MHD oscillatory flow in a composite porous medium channel

By

Muhammad Zaheer Ahmad

A Thesis

Submitted in the partial fulfillment of the requirement for the degree of

MASTER OF SCIENCE

in

MATHEMATICS

Supervised by

Dr. Zaheer Abbas

Department of Mathematics & Statistics

Faculty of Basic & Applied sciences

International Islamic University, Islamabad Pakistan

2013

بسم الله الرحمن الرحيم

In the name of Allah, the most Gracious, the most Merciful

Declaration

I, hereby declare, that this thesis, neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

Muhammad Zaheer Ahmad

MS Mathematics

Reg. No. 50-FBAS/MSMA/F-10

Department of Mathematics & Statistics

Faculty of Basic & Applied sciences

International Islamic University, Islamabad, Pakistan

Certificate

MHD OSCILLATORY FLOW AND HEAT TRANSFER IN A COMPOSITE POROUS MEDIUM CHANNEL

By

MUHAMMAD ZAHEER AHMAD

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF SCIENCES in MATHEMATICS***

We accept this thesis as conforming to the required standard.

1. Dr. IRSHAD AHMAD ARSHAD (Chairman)
2. Dr. ZAHEER ABBAS (Supervisor)
3. Dr. MASOOD KHAN (External Examiner)
4. Dr. NASIR ALI (Internal Examiner)

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2013

Acknowledgements

First and foremost, I am thankful to Almighty Allah, who created human beings, the best creation among all the living species and made them understand to write with pen. He provided me a chance, boldness and capability to achieve this task. I offer countless darood and slaams to my beloved Holy Prophet Hazrat Muhammad (PBUH), for whom this universe has been created. Allah has shown His existence and oneness by sending him as a messenger of Islam and born me as a Muslim.

I offer my most sincere gratitude and thanks to my sincere, affectionate, kind and most respected supervisor **Dr. Zaheer Abbas**, whose kinetic supervision, admonition in a right inclination and inductance of hard work made my task easy and I completed my dissertation well within time. His ideology and concepts have a remarkable impact on my research contrivances. He also arranged some suitable facilities, without which my objective might not be attained. I have learnt a lot from his ability.

I am thankful to all my teachers; they always guided me sincerely and honestly throughout my course work as well as research work. Specially **Dr. Ahmer Mahmood**, who has helped me a lot in the beginning of my research.

I also offer special thanks to all my class fellows and my friends, who really helped me to their best throughout my research period. They helped me throughout my work, whenever I faced any difficulty relating my problem.

Muhammad Zaheer Ahmad

Preface

Many problems involving multiphase flow and heat transfer and multi-component mass transfer arises in a number of scientific and engineering disciplines and is important in the petroleum extraction and transport. Examples include the reservoir rock of an oil field always contains several immiscible fluids in its pores. Zaturaska et.al [1] discussed the flow of viscous fluid driven along a channel by suction at porous walls. King and Cox [2] performed an asymptotic analysis of the steady-state and time-dependent laminar flows in a porous channel. Some theoretical and experimental work on stratified laminar flow of two immiscible fluids in a horizontal pipe as discussed by authors [3]-[6]. Chamkha [7] studied analytical solutions for flow of immiscible fluids in pores and non-porous parallel plates. Later on, Malashetty et al. [8]-[10] analyzed the MHD two-fluid convective flow and heat transfer in composite porous medium. Recently, Umavathi [11] presented an oscillatory flow of unsteady convective fluid in an infinite vertical stratum. Very recently, Umavathi et al. [12] discussed the problem of unsteady oscillatory flow and heat transfer in a horizontal composite porous medium channel.

Motivated by these facts our aim in this dissertation is to study the MHD oscillatory flow in a composite porous medium channel. In chapter 1 basic definitions and flow equations are given. Chapter 2 contains the detail review of the work done by Umavathi et al. [12]. Chapter 3 is carried out an extension of the work by [12] by incorporating a constant magnetic field. The governing flow equations are solved analytically using the perturbation method. The effect of various parameters on the velocity and temperature profiles are analyzed through graphs and discussed.

Contents

Chapter 1

Preliminaries

This chapter deals with some basic definitions and flow equations. The basic idea of perturbation method is also included.

1.1 Basic Definitions

1.1.1 Fluid

Fluids are the substances that can flow from one point to another. Liquids and gases are classified as fluids because they can flow. An important property of fluids is that they possess only bulk modulus and no young's modulus and modulus of rigidity. Fluids play a very important role in many fields of our daily life.

1.1.2 Fluid Flow

Moving fluids have great importance. In order to find the behavior of fluids in motion, we consider their flow through the pipes. When a fluid is in motion, its flow can takes flow in two ways; either steady (laminar) or unsteady (turbulent)

1.1.3 Density

The ratio of mass to the volume is called density of a fluid. The density ρ of a small element of any material is the mass Δm of the element divided by its volume Δv, according to density there are two main types of fluids; compressible and in-compressible fluids.

$$\rho = \Delta m / \Delta v \tag{1.1}$$

1.1.4 Pressure

The pressure p on a planar surface is defined as the compressive normal force applied by the fluid to the surface, FN divided by the area of that surface, A. Mathematically, it is given by

$$p = FN / A \tag{1.2}$$

1.1.5 Internal Energy, Enthalpy, and Specific Heats of a Perfect Fluid

In the model known as a calorically perfect fluid, which we use throughout this text, the specific heats are assumed to be constants. In this model the internal energy change $u_2 - u_1$, and the enthalpy change, $h_2 - h_1$, are related to temperature change $T_2 - T_1$ by the equations

$$\begin{aligned} u_2 - u_1 &= cv(T_2 - T_1) \\ h_2 - h_1 &= cp(T_2 - T_1) \end{aligned} \tag{1.3}$$

The ratio of specific heats occurs so often in gas flow problems that it is given a special symbol

$$\gamma = \frac{cp}{cv} \tag{1.4}$$

Since the specific heats are constants for a calorically perfect fluid, the ratio of specific heats is also a constant.

1.1.6 Viscosity

When one layer of flowing fluid moves relative to another layer, an opposing force comes into play. This internal friction between two layers of a fluid in relative motion is known as fluid friction or viscosity. In other words, the property of fluids due to which they oppose relative motion between their different layers is called viscosity.

$$\mu = \tau_{xy} / \frac{du}{dy} \tag{1.5}$$

1.1.7 Coefficient of Viscosity

Coefficient of Viscosity of a liquid is defined as the tangential force per unit area required to maintain a unit relative velocity between its two layers, its unit distance apart. Its unit in SI is

kg $m^{-1}s^{-1}$

1.2 Types of Flow

A fluid can be classified into different types which are described below

1.2.1 Ideal Fluids

An incompressible fluid having no viscosity is called an ideal fluid.

1.2.2 Laminar Flow

The flow is said to be laminar, if every particle that passes a particular point, moves along exactly the same path, as followed by particles which passed that points earlier.

1.2.3 Steady Flow

A flow in which the fluid properties does not depend on time if ρ is any fluid property then

$$\partial \rho / \partial t = 0 \quad (1.6)$$

1.2.4 Unsteady Flow

A flow which is not steady is called unsteady flow. *i.e.*

$$\partial \rho / \partial t \neq 0 \quad (1.7)$$

1.2.5 In-Compressible Fluids

In great many cases of the flow of liquids (and also of gases) their density may be supposed invariable, *i.e.* constant throughout the volume of the fluid and throughout its motion. In other words, there is no noticeable compression or expansion of the fluid in such cases. We then speak of in-compressible flow.

1.2.6 Compressible Fluids

If the density is variable, *i.e.* not constant throughout the volume of the fluid and throughout its motion, then the flow is called as compressible flow.

1.2.7 Newtonian Fluids

Even among substances commonly accepted as fluids, there is a wide variation in behavior under stress. Fluids obeying Newton's law of viscosity and for which μ has a constant value are called

Newtonian Fluids. Most common fluids fall into this category, for which shear stress is linearly related to velocity gradient.

1.2.8 Non- Newtonian Fluids

Fluids in which the shear stress is not linearly proportional to the deformation rate of the fluid are called non-Newtonian Fluids. *i.e.* they do not possess Newton's law of viscosity

1.2.9 Divergence of a Vector

The divergence of a vector is denoted by $\nabla.\mathbf{V}$ and defined as

$$\nabla.\mathbf{V} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1}e_1 + \frac{\partial v_2}{\partial x_2}e_2 + \frac{\partial v_3}{\partial x_3}e_3 \qquad (1.8)$$

1.2.10 1st Rivillin Erickson Tensor

Strain rate tensor or 1st Rivillin Erickson tensor is denoted by **A** and defined as

$$\mathbf{A} = (\text{grad } \mathbf{V}) + (\text{grad } \mathbf{V})^T \qquad (1.9)$$

Where, $(\text{grad } \mathbf{V}) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ and $(\text{grad } \mathbf{V})^T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$

1.3 Governing Equations

The equations used to study the flow are known as governing equations. The following are the some basic equations used in this phenomenon

1.3.1 Equation of Continuity

Before defining the equation of continuity we define the law of conservation of mass, this law states that the mass of a control volume remains constant. The partial differential equation representing conservation of mass is called the continuity equation.

Its mathematical form is given by

$$\frac{\partial \rho}{\partial t}+\nabla.\rho\,\mathbf{V}=0$$

For incompressible fluids $\frac{\partial \rho}{\partial t}=0$ therefore, the above equation takes the form

$$\nabla.\mathbf{V}=0$$

i.e.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1.10}$$

1.3.2 Navier-Stokes Equations

The Navier-Stokes Equations represents the law of conservation of momentum i.e.

$$\rho_0\,\frac{d}{dt}(\mathbf{V})=div\overline{T}+\rho f \tag{1.11}$$

Where $\overline{T}$ is called Cauchy stress tensor defined as $\overline{T}=\begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{yx} & \tau_{yy}\end{pmatrix}=-PI+\mu\mathbf{A}$, f is called body force per unit mass and t is time, P is the Hydrostatic pressure, **A** is strain rate tensor or 1st Rivillin Erickson tensor defined as

$$\mathbf{A}=(\text{grad }\mathbf{V})+(\text{grad }\mathbf{V})^{\mathbf{T}}$$

1.3.3 Energy Equation

The general form of energy equation is

$$\rho_0 C_p\left(\frac{\partial T}{\partial t}+\mathbf{V}.\text{gradT}\right)=\text{div}\left(\chi_k\ \text{grad}\mathbf{T}\right)+\sigma'_{ik}\frac{\partial v_i}{\partial x_k}$$

$$\rho_0 C_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial x}\left(\chi_k\left(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\chi_k\left(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}\right)\right)+\sigma'_{11}\frac{\partial u}{\partial x}+\sigma'_{12}\frac{\partial u}{\partial y}+\sigma'_{21}\frac{\partial v}{\partial x}+\sigma'_{22}\frac{\partial v}{\partial y}$$

$$\rho_0 C_p\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial y}\left(\chi_k\left(\frac{\partial T}{\partial y}\right)\right)+\sigma'_{12}\frac{\partial u}{\partial y}$$

Where

$$\sigma'_{12}=\chi_{\mu}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-\frac{2}{3}\delta_{12}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\xi_1\delta_{12}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$=> \quad \sigma'_{12}=\chi_{\mu}\left(\frac{\partial u}{\partial y}\right)$$

$$\rho_0 C_p\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}\right)=\chi_k\frac{\partial^2 T}{\partial y^2}+\chi_{\mu}\left(\frac{\partial u}{\partial y}\right)^2+\chi\frac{\mu}{s}u^2 \tag{1.12}$$

Where

C_p is the specific heat at constant pressure, s is porous medium permeability, μ is dynamic viscosity, ρ_0 is the fluid density and T is temperature. Where, $\chi=1$ for porous medium and $\chi=0$ for clear medium

1.4 Perturbation Method

Exact solutions are rare in many branches of science like physics, motion, solid and fluid mechanics because of non-linear, inhomogeneous and general boundary conditions. That is why mathematicians and physicists use approximate solutions. These approximate solutions may be purely numerical, purely analytical or combination of two.

In this method we assume a series solution of the form

$$u(y,\varepsilon)=u_0(y)+\varepsilon u_1(y)+\varepsilon^2 u_2(y)+O(\varepsilon^3) \tag{1.13}$$

Where, u_0, u_1, ... are known function of y. Equation (1.13) is called asymptotic expansion or perturbation of the solution in terms of the parameter ε and assume that the parameter ε is very small but not zero.

In many problems involving a perturbation parameter ε, an expansion of the form $u(y,\varepsilon)=u_0(y)+\varepsilon u_1(y)+\varepsilon^2 u_2(y)+O(\varepsilon^3)$ may not be uniformly valid over the entire interval of interest. Problems leading to non-uniform expansions are known to be singular perturbation or boundary layer problems. These are problems that have multiple length or time scales.

Chapter 2

Unsteady Oscillatory Flow and Heat Transfer in a Composite Porous Medium Channel

This chapter investigates the unsteady oscillatory flow and heat transfer analysis in a horizontal composite porous medium channel. The flow equations are modeled using the Decay-Brinkman equation. The viscous and Darcian dissipation terms are also included in the energy equation. The partial differential equations are solved analytically using two-Term harmonic and non-harmonic functions in both regions of the channel. Effect of the physical parameters on the velocity and temperature fields are shown graphically. In fact, this chapter is a detail review of the work done by J. C. Umavathi et al. [12].

2.1 Mathematical Formulation

Consider unsteady, fully developed, laminar flow of an incompressible viscous fluid through an infinitely long composite channel, as shown in **Fig.2.1**. The region $-h < y < 0$ (region-I) is filled with a porous matrix and the region $0 < y < h$ (region-II) is occupied by a clear viscous fluid. The two walls of the channel are held at constant different temperature T_{w_1} and T_{w_2}, with temperature $T_{w_1} < T_{w_2}$ and the infinite plates are placed horizontally. It should be noted here that since the plates of the channel are assumed to be infinite. All of the physical dependent variables except pressure will only depend on "y" and "t". All the thermo-physical properties of the porous medium are assumed to be constant. In region I, both the fluid and the porous matrix are assumed to be in local thermal equilibrium.

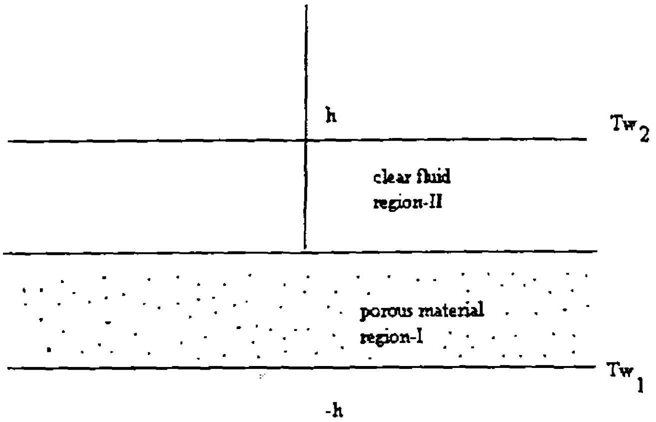

Fig. 2.1: Flow geometry of the problem.

The flow in both regions of the channel is assumed to be driven by a constant pressure gradient $-\frac{\partial p}{\partial x}$ and temperature gradient $\Delta T = T_{w_1} - T_{w_2}$. Under these assumptions, the governing equation of motion and energy are given as:

$$\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0 \qquad (2.1)$$

$$\rho_0 \frac{d}{dt}(\mathbf{V}) = div\ \mathbf{T} + \rho f \qquad (2.2)$$

$$\rho_0 C_p \left(\frac{\partial T}{\partial t} + \mathbf{V}.\mathrm{grad}\mathrm{T}\right) = \mathrm{div}\ (\chi_k\ \mathrm{grad}\mathbf{T}) + \sigma'_{ik} \frac{\partial v_i}{\partial x_k} \qquad (2.3)$$

Where i=1,2 gives the equations for regions I and II, respectively, (u, v) are the velocity component in the x and y directions, T is temperature. The velocity field for the present problem is

$$\mathbf{V} = (u(y,t), v_0(1 + \varepsilon A e^{i\omega t}))$$

p is the Hydrostatic pressure, A is strain rate tensor or 1st Rivillin Erickson tensor defined as

$$A = (\mathrm{grad}\ \mathbf{V}) + (\mathrm{grad}\ \mathbf{V})^{\mathrm{T}} \qquad (2.4)$$

Where, $(\text{grad } \mathbf{V}) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ and $(\text{grad } \mathbf{V})^T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$

$(\text{grad } \mathbf{V}) = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ 0 & 0 \end{pmatrix}$ and $(\text{grad } \mathbf{V})^T = \begin{pmatrix} 0 & 0 \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}$$

Using in equation (2.3) we have

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix} = -p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}$$

$$\tau_{xx} = -p \ , \qquad \tau_{yy} = -p \ , \qquad \tau_{xy} = \tau_{yx} = \mu \frac{\partial u}{\partial y}$$

Now equation (2.2) and (2.3) become

$$\rho_0\left(\frac{\partial u_i}{\partial t} + v_i \frac{\partial u_i}{\partial y}\right) = \chi_\mu \frac{\partial^2 u_i}{\partial y^2} - \frac{\partial P}{\partial x} - \chi \frac{\mu}{s} u_i \qquad (2.5)$$

$$\rho_0 C_p\left(\frac{\partial T_i}{\partial t} + v_i \frac{\partial T_i}{\partial y}\right) = \chi_k \frac{\partial^2 T_i}{\partial y^2} + \chi_\mu \left(\frac{\partial u_i}{\partial y}\right)^2 + \chi \frac{\mu}{s} u_i^2 \qquad (2.6)$$

The other coefficients appearing in equation (2.5) and (2.6) are as follows

$\chi = 1$	for porous matrix region	$\chi = 0$	for clear fluid region
$\chi_\mu = \mu_{eff}$	for porous matrix region	$\chi_\mu = \mu$	for clear fluid region
$\chi_k = K_{eff}$	for porous matrix region	$\chi_k = K$	for clear fluid region

C_p is specific heat at constant pressure , s is porous medium permeability, μ dynamic viscosity and ρ_0 is the fluid density

The appropriate boundary conditions are

$$u_1(-h)=0\,,\quad u_2(h)=0\,,\ u_1(0)=u_2(0)\,,\ \mu_{eff}\frac{\partial u_1}{\partial y}=\mu\frac{\partial u_2}{\partial y}\qquad \text{at}\qquad y=0 \tag{2.7}$$

$$\frac{\mu_{eff}}{\mu}\frac{\partial u_1}{\partial y}=\frac{\partial u_2}{\partial y}\qquad \text{at}\qquad y=0 \tag{2.8}$$

$$T_1(-h)=T_{w_2}\,,\quad T_2(h)=T_{w_2}\,,\qquad T_1(0)=T_2(0)\,, \tag{2.9}$$

$$K_{eff}\frac{\partial T_1}{\partial y}=K\frac{\partial T_2}{\partial y}\qquad \text{at}\qquad y=0 \tag{2.10}$$

Introducing the non dimensional variables

$$u_i=u_0u_i^*\qquad v_i=v_0v_i^*\qquad y=\frac{V}{v_0}y^*\qquad t=\frac{V}{v_0^2}t^*\qquad \theta=\frac{T-T_{w_2}}{T_{w_1}-T_{w_2}} \tag{2.11}$$

Using in equation (2.5) and (2.6)

$$\frac{\partial u_i^*}{\partial t^*}+v_i^*\frac{\partial u_i^*}{\partial y^*}=A_i\frac{\partial^2 u_i^*}{\partial y^{*2}}-\chi\sigma^2u_i^*-P \tag{2.12}$$

$$\frac{\partial \theta_i}{\partial t^*}+v_i^*\frac{\partial \theta_i}{\partial y^*}=B_i\left(\frac{\partial^2\theta_i}{\partial y^{*2}}\right)+A_iEc\left(\frac{\partial u_i^*}{\partial y^*}\right)^2+\chi\alpha^2Ec\left(u_i^*\right)^2 \tag{2.13}$$

and boundary and interface conditions are

$$u_{1i}(-1)=0,\ u_{2i}(1)=0,\quad u_{1i}(0)=u_{2i}(0),\ m\frac{\partial u_{1i}}{\partial y}=\frac{\partial u_{2i}}{\partial y}\ \text{at}\ y=0 \tag{2.14}$$

$$\theta_{1i}(-1)=1-\delta_{1i},\ \theta_{2i}(1)=0,\ \theta_{1i}(0)=\theta_{2i}(0),\ k\frac{\partial\theta_{1i}}{\partial y}=\frac{\partial\theta_{2i}}{\partial y}\ \text{at}\ y=0 \tag{2.15}$$

Where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij}=\begin{cases}0 & for\quad i\neq j\\ 1 & for\quad i=j\end{cases}$$

And all the non-dimensional parameters appearing in (2.12) and (2.13) are

$$A_1 = \frac{\mu_{eff}}{\mu} = m \qquad A_2 = 1 \qquad P = \frac{V^2}{\chi_\mu v_0^2 u_0}\left(\frac{\partial p}{\partial x}\right) \qquad \sigma^2 = \frac{V^2}{s v_0^2} \qquad Ec = \frac{u_0^2}{C_p \Delta T}$$

$$k = \frac{K_{eff}}{K} \qquad \Pr = \frac{\rho_0 C_p V}{K} \qquad B_1 = \frac{k}{\Pr}, \qquad B_2 = \frac{1}{\Pr}$$

2.2 Solution of the Problem

The governing equations (2.12) and (2.13) subject to the boundary conditions (2.14) and (2.15) are solved for the velocity and temperature distribution in both regions using the perturbation technique. For this we assume the solution of the form

$$u_i(y,t) = u_{i0}(y) + \varepsilon e^{i\omega t} u_{i1}(y) + O(\varepsilon^2) + \ldots \tag{2.16}$$

$$\theta_i(y,t) = \theta_{i0}(y) + \varepsilon e^{i\omega t} \theta_{i1}(y) + O(\varepsilon^2) + \ldots \tag{2.17}$$

This is a valid assumption because of choice of v as defined in equation $v = v_0(1+\varepsilon A e^{ti\omega})$ that the amplitude $\varepsilon A \ll 1$. By substituting equation (2.16) and (2.17) in equation (2.12) and (2.13), equating the harmonic and non-harmonic terms and neglecting the higher order terms of $O(\varepsilon^2)$, one obtain the following system of equations

Non-Periodic coefficients

$$A_1 \frac{d^2 u_{10}}{dy^2} + \frac{du_{10}}{dy} - (\chi\sigma^2) u_{10} = P \tag{2.18}$$

$$A_2 \frac{d^2 u_{20}}{dy^2} + \frac{du_{20}}{dy} - (\chi\sigma^2) u_{20} = P \tag{2.19}$$

$$B_1 \frac{d^2 \theta_{i0}(y)}{dy^2} + A_1 Ec\left(\frac{du_{i0}}{dy}\right)^2 + \chi\sigma^2 Ec(u_{10})^2 + \frac{d\theta_{i0}}{dy} = 0 \tag{2.20}$$

$$B_2 \frac{d^2 \theta_{20}}{dy^2} + \frac{d\theta_{20}}{dy} = -Ec\left(C_4 e^{-y} - P\right)^2 \tag{2.21}$$

Periodic coefficients

$$A_1 \frac{d^2 u_{11}}{dy^2} + \frac{du_{11}}{dy} - (\chi\sigma^2 + i\omega)u_{11} = -A\frac{du_{10}}{dy} \tag{2.22}$$

$$A_2 \frac{d^2 u_{21}}{dy^2} + \frac{du_{21}}{dy} - (\chi\sigma^2 + i\omega)u_{21} = -A\frac{du_{20}}{dy} \tag{2.23}$$

$$i\omega\theta_{11} + A\frac{d\theta_{10}}{dy} + \frac{d\theta_{11}}{dy} = B_1 \frac{d^2\theta_{11}}{dy^2} + 2A_1 Ec\frac{du_{10}}{dy}\frac{du_{11}}{dy} + 2\chi\sigma^2 Ecu_{10}u_{11} \tag{2.24}$$

$$i\omega\theta_{21} - A\frac{d\theta_{20}}{dy} + \frac{d\theta_{21}}{dy} = B_2 \frac{d^2\theta_{21}}{dy^2} + 2A_2 Ec\frac{du_{20}}{dy}\frac{du_{21}}{dy} \tag{2.25}$$

The solution of equations (2.18) – (2.25) using the boundary conditions can be written as

$$u_{10} = C_1 e^{m_1 y} + C_2 e^{m_2 y} - \frac{P}{\sigma^2} \tag{2.26}$$

$$u_{20} = C_3 + C_4 e^{-y} + Py \tag{2.27}$$

$$\theta_{10} = C_5 + C_6 e^{m_4 y} + k_{12} e^{2m_1 y} + k_{13} e^{2m_2 y} + k_{14} e^{m_3 y} + k_{10} e^{m_1 y} + k_{11} e^{m_2 y} + k_8 y \tag{2.28}$$

$$\theta_{20} = C_7 + C_8 e^{-\Pr y} + k_{15} e^{-2y} + k_{16} e^{-y} + k_{17} y \tag{2.29}$$

$$\begin{aligned} u_{11} &= e_2 e^{m_1 y} + e_3 e^{m_2 y} + \left(XC_9 \cos F_1 y + XC_{10} \sin F_1 y\right)e^{e_1 y} \\ &+ i\left[(YC_9 \cos F_1 y + YC_{10} \sin F_1 y)e^{e_1 y} + F_2 e^{m_1 y} + F_3 e^{m_2 y}\right] \end{aligned} \tag{2.30}$$

$$u_{21} = (XC_{11} \cos F_4 y + XC_{12} \sin F_4 y)e^{e_4 y} + i\left[(YC_{11} \cos F_4 y + YC_{12} \sin F_4 y)e^{e_4 y} + \frac{A}{\omega}\left(C_4 e^{-y} + P\right)\right] \tag{2.31}$$

$$\begin{aligned} \theta_{11} &= \left(XC_{13} \cos F_5 y + XC_{14} \cos F_5 y\right)e^{e_5 y} + E_7 e^{m_4 y} + P_{23} e^{2my} + P_{24} e^{2m_2 y} + P_{25} e^{m_3 y} + P_{26} e^{m_1 y} + P_{27} e^{m_2 y} \\ &+ e^{m_1 y}\left(P_{28} \cos F_1 y + P_{29} \sin F_1 y\right) + e^{m_2 y}\left(P_{30} \cos F_1 y + P_{31} \sin F_1 y\right) + e^{e_1 y}\left(P_{32} \cos F_1 y + P_{33} \sin F_1 y\right) \end{aligned}$$

$$+i\,[\left(YC_{13}\cos F_5 + YC_{14}\sin F_5 y\right)e^{E_5 y} + F_7 e^{m_4 y} + Q_{23}e^{2m_1 y} + Q_{24}e^{2m_2 y} + Q_{25}e^{m_5 y} + Q_{26}e^{m_1 y} + Q_{27}e^{m_2 y} +$$
$$e^{m_6 y}\left(Q_{28}\cos F_1 y + Q_{29}\sin F_1 y\right) + e^{m_7 y}\left(Q_{30}\cos F_1 y + Q_{31}\sin F_1 y\right) + e^{e_1 y}\left(Q_{32}\cos F_1 y + Q_{33}\sin F_1 y\right)] \qquad (2.32)$$

$$\theta_{21} = e^{e_{27} y}(XC_{15}\cos F_{27} y + XC_{16}\sin F_{27} y) + P_{44}e^{-2y} + P_{45}e^{-y} + e^{m_8 y}(P_{46}\cos F_4 y + P_{47}\sin F_4 y) +$$
$$+e^{e_4 y}\left(P_{48}\cos F_4 y + P_{49}\sin F_4 y\right)] + i[e^{e_{27} y}(YC_{15}\cos F_{27} y + YC_{16}\sin F_{27} y) + Q_{44}e^{-2y} + Q_{45}e^{-y} +$$
$$e^{m_8 y}(Q_{46}\cos F_4 y + Q_{47}\sin F_4 y) + e^{e_4 y}\left(Q_{48}\cos F_4 y + Q_{49}\sin F_4 y\right) + k_{29}] \qquad (2.33)$$

It should be noted that all the constants appearing in the above solutions are defined at the end in the Appendix-I.

2.3 Results and Discussion

The problem of unsteady flow and heat transfer in a composite porous medium channel is investigated analytically. The closed form solutions are reported for small parameter ε such that oscillation amplitude $\varepsilon A \leq 1$. The solution of the periodic and non periodic coefficients of $e^{i\omega t}$ is evaluated for the various parametric conditions. The results are depicted graphically in Figs. 2.2 to 2.8

Figs. 2.2 and Fig. 2.3 display the effect of the porous medium parameter σ on the velocity and temperature profiles, respectively. As the porous medium parameter σ increases, the velocity and temperature decreases in both regions of the channel. This is expected since the porous matrix represents an obstacle to flow and therefore, reduces its velocity and temperature.

Fig. 2.4 depicts the effect of Prandtl number on the temperature profiles. The Prandtl number is the ratio of momentum diffusion to heat diffusion. It is measure of the relative importance of viscosity and heat conduction in a flow field. Thus, as the Prandtl number increases, the viscous force dominate over heat conduction and hence, the temperature decreases. This is obvious from Fig. 2.4.

Fig. 2.5 represents the effect of Eckert number on the temperature profiles. Physically, the Eckert number represents the effect of the viscous and porous medium dissipations. As the Eckert number increases, the temperature field in the channel decreases. The magnitude of the reduction in the temperature field in region-II is larger compared to that in region-I.

The effect of the viscosity ratio *m* on the velocity and temperature profiles is shown in Figs. 2.6 and 2.7, respectively. As the viscosity ratio increases, both the velocity and temperature profiles are decreased. This is due to the fact that as the fluid viscosity increases, the fluid in both regions of the channel becomes thicker and hence the flow velocity is reduced causing the temperature distribution to reduce as well.

Fig. 2.8 displays the influence of the thermal conductivity ratio *k* on the temperature profiles. Increase in the thermal conductivity ratio has the tendency to cool down the thermal state in the channel. This is depicted in the reduction in the fluid temperature as *k* increases as shown in Fig. 2.8.

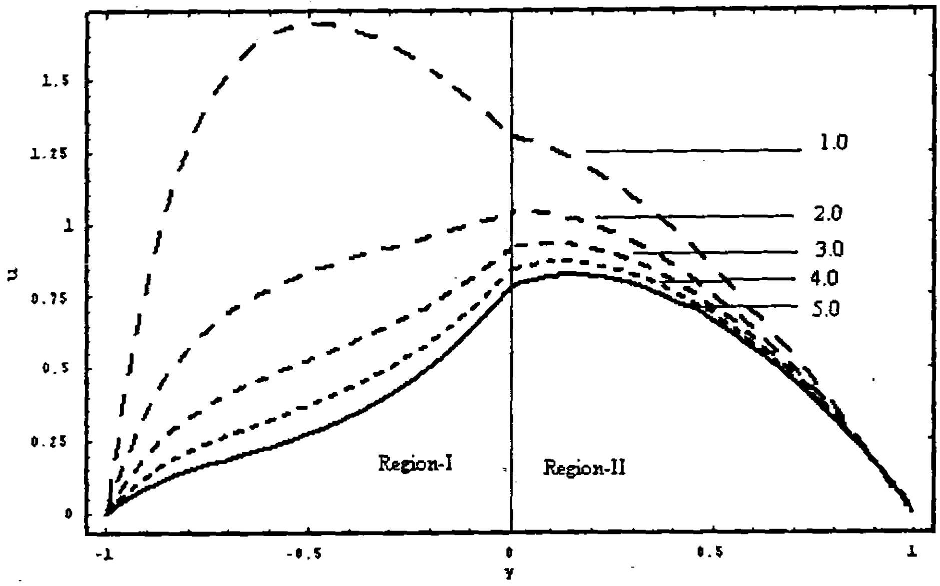

Fig. 2.2: Velocity profile for the different values of the porous medium parameter σ.

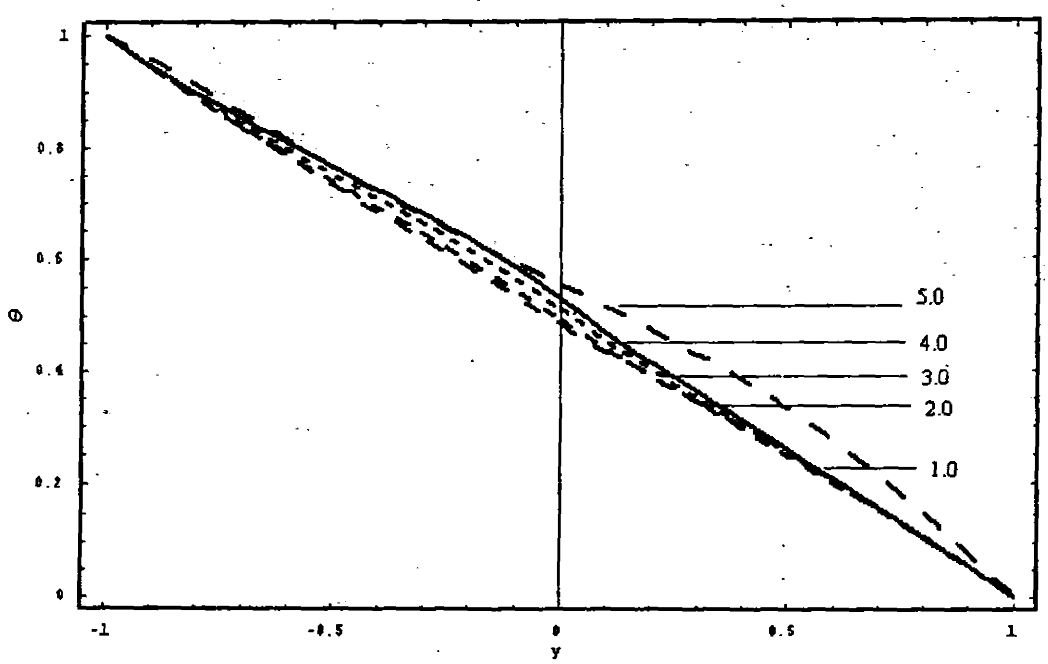

Fig. 2.3: Temperature profile for the different values of the porous medium parameter σ.

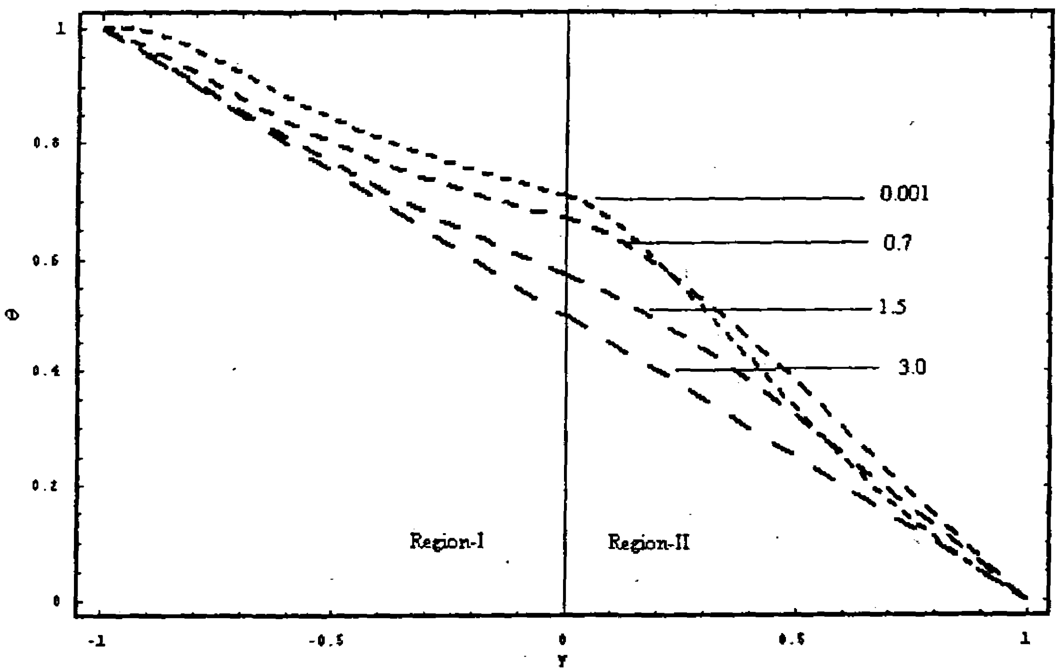

Fig. 2.4: Temperature profile for the different values of the Prandtl number Pr.

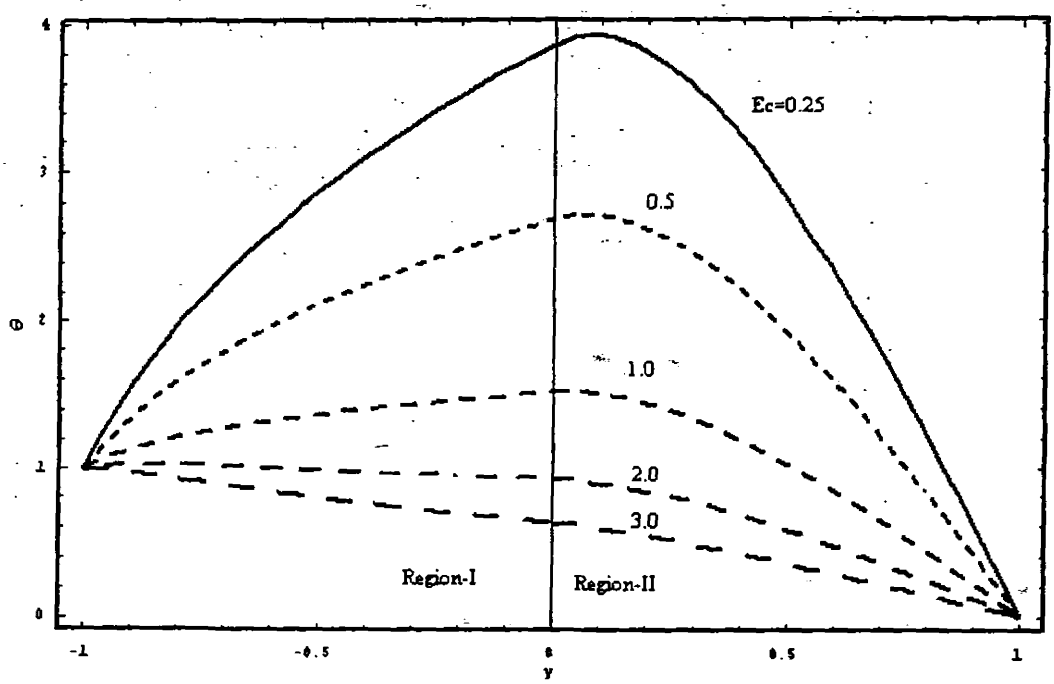

Fig. 2.5: Temperature profile for the different values of the Eckert number Ec

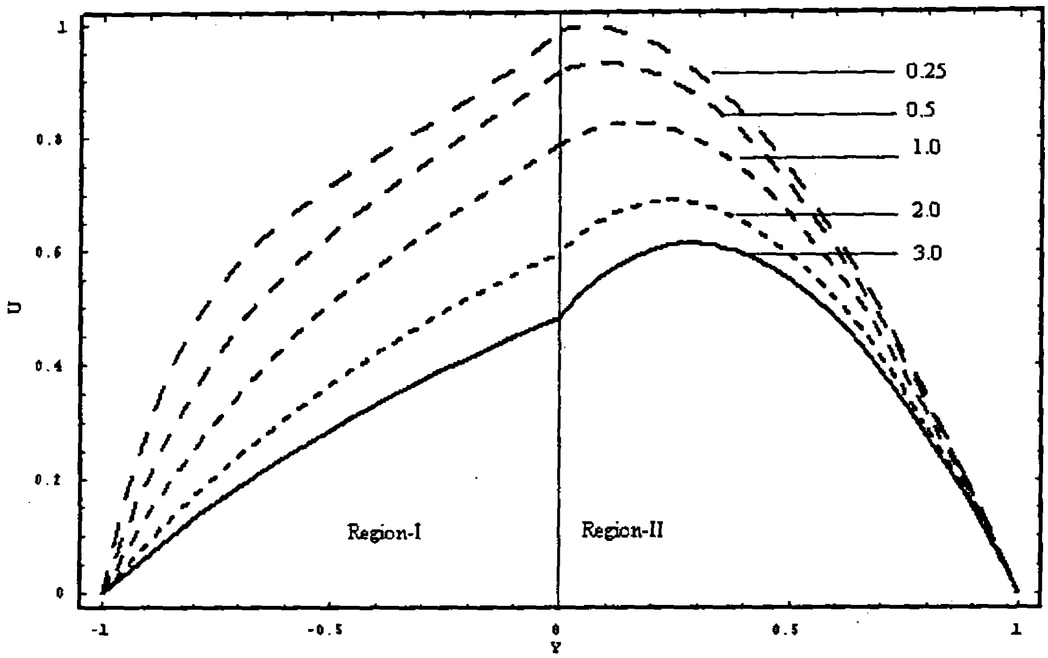

Fig. 2.6: Velocity profile for the different values of the ratio of viscosities m.

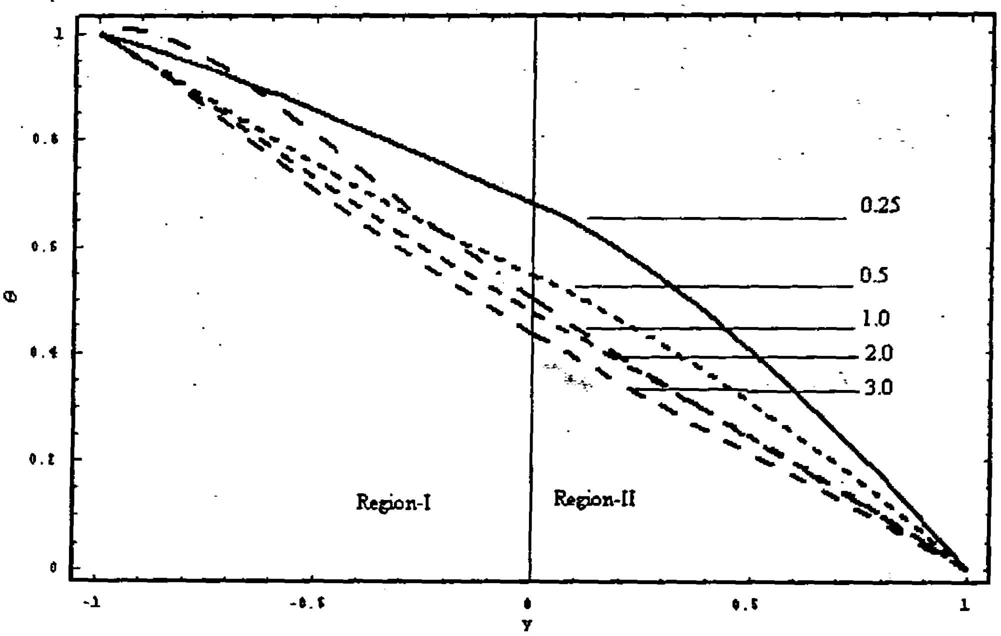

Fig. 2.7: Temperature profile for the different values of the ratio of viscosities m.

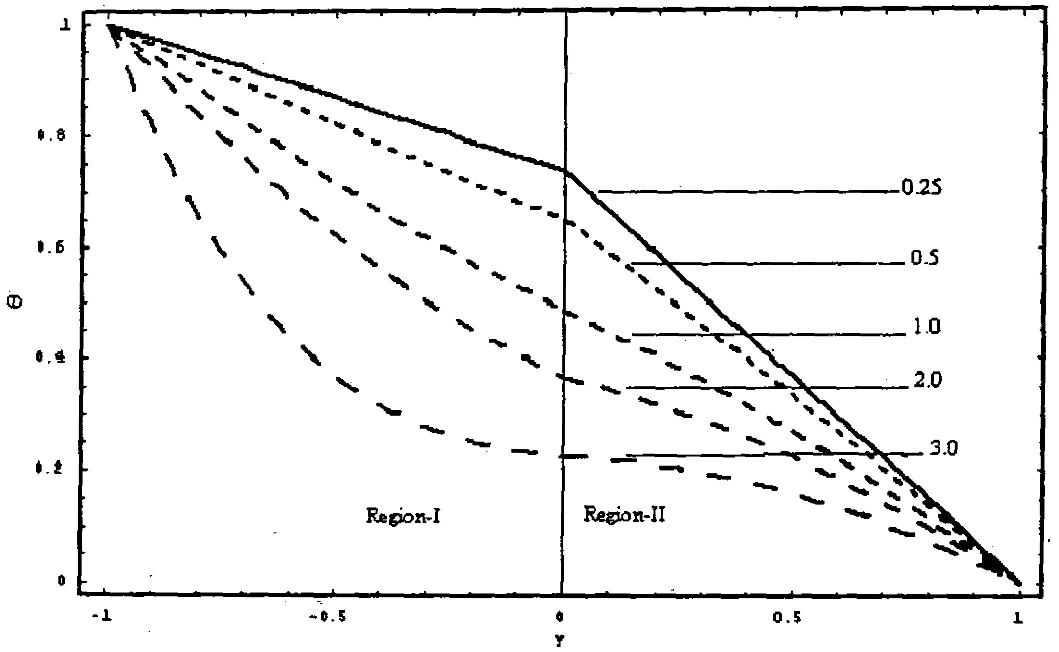

Fig. 2.8: Temperature profile for the different values of the ratio of conductivities k.

2.4 Conclusions

The problem of unsteady flow of a viscous fluid through a horizontal composite channel whose half width is filled with a uniform layer of porous media in the presence of time dependent oscillatory wall transpiration velocity was investigated analytically. Both the fluid and the porous matrix were assumed to have constant physical properties. Separate closed form solution for each region of the channel were obtained taking into consideration suitable interface matching conditions. The closed form results were numerically evaluated and represented graphically for various values of the porous medium parameter, viscosity and thermal conductivity ratios, Prandtl and Eckert numbers.

It was predicted that both the velocity and temperature profiles decreased as either of the porous medium parameter or the viscosity ratio was increased. Furthermore, it was concluded that the temperature field decreased as either of the Prandtl number, Eckert number or the thermal conductivity ratio increased.

It can be concluded that the flow and heat transfer aspects in a horizontal composite channel with permeable walls can be controlled by considering different combinations of fluids and porous media having different viscosities and conductivities.

Chapter 3

MHD Oscillatory Flow in a Composite Porous Medium Channel

This chapter presents the unsteady MHD oscillatory flow of a viscous fluid in a composite porous medium channel. The resultant partial differential equations governing the flow and heat transfer are solved analytically using the same technique as in chapter 2. The influence of the physical parameters on the velocity and temperature profiles are shown graphically and discussed in detail. In fact, this chapter is an extension of the work done by J. C. Umavathi et al. [12].

3.1 Mathematical Formulation

Consider unsteady, fully developed, laminar flow of an incompressible viscous fluid through an infinitely long composite channel, as shown in **Fig. 3.1**. The region $-h < y \leq 0$ (region-I) is filled with a porous material and the region $0 < y < h$ (region-II) is occupied by a clear viscous fluid. Both the walls of the channel are held at constant different temperature T_{w_1} and T_{w_2}, with temperature $T_{w_1} < T_{w_2}$ and the infinite plates are placed horizontally. Here, It should be noted that since the plates of the channel are assumed to be infinite. All of the physical dependent variables except pressure will depend only on "y" and "t". All the thermo-physical properties of the porous medium are assumed to be constant. In region-I, both the fluid and the porous matrix are assumed to be in local thermal equilibrium

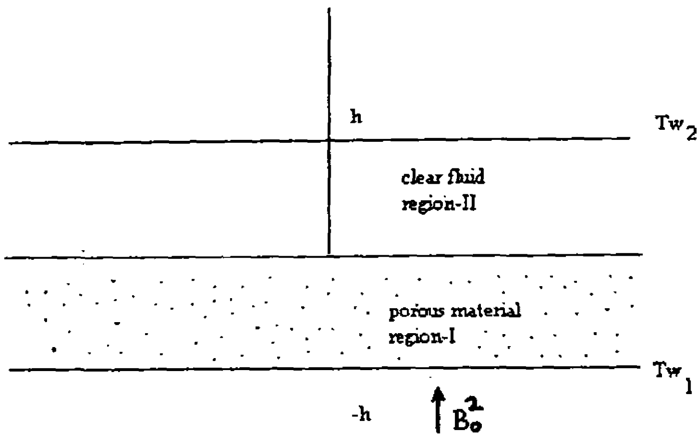

Fig. 3.1: Flow geometry of the problem.

The flow in both regions of the channel is assumed to be driven by a constant pressure gradient $-\frac{\partial P}{\partial x}$ and temperature gradient $\Delta T = T_{w_1} - T_{w_2}$. Under these assumptions, the governing equation of motion and energy are given as:

$$\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0 \qquad (3.1)$$

$$\rho_0 \frac{d}{dt}(\mathbf{V}) = div\ \mathbf{T} - \sigma B_0^2 u_i \qquad (3.2)$$

$$\rho_0 C_p \left(\frac{\partial T}{\partial t} + \mathbf{V}.\mathrm{grad}\mathbf{T}\right) = \mathrm{div}(\chi_k\ \mathrm{grad}\mathbf{T}) + \sigma'_{ik} \frac{\partial v_i}{\partial x_k} \qquad (3.3)$$

Where i=1,2 gives the equations for regions I and II, respectively, (u, v) are the velocity component in the x and y directions, T is temperature. The velocity field for the present problem is

$$\mathbf{V} = (u(y,t), v_0(1 + \varepsilon A e^{i\omega t}))$$

p is the Hydrostatic pressure, σ is the electric conductivity, B_0^2 is the total magnetic field, A is strain rate tensor or 1st Rivillin Erickson tensor defined as

$$A = (\text{grad } \mathbf{V}) + (\text{grad } \mathbf{V})^T \tag{3.4}$$

Where, $(\text{grad } \mathbf{V}) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ and $(\text{grad } \mathbf{V})^T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$

$(\text{grad } \mathbf{V}) = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ 0 & 0 \end{pmatrix}$ and $(\text{grad } \mathbf{V})^T = \begin{pmatrix} 0 & 0 \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}$$

Using in equation (3.3) the above values we have,

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix} = -p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix} - \sigma B_0^2 u_i$$

$$\Rightarrow \quad \tau_{xx} = -p\,, \qquad \tau_{yy} = -p\,, \qquad \tau_{xy} = \tau_{yx} = \mu \frac{\partial u}{\partial y}$$

Now equation (3.2) and (3.3) takes the form

$$\rho_0\left(\frac{\partial u_i}{\partial t} + v_i \frac{\partial u_i}{\partial y}\right) = \chi_\mu \frac{\partial^2 u_i}{\partial y^2} - \frac{\partial P}{\partial x} - \chi \frac{\mu}{s} u_i - \sigma B_0^2 u_i \tag{3.5}$$

$$\rho_0 C_p \left(\frac{\partial T_i}{\partial t} + v_i \frac{\partial T_i}{\partial y}\right) = \chi_k \frac{\partial^2 T_i}{\partial y^2} + \chi_\mu \left(\frac{\partial u_i}{\partial y}\right)^2 + \chi \frac{\mu}{s} u_i^2 \tag{3.6}$$

The other coefficients appearing in equation (3.5) and (3.6) are as follows

$\chi = 1$ for porous matrix region $\chi = 0$ for clear fluid region

$\chi_\mu = \mu_{eff}$ for porous matrix region $\chi_\mu = \mu$ for clear fluid region

$\chi_k = K_{eff}$ for porous matrix region $\chi_k = K$ for clear fluid region

C_p is specific heat at constant pressure , s is porous medium permeability, μ is dynamic viscosity and ρ_0 is the fluid density

The appropriate boundary conditions are

$$u_1(-h) = 0\,, \quad u_2(h) = 0\,, \quad u_1(0) = u_2(0)\,, \quad \mu_{eff}\frac{\partial u_1}{\partial y} = \mu\frac{\partial u_2}{\partial y} \quad \text{at} \quad y = 0 \tag{3.7}$$

$$\frac{\mu_{eff}}{\mu}\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} \quad \text{at} \quad y = 0 \tag{3.8}$$

$$T_1(-h) = T_{w_1}\,, \quad T_2(h) = T_{w_2}\,, \qquad T_1(0) = T_2(0)\,, \tag{3.9}$$

$$K_{eff}\frac{\partial T_1}{\partial y} = K\frac{\partial T_2}{\partial y} \quad \text{at} \quad y=0 \tag{3.10}$$

Introducing the non dimensional variables

$$u_i = u_0 u_i^* \qquad v_i = v_0 v_i^* \qquad y = \frac{V}{v_0} y^* \qquad t = \frac{V}{{v_0}^2} t^* \qquad \theta = \frac{T - T_{w_2}}{T_{w_1} - T_{w_2}} \tag{3.11}$$

Using in equation (3.5) and (3.6) one can obtain

$$\frac{\partial u_i^*}{\partial t^*} + v_i^* \frac{\partial u_i^*}{\partial y^*} = A_i \frac{\partial^2 u_i^*}{\partial y^{*2}} - \chi\alpha^2 u_i^* - Mu_i^* - P \tag{3.12}$$

$$\frac{\partial \theta_i}{\partial t^*} + v_i^* \frac{\partial \theta_i}{\partial y^*} = B_i\left(\frac{\partial^2 \theta_i}{\partial y^{*2}}\right) + A_i Ec\left(\frac{\partial u_i^*}{\partial y^*}\right)^2 + \chi\alpha^2 Ec\left(u_i^*\right)^2 \tag{3.13}$$

and boundary and interface conditions are

$$u_{1i}(-1) = 0, \quad u_{2i}(1) = 0, \quad u_{1i}(0) = u_{2i}(0), \; m\frac{\partial u_{1i}}{\partial y} = \frac{\partial u_{2i}}{\partial y} \text{ at } y = 0 \tag{3.14}$$

$$\theta_{1i}(-1)=1-\delta_{1i},\ \theta_{2i}(1)=0,\ \theta_{1i}(0)=\theta_{2i}(0),\ k\frac{\partial\theta_{1i}}{\partial y}=\frac{\partial\theta_{2i}}{\partial y}\ \text{at}\ y=0 \tag{3.15}$$

Where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij}=\begin{cases}0 & for \quad i\neq j\\ 1 & for \quad i=j\end{cases}$$

And all the non-dimensional parameters appearing in (2.12) and (2.13) are

$$\alpha^2=\frac{V^2}{sv_0^2},\quad Ec=\frac{u_0^2}{C_p\Delta T},\quad k=\frac{K_{eff}}{K},\quad A_1=m,\quad m=\frac{\mu_{eff}}{\mu}\quad \Pr=\frac{\rho_0 C_p V}{K}\quad ,A_2=1$$

$$B_1=\frac{k}{\Pr},\quad B_2=\frac{1}{\Pr}\quad M=\frac{\sigma B_0^2V^2}{\rho_0 v_0^2}\quad P=\frac{V^2}{\chi_\mu v_0^2 u_0}\left(\frac{\partial p}{\partial x}\right)$$

Where, Ec is Eckert number, Pr is Prandtl number and M is the magnetic parameter.

3.2 Solution of the Problem

The governing equations (3.12) and (3.13) subject to the boundary conditions (3.14) and (3.15) are solved for the velocity and temperature distribution in both regions using the perturbation technique. For this we assume the solution of the form

$$u_i(y,t)=u_{i0}(y)+\varepsilon e^{i\omega t}u_{i1}(y)+O(\varepsilon^2)+\ldots \tag{3.16}$$

$$\theta_i(y,t)=\theta_{i0}(y)+\varepsilon e^{i\omega t}\theta_{i1}(y)+O(\varepsilon^2)+\ldots \tag{3.17}$$

This is a valid assumption because of choice of v as defined in equation $v=v_0(1+\varepsilon Ae^{ti\omega})$ that the amplitude $\varepsilon A<<1$. By substituting equation (3.16) and (3.17) in equation (3.12) and (3.13), equating the harmonic and non-harmonic terms and neglecting the higher order terms of $O(\varepsilon^2)$, one obtain the following system of equations

Non-Periodic coefficients

$$A_1\frac{d^2u_{10}}{dy^2}-\frac{du_{10}}{dy}-(\chi\alpha^2+M)u_{10}=P \tag{3.18}$$

$$A_2\frac{d^2u_{20}}{dy^2}-\frac{du_{20}}{dy}-(\chi\alpha^2+M)u_{20}=P \tag{3.19}$$

$$B_1\frac{d^2\theta_{10}}{dy^2}+A_1Ec\left(\frac{du_{10}}{dy}\right)^2+\chi\alpha^2Ec(u_{10})^2-\frac{d\theta_{10}}{dy}=0 \tag{3.20}$$

$$B_2\frac{d^2\theta_{20}}{dy^2}-\frac{d\theta_{20}}{dy}=-Ec\left(C_3m_3e^{m_3y}+C_4m_4e^{m_4y}\right)^2 \tag{3.21}$$

Periodic coefficients

$$A_1\frac{d^2u_{11}}{dy^2}-\frac{du_{11}}{dy}-(\chi\alpha^2+M+i\omega)u_{11}=A\frac{du_{10}}{dy} \tag{3.22}$$

$$A_2\frac{d^2u_{21}}{dy^2}-\frac{du_{21}}{dy}-(\chi\alpha^2+M+i\omega)u_{21}=A\frac{du_{20}}{dy} \tag{3.23}$$

$$k_2\frac{d^2\theta_{11}}{dy^2}-\frac{d\theta_{11}}{dy}-i\omega\theta_{11}=A\frac{d\theta_{10}}{dy}-2mEc\frac{du_{10}}{dy}\frac{du_{11}}{dy}-2\alpha^2Ecu_{10}u_{11} \tag{3.24}$$

$$\frac{1}{\Pr}\frac{d^2\theta_{21}}{dy^2}-\frac{d\theta_{21}}{dy}-i\omega\theta_{21}=A\frac{d\theta_{20}}{dy}-2Ec\frac{du_{20}}{dy}\frac{du_{21}}{dy} \tag{3.25}$$

The solution of equations (3.18) – (3.25) using the boundary conditions can be written as

$$u_{10}=C_1e^{m_1y}+C_2e^{m_2y}-\frac{P}{\alpha^2+M} \tag{3.26}$$

$$u_{20}=C_3e^{m_3y}+C_4e^{m_4y}-\frac{P}{M} \tag{3.27}$$

$$\theta_{10}=C_5+C_6e^{m_6y}+k_8y+k_{10}e^{m_2y}+k_{11}e^{m_1y}+k_{12}e^{2m_1y}+k_{13}e^{2m_2y}+k_{14}e^{m_6y} \tag{3.28}$$

$$\theta_{20}=C_7+C_8e^{\Pr y}+k_{15}e^{2m_3y}+k_{16}e^{2m_4y}+k_{17}e^{m_7y} \tag{3.29}$$

$$u_{11} = E_2 e^{m_1 y} + E_3 e^{m_2 y} + e^{E_1 y}(XC_9 \cos F_1 y + XC_{10} \sin F_1 y)$$
$$+i\left[(YC_9 \cos F_1 y + YC_{10} \sin F_1 y)e^{E_1 y} + F_2 e^{m_1 y} + F_3 e^{m_2 y}\right] \quad (3.30)$$

$$u_{21} = E_4 e^{m_3 y} + E_5 e^{m_4 y} + e^{E_6 y}(XC_{11} \cos F_6 y + XC_{12} \sin F_6 y)$$
$$+i\left[(YC_{11} \cos F_6 y + YC_{12} \sin F_1 y)e^{E_7 y} + F_4 e^{m_3 y} + F_5 e^{m_4 y}\right] \quad (3.31)$$

$$\theta_{11} = \left(XC_{13} \cos F_{6a} y + XC_{14} \sin F_{6a} y\right)e^{E_{6a} y} + e^{E_1 y}\left(P_{13} \cos F_1 y + P_{14} \sin F_1 y\right) + P_{15} e^{m_1 y} + P_{16} e^{m_2 y} +$$
$$+ P_{17} e^{m_6 y} + P_{18} e^{2m_1 y} + P_{19} e^{2m_2 y} + e^{m_5 y}\left(P_{20} \cos F_1 y + P_{21} \sin F_1 y\right) + e^{m_9 y}\left(P_{22} \cos F_1 y + P_{23} \sin F_1 y\right)$$
$$+i[\left(YC_{13} \cos F_{6a} y + YC_{14} \sin F_{6a} y\right)e^{E_{6a} y} + e^{E_1 y}\left(Q_{13} \cos F_1 y + Q_{14} \sin F_1 y\right) + Q_{15} e^{m_1 y} + Q_{16} e^{m_2 y} +$$
$$Q_{17} e^{m_6 y} + Q_{18} e^{2m_1 y} + Q_{19} e^{2m_2 y} + e^{m_8 y}\left(Q_{20} \cos F_1 y + Q_{21} \sin F_1 y\right) + e^{m_9 y}\left(Q_{22} \cos F_1 y + Q_{23} \sin F_1 y\right) + \frac{Ak_8}{\omega}$$
$$] \quad (3.32)$$

$$\theta_{21} = \left(XC_{15} \cos F_{31} y + XC_{16} \sin F_{31} y\right)e^{E_{31} y} + P_{26} e^{2m_3 y} + P_{27} e^{2m_4 y} + P_{28} e^{m_7 y}$$
$$+ e^{m_{10} y}\left(P_{29} \cos F_6 y + P_{30} \sin F_6 y\right) + e^{m_{11} y}\left(P_{31} \cos F_6 y + P_{32} \sin F_6 y\right) + i\,[$$
$$\left(YC_{15} \cos F_{31} y + YC_{16} \sin F_{31} y\right)e^{E_{31} y} + Q_{26} e^{2m_3 y} + Q_{27} e^{2m_4 y} + Q_{28} e^{m_7 y} +$$
$$+ e^{m_{10} y}\left(Q_{29} \cos F_6 y + Q_{30} \sin F_6 y\right) + e^{m_{11} y}\left(Q_{31} \cos F_6 y + Q_{32} \sin F_6 y\right) + e^{\Pr y} E_{32}\,] \quad (3.33)$$

All the constants appearing in the above solutions are presented at the end in the Appendix-II.

3.3 Discussion and Graphical Results

The purpose of this section is to report the effect of various parameters involved in the flow analysis on the temperature and velocity profiles. Fig. 3.2-3.8 illustrate the effects. Special emphasis is given to the effect of magnetic parameter M on the velocity and temperature profiles. Fig. 3.2 and Fig. 3.3 show the effect of the Porous medium parameter α on the velocity and temperature profile, respectively in presence of the non-zero magnetic parameter M. A comparison of these figures with their counter parts in chapter 2 *i.e,* (Fig. 2.2 and 2.3) shows that velocity in case of zero magnetic parameter M is greater than the velocity for non-zero magnetic parameter M. However, for the temperature profile the situation is opposite. Here, the introduction of magnetic field increases the temperature in both the regions.

Fig. 3.4 and 3.5 illustrate the variation of Prandtl and Eckert numbers on temperature profiles respectively, in the presence of magnetic field. The Fig. 3.4 shows that in the presence of magnetic field temperature profile increases when Prandtl number increases. The Fig 3.5 indicates that in presence of magnetic parameter M temperature increases when Eckert number increases, while in absence of M *i.e* comparison with Fig. 2.4 and Fig. 2.5 the situation was opposite.

Similarly magnetic field enhances the effect of ratio of viscosities m on the velocity field as can be seen from Fig. 3.6.

Fig. 3.7 and 3.8 shows the effect of magnetic parameter M on velocity and temperature profiles respectively. As the value of M increases the velocity decreases in both the regions and the increment in the value of magnetic parameter M increases the temperature profile.

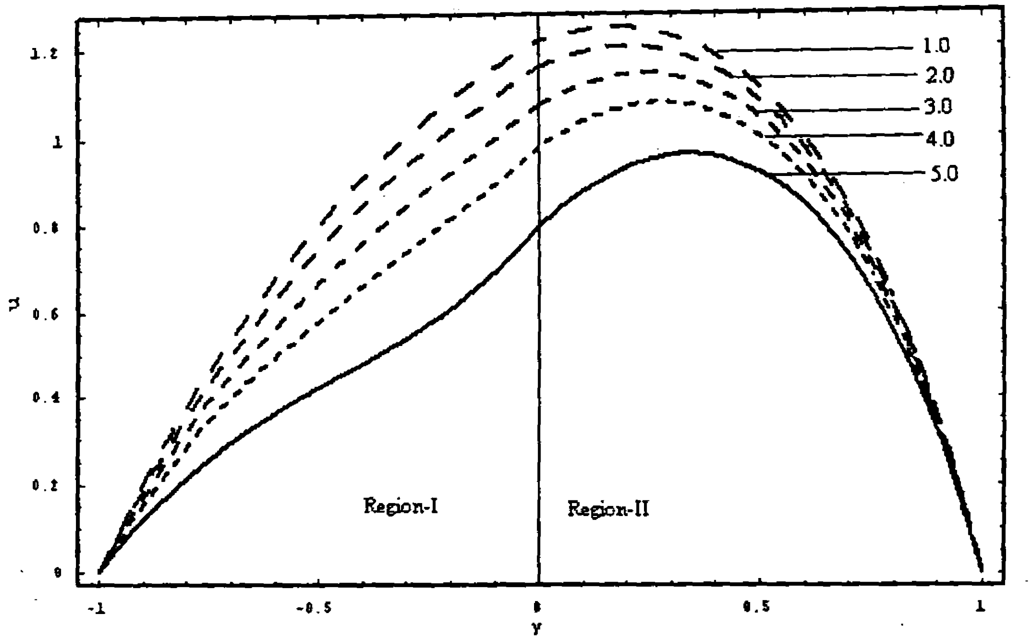

Fig. 3.2: Velocity profile for the different values of the porous medium parameter α.

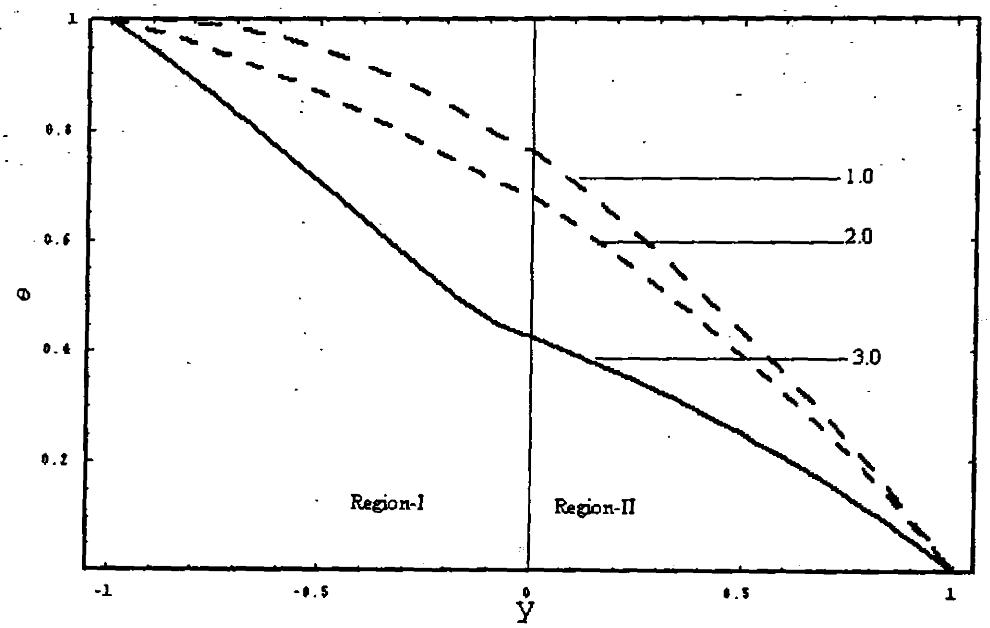

Fig. 3.3: Temperature profile for the different values of the porous medium parameter α.

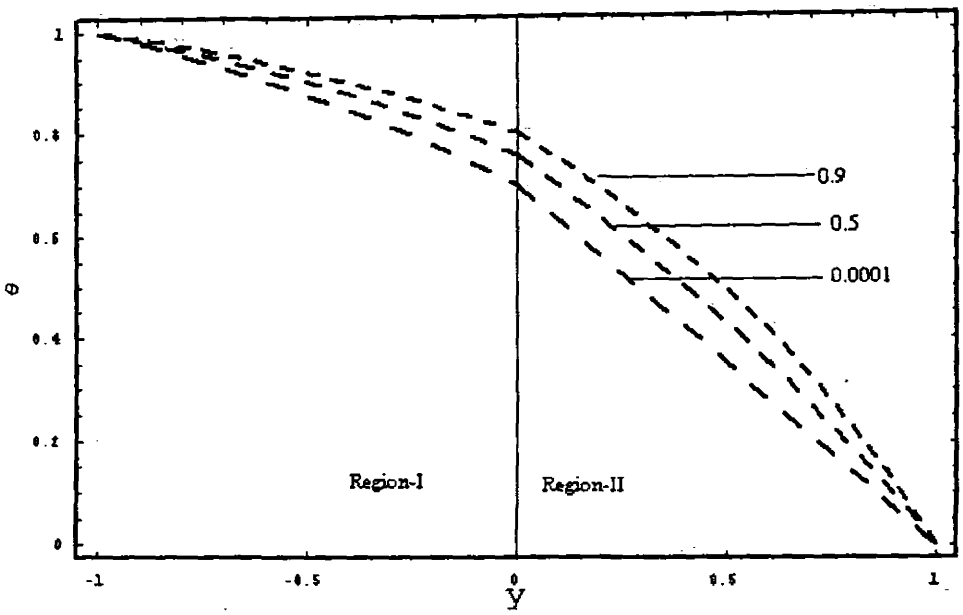

Fig. 3.4: Temperature profile for the different values of the Prandtl number Pr.

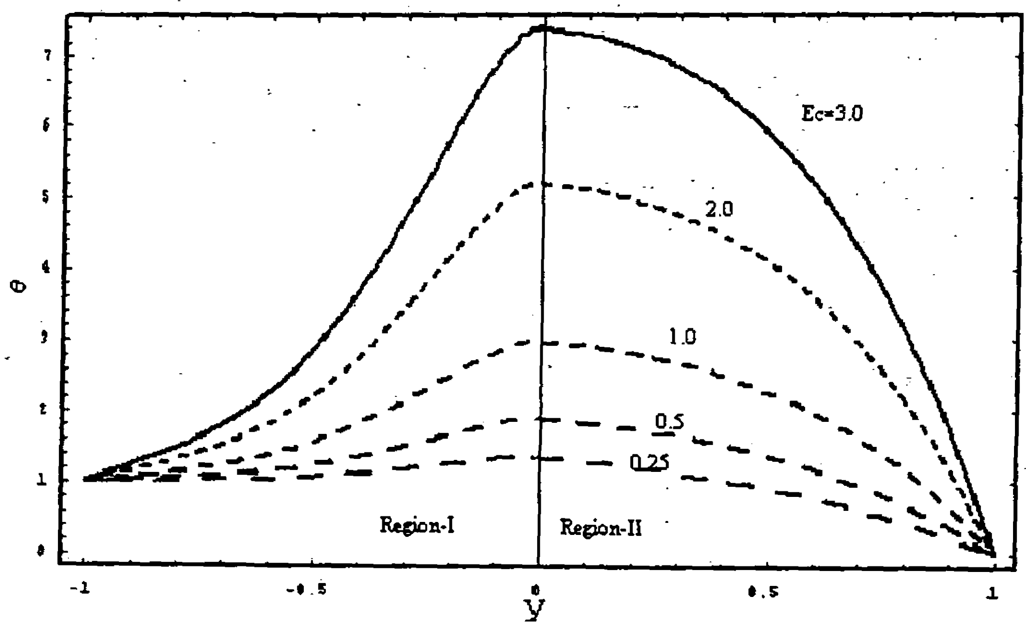

Fig. 3.5: Temperature profile for the different values of the Eckert number Ec

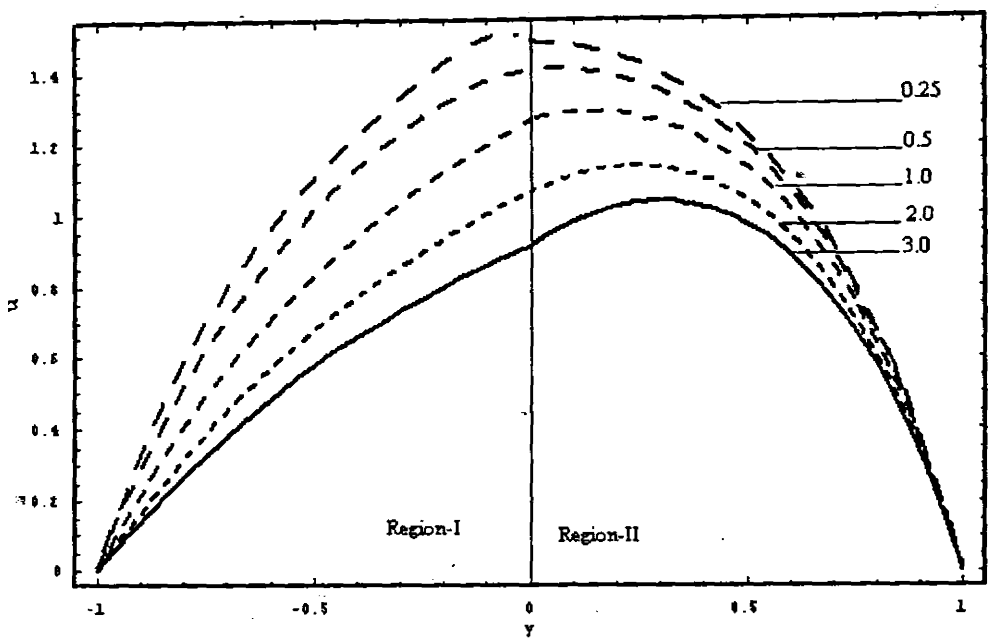

Fig. 3.6: Velocity profile for the different values of the ratio of viscosities m.

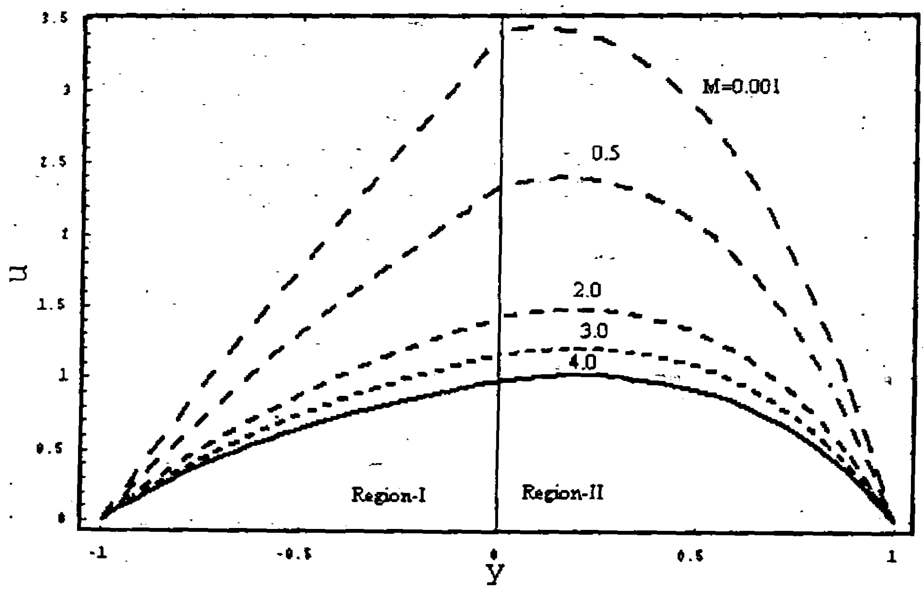

Fig. 3.7: Velocity profile for the different values of magnetic parameter M.

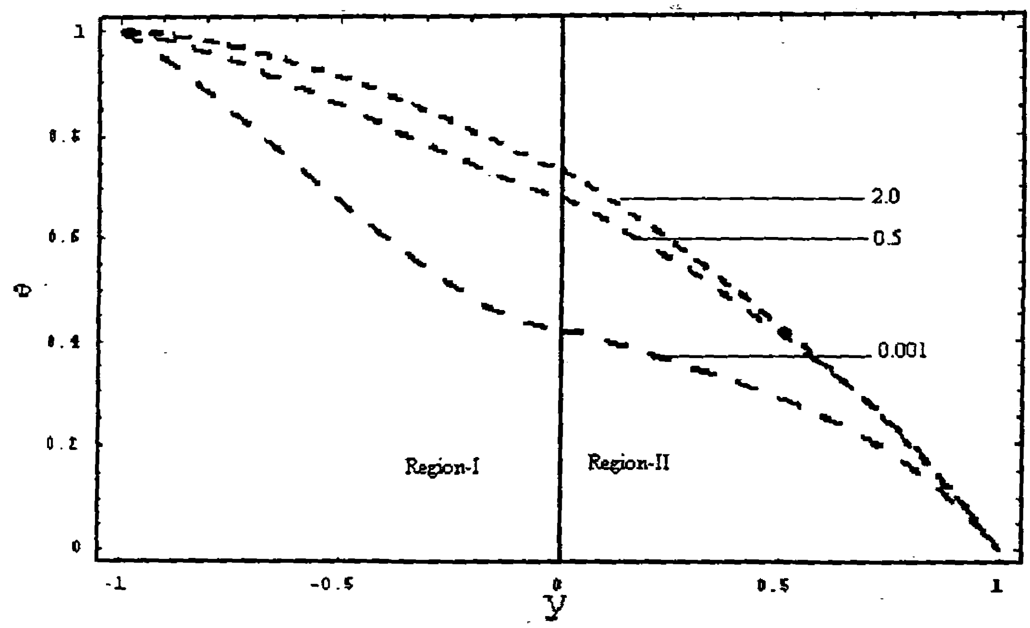

Fig. 3.8: Temperature profile for the different values of magnetic parameter M.

Appendix-I

$B_1 = 2k_2m_6 + 1,\quad B_2 = k_2m_6^2 + m_6 - k_2F_1^2,\quad B_3 = B_2^2 + \omega^2 - B_1^2F_1^2,$

$B_4 = 2k_2m_7 + 1,\quad B_5 = k_2m_7^2 + m_7 - k_2F_1^2,\quad B_6 = B_5^2 + \omega^2 - B_4^2F_1^2,$

$B_7 = 2k_2e_1 + 1,\quad B_8 = k_2e_1^2 + e_1 - k_2F_1^2,\quad B_9 = B_8^2 + \omega^2 - B_7^2F_1^2,$

$B_{10} = 2m_8 + \Pr,\quad B_{11} = m_8^2 + \Pr m_8 - F_4^2,\quad B_{12} = B_{11}^2 + \omega^2 \Pr - B_{10}^2F_4^2,$

$B_{13} = 2e_4 + \Pr,\quad B_{14} = e_4^2 + \Pr e_4 - F_4^2,\quad B_{15} = B_{14}^2 + \omega^2 \Pr - B_{13}^2F_4^2,$

$$C_1 = \frac{-\left(l_3e^{-m_2}\sigma^2 + l_2P\right)}{\sigma^2\left(l_1e^{-m_2} - l_2e^{-m_1}\right)},\quad C_2 = \frac{\left(P - C_1e^{-m_1}\sigma^2\right)}{\sigma^2\left(e^{-m_2}\right)},\quad C_3 = C_1 + C_2 - c_4 - \frac{P}{\sigma^2},$$

$$C_4 = P - m\left(m_1C_1 - m_2C_2\right),\quad C_5 = -C_6e^{-m_4} - l_4,\quad C_6 = \frac{l_9 - l_4}{e^{-m_4} - l_8},$$

$$C_7 = C_5 + C_6 - C_8 - l_6,\quad C_8 = \frac{-\left(km_4C_6 + l_7\right)}{\Pr},$$

$D_1 = 2B_1^2B_2,\quad D_2 = 2B_1B_2^2 - B_1B_3,\quad D_3 = -B_2B_3,\quad D_4 = 2\omega B_2B_1,$
$D_5 = 2B_4^2B_5,\quad D_6 = 2B_4B_5^2 - B_4B_6,\quad D_7 = -B_5B_6,\quad D_8 = 2\omega B_6B_5,$
$D_9 = 2B_7^2B_8,\quad D_{10} = 2B_7B_8^2 - B_7B_9,\quad D_{11} = -B_8B_9,\quad D_{12} = 2\omega B_7B_8,$
$D_{13} = 2B_{10}^2B_{11},\quad D_{14} = 2B_{10}B_{11}^2 - B_{10}B_{12},\quad D_{15} = -B_{11}B_{12},\quad D_{16} = 2\omega \Pr B_{10}B_{11},$
$D_{17} = 2B_{13}^2B_{14},\quad D_{18} = 2B_{13}B_{14}^2 - B_{13}B_{15},\quad D_{19} = -B_{14}B_{15},\quad D_{20} = 2\omega \Pr B_{13}B_{14},$

$$e_1 = \frac{-1 + \sqrt{r_1}\cos\left(\theta_1/2\right)}{2m},\ e_2 = \frac{-2Am_1C_1(mm_1^2 + m_1 - \sigma^2)}{(mm_1^2 + m_1 - \sigma^2)^2 + \omega^2},\ e_3 = \frac{-Am_2C_2(mm_2^2 + m_2 - \sigma^2)}{(mm_2^2 + m_2 - \sigma^2)^2 + \omega^2},$$

$$e_4 = \frac{-1 + \sqrt{r_2}\cos\left(\theta_2/2\right)}{2},\quad e_5 = \frac{-1 + \sqrt{r_3}\cos\left(\theta_3/2\right)}{2k_2},\ e_{5a} = \frac{-2Am_1k_{12}(4k_2m_1^2 + 2m_1)}{(4k_2m_1^2 + 2m_1)^2 \mp \omega^2},$$

$$e_6 = \frac{-2Am_2k_{13}(4k_2m_2^2 + 2m_2)}{(4k_2m_2^2 + 2m_2)^2 + \omega^2},\ e_7 = \frac{-Am_4C_6(k_2m_4^2 + m_4)}{(k_2m_4^2 + m_4)^2 + \omega^2},\ e_8 = \frac{-Am_5k_{14}(k_2m_5^2 + m_5)}{(k_2m_5^2 + m_5)^2 + \omega^2},$$

$$e_9 = \frac{-Am_1k_{10}(k_2m_1^2 + m_1)}{(k_2m_1^2 + m_1)^2 + \omega^2},\ e_{10} = \frac{-2Am_2k_{11}(k_2m_2^2 + m_2)}{(k_2m_2^2 + m_2)^2 + \omega^2},\ e_{11} = \frac{-2Ecmm_1^2C_1e_2(4k_2m_1^2 + 2m_1)}{(4k_2m_1^2 + 2m_1)^2 + \omega^2},$$

$$e_{12}=\frac{-2Ecmm_2^{\,2}C_2e_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2},\qquad e_{13}=\frac{-2Ecmk_{21}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2},$$

$$e_{14}=\frac{-2Ecmm_1^{\,2}C_1F_2(4k_2m_1^2+2m_1)}{(4k_2m_1^2+2m_1)^2+\omega^2},\quad e_{15}=\frac{-2Ecmm_2^{\,2}C_2F_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2},$$

$$e_{16}=\frac{-2Ecmk_{22}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2},\ e_{17}=\frac{-2Ec\sigma^2C_1e_2(4k_2m_1^2+2m_1)}{(4k_2m_1^2+2m_1)^2+\omega^2},\ e_{18}=\frac{-2Ec\sigma^2C_2e_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2},$$

$$e_{19}=\frac{-2Ec\sigma^2k_{26}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2},\ e_{20}=\frac{-2EcPe_2(k_2m_1^2+m_1)}{(k_2m_1^2+m_1)^2+\omega^2},\ e_{21}=\frac{-2EcPe_2(k_2m_2^2+m_2)}{(k_2m_2^2+m_2)^2+\omega^2},$$

$$e_{22}=\frac{-2Ec\sigma^2C_1F_2(4k_2m_1^2+2m_1)}{(4k_2m_1^2+2m_1)^2+\omega^2},\ e_{23}=\frac{-2Ec\sigma^2C_2F_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2},$$

$$e_{24}=\frac{-2Ec\sigma^2k_{27}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2},\ e_{25}=\frac{-2EcPF_2(k_2m_1^2+m_1)}{(k_2m_1^2+m_1)^2+\omega^2},\ e_{26}=\frac{-2EcPF_3(k_2m_2^2+m_2)}{(k_2m_2^2+m_2)^2+\omega^2},$$

$$e_{27}=\frac{-\Pr+\sqrt{r_4}\cos(\theta_4/2)}{2},\ e_{28}=\frac{2\Pr Ak_{15}(4-2\Pr)}{(4-2\Pr)^2+\Pr^2\omega^2},\ e_{29}=\frac{2\Pr Ak_{16}(1-\Pr)}{(1-\Pr)^2+\Pr^2\omega^2},$$

$$e_{30}=\frac{-2Ec\Pr AC_4^2(4-2\Pr)}{(4-2\Pr)^2+\Pr^2\omega^2},\ e_{31}=\frac{-2Ec\Pr AC_4(1-\Pr)}{\omega((1-\Pr)^2+\Pr^2\omega^2)},$$

$$F_1=\frac{\sqrt{r_1}\sin(\theta_1/2)}{2m},\ F_2=\frac{-Am_1C_1\omega}{(mm_1^2+m_1-\sigma^2)^2+\omega^2},\ F_3=\frac{-Am_2C_2\omega}{(mm_2^2+m_2-\sigma^2)^2+\omega^2},$$

$$F_4=\frac{\sqrt{r_2}\sin(\theta_2/2)}{2},\quad F_5=\frac{\sqrt{r_3}\sin(\theta_3/2)}{2k_2},\ F_{5a}=\frac{-2Am_1k_{12}\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},$$

$$F_6=\frac{-2Am_2k_{13}\omega}{(4k_2m_2^2+2m_2)^2+\omega^2},\ F_7=\frac{-Am_4C_6\omega}{(k_2m_4^2+m_4)^2+\omega^2},\ F_8=\frac{-Am_5k_{14}\omega}{(k_2m_5^2+m_5)^2+\omega^2},$$

$$F_9=\frac{-Am_1k_{10}\omega}{(k_2m_1^2+m_1)^2+\omega^2},\ F_{10}=\frac{-Am_2k_{11}\omega}{(k_2m_2^2+m_2)^2+\omega^2},\ F_{11}=\frac{-2Ecmm_1^{\,2}C_1e_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},$$

$$F_{13}=\frac{-2Ecmk_{21}\omega}{(k_2m_5^2+m_5)^2+\omega^2},\ F_{14}=\frac{-2Ecmm_1^{\,2}C_1F_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},\ F_{15}=\frac{-2Ecmm_2^{\,2}C_2F_3\omega}{(4k_2m_2^2+2m_2)^2+\omega^2},$$

$$F_{16}=\frac{-2Ecmk_{22}\omega}{(k_2m_5^2+m_5)^2+\omega^2},\ F_{17}=\frac{-2Ec\sigma^2C_1e_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},\ F_{18}=\frac{-2Ec\sigma^2C_2e_3\omega}{(4k_2m_2^2+2m_2)^2+\omega^2},$$

$$F_{19}=\frac{-2Ec\sigma^2k_{26}\omega}{(k_2m_5^2+m_5)^2+\omega^2},\ F_{20}=\frac{-2EcPe_2\omega}{(k_2m_1^2+m_1)^2+\omega^2},\ F_{21}=\frac{-2EcPe_2\omega}{(k_2m_2^2+m_2)^2+\omega^2},$$

$$F_{22}=\frac{-2Ec\sigma^2C_1F_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},\ F_{23}=\frac{-2Ec\sigma^2C_2F_3\omega}{(4k_2m_2^2+2m_2)^2+\omega^2},\ F_{24}=\frac{-2Ec\sigma^2k_{27}\omega}{(k_2m_5^2+m_5)^2+\omega^2},$$
$$F_{25}=\frac{-2EcPF_2\omega}{(k_2m_1^2+m_1)^2+\omega^2},\ F_{26}=\frac{-2EcPF_3\omega}{(k_2m_2^2+m_2)^2+\omega^2},\ F_{27}=\frac{\sqrt{r_4}\sin(\theta_4/2)}{2},$$
$$F_{28}=\frac{-2\Pr^2 Ak_{15}\omega}{(4-2\Pr)^2+\Pr^2\omega^2},\ F_{29}=\frac{\Pr^2 Ak_{16}\omega}{(1-\Pr)^2+\Pr^2\omega^2},\ F_{30}=\frac{-2Ec\Pr^2 AC_4^2\omega}{(4-2\Pr)^2+\Pr^2\omega^2},$$
$$F_{31}=\frac{-2Ec\Pr^2 AC_4}{(1-\Pr)^2+\Pr^2\omega^2},$$

$$k_2=\frac{k}{\Pr},\quad k_3=\frac{-mEcm_1^2C_1^2}{4m_1^2k_2+2m_1},\quad k_4=\frac{-mEcm_2^2C_2^2}{4m_2^2k_2+2m_2},\quad k_5=\frac{-2mEcm_2m_1C_1C_2}{m_5^2k_2+m_5},$$
$$k_6=\frac{-Ec\sigma^2C_1^2}{4m_1^2k_2+2m_1},\quad k_7=\frac{-Ec\sigma^2C_2^2}{4m_2^2k_2+2m_2},\quad k_8=\frac{-EcP^2}{\sigma^2},\quad k_9=\frac{-2\sigma^2EcC_1C_2}{m_5^2k_2+m_5},$$
$$k_{10}=\frac{2EcC_1P}{m_1^2k_2+m_1},\quad k_{11}=\frac{2EcC_2P}{m_2^2k_2+m_2},\quad k_{12}=k_3+k_6,\quad k_{13}=k_4+k_7,$$

$$k_{14}=k_5+k_9,\quad k_{15}=\frac{-Ec\Pr C_4^2}{4-2\Pr},\quad k_{16}=\frac{2Ec\Pr C_4P}{1-\Pr},\quad k_{17}=-EcP^2,$$

$$k_{18}=\frac{-Ak_8}{\omega},\quad k_{19}=\frac{2mEcm_1C_1}{4(B_1B_2F_1)^2+B_3^2},\ k_{20}=\frac{2mEcm_2C_2}{4(B_4B_4F_1)^2+B_6^2};\ k_{21}=m_2m_1(C_1e_3+C_2e_2),$$
$$k_{22}=m_2m_1(C_1F_3+C_2F_2),\ k_{23}=\frac{2EcC_1\sigma^2}{4(B_1B_2F_1)^2+B_3^2},\ k_{24}=\frac{2mEcC_2\sigma^2}{4(B_4B_5F_1)^2+B_6^2},$$
$$k_{25}=\frac{2PEc}{\sigma^2(4(B_7B_8F_1)^2+B_9^2)},\ k_{26}=C_1e_3+C_2e_2,\ k_{27}=C_1F_3+C_2F_2,\quad k_{28}=\frac{\Pr AC_8}{\omega},$$

$$k_{29}=\frac{-Ak_{17}}{\omega},\quad k_{30}=\frac{-2Ec\Pr C_4}{4(B_{10}B_{11}F_1)^2+B_{12}^2},\quad k_{31}=\frac{-2Ec\Pr P}{4(B_{13}B_{14}F_1)^2+B_{15}^2},$$

$$l_1=1-(e^{-1}-1)mm_1,\quad l_2=1-(e^{-1}-1)mm_2,\quad l_3=Pe^{-1}-P/\sigma^2,$$
$$l_4=-k_8+k_{10}e^{-m_1}+k_{11}e^{-m_2}+k_{12}e^{-2m_1}+k_{13}e^{-2m_2}+k_{14}e^{-m_5}-1,\quad l_5=k_{15}e^{-2}+k_{16}e^{-1}+k_{17},$$
$$l_6=k_{10}+k_{11}+k_{12}+k_{13}+k_{14}-k_{15}-k_{16},$$
$$l_7=k(k_8+m_1k_{10}+m_2k_{11}+2m_1k_{12}+2m_2k_{13}+m_5k_{14})+2k_{15}+k_{16}-k_{17},\quad l_8=1-km_4(e^{-\Pr}-1)/\Pr,$$
$$l_9=l_5+l_6-l_7(e^{-\Pr}-1)/\Pr,\quad l_{10}=e^{-e1}\cos F_1,\quad l_{11}=e^{-e1}\sin F_1,\quad l_{12}=e_2e^{-m1}+e_3e^{-m2},$$
$$l_{13}=F_2e^{-m1}+F_3e^{-m2},\quad l_{14}=e^{e4}\cos F_4,\quad l_{15}=e^{e1}\sin F_4,\quad l_{16}=A(C_4e^{-1}+P)/\omega,$$

$$l_{17}=e_2+e_3, \qquad l_{18}=F_2+F_3-A(C_4+P)/\omega, \qquad l_{19}=m(m_1e_2+m_2e_3),$$

$$l_{20}=m(m_1F_2+m_2F_3)+AC_4/\omega, \qquad l_{21}=l_{14}F_4e_4/l_{15}, \qquad l_{22}=\frac{l_{16}F_4}{l_{15}}+l_{20},$$

$$l_{23}=me_1+l_{21}, \quad l_{24}=l_{21}l_{17}+l_{19}, \quad l_{25}=l_{21}l_{18}+l_{22}, \quad l_{26}=e^{-e5}\cos F_5, \quad l_{27}=e^{-e5}\sin F_5,$$

$$l_{28}=P_{23}e^{-2m1}+P_{24}e^{-2m2}+P_{25}e^{-m5}+P_{26}e^{-m1}+P_{27}e^{-m2}+e^{-m6}\left(P_{28}\cos F_1-P_{29}\sin F_1\right)$$
$$+e^{-m7}\left(P_{30}\cos F_1-P_{31}\sin F_1\right)+e^{-e1}\left(P_{32}\cos F_1-P_{33}\sin F_1\right)+e_7e^{-m4},$$

$$l_{29}=Q_{23}e^{-2m1}+Q_{24}e^{-2m2}+Q_{25}e^{-m5}+Q_{26}e^{-m1}+Q_{27}e^{-m2}+e^{-m6}\left(Q_{28}\cos F_1-Q_{29}\sin F_1\right)$$
$$+e^{-m7}\left(Q_{30}\cos F_1-Q_{31}\sin F_1\right)+e^{-e1}\left(Q_{32}\cos F_1-Q_{33}\sin F_1\right)+F_7e^{-m4}+k_{18},$$

$$l_{30}=e^{e27}\cos F_{27}, \qquad l_{31}=e^{e27}\sin F_{27},$$

$$l_{32}=P_{44}e^{-2}+P_{45}e^{-1}+e^{m8}(P_{46}\cos F_4+P_{47}\sin F_4)+\ +e^{e4}\left(P_{48}\cos F_4+P_{49}\sin F_4\right)$$

$$l_{33}=Q_{44}e^{-2}+Q_{45}e^{-1}+e^{m8}(Q_{46}\cos F_4+Q_{47}\sin F_4)+\ +e^{e4}\left(Q_{48}\cos F_4+Q_{49}\sin F_4\right)+k_{28}e^{-\Pr}+k_{29},$$

$$l_{34}=P_{23}+P_{24}+P_{25}+P_{26}+P_{27}+P_{28}+P_{30}+P_{32}-P_{44}-P_{45}-P_{46}-P_{48}+e_7,$$

$$l_{35}=Q_{23}+Q_{24}+Q_{25}+Q_{26}+Q_{27}+Q_{28}+Q_{30}+Q_{32}-Q_{44}-Q_{45}-Q_{46}-Q_{48}+F_7-k_{28}-k_{29},$$
$$l_{36}=k(2m_1P_{23}+2m_2P_{24}+m_4e_7+m_5P_{25}+m_1P_{26}+m_2P_{27}+m_6P_{28}+F_1P_{29}+m_7P_{30}+F_1P_{31}+e_1P_{32}+F_1P_{33})$$
$$+2P_{44}+P_{45}-m_8P_{46}-F_4P_{47}-e_4P_{48}-F_4P_{49},$$

$$l_{37}=k(2m_1Q_{23}+2m_2Q_{24}+m_4F_7+m_5Q_{25}+m_1Q_{26}+m_2Q_{27}+m_6Q_{28}+F_1Q_{29}+m_7Q_{30}$$
$$+F_1Q_{31}+e_1Q_{32}+F_1Q_{33})+2Q_{44}+Q_{45}-m_8Q_{46}-F_4Q_{47}-e_4Q_{48}-F_4Q_{49}+\Pr Q_{28},$$

$$l_{38}=\frac{l_{30}F_{27}}{l_{31}}-e_{27}, \qquad l_{39}=\frac{l_{32}F_{27}}{l_{31}}+l_{36}, \qquad l_{40}=\frac{l_{33}F_{27}}{l_{31}}+l_{37}, \ l_{41}=ke_5+l_{38}, \ l_{42}=l_{38}l_{34}+l_{39},$$

$$l_{43}=l_{38}l_{35}+l_{40},$$

$$m_1=\frac{-1+\sqrt{1+4\sigma m}}{2m}, \ m_2=\frac{-1-\sqrt{1+4\sigma m}}{2m} \ m_4=\frac{-1}{k_2}, \ m_5=m_1+m_2,$$

$$P_1=e_1XC_9+F_1XC_{10}, \qquad P_2=e_1XC_{10}-F_1XC_9,$$

$$P_3=-D_1F_1^2P_1+D_2F_1P_2+D_3P_1, \qquad P_4=-D_1F_1^2P_2-D_2F_1P_1+D_3P_2,$$
$$P_5=-D_5F_1^2P_1+D_6F_1P_2+D_7P_1, \qquad P_6=-D_5F_1^2P_2-D_5F_1P_1+D_7P_2,$$

$$P_7 = -D_1F_1^2Q_1 + D_2F_1Q_2 + D_3Q_1, \qquad P_8 = -D_1F_1^2Q_2 - D_2F_1Q_1 + D_3Q_2,$$
$$P_9 = -D_5F_1^2Q_1 + D_6F_1Q_2 + D_7Q_1, \qquad P_{10} = -D_5F_1^2Q_2 - D_6F_1Q_1 + D_7Q_2,$$
$$P_{11} = -D_1F_1^2XC_9 + D_2F_1XC_{10} + D_3XC_9, \qquad P_{12} = -D_1F_1^2XC_{10} - D_2F_1XC_9 + D_3XC_{10},$$
$$P_{13} = -D_5F_1^2XC_9 + D_6F_1XC_{10} + D_7XC_9, \qquad P_{14} = -D_5F_1^2XC_{10} - D_6F_1XC_9 + D_7XC_{10},$$
$$P_{15} = -D_9F_1^2XC_9 + D_{10}F_1XC_{10} + D_{11}XC_9, \qquad P_{16} = -D_9F_1^2XC_{10} - D_{10}F_1XC_9 + D_{11}XC_{10},$$
$$P_{17} = -D_1F_1^2YC_9 + D_2F_1YC_{10} + D_3YC_9, \qquad P_{18} = -D_1F_1^2YC_{10} - D_2F_1YC_9 + D_3YC_{10},$$
$$P_{19} = -D_5F_1^2YC_9 + D_6F_1YC_{10} + D_7YC_9, \qquad P_{20} = -D_5F_1^2YC_{10} - D_6F_1YC_9 + D_7YC_{10},$$
$$P_{21} = -D_9F_1^2YC_9 + D_{10}F_1YC_{10} + D_{11}YC_9, \qquad P_{22} = -D_9F_1^2YC_{10} - D_{10}F_1YC_9 + D_{11}YC_{10},$$

$$P_{23} = e_{5a} + e_{11} + e_{17} - F_{14} - F_{22}, \qquad P_{24} = e_6 + e_{12} + e_{18} - F_{15} - F_{23},$$
$$P_{25} = e_8 + e_{13} + e_{19} - F_{16} - F_{24}, \qquad P_{26} = e_9 + e_{20} - F_{25},\ P_{27} = e_{10} + e_{21} - F_{26},$$
$$P_{28} = k_{19}(P_3 - Q_7) + k_{23}(P_{11} - Q_{17}), \qquad P_{29} = k_{19}(P_4 - Q_8) + k_{23}(P_{12} - Q_{18}),$$
$$P_{30} = k_{19}(P_5 - Q_9) + k_{23}(P_{13} - Q_{19}), \qquad P_{31} = k_{19}(P_6 - Q_{10}) + k_{23}(P_{14} - Q_{20}),$$
$$P_{32} = k_{25}(P_{15} - Q_{21}),\ P_{33} = k_{25}(P_{16} - Q_{22}), \qquad P_{34} = e_4XC_{11} + F_4XC_{12},\ P_{35} = e_4XC_{12} - F_4XC_{11},$$
$$P_{36} = -D_{13}F_4^2P_{34} + D_{14}F_4P_{35} + D_{15}P_{34}, \qquad P_{37} = -D_{13}F_4^2P_{35} - D_{14}F_4P_{34} + D_{15}P_{35},$$
$$P_{38} = -D_{17}F_4^2P_{34} + D_{18}F_4P_{35} + D_{19}P_{34}, \qquad P_{39} = -D_{17}F_4^2P_{35} - D_{18}F_4P_{34} + D_{19}P_{35},$$
$$P_{40} = -D_{13}F_4^2Q_{34} + D_{14}F_4Q_{35} + D_{15}Q_{34}, \qquad P_{41} = -D_{13}F_4^2Q_{35} - D_{14}F_4Q_{34} + D_{15}Q_{35},$$
$$P_{42} = -D_{17}F_4^2Q_{34} + D_{18}F_4Q_{35} + D_{19}Q_{34}, \qquad P_{43} = -D_{17}F_4^2Q_{35} - D_{18}F_4Q_{34} + D_{19}Q_{35},$$

$$P_{44} = e_{28} - F_{30}, \qquad P_{45} = e_{29} - F_{31}, \qquad P_{46} = k_{30}(P_{36} - Q_{40}), \qquad P_{47} = k_{30}(P_{37} - Q_{41}),$$
$$P_{48} = k_{31}(P_{38} - Q_{42}), \qquad P_{49} = k_{31}(P_{39} - Q_{43}),$$

$$Q_1 = e_1YC_9 + F_1YC_{10}, \qquad Q_2 = e_1YC_{10} - F_1YC_9, \qquad Q_3 = D_4F_1P_2 - \omega B_3P_1,$$

$$Q_4 = -D_4F_1P_1 - \omega B_3P_2, \qquad Q_5 = D_8F_1P_2 - \omega B_6P_1, \qquad Q_6 = -D_8F_1P_1 - \omega B_6P_2,$$
$$Q_7 = D_4F_1Q_2 - \omega B_3Q_1, \qquad Q_8 = -D_4F_1Q_1 - \omega B_3Q_2, \qquad Q_9 = D_8F_1Q_2 - \omega B_6Q_1,$$
$$Q_{10} = -D_8F_1Q_1 - \omega B_6Q_2, \qquad Q_{11} = D_4F_1XC_{10} - \omega B_3XC_9, \qquad Q_{12} = -D_4F_1XC_9 - \omega B_3XC_{10},$$
$$Q_{13} = D_8F_1XC_{10} - \omega B_6XC_9, \qquad Q_{14} = -D_8F_1XC_9 - \omega B_6XC_{10}, \qquad Q_{15} = D_{12}F_1XC_{10} - \omega B_9XC_9,$$
$$Q_{16} = -D_{12}F_1XC_9 - \omega B_9XC_{10}, \qquad Q_{17} = D_4F_1YC_{10} - \omega B_3YC_9, \qquad Q_{18} = -D_4F_1YC_9 - \omega B_3YC_{10},$$
$$Q_{19} = D_8F_1YC_{10} - \omega B_6YC_9, \qquad Q_{20} = -D_8F_1YC_9 - \omega B_6YC_{10}, \qquad Q_{21} = D_{12}F_1XC_{10} - \omega B_9XC_9,$$
$$Q_{23} = F_{5a} + F_{11} + F_{17} + e_{14} + e_{22},\ Q_{24} = F_6 + F_{12} + F_{18} + e_{15} + e_{23},$$

$$Q_{25} = F_8 + F_{13} + F_{19} + e_{16} + e_{24}, \qquad Q_{26} = -D_{12}F_1YC_9 - \omega B_9YC_{10}, \qquad Q_{27} = F_{10} + F_{21} + e_{26},$$
$$Q_{28} = k_{19}(Q_3 + P_7) + k_{23}(Q_{11} + P_{17}), \qquad Q_{29} = k_{19}(Q_4 + P_8) + k_{23}(Q_{12} + P_{18}),$$
$$Q_{30} = k_{19}(Q_5 + P_9) + k_{23}(Q_{13} + P_{19}), \qquad Q_{31} = k_{19}(Q_6 + P_{10}) + k_{23}(Q_{14} + P_{20}), \qquad Q_{32} = k_{25}(Q_{15} + P_{21}),$$

$$K\Gamma_5 l_{26} - l_{27} l_{41} \qquad l_{27}$$

$$YC_{16} = \frac{-(l_{30} YC_{15} + l_{33})}{l_{31}},$$

$$r_1 = \sqrt{(1+m\sigma^2)^2 + (4\sigma m)^2},\ r_2 = \sqrt{1+16\omega^2},\ r_3 = \sqrt{1+16\omega^2 k_2^2},\ r_4 = \sqrt{\Pr^2 + (4\omega \Pr)^2};$$

$$\theta_1 = \tan^{-1}\frac{4\omega m}{1+m\sigma^2}, \qquad \theta_2 = \tan^{-1} 4\omega, \qquad \theta_3 = \tan^{-1} 4\omega k_2, \qquad \theta_4 = \tan^{-1}\frac{4\omega}{\Pr},$$

$$Q_{33} = k_{25}(Q_{16} + P_{22}), \qquad Q_{34} = e_4 YC_{11} + F_4 YC_{12}, \qquad Q_{35} = e_4 YC_{12} - F_4 YC_{11},$$

$$Q_{36} = -D_{16} F_4 P_{35} - \omega \Pr B_{12} P_{34}, \qquad Q_{37} = -D_{16} F_4 P_{34} - \omega \Pr B_{12} P_{35}, \quad Q_{38} = -D_{20} F_4 P_{35} - \omega \Pr B_{15} P_{34},$$

$$Q_{39} = -D_{20} F_4 P_{34} - \omega \Pr B_{15} P_{35}, \qquad Q_{40} = -D_{16} F_4 Q_{35} - \omega \Pr B_{12} Q_{34},$$

$$Q_{41} = -D_{16} F_4 Q_{34} - \omega \Pr B_{12} Q_{35}, \qquad Q_{42} = -D_{20} F_4 Q_{35} - \omega \Pr B_{15} Q_{34},$$

$$Q_{43} = -D_{20} F_4 Q_{34} - \omega \Pr B_{15} Q_{35}, \qquad Q_{44} = F_{28} + e_{30}, \qquad Q_{45} = F_{29} + e_{31},$$

$$Q_{46} = k_{30}(Q_{36} + P_{40}), \qquad Q_{47} = k_{30}(Q_{37} + P_{41}), \quad Q_{48} = k_{31}(Q_{38} + P_{42}),$$

$$Q_{49} = k_{31}(Q_{39} + P_{43}),$$

$$XC_9 = \frac{l_{11} l_{24} - m_1 F_1 l_{12}}{m F_1 l_{10} - l_{11} l_{23}}, \qquad XC_{10} = \frac{-(l_{10} XC_9 + l_{12})}{l_{11}}, \qquad XC_{11} = XC_9 + l_{17},$$

$$XC_{12} = \frac{-l_{14} XC_{11}}{l_{15}}, \qquad XC_{13} = \frac{l_{27} l_{42} - kF_5 l_{28}}{kF_5 l_{26} - l_{27} l_{41}}, \qquad XC_{14} = \frac{-(l_{26} XC_{13} + l_{28})}{l_{27}},$$

$$XC_{15} = XC_{13} + l_{34}, \qquad XC_{16} = \frac{-(l_{30} XC_{15} + l_{32})}{l_{31}}, \qquad YC_9 = \frac{l_{11} l_{25} - m_1 F_1 l_{13}}{m F_1 l_{10} - l_{11} l_{23}},$$

$$YC_{10} = \frac{-(l_{10} YC_9 + l_{13})}{l_{11}}, \qquad YC_{11} = YC_9 + l_{18}, \qquad YC_{12} = \frac{-l_{14} YC_{11} + l_{16}}{l_{15}},$$

$$YC_{13} = \frac{l_{27} l_{43} - kF_5 l_{29}}{\text{[illegible]}}, \qquad YC_{14} = \frac{-(l_{26} YC_{13} + l_{29})}{l}, \qquad YC_{15} = YC_{13} + l_{35},$$

Appendix-II

$$B_1 = 2k_2m_8 - 1, \quad B_2 = k_2m_8^2 - m_8 - k_2F_1^2, \quad B_3 = B_2^2 + B_1^2F_1^2 + \omega^2, \; B_4 = 2k_2m_9 - 1,$$

$$B_5 = k_2m_9^2 - m_9 - k_2F_1^2, \; B_6 = B_5^2 + B_4^2F_1^2 + \omega^2, \quad B_7 = 2k_2E_1 - 1, \; B_8 = k_2E_1^2 - E_1 - k_2F_1^2,$$

$$B_9 = B_8^2 + B_7^2F_1^2 + \omega^2, \; B_{10} = m_{10}^2 - F_6^2 - m_{10}\,\mathrm{Pr}, \; B_{11} = 2m_{10}F_6 - F_6\,\mathrm{Pr} - \omega\,\mathrm{Pr},$$

$$B_{12} = \frac{B_{10}}{B_{10}^2 - B_{11}^2}, \; B_{13} = \frac{B_{11}}{B_{10}^2 - B_{11}^2}, \quad B_{14} = m_{11}^2 - F_6^2 - m_{11}\,\mathrm{Pr},$$

$$B_{15} = 2m_{11}F_6 - F_6\,\mathrm{Pr} - \omega\,\mathrm{Pr}, \; B_{16} = \frac{B_{14}}{B_{14}^2 - B_{15}^2}, \; B_{17} = \frac{B_{15}}{B_{14}^2 - B_{15}^2},$$

$$C_1 = e^{m_1}\left(\frac{P}{\alpha^2 + M} - C_2e^{-m_2}\right), \quad C_2 = \frac{Pl_3 + C_4l_2}{l_1}, \quad C_3 = e^{-m_3}\left(\frac{P}{M} - C_4e^{m_4}\right), \; C_4 = \frac{P\left(l_1l_6 - l_1l_6\right)}{l_2l_4 - l_1l_5},$$

$$C_5 = -(k_{18} + C_6e^{-m_6}), \; C_6 = \frac{k_{26}\,\mathrm{Pr} + k_{25}\left(e^{\mathrm{Pr}} - 1\right)}{\mathrm{Pr}(1 - e^{-m_6}) + mm_6\left(e^{\mathrm{Pr}} - 1\right)}, \; C_7 = -(k_{19} + C_8e^{\mathrm{Pr}}), \quad C_8 = \frac{mm_6C_6 - k_{25}}{\mathrm{Pr}},$$

$$D_1 = 2\omega B_1F_1, \quad D_2 = \frac{B_2}{B_3 - D_1}, \; D_3 = \frac{B_1F_1 - \omega}{B_3 - D_1}, \; D_4 = 2\omega B_4F_1, \quad D_5 = \frac{B_5}{B_6 - D_4}, \; D_6 = \frac{B_4F_1 - \omega}{B_6 - D_4},$$

$$D_7 = 2\omega B_7F_1, \quad D_8 = \frac{B_8}{B_9 - D_7}, \; D_9 = \frac{B_7F_1 - \omega}{B_9 - D_7},$$

$$E_1 = \frac{1 + \sqrt{r_1}\cos(\theta_1/2)}{2m}, \quad E_2 = \frac{Am_1C_1(mm_1^2 - m_1 - \alpha^2 - M)}{(mm_1^2 - m_1 - \alpha^2 - M)^2 + \omega^2},$$

$$E_3 = \frac{Am_2C_2(mm_2^2 - m_2 - \alpha^2 - M)}{(mm_2^2 - m_2 - \alpha^2 - M)^2 + \omega^2}, \quad E_4 = \frac{Am_3C_3(m_3^2 - m_3 - M)}{(m_3^2 - m_3 - M)^2 + \omega^2}, \; E_5 = \frac{Am_4C_4(m_4^2 - m_4 - M)}{(m_4^2 - m_4 - M)^2 + \omega^2},$$

$$E_6 = \frac{1 + \sqrt{r_2}\cos(\theta_2/2)}{2}, \quad E_{6a} = \frac{1 + \sqrt{r_3}\cos(\theta_3/2)}{2k_2}, \quad E_7 = \frac{Am_6C_6(k_2m_6^2 - m_6)}{(k_2m_6^2 - m_6)^2 + \omega^2},$$

$$E_8 = \frac{Am_2k_{10}(k_2m_2^2 - m_2)}{(k_2m_2^2 - m_2)^2 + \omega^2}, \quad E_9 = \frac{Am_1k_{11}(k_2m_1^2 - m_1)}{(k_2m_1^2 - m_1)^2 + \omega^2}, \quad E_{10} = \frac{2Am_1k_{12}(4k_2m_1^2 - 2m_1)}{(4k_2m_1^2 - 2m_1)^2 + \omega^2},$$

$$E_{11} = \frac{2Am_2k_{13}(4k_2m_2^2 - 2m_2)}{(4k_2m_2^2 - 2m_2)^2 + \omega^2}, \; E_{12} = \frac{Am_6k_{14}(k_2m_6^2 - m_6)}{(k_2m_6^2 - m_6)^2 + \omega^2}, \quad E_{13} = \frac{-2mm_1m_2EcP_3(k_2m_6^2 - m_6)}{(k_2m_6^2 - m_6)^2 + \omega^2},$$

$$E_{14} = \frac{-2mm_1^2EcC_1E_2(4k_2m_1^2 - 2m_1)}{(4k_2m_1^2 - 2m_1)^2 + \omega^2}, \quad E_{15} = \frac{-2mEcC_2E_3(4k_2m_2^2 - 2m_2)}{(4k_2m_2^2 - 2m_2)^2 + \omega^2},$$

$$E_{16}=\frac{-2mm_1m_2EcQ_3(k_2m_6^2-m_6)}{(k_2m_6^2-m_6)^2+\omega^2},\qquad E_{17}=\frac{-2mm_1{}^2EcC_1F_2(4k_2m_1^2-2m_1)}{(4k_2m_1^2-2m_1)^2+\omega^2},$$

$$E_{18}=\frac{-2mEcC_2F_3(4k_2m_2^2-2m_2)}{(4k_2m_2^2-2m_2)^2+\omega^2},\qquad E_{19}=\frac{-2\alpha^2EcC_1E_2(4k_2m_1^2-2m_1)}{(4k_2m_1^2-2m_1)^2+\omega^2},$$

$$E_{20}=\frac{-2\alpha^2EcC_1E_3(k_2m_6^2-m_6)}{(k_2m_6^2-m_6)^2+\omega^2},\qquad E_{21}=\frac{-2\alpha^2EcC_2E_2(k_2m_6^2-m_6)}{(k_2m_6^2-m_6)^2+\omega^2},$$

$$E_{22}=\frac{-2\alpha^2EcC_2E_3(4k_2m_2^2-2m_2)}{(4k_2m_2^2-2m_2)^2+\omega^2},\qquad E_{23}=\frac{2\alpha^2EcPE_2(k_2m_1^2-m_1)}{(\alpha^2+M)[(k_2m_1^2-m_1)^2+\omega^2]},$$

$$E_{24}=\frac{2\alpha^2EcPE_3(k_2m_1^2-m_1)}{(\alpha^2+M)[(k_2m_2^2-m_2)^2+\omega^2]},\qquad E_{25}=\frac{-2\alpha^2EcC_1F_2(k_2m_1^2-m_1)}{(k_2m_1^2-m_1)^2+\omega^2},$$

$$E_{26}=\frac{-2\alpha^2EcC_1F_3(k_2m_6^2-m_6)}{(k_2m_6^2-m_6)^2+\omega^2},\qquad E_{27}=\frac{-2\alpha^2EcC_2F_2(k_2m_6^2-m_6)}{(k_2m_6^2-m_6)^2+\omega^2},$$

$$E_{28}=\frac{-2\alpha^2EcC_2F_34k_2m_2^2-2m_2)}{(4k_2m_2^2-2m_2)^2+\omega^2},\qquad E_{29}=\frac{2\alpha^2EcPF_2(k_2m_1^2-m_1)}{(\alpha^2+M)[(k_2m_1^2-m_1)^2+\omega^2]},$$

$$E_{30}=\frac{2\alpha^2EcPF_3(k_2m_2^2-m_2)}{(\alpha^2+M)[(k_2m_2^2-m_2)^2+\omega^2]},\qquad E_{31}=\frac{\Pr+\sqrt{r_4}\cos(\theta_4/2)}{2},\ E_{32}=\frac{A\Pr C_8}{\omega},$$

$$E_{33}=\frac{2A\Pr m_4k_{15}(4m_3^2-2\Pr m_3)}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},\qquad E_{34}=\frac{2A\Pr m_4k_{16}(4m_4^2-2\Pr m_4)}{(4m_4^2-2\Pr m_4)^2+(\omega\Pr)^2},$$

$$E_{35}=\frac{A\Pr m_7k_{17}(4m_7^2-2\Pr m_7)}{(4m_7^2-2\Pr m_7)^2+(\omega\Pr)^2},\qquad E_{36}=\frac{-2\Pr Ecm_3{}^2C_3E_4(4m_3^2-2\Pr m_3)}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},$$

$$E_{37}=\frac{A\Pr Ecm_3m_4C_4E_5(m_7^2-\Pr m_7)}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},\qquad E_{38}=\frac{A\Pr Ecm_3m_4C_4E_4(m_7^2-\Pr m_7)}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},$$

$$E_{39}=\frac{-2\Pr Ecm_4{}^2C_4E_5(4m_4^2-2\Pr m_4)}{(4m_4^2-2\Pr m_4)^2+(\omega\Pr)^2},\qquad E_{40}=\frac{-2\Pr Ecm_3{}^2C_3F_4(4m_3^2-2\Pr m_3)}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},$$

$$E_{41}=\frac{-2\Pr Ecm_3m_4C_4F_5(m_7^2-\Pr m_7)}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},\qquad E_{42}=\frac{-2\Pr Ecm_3{}^2C_3F_4(4m_3^2-2\Pr m_3)}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},$$

$$E_{43}=\frac{A\Pr Ecm_3m_4C_4F_5(m_7^2-\Pr m_7)}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},$$

$$F_1=\frac{\sqrt{r_1}\sin(\theta_1/2)}{2m},\quad F_2=\frac{Am_1C_1\omega}{(mm_1^2-m_1-\alpha^2-M)^2+\omega^2},\quad F_3=\frac{Am_2C_2\omega}{(mm_2^2-m_2-\alpha^2-M)^2+\omega^2},$$

$$F_4=\frac{Am_3C_3\omega}{(m_3^2-m_3-M)^2+\omega^2},\ F_5=\frac{Am_4C_4\omega}{(m_4^2-m_4-M)^2+\omega^2},\quad F_6=\frac{\sqrt{r_2}\sin(\theta_2/2)}{2},$$

$$F_{6a}=\frac{\sqrt{r_3}\sin(\theta_3/2)}{2k_2},\quad F_7=\frac{Am_6C_6\omega}{(k_2m_6^2-m_6)^2+\omega^2},\quad F_8=\frac{Am_2k_{10}\omega}{(k_2m_2^2-m_2)^2+\omega^2},$$

$$F_9=\frac{Am_1k_{11}\omega}{(k_2m_1^2-m_1)^2+\omega^2},\quad F_{10}=\frac{2Am_1k_{12}\omega}{(4k_2m_1^2-2m_1)^2+\omega^2},\quad F_{11}=\frac{2Am_2k_{13}\omega}{(4k_2m_2^2-2m_2)^2+\omega^2},$$

$$F_{12}=\frac{Am_6k_{14}\omega}{(k_2m_6^2-m_6)^2+\omega^2},\quad F_{13}=\frac{-2mm_1m_2EcP_3\omega}{(k_2m_6^2-m_6)^2+\omega^2},\quad F_{14}=\frac{-2mm_1^2EcC_1E_2\omega}{(4k_2m_1^2-2m_1)^2+\omega^2},$$

$$F_{15}=\frac{-2mEcC_2E_3\omega}{(4k_2m_2^2-2m_2)^2+\omega^2},\quad F_{16}=\frac{-2mm_1m_2EcQ_3\omega}{(k_2m_6^2-m_6)^2+\omega^2},\quad F_{17}=\frac{-2mm_1^2EcC_1F_2\omega}{(4k_2m_1^2-2m_1)^2+\omega^2},$$

$$F_{18}=\frac{-2mEcC_2F_3\omega}{(4k_2m_2^2-2m_2)^2+\omega^2},\quad F_{19}=\frac{-2\alpha^2EcC_1E_2\omega}{(4k_2m_1^2-2m_1)^2+\omega^2},\quad F_{20}=\frac{-2\alpha^2EcC_1E_3\omega}{(k_2m_6^2-m_6)^2+\omega^2},$$

$$F_{21}=\frac{-2\alpha^2EcC_2E_2\omega}{(k_2m_6^2-m_6)^2+\omega^2},\quad F_{22}=\frac{-2\alpha^2EcC_2E_3\omega}{(4k_2m_2^2-2m_2)^2+\omega^2},\quad F_{23}=\frac{2\alpha^2EcPE_2\omega}{(\alpha^2+M)[(k_2m_1^2-m_1)^2+\omega^2]}$$

$$F_{24}=\frac{2\alpha^2EcPE_3\omega}{(\alpha^2+M)[(k_2m_2^2-m_2)^2+\omega^2]},\quad F_{25}=\frac{-2\alpha^2EcC_1F_2\omega}{(k_2m_1^2-m_1)^2+\omega^2},\quad F_{26}=\frac{-2\alpha^2EcC_1F_3\omega}{(k_2m_6^2-m_6)^2+\omega^2},$$

$$F_{27}=\frac{-2\alpha^2EcC_2F_2\omega}{(k_2m_6^2-m_6)^2+\omega^2},\quad F_{28}=\frac{-2\alpha^2EcC_2F_3\omega}{(4k_2m_2^2-2m_2)^2+\omega^2},\quad F_{29}=\frac{2\alpha^2EcPF_2\omega}{(\alpha^2+M)[(k_2m_1^2-m_1)^2+\omega^2]},$$

$$F_{30}=\frac{2\alpha^2EcPF_3\omega}{(\alpha^2+M)[(k_2m_2^2-m_2)^2+\omega^2]},\quad F_{31}=\frac{\sqrt{r_4}\sin(\theta_4/2)}{2},\quad F_{33}=\frac{2A\Pr m_3k_{15}\omega\Pr}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},$$

$$F_{34}=\frac{2A\Pr m_4k_{16}\omega\Pr}{(4m_4^2-2\Pr m_4)^2+(\omega\Pr)^2},\quad F_{35}=\frac{A\Pr m_7k_{17}\omega\Pr}{(4m_7^2-2\Pr m_7)^2+(\omega\Pr)^2},$$

$$F_{36}=\frac{-2\Pr Ecm_3^2C_3E_4\omega\Pr}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},\quad F_{37}=\frac{A\Pr Ecm_3m_4C_4E_5\omega\Pr}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},$$

$$F_{38}=\frac{A\Pr Ecm_3m_4C_4E_4\omega\Pr}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},\quad F_{39}=\frac{-2\Pr Ecm_4^2C_4E_5\omega\Pr}{(4m_4^2-2\Pr m_4)^2+(\omega\Pr)^2},$$

$$F_{40}=\frac{-2\Pr Ecm_3^2C_3F_4\omega\Pr}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},\quad F_{41}=\frac{-2\Pr Ecm_3m_4C_4F_5\omega\Pr}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},$$

$$F_{42}=\frac{-2\Pr Ecm_3^2C_3F_4\omega\Pr}{(4m_3^2-2\Pr m_3)^2+(\omega\Pr)^2},\quad F_{43}=\frac{A\Pr Ecm_3m_4C_4F_5\omega\Pr}{(m_7^2-\Pr m_7)^2+(\omega\Pr)^2},$$

$$k_2 = \frac{k}{\Pr}, \quad k_3 = \frac{-mEcC_1^2 m_1^2}{4k_2 m_1^2 - 2m_1}, \quad k_4 = \frac{-mEcC_2^2 m_2^2}{4k_2 m_2^2 - 2m_2}, \quad k_5 = \frac{-2mEcC_1 m_1 C_2 m_2}{k_2 m_6^2 - m_6},$$

$$k_6 = \frac{-\alpha^2 EcC_1^2}{4k_2 m_1^2 - 2m_1}, \quad k_7 = \frac{-\alpha^2 EcC_2^2}{4k_2 m_2^2 - 2m_2}, \quad k_8 = \frac{\alpha^2 EcP^2}{(\alpha^2 - M)}, \quad k_9 = \frac{-2\alpha^2 EcC_1 C_2}{k_2 m_6^2 - m_6},$$

$$k_{10} = \frac{2\alpha^2 EcPC_2}{(\alpha^2 + M)[k_2 m_2^2 - m_2]}, \quad k_{11} = \frac{2\alpha^2 EcPC_1}{(\alpha^2 + M)[k_2 m_1^2 - m_1]}, \quad k_{12} = k_3 + k_6, \quad k_{13} = k_4 + k_7,$$

$$k_{14} = k_5 + k_9, \quad k_{15} = \frac{-\Pr Ecm_3^2 C_3^2}{4m_3^2 - 2\Pr m_3}, \quad k_{16} = \frac{-\Pr Ecm_4^2 C_4^2}{4m_4^2 - 2\Pr m_4}, \quad k_{17} = \frac{-2\Pr Ecm_3 m_4 C_3 C_4}{m_7^2 - \Pr m_7},$$

$$k_{18} = -k_8 + k_{10}e^{-m_2} + k_{11}e^{-m_1} + k_{12}e^{-2m_1} + k_{13}e^{-2m_2} + k_{14}e^{-m_6} - 1, \quad k_{19} = k_{15}e^{2m_3} + k_{16}e^{2m_4} + k_{17}e^{m_7},$$

$$k_{20} = k_{10} + k_{11} + k_{12} + k_{13} + k_{14}, \quad k_{21} = k_{17} + k_{15} + k_{16}, \quad k_{22} = k_{21} - k_{20},$$

$$k_{23} = k_8 + m_2 k_{10} + m_1 k_{11} + 2m_1 k_{12} + 2m_2 k_{13} + m_6 k_{14}, \quad k_{24} = 2m_3 k_{15} + 2m_4 k_{16} + m_7 k_{17},$$

$$k_{25} = k_{24} - mk_{23}, \quad k_{26} = k_{22} + k_{18} - k_{19}, \quad k_{27} = e^{-E6a}\cos F_{6a}, \quad k_{28} = e^{-E6a}\sin F_{6a},$$

$$k_{29} = P_{15}e^{-m1} + P_{16}e^{-m2} + P_{17}e^{-m6} + P_{18}e^{-2m1} + P_{19}e^{-2m2} + e^{-m8}\left(P_{20}\cos F_1 - P_{21}\sin F_1\right)$$
$$+e^{-m9}\left(P_{22}\cos F_1 - P_{23}\sin F_1\right) + e^{-E1}\left(Q_{13}\cos F_1 - Q_{14}\sin F_1\right) + \frac{Ak_8}{\omega},$$

$$k_{30} = Q_{15}e^{-m1} + Q_{16}e^{-m2} + Q_{17}e^{-m6} + Q_{18}e^{-2m1} + Q_{19}e^{-2m2} + e^{-m8}\left(Q_{20}\cos F_1 - Q_{21}\sin F_1\right)$$
$$+e^{-m9}\left(Q_{22}\cos F_1 - Q_{23}\sin F_1\right) + e^{-E1}\left(Q_{13}\cos F_1 - Q_{14}\sin F_1\right), \quad k_{31} = \cos F_{31}e^{E_{31}},$$

$$k_{32} = \sin F_{31}e^{E_{31}}, \quad k_{33} = P_{26}e^{2m_3} + P_{27}e^{2m_4} + P_{28}e^{m_7} + e^{m_{10}}\left(P_{29}\cos F_6 + P_{30}\sin F_6\right)$$
$$+e^{m_{11}}\left(P_{31}\cos F_6 + P_{32}\sin F_6\right),$$

$$k_{34} = Q_{26}e^{2m_3} + Q_{27}e^{2m_4} + Q_{28}e^{m_7} + e^{m_{10}}\left(Q_{29}\cos F_6 + Q_{30}\sin F_6\right) + e^{m_{11}}\left(Q_{31}\cos F_6 + Q_{32}\sin F_6\right),$$

$$k_{35} = P_{15} + P_{16} + P_{17} + P_{18} + P_{19} + P_{20} + P_{22} + P_{13} - P_{26} - P_{27} - P_{28} - P_{29} - P_{31},$$

$$k_{36} = Q_{15} + Q_{16} + Q_{17} + Q_{18} + Q_{19} + Q_{20} + Q_{22} + Q_{13} - Q_{26} - Q_{27} - Q_{28} - Q_{29} - Q_{31} - E_{32} + \frac{Ak_8}{\omega},$$

$$k_{37} = m_1 P_{15} + m_2 P_{16} + m_6 P_{17} + 2m_1 P_{18} + 2m_2 P_{19} + m_8 P_{20} + F_1 P_{21} + E_1 P_{13} + m_9 P_{22} + F_1 P_{23} + F_1 P_{14},$$

$$k_{38} = m_1 Q_{15} + m_2 Q_{16} + m_6 Q_{17} + 2m_1 Q_{18} + 2m_2 Q_{19} + m_8 Q_{20} + F_1 Q_{21} + E_1 Q_{13} + m_9 Q_{22} + F_1 Q_{23} + F_1 Q_{14},$$

$$k_{39} = 2m_3 P_{26} + 2m_4 P_{27} + m_7 P_{28} + m_{10} P_{29} + F_6 P_{30} + m_{11} P_{31} + F_6 P_{32},$$

$$k_{40} = 2m_3 Q_{26} + 2m_4 Q_{27} + m_7 Q_{28} + m_{10} Q_{29} + F_6 Q_{30} + m_{11} Q_{31} + F_6 Q_{32}, \quad k_{41} = E_{6a} + \frac{F_{6a}k_{27}}{k_{28}},$$

$k_{42} = k_{37} + \frac{F_{6a} k_{29}}{k_{28}}, \qquad k_{43} = k_{38} + \frac{F_{6a} k_{30}}{k_{28}}, \qquad k_{44} = E_{31} - \frac{F_{31} k_{31}}{k_{32}}, \qquad k_{45} = k_{39} - \frac{F_{31} k_{33}}{k_{32}},$

$k_{46} = k_{40} - \frac{F_{31} k_{34}}{k_{32}}; \qquad k_{47} = k_{42} k - k_{45}, \qquad k_{48} = k_{43} - k_{46}, \qquad k_{49} = k_{41} k - k_{44},$

$k_{50} = k_{47} - k_{35} k_{44}, \qquad k_{51} = k_{48} - k_{36} k_{44},$

$l_1 = 1 - e^{m_1 - m_2}, \qquad l_2 = 1 - e^{m_4 - m_3}, \qquad l_3 = \frac{\left(\alpha^2 + M\right) e^{-m_3} - M e^{m_1} - \alpha^2}{M\left(\alpha^2 + M\right)},$

$l_4 = m m_2 - m m_1 e^{m_1 - m_2}, \; l_5 = m_4 - m_3 e^{-m_4 - m_3}, \qquad l_6 = \frac{m_3 e^{-m_3}}{M} - \frac{m m_1 e^{m_1}}{\left(\alpha^2 + M\right)}, \qquad l_7 = E_2 + E_3 - E_4 - E_5,$

$l_8 = F_2 + F_3 - F_4 - F_5, \; l_9 = m\left(m_1 E_2 + m_2 E_3\right), \; l_{9a} = m\left(m_1 F_2 + m_2 F_3\right), \qquad l_{10} = \cos F_1 e^{-E_1},$

$l_{11} = \sin F_1 e^{-E_1}, \qquad l_{12} = e^{-m_1} E_2 + e^{-m_2} E_3, \quad l_{13} = e^{-m_1} F_2 + e^{-m_2} F_3, \qquad l_{14} = \cos F_6 e^{E_6},$

$l_{15} = \sin F_6 e^{E_6}, \qquad l_{16} = e^{m_3} E_4 + e^{m_4} E_5, \quad l_{17} = e^{m_3} F_4 + e^{m_4} F_5, \qquad l_{18} = m_3 E_4 + m_4 E_5,$

$l_{19} = m_3 F_4 + m_4 F_5, \qquad l_{20} = l_{18} - l_9, \qquad l_{21} = l_{19} - l_{9a}, \qquad l_{22} = l_{20} - \frac{F_{6a} l_{16}}{l_{15}} - \frac{m_1 F_1 l_{12}}{l_{11}},$

$l_{23} = l_{21} - \frac{F_6 l_{17}}{l_{15}} - \frac{m_1 F_1 l_{13}}{l_{11}}, \qquad l_{24} = \frac{m\left(E_1 l_{11} + F_1 l_{10}\right)}{l_{11}}, \qquad l_{25} = \frac{\left(F_6 l_{14} - E_6 l_{15}\right)}{l_{15}},$

$m_1 = \frac{1 + \sqrt{1 + 4m\left(\alpha^2 + M\right)}}{2m}, \qquad m_2 = \frac{1 - \sqrt{1 + 4m\left(\alpha^2 + M\right)}}{2m}, \qquad m_3 = \frac{1 + \sqrt{1 + 4M}}{2},$

$m_4 = \frac{1 - \sqrt{1 + 4M}}{2}, \qquad m_5 = \frac{1}{k_2}, \qquad m_6 = m_1 + m_2, \qquad m_7 = m_3 + m_4,$

$m_8 = m_1 + E_1, \qquad m_9 = m_2 + E_1, \qquad m_{10} = m_3 + E_6, \qquad m_{11} = m_4 + E_6,$

$P_1 = E_1 XC_9 + F_1 XC_{10}, \qquad P_2 = E_1 XC_{10} - F_1 XC_9, \qquad P_3 = E_3 C_1 + E_2 C_2,$

$P_4 = -2 m m_1 Ec C_1 P_1 D_2 - 2\alpha^2 Ec C_1 XC_9 D_2, \qquad P_5 = -2 m m_1 Ec C_1 P_1 D_3 - 2\alpha^2 Ec C_1 XC_9 D_3,$

$P_6 = -2 m m_2 Ec C_2 P_1 D_5 - 2\alpha^2 Ec C_2 XC_9 D_5, \qquad P_7 = -2 m m_2 Ec C_2 P_1 D_6 - 2\alpha^2 Ec C_2 XC_9 D_6,$

$$P_8 = 2mm_1 EcC_1 P_2 D_3 + 2\alpha^2 EcC_1 XC_{10} D_3, \qquad P_9 = -2mm_1 EcC_1 P_2 D_2 - 2\alpha^2 EcC_1 XC_{10} D_2,$$

$$P_{10} = 2mm_2 EcC_2 P_2 D_6 + 2\alpha^2 EcC_2 XC_{10} D_6, \qquad P_{11} = -2mm_2 EcC_2 P_2 D_5 - 2\alpha^2 EcC_2 XC_{10} D_5,$$

$$P_{12} = \frac{2\alpha^2 EcPXC_9}{\alpha^2 + M}, \quad P_{13} = P_{12} D_8 - P_{12} D_9, \quad P_{14} = P_{12} D_9 + P_{12} D_8, \quad P_{15} = E_9 + E_{23} - F_{29},$$

$$P_{16} = E_8 + E_{24} - F_{30}, \quad P_{17} = E_7 + E_{12} + E_{13} + E_{20} + E_{21} - F_{16} - F_{26} - F_{27},$$

$$P_{18} = E_{10} + E_{14} + E_{19} - F_{17} - F_{25}, \qquad P_{19} = E_{11} + E_{15} + E_{22} - F_{18} - F_{28}, \qquad P_{20} = P_4 + P_8,$$

$$P_{21} = P_5 + P_9, \qquad P_{22} = P_6 + P_{10}, \quad P_{23} = P_7 + P_{11}, \qquad P_{24} = E_6 XC_{11} + F_6 XC_{12},$$

$$P_{25} = F_6 XC_{11} - E_6 XC_{12}, \qquad P_{26} = E_{33} + E_{36} - F_{40} - F_{42}, \qquad P_{27} = E_{34} + F_{39},$$

$$P_{28} = E_{35} + E_{37} + E_{38} - F_{41} - F_{43}, \qquad P_{29} = -2\Pr Ecm_3 C_3 \left(P_{24} B_{12} + P_{25} B_{13}\right),$$

$$P_{30} = -2\Pr Ecm_3 C_3 \left(P_{24} B_{13} - P_{25} B_{12}\right), \quad P_{31} = -2\Pr Ecm_4 C_4 \left(P_{24} B_{16} + P_{25} B_{17}\right),$$

$$P_{32} = -2\Pr Ecm_4 C_4 \left(P_{24} B_{17} - P_{25} B_{16}\right),$$

$$Q_1 = E_1 YC_9 + F_1 YC_{10}, \qquad Q_2 = E_1 YC_{10} - F_1 YC_9, \qquad Q_3 = F_3 C_1 + F_2 C_2,$$

$$Q_4 = -2mm_1 EcC_1 Q_1 D_2 - 2\alpha^2 EcC_1 YC_9 D_2, \qquad Q_5 = -2mm_1 EcC_1 Q_1 D_3 - 2\alpha^2 EcC_1 YC_9 D_3,$$

$$Q_6 = -2mm_2 EcC_2 Q_1 D_5 - 2\alpha^2 EcC_2 YC_9 D_5, \qquad Q_7 = -2mm_2 EcC_2 Q_1 D_6 - 2\alpha^2 EcC_2 YC_9 D_6,$$

$$Q_8 = 2mm_1 EcC_1 Q_2 D_3 + 2\alpha^2 EcC_1 YC_{10} D_3, \qquad Q_9 = -2mm_1 EcC_1 Q_2 D_2 - 2\alpha^2 EcC_1 YC_{10} D_2,$$

$$Q_{10} = 2mm_2 EcC_2 Q_2 D_6 + 2\alpha^2 EcC_2 YC_{10} D_6, \qquad Q_{11} = -2mm_2 EcC_2 Q_2 D_5 - 2\alpha^2 EcC_2 YC_{10} D_5,$$

$$Q_{12} = \frac{2\alpha^2 EcPYC_9}{\alpha^2 + M}, \quad Q_{13} = Q_{12} D_8 - Q_{12} D_9, \quad Q_{14} = Q_{12} D_9 + Q_{12} D_8, \quad Q_{15} = F_9 + F_{23} + E_{29},$$

$$Q_{16} = F_8 + F_{24} + E_{30}, \quad Q_{17} = F_7 + F_{12} + F_{13} + F_{20} + F_{21} + E_{16} + E_{26} + E_{27},$$

$$Q_{18} = F_{10} + F_{14} + F_{19} + E_{17} + E_{25}, \qquad Q_{19} = F_{11} + F_{15} + F_{22} + E_{18} + E_{28}, \qquad Q_{20} = Q_4 + Q_8,$$

$$Q_{21} = Q_5 + Q_9, \qquad Q_{22} = Q_6 + Q_{10}, \qquad Q_{23} = Q_7 + Q_{11}, \qquad Q_{24} = E_6 YC_{11} + F_6 YC_{12},$$

$$Q_{25} = F_6 YC_{11} - E_6 YC_{12}, \qquad Q_{26} = F_{33} + F_{36} + E_{40} + E_{42}, \qquad Q_{27} = F_{34} + F_{39},$$

$$Q_{28} = F_{35} + F_{37} + F_{38} + E_{41} + E_{43}, \qquad Q_{29} = -2\Pr Ecm_3 C_3 \left(Q_{24} B_{12} + Q_{25} B_{13}\right),$$

$$Q_{30} = -2\Pr Ecm_3C_3\left(Q_{24}B_{13} - Q_{25}B_{12}\right), \qquad Q_{31} = -2\Pr Ecm_4C_4\left(Q_{24}B_{16} + Q_{25}B_{17}\right),$$

$$Q_{32} = -2\Pr Ecm_4C_4\left(Q_{24}B_{17} - Q_{25}B_{16}\right),$$

$$r_1 = \sqrt{\left(1 + 4m\alpha^2 + 4mM\right)^2 + \left(4m\omega\right)^2}, \qquad r_2 = \sqrt{\left(1+4M\right)^2 + \left(4\omega\right)^2}, \qquad r_3 = \sqrt{1 + \left(4\omega k_2\right)^2},$$

$$r_4 = \sqrt{\Pr^4 + \left(4\omega \Pr\right)^2}, \qquad \theta_1 = Tan^{-1}\left[\frac{4m\omega}{1 + 4m\alpha^2 + 4mM}\right], \qquad \theta_2 = Tan^{-1}\left[\frac{4\omega}{1+4M}\right],$$

$$\theta_3 = Tan^{-1}\left[4\omega k_2\right], \qquad \theta_4 = Tan^{-1}\left[\frac{4\omega}{\Pr}\right],$$

$$XC_9 = \frac{l_{22} - l_7 l_{25}}{ml_{24} + l_{25}}, \qquad YC_9 = \frac{l_{23} - l_8 l_{25}}{ml_{24} + l_{25}}, \qquad XC_{10} = \left(\frac{l_{13} + l_{10}YC9}{l_{11}}\right),$$

$$YC_{10} = -\left(\frac{l_{12} + l_{10}XC9}{l_{11}}\right), \qquad XC_{11} = l_7 + XC_9, \qquad YC_{11} = l_8 + YC_9,$$

$$XC_{12} = -\left(\frac{l_{16} + XC_{11}}{l_{15}}\right), \qquad YC_{12} = -\left(\frac{l_{17} + YC_{11}}{l_{15}}\right), \qquad XC_{13} = \frac{-k_{50}}{k_{49}},\ YC_{13} = \frac{-k_{51}}{k_{49}},$$

$$XC_{14} = \frac{k_{29} - k_{27}k_{50}}{k_{28}}, \qquad YC_{14} = \frac{k_{30}k_{49} - k_{27}k_{51}}{k_{28}k_{49}}, \qquad XC_{15} = k_{35} + XC_{13},$$

$$YC_{15} = k_{36} - YC_{13}, \qquad XC_{16} = -\left(\frac{k_{33} + k_{31}XC_{15}}{k_{32}}\right), \qquad YC_{16} = \left(\frac{-k_{34} + k_{31}XC_{15}}{k_{32}}\right),$$

Bibliography

1. M. V. Zatursaka, P.G. Drazin, W.H.H. Banks, On the flow of viscous fluid driven along a channel by suction at porous walls, *Fluid Dyn. Res.*, 4, pp. 151-178, 1998.

2. J. R. King, S. M. Cox, Asymtotic analysis of the steady-state and time-dependent Berman problem, *J. Eng. Math.*, 39, pp. 87-130, 2001.

3. B. A. Packham, R. Shail, Stratified Laminar flow of two immiscible fluids, *P. camb. Philos. Soc.*, 69, pp. 443-448, 1974.

4. A. Setayesh, V. Sahai, Heat transfer in developing MHD poiseuille flow and variable transport properties, *Int. J. Heat Mass Tran.*, 33(8), pp. 1711-1720, 1990.

5. Malashetty M. S, V. Leela MHD heat transfer in two fluid flow. In: *Proc. Of National Heat Transfer Conference sponsored by AICHE and ASME HTD*, 159, pp. 171-175, 1991.

6. Malashetty M. S, V. Leela, MHD heat transfer in two phase flow, *Int. J. Eng. Sci.*, 30, pp. 371-377, 1992.

7. AJ Chamkha, Flow of two immiscible fluids in porous and non porous channel. *J. Fluid. Eng.-T. ASME*, 122(1), pp. 117-124, 2000.

8. Malashetty M. S, J.C. Umavathi, J. P. Kumar, Two fluid magneto convection flow in an inclined channel, *Int. J. Transport Phenomena*, 3, pp.73-84, 2001.

9. Malashetty MS, J. C. Umavathi, J. P. Kumar, Convective MHD two fluid flows and heat transfer in an inclined channel, *Heat Mass Transfer*, 37, pp. 259-264, 2001.

10. Malashetty MS, J. C. Umavathi, J. P. Kumar, Convective flow and heat transfer in an inclined composite porous medium, *J. Porous Media*, 4. 15-22, 2001.

11. J. C. Umavathi, D. Palaniappan, Oscilatory flow of unsteady oberbeck convection fluid in an infinite vertical porous... *J. Appl. Mech.*, 69(2), pp. 35-60, 2000.

12. J.C. Umavathi, A.J. Chamkha, A Mateen, A. Al-Mudhaf, Unsteady oscillatory flow and heat transfer in a horizontal composite porous medium channel. *Non-linear Analysis: Modeling and Controle* 2009 vol 14. N o 3. 397-415, 2009