Bayesian Estimation of Money Demand Function of Pakistan Economy

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بسم الله الرحمن الرحيم

In the Name of ALLAH

The compassionate the merciful

He is the most omniscient.

(And he to whom wisdom is grante

reciveth indeed a benefit overflowing)

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A Thesis

Submitted in the Partial Fulfillment of the Requirements for the Degree of

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In
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Supervised By

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN STATISTICS

We accept this dissertation as conforming to the required standard.

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Dedicated

To

My beloved

Papa and Mama
and
Respected teachers

With their efforts and support I would not be able to complete this task, without their Love, Prayers, and Support, I am nothing.

Forwarding Sheet by Research Supervisor

The thesis entitled '**Bayesian Estimation of Money Demand Function of Pakistan Economy**' Submitted by **Qasim Jan** (Registration # **42-FBAS/MSST/S14**) in partial fulfillment of MS degree in Statistics has been completed under my guidance and supervision. I am satisfied with the quality of his research work and allow him to submit this thesis for further process to graduate with Master of Science degree from Department of Mathematics and Statistics, as per IIU Islamabad rules and regulations.

Dated ____________________

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Qasim jan

DECLARTION

I hereby declare that this thesis neither as a whole nor as a part there of, has been copied out from any source It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my supervisor **Dr. Muhammad Akbar** No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning

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TABLE OF CONTENTS

List of Tables v

List of Figures vi

List of Abbreviation vii

Abstract viii

CHAPTER 1 .... **1-4**

Introduction **1**

Chapter 2 .... **5-24**

Literature review **5**

2.1 Introduction to Bayesian Analysis 5

2.2 Bayes theorem and Bayesian Methodology 5

2.3 Likelihood Function 7

2.4 Prior distribution 7

2.4.1 Informative Prior 8

2.4.2 Non-informative 8

2.5 Choice Prior Distribution 9

2.5 MCMC Simulation and Bayesian Methodology 10

2.7 Gibbs Sampling 10

2.8 Elicitation 11

2.9 Burn in and Thinning 11

2.10 Posterior distribution 12

2.11 Posterior Predictive distribution 13

2.12 What is Money 13

2.13 What is the role of Money in an Economy 14

2.14 Money Demand 15

2.15 Literature Regarding Bayesian regression 15

2.16 Literature Regarding Money Demand in Pakistan 20

Chapter 3 25-47

Materials and Methods 25

3.1 Introduction 25

3.2 Specification of the Model 25

3.3 Bayesian multiple linear Regression model in matrix form 26

3.4 Likelihood Function 27

3.5 Non-informative prior for linear model 29

3.6 Normal linear Regression model through with independent NG prior 30

3.7 Posterior distribution under A Normal-Gamma prior 31

3.8 Building elicited priors utilizing linear regression 32

3.9 Construction of elicited priors on τ utilizing in linear regression 34

3.10 Elicitation of hyper-parameters 37

3.11 Gibbs sampling designed for linear regression model 38

3.12 Convergence diagnostic tests for BLR 42

3.12.1 Trace plots 42

3.12.2 BGR convergence diagnostic test 43

3.12.3 MC standard error 44

3.12.4 Autocorrelation function 45

3.12.5 Overall goodness-of-fit of model 45

3.13 Credible interval 45

Chapter 4 .. .. **48-69**

Result and Discussion **46**

4.1 Introduction 48

4.2 Posterior results using non-informative prior through UP 48

4.3 Convergence diagnostic plots of UP 50

4.3.1 Trace plots of UP 50

4.3.2 Kernel density plots of UP 52

4.3.3 Autocorrelation Function plots of UP 53

4.3.4 BGR plots of UP 54

4.4 Posterior results using informative prior through NG 55

4.5 Convergence diagnostic plots of NG prior 57

4.5.1 Trace plots of NG prior 57

4.5.2 Kernel density plots of NG prior 59

4 5 3 Autocorrelation plots of NG prior 60

4 5 4 BGR plots of NG prior 61

4 6 Posterior predictive diatribution and Simulation analysis of interest rate 62

Chapter 5 . . . 70-71

Summary, Conclusion and Further Research 70

References . . . 72-74

LISIT OF TABLES

Table 3 1 Elicited values of Hyper parameters 38

Table 4 1 Posterior Results Using Non-Information Through UP 49

Table 4 2 Posterior Results Using Informative prior Through NG 56

Table 4 3 Prediction Results 64

LIST OF FIGURES

Figure 4 1	Trace plots of UP	51
Figure 4 2	Kernel density plots of UP	53
Figure 4 3	ACF plots of UP	54
Figure 4 4	BGR plots of UP	55
Figure 4 5	Trace plots of NG	58
Figure 4 6	Kernel density plots of NG	60
Figure 4 7	ACF plots of NG	61
Figure 4 8	BGR plots of NG	62

LIST OF ABBREVATIONS

MCMC	Markov Chain Monte Carlo
MCSE	Monte Carlo standard Errors
L	Likelihood
UP	Uniform Prior
NG	Normal Gamma
E	Expectation
BGR	Brooks Gelman Rubin
RIP	Reference Information Prior
ML	Maximum Likelihood
DF	Degree of Freedom
ACF	Autocorrelation Function
X^T	X Transpose
OLS	Ordinary Least Square
BLR	Bayesian linear regression

ABSTRACT

Econometric analysis can be done by either classical inferential approach or Bayesian inferential approach. Review of literature shows that Classical approach is popular while studies related to application of Bayesian econometric techniques are rarely available in the literature of applied econometrics. Hence application of Bayesian inferential approach for estimation of money demand function is the main objective of this study. Money demand function is specified and then data of the relevant variables of Pakistan economy are collected from various issues of Pakistan Economic Survey. Model's parameters are simulated under non-informative and informative priors while conducting MCMC simulations using Gibbs sampling. Uniform priors are taken as non-informative priors while Normal-Gamma priors are taken as informative priors. Hyper parameters of informative prior density are elicited using PV method of elicitation. Simulated results show that aggregate demand and price level positively affect money demand in Pakistan economy. Interest rate inversely affects money demand. Size of parameters may be considered for monetary policy actions. Moreover, precision of model is larger in case of informative priors as compare to non-informative priors. It implies that elicitation of hyper parameters on the basis of experts' opinion is useful for the model under consideration. At the end posterior predictive distribution is derived and money demand is predicted for different levels of interest rate.

CHAPTER 1

INTRODUCATION

Statistics is the science that communicates information to particular inquiries of interest. It has spread applications in different fields like commerce, engineering, medical sciences, social sciences, applied and natural sciences etc. It is not the science of only gathering information, but it also provides a set of tools for analyzing, clarifying, modeling and understanding the phenomena on the basis of information.

Research in statistics can be categorized into two types, i.e. theoretical and applied statistics. Theoretical statistics deals with the development of rules and techniques for collection, presentation and analysis of the data. While application of these techniques in order to analyze a particular phenomenon lies in the area of applied statistics. Each of these two areas can further be classified by two approaches that is classical approach and Bayesian approach. Bayesian and frequentist methodologies have distinct points of view. In Bayesian procedure, we presume that the data under the observation is fixed and model parameters are random, whereas in frequentist procedure presumed model's parameters to be constant.

In Bayesian procedure, prior information is incorporated into existing information to acquired improved information known as posterior information and entire examination is accomplished on the basis of this improved knowledge. Whereas the entire results in frequentist approach are based on only the current information. In frequentist analysis, estimators are employed to estimate the true values of the unknown parameters. Its inference gives a complete distribution of the parameters. Bayesian analysis depends on the parameters of the posterior distribution and provides summaries of this distribution on the basis of probability On the other hand we are unable to have direct probabilistic analyses in frequentist procedure Frequentist procedure is based on the sampling distribution of estimators of parameters and provides parameters point estimates and their standard errors as well as confidence interval In frequentist procedure confidence interval do not have uncomplicated enunciation as does Bayesian credible interval

Hence it may be concluded here that application of Bayesian inference may provide better results as compared to classical inference in order to analyze any particular phenomenon However the literature shows very limited such studies that contain application of Bayesian inferential procedure to general linear regression model Hence, application of Bayesian approach for modeling of a real phenomenon may be a significant contribution in research literature

Monetary policy is considered as the most important demand management macroeconomic policy which plays an important role to counter short run fluctuations in the economy Successful monetary policy actions are important to stabilize the economy while misleading actions of monetary policy makers destabilize the economy State Bank of Pakistan has been following monetary targeting strategy where money supply is adjusted to meet money demand in order to achieve equilibrium in the economy Hence role of money demand has significance for successful monetary policy actions in Pakistan and accurate knowledge about determinants of money demand in the economy is very important A number of studies available in the literature containing money demand functions of the Pakistan economy as well as of other economies of the world In all these studies classical econometric techniques have been used to estimate the models There is no such study available in the literature that contains money demand function estimated by Bayesian approach. Hence, application of Bayesian inference to estimate and analysis of money demand function using data of Pakistan economy may be a significant contribution in the literature of applied econometrics and Statistics. Keeping in view the above discussion, the following objectives are specified in the proposed study.

Objectives of the study

The key objectives of our study are follows:

- Estimation of Money Demand Function of Pakistan Economy by Employing Bayesian Econometric Techniques
- Econometric Analysis of the Phenomena by Using Non-informative and informative prior
- Comparison of the results under informative and non-informative priors
- Policy Analysis using MCMC simulations
- To give policy implication

To achieve these targets, outlines of the present study are as follows.

Chapter 2 is concerned with the basic elements of Bayesian statistics. We present Bayes theorem and Bayesian methodology, MCMC simulation and Bayesian approach and prior distribution (informative and non-informative). We similarly define the MCMC simulation through Gibbs sampling and elicitation. Posterior distribution and posterior predictive distribution are also explained. The definition of money money demand and the importance of money are also provided in this chapter. Comprehensive review of the existing literature of the Bayesian regression and estimation of money demand of Pakistan is presented.

In Chapter 3, we present specification of the model and description of the money demand model with notation Posterior distributions using informative and non-informative priors are derived Moreover MCMC simulations using Gibbs sampling design elicitation of hyper parameters and different diagnostic tests are explained in this chapter Chpater-4 contains results and discussion while the last chapter contains conclusion of the study

CHAPTER 2

LITERATURE REVIEW

2.1. Introduction to Bayesian Analysis

Bayesian procedure is considered superior to the classical approach due to incorporation of prior information Bayesian perspective is basically based upon implementation of Bayes theorem Under Bayesian inference parameters are considered as random variables and hence, their estimates are simulated through constructing probability density functions on the basis of all available information While in classical approach parameters are considered to be constant

2.2. Bayes Theorem and Bayesian Methodology

To introduce the Bayesian methodology suppose A & B are two random variables According to the conditional probability we can write as

$$p(A,B) = p(B)\,p(A|B)$$

or

$$p(A,B) = p(A)p(B|A)$$

i-e

$$p(B)p(A|B) = p(B|A)p(A)$$

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Comparing this dual statement for $p\,(A,B)$ and reordering give us with Bayes rule

Suppose y be a vector or matrix of data and θ be a vector of the parameters for a model which search to describe an explanation of y We have the information about θ based on y This can be achieved by employing Bayes rule In Bayesian we would substitute B by y and A by θ to obtain the following form

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

The kernel density can be expressed as

$$p(\theta|y) \propto p(y|\theta)p(\theta),$$

where $p(\theta|y)$ is the posterior density $p(y|\theta)$ is the likelihood function and $p(\theta)$ is the prior density This is frequently mentioned as posterior is proportional to likelihood times prior koop' (2003)

The modelling estimation and inferences in Bayesian methodology contains the following steps

- Derivation of likelihood function
- Construction of a prior distribution for the parameters of the model
- Derivation of the posterior distribution for the model parameters
- Application of simulation procedure to estimate parameters of the model
- Prediction through posterior predictive distribution

In usual Bayesian statistical inference the main objective is that to acquire the posterior distribution of model parameters The posterior distribution can be best comprehend as a weighted average among knowledge about the parameters earlier data is observed (which is represented by the prior distribution) and the information about the parameters hold inside in the observed data (which is represented by the likelihood function) From a Bayesian point of view any inferential question can be responded by a suitable analysis of the posterior distribution By gaining the posterior distribution and also calculated that point estimate and interval estimates of parameters, prediction outcomes for future knowledges and probabilistic inference of hypothesis

2.3. Likelihood Function

The likelihood function is a collection of observations i e X_1 X_2 X_n which is joint probability density function this function is considered for unidentified parameter such as π The likelihood is due to the value of parameter π for which the function $L(\pi)$ has a large value respectively which is further likely to be the right value of un-known parameter This function consists of all important information and is represented by $L(\pi, X_1\ X_2,\ \ X_n)$ It shows that the probability density function of information is conditional on π by $f(X_1, X_2\ \ X_n/\pi)$

Mathematically the joint density can be expressed as

$$L(\pi, X_1\ X_2\ \ X_n) = f(X_1\ X_2\ \ X_n/\pi)$$

2.4. Prior Distribution

Prior distribution is the core concept of Bayesian statistics It is constructed in order to incorporate extra information about uncertain parameters When prior distribution is combined with current information represented by likelihood it results in posterior distribution The major difference among classical and Bayesian approach is the utilization of prior distribution in various analysis Prior distribution is the probability distribution of the parameters providing statistical information about parameters, before detecting or examining the information which is very critical for the model parameters This information predominantly depends upon researcher's preceding knowledge personal opinions or expert judgement theoretical information historical information or may be these facts are derived from literature In Bayesian concept, the Prior distribution has a major role in estimation, particularly when the data doesn't provide clear information Prior distribution is mainly categorized as informative and non-informative priors Some authors divide it into four major branches which are named as informative non-informative least informative and weakly informative

2.4.1. Informative Prior

Informative priors are those which are not controlled by likelihood and has a major effect on posterior distribution These priors are proper prior and take a well-known density form The outcomes of informative prior are best appropriate and reliable as compared to non-informative priors The utilization of informative priors recognizes that estimation not only affect the current knowledge but also have some effect on prior information The main sources of informative priors are preceding studies researcher's perception interviewing essential experts published work closeness through conjugacy and other records resulting sources By increasing the preceding information the precision will be increased if the current information matched with the preceding

2.4.2. Non-Informative Prior

Non-informative priors are used when we do not use subjective prior due to some reasons as mentioned by many authors in literatures These priors don t concern any controversial information but these priors give a solid proof to change data from incorrect inference

Similarly these prior are also called reference prior uniform prior Jeffery prior or weak prior Uniform prior distribution is purely flat distribution that allocate identical probability to every possible parameters In many condition exiting information may be problematic to justify using informative priors. A non-informative uniform prior has very minimal impact on the outcome of posterior distribution. Consider in the normal case of an un-informative uniform prior for the mean of a normal distribution. It would essentially have a uniform mass over the interval $p(\theta) = c, (-\infty \leq \theta \leq \infty)$ wherever $c > 0$ is a constant, subsequently this places identical weight on totally probable values of θ.

Jeffery's rule or prior is defined as "the density of parameters proportional to square root of the determinant of the fisher information metric representatively.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^T$ is a vector of parameters $\beta_1, \beta_2, \ldots, \beta_n$

This can be achieved as

$$P_1(\beta) \propto \sqrt{\det\{I(\beta)\}} \qquad (2.1)$$

where "det" indicates the determinant and $I(\beta)$ indicates the $(n \times n)$ fisher information matrix which is the logarithm of maximum likelihood function of parameter β and partially differentiating two times with respect to the parameter β, is given by

$$I(\beta) = -E\left\{\frac{\partial^2 ln\, L(\beta)}{\partial \beta^2}\right\},$$

where "E" is the expectation.

2.5. Choice of Prior Distribution

The prior gives the prior knowledge approximately unidentified parameters that are adequately jointed into the result of the Bayes theorem. The selection of the prior distribution rely on the nature and the limit of the parameters occurring measured over the Bayesian investigation.

If it varies from 0 to 1, we often employ the range 0 to ∞ be beta prior or Gamma prior and for normal prior range is -∞ to ∞ frequently We enumerate the uncertainty approximately not familiar parameters in the shape of probability distribution in prior distribution

2.6. MCMC Simulation and Bayesian Methodology

Ulam and von-Neumann (1958) introduced the term 'Monte Carlo' for the stochastic simulation and used this idea for conducting experiments in the process of Atomic Bomb It is entirely trouble to look up concise definition of word MC Approximating an expectation using sample mean as a function of random variable In situation where normalizing constant is analytically not controlled, we used the MCMC simulation method We can tell that MCMC simulation is a process in which sampled values are drawn sequentially and the posterior density for each uncertainty depends on former samples '

In MCMC techniques random samples are obtained from an arbitrary distribution These samples are employed to round off the expectation of parameter importance Wherever the chain was ongoing the MCMC will touch the desired distribution The MCMC method is useful when the distribution has many parameters

2.7. Gibbs Sampling

Markov chain Monte Carlo technique is generally known as Gibbs sampling In this technique, we obtain samples from posterior distribution where every iteration of the MC algorithm sample is generated In direction to estimate vector of parameter or there is a difficulty in algebraic expression we generally need to evaluate the multiple integral Gibbs sampling technique is utilized when joint density is unidentified while the conditional distribution is clearly identified When the direct sampling is not possible, we usually used Gibbs sampling where sample is generated from joint density The sampler generate an MC sample where every sample is associated with others Gibbs sampling is one of the most beneficial technique of MCMC simulation, when the conditional distribution is well-known parametric form The Gibbs sampling technique mainly decomposes posterior distribution into simpler method such as full conditional distribution for every unidentified parameter

2.8. Elicitation

Elicitation of hyper-parameter is very significant research factor in Bayesian statistics It is the process of about knowledge, taught or undetermined quantities in a density function Elicitation has gained a lot of criticism from the questioners and they said that elicitation should not be done Elicitation is a way through which views and information of a person approximately uncertain quantities are computed into a probability which means a technique of identifying prior distribution of a statistical model for one or more unknown parameters Subsequently the description of prior distribution is then joined with the likelihood function through Bayes rule to achieve the posterior distribution

2.9. Burn-In and Thinning

In burn-in process initial imperfect estimates are not familiar in MCMC analysis and we cannot identify the favorable initial values We cannot utilize these primary estimates to use in any type of analysis So we discard these initial values as burn-in period For each chain the length of the burn-in period is distinct If we identify how rapidly the Markov chain converges on the desired distribution Thinning is a process of reduction of autocorrelation in the MCMC sample through sub-sampling the

MCMC chain each pre-specified number of iterations measured through thinning interval

For example thinning interval 1 indicates that using the complete MCMC sample the thinning interval 2 indicates using each 2nd sample values formerly the thinning interval 4 indicates that to using every 4th values from iterations for example 1 5 9 12 and so on Thinning must be applied with care when used to minimize auto-correlation as it may not be the most appropriate way of enhancing the precision of estimates

2.10. Posterior Distribution

In Bayesian inference, posterior distribution is derived by multiplying likelihood function with prior distribution Posterior distribution gets many appreciations in Bayesian theory due to its updating nature and summarizing entire information available in prior distribution and sample The posterior knowledge is proportional to the product of prior knowledge and sample knowledge The likelihood function $L(X_i, \lambda)$ and the prior distribution $p(\lambda)$, if the parameter is continued then the posterior distribution is

$$P(\lambda|X) = \frac{L(X_i\ \lambda)P(\lambda)}{\int_{-\infty}^{\infty} L(X_i\ \lambda)P(\lambda)d\lambda} \tag{2.3}$$

where

$P(\lambda)$ is the prior density of λ

$L(X_i\ \lambda)$ is the likelihood function as a function of x

$\int_{-\infty}^{\infty} L(X_i\ \lambda)\ P(\lambda)d\lambda$ is the standardizing constant and

$P(\lambda/X)$ is the posterior density of λ given the data $X = x$

2.11. Posterior Predictive Distribution

In Bayesian statistics posterior predictive distribution is an inspection device used to examine the consistency of model with data To evaluate the consistency of PPD produce simulated data called y^* compare this data with only observed data The PPD is found through integrating parameters out of posterior distribution of next observation M^* and the parameter X and the data which equals previous observations (x_1, M_1) (x_2, M_2) (x_n, M_n)

$$p(M^* | x^*, \underline{x}) = \int p(M^*, \pi | x^*, \underline{x})\, d\pi$$

$$= \int p(M^* | \pi, x^*, \underline{x})\, p(\pi | \underline{x}) d\pi$$

$$= \int p(M^* | \pi)\, p(\pi | \underline{x})\, d\pi$$

where

$p(M^* | x^*, \underline{x})$ is the posterior predictive distribution of M^* showing future and past data $p(\pi | \underline{x})$, is the posterior distribution of π given X (previous data) $p(M^*)$ is the given distribution of M^*

2.12. What is Money?

Money is something that people use everyday life We earn and spend it but often don't think much about it Economists define money as any good which is widely accepted as final payment for goods and services Through the ages, Money has taken different forms examples include the cowry shells in Africa, large strings of beads called wampum used by Native Americans and early American settlers and stone wheels on the Pacific island of Yap What do these forms of money have in common? They share the three functions of money First, Money is a store of value Second Money is a unit of account Third Money is a medium of exchange

2.13. What is The Role of Money in an Economy

The role of money in economy can be judged from the following aspects

- Money has replaced barter system Under the barter system the consumer have limited option of trade As Robertson in his book Money writes one can see what do people want and in how much quantity they want with the help of money in an economy
- Production consumption and distribution of money is relatively cooperative for consumer and producers All the economic activates of a market are dependent on money According to Milton in his book The financial Organization of Society ' writes " to start the process of production money is a basic and inevitable factor of production ' The producer need money to start business The inputs as well as labor are purchased with the help of money Cost and profit are defined in terms of money
- The advance payment are made through money where as there was no such concept in barter system The producer has to make advance payments at the beginning of production process The loan transactions take place in advance and thus all such is possible because of money
- Economic and social changes are possible due to money Money as the prime source for the industrialization new inventions and techniques are adopted to earn more money Devonport in his book Economics of Enterprise" writes all economic comparisons are made in money People of all fields are performing

their services for money Money is the base for running all the economic activities of the country

2.14. Money Demand

Money demand is defined as the quantity of monetary assets that people choose to hold money in their portfolios Money demand is important for monetary policy makers while making policy decisions as it is managed to control short-run fluctuations in the economy

Suppose I have a bank account In that account I have Rupee 1 000 My potential spending could be represented by that Rupee 1000 However I only ever hold Rupee 50 as cash at any time My demand for money is therefore different to the amount of money I have at my disposal to spend The demand for money is dependent on the price of money

2.15. Literature Regarding Bayesian Regression

Chen and Deely (1996) discussed Bayesian model for a constrained linear regression serious problem The constrained arising as estimated in the environment of predicting the fresh crop of apples for the year onward The Bayesian approach with the Gibbs sampler is presented to be generally appropriate to the constrained serious issues It is probable to achieve the Bayesian approximations of model parameters marginal posterior density estimates and Bayesian predictions Alternate methods such as Bayesian OLS and ICLS approximations are too discussed for comparisons The conclusion of this study depicts that for the predicting groups of 29 the percentage errors were 10 17 13 5 for the Bayesian OLS and ICLS methods respectively The

Bayesian errors remained minor than the OLS errors for 21 growers and smaller than the ICLS errors for 18 growers In overall the Bayesian errors were minor as well

Cowls and Carlin (1996) in paper explain that a serious problem for using MCMC methods in which condition stop our sampling Therefore various MCMC operators address the convergence issues by applying diagnostic devices to the output created by running their samples After complete review, this study explains the various convergence diagnostic definition of the theoretical source and practical implementation separately and highlightes the MCMC convergence diagnostic are Gelman and Rubin, Raftery and Lewis Geweke Roberts Ritter and Tanner, Zellner and Min Liu Liu and Rubin Garrer and smith Johnson, Heidelberger and Weich Mykland Tierney and Yu Yu Yu and Mykland These test are Quantitive Graphical, Theoretical basis univariate or full joint distribution, bias or variance applicapibilty are all characteristic different from each other in application

In this process two quantiles x_1 and x_2 were introduced to match z_1 and z_2 accurately identify the form of pdf Here we have two unidentified parameters and two equations In these two equations α and β are mean and standard deviation parameters of the normal form Solving the simultaneous way for α and β, prior distribution is entirely calculated

Strickland and Alsto (2013) explained that the linear regression model is going through to examine the monthly production of cars and station wagons (thousands of cars per month) in Australia since November 1969 to August 1995 The 1st column indicates the names of the variables, whereas the next columns describes the marginal posterior standard error the upper and lower 95% HPD intervals The estimated values used in the summer months of Dec Jan and Feb indicate a drop in manufacture relation to Oct which is probably partially by the national leave through that period The inference also catches the modelled outlier is statistically significant and specifies an enormous drop in production used for the month of Nov 1977 It is clear that there is no one explanatory variables involved in the model which have zero in the 95% HPD interval estimated The IF factors are very small representing the MCMC algorithm mixes Good But the other diagnostic tests are not being explained in his paper

Sinai and Hsu (2014) explored in paper Bayesian multivariate linear regression model conclusion with the use of flexible prior used for the covariance construction The normally assumed Bayesian format includes the conjugate prior MND used for regression coefficients and inverse Wishart conditions for the covariance matrix In this study leave these techniques and used Bayesian estimator used for the covariance A multivariate normal prior for the unique components of the matrix logarithm of the covariance matrix is examined Such construction examined for a comfortable period of prior distribution for covariance, regarding quality of views in prior position hyper parameters and also additional capacity to model the potential association between the covariance structures The posterior moments are calculated through MCMC procedure Gibbs sampling is used for the reaching of the target posterior distribution and concluded from the result that the flexible prior description used for the covariance building of a multivariate multiple regression can deliver a comfortable period of distributions when compared with inverse Wishart family

Tiao and Zellner (1964) in their article showed that how the prior information can be employed in combining with sample data in building inference about the parameters of regression model The key concern of the paper is to establish the techniques for using information from the one sample as a prior information in the analysis of following sample

The two samples are assumed from an ordinary population with unequal fluctuations The posterior distribution shaped in Section 2 is the result of multivariate normal and multivariate-t procedure In Section 3 the joint posterior and marginal posterior density are transformed into an asymptotic terms Furthermore, the outcome found in Sections 2 and 3 are presented numerically

Zellner (1983) presented in his paper numerous testing estimation and forecasting issues have been examined in the econometric literature This study giving consideration to that reference diffuse priors have been extremely valuable and proposed that reference informative priors (RIPs) will probably as suitable well A methodology for detailing RIPs for regression models was designated that provided normally straightforward RIPs and simple posterior distribution For a basic structural regression model it was demonstrated how these RIPs can be utilized to examine the influence of prior s estimates At last completion of likelihood function it was pointed out that numerical integration procedures have been cooperative in investigating posterior pdf and testing the validity of asymptotic and different approximations In framework specified that the past record of strong achievement in applying Bayesian examination in Econometrics it shows up very likely that there will be frequent uses of Bayesian examination in Econometrics future

Zellner (1976) presented in the study the conventional multiple regression model has been examined in the supposition that error expressions take a shared multivariate Student-t pdf by zero mean vector and covariance matrix as a scalar time a unit matrix

In Bayesian estimation of the model through a diffuse prior pdf used for the regression coefficients and multivariate student-t error terms the study established the joint posterior distribution used for the regression coefficients is inaccurately the similar multivariate student-t method as appearing from the joint normal model Discussion of the posterior distribution is generate to exist in the formula of an F-distribution of the scale parameter σ^2 When the df parameter in the error term distribution huge grows and therefore the error terms distribution move toward normality the posterior distribution of vs^2/σ^2 move toward the typical and $\chi 2$ pdf with v df Finally a natural conjugate prior distribution used for the multivariate Student-t regression model was existing

Zellner (1979) in this article explained that of the many research advances on the statistical investigation of regression models are being revised Numerous estimation testing and forecasting procedures utilized in econometric models have exactly large samples explanation Particular Bayesian conclusion identifying with econometric models are reviewed Many statistical issues needing further analysis are marked It is determined that superior solutions to these issues better data, additional sophisticated use of economic theory application of further rigorous diagnostic checks containing forecasting checks and use of expertly-designed simulation research probably will construct developed macro-econometric models

In article the author also recommended that Bayesian techniques compromised good clarification for various difficulties Further formal techniques utilizing for prior information in the examination of assumed SEM are necessary a problematic zone that can move toward best suitably at present by the usage of Bayesian analysis This study also highlights the significance of using the prior methodology carefully in desiring for forecasting

2.16. Literature Regarding Money Demand function of Pakistan

Azim *et al* (2010) explored the money demand function for Pakistan by utilizing ARDL procedure for yearly data for the period 1973 to 2007 By using money demand (M1 or M2) real income inflation rate and exchange rate variables The study observed the positively association of the real income and inflation with money demand but the exchange rate positive effects on money demand

Anwar and Asghar (2012) estimated that long run association for money demand that real income inflation rate and exchange rate is taking as explanatory variables by utilizing the ARDL method over the annual period of 1975 to 2009 By using wanted holding of real money balances (M1 or M2) are demand for money GDP deflator is utilized for finding inflation rate The study concluded that in Pakistan M1 is cointegrated with its elements but estimates are not constant over the time period while for M2 the ECM moves the expected significant sign which shows that M2 is cointegrated with its determinants The study suggested to focus only on the long run stabilization policy for Pakistan to the monetary authorities and policy makers

Asad *et al* (2011) investigated money demand function (M2) of Pakistan over the period 1980.Q1 to 2009 Q2 by employing the ARDL method and by using money demand real GDP, interest rate inflation foreign rate of interest real effective exchange rate The study concluded that association between M2 and a set of explanatory variables is stable in long run Inflation is very high and increased interest rate causing a decrease in money demand Foreign interest rate indicate a small impact and negative signs and suggest that inflation has a large impact on money demand of Pakistan which creates problems for achieving the target M2 level in Pakistan

Ahmad and Munir (2000) estimated money demand function of Pakistan by taking real income, price level and inflation rate as explanatory variables The study employed OLS and cointegratin estimation while using the quarterly data over the period 1972 II to 1996 I by using the desired money balances (M1 and M2) current income, index of industrial production in the place of GDP Inter-bank call money rate as short term interest rate CPI as inflation rate The study concluded that inflation rate is more significant variable as compared to the nominal interest rate in determining the money demand Money demand in the short run is not very sensitive to the shocks so the monetary authority need to take into the account the speed of adjustment in money demand in order to make the outcomes consistent with the targets

Bahmani-Oskooee and Shin Sungwon (2002) examined that the stability of short run as well as long run dynamics of M1 M2 M3 money demand function in Korea by using the Johansen and Juselius Cointegration along with the CUSUM and CUSUMSQ test by using quarterly data over the period 1973 I to 1997 III by using real monetary aggregate (M1 M2 M3) real income as *Y* nominal effective exchange rate as NEX and interest rate as R This study established that none of the monetary aggregates have a stable relation with interest rate income and exchange rate in KOREA

Das and Mandal (2000) explored that whether money demand function can be estimated a partial model (by a single equation) or a full system method like Vector autoregressive model by employing Johansen's Vector Auto-Regressive (VAR) approach and Hansen s methodology and by using monthly data over the period 1981 4 to 1998 3 and by using Broad Money (M3) IIP as index of industrial Production for income WPI as price level Call money rate as a proxy for short term interest rate Index of stock price 36 country trade weighted real effective exchange rate index as exchange rate variables The study determined that Long run demand for M3 is stable in spite of large shocks due to financial liberalization The results indicated that the common practice of having long run parameters from the short run parameters that were obtained from partial adjustment model as misleading

Khan (1997) in paper evaluated the cointegration applying Engle-Granger method and ECM over the class 1972 to 1992 and by applying the actual money balances (M1 and M2) real profits insignificant interest rate and probable level of inflation variables The initiate income flexibility of M2 in the region of 1 1 which indicates that for money demand has remained growing at an amount better or not better proportional to fluctuations in the income decay His initiate actual interest rate constant to be right low but statistically important whereas the increase ratio as an chance price variables is get significant with probable non-positive sign

Khan and Sajjid (2005) explored the mutually long and short run association among the real money balances and their determinants for Pakistan over the Period 1982 Q2 to 2002 Q4 by using Cointegration test (ARDL approach which is a recently developed technique) and ECM and by using broad money M2 CPI as the Price level Real GDP, money market rate as interest rate, log difference of CPI as inflation rate US Federal fund rate as foreign interest rate trade weighted real effective exchange rate index composed by IMF as exchange rate The study found stationary long run relationship between money demand and the explanatory variables

Kumari and Mahakud (2012) explored the relationship amongst demand for money (M1 and M3) and its determinants during the post liberalization period in India by employing the Johansen and Juselius Cointegration model, VECM model and Granger causality test over the monthly period of 1961 1 to 2010 8 This study concluded that there is presence of more than two cointegrating vector for each of money demand specification The exchange rate have negative effect on M1 and stock prices have negative and significant effect on money demand inflation is positively related to M3 and IIP are negatively significant in the first lag difference All the five variables affects the demand for money function (M1 and M3)

Qayyum and Nishat (2001) in paper using the long run cointegration and error correction model of the actual money demand in the disaggregated commercial and private sector he determined that in together sectors there be present a long run association amongst the actual money demand and its determinates The long run profits flexibility of money for business region sales flexibility of money is better as equate to the personal sector Such as the interest rate is anxious, the business region have reacted the interest amount on bank loans while people are affect by long term rates of pledge profit Inflation rate effect on money demand of household region is greater than business region The actual assets replacement in the long run is powerful across the regions however this occasion is powerful in personal region The short run variation in inflation has powerful effect on complete and personal region while in business region are not

Singh and Pandey (2012) examined the behaviour of money demand function in India by employing Gregory and Hansen (1996) cointegration over the annual period of 1953 to 2008 and by using Demand for money real GDP and nominal interest rate The study established that there is presence of cointegration between money demand and its determinants with a structural break in year 1965 The interest rate and income elasticities are significant with expected signs During 1975 to 1998 the demand for money became unstable due to various factors

Tang (2007) analyzed the money demand function for the five southeast Asian countries (Malaysia Singapore Thailand Indonesia and Philippines) by employing the

ARDL approach of Cointegration over the period 1961 to 2005 for Malaysia Philippines Thailand and Indonesia and 1972 to 2005 for Singapore and by using M2 aggregate macroeconomic component of real income (Real GDP GDP) exchange rate and inflation variables They concluded that real M2, disaggregated components of real income (final consumption expenditure expenditure on investment goods, export) inflation rate and exchange rate are cointegrated for Philippines Malaysia and Singapore but not for Thailand and Indonesia Money demand function is stable for five Southeast Asian economies under ARDL, except for the short run money demand equation for Indonesia

CHAPTER 3

MATRIALES AND METHODS

3.1. Introduction

In this chapter we explain the specification of the model We present the complete form of the Bayesian linear regression model in matrix form and likelihood function of the model Section 5 presents the complete steps of the derivation of posterior distribution using non-informative priors Sections 6 and 7 present the prior through Normal-gamma and the derivation of posterior distribution through informative prior Section 8 presents elicitation of priors on parameters of the model and Section 9 explains the construction of elicited priors on τ in linear regression Numerical results of elicited prior are present in Section 10 Section 11 gives the Gibbs sampling design for linear regression model Section 12 contains discussion about diagnostic tests for Bayesian linear regression

3.2. Specification of The Model

Money demand function is specified on the basis of Liquidity demand theory of macroeconomics According to liquidity demand theory money demand depends on the aggregate demand price level in the economy and domestic interest rate Hence the model may be written as follows

$$M = \beta_0 + \beta_1 Y + \beta_2 P + \beta_3 R + \mu_i \qquad (3\ 1)$$

where

M = money demand

Y = is the aggregate demand

P = is the price level.

R = is the interest rate

μ_t = is the disturbance term

Data of all the above four variables are taken from various issues of Pakistan Economic Survey ranging from 1960 to 2014

3.3. Bayesian Multiple Regression Model In Matrix Form

The specified model is as follows

$$M_t = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \beta_3 x_{t3} + \mu_t \qquad (3.2)$$

where i = 1, 2, ..., N

The above model can be written in matrix form as follows

$$M = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \\ \\ m_N \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \\ \\ \mu_N \end{bmatrix}$$

The (t × 1) vector

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

And $(N \times t)$ matrix is

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} & x_{13} \\ 1 & x_{21} & x_{22} & x_{23} \\ 1 & x_{31} & x_{32} & x_{33} \\ & & & \\ 1 & x_{N1} & x_{N2} & x_{N3} \end{bmatrix}$$

Also we can write in complete structure model

$$M = X\beta + \mu_i \qquad (3\ 3)$$

3.4. Likelihood Function

Assumption about μ and X determine the form of the likelihood function

- μ has a multivariate normal distribution by mean 0_N and covariance matrix $\sigma^2 I_N$ Where 0_N an N-vector whose elements are equivalent to 0 and I_N is the $(N \times N)$ identity matrix i e $\mu \sim N(0_N\ H^{-1} I_N)$ where $H = \sigma^{-2}$
- Each elements of X are fixed (not random variables) For all components μ they are independent through probability density function s $(X|\lambda)$ and λ is a vector of parameters that does not contain β and H

Variance-covariance matrix of the residuals are as follows

$$var(\mu) = \begin{bmatrix} var(\mu_1) & cov(\mu_1, \mu_2) & & cov(\mu_1, \mu_N) \\ cov(\mu_1, \mu_2) & var(\mu_2) & & \\ & cov(\mu_2, \mu_3) & var(\mu_3) & \\ & & & cov(\mu_{N-1}, \mu_N) \\ cov(\mu_1, \mu_N) & & & var(\mu_N) \end{bmatrix}$$

$$= \begin{bmatrix} H^{-1} & 0 & 0 & 0 & 0 \\ 0 & H^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H^{-1} \end{bmatrix}$$

In new arguments, the statement that $var(\mu_i) = H^{-1} I_N$ is solid notation for

$var(\mu_i) = H^{-1}$ and $cov(\mu_i, \mu_j) = 0$ for $i, j = 1,2 \quad N$ and $i \neq j$

The second assumption describes that we can remain conditionally on X and provide $p(M|X, \beta, H)$ purpose for likelihood distribution Utilizing the classification of the multivariate normal density we can mark the likelihood function

$$p(M|\beta, H) = \frac{H^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\left\{exp\left[-\frac{H}{2}(M - X\beta)^T(M - X\beta)\right]\right\} \quad (3\ 4)$$

Taking part of the Equation (3 4)

$$(M - X\beta)^T(M - X\beta) \quad (3\ 5)$$

Add and subtract "$X\hat{\beta}$"

$$\left(M - X\hat{\beta} + X\hat{\beta} - X\beta\right)^T\left(M - X\hat{\beta} + X\hat{\beta} - X\beta\right)$$

$$\left(M - X\hat{\beta} - X(\beta - \hat{\beta})\right)^T\left(M - X\hat{\beta} - X(\beta - \hat{\beta})\right)$$

$$\left(M - X\hat{\beta}\right)^T\left(M - X\hat{\beta}\right) + \left(\beta - \hat{\beta}\right)^T XX^T\left(\beta - \hat{\beta}\right) \quad (3\ 6)$$

As well as the cross product expressions

$$\left(\beta - \hat{\beta}\right)^T X^T\left(M - X\hat{\beta}\right) = \left(\beta - \hat{\beta}\right)^T\left(X^T M - X^T X(X^T X)^{-1} X^T M\right) = 0$$

And

$$\left(M - X\hat{\beta}\right)^T\left(M - X\hat{\beta}\right) = SSE$$

$$\left(M - X\hat{\beta}\right)^T\left(M - X\hat{\beta}\right) + \left(\beta - \hat{\beta}\right)^T X^T X(\beta - \hat{\beta})$$

So

$$(M - X\beta)^T(M - X\beta) = SSE + \left(\beta - \hat{\beta}\right)^T X^T X\left(\beta - \hat{\beta}\right) \quad (3\ 7)$$

Put Equation (3 7) in to Equation (3 4) we have

$$p(M|\beta, H) = \frac{H^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\left\{exp\left[-\frac{H}{2}\left(SSE + (\beta - \hat{\beta})^T X^T X(\beta - \hat{\beta})\right)\right]\right\} \quad (3.8)$$

3.5. Non-Informative Prior for Linear Model

By the definition of the uniform prior

$$p(\beta) \propto c \quad \text{and} \quad p(H) = 1 \quad (3.9)$$

Over the support of $(-\infty, \infty)$ and $(0, \infty)$ respectively we are assuming independence between β and σ^2

Using the likelihood function (3.4) and prior distribution (3.9) the posterior distribution of the model is

$$p(M|\beta, H) = \frac{H^{N/2}}{(2\pi)^{N/2}} exp\left\{-\frac{H}{2}(M - X\beta)^T(M - X\beta)\right\} \quad (3.10)$$

By using rules of OLS

$$\hat{\beta} = (X^T X)X^T M \qquad \hat{\sigma}^2 = \frac{(M-X\beta)^T(M-X\beta)}{(n-k)} \quad (3.11)$$

By using the completing square we write

$$= \frac{H^{N/2}}{(2\pi)^{N/2}} exp\left\{-\frac{H}{2}\left(SSE + (\beta - \hat{\beta})^T X^T X(\beta - \beta)\right)\right\}$$

Therefore the joint posterior from the likelihood function (2.3) is provided by

$$p(\beta, H|X, M) \propto p(\beta, H|X, M)p(\beta)p(H)$$

$$\propto \frac{H^{N/2}}{(2\pi)^{N/2}} exp\left\{-\frac{H}{2}\left(SSE + (\beta - \hat{\beta})^T X^T X(\beta - \hat{\beta})\right)\right\} \quad (3.12)$$

With kernel density

$$\propto H^{N/2} exp^{-\frac{H}{2}(SSE)} \times exp^{\left(-\frac{1}{2}(\beta-\hat{\beta})^T H X^T X(\beta-\hat{\beta})\right)}$$

$$\propto H^{a^*-1} exp^{-Hb^*} \times exp^{\left(-\frac{1}{2}(\beta-\hat{\beta})^T Q^*(\beta-\hat{\beta})\right)} \tag{3.13}$$

The posterior parameters are

$$a^* = \frac{N+2}{2} \qquad b^* = \frac{SSF}{2} \qquad \hat{\beta} = (X^T X) X^T M\,, \qquad Q^* = H(X^T X)$$

3.6. Normal Linear Regression Model through Independent Normal-Gamma Prior.

By using natural conjugate prior wherever $p(\beta/H)$ existed a normal density and $p(H)$ a gamma density Here we utilize a same prior then one which expects prior independence amongst β and H

Specifically we adopt $p(\beta, H) = p(\beta)\ p(H)$ with $p(\beta)$ existence normal distribution and $p(H)$ being gamma distribution PDF

$$p(\beta) = \frac{1}{(2\pi)^{\frac{k}{2}}} \left|\underline{Q}\right|^{-\frac{1}{2}} exp\left[-\frac{1}{2}\left(\beta - \underline{\beta}\right)^T \underline{Q}^{-1}\left(\beta - \underline{\beta}\right)\right] \tag{3.14}$$

and

$$p(H) = c_G^{-1} H^{a-1} exp(-Hb) \tag{3.15}$$

where "c_G" is the integrating constant used for the gamma pdf That is $\underline{\beta} = E\,(\beta|M)$ is still the prior mean of β and $\underline{Q}$ is the variance covariance matrix of β Where $var(\beta|H) = H^{-1}\underline{Q}$ For clearness we are utilizing the similar illustration as in the likelihood function in matrix procedure Where in $p(H)$ "a" is the scale parameter and "b" shape parameter The parameter in $p(\beta)$ and $p(H)$ can be find through Elicitation process

By the combination of Equation (3 14) and Equation (3 15) we develop the concluding result of normal gamma prior is

$$p(\beta, H) \propto \left\{exp\left[-\frac{1}{2}(\beta - \underline{\beta})^T \underline{Q}^{-1}(\beta - \underline{\beta})\right]\right\}\{H^{a-1} exp(-Hb)\} \tag{3.16}$$

3.7. Posterior Distribution Under Normal-Gamma Prior

Posterior distribution is obtained as follows

$$p(\beta, H|M) \propto p(\beta, H)L(\beta, H) \tag{3.17}$$

$$p(\beta, H|M) \propto \left\{exp\left[-\tfrac{1}{2}(\beta-\underline{\beta})^T\underline{Q}^{-1}(\beta-\underline{\beta})\right]\right\}\{H^{a-1}\exp(-Hb)\} \times$$

$$H^{\frac{N}{2}}\left\{exp\left[-\tfrac{H}{2}\left(SSE+(\beta-\hat{\beta})^T X^T X(\beta-\hat{\beta})\right)\right]\right\} \tag{3.18}$$

$$= H^{a+\frac{N}{2}-1}\, exp^{\left(-H\left(b-\frac{SSE}{2}\right)\right)}\, exp\left[-\tfrac{1}{2}\left(\left(\beta-\underline{\beta}\right)^T\underline{Q}^{-1}\left(\beta-\underline{\beta}\right)+(\beta-\right.\right.$$

$$\left.\left.\hat{\beta})^T HX^T X(\beta-\hat{\beta})\right)\right]$$

$$= H^{a_1\ -1}\, exp^{(-Hb_1\ )}\, exp\left[-\tfrac{1}{2}\left(\left(\beta-\underline{\beta}\right)^T\underline{Q}^{-1}\left(\beta-\underline{\beta}\right)+(\beta-\hat{\beta})^T HX^T X(\beta-\hat{\beta})\right)\right] \tag{3.19}$$

Taking the part

$$\left(\beta-\underline{\beta}\right)^T\underline{Q}^{-1}\left(\beta-\underline{\beta}\right)+(\beta-\hat{\beta})^T HX^T X(\beta-\hat{\beta})$$

After simplification we get that

$$= H^{a\ \ -1}\, exp^{(-Hb\ \ )}\, exp\left[-\tfrac{1}{2}\left(\underline{Q}^{-1}+HX^T X\right)\left(\beta-\frac{\underline{Q}^{-1}\underline{\beta}+HX^T X\hat{\beta}}{\underline{Q}^{-1}+HX^T X}\right)^2\right] \tag{3.20}$$

$$a_1^* = a+\frac{N}{2} \qquad \text{and} \qquad b_1^* = b+\frac{SSE}{2}$$

As we know that

$$\bar{Q}^{-1} = \left(\underline{Q}^{-1}+HX^T X\right)$$

$$\bar{Q} = \left(\underline{Q}^{-1}+HX^T X\right)^{-1} \tag{3.20}$$

$$\bar{\beta} = \frac{\underline{Q}^{-1}\underline{\beta}+HX^T X\hat{\beta}}{\underline{Q}^{-1}+HX^T X} = \frac{\underline{Q}^{-1}\underline{\beta}+HX^T X\hat{\beta}}{\bar{Q}^{-1}}$$

$$\bar{\beta} = \bar{Q}\left(\underline{Q}^{-1}\underline{\beta} + HX^{T}X\hat{\beta}\right) \quad (3.21)$$

Where $\bar{\beta}$ is the mean and $\bar{Q}$ is var-cov matrix of posterior distribution. However, it must be stressed that both distribution of normal Gamma prior and likelihood function do not relate directly to the posterior of interest $p(\beta, H|M)$, but moderately to the conditional posterior $p(\beta|M, H)$ and $p(H|M, \beta)$. Since $p(\beta, H|M) \neq p(\beta|M, H)\, p(H|M, \beta)$ the conditional posteriors of kernel multivariate normal distribution and gamma distribution do not straightforward. However, here is a posterior simulator known as the Gibbs sampler which utilized conditional posteriors like kernel multivariate normal distribution and gamma distribution to create random draws β^{u} and H^{u} for $u = 1, 2 \ldots U$, which can be a midpoint to create approximations of posterior properties only as through Monte Carlo integration.

3.8. Building Elicited Priors Utilizing Linear Regression

Spetzler and Holstein (1975) categorized the elicitation process in three stages.

Deterministic elicitation requires experts for the elicitation process which can carried out the procedure by specifying the explanatory variables and prior parameters in the exiting model.

Probabilistic elicitation where experts are being interviewed and assessor used two approaches and assessor would asked fixed value query with probability answer as fixed probabilistic query with a value answer.

Informational elicitation where the assessor determines the internal consistency.

The elicited priors are established from experts. The important task is the interpretation of statements into precise probability statements. This method expands from familiar assignments to explanatory elicitation ideas and even regression analysis.

It is a method to enquire the assessors to provide conclusion variable levels associated with suitable cumulative probability degrees (V-method) These assessors give values for the desired median the 5% level the 95% level or other natural limits The results are then non-parametrically summarized which are utilized as input into parametric family of priors for example the normal/students-t

Suppose a general example where the examiner asks an expert for predictions on a probable results for an interval-measured incident The (V-method) question is generally what will be a probable short value outcome in the form of a 0 25 quantile (x_1) and a probable high value in the form of a 0 75 quantile (x_2) These outcome helps the researcher to identify a normal distribution for this incident The two provided quantile values x_1 and x_2 correlated to z_1= 0 25 and z_2= 0 75 completely identify the nature of a normal pdf Here, we have two equations and two unidentified parameters

$$z_1 = \frac{x_1 - \alpha}{\beta}, \qquad z_2 = \frac{x_2 - \alpha}{\beta}$$

Here α and β represent the mean and standard deviation of parameters of the normal form

$$f({}^{x}/_{\alpha}, \beta) = \frac{1}{(2\pi\beta^2)^{\frac{1}{2}}} exp\left(\frac{-(x-\alpha)^2}{2\beta^2}\right)$$

Solving these equation for α and β we have a well-defined prior distribution from the elicitation Individual expert is usually insufficient so we query from j experts where $j = 1, 2, \quad J$ Constructing an over-specified series of equations as there are $j \times 2$ equations and only two unidentified We may ask from each expert to provide us just more than two quantiles and it is always considered more reliable to have many assessed points Here every assessor is asked to provide three quantile values at M = [0 90 0 95 0 99] corresponding to standard normal points Z_M

Now we are able to re-expressed (1) used for the quantile level M specified by experts $j = x_{jM} = \alpha + \beta z_{jM}$ Consequently the total number of experts elicited the results constitute over-specification (M × 3 equations and 2 unknowns) of a normal distribution

$$x_{11} = \alpha + \beta_{z11},\ x_{21} = \alpha + \beta_{z21} \quad ,x_{(M-1)1} = \alpha + \beta_{z(M-1)1}, x_{M1} = \alpha + \beta_{zM1}$$

$$x_{12} = \alpha + \beta_{z12},\ x_{22} = \alpha + \beta_{z22} \quad ,x_{(M-1)2} = \alpha + \beta_{z(M-1)2}, x_{M2} = \alpha + \beta_{zM2}$$

$$x_{13} = \alpha + \beta_{z13},\ x_{23} = \alpha + \beta_{z23} \quad ,x_{(M-1)3} = \alpha + \beta_{z(M-1)3}, x_{M3} = \alpha + \beta_{zM3}$$

The solution suggested by this arrangement is to run a simple bivariate linear regression as α is the intercept and β is slope

3.9. Elicited Priors on τ (Precision)

It is not easy for researcher to understand directly about the standard deviation variance or precision but it is comparatively easy to think about percentiles of distribution of data values y_i (Not to be confused with the percentiles of the distribution for μ that we just used)

For example a cotton grower will be capable to propose about how many bushels per acre of land he would go beyond in one year out of 10 or 9 years out of 10 years It will be easy for instructor to propose about the 90th percentile of exam score i e $\gamma_{0\,90}$ shows that 10% students score above this level It will be better to think about lower percentile i e $\gamma_{0\,10}$ To provide a prior guess for mean the assumption of independence of mean and variance information about these percentiles provide us the value of variability It is essential for researcher to investigate a compatible guess for percentile suppose $\tilde{\gamma}_{0\,10}$ and to analyze how best that guess is

The most favorable guess for $\gamma_{0.90}$ gives us a favorable guess for τ and σ as τ_0 and σ_0. As we know that $\gamma_{0.90} = a + 1.28\sigma$ usually we marked α percentile of data as γ_α and α percentile of a standard normal as z_α using a and $\tilde{\gamma}_\alpha$ i.e. our best guesses used for a and γ_α. We have

$$\tilde{\gamma}_\alpha = a + z_\alpha\sigma_0 = a + z_\alpha\sqrt{1/\tau_0}$$

Which contributes our best guesses used for the standard deviation and precision as

$$\sigma_0 = (\tilde{\gamma}_\alpha - a)/z_\alpha, \quad \tau_0 = \left[z_\alpha/(\tilde{\gamma}_\alpha - a)\right]^2$$

For our instructor the best guess for the mean grade μ was $a = \delta$. Suppose that the best guess for the 90th percentile of individual test scores is $\tilde{\gamma}_{0.90} = \bar{\vartheta}$. That corresponds to a best guess for the standard deviation σ and the precision τ

$$\sigma_0 = \delta_1 \qquad \tau_0 = \vartheta_1$$

Correspondingly a best guess used for τ or σ is insufficient so we need some idea of uncertainty about τ or σ. For these prior Gamma distributions to model best guess for τ or σ we adjust prior guess to prior mode. (Note that if the prior mode is very close to 0 as is $(\tau_0 = \vartheta_1)$ it will be very complicated to find a Gamma distribution numerically with that mode and specified small percentile (say 0.10).

As a best Bayesian we assumed that parameter $\tilde{\gamma}_\alpha$ is modeled with a distribution and $\tilde{\gamma}_\alpha$ is measures of the midpoint of that distribution. Now we inquire the expert to provide us a percentile for the distribution of γ_α. We remember the instructor that his best guess for $\gamma_{0.90}$ was $\tilde{\gamma}_{0.90} = \omega$ and how more than ω considers $\gamma_{0.90}$ possibly. In common we proceed this upper bounds say $\tilde{u}_\alpha$ to be the 90th, 95th and 99th percentile of the distribution. Certainly we should work with a lower limit if possible.

If we take $P(\gamma_\alpha - \tilde{u}_\alpha) = 0.95$ so we then argue that

$$0.95 = p(\mu + z_\alpha \sigma \leq \tilde{u}_\alpha / \mu = a)$$

$$= p(a + z_\alpha \sigma \leq \tilde{u}_\alpha)$$

$$= p(\sigma \leq (\tilde{u}_\alpha - a)/z_\alpha)$$

$$= p(\tau \geq \{z_\alpha/(\tilde{u}_\alpha - a)\}^2)$$

Therefore the 95[th] percentile used for σ and the 5[th] percentile used for τ will be,

$$\tilde{\sigma}_{0.95} = (\tilde{u}_u - a)/z_\alpha \qquad \tilde{\tau}_{0.05} = [z_\alpha/(\tilde{u}_\alpha - a)]^2$$

If our instructor trusts that the 90[th] percentile of test scores may possibly be no longer higher than $\tilde{u}_{0.90}$ then with a we have

$$\tilde{\sigma}_{0.95} = \omega \quad \tilde{\tau}_{0.05} = \omega$$

Firstly find a Gamma (j, f) distribution for σ that is concentrated close σ_0 and has 95[th] percentile $\tilde{\sigma}_{0.95}$. Compare the mode of the Gamma (j, f) to σ_0, so

$$\sigma_0 = (j - 1)/f \qquad \text{or} \qquad j = 1 + \sigma_0 f$$

If we can specify a value for j, the process will be complete

We have to discover f so that the Gamma $(1 + \sigma_0 f\ f)$ distribution has 95[th] percentile $\tilde{\sigma}_{0.95}$. This can be developed by utilizing any PC schedule that measure percentile of the Gamma distribution. Simply continuous attempting for distinctive estimations of f until you calculate $\tilde{\sigma}_{0.95}$ as here 95[th] percentile. For our illustration we require a mode δ_1 and 95[th] percentile of ϑ. This happens with a Gamma (x, y) distribution. General our independence prior for the instructor is

$$\sigma \sim \text{Gamma}(x, y)$$

Otherwise we can find out a $\text{Gamma}(c, d)$ distribution used for τ that has mode

$$\tau_0 = (c - 1)/d \qquad \text{or} \qquad c = 1 + \tau_0 d$$

Furthermore 5th percentile $\tilde{\tau}_{0.05}$ which is again refined by trail and mistake. For better understanding, we have a mode of $\tilde{\tau}_{0.05}= \vartheta_1$ and 5th percentile of $\tilde{\tau}_{0.05}= \vartheta$. These amounts are near 0 and Gamma distribution with minor modes that will be extremely skewed. It is difficult to find out a Gamma distribution that justifies these limits. We suggest that data elicited about the 5th percentile of γ_u instead of the 95th percentile.

3.10. Elicitation of Hyper-parameters

Elicitation of hyper-parameters modifies the expert opinions for special objective into a probability model. It is a scheme to identify the hyper-parameters of the prior distribution. It can be utilized for acquiring the posterior distribution and advance analysis. Aslam (2003) recommended distinct approaches of elicitation constructed on prior predictive distribution for the hyper-parameters.

In this study, PV-method is used which is based on distinct expert values according to distinct probabilities of parameters means and variance (precision). For single expert and single probabilities we can solve it through simultaneous equation and for numerous experts and numerous probabilities we can take the help of R-Package and also find in this way hyper-parameter of precision of the Normal Gamma distribution.

We have completely explained these two methods in section 3.8 and 3.9. Based on the above method, the following priors of hyper-parameters have been elicited.

Table 3 1 Elicited values of hyper parameters

Node	Mean	Variance	Precision
Intercept	-25 09	0 80496	1 543
Aggregate demand	2 455	0 0456	480 917
Price	0 52714	0 06786	217 54
Interest rate	-1 90505	0 4824	4 304

Hyper-parameter for precision are elicited as 1 0603 for shape parameter and 0 009 for scale parameter

3.11. Gibbs Sampling Designed For Linear Regression Model

We presented the procedure of estimating linear regression Model by utilizing Gibbs sampling Consider the estimation of the preceding regression model through Gibbs sampling

$$M_t = BX_t + \mu_t \qquad (3\ 24)$$

where $\mu_t \sim N(0\ \tau)$

$$B = [\beta_0, \beta_1, \beta_2, \beta_3]$$

$$X_t = [1,\ Y,\ P,\ R\ ]$$

Where M_t is the money demand function in Pakistan over the period 1960 to 2014 Let $X_t = [1,\ Y,\ P,\ R\ ]$ designate the right hand side variables in the equation (3 24) and $B = [\beta_0, \beta_1, \beta_2, \beta_3]$ the coefficient of vector Our target is to estimate the marginal posterior distribution of $\beta_0, \beta_1, \beta_2, \beta_3$ and τ are discussed above, these marginal distributions are analytically more problematic to derive We derived the posterior distribution of B= $[\beta_0, \beta_1, \beta_2, \beta_3]$ Conditional on τ and the posterior distribution of τ conditional on B= $[\beta_0, \beta_1, \beta_2, \beta_3]$ An approximation of this model continues in the following phases

The posterior distribution of τ conditional on B= $[\beta_0, \beta_1, \beta_2, \beta_3]$ estimation of this model continues in the following phases

Phase 1 adjust priors and initial values We adjust a normal priors for the coefficients B

$$P(B) \sim N\left(\begin{pmatrix}\alpha^0\\ \beta_1^0\\ \beta_2^0\\ \beta_3^0\end{pmatrix}\begin{pmatrix}\Sigma_{B0} & 0 & 0 & 0\\ 0 & \Sigma_{B1} & 0 & 0\\ 0 & 0 & \Sigma_{B2} & 0\\ 0 & 0 & 0 & \Sigma_{B3}\end{pmatrix}\right)$$

$$p(\sigma^2) \sim \Gamma\left(\frac{f_0}{2}, \frac{f_1}{2}\right)$$

To initialize the Gibbs sampler we need initial value for τ or B In this model we assumed that the initial value for $\tau = \tau_{ols}$ Where τ_{ols} is the OLS estimates of τ

Phase 2. Specified a value for τ we sampled from the conditional posterior distribution of B The normal distribution with a known mean and variance specified

$$H(B/\tau, M_t) \sim N(M^*, V^*)$$

Where

$$M^* = \bar{\beta}$$

$$V^* = \frac{\bar{v}\bar{s}^2}{\bar{v} - 2}\bar{V}$$

We have all the ingredients to calculate that M^* and V^* are 4×1 and 4×4 matrix respectively We now need a sample from the normal distribution with mean M^* and variance V^*

For this we can use the following algorithm

Algorithm. 1 To sample a $k \times 1$ vector denoted by z following from the N (m, v) distribution First generate $k \times 1$ number from the standard normal distribution (call the z^0) These standard normal numbers can then be transformed such that the mean is equal to m and variance equals v utilizing the following transformation

$$z = m + z^0 \times v^{\frac{1}{2}}$$

We added the mean and multiplies z^0 by the square root of the variance The procedure in algorithm 1 recommended that once we have calculated M^* and V^*, the draw for B is acquire as

$$B^1 = M^* + [\bar{\bar{B}} \times (V^*)^{\frac{1}{2}}]^T$$

Where $\bar{\bar{B}}$ is a 1×4 vector from the standard normal distribution The superscript 1 in B^1 represents the first Gibbs iteration $\bar{\bar{B}}$

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} E(\hat{\beta}_0) \\ E(\hat{\beta}_1) \\ E(\hat{\beta}_2) \\ E(\hat{\beta}_3) \end{bmatrix} + [Z_0 \quad Z_1 \quad Z_2 \quad Z_3] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{bmatrix}$$

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} E(\hat{\beta}_0) \\ E(\hat{\beta}_1) \\ E(\hat{\beta}_2) \\ E(\hat{\beta}_3) \end{bmatrix} + [S_1 \quad S_2 \quad S_3 \quad S_4]^T$$

$$\begin{bmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\end{bmatrix} = \begin{bmatrix}E(\hat{\beta}_0)\\ E(\hat{\beta}_1)\\ E(\hat{\beta}_2)\\ E(\hat{\beta}_3)\end{bmatrix} + \begin{bmatrix}S_1\\ S_2\\ S_3\\ S_4\end{bmatrix}$$

$$\begin{bmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\end{bmatrix} = \begin{bmatrix}E(\hat{\beta}_0)+S_1\\ E(\hat{\beta}_1)+S_2\\ E(\hat{\beta}_2)+S_3\\ E(\hat{\beta}_3)+S_4\end{bmatrix} = \begin{bmatrix}\beta_0^1\\ \beta_1^1\\ \beta_2^1\\ \beta_3^1\end{bmatrix}$$

This is the 1st iteration of mean

Algorithm 2, To a sample a scalar from Z the normal gamma distribution with degree of freedom $\frac{f_0}{2}$ and scale parameter $\frac{f_1}{2}$ i e, $\Gamma\left(\frac{f_0}{2},\frac{f_1}{2}\right)$ Generates f_1 numbers from the standard normal distribution $z^0 \sim N(0,1)$ Then

$$Z = \frac{v}{z^{0\prime}z^0}$$

$$Z_0 = \begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix} \quad Z_0' = \begin{bmatrix}z_1 & z_2 & z_3 & z_4\end{bmatrix}$$

$$\hat{\Sigma}_1 = \hat{\sigma}_1^2\Sigma_0$$

$$= \hat{\sigma}_1^2\begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14}\\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24}\\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34}\\ \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44}\end{bmatrix}$$

$$= \hat{\sigma}_1^2\begin{bmatrix}\Sigma_{11} & 0 & 0 & 0\\ 0 & \Sigma_{22} & 0 & 0\\ 0 & 0 & \Sigma_{33} & 0\\ 0 & 0 & 0 & \Sigma_{44}\end{bmatrix}$$

This is the 1st iteration of $\hat{\sigma}_1^2$ Therefore the 1st complete iteration in model form is $\hat{\beta}_0, \hat{\beta}_1\ \hat{\beta}_2\ \hat{\beta}_3$ and $\hat{\sigma}_1^2$

Phase 3 Repeat steps 2 and 3 M times to obtain B^1 B^2 , B^M and $(\sigma^2)^1, (\sigma^2)^2,$ $(\sigma^2)^M$ The last H values of B and σ^2 from these iterations is used from the empirical distribution of these parameters This empirical distribution is an approximation to the marginal posterior distribution The initials iterations which are discarded are referred to burn-in-iterations These are the number of iterations required for the Gibbs sample converges

3.12. Convergence Diagnostic Tests for BLR

For summarization of the posterior distribution to compute any applicable amount we used simulated draws In Bayesian inference the Markov Chain Monte Carlo (MCMC) methods generate samples and utilized these samples to estimate expectations of measures of interest from the posterior distribution For the procedure you need to choose whether Markov Chain achieved its stationary and you need to indicate the number of iterations at the point where the Markov Chain has achieved stationarity Still before building any estimation you must check the convergence of entirely parameters in your model by utilizing the convergence diagnostics

There are various statistical diagnostic tests introduce trace plots autocorrelation function Brooks Gelman Rubin and Monte Carlo Standard Errors

3.12.1 Trace plot

In the assessing convergences of trace plot are very cooperative the trace plot on the X-axis depicts the iteration numbers and X-axis is the estimated value of each iterations The Trace plot also depicts that your chain converges to the desired distribution if it's required to longer burn-in period A trace plot also interconnects where the chain is mixing sufficient The feature of stationery that are best identified from a trace plot are reasonably constant mean and variance The chain is a mixed good cover its posterior distribution are quickly. Sometimes the initial values of the chains are very different from each other. In this situation the convergence is very problematic, for this solution we discarded the 1st few thousand iterations. In Fig.(4.1) the trace plot specifies that the chain is reached to the desired distribution.

3.12.2 Brooks Gelman-Rubin Test

This test can be used for multiple chains for each chain has different initial values. This test depends on running parallel chains from joint runs is green, the central size of the 80 % interval within the single runs is blue and the proportion R = (joint/within) is red. R would normally be estimated to be larger than 1. If beginning values are appropriate completed dispread Brooks and Gelman (1998) highlighted that one should be related mutually the pooled and inside interval sizes to stability.

The statistic R can be estimated by

$$\hat{R} = \frac{\hat{V}}{WSS} = \frac{T-1}{T} + \frac{BSS/T}{WSS}\,\frac{k+1}{k}$$

Where k is the number of generating samples (chains). T Is the number of iterations retained in each sample (chain). BSS/T does the variance of the posterior mean values over all generated samples/chains (b/w-sample variance). WSS is the mean of the variance within each sample (with in sample variability).

$$\hat{V} = \frac{T-1}{T}\,WSS + \frac{BSS}{T}\,\frac{k+1}{k}$$

Where $\hat{V}$ is the pooled posterior variance estimate. When convergence is achieved and the size of the generated data is large. The values $\hat{R}$ is approaching to 1 designate convergence.

3.12.3 Monte Carlo Standard Errors

In the inference of the MCMC out comes a significant measure that must be testified and shown is the Monte Carlo Error (MC error) which measures the variability of every estimate due to the simulation MC Error must be small in the direction to estimate the parameter of importance with increase precision In the event that you run the chain until the Monte Carlo Standard deviance is under 5% of the for all parameters of awareness, the chain has no bad mixing For further complete explanation of MC Error see lines Ntzofras (2009) Bayesian Modelling using Winbugs

3.12.4 Autocorrelation Function

By checking the level of dependence expression at the estimated ACF The purpose of integers J that provides evaluated association b/w θ^k And θ^{k+J} After burning this correlation must depend on the lag J but not on k It is calculated as the sample correlation between the pairs (θ^L, θ^{L+J}) $L = 1,2$ s^{-J} If the autocorrelations are close to zero except to say the 1st two $J = 1, 2$ then we could take each third $\theta^{3k}, L = 1, 2$ s after the burn in this sample be nearly uncorrelated its throw away information unless there is extreme autocorrelation e g high association even with say $J = 30$ By drawing the graph in winbugs of ACF on the top to look good If the ACF rapidly approaches to zero and stay there the chain is also good mixing then there is no autocorrelation

The sample autocorrelation of lag h is defined in terms of the sample auto covariance function

$$\hat{\rho}(h) = \frac{\gamma(h)}{\hat{\gamma}(0)} \ , \ |h| < n$$

The sample autocorrelation function of lag h is defined

$$\hat{\gamma}(h) = \frac{1}{n-h}\sum_{t=1}^{n-h}(\theta_i^{t+h} - \bar{\theta}_i) \qquad 0 \leq h < n$$

3.12.5 Overall Goodness-of-Fit of The Model

With a specific end goal to check the whole goodness-of-fit we can compute "R^2" measure to identify the reduction of the error variance because of the explanatory variables involved in the model Therefore we can recycle this amount utilizing the sample variance of the reaction variable M namely "S_M^2", utilizing "R^2" Statistics specified by

$$R^2 = 1 - \frac{\tau^{-1}}{S_M^2} = 1 - \frac{\sigma^2}{S_M^2}$$

Anywhere is "S_M^2" The sample variance of "M" This capacity can be understood as the proportional decrease of uncertainty regarding the response variable M completed by including explanatory variables X_j In the Model Moreover it can be viewed as the Bayesian investigation of the adjusted coefficient of determination

$$R^2_{adj} = 1 - \frac{\hat{\sigma}^2}{S_M^2}$$

where

$$\hat{\sigma}^2 = \frac{1}{n-p}\sum(y_i - \hat{y}_i)^2 \quad \text{with} \quad \hat{y}_i = \beta_0 + \sum X_{ij}\beta_j$$

where

$\hat{\beta}_j$ are the Maximum likelihood estimate of β_j

3.13. Credible interval

By calculating Significant posterior Model probabilities usually requires the elicitation of informative prior For the Bayesian desires to do Model testing or comparison with a non-informative prior there are some other techniques which can be utilized By outlining the ideas in the situation of the parameter vector β in the Normal linear regression Model but they are quite general and can be used with the parameters of any Model Let the components of the vector of regression coefficients β can each lie anywhere in the interval $(-\infty, \infty)$ which is denoted by $\beta \in \mathcal{F}^k$

Let $X = g(\beta)$ be some m-vector of functions of "β" which is well-defined over Region Ω, where $m \leq \bar{k}$ Let N be a region inside Ω denoted by $N \leq \Omega$

Credible set:

Let $X = g(\beta) = \beta_j$ a single regression coefficient Formerly a 95 % credible interval for β_j is any interval $[\acute{a}, \acute{b}]$ similarly

$$p(a \leq \beta_j \leq \acute{b}/M) = \int_a^b p(\beta_j/M) d\beta_j = 0.95$$

Several possible credible intervals are normal

For illustration that $\beta_j/M \sim N(0,1)$ By using statistical tables for the standard normal we find that $[-1.96, 1.96]$ is a 95% credible interval as is $[-1.23, 3.45]$ and $[-1.64, \infty]$ etc We can pick the last interval

HPDI

A 100 (1 - α) % highest posterior density interval for X is a 100 (1 - α) % credible interval for X with the property that it has a minor range any other 100 (1 - α) % credible interval for X Though doing Bayesian approximation for introducing the highest posterior density it is recognizable in count to point the estimation

For illustration the researcher impact data a posterior mean in addition to a 95% HPD of β_j The researcher is 95% confident that in the HPDI β_j lies within the HPDI Consider for illustration two normal linear regression models as $(Y = X\beta + \mu)$ and that point of significant whether the explanatory variable must be involved Consequently the two models under attention remain

$$w_0 \quad \beta_j = 0$$

$$w_1 \quad \beta_j \neq 0$$

By using t-distribution properties HPDI can be designed for β_j If this HPDI does not contain zero, then this is indicated against w_0 A conclusion that the HPDI does contain zero is taken as indication in preferring of w_0

The scholar who identifies frequentist Econometrics will identify the comparison of this methodology with the common hypothesis that $\beta_j = 0$ can be done by computing a confidence interval for β_j If zero lies in this confidence interval then the hypostsis is accepted If it doesn t the hypothesis is rejected Confidence intervals have a many different interpretation from HPDIs

CHAPTER 4

RESULTS AND DISCUSSION

4.1. Introduction

This chapter contains posterior estimates along with results of diagnostic tests under uniform priors Posterior estimates along with results of diagnostic tests under Normal-Gamma priors are also presented These two types of results are compared At the end, prediction results of money demand due to change in interest rate are given

4.2. Posterior Results Using Non-Informative Prior Through UP

Posterior summaries of the model using the non-informative uniform prior have been calculated (Table 4 1) after running the MCMC algorithm for 2500000 iterations and discarding additional 900000 iterations as a burn in period and thinning interval is 200 The final posterior summaries on 24000 samples are provided as a 1 % increase in aggregate demand causes a 2 481% increase on the average in money demand Whereas the average effect may vary from 2 393 to 2 569 which is 95% credible interval On the basis of 95% credible interval coefficient estimate of aggregate demand is significant as the interval doesn t contain 0

1 unit change in price causes 0 5323% increases on the average in money demand Whereas the average effect may vary from 0 4231 to 0 6416 which is 95% credible interval for the average effect of price on money demand

Table 4.1 Posterior result using non-informative through UP

Node	Estimate	Standard Error	MC-Error	Median	95 % Credible Interval		Significance
					Lower Bound	Upper Bound	
Intercept	-24.94	0.06325	0.01259	-24.95	-26.19	-23.70	Sig
Aggregate Demand	2.481	0.04488	8.959E-4	2.481	2.393	2.569	Sig
Price	0.5323	0.05552	8.657E-4	0.5321	0.4231	0.6416	Sig
Interest Rate	-1.401	0.8447	0.00744	-1.405	-3.044	0.2597	In-Sig
Precision.M	48.36	9.328	0.05995	47.65			
R^2_B	0.995	8.766E-4	3.943E-6	0.995			

On the basis of 95% Credible interval coefficient estimate of price is significant as the interval doesn't contain 0.

1 unit change in interest rates causes 1.401% decreases on the average in money demand. Whereas the average effect may vary from -3.044 to 0.02597 which is 95% credible interval for the average effect of interest rate on money demand. On the basis of 95% credible interval coefficient estimate of the interest rate is insignificant as the interval contains 0.

From the 5th column the estimated posterior mean and median is actually close signifying that the posterior distribution of all parameters may be symmetric. In fact the posterior distribution with a mean in this model is known to be a normal R_B^2 indicates that the overall model is a good fit

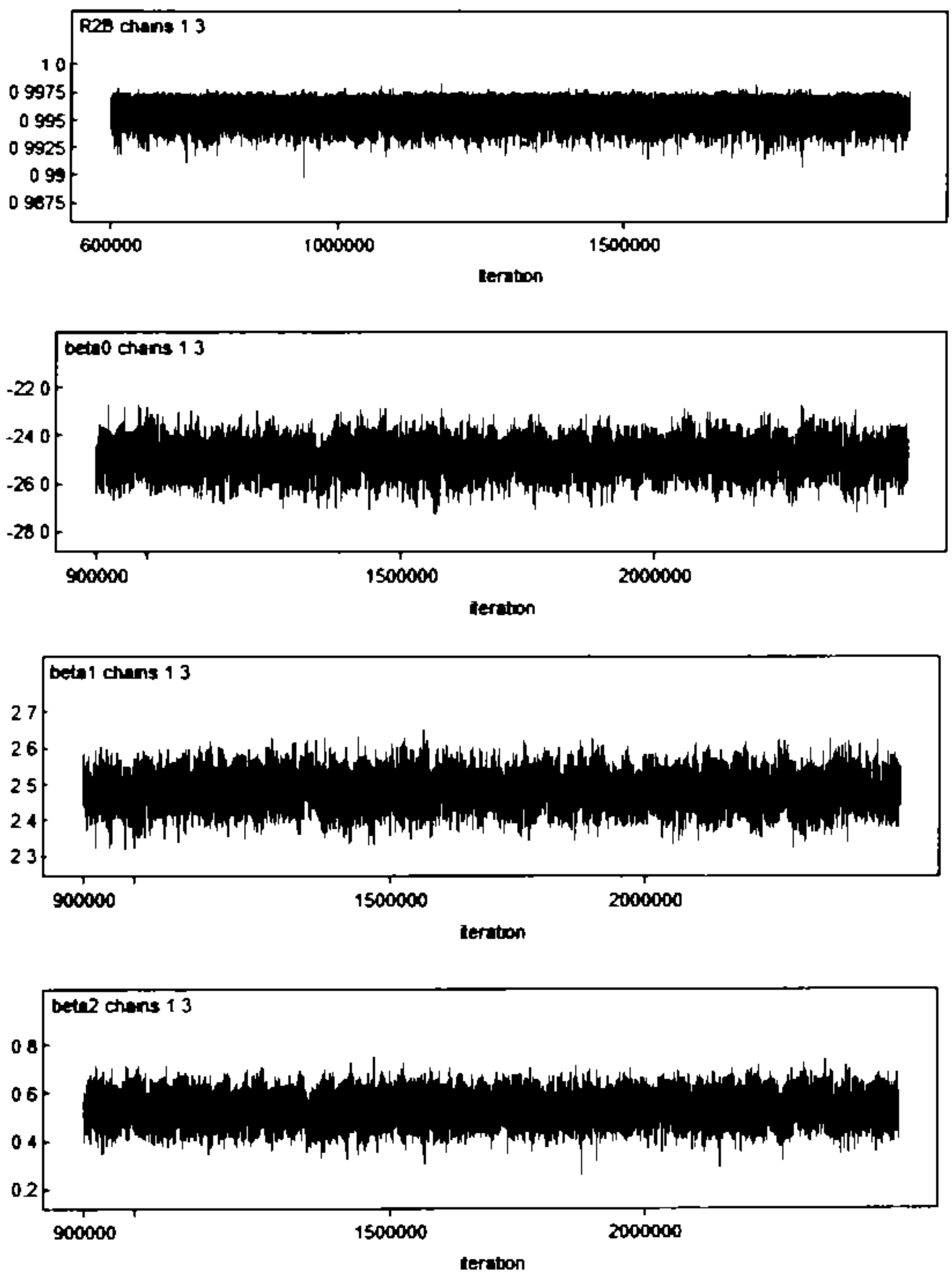
4.3. Convergence Diagnostic Plots of UP

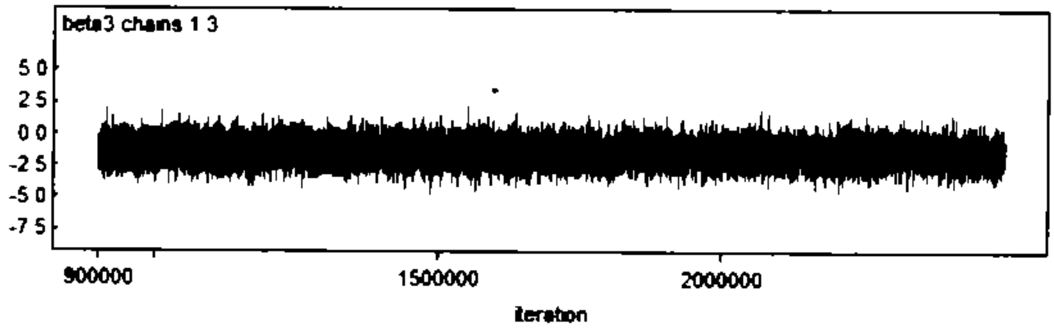
There are multiples convergence diagnostic plots of UP prior model which can be interpreted in convergence of the target distribution which are given below

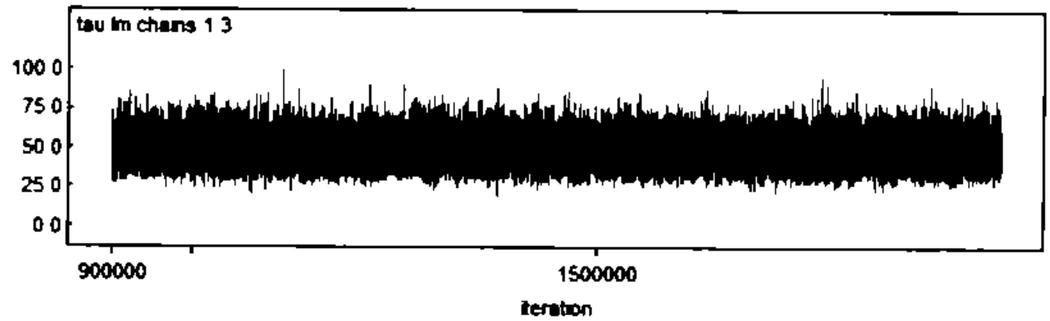
4.3.1. Trace Plots of UP

From the Fig (4 1) we illustrated that the trace plot of samples viruses the simulation numbers (index) running the multiple chain 3 each chain indicating a different color In this situation we can be reasonably confident that convergence has been achieved because all chains appears to be overlapping one another The trace tells that the chain is converted to the stationary distribution after the longer burns in the period The feature of stationary that most familiar from a trace plot is a relatively constant mean and variance The chain that mixes well converges to its posterior space rapidly This figure also shows a perfect trace plot Because the center of the chain appears to be around constant mean values with very small fluctuations This indicates that the chain would reach the target (right stationary) distribution We concluded that the mixing is sufficient good for each parameters

Fig 4 1 Trace plots of UP



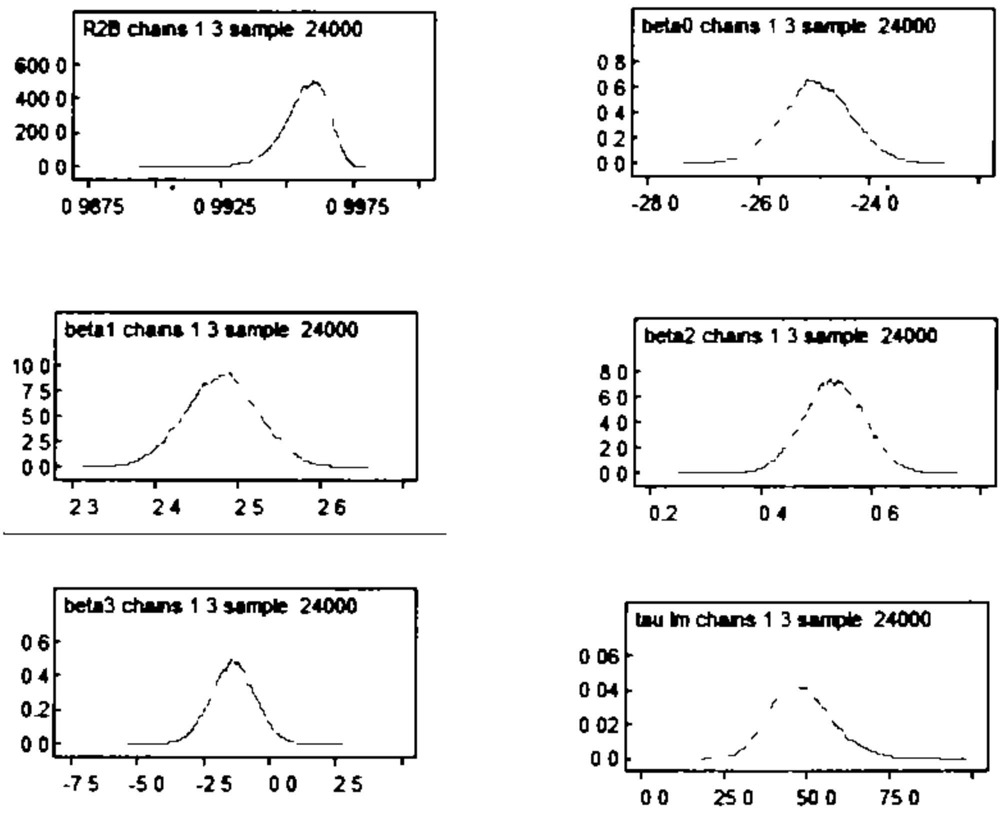




4.3.2. Kernel Density plots of UP

From Fig (4 2) the kernel density plots alternative visualizations of the simulated Marginal posterior distribution of the parameters The marginal posterior densities of β_0, β_1 β_2, β_3 and τ Under Non-informative prior look normal Kernel density plots indicate that the Bayesian point estimate typical posterior mean (or posterior median) and the range between the 2 5th and 95 5th percentile represents 95% Bayesian confidence interval also called credible interval The numerical outcomes of β_0, β_1, β_2, β_3 and τ (posterior means or medians) and graphical representation gives similar results Hence the posterior kernel density designed to stabilize and converges for all parameters β_0, β_1, β_2, β_3 and τ

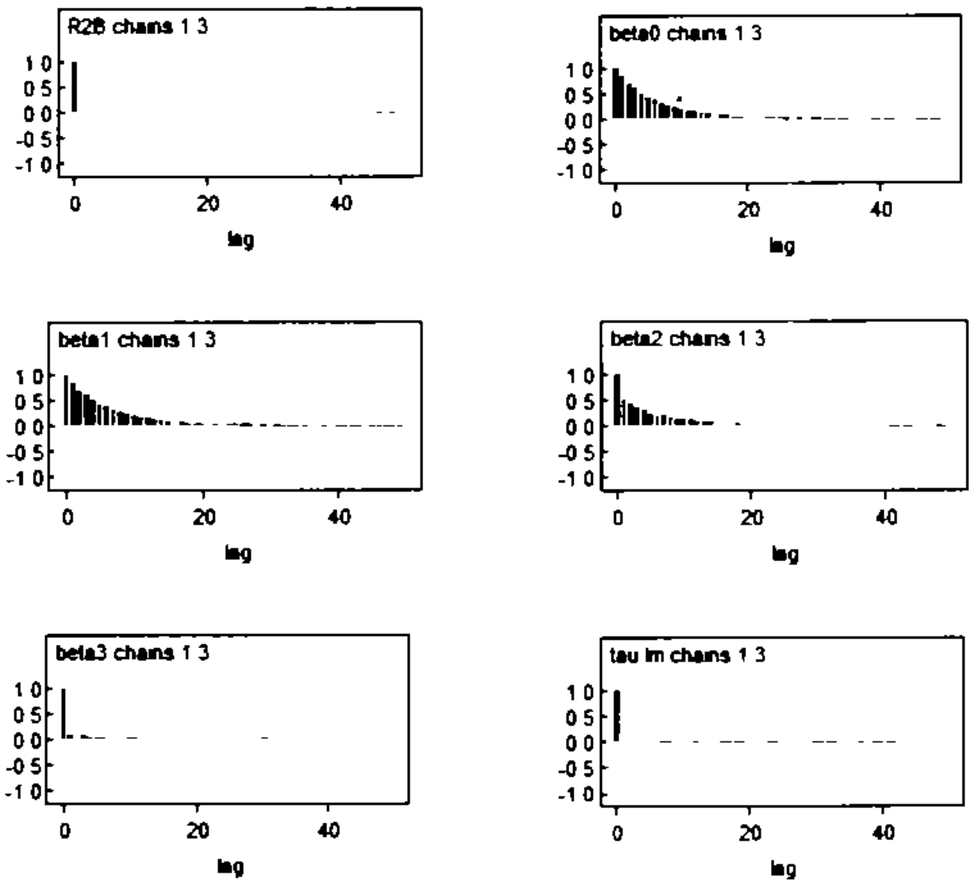
Fig 4 2 Kernel density plots of UP



4.3.3. Autocorrelation Function Plots of UP

The autocorrelation function plots from Fig (4 3) for the chain of each parameter as shown can also indicate the dimension of the posterior distribution that are mixing rapidly Rapidly mixing is often associated with low posterior correlation between parameters The plots indicate that all parameters are mixing well with autocorrelation vanishing before 5 lags in each case

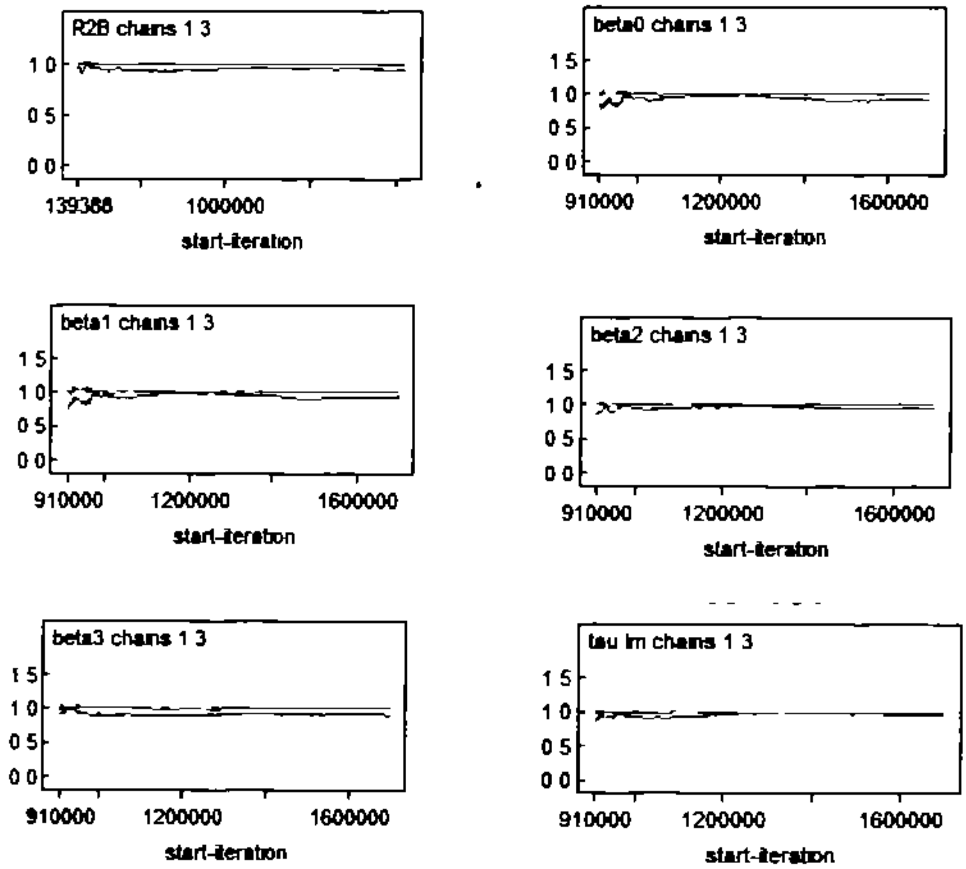
Fig 4 3 Autocorrelation function plots of UP



4.3.4. BGR Plots of UP

Fig (4 4) BGR-plot indicates that the diagnostic plot generated for the 2500000 values of β's sampled from the 3 coins and discarding the 1st 900000 The blue line represents the average width of 80% credible intervals computed from the 3 separate chains The green lines represent computed from the pooled data and the red lines is the ratio of these two values The fig indicates that the ratio is 1 and the 3 chains converges to its desired distribution

Fig 4 4 BGR plots of UP



4.4. Posterior Results Using Informative Prior Through NG

From table (4 2) specified that the Posterior summaries and densities when running the MCMC algorithm utilized for 2000000 iterations and discard the initial 900000 ones and taking the thinning interval as 99 The final posterior summaries on 33333 samples are supplied as a 1 % increases in aggregate demand causes 2 48 % increases on the average in money demand

Table 4 2 Posterior result using informative prior through NG

Node	Estimate	Standard Error	MC - Error	Median	95 % Credible Interval		Significance
					Lower Bound	Upper Bound	
Intercept	24 9	0 0 3701	0 005693	-24 9	-24 63	-24 18	Sig
Aggregate Demand	2 48	0 02564	3 939E-4	2 479	2 43	2 53	Sig
Price	0 537	0 03688	3 664E-4	0 538	0 4655	0 610	Sig
Interest Rate	-1 753	0 4134	0 002315	-1 752	-2 563	-0 941	Sig
Precision M	49 13	9 365	0 05057	48 5			
R^2_B	0 995	8 766E -4	3 943E -6	0 995			

Whereas the average effect may fluctuate from 2 43 to 2 53 which is 95 % credible interval aggregate demand effected on money demand On the basis of 95% credible interval coefficient estimate of aggregate demand is significant as the interval doesn t contain 0

1 unit change in price causes 0 537 % increases on the average in money demand Whereas the average effect may fluctuate from 0 4655 to 0 610 which is 95 % credible interval for the average effect of price on money demand On the basis of 95% credible interval coefficient estimated of price is significant as the interval doesn't contain 0

1 unit change in interest rate causes -1.753% decreases on the average in money demand. Whereas the average effect may vary from -2.563 to -0.941 which is 95% credible interval for the average effect of interest rate on money demand. On the basis 95% credible interval coefficient estimate of interest rate is significant as the interval does not contain 0.

From the 5th column the estimated posterior mean and median is actually close specifying that the posterior distribution of (all parameters) may be symmetric. In fact the posterior distribution with a mean in this model is known to be a normal. From the table we also depict that the informative prior Model has a significant improvement as compared to non-informative prior Model of the precision effects in the production of Money demand while containing in the Model as other explanatory variables as Aggregate demand, Price and Interest rate from the result of R^2_a indicate that the model is the best fit.

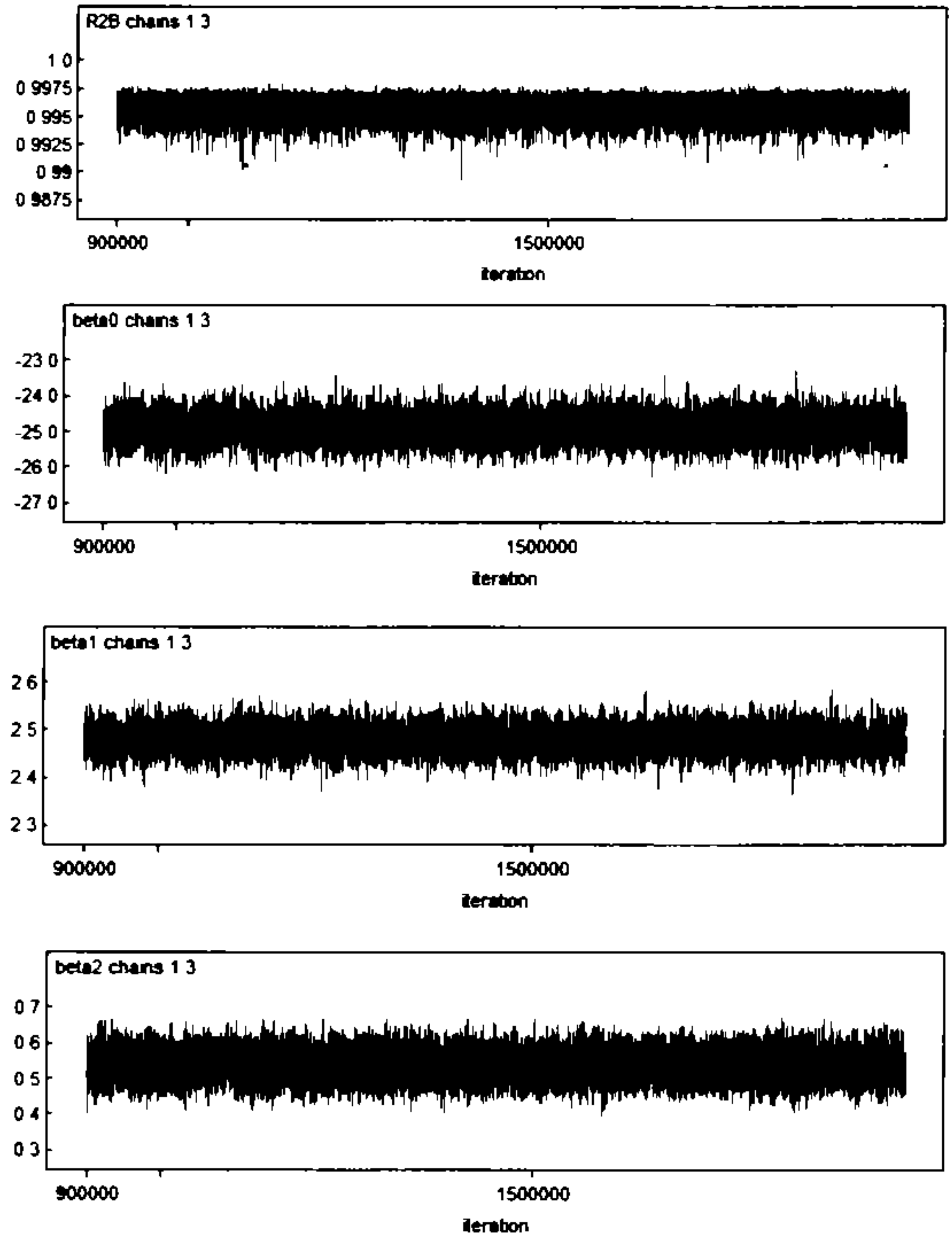
4.5. Convergence Diagnostic Plots of NG Prior

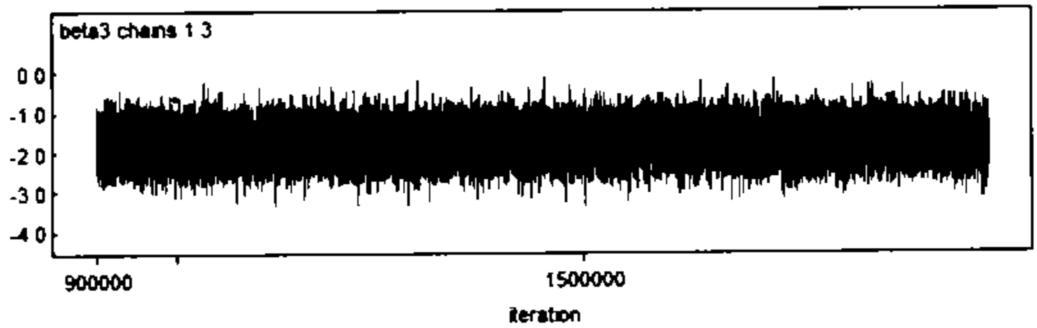
There are multiples convergence diagnostic plots of NG prior model which can be interpreted in convergence of the target distribution which are given below.

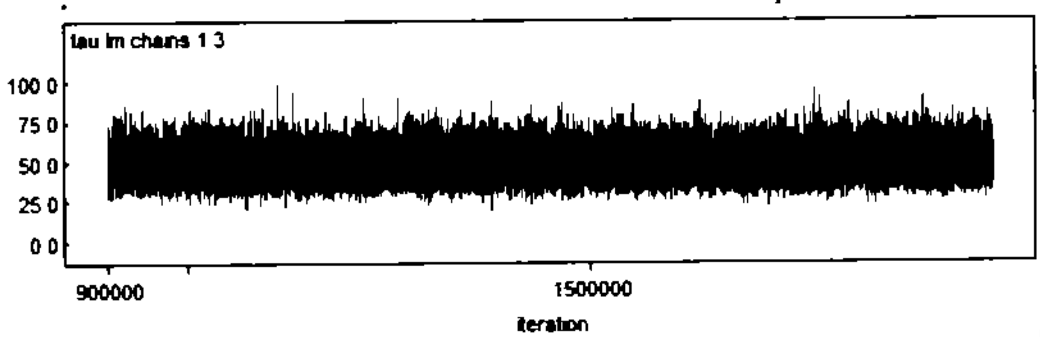
4.5.1. Trace Plots of NG Prior

From Fig (4.5) we depict that the trace plot of samples viruses the simulation numbers (index) running the multiple chain 3 indicating each chain has a different color. In this situation, we can be practically self-possessed that convergence has been realized because all chains appears to be intersecting one another. The trace tells that the chain is converted to the stationary distribution after the longer burns in the period. The feature of stationary that most identifiable from a trace plot is a considerably constant mean and variance. A chain that mixes well crosses its posterior space quickly

Fig 4 5 Trace plots NG prior







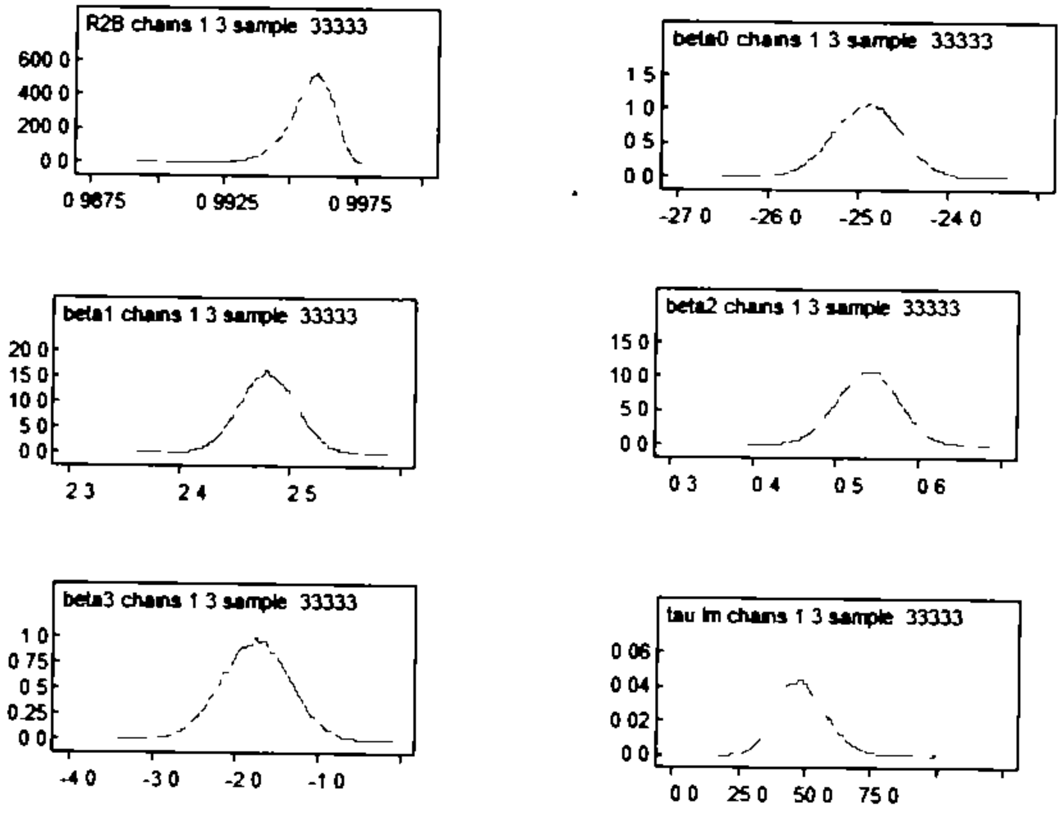
This figure also shows a perfect trace plot Because the center of the chain performs to be around constant mean values with very small fluctuations

This indicates that the chain would reach the target (right, stationary) distribution We concluded that the mixing is quite good here for each parameter

4.5.2. Kernel Density Plots of NG Prior

By obtaining from the Fig (4 6) that the kernel density plots alternative visualizations of the simulated Marginal posterior distribution of the parameters The marginal posterior densities of β_0 β_1, β_2, β_3 and τ Under informative prior look normal Kernel density plots indicate that the Bayesian point estimate typical posterior mean (or posterior median) and the range between the 2 5th and 95 5th percentile represents 95% Bayesian confidence interval and is also called credible interval The numerical outcome of $\beta_0, \beta_1, \beta_2, \beta_3$ and τ (posterior means or medians) and graphical

Fig 4 6 Kernel density plots of NG prior

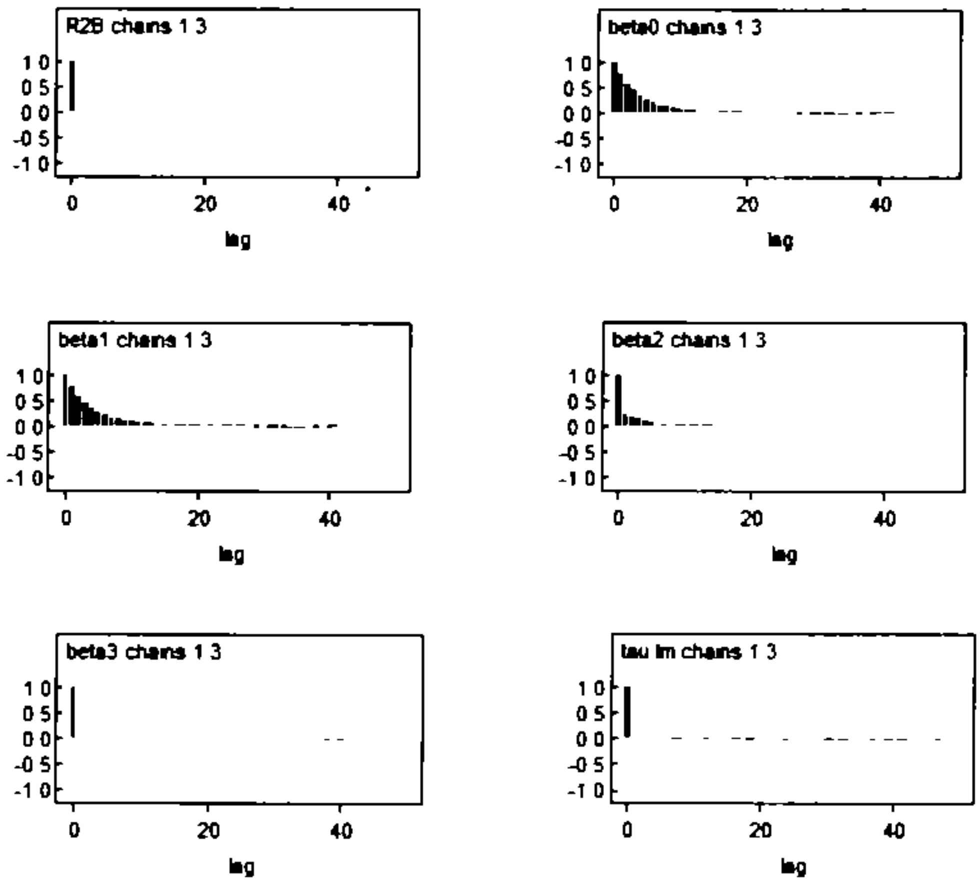


representation give the similar results Hence the posterior kernel density designed to stabilize and convergence for all parameters of β_0. β_1 β_2 β_3 and precision

4.5.3. Autocorrelation Function Plots of NG Prior

The autocorrelation function from Fig (4 7) for the chain of each parameter as shown also indicate the dimension of the posterior distribution that are mixing rapidly Rapidly mixing is often associated with low posterior correlation between parameters The plots indicates that all parameters are mixing well with autocorrelation vanishing before the starting lags in each case

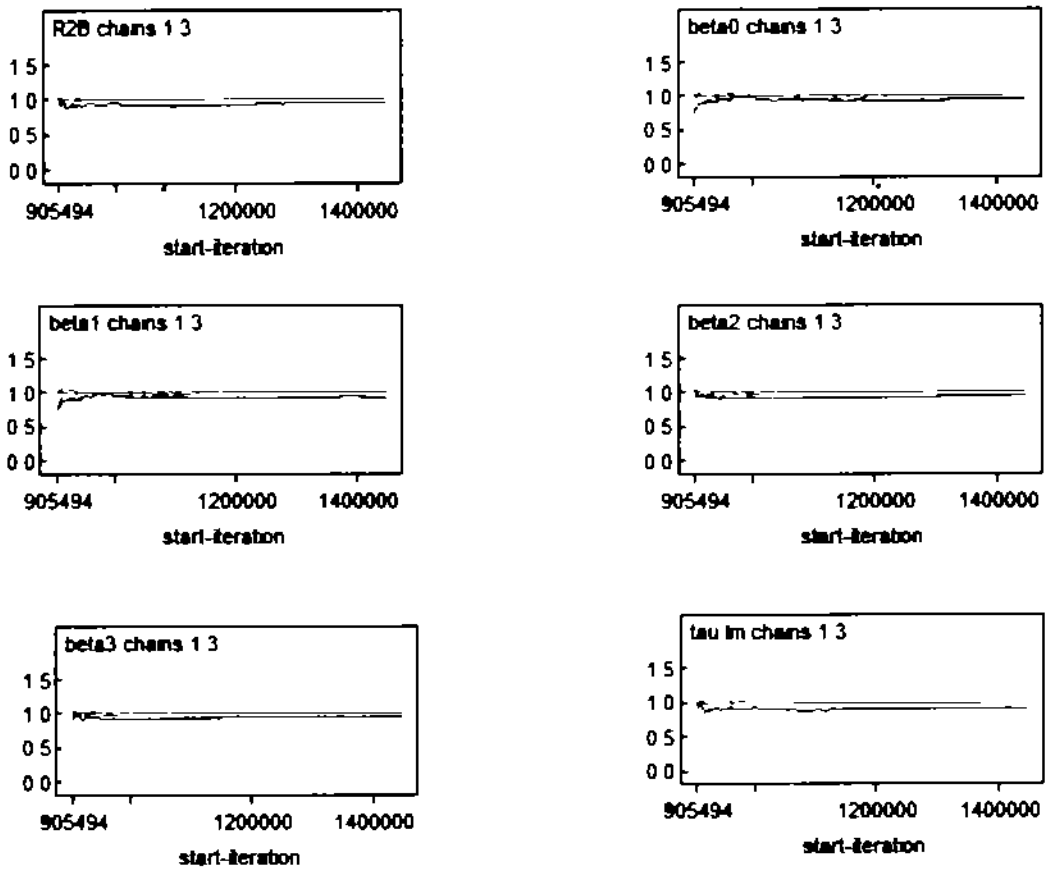
Fig 4 7 Autocorrelation function plots of NG prior



4.5.4. BGR Plots of NG Prior

Fig (4 8) BGR-plot indicates that the diagnostic plot generated for the 2000000 values of β's sampled from the 3 chains and discarding the 1^{st} 900000 The blue line represents the average width of 80% credible intervals computed from the 3 separate chains The green lines represent computed from the pooled data and the red lines is the ratio of these two values The fig indicates that the ratio is 1 and the 3 chains are converged

Fig 4 8 BGR plots of NG prior



4.6. Posterior Predictive Distribution and Simulation Analysis of Interest Rate.

In the previous section, estimated model is presented along with diagnostic tests Results of diagnostic tests establish validity of the model and hence, it can be used for prediction and forecasting Under Bayesian framework prediction and forecasting is done by constructing posterior predictive distribution If we have a normal linear regression model as in Equation (3 3) by likelihood and prior assumed in Equation (3 4) and Equation (3 16) Posterior inference can be completed using Equation (3 21) and Equation (3 23) We need predictive inference on T unobserved estimation of the dependent variable which we designate by $M^* = (M_1^* \quad M_T^*)$ which produces as indicated by

$$M^* = X^*\beta + \mu^* \tag{4 1}$$

where μ^* Is independent of μ and is $N(0,\ H^{-1}I_T)$ and X^* is a $T \times K$ matrix similarities to X, holding the K explanatoray variables for each of the T out-of- sample information points

One method for deriving the predictive pdf is to record the joint pdf of $p(M^*, \beta,\ H|X,\ X^*, M)$ and integrate regarded β and H to achieve the marginal pdf to M^* which is the predictive pdf

$$p(M^*, \beta, H|X,\ X^*, M) = \int \int p(M^*|\beta, H, M)\, p(\beta, H|M, X) d\beta dH \tag{4 2}$$

The statement that μ^* is independent of μ suggests that M and M^* are independent of each other and therefore $p(M^*|\beta, H, M) = p(M^*|\beta, H)$ The concluding expressions can be written as

$$p(M^*|\beta, H, M) = \frac{H^{\frac{t}{2}}}{(2\pi)^{\frac{t}{2}}} exp\left[-\frac{H}{2}(M^* - X^*\beta)^T(M^* - X^*\beta)\right] \tag{4 3}$$

Multiplying Equation (4 3) by the posterior specified in Equation (3 16) and Equation (3 18) and integrating produces a multivariate-t predictive density of the form

$$M^*|M \sim t\left(X^*\bar{\beta}, \bar{s}^{-2}\{I_T + X^*\bar{Q}X^{*T}\}, \bar{v}\right) \tag{4 4}$$

Using the above posterior predictive distribution money demand is predicted for various values of interest rate and the results are presented in Table (4 3) Results show that as the rate of interest increases money demand decreases in the economy For every change of 0 5 percentage point, there is 1 0052 percent inverse change in money

Table 4 3 Simulation analysis of interest rate

IR %	Δ IR	Predicted log (M2)	% Δ in log (M2)	M2 (Rs. Millions)	95% Credible Interval of M2	
3		16 58		15870934 89	11640488 89	15870934 89
4	1	16 56	-2 02013	15556669 32	11409991 76	15556669 32
5	1	16 55	-1 00502	15401877 88	11184058 79	15401877 88
6 1	1 1	16 53	-2 02013	15096900 26	11072775 55	15096900 26
6 2	0 1	16 53	0	15096900 26	10962599 59	15096900 26
6 3	0 1	16 52	-1 00502	14946683 59	10962599 59	14946683 59
6 4	0 1	16 52	0	14946683 59	10962599 59	14946683 59
6 5	0 1	16 52	0	14946683 59	10962599 59	14946683 59
6 6	0 1	16 52	0	14946683 59	10962599 59	14946683 59
6 7	0 1	16 52	0	14946683 59	10853519 9	14946683 59
6 8	0 1	16 51	-1 00502	14797961 61	10853519 9	14797961 61
6 9	0 1	16 51	0	14797961 61	10853519 9	14797961 61
7	0 1	16 51	0	14797961 61	10853519 9	14797961 61

7 1	0 1	16 51	0	14797961 61	10853519 9	14797961 61
7 2	0 1	16 51	0	14797961 61	10853519 9	14797961 61
7 3	0 1	16 51	0	14797961 61	10745525 57	14797961 61
7 4	0 1	16 5	1 00502	14650719 43	10745525 57	14650719 43
7 5	0 1	16 5	0	14650719 43	10745525 57	14650719 43
7 6	0 1	16 5	0	14650719 43	10745525 57	14650719 43
7 7	0 1	16 5	0	14650719 43	10745525 57	14650719 43
7 8	0 1	16 5	0	14650719 43	10638605 81	14650719 43
7 9	0 1	16 49	1 00502	14504942 33	10638605 81	14504942 33
8	0 1	16 49	0	14504942 33	10638605 81	14504942 33
8 1	0 1	16 49	0	14504942 33	10638605 81	14504942 33
8 2	0 1	16 49	0	14504942 33	10638605 81	14504942 33
8 3	0 1	16 49	0	14504942 33	10638605 81	14504942 33
8 4	0 1	16 49	0	14504942 33	10532749 91	14504942 33
8 5	0 1	16 49	0	14504942 33	10532749 91	14504942 33
8 6	0 1	16 49	0	14504942 33	10532749 91	14504942 33
8 7	0 1	16 49	0	14504942 33	10532749 91	14504942 33
8 8	0 1	16 49	0	14504942 33	10532749 91	14504942 33

8 9	0 1	16 48	-1 00502	14360615 75	10427947 3	14360615 75
9	0 1	16 48	0	14360615 75	10427947 3	14360615 75
9 1	0 1	16 47	1 00502	14217725 23	10427947 3	14217725 23
9 2	0 1	16 47	0	14217725 23	10427947 3	14217725 23
9 3	0 1	16 47	0	14217725 23	10427947 3	14217725 23
9 4	0 1	16 47	0	14217725 23	10427947 3	14217725 23
9 5	0 1	16 47	0	14217725 23	10427947 3	14217725 23
9 6	0 1	16 46	1 00502	14076256 5	10324187 49	14076256 5
9 7	0 1	16 46	0	14076256 5	10324187 49	14076256 5
9 8	0 1	16 46	0	14076256 5	10324187 49	14076256 5
9 9	0 1	16 46	0	14076256 5	10324187 49	14076256 5
10	0 1	16 46	0	14076256 5	10324187 49	14076256 5
10 1	0 1	16 46	0	14076256 5	10324187 49	14076256 5
10 2	0 1	16 45	-1 00502	13936195 41	10221460 11	13936195 41
10 3	0 1	16 45	0	13936195 41	10221460 11	13936195 41
10 4	0 1	16 45	0	13936195 41	10221460 11	13936195 41
10 5	0 1	16 45	0	13936195 41	10221460 11	13936195 41
10 6	0 1	16 45	0	13936195 41	10221460 11	13936195 41

10 7	0 1	16 45	0	13936195 41	10221460 11	13936195 41
10 8	0 1	16 44	-1 00502	13797527 95	10119754 88	13797527 95
10 9	0 1	16 44	0	13797527 95	10119754 88	13797527 95
11	0 1	16 44	0	13797527 95	10119754 88	13797527 95
11 1	0 1	16 44	0	13797527 95	10119754 88	13797527 95
11 2	0 1	16 44	0	13797527 95	10119754 88	13797527 95
11 3	0 1	16 44	0	13797527 95	10019061 63	13797527 95
11 4	0 1	16 43	-1 00502	13660240 25	10019061 63	13660240 25
11 5	0 1	16 43	0	13660240 25	10019061 63	13660240 25
11 6	0 1	16 43	0	13660240 25	10019061 63	13660240 25
11 7	0 1	16 43	0	13660240 25	10019061 63	13660240 25
11 8	0 1	16 43	0	13660240 25	10019061 63	13660240 25
11 9	0 1	16 42	-1 00502	13524318 59	9919370 306	13524318 59
12	0 1	16 42	0	13524318 59	9919370 306	13524318 59
12 5	0 5	16 41	-1 00502	13389749 37	9820670 922	13389749 37
13	0 5	16 41	0	13389749 37	9722953 614	13389749 37
13 5	0 5	16 4	1 00502	13256519 14	9722953 614	13256519 14
14	0 5	16 39	-1 00502	13124614 57	9722953 614	13124614 57

14 5	0 5	16 38	-1 00502	12994022 47	9530426 233	12994022 47
150	135 5	16 37	-1 00502	12864729 79	9435596 907	12864729 79
15 5	134 5	16 36	-1 00502	12736723 59	9341711 149	12736723 59
16	0 5	16 35	-1 00502	12609991 07	9248759 57	12609991 07
16 5	0 5	16 34	-1 00502	12484519 57	9156732 875	12484519 57
17	0 5	16 34	0	12484519 57	9065621 861	12484519 57
17 5	0 5	16 33	-1 00502	12360296 52	8975417 416	12360296 52
18	0 5	16 32	-1 00502	12237309 51	8975417 416	12237309 51
18 5	0 5	16 31	-1 00502	12115546 25	8886110 521	12115546 25
19	0 5	16 3	-1 00502	11994994 55	8797692 244	11994994 55
19 5	0 5	16 29	-1 00502	11875642 36	8710153 743	11875642 36
20	0 5	16 28	-1 00502	11757477 75	8623486 265	11757477 75
20 5	0 5	16 27	-1 00502	11640488 89	8537681 143	11640488 89
21	0 5	16 27	0	11640488 89	8452729 796	11640488 89
21 5	0 5	16 26	-1 00502	11524664 09	8368623 73	11524664 09
22	0 5	16 25	-1 00502	11409991 76	8285354 532	11409991 76
22 5	0 5	16 23	-2 02013	11184058 79	8202913 877	11184058 79
23	0 5	16 23	0	11184058 79	8121293 52	11184058 79

23 5	0 5	16 22	-1 00502	11072775 55	8040485 3	11072775 55
24	0 5	16 21	-1 00502	10962599 59	7960481 134	10962599 59
24 5	0 5	16 2	1 00502	10853519 9	7881273 024	10853519 9
25	0 5	16 2	0	10853519 9	7881273 024	10853519 9
25 5	0 5	16 19	-1 00502	10745525 57	7802853 047	10745525 57
26	0 5	16 18	-1 00502	10638605 81	7725213 362	10638605 81
26 5	0 5	16 17	-1 00502	10532749 91	7648346 204	10532749 91
27	0 5	16 16	-1 00502	10427947 3	7572243 888	10427947 3
27 5	0 5	16 15	1 00502	10324187 49	7496898 803	10324187 49
28	0 5	16 14	-1 00502	10221460 11	7422303 413	10221460 11
28 5	0 5	16 13	-1 00502	10119754 88	7348450 26	10119754 88
29	0 5	16 12	-1 00502	10019061 63	7275331 958	10019061 63
29 5	0 5	16 12	0	10019061 63	7202941 196	10019061 63
30	0 5	16 11	-1 00502	9919370 306	7131270 737	9919370 306

demand This response of money demand to change in interest rate remains almost same for all levels of interest rate, from 3% to 30%

CHAPTER 5

SUMMARY, CONCLUSION AND FURTER RESERCH

This thesis comprises five chapters Chapter 1 provides the explanation of our topic and objectives of our study The core objective of our study is the estimation of Money Demand Function of Pakistan Economy by Employing Bayesian approach

Chapter 2 provides brief discussion of the basic elements of Bayesian Inference including Bayes theorem likelihood function, prior distribution informative and non-informative prior, choice of prior distribution, MCMC simulation Gibbs sampling Elicitation procedures, posterior distribution, and posterior predictive distribution This chapter also discusses the concept of money the role of money in an Economy, money demand At the end of the chapter review of literature related to Bayesian regression and Money demand models is presented We concluded that literature contains very limited studies containing the application of Bayesian analysis approaches to general linear regression Model Moreover there is no such study that contains money demand function through Bayesian approach

Chapter 3 present specification of the model, Bayesian multiple regression model in matrix form derivation of posterior distribution through informative and non-informative prior The model is specified on the basis of liquidity demand theory Elicitation of hyper parameters is done by using PV-method At the end of this chapter convergence diagnostic tests such as Trace plot BGR test MCSE ACF and overall goodness of fit for Bayesian linear regression are presented

Chapter 4 contains the numerical results of our model The same data set is used for the Analysis of the Model using Non-informative (Uniform priors) and informative (Normal-Gamma) priors to simulate parameters of the model Diagnostic tests establish validity of both models We concluded that the estimated results are found by using informative prior superior from non-informative prior We have observed that the results obtain via the Non-informative prior are smaller precision as compare to informative prior due to updated information provided Moreover standard errors of all parameters are smaller in the results based on informative priors as compare to non-informative priors It implies that the incorporated information through prior distribution are useful as the results show larger precision Parameters estimates show that aggregate demand and price level directly and significantly affect money demand in Pakistan while interest rate negatively affect money demand in Pakistan The results match to the theory of economics Moreover simulation results show that 0 5 percentage point change in interest rate causes 1 0052 percent inverse change in money demand Hence monetary policy makers may use this prediction as guideline while changing level of interest rate in order to meet their specified goals about money demand in the economy

For further research this work can be extended to include other macroeconomic theories Moreover the model may be re-estimated on the basis of other non-informative and informative priors and then results may be compared

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