i\left(r,p\right)\right)=\frac{\partial^2\varphi_i\left(r,p\right)}{\partial r^2}, \tag{3.15}$$

$$\begin{aligned} \mathcal{N}_1\left[\varphi_1\left(r,p\right)\right] &= \frac{1}{r}\frac{du}{dr}+\frac{\Lambda}{r}(\frac{du}{dr})^3+3\Lambda(\frac{du}{dr})^2\frac{d^2u}{dr^2}+G_r\theta \\ &\quad +B_r\phi, \end{aligned} \tag{3.16}$$

$$\mathcal{N}_2[\varphi_2(r,p)] = \frac{1}{r}\frac{d\theta}{dr} + \frac{N_b}{\alpha}\frac{d\theta}{dr}\frac{d\phi}{dr} + \frac{N_t}{\alpha}\left(\frac{d\theta}{dr}\right)^2, \tag{3.17}$$

and

$$\mathcal{N}_3[\varphi_3(r,p)] = \frac{1}{r}\frac{d\phi}{dr} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta}{dr} + \frac{d^2\theta}{dr^2}\right). \tag{3.18}$$

The corresponding conditions are

$$\begin{aligned}
\varphi_1(1) &= 1+\gamma\left(\frac{d\varphi_1}{dr}(1) + \Lambda\left(\frac{d\varphi_1}{dr}(1)\right)^3\right), \\
\varphi_1(k) &= \gamma\left(\frac{d\varphi_1}{dr}(k) + \Lambda\left(\frac{du}{dr}(k)\right)^3\right), \\
\varphi_2(1) &= 1,\ \varphi_2(k)=0;\ \varphi_3(1)=1,\ \varphi_3(k)=0.
\end{aligned} \tag{3.19}$$

The **zeroth-order** deformation problems are of the form

$$\begin{aligned}
\frac{d^2u_0}{dr^2} &= 0, \\
u_0(1) &= 1+\gamma\left(\frac{du_0}{dr}(1) + \frac{\Lambda}{\mu}\left(\frac{du_0}{dr}(1)\right)^3\right), \\
u_0(k) &= \gamma\left(\frac{du_0}{dr}(k) + \frac{\Lambda}{\mu}\left(\frac{du_0}{dr}(k)\right)^3\right),
\end{aligned} \tag{3.20}$$

$$\frac{d^2\theta_0}{dr^2} = 0, \qquad \theta_0(1)=1,\ \theta_0(k)=0, \tag{3.21}$$

$$\frac{d^2\phi_0}{dr^2} = 0, \qquad \phi_0(1)=1,\ \phi_0(k)=0. \tag{3.22}$$

First and second order problems are defined as

$$\frac{d^2v_1}{dr^2} = C_7\left\{\begin{array}{c} \frac{1}{r}\frac{du_0}{dr} + \frac{\Lambda}{r}\left(\frac{du_0}{dr}\right)^3 + 3\Lambda\left(\frac{du_0}{dr}\right)^2\frac{d^2u_0}{dr^2} + \\ G_r\theta_0 + B_r\phi_0 \end{array}\right\}, \tag{3.23}$$

$$\frac{d^2\theta_1}{dr^2} = C_7\left\{\frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2\right\}, \tag{3.24}$$

$$\frac{d^2\phi_1}{dr^2} = C_7\left\{\frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_b}{N_t}\left(\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2}\right)\right\}, \tag{3.25}$$

$$\begin{aligned}\frac{d^2u_2}{dr^2} &= \frac{d^2u_1}{dr^2} + C_8\left\{\begin{array}{c}\frac{1}{r}\frac{du_0}{dr} + \frac{\Lambda}{r}\left(\frac{du_0}{dr}\right)^3 + 3\Lambda\left(\frac{du_0}{dr}\right)^2\frac{d^2u_0}{dr^2} \\ +G_r\theta_0 + B_r\phi_0\end{array}\right\} \\ &+C_7\left\{\begin{array}{c}\frac{d^2u_1}{dr^2} + \frac{1}{r}\frac{du_1}{dr} + \frac{3\Lambda}{r}\left(\frac{du_0}{dr}\right)^2\frac{du_1}{dr} + \\ 3\Lambda\left(\left(\frac{du_0}{dr}\right)^2\frac{d^2u_1}{dr^2} + 2\frac{du_0}{dr}\frac{du_1}{dr}\frac{d^2u_0}{dr^2}\right) - \\ +G_r\theta_1 + B_r\phi_1\end{array}\right\},\end{aligned} \tag{3.26}$$

$$\begin{aligned}\frac{d^2\theta_2}{dr^2} &= \frac{d^2\theta_1}{dr^2} + C_8\left\{\frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2\right\} + \\ &+C_7\left\{\frac{1}{r}\frac{d\theta_1}{dr} + \frac{N_b}{\alpha}\left(\frac{d\theta_1}{dr}\frac{d\phi_0}{dr} + \frac{d\theta_0}{dr}\frac{d\phi_1}{dr}\right) + \frac{2N_t}{\alpha}\left(\frac{d\theta_0}{dr}\frac{d\theta_1}{dr}\right)\right\},\end{aligned} \tag{3.27}$$

$$\begin{aligned}\frac{d^2\phi_2}{dr^2} &= \frac{d^2\phi_1}{dr^2} + C_8\left\{\frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_b}{N_t}\left(\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2}\right)\right\} \\ &+C_7\left\{\frac{d^2\phi_1}{dr^2} + \frac{1}{r}\frac{d\phi_1}{dr} + \frac{N_b}{N_t}\left(\frac{1}{r}\frac{d\theta_1}{dr} + \frac{d^2\theta_1}{dr^2}\right)\right\}\end{aligned} \tag{3.28}$$

and so forth.

The solutions of the above deformation problems up to second order are

$$\left.\begin{array}{c}u = u_0 + u_1 + u_2 + ... \\ \theta = \theta_0 + \theta_1 + \theta_2 + ... \\ \phi = \phi_0 + \phi_1 + \phi_2 + ...\end{array}\right\}, \tag{3.29}$$

Case II: For variable viscosity model, we take

$$\mu = r, \tag{3.30}$$

such that

$$L_i\left(\varphi_i\left(r,p\right)\right)=\frac{\partial^2\varphi_i\left(r,p\right)}{\partial r^2}. \tag{3.31}$$

Defining non-linear operator as

$$\left.\begin{array}{c} N_1\left(\varphi_1\left(r,p\right)\right)=\frac{2}{r}\frac{du}{dr}+\frac{\Lambda}{r^2}(\frac{du}{dr})^3+\frac{3\Lambda}{r}(\frac{du}{dr})^2\frac{d^2u}{dr^2} \\ +\frac{G_r}{r}\theta+\frac{B_r}{r}\phi \end{array}\right\}, \tag{3.32}$$

$$\mathcal{N}_2[\varphi_2\left(r,p\right)]=\frac{1}{r}\frac{d\theta}{dr}+\frac{N_b}{\alpha}\frac{d\theta}{dr}\frac{d\phi}{dr}+\frac{N_t}{\alpha}\left(\frac{d\theta}{dr}\right)^2, \tag{3.33}$$

$$\mathcal{N}_3[\varphi_3\left(r,p\right)]=\frac{1}{r}\frac{d\phi}{dr}+\frac{N_b}{N_t}\left(\frac{1}{r}\frac{d\theta}{dr}+\frac{d^2\theta}{dr^2}\right) \tag{3.34}$$

along with the boundary conditions

$$\left.\begin{array}{c} \varphi_1(1)=1+\gamma\left(\frac{d\varphi_1}{dr}+\frac{\Lambda}{r}\left(\frac{d\varphi_1}{dr}\right)^3\right),\ \varphi_1(k)=\gamma\left(\frac{d\varphi_k}{dr}+\frac{\Lambda}{r}\left(\frac{d\varphi_k}{dr}\right)^3\right) \\ \varphi_2(1)=1,\ \varphi_2(k)=0 \\ \varphi_3(1)=1,\ \varphi_3(k)=0 \end{array}\right\}. \tag{3.35}$$

The **zeroth-order** problem is given by

$$\begin{aligned} \frac{d^2u_0}{dr^2} &= 0, \\ u\left(1\right) &= 1+\gamma\left(\frac{du}{dr}\left(1\right)+\frac{\Lambda}{r}\left(\frac{du}{dr}\left(1\right)\right)^3\right), \\ u\left(k\right) &= \gamma\left(\frac{du}{dr}\left(k\right)+\frac{\Lambda}{r}\left(\frac{du}{dr}(k)\right)^3\right), \end{aligned} \tag{3.36}$$

$$\frac{d^2\theta_0}{dr^2}=0,\ \theta_0(1)=1,\ \theta_0(k)=0, \tag{3.37}$$

$$\frac{d^2\phi_0}{dr^2}=0,\ \phi_0(1)=1,\ \phi_0(k)=0. \tag{3.38}$$

First order and second order problems are given by

$$\frac{d^2u_1}{dr^2} = C_9 \left\{ \begin{array}{c} \frac{2}{r}\frac{du_0}{dr} + \frac{\Lambda}{r^2}\left(\frac{du_0}{dr}\right)^3 + \frac{3\Lambda}{r}\left(\frac{du_0}{dr}\right)^2 \frac{d^2u_0}{dr^2} \\ +\frac{G_r}{r}\theta_0 + \frac{B_r}{r}\phi_0 - \frac{c}{r} \end{array} \right\}, \tag{3.39}$$

$$\frac{d^2\theta_1}{dr^2} = C_9 \left\{ \frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2 \right\}, \tag{3.40}$$

$$\frac{d^2\phi_1}{dr^2} = C_9 \left\{ \frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_b}{N_t}\left(\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2}\right) \right\}, \tag{3.41}$$

$$\begin{array}{l} \frac{d^2u_2}{dr^2} = \frac{d^2u_1}{dr^2} + C_{10} \left\{ \begin{array}{c} \frac{2}{r}\frac{du_0}{dr} + \frac{\Lambda}{r^2}\left(\frac{du_0}{dr}\right)^3 + \frac{3\Lambda}{r}\left(\frac{du_0}{dr}\right)^2 \frac{d^2u_0}{dr^2} \\ +\frac{G_r}{r}\theta_0 + \frac{B_r}{r}\phi_0 \end{array} \right\} \\ +C_9 \left\{ \begin{array}{c} \frac{d^2u_1}{dr^2} + \frac{2}{r}\frac{du_1}{dr} + \frac{3\Lambda}{r^2}\left(\frac{du_0}{dr}\right)^2 \frac{du_1}{dr} \\ \frac{3\Lambda}{r}\left(\left(\frac{du_0}{dr}\right)^2 \frac{d^2u_1}{dr^2} + 2\frac{du_0}{dr}\frac{du_1}{dr}\frac{d^2u_0}{dr^2}\right) \\ +\frac{G_r}{r}\theta_1 + \frac{B_r}{r}\phi_1 \end{array} \right\}, \end{array} \tag{3.42}$$

$$\begin{array}{l} \frac{d^2\theta_2}{dr^2} = \frac{d^2\theta_1}{dr^2} + C_{10} \left\{ \frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2 \right\} + \\ +C_9 \left\{ \frac{1}{r}\frac{d\theta_1}{dr} + \frac{N_b}{\alpha}\left(\frac{d\theta_1}{dr}\frac{d\phi_0}{dr} + \frac{d\theta_0}{dr}\frac{d\phi_1}{dr}\right) + \frac{2N_t}{\alpha}\left(\frac{d\theta_0}{dr}\frac{d\theta_1}{dr}\right) \right\}, \end{array} \tag{3.43}$$

$$\begin{array}{l} \frac{d^2\phi_2}{dr^2} = \frac{d^2\phi_1}{dr^2} + C_{10} \left\{ \frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2} \right\} \\ +C_9 \left\{ \frac{d^2\phi_1}{dr^2} + \frac{1}{r}\frac{d\phi_1}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_1}{dr} + \frac{d^2\theta_1}{dr^2} \right\}. \end{array} \tag{3.44}$$

The solutions of the above deformation problems up to second order are

$$\left. \begin{array}{c} u = u_0 + u_1 + u_2 + \ldots \\ \theta = \theta_0 + \theta_1 + \theta_2 + \ldots \\ \phi = \phi_0 + \phi_1 + \phi_2 + \ldots \end{array} \right\}, \tag{3.45}$$

3.3 Graphs

The solution is obtained by Optimal Homotopic Asymptotic Method and we fixed value of $k = 2$. The investigation of the effect of slip parameter γ on velocity for both constant and variable viscosity are shown in Figs. 3.1 to 3.2. The Figs. 3.3 to 3.7, have been prepared to explain the effect of N_t and N_b on temperature.The effects of N_t and N_b on velocity nanoparticles concentration are display in Figs. 3.7 to 3.10.

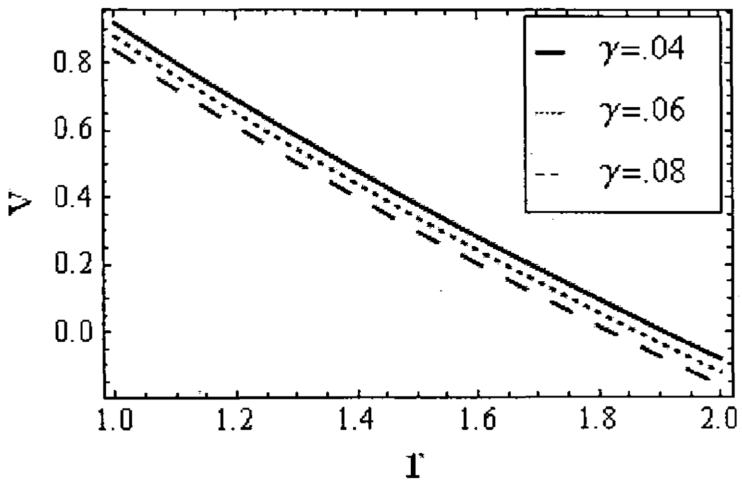

Fig. 3.1 :Effect of γ on velocity profile when $N_b = 1$ and $N_t = 1$ for constant viscosity.

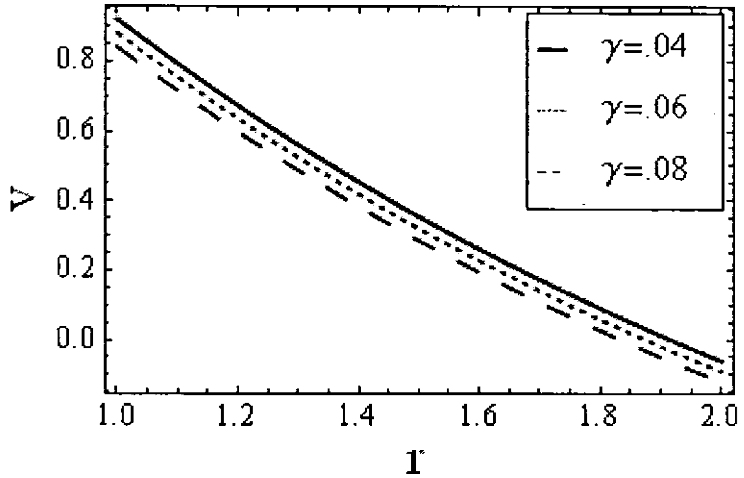

Fig. 3.2 :Effect of γ on velocity profile when $N_b = 1$ and $N_t = 1$ for variable viscosity.

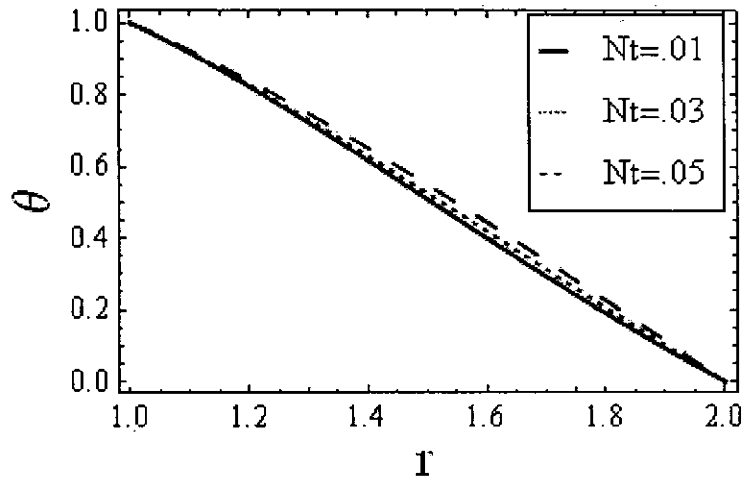

Fig. 3.3 :Effect of N_t on temperature profile when $N_b = 0.1$ for constant viscosity.

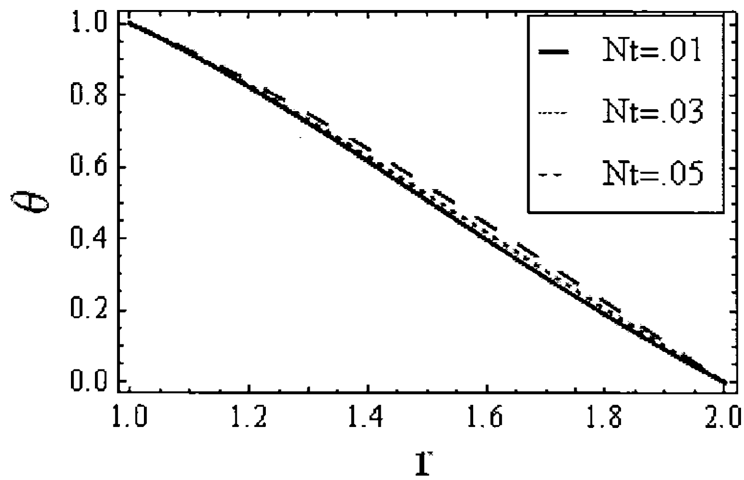

Fig. 3.4 :Effect of N_t on temperature profile when $N_t = 0.1$ for constant viscosity.

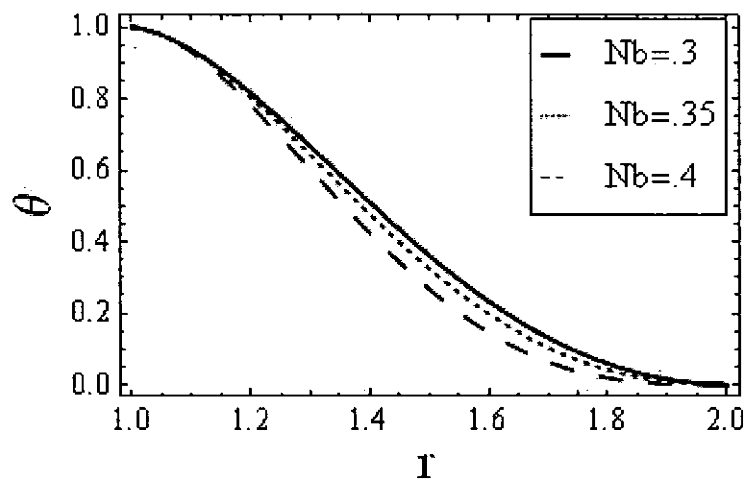

Fig. 3.5 :Effect of N_b on temperature profile when $N_t = 0.1$ for constant viscosity.

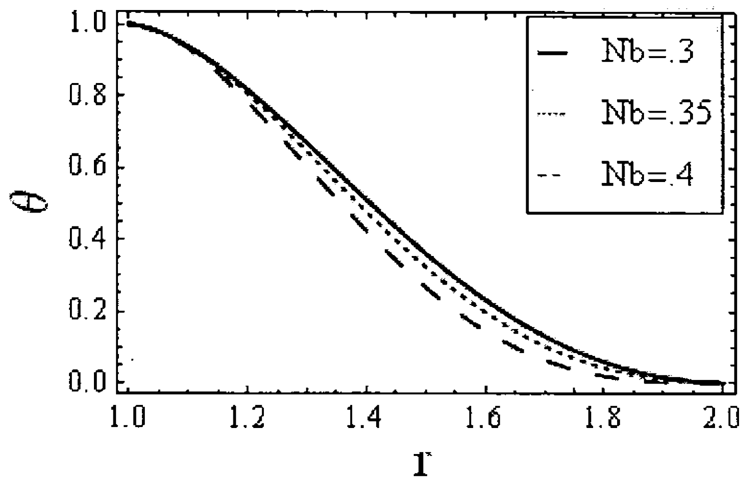

Fig. 3.6 :Effect of N_b on temperature profile when $N_t = 0.1$ for constant viscosity.

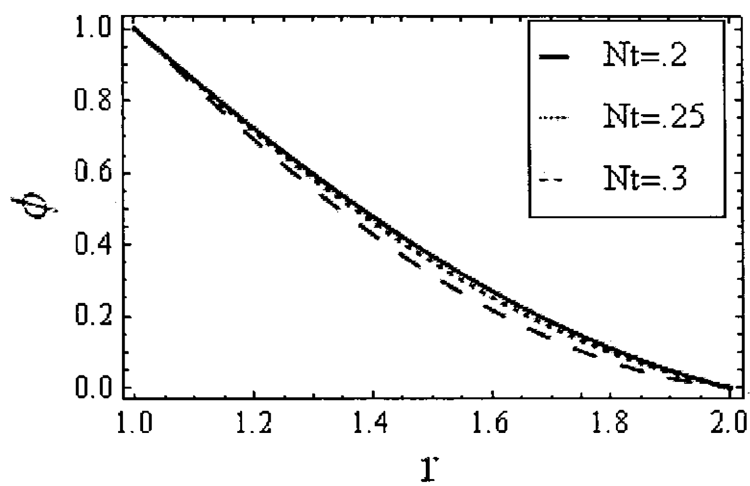

Fig. 3.7 :Effect of N_t on nanoparticles profile when $N_b = 0.1$ for constant viscosity.

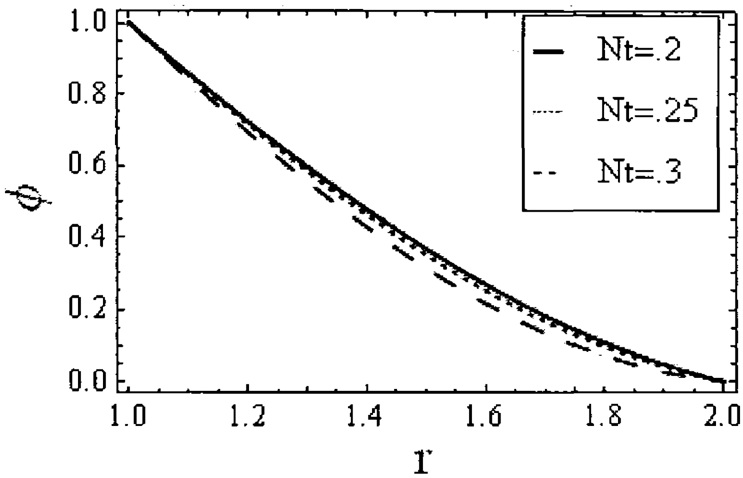

Fig. 3.7 :Effect of N_t on nanoparticles profile when $N_b = 0.1$ for constant viscosity.

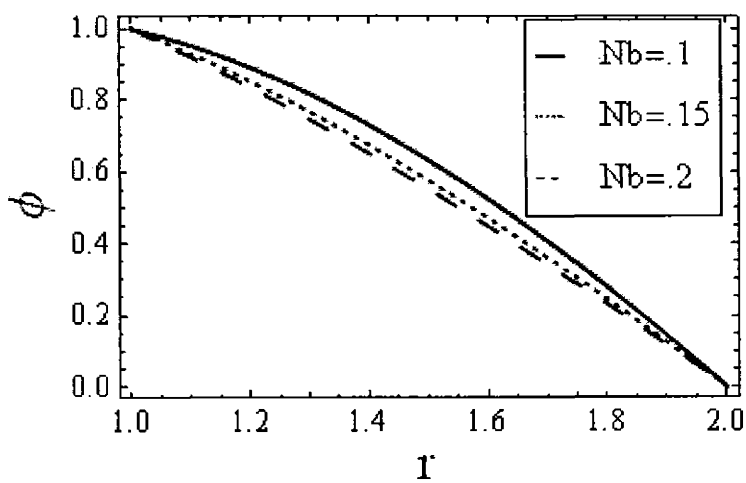

Fig. 3.9 :Effect of N_b on nanoparticles profile when $N_t = 0.1$ for constant viscosity.

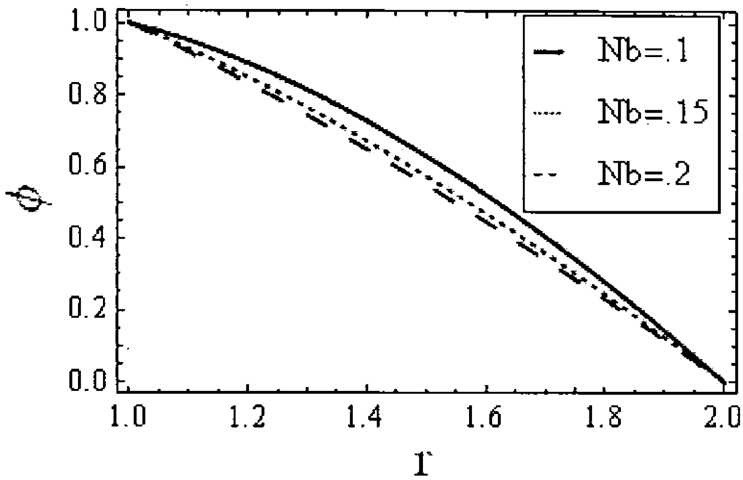

Fig. 3.10 :Effect of N_b on nanoparticles profile when $N_t = 0.1$ for constant viscosity.

3.4 Results and Discussion

In this chapter, flow of third grade nanofluid in a coaxial cylinders is examined. A new method is proposed to solve the nonlinear boundary value problem. In this method we control the convergence using a number j of auxiliary constants $C_1, C_2, ...C_j$ which are optimally determined. Moreover, this method converges quickly to exact solution than other methods. The results of problem are showen in graphical form. The graphs display the behavior of the velocity, temperature and nanoparticles concentration are plotted against r. To see the effects of emerging parameters for constant and variable viscosity Figs. 3.1 to 3.10 have been displayed. In Figs. 3.1 to 3.2, it is found that the velocity decreases with an increase in the values of γ. Figs. 3.3 to 3.6 explain the variation of N_t and N_b on the temperature distribution. Here, it is revealed that the temperature profile increases when large values of N_b have been taken into account and the temperature profile decreases with decreases N_t. Figs. 3.7 to 3.10 bring out the influence of nanoparticles concentration for constant and variable viscosity. It is observed that the nanoparticles concentration increases with the decrease in N_b and decreases by increasing N_t.

Appendix A

The related coefficients are given by

$$A_1 = \frac{1}{64}c^2\Gamma,$$

$$A_2 = \frac{1}{576}c^4\Gamma\Lambda + \frac{1}{96}c^4\hbar\Gamma\Lambda + \frac{1}{144}c^4\hbar^2\Gamma\Lambda,$$

$$A_3 = \frac{3}{1024}c^6\hbar\Gamma\Lambda^2 + \frac{29}{4096}c^6\hbar^2\Gamma\Lambda^2 + \frac{3}{1024}c^6\hbar^3\Gamma\Lambda^2 + \frac{1}{1024}c^6\hbar^4\Gamma\Lambda^2,$$

$$A_4 = \frac{39}{12800}c^8\hbar^2\Gamma\Lambda^3 + \frac{27}{6400}c^8\hbar^3\Gamma\Lambda^3 + \frac{3}{1600}c^8\hbar^4\Gamma\Lambda^3,$$

$$A_5 = \frac{9}{4096}c^{10}\hbar^3\Gamma\Lambda^4 + \frac{11}{4096}c^{10}\hbar^4\Gamma\Lambda^4 + \frac{1}{1024}c^{10}\hbar^5\Gamma\Lambda^4 + \frac{1}{4608}c^{10}\hbar^6\Gamma\Lambda^4,$$

$$A_6 = \frac{135}{114688}c^{12}\hbar^4\Gamma\Lambda^5 + \frac{135}{100352}c^{12}\hbar^5\Gamma\Lambda^5 + \frac{9}{14336}c^{12}\hbar^6\Gamma\Lambda^5 + \frac{3}{25088}c^{12}\hbar^7\Gamma\Lambda^5 + \frac{1}{50176}c^{12}\hbar^8\Gamma\Lambda^5,$$

$$A_7 = \frac{243}{524288}c^{14}\hbar^5\Gamma\Lambda^6 + \frac{135}{262144}c^{14}\hbar^6\Gamma\Lambda^6 + \frac{27}{131072}c^{14}\hbar^7\Gamma\Lambda^6 + \frac{3}{65536}c^{14}\hbar^8\Gamma\Lambda^6,$$

$$A_8 = \frac{13}{98304}c^{16}\hbar^6\Gamma\Lambda^7 + \frac{1}{8192}c^{16}\hbar^7\Gamma\Lambda^7 + \frac{1}{24576}c^{16}\hbar^8\Gamma\Lambda^7,$$

$$A_9 = \frac{81}{3276800}c^{18}\hbar^7\Gamma\Lambda^8 + \frac{27}{1638400}c^{18}\hbar^8\Gamma\Lambda^8,$$

$$A_{10} = \frac{81}{31719424}c^{20}\hbar^8\Gamma\Lambda^9,$$

$$A_{11} = \frac{1}{4}c^4\hbar^4\Lambda\Gamma + \frac{1}{2}c^4\hbar^5\Lambda\Gamma + \frac{3}{8}c^4\hbar^6\Lambda\Gamma + \frac{1}{8}c^4\hbar^7\Lambda\Gamma + \frac{1}{64}c^4\hbar^8\Lambda\Gamma,$$

$$\begin{aligned} A_{12} \;=\; & \frac{1}{9}c^2\hbar^2\Gamma + \frac{1}{9}c^2\hbar^3\Gamma + \frac{1}{36}c^2\hbar^4\Gamma - \frac{4}{27}c^4\hbar^3\Lambda\Gamma - \frac{14}{27}c^4\hbar^4\Lambda\Gamma - \frac{19}{27}c^4\hbar^5\Lambda\Gamma - \frac{25}{54}c^4\hbar^6\Lambda\Gamma - \\ & \frac{4}{27}c^4\hbar^7\Lambda\Gamma - \frac{1}{54}c^4\hbar^8\Lambda\Gamma + \frac{2}{27}c^6\hbar^5\Lambda^2\Gamma + \frac{1}{9}c^6\hbar^6\Lambda^2\Gamma + \frac{1}{18}c^6\hbar^7\Lambda^2\Gamma + \frac{1}{108}c^6\hbar^8\Lambda^2\Gamma, \end{aligned}$$

$$
\begin{aligned}
A_{13} &= -\frac{1}{24}c^2\hbar\Gamma - \frac{5}{48}c^2\hbar^2\Gamma - \frac{1}{12}c^2\hbar^3\Gamma - \frac{1}{48}c^2\hbar^4\Gamma + \frac{1}{24}c^4\hbar^2\Lambda\Gamma + \frac{11}{48}c^4\hbar^3\Lambda\Gamma + \frac{7}{16}c^4\hbar^4\Lambda\Gamma + \\
&\quad \frac{11}{24}c^4\hbar^5\Lambda\Gamma + \frac{13}{48}c^4\hbar^6\Lambda\Gamma + \frac{1}{12}c^4\hbar^7\Lambda\Gamma + \frac{1}{96}c^4\hbar^8\Lambda\Gamma - \frac{13}{216}c^6\hbar^4\Lambda^2\Gamma - \frac{41}{216}c^6\hbar^5\Lambda^2\Gamma - \\
&\quad \frac{59}{288}c^6\hbar^6\Lambda^2\Gamma - \frac{5}{54}c^6\hbar^7\Lambda^2\Gamma - \frac{13}{864}c^6\hbar^8\Lambda^2\Gamma + \frac{1}{96}c^8\hbar^6\Lambda^3\Gamma + \frac{1}{96}c^8\hbar^7\Lambda^3\Gamma + \frac{1}{384}c^8\hbar^8\Lambda^3\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
A_{14} &= \frac{1}{225}c^2\Gamma + \frac{4}{225}c^2\hbar\Gamma + \frac{2}{75}c^2\hbar^2\Gamma + \frac{4}{225}c^2\hbar^3\Gamma + \frac{1}{225}c^2\hbar^4\Gamma - \frac{4}{675}c^4\hbar\Lambda\Gamma - \frac{11}{225}c^4\hbar^2\Lambda\Gamma - \\
&\quad \frac{88}{675}c^4\hbar^3\Lambda\Gamma - \frac{13}{75}c^4\hbar^4\Lambda\Gamma - \frac{4}{27}c^4\hbar^5\Lambda\Gamma - \frac{2}{25}c^4\hbar^6\Lambda\Gamma - \frac{16}{675}c^4\hbar^7\Lambda\Gamma - \frac{2}{675}c^4\hbar^8\Lambda\Gamma + \\
&\quad \frac{14}{675}c^6\hbar^3\Lambda^2\Gamma + \frac{271}{2700}c^6\hbar^4\Lambda^2\Gamma + \frac{122}{675}c^6\hbar^5\Lambda^2\Gamma + \frac{106}{675}c^6\hbar^6\Lambda^2\Gamma + \frac{44}{675}c^6\hbar^7\Lambda^2\Gamma + \\
&\quad \frac{7}{675}c^6\hbar^8\Lambda^2\Gamma - \frac{1}{81}c^8\hbar^5\Lambda^3\Gamma - \frac{43}{1350}c^8\hbar^6\Lambda^3\Gamma - \frac{16}{675}c^8\hbar^7\Lambda^3\Gamma - \frac{11}{2025}c^8\hbar^8\Lambda^3\Gamma + \\
&\quad \frac{1}{1350}c^{10}\hbar^7\Lambda^4\Gamma + \frac{1}{2700}c^{10}\hbar^8\Lambda^4\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
A_{15} &= \frac{1}{2916}c^4\Lambda\Gamma + \frac{1}{243}c^4\hbar\Lambda\Gamma + \frac{11}{729}c^4\hbar^2\Lambda\Gamma + \frac{19}{729}c^4\hbar^3\Lambda\Gamma + \frac{13}{486}c^4\hbar^4\Lambda\Gamma + \frac{14}{729}c^4\hbar^5\Lambda\Gamma + \\
&\quad \frac{7}{729}c^4\hbar^6\Lambda\Gamma + \frac{2}{729}c^4\hbar^7\Lambda\Gamma + \frac{1}{2916}c^4\hbar^8\Lambda\Gamma - \frac{5}{1458}c^6\hbar^2\Lambda^2\Gamma - \frac{17}{729}c^6\hbar^3\Lambda^2\Gamma - \\
&\quad \frac{19}{324}c^6\hbar^4\Lambda^2\Gamma - \frac{56}{729}c^6\hbar^5\Lambda^2\Gamma - \frac{83}{1458}c^6\hbar^6\Lambda^2\Gamma - \frac{16}{729}c^6\hbar^7\Lambda^2\Gamma - \frac{5}{1458}c^6\hbar^8\Lambda^2\Gamma + \\
&\quad \frac{97}{17496}c^8\hbar^4\Lambda^3\Gamma + \frac{101}{4374}c^8\hbar^5\Lambda^3\Gamma + \frac{25}{729}c^8\hbar^6\Lambda^3\Gamma + \frac{46}{2187}c^8\hbar^7\Lambda^3\Gamma + \frac{79}{17496}c^8\hbar^8\Lambda^3\Gamma - \\
&\quad \frac{1}{648}c^{10}\hbar^6\Lambda^4\Gamma - \frac{2}{729}c^{10}\hbar^7\Lambda^4\Gamma - \frac{1}{972}c^{10}\hbar^8\Lambda^4\Gamma + \frac{1}{46656}c^{12}\hbar^8\Lambda^5\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
A_{16} &= \frac{8}{35721}c^6\hbar\Lambda^2\Gamma + \frac{74}{35721}c^6\hbar^2\Lambda^2\Gamma + \frac{244}{35721}c^6\hbar^3\Lambda^2\Gamma + \frac{422}{35721}c^6\hbar^4\Lambda^2\Gamma + \frac{440}{35721}c^6\hbar^5\Lambda^2\Gamma + \\
&\quad \frac{32}{3969}c^6\hbar^6\Lambda^2\Gamma + \frac{104}{35721}c^6\hbar^7\Lambda^2\Gamma + \frac{16}{35721}c^6\hbar^8\Lambda^2\Gamma - \frac{40}{35721}c^8\hbar^3\Lambda^3\Gamma - \frac{233}{35721}c^8\hbar^4\Lambda^3\Gamma - \\
&\quad \frac{520}{35721}c^8\hbar^5\Lambda^3\Gamma - \frac{572}{35721}c^8\hbar^6\Lambda^3\Gamma - \frac{304}{35721}c^8\hbar^7\Lambda^3\Gamma - \frac{62}{35721}c^8\hbar^8\Lambda^3\Gamma + \frac{34}{35721}c^{10}\hbar^5\Lambda^4\Gamma + \\
&\quad \frac{38}{11907}c^{10}\hbar^6\Lambda^4\Gamma + \frac{40}{11907}c^{10}\hbar^7\Lambda^4\Gamma + \frac{38}{35721}c^{10}\hbar^8\Lambda^4\Gamma - \frac{4}{35721}c^{12}\hbar^7\Lambda^5\Gamma - \frac{1}{10206}c^{12}\hbar^8\Lambda^5\Gamma,
\end{aligned}
$$

$$A_{17} = \frac{1}{11664}c^8\hbar^2\Lambda^3\Gamma + \frac{23}{34992}c^8\hbar^3\Lambda^3\Gamma + \frac{23}{11664}c^8\hbar^4\Lambda^3\Gamma + \frac{1}{324}c^8\hbar^5\Lambda^3\Gamma + \frac{97}{34992}c^8\hbar^6\Lambda^3\Gamma + \frac{23}{17496}c^8\hbar^7\Lambda^3\Gamma + \frac{1}{3888}c^8\hbar^8\Lambda^3\Gamma - \frac{151}{629856}c^{10}\hbar^4\Lambda^4\Gamma - \frac{47}{39366}c^{10}\hbar^5\Lambda^4\Gamma - \frac{229}{104976}c^{10}\hbar^6\Lambda^4\Gamma - \frac{34}{19683}c^{10}\hbar^7\Lambda^4\Gamma - \frac{307}{629856}c^{10}\hbar^8\Lambda^4\Gamma + \frac{5}{46656}c^{12}\hbar^6\Lambda^5\Gamma + \frac{1}{3888}c^{12}\hbar^7\Lambda^5\Gamma + \frac{13}{93312}c^{12}\hbar^8\Lambda^5\Gamma - \frac{1}{279936}c^{14}\hbar^8\Lambda^6\Gamma,$$

$$A_{18} = \frac{104}{4782969}c^{10}\hbar^3\Lambda^4\Gamma + \frac{697}{4782969}c^{10}\hbar^4\Lambda^4\Gamma + \frac{1808}{4782969}c^{10}\hbar^5\Lambda^4\Gamma + \frac{2368}{4782969}c^{10}\hbar^6\Lambda^4\Gamma + \frac{1544}{4782969}c^{10}\hbar^7\Lambda^4\Gamma + \frac{400}{4782969}c^{10}\hbar^8\Lambda^4\Gamma - \frac{172}{4782969}c^{12}\hbar^5\Lambda^5\Gamma - \frac{230}{1594323}c^{12}\hbar^6\Lambda^5\Gamma - \frac{304}{1594323}c^{12}\hbar^7\Lambda^5\Gamma - \frac{386}{4782969}c^{12}\hbar^8\Lambda^5\Gamma + \frac{4}{531441}c^{14}\hbar^7\Lambda^6\Gamma + \frac{5}{531441}c^{14}\hbar^8\Lambda^6\Gamma,$$

$$A_{19} = \frac{107}{26572050}c^{12}\hbar^4\Lambda^5\Gamma + \frac{302}{13286025}c^{12}\hbar^5\Lambda^5\Gamma + \frac{43}{885735}c^{12}\hbar^6\Lambda^5\Gamma + \frac{614}{13286025}c^{12}\hbar^7\Lambda^5\Gamma + \frac{443}{26572050}c^{12}\hbar^8\Lambda^5\Gamma - \frac{11}{2952450}c^{14}\hbar^6\Lambda^6\Gamma - \frac{16}{1476225}c^{14}\hbar^7\Lambda^6\Gamma - \frac{23}{2952450}c^{14}\hbar^8\Lambda^6\Gamma + \frac{1}{3936600}c^{16}\hbar^8\Lambda^7\Gamma,$$

$$A_{20} = \frac{104}{192913083}c^{14}\hbar^5\Lambda^6\Gamma + \frac{160}{64304361}c^{14}\hbar^6\Lambda^6\Gamma + \frac{248}{64304361}c^{14}\hbar^7\Lambda^6\Gamma + \frac{400}{192913083}c^{14}\hbar^8\Lambda^6\Gamma - \frac{16}{64304361}c^{16}\hbar^7\Lambda^7\Gamma - \frac{26}{64304361}c^{16}\hbar^8\Lambda^7\Gamma,$$

$$A_{21} = \frac{1}{19131876}c^{16}\hbar^6\Lambda^7\Gamma + \frac{5}{28697814}c^{16}\hbar^7\Lambda^7\Gamma + \frac{1}{6377292}c^{16}\hbar^8\Lambda^7\Gamma - \frac{1}{114791256}c^{18}\hbar^8\Lambda^8\Gamma,$$

$$A_{22} = \frac{8}{2424965283}c^{18}\hbar^7\Lambda^8\Gamma + \frac{16}{2424965283}c^{18}\hbar^8\Lambda^8\Gamma,$$

$$A_{23} = \frac{1}{8437157316}c^{20}\hbar^8\Lambda^9\Gamma.$$

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