SIMILARITY SOLUTION FOR FLOW OVER A HEATED FLAT PLATE

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1. Similarity analysis

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
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Certificate

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We accept this dissertation as confirming to the required standard.

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2016

DEDICATION

This work is dedicated to

My beloved Parents

&

Iqra Batool

Who has enriched my knowledge and become beacon of light at the time of distress.

DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor **Dr. Tariq Javed**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

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Preface

Convection in generally the most dominant mode of heat transfer in gases and liquids which is due to movement of its particles [1]. Convective heat transfer covers the combined processes of conduction and advection and is therefore occurs in almost all branches of engineering [2,3]. The knowledge of the methods used to model convective heat transfer is therefore required by practicing engineers in the laboratories. External flows involve a flow which is essentially over the geometry of infinite in extent or over the outer surface of the body. In order to predict the convective heat transfer rate, the three basic properties of the fluids namely the pressure, the velocity vector and the temperature play a major role [4,5]. Once the distribution of these quantities are determined, the variation of heat transfer rate can be obtained. In order to determine the distribution of pressure, velocity and the temperature, the principle of conservation of mass, conservation of momentum and the conservation of energy represented by equation of continuity, Navier-Stokes and energy equations respectively.

The aim of the present study is to predict the heat transfer rates for situations involving external laminar flows. In all the investigation the fluid properties will be assumed to be constant and flow is assumed to be two dimensional [6,7]. In addition, dissipation effect in the energy equation will be neglected. The three chapters are arranged in the following sense.

Chapter one includes some basic definitions and information related to the flow and convective heat transfer [8]. The basic equation of continuity, Navier-Stokes and energy is also presented for basic understanding of the readers. Chapter two investigates the heat transfer analysis in the flow of a fluid over a flat plate with constant velocity whose entire surface is held at a uniform temperature which is different from that of the fluid ahead of the boundary layer. These problems are commonly known as Blasius flow and Pohlhausen flow discussed in [1,4]. The heat transfer rate at the wall is predicted against Reynolds number and compared with the experimental date. The transition from laminar to the turbulence region is also shown through the graphs. In the last chapter, we presented the similarity solutions for a fluid flow which is discussed in previous chapter but here we assumed that the temperature of the flat plate varies with x. Another very important flow problem namely Falkner-Skan flow [9,10] which is basically flow over wedge-shaped body with an inclined angle is discussed in detail in this chapter. The numerical computation is made with the help of shooting method [11,14] throughout the study which is also discussed briefly wherever it is desired. The heat transfer rate for accelerating, constant and decelerating flow case is predicted and shown through the graphs.

Contents

Chapter 1

Preliminaries

This chapter includes some basic definitions related to fluid flow and heat transfer analysis. Numerical method of Shooting technique with Runge-Kutta fourth order scheme is also discussed in detail for better understanding of the readers for computation purpose.

1.1 Fluid

A substance that sustains no fixed shape and deforms easily due to external pressure is called fluid.

1.2 Flow

A phenomenon of continuous deformation under the action of applied forces is called flow.

1.3 Properties of Fluid

The fluids in general can be described through the following major properties.

Density

Density of a fluid is defined as the amount of mass per unit volume. Mathematically, the density ρ at a point can be defined as

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}, \tag{1.1}$$

where δV is volume element around the point P and δm is mass of the fluid within δV.

Pressure

The pressure P is the magnitude of the normal force F acting per unit area on a surface S. The mathematical form of pressure at a point is

$$P = \lim_{S\to 0}\left(\frac{F}{S}\right). \tag{1.2}$$

Temperature

Temperature is a physical quantity that measures degree of hotness or coldness on numerical scale. In other words, it is a measurement of the average heat or thermal energy of the particles in a substance.

Viscosity

Viscosity or dynamic viscosity is defined as the resistance of the fluid particles against the motion. It is mathematically represented by symbol μ and defined as the ratio of shear stress to the rate of shear strain, i.e.

$$\mu = \frac{\text{Shear Stress}}{\text{Rate of Shear Strain}} = \frac{\tau_{yx}}{du/dy}. \tag{1.3}$$

Kinematic Viscosity

It is defined as the ratio of dynamic viscosity μ to the density ρ. Mathematically, it is denoted by ν and defined as

$$\nu = \frac{\mu}{\rho}. \tag{1.4}$$

1.4 Classification of Fluid

In general, the fluid can be theoretically classified into the following categories:

1.4.1 Ideal Fluid

A fluid having zero or negligible viscosity is termed as ideal fluid. It does not actually exists in nature.

1.4.2 Real Fluid

Real fluids are those in which the role of viscosity is non-negligible. The flow of real fluid is called viscous flow. Real fluids can be further subdivided into two categories, Newtonian fluid and non-Newtonian fluid. The details of which are explained as follows:

1.4.2.1 Newtonian Fluid

The fluids which obey the Newton's law of viscosity are termed as Newtonian Fluids. According to this law, stress is directly and linearly proportional to the rate of deformation, i.e.

$$\tau_{yx} \propto \frac{du}{dy},$$

or

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.5}$$

where μ is the constant of proportionality called dynamic viscosity.

1.4.2.2 Non-Newtonian Fluids

The fluids in which shear stress is non linearly proportional to the deformation rate, are termed as Non-Newtonian fluids. For such fluids power law model holds and are defined by

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n, \tag{1.6}$$

where k denote consistency index and n is the flow behavior index. It can also be written as

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^{n-1}\frac{du}{dy} = \eta\frac{du}{dy}, \tag{1.7}$$

with $\eta = k\left(\frac{du}{dy}\right)^{n-1}$ is treated as apparent viscosity.

1.5 Types of Flow

In fluid mechanics, the following flows have been studies:

Internal Flow

A fluid flow within confined boundaries is termed as internal flow. Flow in pipe or in duct are common examples of internal flows.

External Flow

In this way, external flows are defined as the flow over a geometry which are infinite in extent or over the body. Examples of external flows are flow of air over cars and around aero planes.

Laminar Flow

Flows in which each fluid particle possesses definite path and the individual particles do not cross each other are termed as laminar flow.

Turbulent Flow

Flows in which each fluid particle does not have a definite path due to which they cross each other are termed as turbulent flow.

Compressible Flow

A flow in which the density of the considered fluid changes during the flow is termed as compressible flow. All gases are generally considered as compressible fluid.

Incompressible Flow

A flow in which density of flowing fluid do not change during the flow is termed as incompressible flow. All liquids are generally considered to have incompressible fluid.

Steady Flow

A flow in which properties associated with the motion of fluid are independent of time is termed as steady flow. Mathematically, it can be written as

$$\frac{\partial \eta}{\partial t} = 0. \tag{1.8}$$

Unsteady Flow

If the properties of the flow change with respect to time, then the flow is termed as unsteady flow. Mathematically, it can be written as

$$\frac{\partial \eta}{\partial t} \neq 0. \tag{1.9}$$

1.6 Conservation Laws

The following conservation laws take part primary role in studying the fluid flows.

1.6.1 Continuity Equation

The law of conservation of mass for a compressible fluid in term of continuity equation can be written as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{V}) = 0. \tag{1.10}$$

In which $\boldsymbol{V}$ is the velocity of the fluid, ρ is the density and t denotes the time. For an incompressible fluid with ρ being constant, the above equation becomes

$$\nabla.\boldsymbol{V} = 0. \tag{1.11}$$

1.6.2 Momentum Equation

For an infinitesimal element, the basic momentum equation as a partial differential equation in vector form is

$$\rho \frac{d\boldsymbol{V}}{dt} = -\nabla p + \nabla.\boldsymbol{T} + \rho \boldsymbol{b}, \tag{1.12}$$

where p is pressure, $\boldsymbol{b}$ is body force, $\frac{d}{dt}$ is the convective derivative defined as $\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \vec{V}.\nabla\right)$, the $\boldsymbol{T}$ is a Cauchy stress tensor defined as

$$\boldsymbol{T} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}, \tag{1.13}$$

where τ_{xx}, τ_{yy} *and* τ_{zz} are normal stresses and the remaining components in $\boldsymbol{T}$ are known as shear stresses.

1.6.3 Energy Equation

The energy equation is described as

$$\rho c_p \frac{dc}{dt} = \boldsymbol{T}.\boldsymbol{L} - \nabla.\boldsymbol{q}. \tag{1.14}$$

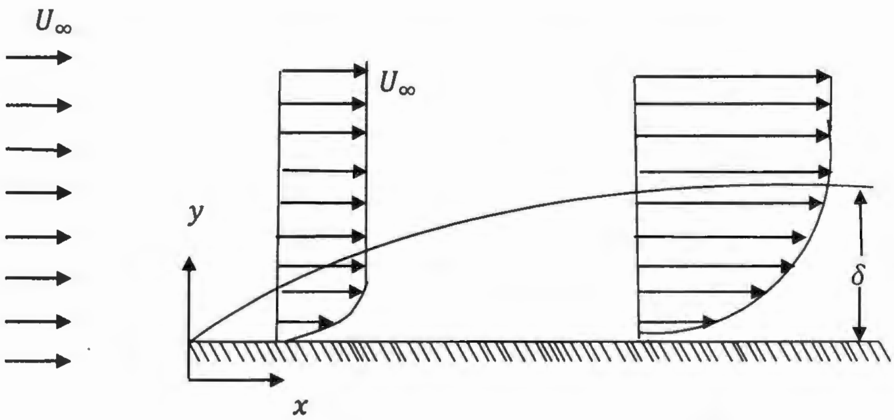
In Cartesian coordinates, it is given as

$$\rho c_p\left(\frac{\partial T}{\partial t}+\frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial^2 x}+\frac{\partial^2 T}{\partial^2 y}\right)+\mu\varphi, \tag{1.15}$$

where φ is viscous dissipation function.

1.7 Boundary layer

A boundary layer is the small layer of fluid of thickness δ adjacent to the boundary where the effects of viscosity are important. The method of dividing the fluid into two parts namely boundary layer and free stream was first suggested by Prandtl and is elaborated through figure below. Due to which the entire flow field can be described in the following ways:



i. A very thin layer adjacent to the plate in which velocity gradient i.e. $\frac{\partial u}{\partial y}$ normal to the wall is extremely large and hence the effects of viscosity cannot be neglected even if the viscosity μ is small.

ii. In the remaining region, velocity gradient $\frac{\partial u}{\partial y}$ is extremely small so the viscous forces may be considered negligible. In this region flow is treated as in-viscid flow.

1.8 Heat Transfer

Heat is defined as energy in transit from high temperature substance to a lower temperature substance. Heat transfer occurs whenever there is temperature difference, then two bodies at different temperatures, attained in contact with each other.

1.8.1 Modes of Heat Transfer

Heat transfer phenomena can be expressed in the following three modes as follows:

Conduction

The transfer of heat from one part of system to another part by inter collisions of interconnected molecules is termed as conduction. The transfer of heat in solids is known as conduction.

Convection

Convection is the way in which the heat is transferred by motion of heated molecules in the system. The heat transfer in liquids and gases are known as convection.

Radiation

Transfer of heat through electromagnetic waves is known as radiation.

1.9 Types of Convection

The convection phenomena can be divided further in three categories as follows:

Natural or Free Convection

Natural convection, is the phenomenon in which fluid motion does not require any external agent or source to transfer heat. It occurs only due to the temperature difference from place to place.

Forced Convection

If the heat transfer occurs only due to an external agent, then this phenomenon is known as force convection.

Mixed Convection

If the heat transfer is due to both force and natural convection, then it is termed as mixed convection.

1.10 Dimensionless Numbers

A dimensionless number is the number without any dimensions associated with it. It is the ratio of the quantities having same dimensions. There is a lot of dimensionless numbers but here we mention only those which are being used in this work.

Reynolds Number

The ratio which approximate the relationship between inertia force to the viscous force is called Reynolds number. In mathematical notation, this number is denoted by *Re* and is defined by

$$Re = \frac{UL}{\nu}, \tag{1.16}$$

where U is reference velocity and L is the characteristics length.

Prandtl Number

Prandtl number Pr is another very important dimensionless number which is defined as the ratio of momentum diffusivity to thermal diffusivity. Mathematically, it is defined as

$$Pr = \frac{\nu}{\alpha} = \frac{\text{Momentum diffusivty}}{\text{Thermal diffusivty}} = \frac{c_p\mu}{k}, \tag{1.17}$$

where $\nu,\ \alpha,\ \mu,\ k,\ c_p,\ \rho$ are kinematic viscosity, thermal diffusivity, dynamic viscosity, thermal conductivity, specific heat and density of the fluid respectively.

1.11 Solution Methodologies

Most of the problems arise in the field of science and engineering can be represented by non-linear ordinary/partial differential equations and to find the exact solution of these problems is quiet tough and sometimes impossible. Therefore, in order to overcome such types of difficulty, many numerical and analytical techniques are developed. Shooting method is one of them which is widely used numerical technique applicable for system of non-linear boundary value problems. We have used this method in the subsequent chapter to get the numerical solution.

1.11.1 Shooting Method

Consider a general second-order boundary value problem as ordinary differential equation with one condition specified at $x = a$ and the other at $x = b$ (say) as discussed in [14]. In Mathematical form, it can be written as

$$y'' = f(x, y, y'), \tag{1.18}$$

$$y(a) = \alpha, y(b) = \beta. \quad (1.19a,1.19b)$$

In order to solve the above problem by shooting method, it is required to reduce the boundary value problem into two first order initial value problems. It is reduced as follows

$$y' = z,\ \ z' = f(x, y, z), \tag{1.20}$$

subject to the initial conditions

$$y(a) = \alpha, \;\; y'(a) = u^{(i)}, \tag{1.21}$$

where $u^{(i)}$ is the missing initial slope which is to be determine by using boundary condition $(1.19b)$. To solve the problem with shooting method, the initial value problem is integrated numerically using Runge-Kutta Method from $x = a$ to the terminal point $x = b$. The accuracy of the supposed missing initial level is assessed by comparing the calculated value of y at the end point with the given value there $y(b) = \beta$. In case of difference in values, another value of missing initial level is assumed and the same process will be done again and again until the accuracy at the end point is achieved. Here to find the initial guess, Newton Raphson method, is used once an initial slope $u^{(i)}$is assumed, the next slope $u^{(i+1)}$ is calculated by the following formula

$$u^{(i+1)} = u^{(i)} - \frac{y(b) - \beta}{\frac{dy(b)}{du}}. \tag{1.22}$$

For integration of the system of initial value problems, we have used Fourth order Runge-Kutta scheme, which is explained as follows:

1.11.2 Fourth Order Runge-Kutta Method

Let us consider the second order initial value problem as

$$y'' = f(x, y, y'), \tag{1.23}$$

subject to initial conditions

$$y(x_0) = a, \qquad y'(x_0) = b. \tag{1.24a,1.24b}$$

In order to integrate the above problem, it is required to convert the above second order initial value problem to the system of two first order initial value problem by introducing new dependent variable z as

$$y' = z(= g(x, y, z)), \tag{1.25}$$

$$z' = f(x, y, z) \tag{1.26}$$

and the initial conditions become

$$y(x_0) = a, \;\; z(x_0) = b. \tag{1.27}$$

Now the solution of the system of two first orders ordinary differential equation Eqs.(1.25) and (1.26) subject to the initial conditions Eq.(1.27) can be computed by the formula

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{1.28}$$

$$z_{n+1} = z_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \tag{1.29}$$

where

$$k_1 = hg(x_n, y_n, z_n),\ l_1 = hf(x_n, y_n, z_n), \tag{1.30}$$

$$k_2 = hg\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right), \tag{1.31}$$

$$l_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right), \tag{1.32}$$

$$k_3 = hg\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right), \tag{1.33}$$

$$l_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right), \tag{1.34}$$

$$k_4 = hg(x_n + h, y_n + k_3, z_n + l_3), \tag{1.35}$$

$$l_4 = hf(x_n + h, y_n + k_3, z_n + l_3), \tag{1.36}$$

where $h = \frac{b-a}{n}$ is uniform step size and n is total number of steps.

Chapter 2

Similarity Solution of Blasius and Pohlhausen flow

In this chapter, we revised the two very important boundary layer flows named as Blasius and Pohlhausen flows [4]. The governing partial differential equations are converted into system of ordinary differential equations by using similarity variables which are then solved numerically by well-known shooting technique [11], with fourth order Runge-Kutta integration scheme [11,12].

2.1 Blasius Flow

In this section, we discussed the Blasius flow problem.

2.1.1 Mathematical formulation

Consider an infinite long flat plate submerged in a steady, incompressible, two-dimensional flow, whose free stream velocity U and the free stream temperature T^* is uniform and constant. Let the Cartesian coordinate system coincides with the leading edge of the plate in such a way that, x-axis lies along the plate parallel to U and $T^*$and y-axis is perpendicular to the plate, as shown in figure below

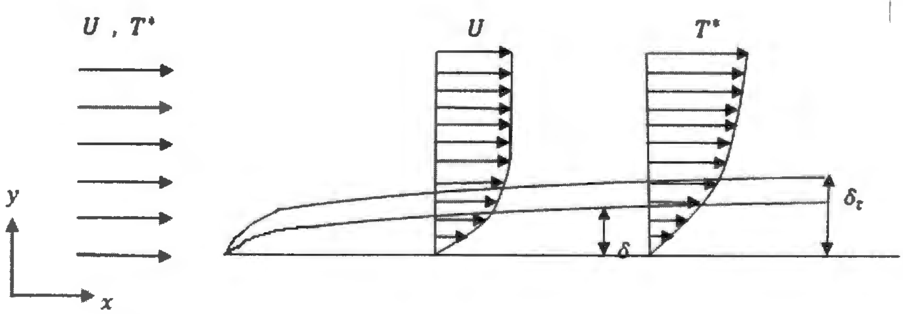


Figure : *Physcial model of Blasius and Pohlhausen visous flow*

The governing boundary layer equations are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial^2 y}\,, \tag{2.1}$$

$$\frac{dp}{dx}=-\rho U\frac{dU}{dx}\,, \tag{2.2}$$

and the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.3}$$

Let us consider the velocity U of the potential flow is constant, so that

$$\frac{dU}{dx}=0\,, \tag{2.4}$$

Consequently, Eq. (2.2) becomes

$$\frac{dp}{dx}=0. \tag{2.5}$$

Thus Eq. (2.1) reduces to the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial^2 y}\,, \tag{2.6}$$

in which u and v are the velocity components along the horizontal and vertical directions and ν is the kinematic viscosity of the fluid.

The Energy equation is

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\left(\frac{\nu}{Pr}\right)\frac{\partial^2 T}{\partial^2 y}\,, \tag{2.7}$$

where $\frac{\nu}{Pr}=\frac{k}{\rho c_p}$, k denotes thermal conductivity, ρ is density and c_p is the specific heat.

The boundary conditions are,

$$\begin{aligned} &u=0,\quad T=T_w,\ \text{when}\ \ y=0,\\ &u\to U,\ T\to T^*,\quad \text{when}\ \ y\to\infty. \end{aligned} \tag{2.8}$$

where U denotes free stream velocity and T^* denotes free stream temperature of the flow outside the boundary layer. Introducing the stream function $\Psi(x,y)$ satisfying the following relationship:

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{2.9}$$

We introduce non dimensional similarity variables as

$$\eta = y\sqrt{\frac{U}{x\upsilon}}, f(\eta) = \frac{\Psi(x,y)}{\sqrt{x\nu U}}. \tag{2.10}$$

Using Eqs.(2.9, 2.11) , Eq. (2.3) is identically satisfied and Eq. (2.6) reduces to the form

$$f''' + \frac{1}{2} f f'' = 0, \tag{2.11}$$

where the prime denotes differentiation w.r.t η. The boundary conditions are given in Eq. (2.8) becomes

$$f(0) = 0, f'(0) = 0, \text{ when } \eta = 0,$$
$$f'(\eta) \to 1, \text{ when } \eta \to \infty. \tag{2.12}$$

Eq. (2.12) together with boundary conditions (2.13) is called Blasius problem. In the next section, the solution of boundary value problem (2.11), (2.12) obtained by using shooting technique is explained in detail.

2.1.2 Numerical Solution of Problem

2.1.2.1 Shooting Method

Since the governing equation of the problem (2.11) subject to boundary condition (2.12) is nonlinear boundary value problem, the exact solution of which cannot be obtained by any mean. It is therefore, we used numerical scheme, i.e. the shooting technique in combination of fourth order Runge Kutta scheme. For this purpose, we need to convert modeled boundary value problem into a system of first order initial value problem as follow:

$$
\begin{aligned}
f &= x_1, \\
f' &= x_2, \\
f'' &= x_3, \\
x'_3 &= -\frac{1}{2}x_1x_3,
\end{aligned}
\tag{2.13}
$$

with initial conditions

$$
\begin{aligned}
x_1(0) &= 0, \\
x_2(0) &= 0, \\
x_3(0) &= s,
\end{aligned}
\tag{2.14}
$$

where s is the unknown missing initial condition which is to be determined in such a way that solution must meet the outer boundary condition $x_2\,(\infty) = 1$ i.e. $f'(\infty) = 1$.

2.1.3 Results and Discussion

The ordinary differential equation, (2.11) subject to the boundary conditions (2.12) is solved numerically by using shooting technique with fourth order Runge-Kutta algorithm. The obtained solution is expressed through figures from 2.1 to 2.3 against similarity variable η.

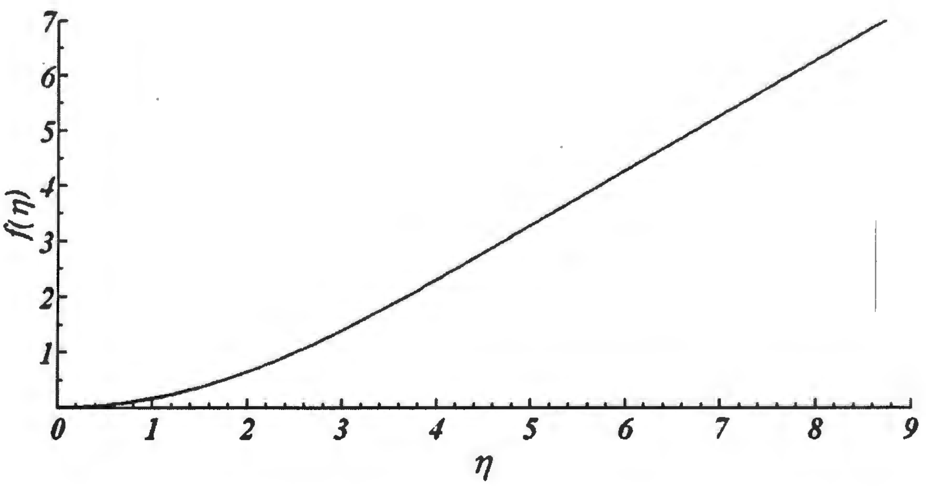


Figure 2.1: Graph of f against, similarity variable η.

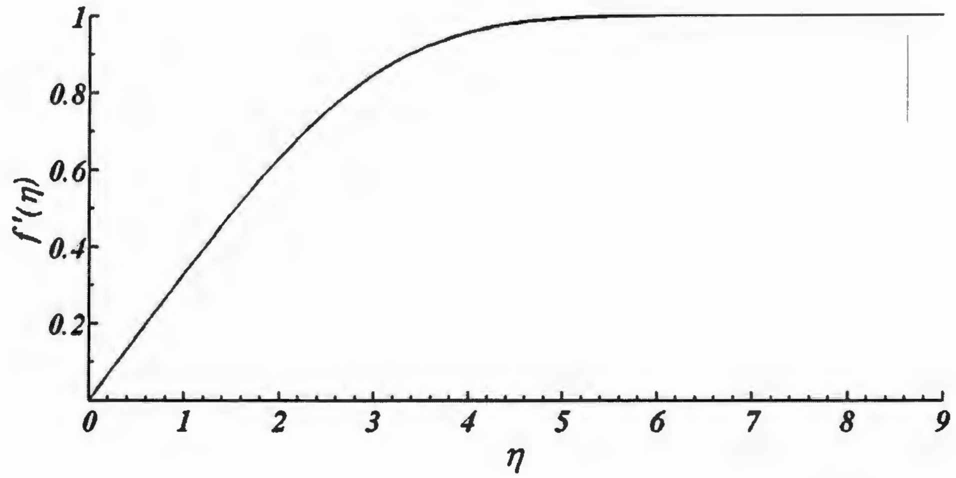


Figure 2.2: Velocity profile against similarity variable, η.

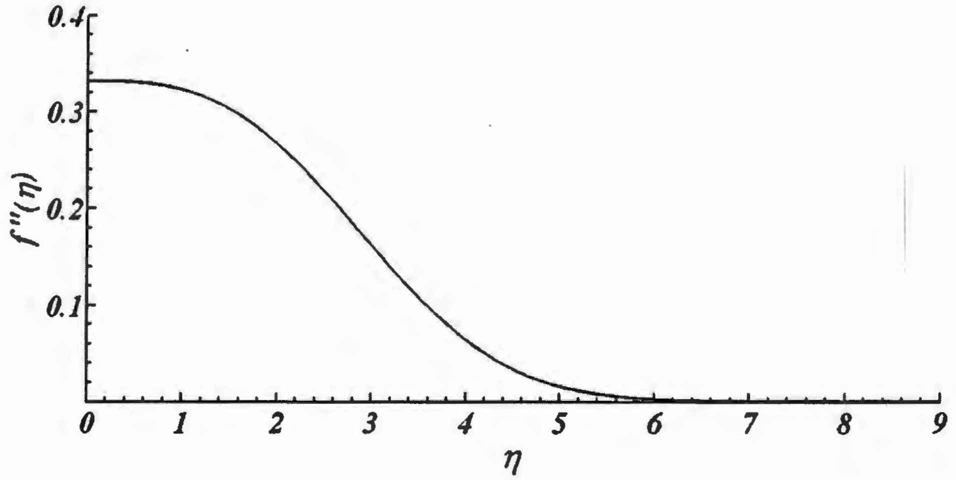


Figure 2.3: Graph of f'' against similarity variable, η.

2.2 Pohlhausen Problem

In this section, we revised the Pohlhausen problem

2.2.1 Mathematical Formulation

The geometry of the flow problem is already described in previous section. Introducing the similarity variable for temperature θ as follows with η defined in Eq.(2.10).

$$\theta(\eta) = \frac{T_w - T}{T_w - T^*}, \tag{2.15}$$

where T denotes the dimensional temperature and T_w and $T^*$are constant temperature at boundary and free stream respectively. Introducing the similarity variable (2.10) and (2.15), we get

$$u\frac{\partial\theta}{\partial x}+v\,\frac{\partial\theta}{\partial y}=\frac{\nu}{Pr}\,\frac{\partial^2\theta}{\partial^2 y}\,, \tag{2.16}$$

the boundary conditions (2.8) becomes

$$\begin{aligned}&\theta=0\ \ , \text{ when }\ y=0,\\ &\theta\rightarrow 1\ \ , \text{ when }\ y\rightarrow\infty.\end{aligned} \tag{2.17}$$

Using Eqs.(2.9 – 2.11) into Eq. (2.16), we get

$$\theta''+\frac{Pr}{2}\,\theta' f=0\,, \tag{2.18}$$

where prime denote derivative w.r.t η. The boundary condition in Eq. (2.17) can be written as

$$\begin{aligned}&\theta=0\ \ , \text{ when }\ \eta=0,\\ &\theta\rightarrow 1\,, \text{ when }\ \eta\rightarrow\infty.\end{aligned} \tag{2.19}$$

in which Pr is the Prandtl numbers. In order to solve Eq. (2.18) subject to the boundary conditions (2.19), shooting method is used explained as follows:

2.2.2 Numerical Solution of Problem

2.2.2.1 Shooting Method

Pohlhausen problem is like previous nonlinear boundary value problem and shooting technique with Runge-Kutta forth order method as integrator is used to construct its solution. The modeled boundary value problem (2.18) is converted into initial value system as

$$\begin{aligned}\theta &= y_1,\\ \theta' &= y_2,\\ y'_2 &= -\frac{Pr}{2}y_2 x_1,\end{aligned} \tag{2.20}$$

with initial conditions

$$y_1(0) = 0,$$
$$y_2(0) = t, \tag{2.21}$$

where t is the missing initial condition which is to be determined subject to satisfy the outer boundary condition i.e. $y_1(\infty) = 1$.

2.2.3 Results and Discussion

The nonlinear ordinary differential equation (2.18) subject to the boundary conditions (2.19) is solved numerically by using shooting method.

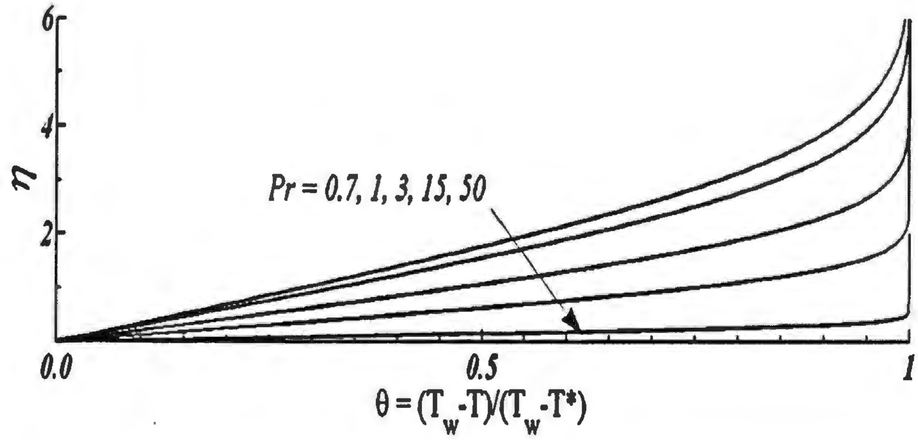


Figure 2.4: Graph of, θ with η, for distinct values of Pr.

The computed temperature profile θ against η, is shown in Figure 2.4 for different values of Prandtl number Pr. It is seen from Figure 2.4 that the trajectory of temperature profile θ against similarity variable η at $Pr = 1$, is identical to the trajectory of velocity profile f' against η which is shown in Figure 2.2. Table 2.1 is drawn to show the numerical values of $\theta'(0)$ and $0.332Pr^{1/3}$ against different values of Pr. It is shown that with the increase in the values of Pr, the value of $\theta'(0)$ also increases.

Pr	$A = \theta'\|_{\eta=0}$	$0.332Pr^{1/3}$
0.6	0.275	0.280
0.7	0.291	0.295
0.8	0.305	0.308
0.9	0.323	0.321
1.0	0.332	0.332
1.1	0.340	0.343
7.0	0.644	0.635
10.0	0.724	0.715
15.0	0.826	0.819

Table 2.1: Values of A for different values of Pr .

If a mean heat transfer coefficient for whole plate, $\bar{h}$ is defined as

$$\bar{h} = \frac{Q_w}{L(T_w - T_1)}\ , \tag{2.22}$$

where $Q_w = 2Ak(T_w - T_1)\sqrt{\frac{U}{x\upsilon}}$, then Eq. (2.22) becomes

$$\bar{h} = \frac{2Ak}{L}\sqrt{\frac{U}{x\upsilon}}\ , \tag{2.23}$$

where $A = 0.332Pr^{1/3}$. The mean Nusselt number for the whole plate $Nu_L(= \frac{\bar{h}L}{k})$, is therefore given as

$$Nu_L = 2ARe_L{}^{1/2}, \tag{2.24}$$

where Re_L is the Reynolds number based on the plate length L. Using value of A into Eq. (2.24) we get mean Nusselt number,

$$Nu_L = 0.664Pr^{1/3}Re_L{}^{1/2}. \tag{2.25}$$

The mean Nusselt number for the whole plate is drawn in Figure 2.5 to show the behavior of predicted and experimental mean Nusselt number against Reynolds number when Pr = 1 is fixed.

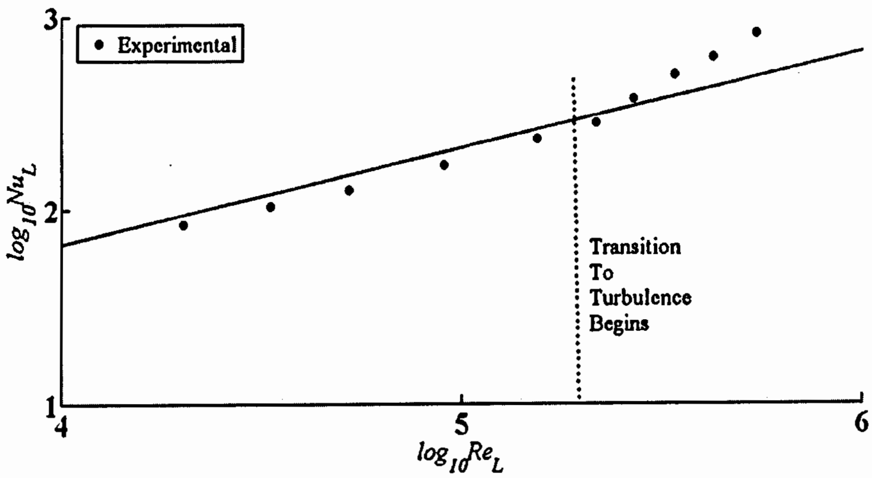


Figure 2.5: Behavior of Nusselt number Nu_L against Reynolds number Re_L.

2.3 Example

The similarity solution of the laminar boundary layer flow can be used in different ways. In laminar boundary layer equations, similarity solution can be used as derived above. We can plot the local and mean heat transmission and also velocity and temperature profiles in the boundary layer, at the edge of a plate. We are given, air flows at a velocity of 5m/s, length is 20cm and the mean temperature is kept at 50°C.

Here, in the air temperature

$$Ꝁ = \frac{0.0278w}{m}°C\ , \qquad \nu = \frac{0.0000179m^2}{s},$$

and

$$A = \theta'|_{\eta=0} = 0.293,$$

where Ꝁ is thermal conductivity and ν denote the viscosity of the fluid. Local heat transfer rate is

$$q_w = Ak(T_w - T_1)\sqrt{\frac{u_1}{x\nu}}\ . \tag{2.26}$$

The variation of q_w with x is shown in Figure 2.6, twice the local heat transfer rate is the mean heat transfer rate at the end of Plate as

$$\overline{q}_w = 2q_w. \tag{2.27}$$

In the present case, the velocity and temperature profiles at the end of the plate are required so using the similarity variable η and θ as defined in Eq. (2.10). At the trailing edge of the plate

$$y = \eta\sqrt{\frac{x\nu}{U}}\ . \tag{2.28}$$

The variation of values of y is also in Figure 2.7. We get velocity and temperature profile, using similarity variables as

$$\frac{u}{u_1} = f'. \tag{2.28}$$

and

$$T = 80 - 60\theta. \tag{2.29}$$

2.3.1 Result and discussion

We made comparison between local and mean heat transfer rates, shown in figure 2.6.

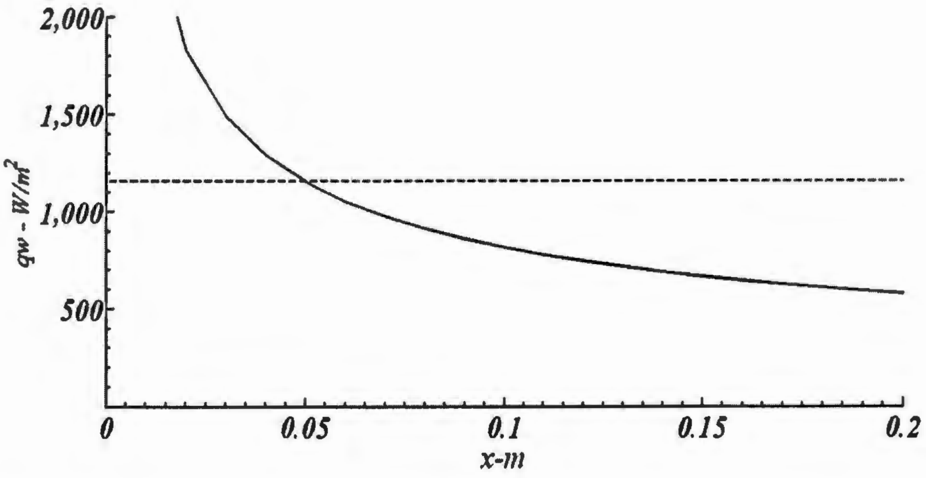


Figure 2.6: Variation of q_w (solid line) and $\overline{q_w}$ (dashed line) against x.

Table 2.2: Using the variation of f' with η and θ with η for Pr = 0.7, a table of the following form can be formulated as:

η	f'	θ	$y(cm)$	$u(m/s)$	$T(°C)$
0	0	0	0	0	80.0
0.5	0.166	0.145	0.0423	0.829	71.28
1.0	0.330	0.290	0.0846	1.649	62.6
2.0	0.630	0.561	0.169	3.149	46.4
3.0	0.846	0.780	0.254	4.230	33.4
4.0	0.956	0.912	0.338	4.778	25.3
5.0	0.992	0.974	0.423	4.958	21.6
6.0	0.999	0.995	0.508	4.995	20.3
8.0	1.000	1.000	0.677	5.000	20.0

The variation of velocity and temperature profiles, that are given in the above table are plotted in Figures 2.7 and 2.8.

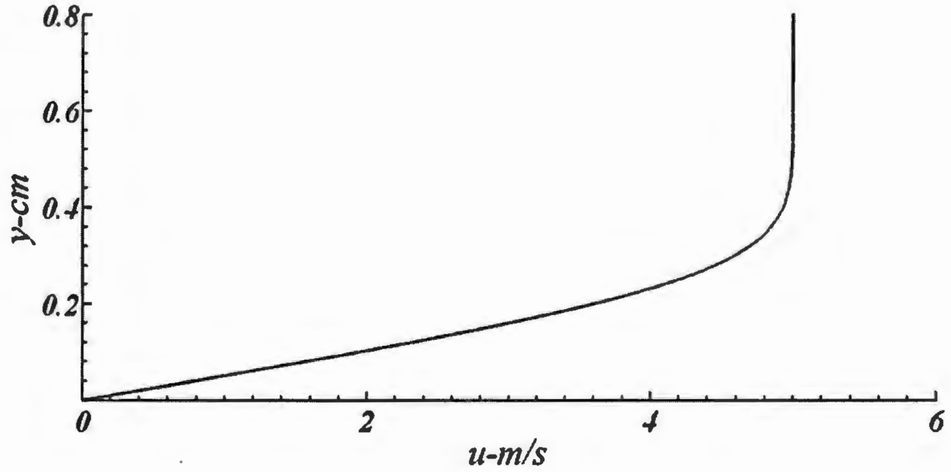


Figure 2.7: Variation of velocity u against y.

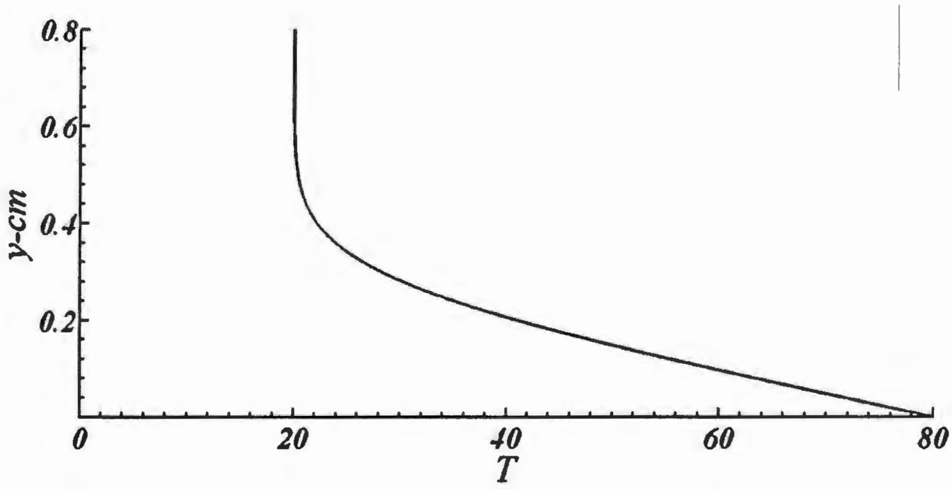


Figure 2.8: Graph of temperature profile, T against y.

In present case, we are using the similarity solutions of the boundary layer flow of Blasius and Pohlhausen flow which are derived above. We have plotted local and mean heat transfer rate as shown in figures 2.7 and 2.8 along with the plate and also velocity and temperature profiles in the boundary layer at the end of the plate. We have drawn table 2.2, using similarity variables η and θ described in Eq.(2.10) and using the values that are given above in example, and then using these similarity variables we made relation for velocity and temperature profiles.

Chapter 3

Similarity Solution For Flow over Flat Plate with other Thermal Boundary Conditions

3.1 Mathematical Modeling

In previous chapter, we are done with the case in which plate has a uniform temperature (i.e. Pohlhausen flow). In this chapter, we revised some study subjected to the flat plate in which temperature varies with x. Moreover, the heat transfer analysis of Falkner-Skan flow is also investigated in this chapter. Numerical solutions of both problem is computed with the help of shooting method. The complete procedure of the method is elaborated in detail in this chapter. The graphs are prepared for accelerating, decelerating and constant flow. The numerical values of heat transfer rate at the wall is compared with the results available in the literature. In this case thermal condition will be

$$T_w - T^* = Cx^n \tag{3.1}$$

the following non-dimensional temperature is introduced as

$$\theta(\eta) = \frac{T_w - T}{T_w - T^*} = 1 - \frac{T - T^*}{T_w - T^*}\,, \tag{3.2}$$

we consider that θ depend on similarity variable, η as prescribed above.

The energy equation (2.7) can be written in form of θ as

$$-u\frac{\partial}{\partial x}[(1-\theta)(T_w - T^*)] + v\frac{\partial\theta}{\partial y}(T_w - T^*) = \left(\frac{\nu}{Pr}\right)\frac{\partial^2\theta}{\partial^2 y}(T_w - T^*). \tag{3.3}$$

and the boundary conditions become

$$\theta(0) = 0, \text{ as } \eta = 0,$$
$$\theta(\eta) \to 1, \text{ as } \eta \to \infty. \qquad (3.4)$$

After using Eq.(2.9), (2.10), (2.11) and (3.1), reduces to Eq. (3.3),

$$\theta'' + nPrf'(1-\theta) + \frac{Pr}{2}\theta' f = 0\,, \qquad (3.5)$$

where n is a parameter and boundary conditions in Eq.(3.4) becomes:

$$\eta = 0 : \theta = 0,$$
$$\eta \to \infty : \theta \to 1. \qquad (3.6)$$

3.1.1 Result and discussion

The nonlinear ordinary differential equation (3.4) subject to the boundary conditions (3.5) is solved by using MATLAB built in function Bvp4c. The computed temperature profile for pertinent values of Pr against n can be predicted from the Figure 3.1. It is shown that with the increase in Prandtl number, heat transfer rate or flat plate increases. Similarly, the parameter n is also responsible to enhance the heat transfer rate at the surface.

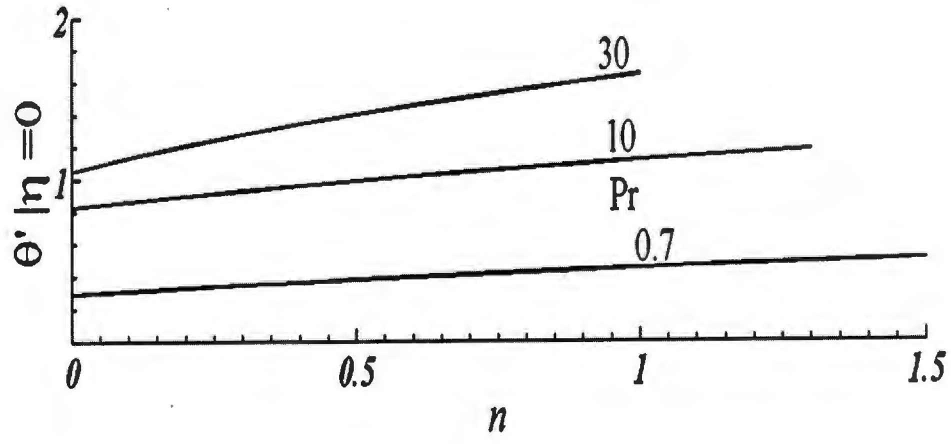


Figure 3.1: Variation of $\theta'|_{\eta=0}$, with n, for distinct values of Pr.

3.2 Falkner-Skan Problem

In this section we shall revise Falkner-Skan flow [9]. Mathematical formulation is made for the reader, then the solution is obtained by shooting method.

3.2.1 Mathematical Modeling

Consider a flow, around a wedge submerged in a fluid of a small viscosity. At a leading edge (i.e. stagnation point) O, thickness of boundary layer is zero and velocity increases from zero at the wall to the value of potential flow at the edge of the boundary layer. It is assumed that free stream velocity U is constant and uniform at the edge of the boundary layer. Suppose that the x-axis lying along the wall of the wedge and y-axis is perpendicular to it as shown in figure.

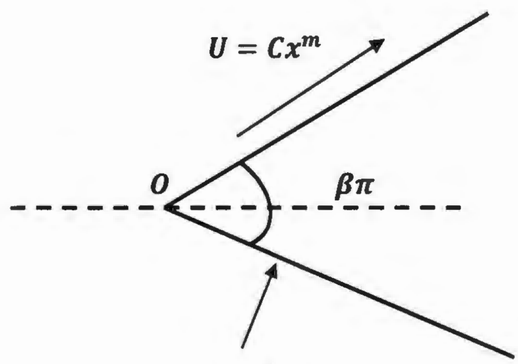


Figure I: Physical model of Falkner – Skan flow.

For the case of steady flow, the Prandtl boundary layer equation are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \qquad (3.7)$$

with continuity equation given in Eq. (2.3), where u and v are the velocity components in x and y directions of fluid flow respectively, U is the reference velocity at the edge of the boundary layer. Furthermore, the velocity of potential flow is assumed to be proportional to a power of the length coordinate x along the wall. The boundary conditions are

$$\begin{aligned} y = 0 &: u = 0, v = 0, \\ y \to \infty &: u \to U(x) = Cx^m, \end{aligned} \qquad (3.8)$$

where m is the power law parameter of Falkner-Skan and x is measured from tip of wedge. The stream function $\Psi(x, y)$ is introduce given in Eq. (2.9) and using following similarity transformation

$$\eta = y\sqrt{\frac{(m+1)U}{2x\nu}}, \theta(\eta) = \frac{T_W - T}{T_W - T^*} \tag{3.9}$$

and

$$f(\eta) = \sqrt{\frac{(m+1)}{2\nu Cx}}\Psi. \tag{3.10}$$

in Eqs. (3.9) and (3.10), we get

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0, \tag{3.11}$$

subject to the boundary conditions

$$\begin{aligned} f(0) &= 0, f'(0) = 0, \\ f'(\eta) &\to 1, \text{ as } \eta \to \infty, \end{aligned} \tag{3.12}$$

where prime denotes derivative w.r.t η, and β is related to m through relation

$$\beta = \frac{2m}{m+1}. \tag{3.13}$$

The energy equation (2.7)can be written as,

$$\theta'' + \left(\frac{m+1}{2}\right) Pr\, \theta' f = 0, \tag{3.14}$$

The boundary conditions on θ, are same as was defined in flat plate problem, i.e.

$$\begin{aligned} &\theta = 0, \text{ as } \eta = 0, \\ &\theta \to 1, \text{ as } \eta \to \infty. \end{aligned} \tag{3.15}$$

The parameter Pr is the Prandtl numbers, and m is constant. In order to solve Eqs. (3.11), (3.14) subject to the boundary conditions (3.12), (3.15) , shooting method is used which is explained in the following section.

3.2.2 Method of Solution

3.2.1.1 Shooting Method

Since boundary value problem is non-linear like discussed in previous chapter, therefore we used shooting method with Runge Kutta fourth order as integrator to construct its solution. The boundary value problem Eq.(3.11) – (3.12) is reduced into initial value problem as

$$\begin{aligned} f &= z_1, \\ f' &= z_2, \\ f'' &= z_3, \\ z'_3 &= -\left(z_1 z_3 + \beta(1 - z_2^2)\right), \end{aligned} \tag{3.16}$$

with initial conditions

$$\begin{aligned} z_1(0) &= 0, \\ z_2(0) &= 0, \\ z_3(0) &= t, \end{aligned} \tag{3.17}$$

where t is the missing initial condition. It is found in such a way that the solution satisfies the outer boundary condition i.e. $z_2(\infty) = 1$.

Now the modeled boundary value problem (3.13) is converted into initial value system as

$$\begin{aligned} \theta &= y_1, \\ \theta' &= y_2, \\ y'_2 &= -\frac{Pr}{2} y_2 x_1, \end{aligned} \tag{3.18}$$

with initial conditions

$$\begin{aligned} y_1(0) &= 0, \\ y_2(0) &= s, \end{aligned} \tag{3.19}$$

where s is the missing initial condition. It is found in this manner that the solution must meet the outer boundary condition i.e. $y_1(\infty) = 1$.

3.2.3 Results and Discussion

The ordinary differential equation, (3.11) subject to the boundary conditions (3.12) is obtained numerically using shooting scheme. The solution of the Falkner Skan equation corresponding to $\beta > 0$ are known as accelerating flows, those corresponding to $\beta = 0$ are knows as the constant flows and those corresponding to $\beta < 0$ are known as decelerating flows. Physically relevant solution exists for $-0.987 \leq \beta \leq 1.6$.

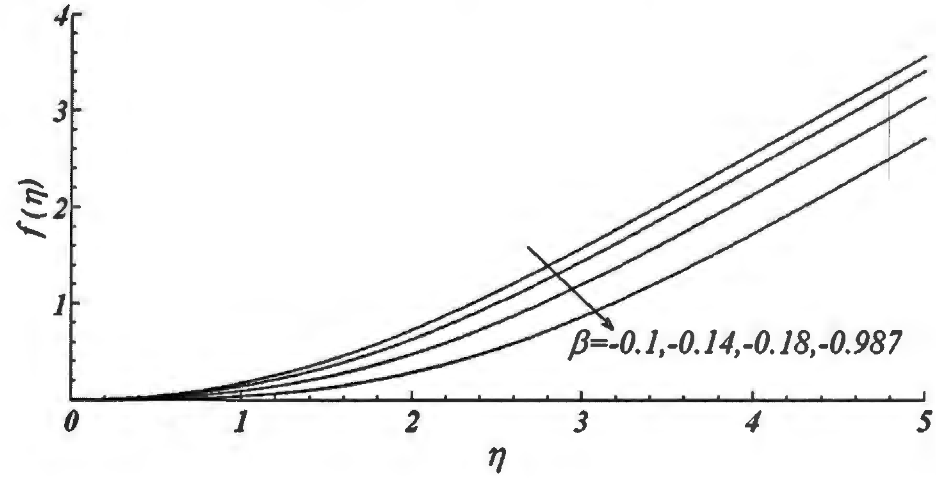


Figure 3.2: Variation of f, against similarity variable η for $\beta < 0$ in case of decelerating flows.

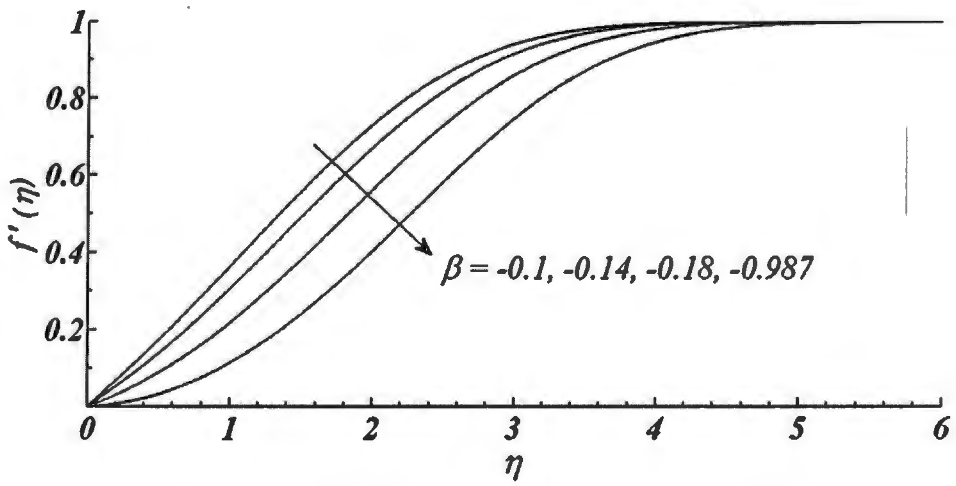


Figure 3.3: Variation of velocity f' ,against similarity variable η for $\beta < 0$ in case of decelerating flows.

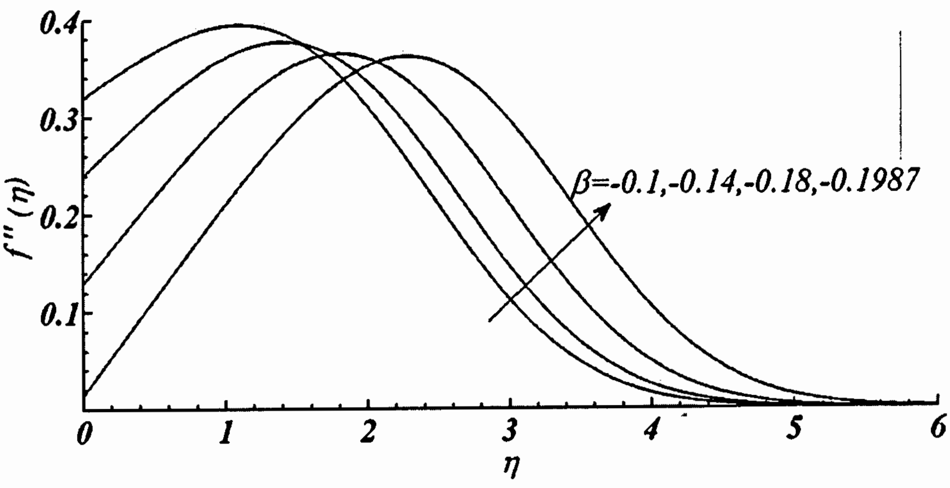


Figure 3.4: Variation of f'', against similarity variable η for $\beta < 0$ in case of decelerating flows.

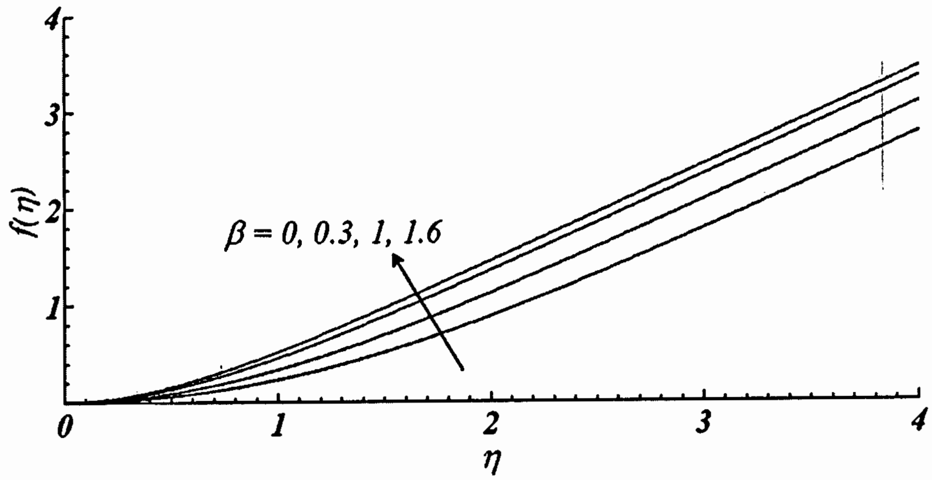


Figure 3.5: Variation of f, against similarity variable η for $\beta > 0$ in case of accelerating flows.

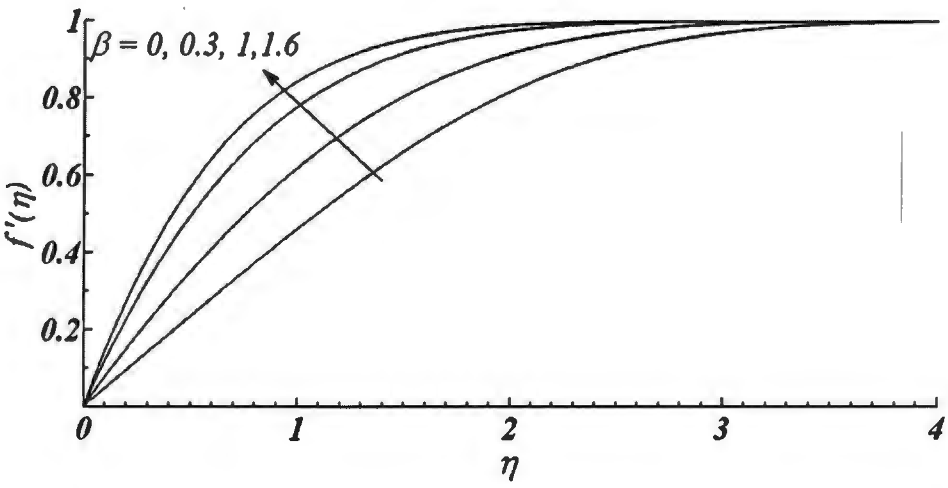


Figure 3.6: Variation of f', against similarity variable η for $\beta > 0$ in case of accelerating flows.

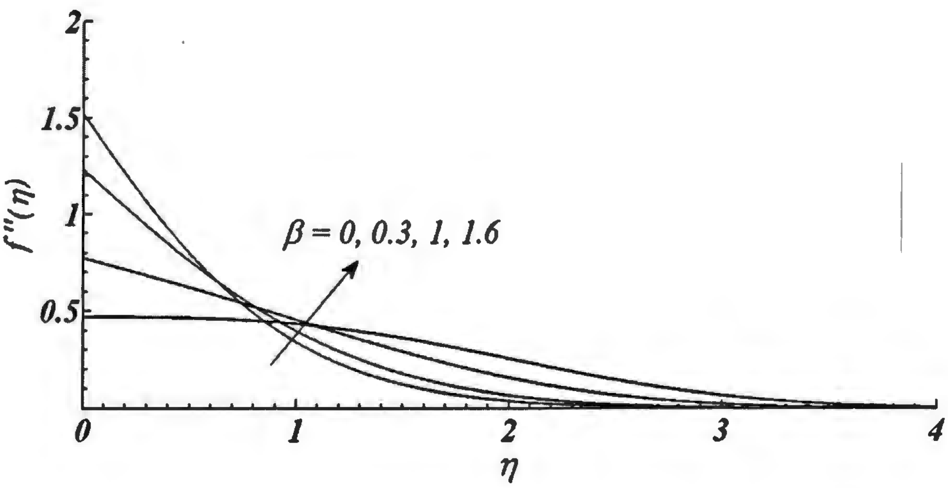


Figure 3.7: Variation of f'', against similarity variable η for $\beta > 0$ in case of accelerating flows.

The ordinary differential equation, (3.13) subject to the boundary conditions (3.14), for selected values of m and Pr, is solved numerically using shooting scheme.

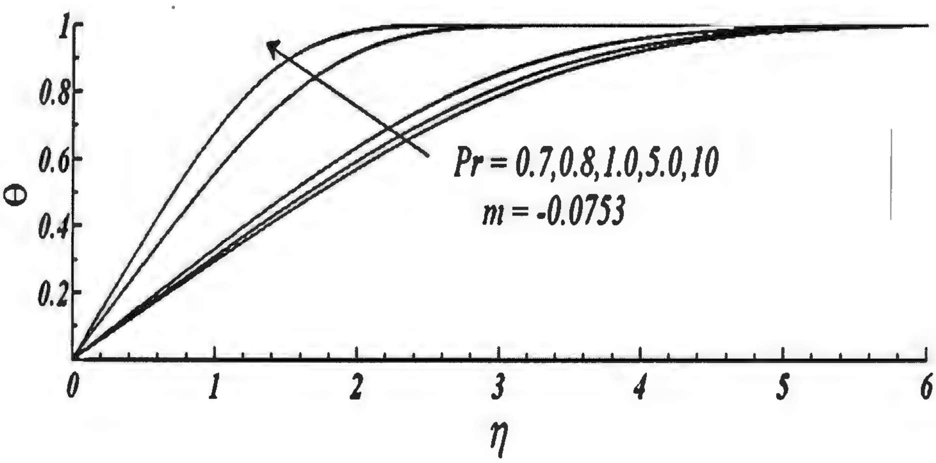


Figure 3.8: Graph of θ with η, for distinct values of Pr with fixed value of m.

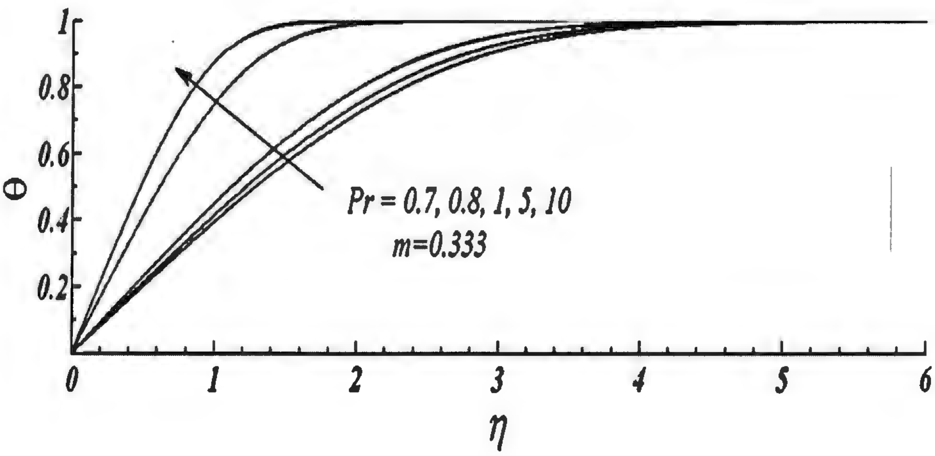


Figure 3.9: Graph of θ with η, for distinct values of m with fixed value of Pr.

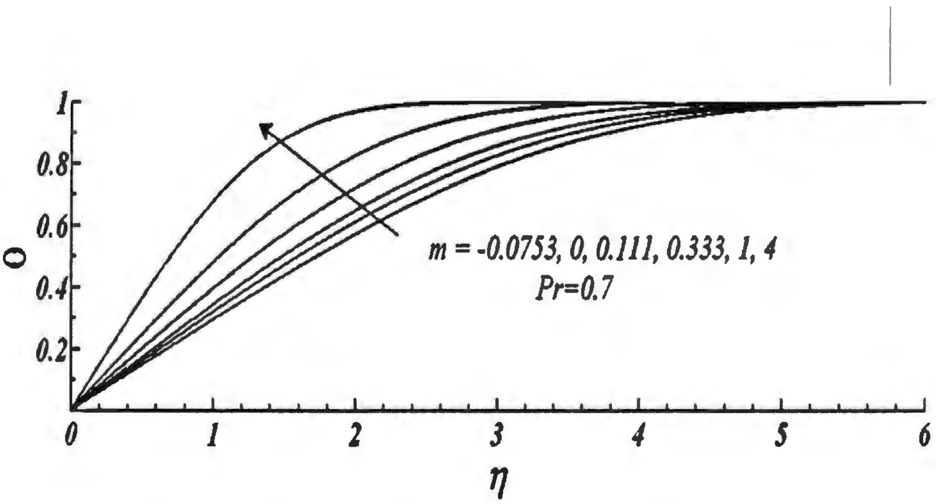


Figure 3.10: Graph of θ with η, for distinct values of m with fixed value of Pr.

	Pr				
m	0.7	0.8	1.0	5.0	10.0
−0.0753	0.297	0.310	0.335	0.576	0.722
0.0	0.322	0.338	0.366	0.643	0.812
0.111	0.347	0.364	0.395	0.702	0.890
0.333	0.401	0.423	0.461	0.836	1.069
1.0	0.495	0.521	0.569	1.038	1.329
4.0	0.737	0.776	0.845	1.523	1.945

Table 3.1: Variation of $\theta'(0)$ for various values of m and Pr.

Figures 3.2 to 3.10 are drawn to show the effects of different parameters including β, m and Pr on the velocity and temperature profiles against similarity variable η. The effects of β for decelerating flow case ($\beta < 0$) on f , f' and f'' against η are shown through Figures 3.2-3.4 respectively. It is seen that the effect of β on f as a decelerating flow case is considerable only in the region $0 \leq \eta \leq 3$ and after this region, f grows linearly along η for all $\beta < 0$ as shown in Figure 3.2. It is observed from Figure 3.3 that with the increase in the magnitude of β for decelerating flow case, the velocity f' decreases in the boundary layer region. However, the momentum boundary layer thickness increases with the increase in the magnitude of β for decelerating flow case as shown in the figure. The behavior of $f''(\eta)$ against η for different β is shown in Figure 3.4. Similarly, the effect of β as accelerating flow case on f , f' and f'' are shown in Figures 3.5-3.7 respectively. It is noted from the Figure 3.5 that the region for which the effects of β on f as accelerating flow case is small as compared to that of decelerating flow case as shown in Figure 3.2. It is further seen that f decreases with the increase in β for accelerating flow case. It is pertinent to mention that the boundary layer thickness is minimum in case of accelerating flow as shown in Figure 3.6 as compared to that of deceleration flow case shown in Figure 3.3. The increase in magnitude of β in this case helps to increase the velocity within the boundary layer. The momentum boundary layer thickness decreases with the increase in magnitude of β for accelerating flow case which is quiet opposite to that of deceleration flow case. The behavior of f'' against η for different β as accelerating flow case is shown in Figure 3.7. The effects of different values of Pr on the temperature profile for $m = -0.0753$ and $m = 0.333$ are shown in Figures 3.8 and 3.9 respectively. It is noted that with the increase in Prandtl number Pr, the temperature within the boundary layer increases, but thermal boundary layer thickness reduces against Pr as shown in Figures 3.8 and 3.9.

Figure 3.10 is drawn to show the effect of parameter m on the temperature profile. It is seen that this parameter is also responsible to augments heat within boundary layer, but thermal boundary thickness is found as decreasing function of the parameter m. Table 3.1 is drawn to show the numerical values of $\theta'(0)$ as heat transfer rate at the surface at different values of parameter m and Pr. It is noted that heat transfer rate increases due to increase in both parameters m and Pr.

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