Application of Legendre and Chebyshev wavelets method to some nonlinear boundary value problems

By

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1 Legendre's Functions

بسم الله الرحمن الرحيم

In The Name of Almighty ALLAH,
The Most Beneficent, The Most Merciful

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A Thesis
Submitted in the Partial Fulfillment of the
Requirements for the Degree of

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In
MATHEMATICS

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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2016

Dedicated

Specially
To My Father, Mother [Late]
All family members
And my Teachers

Whose prayers are always a great source
of motivation for me.

Declaration

I hereby declare and affirm that this research work neither as a whole nor as a part has been copied out from any source It is further declared that I have developed this research work entirely on the basis of my personal efforts.

Moreover, no portion of the work presented in this thesis has been submitted in support of an application for other degree of qualification in this or any other university or institute of learning.

Signature: ____________

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Preface

The boundary value problems arising in science and engineering are generally non-linear. Since the exact solutions for such problems are rare therefore different analytic and numerical methods have been proposed in the past two decades to obtain the approximate solutions. The methods which gained considerable attention include Keller-Box method [1], shooting and finite difference methods [2], hybrid numerical method [3], homotopy analysis method [4], homotopy perturbation method [5, 6], variational iteration method [7, 8], Adomian decomposition method [9], differential transformation method [10, 11] etc. These methods have been used for solving several boundary value problems arising in fluid flows and heat transfer [12, 13, 14, 15, 16, 17, 18, 19, 20].

Legendre wavelets method based on wavelet theory is an active area of research for solving differential equations. For the basic idea of wavelets the readers are referred to the book by Chui [21]. Various types of wavelets have been used by researchers for estimating the solution. One of the important tools is the Haar wavelets [22]. Yousefi and Razzaghi [23] implemented the Legendre wavelets method for nonlinear Volterra-Fredholm integral equations. Dizicheh et al. [24] proposed an algorithm based on Legendre wavelets for solving initial value problems in large domains. Very recently, Yang and Hou [25] presented Chebyshev wavelets method for solving Bratu's problem. Though Legendre and Chebyshev wavelets method are not new but the application of these methods to the solution of nonlinear problems in fluid mechanics and heat transfer is limited. For a particular choice of involved parameters the Legendre and Chebyshev wavelets methods yield an explicit solution expression which might be of some interest to the experimentalists and numerical analysts working on code development.

Keeping the above fact in mind, we have solved the classical problems of parallel plate flow of third grade fluid and forced convection in a porous duct using Legendre wavelets in chapter 2. The obtained results are in excellent agreement with existing results. Numerical values of the solutions are tabulated to have estimate of error between present solution and available results It is mentioned that governing equations of these problems are nonlinear and it is difficult to obtain their exact solutions. Chapter 3 provides Chebyshev wavelets solution to some classical nonlinear problems arising in fluid mechanics and heat transfer The quadratic Riccati's equation and nonlinear sixth order equation are also dealt by this method in chapter 3 The solution obtained in each case compared with exact or numerical solution It is hoped that these methods can be implemented for finding the solution of complex boundary value problems in other disciplines of science and engineering.

Contents

Chapter 1: Preliminaries

The objective of this chapter is to introduce the reader with the basics of wavelets The chapter starts with the definition of wavelets The Legendre wavelets and their applications to the solution of second order boundary value problems is explained in the next section The last section provide the definition and implementation of Chebyshev wavelets to nonlinear second order boundary value problems

1.1 Review of wavelet theory

Wavelets constitutes a family of functions constructed from a single function called the mother wavelet by scaling and translating A family of continuous wavelets is defined as [26, 27]

$$\psi_{\alpha,\beta}(\xi)=|\alpha|^{\frac{-1}{2}}\psi\left(\frac{\xi-\beta}{\alpha}\right), \qquad \alpha,\beta\in\mathbb{R},\alpha\neq 0, \qquad (1\ 1)$$

where α,β are the dilation and translation parameters respectively The family of discrete wavelets is defined by restricting α,β to discrete values $\alpha=\alpha_0^{-k}, \beta=n\beta_0\alpha_0^{-k}, \alpha_0>1$, $\beta_0>0$ and n and k are positive integers, that is

$$\psi_{k,n}(\xi)=|\alpha_0|^{\frac{k}{2}}\psi\left(\alpha_0^k\xi-n\beta_0\right). \qquad (1\ 2)$$

Here $\psi_{k,n}(\xi)$ forms basis for $L^2(\mathbb{R})$ In particular, when $\alpha_0=2$, $\beta_0=1$ then $\psi_{k,n}(\xi)$ forms an orthonormal basis [26]

1.2 Legendre Wavelets

Legendre wavelets are defined on the interval [0, 1) as

$$\psi_{nm}(\xi)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}P_m\left(2^k\xi-\hat{n}\right) & \frac{\hat{n}-1}{2^k}\leq\xi<\frac{\hat{n}+1}{2^k}\\ 0 & \text{otherwise,}\end{cases} \qquad (1\ 3)$$

where $\hat{n}=2n-1$, $n=1,2,3,\ ,2^{k-1}$, $m=0,1,2,3,\ ,M-1$ The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $\alpha=2^{-k}$ and the translation parameter is $\beta=\hat{n}2^{-k}$ $P_m(\xi)$ are the well-known Legendre polynomials of order m, which are defined on the interval [-1,1], and can be determined through the following recurrence relation

$$P_0(\xi)=1,\ P_1(\xi)=\xi,$$

$$P_{m+1}(\xi)=\left(\frac{2m+1}{m+1}\right)\xi P_m(\xi)-\left(\frac{m}{m+1}\right)P_{m-1}(\xi), \qquad m=2,3,\ . \tag{1 4}$$

1.2.1 Legendre wavelets and function approximation

We may expand function $f(\xi)$ using Legendre wavelets as

$$f(\xi)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}C_{nm}\psi_{nm}(\xi), \tag{1 5}$$

where $C_{nm}=\langle f(\xi),\psi_{nm}(\xi)\rangle$ In truncated form the infinite series in (1 5) can be written as

$$f(\xi)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}C_{nm}\psi_{nm}(\xi)=C^T\Psi(\xi), \tag{1 6}$$

where C and $\Psi(\xi)$ are $2^{k-1}M\times 1$ matrices given by

$$C=\left[c_{10}\quad c_{11}\qquad c_{1M-1}\quad c_{20}\qquad c_{2M-1}\quad c_{2^{k-1}0}\quad .\qquad c_{2^{k-1}M-1}\right]^T \tag{1 7}$$

$$\Psi(\xi)=[\psi_{10}(\xi)\quad \psi_{11}(\xi)\qquad \psi_{1M-1}(\xi)\quad \psi_{20}(\xi)\qquad \psi_{2M-1}(\xi)\quad \psi_{2^{k-1}0}(\xi)$$

$$\psi_{2^{k-1}M-1}(\xi)]^T \tag{1 8}$$

We note that the number of elements of C depends on the choice of k and M For instance if we choose $k=1$ and $M=6$, then the Legendre wavelet series with six unknowns is sought on the interval [0, 1] For $k = 2$ and $M = 6$, the interval [0, 1] is subdivided into two intervals [0, 1/2] and [1/2, 1] and in each subinterval the Legendre wavelet series contains six unknowns

1.2.2 Implementation of Legendre wavelets method to B.V.Ps

Consider a second order nonlinear boundary value problem

$$y'' + \varphi(\eta, y, y', y'') = 0 \tag{1 9}$$

subject to the boundary conditions $y(a) = a_0, y(b) = b_0$ The Green's function corresponding to homogeneous problem is

$$G(\eta,\xi) = \begin{cases} \left(\dfrac{b-\xi}{b-a}\right)(\eta - a), & \text{for } a \le \xi < \eta, \\ \left(\dfrac{b-\xi}{b-a}\right)(\eta - b), & \text{for } \eta < \xi \le b \end{cases} \tag{1 10}$$

with the help of (1 10) the boundary value problem can be transformed into following integral equation

$$y(\eta) = P(\eta) + \int_a^b G(\eta,\xi)\varphi(\xi)d\xi\,, \tag{1 11}$$

where $\varphi(\xi) = \varphi(\xi, y, y', y'')$ and $P(\eta)$ is the function established from the nonhomogeneous boundary value problem $P''(\eta) = 0$, $P(a) = a_0, P(b) = b_0$. In order to apply Legendre wavelets, we expand the solution of (1 9) in terms of Legendre wavelets as follows

$$y(\eta) = C^T\Psi(\eta)\,, \tag{1 12}$$

where C and $\Psi(\eta)$ are defined in Eqs (1 7) and (1 8) Substitution of (1 12) into (1 11) yields

$$C^T\Psi(\eta) = P(\eta) + \int_a^b G(\eta,\xi)\varphi(C^T\Psi(\xi))d\xi \tag{1 13}$$

Collocating Eq (1 13) at $2^{k-1}M$ points η_i gives

$$C^T\Psi(\eta_i) = P(\eta_i) + \int_a^b G(\eta_i,\xi)\varphi(C^T\Psi(\xi))d\xi, \tag{1 14}$$

where collocation points η_i's are Gauss-Lobatto points defined as

$$\eta_i = \cos\left(\frac{i\pi}{2^k M}\right), \qquad i = 1,2,3,\ldots,2^{k-1}M \tag{1 15}$$

Application of Gaussian quadrature formula to Eq (1 14) requires the transformation of ξ-interval to τ-interval [-1, 1] This can be done by the change of variable as

$$\xi = \frac{1}{2}(1-\tau)a + \frac{1}{2}(1+\tau)b \tag{1 16}$$

Now using the definition of Gaussian quadrature [28]

$$\int_a^b f(\xi)d\xi = \frac{b-a}{2}\int_{-1}^{1}\hat{f}(\tau)d\tau = \sum_{j=1}^{s}\omega_j\hat{f}(\tau_j), \tag{1 17}$$

where 'τ_j' are the zeros of Legendre polynomial P_{s+1} and 'ω_j' are corresponding weights, thus Eq (1 14) can be written as

$$C^T\Psi(\eta_i) = P(\eta_i) + \frac{b-a}{2}\int_{-1}^{1} H\left(\eta_i, \frac{1}{2}(1-\tau)a + \frac{1}{2}(1+\tau)b\right)d\tau,$$

or
$$C^T\Psi(\eta_i) = P(\eta_i) + \frac{b-a}{2}\sum_{j=1}^{s}\omega_j H\left(\eta_i, \frac{1}{2}(1-\tau_j)a + \frac{1}{2}(1+\tau_j)b\right). \tag{1 18}$$

where $H(\eta_i,\xi) = G(\eta_i,\xi)\varphi\left(C^T\Psi(\xi)\right)$ Eq (1 18) will gives $2^{k-1}M$ nonlinear equations which can be solved for the elements of C in equation (1 7) using Newton's iterative method or by any computational software like Mathematica or MATLAB We point out here that the choice of initial values to compute the element of C may affect the convergence of the method to real roots

Therefore care must be taken while choosing these initial values However the built-in routine in Mathematica give all possible solutions of simultaneous nonlinear algebraic equations which might be helpful in locating the appropriate real solution

1.2.3 Error Analysis for Legendre wavelets method

Since the truncated Legendre wavelets series is an approximate solution of nonlinear Fredholm integral equation, so one has an error function $E(\eta)$ for $y(\eta)$ as follows

$$E(\eta)=\left\|y(\eta)-C^T\Psi(\eta)\right\| \tag{1 19}$$

The error bound for the Legendre wavelets method approximation is demonstrated in following lemma [29]

Lemma: Suppose that the function $y\ [0,1]\rightarrow R$ is m times continuously differentiable i e $y\in C^m[0,1]$ Then $C^T\Psi(\eta)$ approximate y with mean error bound as follows

$$\left\|y(\eta)-C^T\Psi(\eta)\right\|\le\left\|\frac{1}{m!2^{mk}}\operatorname{Sup}_{\eta\in[0,1)}\left|y^m(\eta)\right|\right\| \tag{1 20}$$

For proof of above lemma we refer the reader to [29] It is has been also proved in the literature that the truncated series given by Eq (1 6) converges towards the exact solution $y(\eta)$ [30, 31]

1.3 Chebyshev wavelets

Chebyshev wavelets are defined on the interval [0, 1) as

$$\psi_{nm}(\xi)=\begin{cases}\dfrac{\alpha_m 2^{\frac{k-1}{2}}}{\sqrt{\pi}}P_m\left(2^k\xi-\hat{n}\right) & \dfrac{\hat{n}-1}{2^k}\le\xi<\dfrac{\hat{n}+1}{2^k},\\ 0 & \text{otherwise},\end{cases} \tag{1 21}$$

where $\hat{n} = 2n-1$ and $\alpha_m = \begin{cases} \sqrt{2} & m=0 \\ 2 & m \geq 0 \end{cases}$, $n = 1,2,3, \ldots, 2^{k-1}$, $m = 0,1,2,3, \ldots, M$. $P_m(\xi)$ are the well-known Chebyshev polynomials of the first kind of degree m, which are defined on the interval [-1,1], and can be determined through the following recurrence relation

$$P_0(\xi) = 1, \; P_1(\xi) = \xi,$$

$$P_{m+1}(\xi) = 2\xi P_m(\xi) - P_{m-1}(\xi), \qquad m = 2,3,4,\ldots \tag{1.22}$$

We should note that the set of Chebyshev wavelets are orthogonal with respect to the weight function $w_n(x) = w(2^k x - \hat{n})$.

1.3.1 Chebyshev wavelets and function approximation

The orthogonality of Chebyshev wavelets enables us write

$$f(\xi) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \psi_{nm}(\xi) \tag{1.23}$$

where $C_{nm} = < f(\xi), \psi_{nm}(\xi) >$. The above infinite series in truncated form can be written as

$$f(\xi) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm} \psi_{nm}(\xi) = C^T \Psi(\xi), \tag{1.24}$$

where C and $\psi(\xi)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \left[c_{10} \quad c_{11} \quad \ldots \quad c_{1M-1} \quad c_{20} \quad \ldots \quad c_{2M-1} \quad c_{2^{k-1}0} \quad \ldots \quad c_{2^{k-1}M-1} \right]^T \tag{1.25}$$

$$\Psi(\xi) = \left[\psi_{10}(\xi) \quad \psi_{11}(\xi) \quad \ldots \quad \psi_{1M-1}(\xi) \quad \psi_{20}(\xi) \quad \ldots \quad \psi_{2M-1}(\xi) \quad \psi_{2^{k-1}0}(\xi) \quad \ldots \quad \psi_{2^{k-1}M-1}(\xi) \right]^T \tag{1.26}$$

1.3.2 Operational matrix of derivatives

The derivative of a Chebyshev polynomial can be expressed as a linear combination of lower-order Chebyshev polynomials, that is

$$\begin{cases} P'_m(\xi) = 2m\sum_{k=1}^{m-1} P_m(\xi) & m \text{ is even,} \\ P'_m(\xi) = 2m\sum_{k=1}^{m-1} P_m(\xi) + mP_0(\xi) & m \text{ is odd} \end{cases} \tag{1.27}$$

In the view of (1.27), the derivative of $\psi_{nm}(\xi)$ is given as,

$$\psi'_{nm}(\xi) = \begin{cases} \dfrac{\alpha_m 2^{\frac{k-1}{2}}}{\sqrt{\pi}} 2^k\, 2m\sum_{k=1}^{m-1} P_m\left(2^k\xi - \hat{n}\right) & m \text{ even,} \\ \dfrac{\alpha_m 2^{\frac{k-1}{2}}}{\sqrt{\pi}} .2^k\, 2m\sum_{k=1}^{m-1} P_m\left(2^k\xi - \hat{n}\right) + mP_0\left(2^k\xi - \hat{n}\right) & m \text{ odd} \end{cases} \tag{1.28}$$

The function $\psi_i(\xi)$ is zero outside the interval $\left[\dfrac{\hat{n}-1}{2^k}, \dfrac{\hat{n}+1}{2^k}\right)$, so

$$\psi'_i(\xi) = \psi_i(\xi)\,W \qquad i = 1,2,\ldots,2^{k-1} \tag{1.29}$$

where

$$W = 2^k \begin{bmatrix} 0 & \sqrt{2} & 0 & 3\sqrt{2} & 0 & 5\sqrt{2} & \cdot & (M-1)\sqrt{2} \\ 0 & 0 & 4 & 0 & 8 & 0 & & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & & 2(M-1) \\ & \cdot & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & 2(M-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{M\times M} \quad \text{when } M \text{ is even,}$$

$$W = 2^k \begin{bmatrix} 0 & \sqrt{2} & 0 & 3\sqrt{2} & 0 & 5\sqrt{2} & & 0 \\ 0 & 0 & 4 & 0 & 8 & 0 & & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & & 2(M-1) \\ & & \cdot & & \cdot & & & \\ & & & \cdot & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 2(M-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{M\times M} \quad \text{when } M \text{ is odd,}$$

Defining $D = diag(W^T, W^T, W^T, \ldots, W^T)$, one can write

$$\Psi'(\xi) = D\Psi(\xi) \tag{1.30}$$

The above relation can be generalized to nth derivative as

$$\Psi^{(n)}(\xi) = D^n\Psi(\xi) \qquad n = 1,2,3,\ldots \tag{1.31}$$

1.3.3 Implementation of Chebyshev wavelets method to B.V.Ps

In order to apply Chebyshev wavelets to a second order nonlinear boundary value problem defined in Eq (1 9), we expand the solution of (1.9) in terms of Chebyshev wavelets as follows

$$y(\eta) = C^T\Psi(\eta), \tag{1.32}$$

where C and $\psi(\eta)$ are defined in Eqs (1 25) and (1 26) Inserting (1 32) into (1 9), one gets

$$C^T D^2\Psi(\eta) + \varphi(\eta, C^T D\Psi(\eta), C^T D^2\Psi(\eta)) = 0, \tag{1 33}$$

Defining collocation points η_i's as

$$\eta_i = \frac{1}{2}\left[1+\cos\left(\frac{(i-1)\pi}{2^{k-1}M-1}\right)\right], \qquad i = 2,3, \ ,2^{k-1}M-1, \tag{1 34}$$

Eq (1 33) can be transformed into the following system,

$$C^T D^2\Psi(\eta_i) + \varphi(\eta_i, C^T D\Psi(\eta_i), C^T D^2\Psi(\eta_i)) = 0 \tag{1 35}$$

The boundary conditions on Eq (1 9) can be written as

$$C^T\Psi(a) = a_0, C^T\Psi(b) = b_0 \tag{1 36}$$

The system of $2^{k-1}M-2$ nonlinear equations along with the (1 36) can be solved for the elements of C in equation (1 25) using Newton's iterative method or by any computational software like Mathematica or MATLAB

1.3.4 Error Analysis for Chebyshev wavelets method

The error bound for the Chebyshev approximation is demonstrated in following lemma, given in [25]

Lemma: Suppose that the function $f:[0,1)\to R$ is m times continuously differentiable $f\in C^{m}[0,1]$ Then $C^{T}\Psi(\eta)$ approximate f with mean error bound as follows

$$\left\|f(\eta)-C^{T}\Psi(\eta)\right\|\leq\left\|\frac{1}{m!4^{m}2^{m(k-1)}}\max_{\eta\in[0,1]}\left|f^{m}(\eta)\right|\right\| \qquad (1.37)$$

Chapter 2: Solution of some flow problems using Legendre wavelets

A method based on Legendre wavelets is presented in this chapter to discuss the flow of a third grade fluid between parallel plates and forced convection in a porous duct The flow problems are modeled in terms of integral equations and then solved by Legendre wavelets method The comparison between present results and existing solutions shows that the Legendre wavelets method is a powerful tool for solving nonlinear boundary value problems We hope this method can be used for solving many interesting problems arising in flows of non-Newtonian fluids

2.1 Parallel plate flow of third grade fluid

2.1.1 Plane Couette Flow

We consider the steady laminar flow of an incompressible third grade fluid between two horizontal infinite parallel plates separated by a distance h The upper plate ($y=h$) is moving with uniform velocity U while the lower plate ($y = 0$) is stationary The motion of the upper plate sets the fluid particles moving in the direction parallel to the plates Let x-axis be taken in the direction of flow and y in the direction normal to the flow Assume that there is no pressure gradient in the direction of x-axis The resulting normalized differential equation derived in [14] for such a flow in the absence of pressure gradient, reduces to

$$y''(\eta)+6\beta y''(\eta)y'^2(\eta)=0\,,\ y(0)=0\,,\ y(1)=1 \qquad (2\ 1)$$

The Green's function corresponding to Eq (2 1) is given by

$$G(\eta,\xi)=\begin{cases}\xi(1-\eta), & \text{for } 0\le\xi<\eta,\\ \eta(1-\xi), & \text{for } \eta<\xi\le 1,\end{cases} \tag{2 2}$$

and the integral equation is

$$y(\eta)=P(\eta)+\int_0^1 G(\eta.\xi)\varphi(\xi)d\xi\,, \tag{2 3}$$

where $\varphi(\xi)=6\beta y'^2(\xi)y''(\xi)$ and $P(\eta)=\eta$

Substitution of Eq (2 2) into Eq (2 3) yields

$$y(\eta)=\eta+\int_0^\eta \xi(1-\eta)\left[6\beta y'^2(\xi)y''(\xi)\right]d\xi+\int_\eta^1 \eta(1-\xi)\left[6\beta y'^2(\xi)y''(\xi)\right]d\xi \tag{2 4}$$

Now to apply Legendre wavelets, we proceed with $k=1, M=1$ Thus the solution of (2 1) will be of the form

$$y(\eta)=c_{10}\psi_{10}(\eta)+c_{11}\psi_{11}(\eta)$$

or $$y(\eta)=c_{10}+\sqrt{3}c_{11}(2\eta-1), \tag{2 5}$$

where the values of $\psi_{10}(\eta)$ and $\psi_{11}(\eta)$ are obtained through the definition of Legendre wavelets Invoking (2.5) into (2 4) gives

$$c_{10}+\sqrt{3}c_{11}(2\eta-1)=\eta \tag{2 6}$$

We choose the collocation points as $\eta=0,\frac{1}{\sqrt{2}}$. Substituting these collocation points in (2 6) and solving the resulting equation for unknowns c_{10} and c_{11}, we get $c_{10}=\frac{1}{2}$, $c_{11}=\frac{1}{2\sqrt{3}}$ and thus

$$y(\eta)=c_{10}\psi_{10}(\eta)+c_{11}\psi_{11}(\eta)=\eta\,, \tag{2 7}$$

which is the exact solution

2.1.2 Fully-developed plane Poiseuille flow

We now consider a homogeneous third grade fluid occupying the space between two stationary infinite parallel plates which are distant *2h* apart The flow is generated due to a constant pressure gradient Let the origin of the rectangular coordinates be mid-way between the plates The differential equation which governs the problem under consideration is [**14**]

$$y''(\eta)+6\beta y''(\eta)y'^2(\eta)=p\,,\ y'(0)=0\,,y(1)=0\,. \tag{2 8}$$

Eq. (2.8) along with boundary conditions can be transformed into following integral equation

$$y(\eta)=\frac{1}{2}p(-1+\eta^2)+\int_0^1 G(\eta,\xi)\left[6\beta y'^2(\xi)y''(\xi)\right]d\xi\,, \tag{2 9}$$

where $G(\eta,\xi)$ is the associated Green's function given by

$$G(\eta,\xi)=\begin{cases}1-\eta, & \text{for } 0\le\xi<\eta,\\ 1-\xi, & \text{for } \eta<\xi\le 1\end{cases} \tag{2.10}$$

For application of Legendre wavelets method we choose $k=1,M=3$ Thus we can write the solution of (2.9) as

$$y(\eta)=C^T\Psi(\eta)$$

or $$y(\eta)=c_{10}\psi_{10}(\eta)+c_{11}\psi_{11}(\eta)+c_{12}\psi_{12}(\eta)$$

or $$y(\eta)=c_{10}+\sqrt{3}c_{11}(2\eta-1)+\sqrt{5}c_{12}(6\eta^2-6\eta+1) \tag{2 11}$$

Substituting (24) into (22) yields

$$y(\eta)=\frac{1}{2}p(-1+\eta^2)+\int_0^{\eta}(1-\eta)6\beta\left(12\sqrt{5}c_{12}\right)\left(2\sqrt{3}c_{11}+\sqrt{5}c_{12}\left(12\xi-6\right)\right)^2 d\xi$$

$$+\int_{\eta}^{1}(1-\xi)6\beta\left(12\sqrt{5}c_{12}\right)\left(2\sqrt{3}c_{11}+\sqrt{5}c_{12}\left(12\xi-6\right)\right)^2 d\xi \qquad (2\ 12)$$

Here we collocate (2 12) at $\eta=0,\frac{1}{2},1$ Now transforming the domain of integration in (2 12) to [-1, 1] and substituting the value of collocation points, we get the following three equations

$$c_{1\,0}-\sqrt{3}c_{1\,1}+\sqrt{5}c_{1,2}=0\,, \qquad (2\ 13)$$

$$c_{10}\text{-}1\ 11803\ c_{12}=0\ 25\ (2\ -\ 0\ 5\,p)$$

$$+\int_{-1}^{1}\left[(1+\tau_1)\beta c_{12}\begin{pmatrix}241\ 495c_{11}^2+(\text{-}935\ 307+935\ 307\tau_1)c_{11}c_{12}\\+\left(905\ 608\text{-}1811\ 22\tau_1+905\ 608\tau_1^{\,2}\right)c_{12}^2\end{pmatrix}\right]d\tau_1$$

$$+\int_{-1}^{1}\begin{bmatrix}(241\ 495\text{-}241\ 495\tau_2)\beta c_{11}^2c_{12}+\left(935\ 307\text{-}935\ 307\tau_2^{\,2}\right)\beta c_{11}c_{12}^2\\+\left(905\ 608+905\ 608\tau_2\ \text{-}905\ 608\tau_2^{\,2}\ \text{-}905\ 608\tau_2^{\,3}\right)\beta c_{12}^3\end{bmatrix}d\tau_2\,.$$

(2 14)

$$c_{10}+\sqrt{3}c_{11}+\sqrt{5}c_{12}=1 \qquad (2\ 15)$$

Now we apply Gaussian integration formulae to evaluate the integrals appearing in the above three equations Since the first integral in (2 14) involves a third degree polynomial, therefore we use zeros of Legendre polynomial P_4 and associated weights to evaluate it Similarly, the zeros of Legendre polynomial P_4 and the associated weight functions will be used to evaluate the second integral in (2 14) This results in the following three equations

$$c_{1\,0}-\sqrt{3}c_{1\,1}+\sqrt{5}c_{1\,2}=0\,, \qquad (2\ 16)$$

$$0\ 125p+c_{10}+\left(-1\ 11803-241\ 495\beta c_{11}^{2}\right)c_{12}-603\ 738\beta c_{12}^{3}=\frac{1}{2}, \qquad (2\ 17)$$

$$c_{10}+\sqrt{3}c_{11}+\sqrt{5}c_{12}=1, \qquad (2\ 18)$$

Now solving Eqs (216-2 18) for a fixed value of β and p using Newton's iterative method, one can easily get the values of c_{10}, c_{11} and c_{12} The comparison of Legendre wavelet solution and numerical solution by using BVP4C routine of MATLAB is shown in **Fig. 1** **Table 1** shows the absolute error between numerical solution and Legendre wavelet solution for $\beta=0\ 5$, $p=-1\ 5$ Using lemma in chapter 1 the error bound for this problem turns out to be 0 02742 This clearly indicates that absolute error does not exceed the value predicted by the lemma

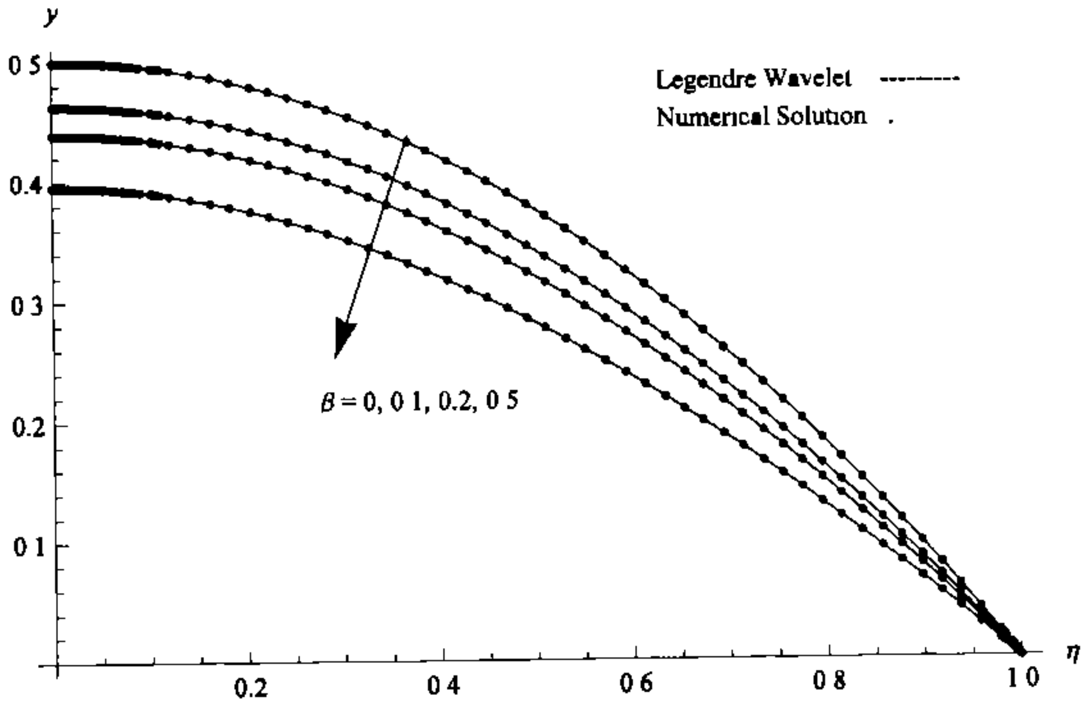

Fig.1: Comparison of numerical solution with Legendre wavelet solution for the velocity profile $y(\eta)$ of fully-developed plane Poiseuille flow when p =-1

Table 1: Comparison of Legendre wavelet solutions with the numerical solution fully-developed plane Poiseuille flow for values of β and p

	$\beta = 0.5, p = -1.5$		
η	Legendre Wavelet Solution	Numerical Solution	Absolute Error
0.1	0.5150	0.5148	0.000125439
0.3	0.4598	0.4598	0.000020202
0.5	0.3631	0.3631	6.3916×10^{-6}
0.7	0.2356	0.2356	4.8202×10^{-6}
0.9	0.0837	0.0837	5.3032×10^{-6}

2.1.3 Plane Couette–Poiseuille flow

Here the flow of third grade fluid is generated by simultaneous application of the constant pressure gradient and motion of the upper plate. The resulting differential equation and the corresponding boundary conditions are [14]

$$y''(\eta) + 6\beta y''(\eta) y'^2(\eta) = p,\ y(0) = 0,\ y(1) = 1 \quad (2.19)$$

The Green's function corresponding to (32) is given by

$$G(\eta,\xi) = \begin{cases} \xi(1-\eta), \text{ for } 0 \le \xi < \eta, \\ \eta(1-\xi), \text{ for } \eta < \xi \le 1, \end{cases} \quad (2.20)$$

and the integral equation associated with boundary value problem (32) is

$$y(\eta) = P(\eta) + \int_0^1 G(\eta,\xi)\varphi(\xi) d\xi, \quad (2.21)$$

where $\varphi(\xi) = 6\beta y'^2(\xi) y''(\xi) - p$ and $P(\eta) = \eta$

We have applied Legendre wavelet method for $k = 1, M = 3$ and compared the results with numerical solution in **Table 2**. Graphical results obtained through both solutions are also displayed in **Figs. 2-4**. We have computed the error bound using lemma of chapter 1 for $\beta = 0\ 5 \quad p = -1\ 5$ just to show that the absolute error does not exceeds its upper bound It is found that the error bound for this case is 0 002520 Clearly the absolute error does not exceed this value

Table 2: Comparison of Legendre wavelet solutions with the numerical solution of plane Couette–Poiseuille flow for various values of β and p

β =0, p = -1 5				β =0 5, p = -1 5			
η	Legendre Wavelet Solution	Numerical Solution	Absolute Error	η	Legendre Wavelet Solution	Numerical Solution	Absolute Error
0 1	**0.1675**	**0.1675**	**3.812×10^{-9}**	0 1	**0.115786**	**0.115786**	$1\ 55199 \times 10^{-6}$
0 3	**0.4575**	**0.4575**	**3.812×10^{-9}**	0 3	**0.338073**	**0.338073**	**5.7949×10^{-7}**
0 5	**0.6875**	**0 6875**	**3.812×10^{-9}**	0 5	**0.546980**	**0.546980**	**6.0232×10^{-8}**
0 7	**0.8575**	**0.8575**	**3.812×10^{-9}**	0 7	**0.741048**	**0.741048**	**$6\ 1000 \times 10^{-7}$**
0 9	**0.9675**	**0.9675**	**3.812×10^{-9}**	0 9	**0.918379**	**0.918379**	**5.3475×10^{-8}**
$\beta = 1, p =$ -1 5				$\beta = 1\ 5, p =$ -1 5			
0 1	0 109219	0 109219	**5.2309×10^{-9}**	0.1	0 106526	0 106526	**$1\ 9325 \times 10^{-10}$**
0 3	0 321998	0 321998	**1.5921×10^{-9}**	0 3	0 315486	0 315486	**1.7781×10^{-9}**
0 5	0 526818	0 526818	**1.0310×10^{-9}**	0 5	0 518763	0 518763	**2.4963×10^{-9}**
0 7	0 723105	0 723105	**7.7161×10^{-10}**	0 7	0 716054	0 716054	**5.9340×10^{-10}**
0 9	0 910175	0 910175	**2.2304×10^{-8}**	0 9	0 907015	0 907015	**5.4273×10^{-9}**
$\beta = 2, p =$ -1 5							
0 1	0 105054	0 105054	**1.0764×10^{-9}**				
0 3	0 311953	0 311953	**4.2288×10^{-9}**				
0 5	0 514429	0 514429	**5.2297×10^{-9}**				
0 7	0 712298	0 712298	**5.3988×10^{-10}**				
0 9	0 90535	0 90535	**9.0192×10^{-10}**				

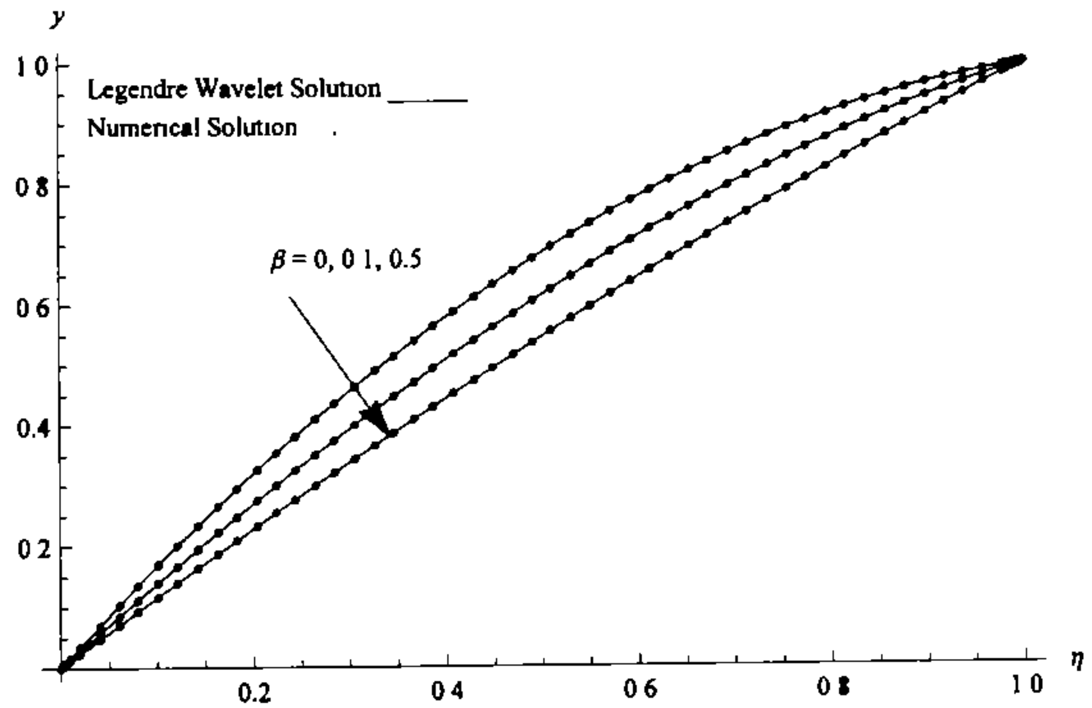

Fig. 2: Comparison of Legendre wavelet solutions with the numerical solution of plane Couette–Poiseuille flow for various values of β when p = -1 5

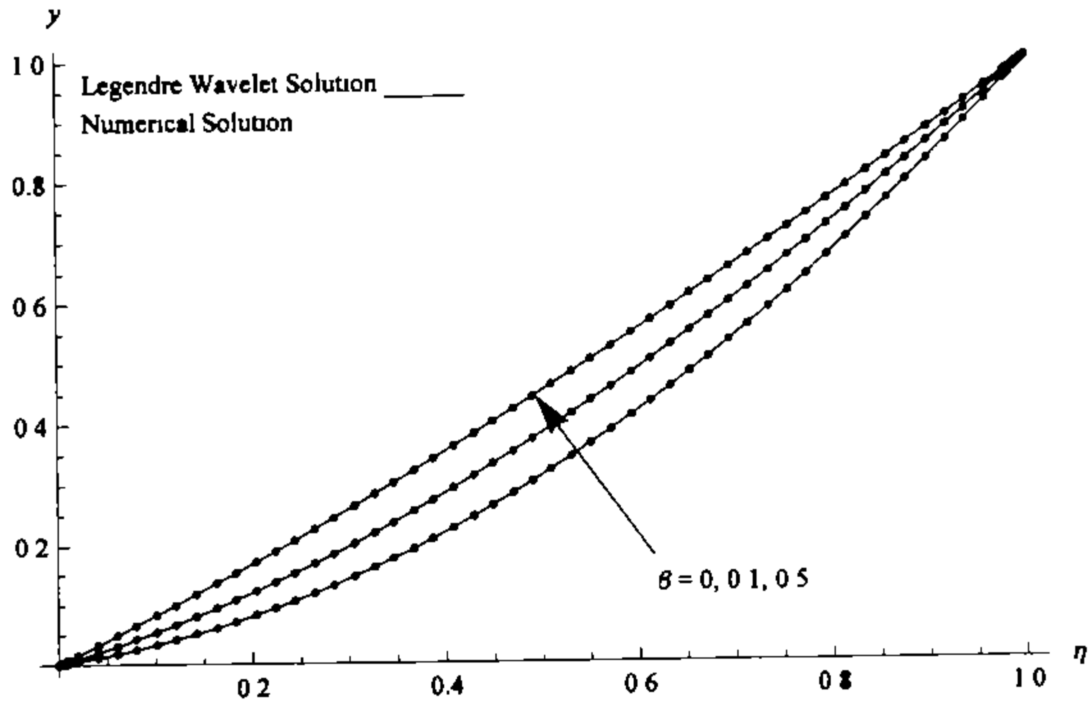

Fig. 3: Comparison of Legendre wavelet solutions with the numerical solution of plane Couette–Poiseuille flow for various values of β when p = 1 5

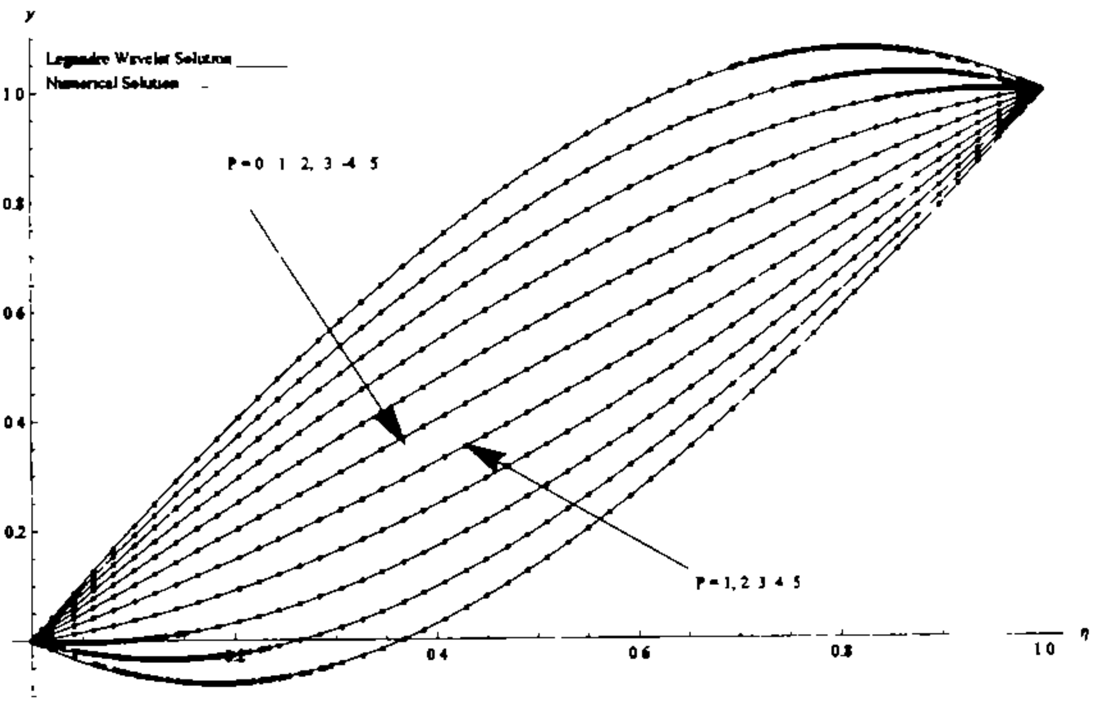

Fig. 4: Comparison of Legendre wavelet solutions with the numerical solution of plane Couette-Poiseuille flow for various values of p when $\beta = 0.1$

2.2 Forced convection in a porous saturated duct

Let us consider steady pressure driven flow of a Newtonian fluid through a horizontal channel filled with porous medium The governing equation in dimensionless form for such flow is [32]

$$y''(x) - s^2 y(x) - Fsy^2(x) + \frac{1}{B} = 0 \tag{2.22}$$

subject to the boundary conditions $y(-1) = 0, y(1) = 0$ To apply Legendre wavelet method, we use transformation $x = 2\eta - 1$ for our convenience Thus we can write (2.22) and boundary conditions as

$$y''(\eta) - 4s^2 y(\eta) - 4Fsy^2(\eta) + \frac{4}{B} = 0 \, . \; y(0) = 0, y(1) = 0 \tag{2.23}$$

In the same manner as described previously we can transform (2 23) into following integral equation

$$y(\eta)=\int_0^1 G(\eta,\xi)\varphi(\xi)d\xi\,, \tag{2 24}$$

where
$$G(\eta,\xi)=\begin{cases}\xi(1-\eta), & \text{for } 0\le\xi<\eta,\\ \eta(1-\xi), & \text{for } \eta<\xi\le 1,\end{cases} \tag{2 25}$$

and $\varphi(\xi)=-4s^2y(\xi)-4Fsy^2(\xi)+\dfrac{4}{B}$

Taking $k=1$, $M=6$, we have obtained the solution and results shown in Table and Figs 5 and 6

Table 3: Comparison of the values of the Legendre wavelet approximation solutions with the numerical solution of forced convection in a porous saturated duct

F=1,B=1

	s=1	*s*=2	*s*=3	*s*=4
Legendre wavelets Solution y'(1)	-0 72124	-0 46913	-0 32803	-0 24857
Numerical Solution y'(1)	-0.72124	-0 46913	-0.32803	-0 24857
SHAM Solution (15th Order) y'(1)	-0 72120	-0 46913	-0 32803	-0 24857

s =1,B=1

	F=0 2	*F*=0 4	*F*=0 6	*F*=0 8	*F*=1
Legendre wavelet Solution y'(1)	-0 75248	-0 74395	-0.73594	-0 72838	-0 72124
Numerical Solution y'(1)	-0 75248	-0 74395	-0 73594	-0 72838	-0 72124
SHAM Solution (20th Order) y'(1)	-0.75248	-0.74396	-0 73594	-0.72838	-0.72124

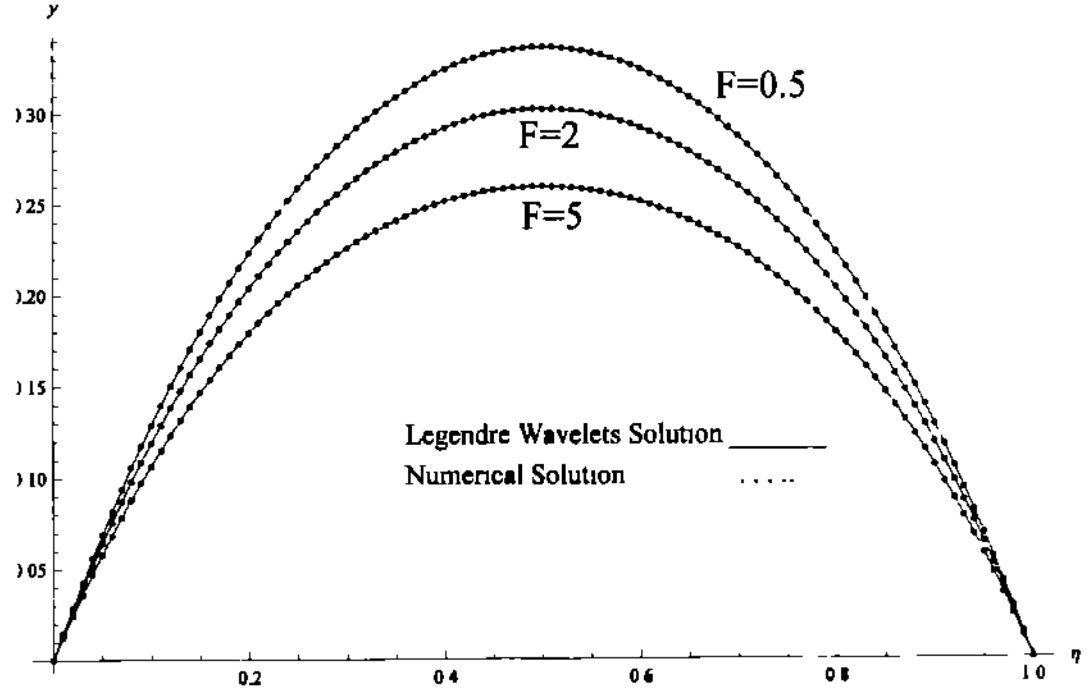

Fig. 5: Comparison of Legendre wavelet solutions with the numerical solution of forced convection in a porous saturated duct for different values of F when $s = 1$, $B = 1$

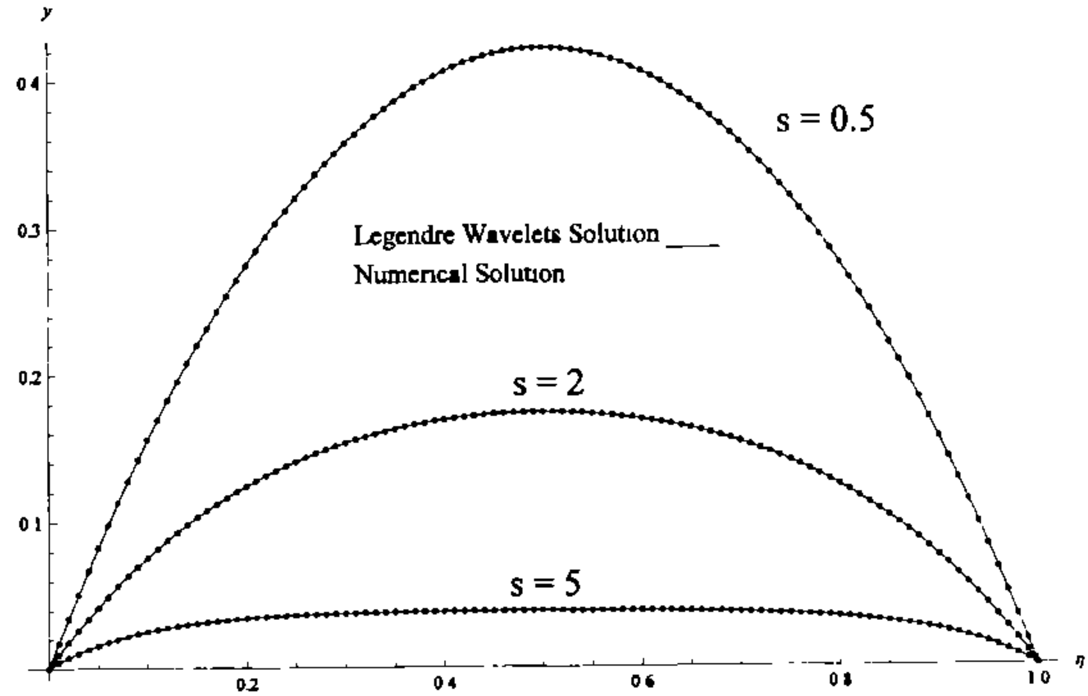

Fig. 6: Comparison of Legendre wavelet solutions with the numerical solution of forced convection in a porous saturated duct for different values of s when $F = 1$, $B = 1$

The Table and Figures shows excellent agreement between both the solutions. The same problem is treated in [32] by Spectral homotopy analysis method (SHAM) It is observed that our solution is in excellent agreement with the solution obtained by SHAM This also demonstrates the efficiency and accuracy of Legendre wavelet method The simplicity of the algorithm just requires few seconds for a personal computer to run the code and get the unknown values of matrix C The only computational part involved in this method is the calculation of the elements of matrix C which can be done without much effort There are several solvers available which can solve large system of nonlinear algebraic equations The procedure of SHAM is much different In SHAM a nonlinear differential equation is reduced into an infinite number of differential equations which are then solved using Chebyshev pseudospectral method We do not claim the efficiency of Legendre wavelet method for all type of nonlinear boundary value problems but for this specific problem this method is found to be efficient than SHAM. For instance when s=1, SHAM gives fifth order accurate solution at 10th order of approximation while this accuracy is achieved with Legendre wavelet method by taking k=1 and M=6 This clearly demonstrates that the required accuracy for this particular problem can be achieved by Legendre wavelet method at very low computational cost

Conclusion:

In this chapter a Legendre wavelets method is employed to solve some two-point boundary value problems in fluid mechanics Based on developed algorithm the results are obtained for Couette flow, Poiseuilleflow, Couette- Poiseuille flow and forced convection in a porous duct The graphical results for both numerical solutions and Legendre wavelets solution are presented The absolute error for both the solutions illustrates that solutions are in excellent agreement We hope, it will open a window to implement this method to other complicated nonlinear boundary value problems in non-Newtonian fluids

Chapter 3: Solution of some nonlinear problems using Chebyshev wavelets

This chapter is concerned with the solution of nonlinear problems using Chebyshev wavelets Five representative problems are solved The Chebyshev wavelets solution for each problem is compared with the already available approximate or exact solution.

3.1 Cooling of lumped system with variable specific heat

For this problem, the nondimensional equation and initial condition are

$$\left(1+\beta y(\eta)\right)\frac{dy}{d\eta}+y(\eta)=0\,,\ y(0)=1 \tag{3 1}$$

Chebyshev wavelet method suggests to write

$$y(\eta)=C^{T}\Psi(\eta)\,, \tag{3 2}$$

where C and $\Psi(\eta)$ are defined in (1 25) and (1 26), respectively Substituting (3 2) in (3 1), we get

$$\left(1+\beta C^{T}\Psi(\eta)\right)C^{T}D\Psi(\eta)+C^{T}\Psi(\eta)=0\,, \tag{3 3}$$

To proceed further, we choose $k=1$, $M=3$ and then $y(\eta)$ becomes

$$y(\eta)=c_{10}\psi_{10}(\eta)+c_{11}\psi_{11}(\eta)+c_{12}\psi_{12}(\eta)\,, \tag{3 4}$$

Using the definition of Chebyshev wavelets, the above equation becomes

$$y(\eta)=\frac{\sqrt{2}c_{10}}{\sqrt{\pi}}+\frac{2(2\eta-1)c_{11}}{\sqrt{\pi}}+\frac{2\left(8\eta^{2}-8\eta+1\right)c_{12}}{\sqrt{\pi}} \tag{3 5}$$

Invoking (3 5) into (3 3) gives

$$4\beta\left(c_{11}(4\eta-2)+2c_{12}(8(\eta-1)\eta+1)+\sqrt{2}c_{10}\right)\left(4c_{12}(2\eta-1)+c_{11}\right)$$
$$+\sqrt{\pi}\left(c_{11}(4\eta+2)+2c_{12}(8\eta(\eta+1)-7)+\sqrt{2}c_{10}\right)=0 \qquad (3\ 6)$$

Substituting the collocation points $\eta=\frac{1}{4},\frac{3}{4}$ in (3 6), we get the following algebraic equations

$$\frac{\sqrt{2}c_{10}-2c_{11}+2c_{12}}{\sqrt{\pi}}=0 \qquad (3\ 7)$$

$$\frac{4\beta\left(\sqrt{2}c_{10}+c_{11}-c_{12}\right)\left(c_{11}+2c_{12}\right)+\sqrt{\pi}\left(\sqrt{2}c_{10}+5c_{11}+7c_{12}\right)}{\pi}=0, \qquad (3\ 8)$$

$$\frac{\sqrt{\pi}\left(\sqrt{2}c_{10}+3c_{11}-9c_{12}\right)-4\beta\left(c_{11}-2c_{12}\right)\left(-\sqrt{2}c_{10}+c_{11}+c_{12}\right)}{\pi}=0, \qquad (3\ 9)$$

where the Eq (3 7) is the consequence of the initial condition $y(0)=1$ Solving the above system for $\beta=0\ 1$, we get

$$c_{10}=0\ 831828,\ c_{11}=-0\ 26843,\ c_{12}=0\ 0296056 \qquad (3\ 10)$$

Thus

$$y(\eta)=1-0\ 87303\eta+0\ 26725\eta^2 \qquad (3\ 11)$$

Similar solution expressions can be obtained for other values of β The solution obtained by above defined procedure is compared with the numerical solution in **Figs. 3.1-3.3** and **Table 3.1** for different values of β An excellent correlation between both the solutions is observed Due to separable nature of the equation (3 1), an exact solution can be easily obtained and given by **[33]**

$$\ln y+\beta(y-1)+\eta=0 \qquad (3\ 12)$$

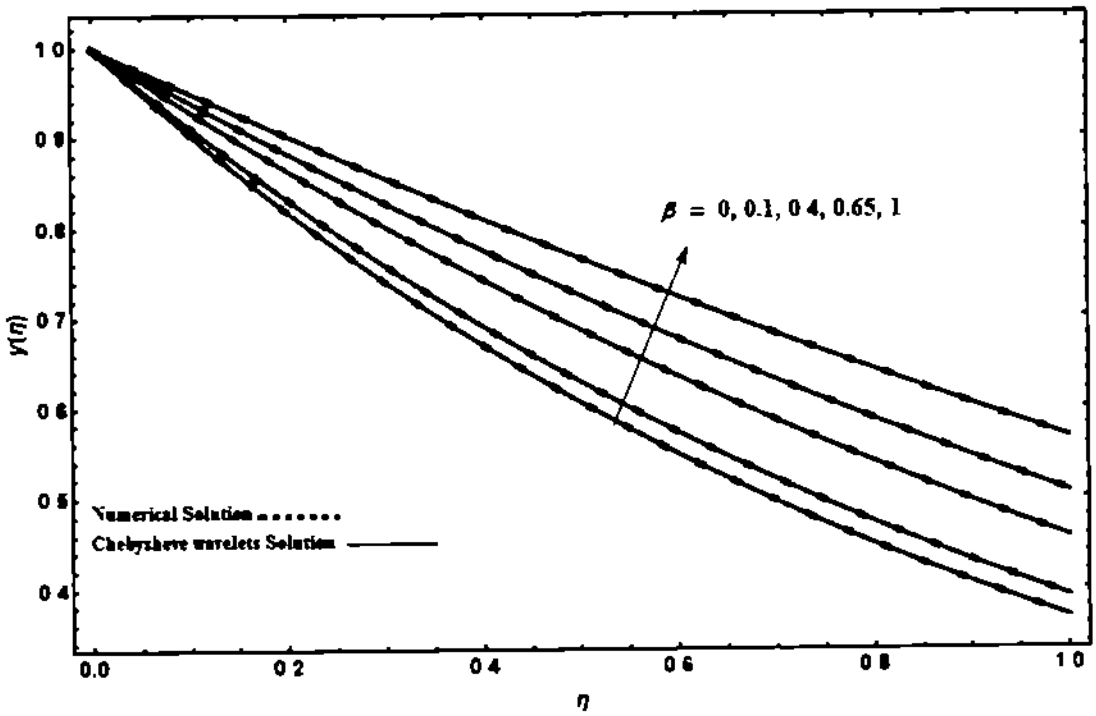

Fig. 3.1: Comparison of Chebyshev wavelets solution ($k = 1$, $M = 6$) with numerical solution in the interval [0, 1]

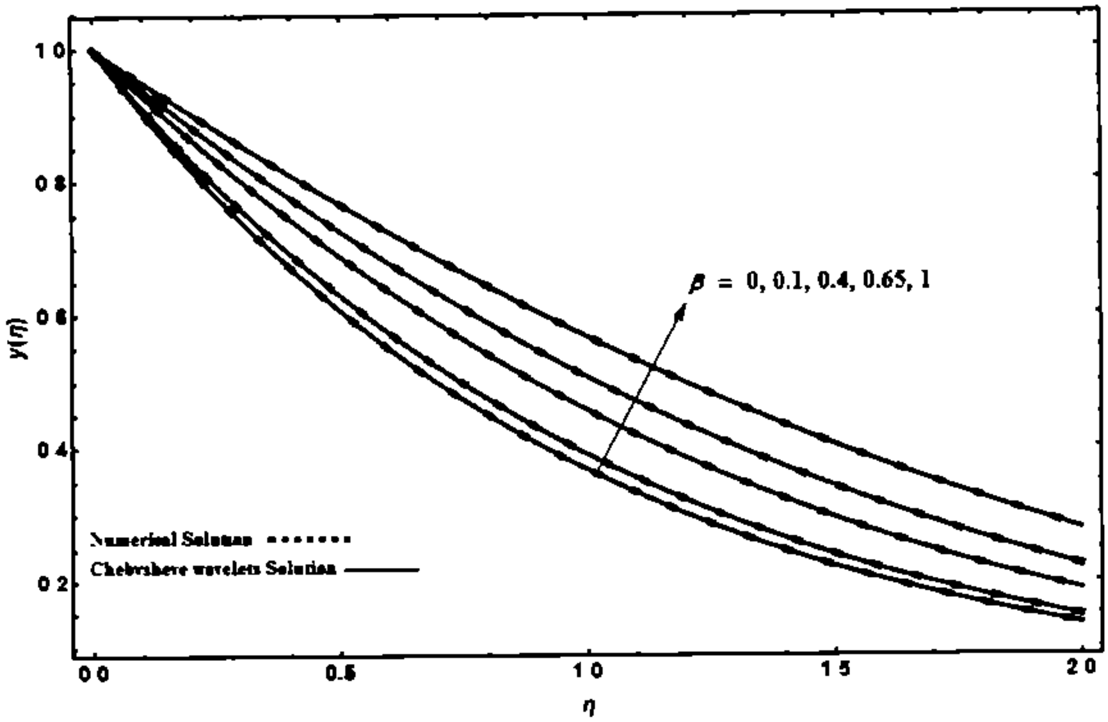

Fig. 3.2: Comparison of Chebyshev wavelets solution ($k = 1$, $M = 6$) with numerical solution in the interval [0, 2]

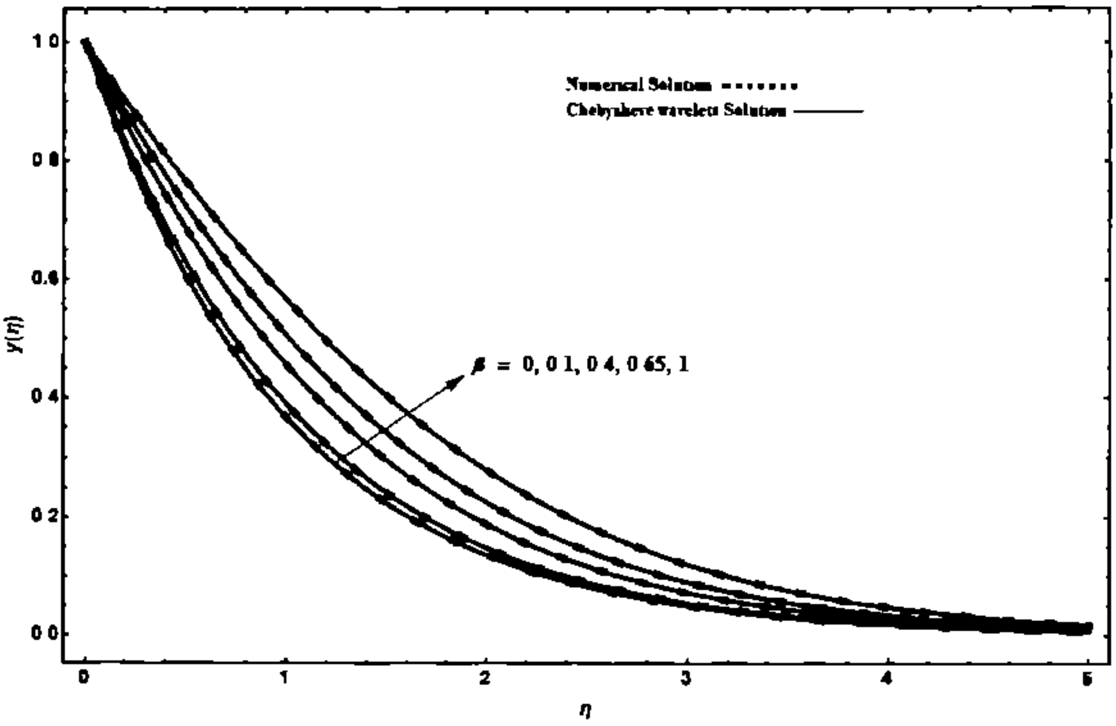

Fig. 3.1: Comparison of Chebyshev wavelets solution (k = 1, M = 12) with numerical solution in the interval [0, 5]

Table 3.1: Comparison of Chebyshev wavelets solution (k = 1, M = 12) with numerical solution and the absolute error is computed.

η	Chebyshev wavelets solution, k = 1, M = 12	Numerical Solution	Absolute Error
0.1	0.912765	0.912765	9.32343×10^{-9}
0.3	0.758897	0.758897	5.36144×10^{-9}
0.5	0.629429	0.629429	7.0725×10^{-9}
0.7	0.520953	0.520953	$3.40449\times10^{-}$[illegible]
0.9	0.4304	0.4304	1.63547×10^{-9}

3.2 The reaction diffusion Equation

The reaction diffusion equation with boundary conditions in nondimensional form is given by

$$\frac{d^2 y}{d\eta^2} - \beta^2 y^n(\eta) = 0, \text{ with } \left.\frac{dy}{d\eta}\right|_{\eta=0} = 0, \; y(1) = 1, \qquad (3\ 13)$$

where $\eta > 0$ substituting $y(\eta) = C^T\Psi(\eta)$ into (3 13), we get

$$C^T D^2\Psi(\eta) + \beta^2\left(C^T\Psi(\eta)\right)^n = 0 \qquad (3\ 14)$$

Now, for $k = 1$, $M = 3$, we can write

$$y(\eta) = \frac{\sqrt{2}c_{10}}{\sqrt{\pi}} + \frac{2(2\eta - 1)c_{11}}{\sqrt{\pi}} + \frac{2\left(8\eta^2 - 8\eta + 1\right)c_{12}}{\sqrt{\pi}} \qquad (3\ 15)$$

Substitution of (3 15) into (3 14) gives

$$\frac{32c_{12}}{\sqrt{\pi}} - \beta^2\left(\frac{16\eta^2 c_{12}}{\sqrt{\pi}} + \frac{4\eta c_{11}}{\sqrt{\pi}} - \frac{16\eta c_{12}}{\sqrt{\pi}} + \sqrt{\frac{2}{\pi}}c_{10} + \frac{2c_{12}}{\sqrt{\pi}} - \frac{2c_{11}}{\sqrt{\pi}}\right)^n = 0 \qquad (3\ 16)$$

The choice of collocation point $\eta = \frac{1}{2}$ yield the following equation

$$\frac{32c_{12}}{\sqrt{\pi}} - \beta^2\left(\sqrt{\frac{2}{\pi}}c_{10} - \frac{2c_{12}}{\sqrt{\pi}}\right)^n = 0 \qquad (3\ 17)$$

The boundary conditions $\left.\frac{dy}{d\eta}\right|_{\eta=0} = 0$ and $y(1) = 1$ yield

$$\frac{4\left(c_{11} - 4c_{12}\right)}{\sqrt{\pi}} = 0 \qquad (3\ 18)$$

$$\frac{\sqrt{2}c_{10} + 2\left(c_{11} + c_{12}\right)}{\sqrt{\pi}} = 1, \qquad (3\ 19)$$

Eqs (3 17-3 19) can be solved for c_{10}, c_{11} and c_{12} for any specific value of β and n Hence the solution is complete For $\beta = 1$ and $n = 2$

$$y(\eta) = 0\ 30020z^2 + 0\ 6998 \qquad (3\ 20)$$

A comparison of Chebyshev wavelets solution for corresponding numerical solution is presented in **Figs. 3.4-3.6** Again an excellent agreement is observed between both the solutions

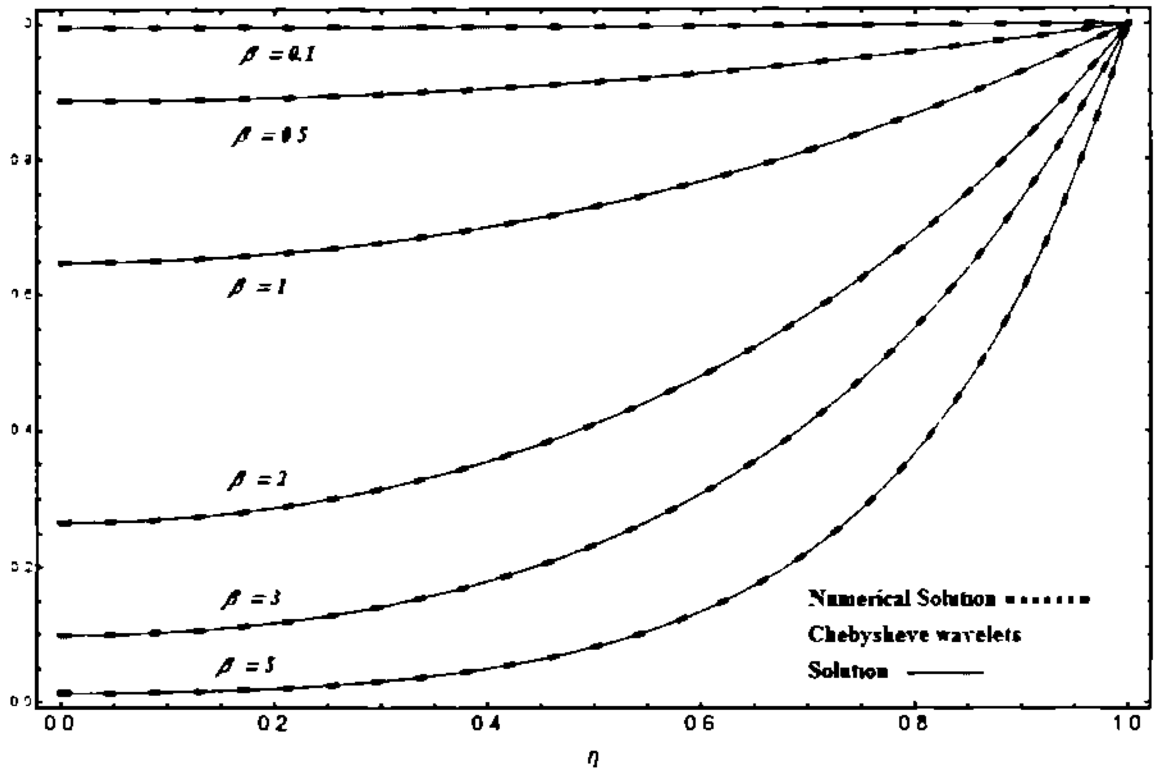

Fig. 3.4: Comparison of Chebyshev wavelets solution (k = 1, M = 12) with numerical solution for n=1

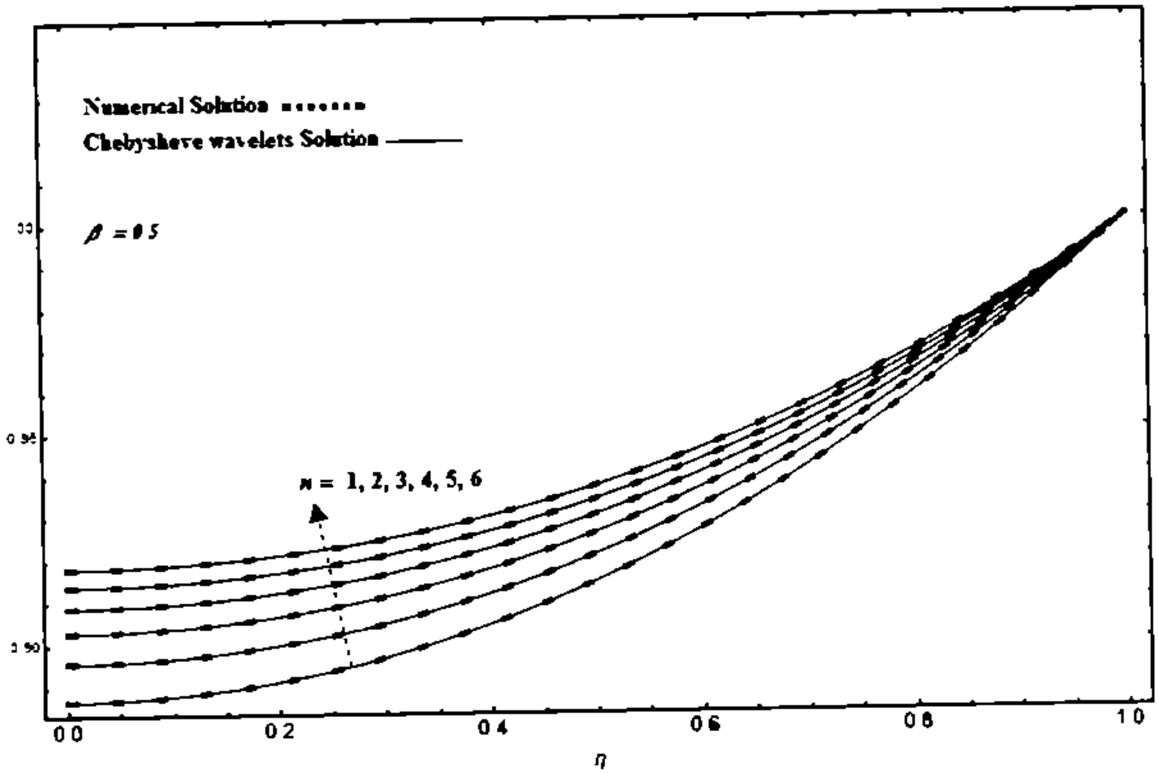

Fig. 3.5: Comparison of CWS (k = 1, M = 12) with numerical solution for β =0 5

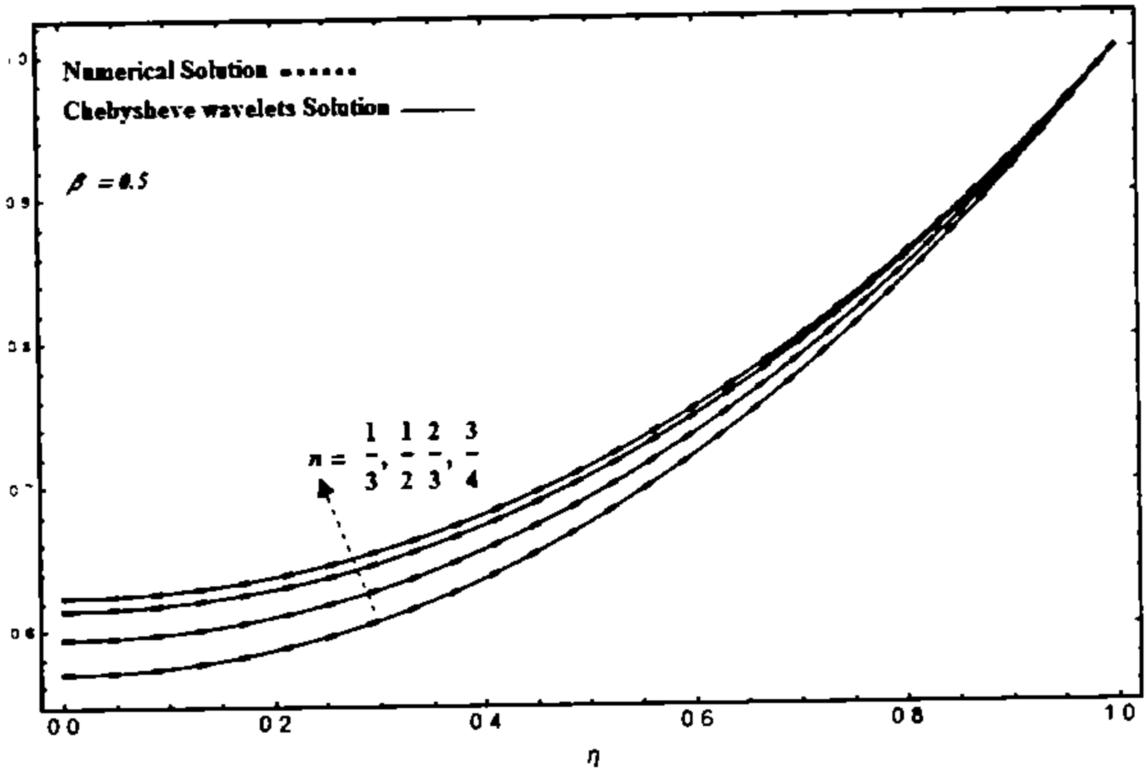

Fig. 3.6: Comparison of CWS (k = 1, M = 12) with numerical solution for β =0 5

3.3 Quadratic Riccati's equation

The quadratic Riccati's equation with imposed initial condition $y(0) = 0$ reads

$$\frac{dy}{d\eta} = 2y(\eta) - y^2(\eta) + 1 \tag{3.21}$$

An exact solution of this equation is

$$y(\eta) = 1 + \sqrt{2}\tanh\left[\sqrt{2}\eta + \frac{1}{2}\log\left[\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]\right] \tag{3.22}$$

The Chebyshev wavelets solution in the domain [0, 1] for $k = 1$, $M = 24$ is

$$\begin{aligned} y(\eta) = {} & 0.037874\eta^{23} - 0.39607\eta^{22} + 1.8708\eta^{21} - 5.270\eta^{20} + 9.882\eta^{19} \\ & - 13.123\eta^{18} + 12.917\eta^{17} - 9.736\eta^{16} + 5.619\eta^{15} - 2.4294\eta^{14} \\ & + 0.8706\eta^{13} - 0.22764\eta^{12} - 0.080051\eta^{11} - 0.08550\eta^{10} + 0.068311\eta^{9} \\ & + 0.22572\eta^{8} + 0.16820\eta^{7} - 0.15555\eta^{6} - 0.46667\eta^{5} - 0.33333\eta^{4} \\ & + 0.33333\eta^{3} + 1.0000\eta^{2} + 1.000\eta \end{aligned} \tag{3.23}$$

The above solution diverges for $\eta > 1$. To obtain the solution expression valid for $\eta > 1$. We first transform the interval $[0, \lambda], \lambda > 1$ into the interval [0, 1] by using the transformation $\xi = \frac{\eta}{\lambda}$ and then solve the transformed equation in the interval $\xi \in [0,1]$. Finally the solution obtained in this way is reverted to the original variable η. Following this process the solutions for $\lambda = 2, 3$ and 6 are given by

$\lambda = 2$,

$$\begin{aligned} y(\eta) = {} & -0.0019692\eta^{23} + 0.046603\eta^{22} - 0.5136\eta^{21} + 3.4973\eta^{20} \\ & - 16.458\eta^{19} + 56.73\eta^{18} - 148.13\eta^{17} + 299.09\eta^{16} - 472.79\eta^{15} \\ & + 589.7\eta^{14} - 583.1\eta^{13} + 458.50\eta^{12} - 286.57\eta^{11} + 141.49\eta^{10} \\ & - 54.81\eta^{9} + 16.695\eta^{8} - 3.5839\eta^{7} + 0.47586\eta^{6} - 0.5422\eta^{5} \\ & - 0.32728\eta^{4} + 0.33303\eta^{3} + 1.0000\eta^{2} + 1.000\eta \end{aligned} \tag{3.24}$$

$\lambda = 3$,

$$
\begin{aligned}
y(\eta) = & -4.9630\times10^{-6}\eta^{23} + 0.00020555\eta^{22} - 0.0039392\eta^{21} + 0.046470\eta^{20} \\
& - 0.37836\eta^{19} + 2.2580\eta^{18} - 10.232\eta^{17} + 35.970\eta^{16} - 99.33\eta^{15} \\
& + 216.80\eta^{14} - 374.35\eta^{13} + 509.8\eta^{12} - 544.4\eta^{11} + 452.24\eta^{10} \\
& - 290.08\eta^{9} + 142.11\eta^{8} - 51.78\eta^{7} + 13.718\eta^{6} - 3.0696\eta^{5} \\
& - 0.008837\eta^{4} + 0.30866\eta^{3} + 1.0010\eta^{2} + 1.000\eta
\end{aligned}
\tag{3.25}
$$

$\lambda = 4$,

$$
\begin{aligned}
y(\eta) = & \; 2.1039\times10^{-7}\eta^{23} - 0.000010480\eta^{22} + 0.00024473\eta^{21} - 0.0035581\eta^{20} \\
& + 0.036077\eta^{19} - 0.27076\eta^{18} + 1.5579\eta^{17} - 7.0217\eta^{16} + 25.110\eta^{15} \\
& - 71.714\eta^{14} + 163.80\eta^{13} - 298.29\eta^{12} + 429.85\eta^{11} - 484.56\eta^{10} + 421.15\eta^{9} \\
& - 277.93\eta^{8} + 137.24\eta^{7} - 49.272\eta^{6} + 11.867\eta^{5} - 2.3885\eta^{4} + 0.54205\eta^{3} \\
& + 0.98882\eta^{2} + 1.0002\eta
\end{aligned}
\tag{3.26}
$$

$\lambda = 5$,

$$
\begin{aligned}
y(\eta) = & \; 7.0724\times10^{-9}\eta^{23} - 3.9978\times10^{-7}\eta^{22} + 0.000010493\eta^{21} - 0.00016952\eta^{20} \\
& + 0.0018828\eta^{19} - 0.015195\eta^{18} + 0.091722\eta^{17} - 0.41867\eta^{16} + 1.4360\eta^{15} \\
& - 3.5748\eta^{14} + 5.7569\eta^{13} - 2.8403\eta^{12} - 13.799\eta^{11} + 46.868\eta^{10} - 81.241\eta^{9} \\
& + 91.175\eta^{8} - 69.091\eta^{7} + 35.974\eta^{6} - 13.130\eta^{5} + 2.5205\eta^{4} - 0.049059\eta^{3} \\
& + 1.0265\eta^{2} + 0.99928\eta
\end{aligned}
\tag{3.27}
$$

$\lambda = 6$,

$$
\begin{aligned}
y(\eta) = & -8.6109\times10^{-10}\eta^{23} + 6.1263\times10^{-8}\eta^{22} - 2.0402\times10^{-6}\eta^{21} \\
& + 0.000042252\eta^{20} - 0.00060957\eta^{19} + 0.0065044\eta^{18} \\
& - 0.053189\eta^{17} + 0.34076\eta^{16} - 1.7336\eta^{15} + 7.0564\eta^{14} \\
& - 23.037\eta^{13} + 60.205\eta^{12} - 125.16\eta^{11} + 204.70\eta^{10} - 259.07\eta^{9} \\
& + 248.30\eta^{8} - 175.87\eta^{7} + 90.270\eta^{6} - 33.001\eta^{5} + 7.4652\eta^{4} \\
& - 0.81571\eta^{3} + 1.0905\eta^{2} + 0.99710\eta
\end{aligned}
\tag{3.28}
$$

Graphical results shown in **Figs. 3.7-3.12** present a comparison of CWS with numerical solution in the intervals $[0,\lambda], \lambda = 2,3,4,5,6$. Each figure clearly demonstrates an excellent agreement between both solutions.

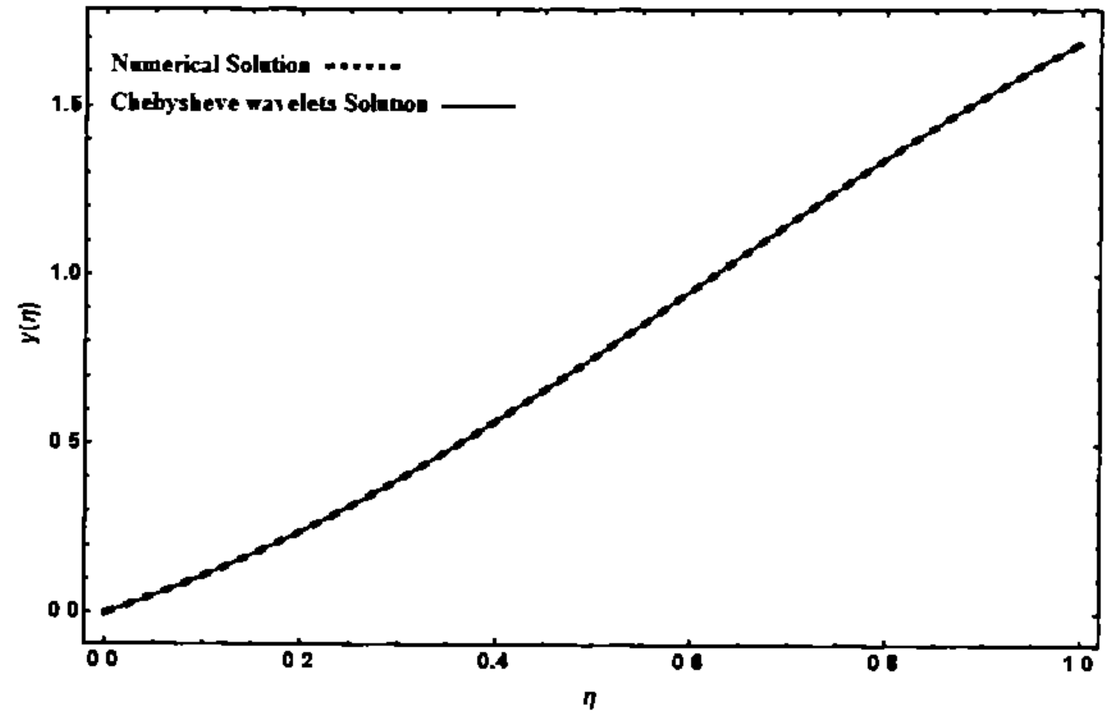

Fig. 3.7: Comparison of CWS ($k = 1, M = 24$) with numerical solution in the interval [0, 1]

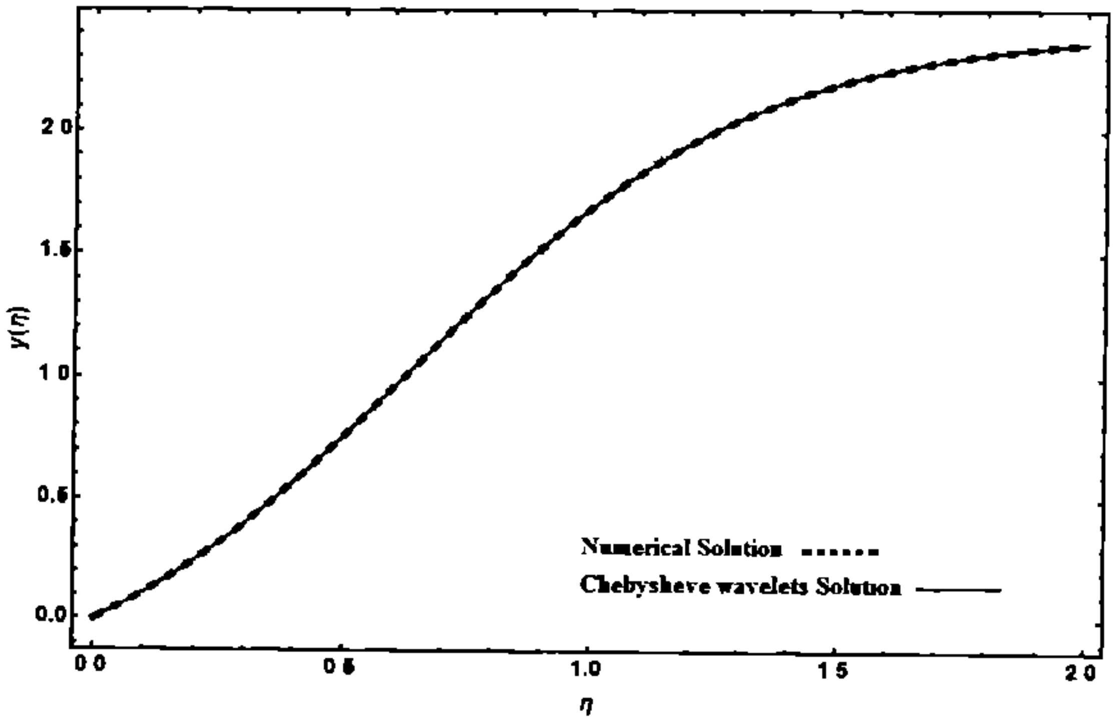

Fig. 3.8: Comparison of CWS ($k = 1, M = 24$) with numerical solution in the interval [0, 2]

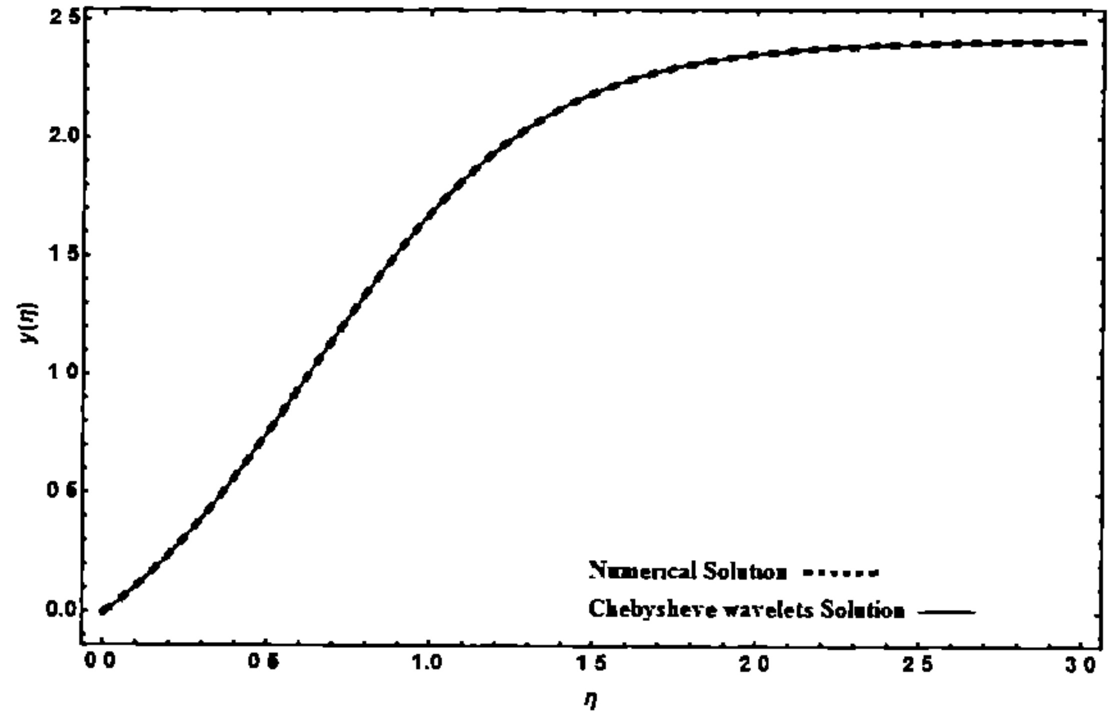

Fig. 3.9: Comparison of CWS ($k = 1$, $M = 24$) with numerical solution in the interval [0, 3]

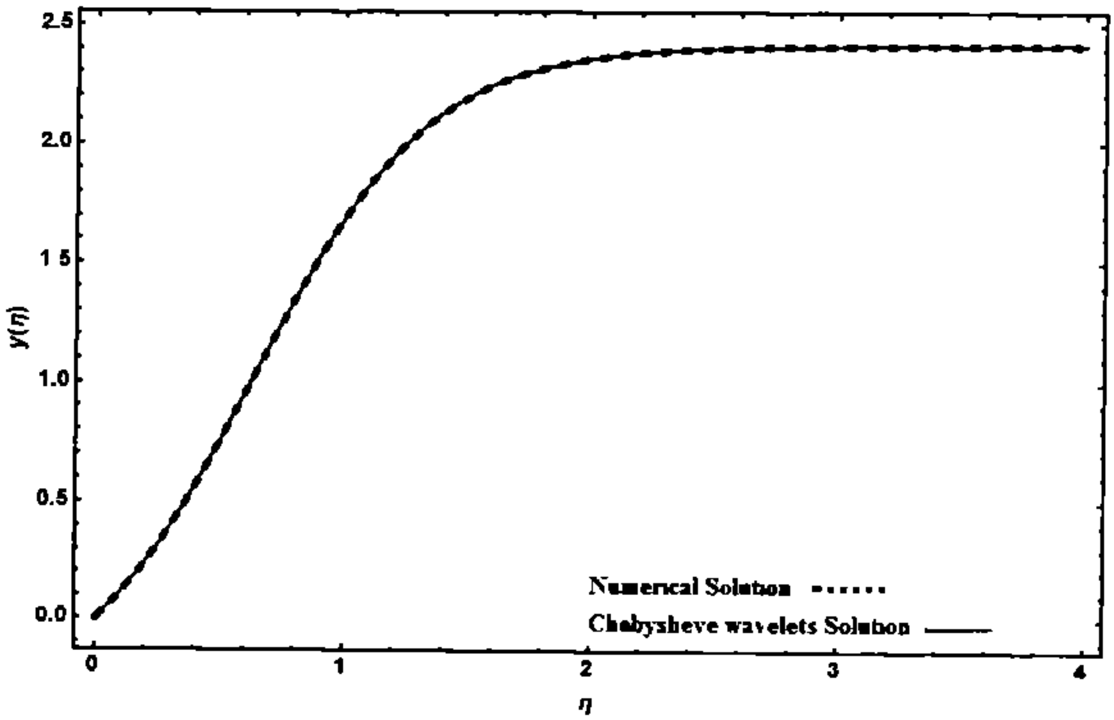

Fig. 3.10: Comparison of CWS ($k = 1$, $M = 24$) with numerical solution in the interval [0, 4]

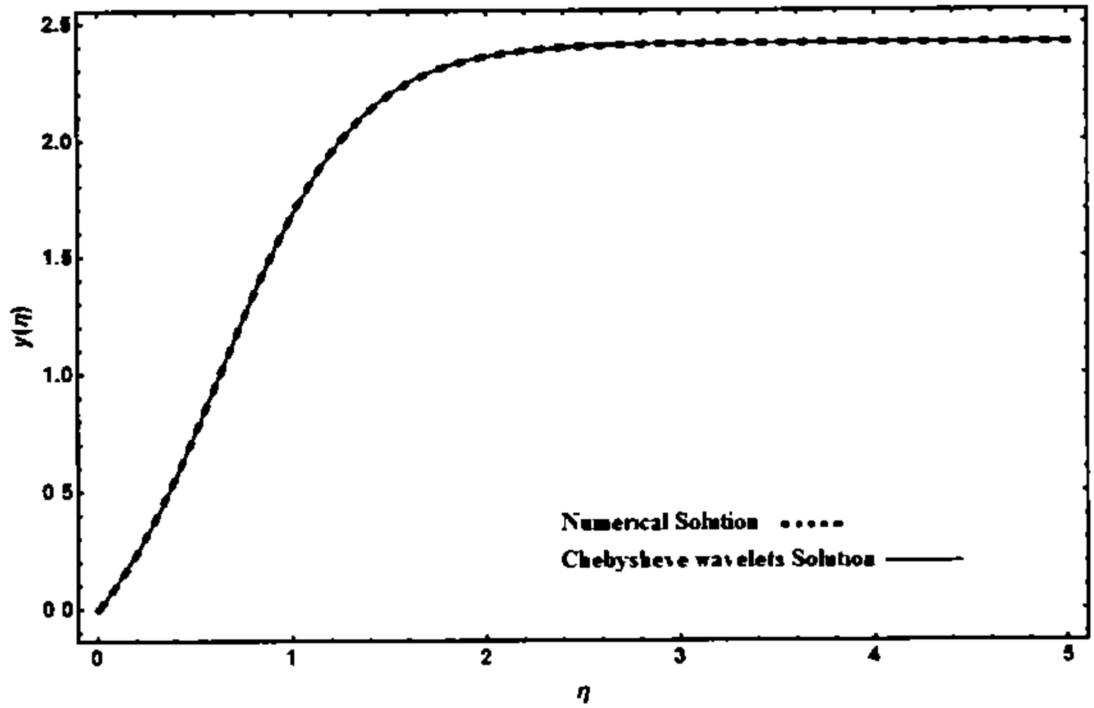

Fig. 3.11: Comparison of CWS ($k = 1$, $M = 24$) with numerical solution in the interval [0, 5]

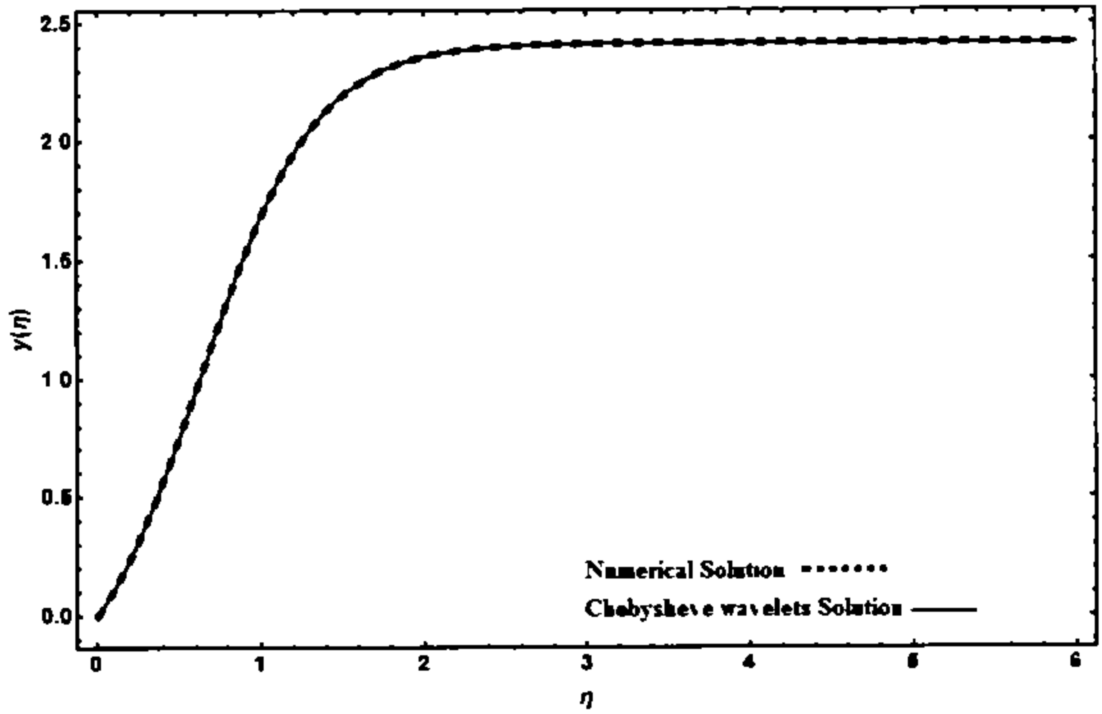

Fig. 3.12: Comparison of CWS ($k = 1$, $M = 24$) with numerical solution in the interval [0, 6]

3.4 Thin film flow of a third grade fluid

Here, we shall solve a nonlinear differential equation which represent the phenomena of thin film flow of a third grade fluid down an inclined plane [34]. The dimensionaless form of the problem is represented by the following nonlinear differential equation and boundary conditions

$$\frac{d^2y}{d\eta^2}+6\beta\left(\frac{dy}{d\eta}\right)^2\frac{d^2y}{d\eta^2}+1=0,\ \text{with}\ y(0)=0,\ y'(1)+2\beta\left(y'(1)\right)^3=0 \qquad (3.29)$$

Following the procedure of Chebyshev wavelets method, the solution for $k = 1$, $M = 6$ is shown for several values of β in **Fig. 3.13**

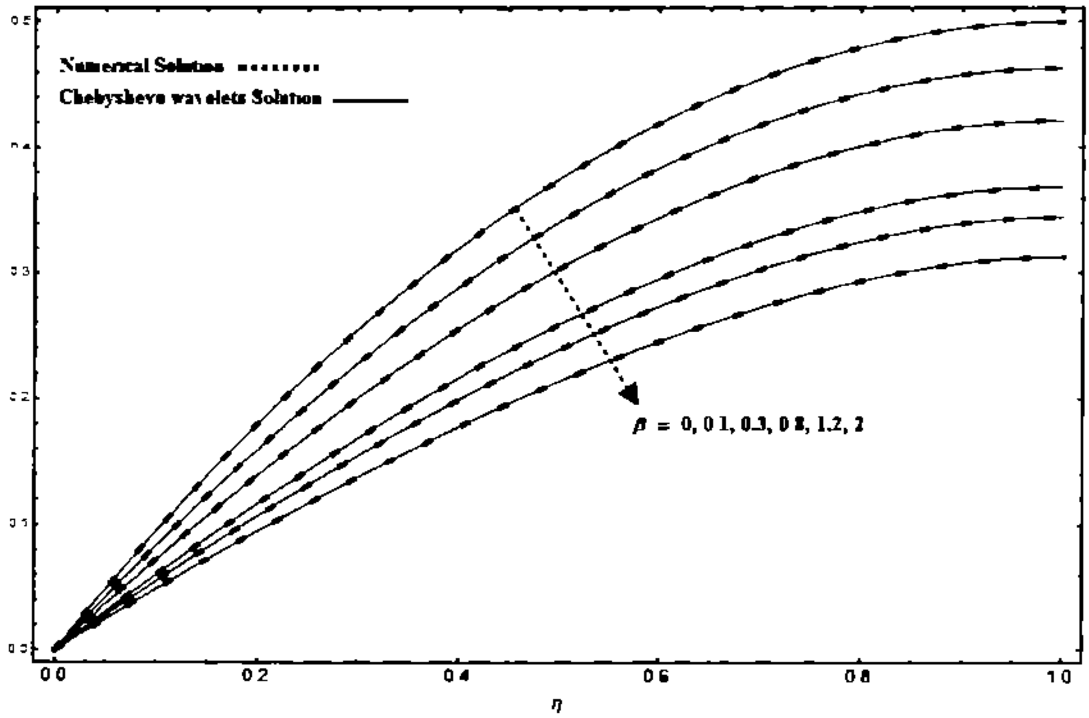

Fig. 3.13: Comparison of CWS (k = 1, M = 6) with numerical solution

3.5 Sixth order nonlinear boundary value problem

Consider the following nonlinear boundary value problem

$$\frac{d^6 y}{d\eta^6} = e^{-\eta} y^2(\eta), \qquad 0 < \eta < 1, \tag{3 30}$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = 1, \; y(1) = y''(1) = y^{(iv)}(1) = e \tag{3 31}$$

The above equation admits the exact solution e^{η}

To employ Chebyshev wavelet method to the sixth order boundary value problem, we first transform the Eq (3 30) into second order simultaneous differential equations To this end, we use the following transformations·

$$\frac{d^2 y}{d\eta^2} = q(\eta), \qquad \frac{d^2 q}{d\eta^2} = r(\eta), \tag{3 32}$$

The above transformations yield the following set of simultaneous differential equations

$$\begin{cases} \dfrac{d^2 y}{d\eta^2} = q(\eta) & y(0) = 1, y(1) = e, \\ \dfrac{d^2 q}{d\eta^2} = r(\eta) & q(0) = 1, q(1) = e, \\ \dfrac{d^2 r}{d\eta^2} = e^{-\eta} y^2(\eta) & r(0) = 1, r(1) = e \end{cases} \tag{3 33}$$

Now, we substitute $y(\eta) = C^T \psi(\eta)$, $q(\eta) = E^T \psi(\eta)$ and $r(\eta) = F^T \psi(\eta)$ in (3 32) and get

$$\begin{cases} C^T D^2 \Psi(\eta) = E^T \Psi(\eta) & C^T \Psi(0) = 1, C^T \Psi(1) = e, \\ E^T D^2 \Psi(\eta) = F^T \Psi(\eta) & E^T \Psi(0) = 1, E^T \Psi(1) = e, \\ F^T D^2 \Psi(\eta) = e^{-\eta} \left(C^T \Psi(\eta)\right)^2 & F^T \Psi(0) = 1, F^T \Psi(1) = e \end{cases} \tag{3 34}$$

A typical choice $k=1, M=12$ yields

$$
\begin{aligned}
y(\eta) &= c_{10}\psi_{10}(\eta)+c_{11}\psi_{11}(\eta)+c_{12}\psi_{12}(\eta)+\ldots+c_{111}\psi_{111}(\eta)\\
&=\sqrt{\frac{2}{\pi}}c_{10}+\frac{4\eta c_{11}}{\sqrt{\pi}}-\frac{2c_{11}}{\sqrt{\pi}}+\frac{16\eta^2 c_{12}}{\sqrt{\pi}}-\frac{16\eta c_{12}}{\sqrt{\pi}}+\frac{2c_{12}}{\sqrt{\pi}}+\frac{64\eta^3 c_{13}}{\sqrt{\pi}}-\frac{96\eta^2 c_{13}}{\sqrt{\pi}}+\frac{36\eta c_{13}}{\sqrt{\pi}},\\
&-\frac{2c_{13}}{\sqrt{\pi}}+\frac{256\eta^4 c_{14}}{\sqrt{\pi}}-\frac{512\eta^3 c_{14}}{\sqrt{\pi}}+\frac{320\eta^2 c_{14}}{\sqrt{\pi}}-\frac{64\eta c_{14}}{\sqrt{\pi}}+\frac{2c_{14}}{\sqrt{\pi}}+\frac{1024\eta^5 c_{15}}{\sqrt{\pi}}\\
&-\frac{2560\eta^4 c_{15}}{\sqrt{\pi}}+\frac{2240\eta^3 c_{15}}{\sqrt{\pi}}-\frac{800\eta^2 c_{15}}{\sqrt{\pi}}+\frac{100\eta c_{15}}{\sqrt{\pi}}-\frac{2c_{15}}{\sqrt{\pi}}+\frac{4096\eta^6 c_{16}}{\sqrt{\pi}}-\frac{12288\eta^5 c_{16}}{\sqrt{\pi}}\\
&+\frac{13824\eta^4 c_{16}}{\sqrt{\pi}}-\frac{7168\eta^3 c_{16}}{\sqrt{\pi}}+\frac{1680\eta^2 c_{16}}{\sqrt{\pi}}-\frac{144\eta c_{16}}{\sqrt{\pi}}+\frac{2c_{16}}{\sqrt{\pi}}+\frac{16384\eta^7 c_{17}}{\sqrt{\pi}}\\
&-\frac{57344\eta^6 c_{17}}{\sqrt{\pi}}+\frac{78848\eta^5 c_{17}}{\sqrt{\pi}}-\frac{53760\eta^4 c_{17}}{\sqrt{\pi}}+\frac{18816\eta^3 c_{17}}{\sqrt{\pi}}-\frac{3136\eta^2 c_{17}}{\sqrt{\pi}}+\frac{196\eta c_{17}}{\sqrt{\pi}}\\
&-\frac{2c_{17}}{\sqrt{\pi}}+\frac{65536\eta^8 c_{18}}{\sqrt{\pi}}-\frac{262144\eta^7 c_{18}}{\sqrt{\pi}}+\frac{425984\eta^6 c_{18}}{\sqrt{\pi}}-\frac{360448\eta^5 c_{18}}{\sqrt{\pi}}+\frac{168960\eta^4 c_{18}}{\sqrt{\pi}}\\
&-\frac{43008\eta^3 c_{18}}{\sqrt{\pi}}+\frac{5376\eta^2 c_{18}}{\sqrt{\pi}}-\frac{256\eta c_{18}}{\sqrt{\pi}}+\frac{2c_{18}}{\sqrt{\pi}}+\frac{262144\eta^9 c_{19}}{\sqrt{\pi}}-\frac{1179648\eta^8 c_{19}}{\sqrt{\pi}}\\
&+\frac{2211840\eta^7 c_{19}}{\sqrt{\pi}}-\frac{2236416\eta^6 c_{19}}{\sqrt{\pi}}+\frac{1317888\eta^5 c_{19}}{\sqrt{\pi}}-\frac{456192\eta^4 c_{19}}{\sqrt{\pi}}+\frac{88704\eta^3 c_{19}}{\sqrt{\pi}}\\
&-\frac{8640\eta^2 c_{19}}{\sqrt{\pi}}+\frac{324\eta c_{19}}{\sqrt{\pi}}-\frac{2c_{19}}{\sqrt{\pi}}+\frac{2c_{110}}{\sqrt{\pi}}+\frac{1048576\eta^{10} c_{110}}{\sqrt{\pi}}-\frac{5242880\eta^9 c_{110}}{\sqrt{\pi}}\\
&+\frac{11141120\eta^8 c_{110}}{\sqrt{\pi}}-\frac{13107200\eta^7 c_{110}}{\sqrt{\pi}}+\frac{9318400\eta^6 c_{110}}{\sqrt{\pi}}-\frac{4100096\eta^5 c_{110}}{\sqrt{\pi}}\\
&+\frac{1098240\eta^4 c_{110}}{\sqrt{\pi}}-\frac{168960\eta^3 c_{110}}{\sqrt{\pi}}+\frac{13200\eta^2 c_{110}}{\sqrt{\pi}}-\frac{400\eta c_{110}}{\sqrt{\pi}}+\frac{4194304\eta^{11} c_{111}}{\sqrt{\pi}}\\
&-\frac{23068672\eta^{10} c_{111}}{\sqrt{\pi}}+\frac{54788096\eta^9 c_{111}}{\sqrt{\pi}}-\frac{73531392\eta^8 c_{111}}{\sqrt{\pi}}+\frac{61276160\eta^7 c_{111}}{\sqrt{\pi}}\\
&-\frac{32800768\eta^6 c_{111}}{\sqrt{\pi}}+\frac{11275264\eta^5 c_{111}}{\sqrt{\pi}}-\frac{2416128\eta^4 c_{111}}{\sqrt{\pi}}+\frac{302016\eta^3 c_{111}}{\sqrt{\pi}}\\
&-\frac{19360\eta^2 c_{111}}{\sqrt{\pi}}+\frac{484\eta c_{111}}{\sqrt{\pi}}-\frac{2c_{111}}{\sqrt{\pi}},
\end{aligned}
\tag{3.35}
$$

$$q(\eta)=e_{10}\psi_{10}(\eta)+e_{11}\psi_{11}(\eta)+e_{12}\psi_{12}(\eta)+\ldots+e_{111}\psi_{111}(\eta)$$
$$=\sqrt{\frac{2}{\pi}}e_{10}+\frac{4\eta e_{11}}{\sqrt{\pi}}-\frac{2e_{11}}{\sqrt{\pi}}+\frac{16\eta^2 e_{12}}{\sqrt{\pi}}+\quad -\frac{19360\eta^2 e_{111}}{\sqrt{\pi}}+\frac{484\eta e_{111}}{\sqrt{\pi}}-\frac{2e_{111}}{\sqrt{\pi}}. \quad (3\ 36)$$

$$r(\eta)=f_{10}\psi_{10}(\eta)+f_{11}\psi_{11}(\eta)+f_{12}\psi_{12}(\eta)+\ldots+f_{111}\psi_{111}(\eta)$$
$$=\sqrt{\frac{2}{\pi}}f_{10}+\frac{4\eta f_{11}}{\sqrt{\pi}}-\frac{2f_{11}}{\sqrt{\pi}}+\frac{16\eta^2 f_{12}}{\sqrt{\pi}}+\quad -\frac{19360\eta^2 f_{111}}{\sqrt{\pi}}+\frac{484\eta f_{111}}{\sqrt{\pi}}-\frac{2f_{111}}{\sqrt{\pi}} \quad (3\ 37)$$

Substituting Eqs (3 35)-(3 37) into (3 34), one gets

$$\frac{20480\eta^3 c_{15}}{\sqrt{\pi}}-\frac{245760\eta^3 c_{16}}{\sqrt{\pi}}+\frac{3072\eta^2 c_{14}}{\sqrt{\pi}}+\frac{165888\eta^2 c_{16}}{\sqrt{\pi}}-\frac{30720\eta^2 c_{15}}{\sqrt{\pi}}+\frac{384\eta c_{13}}{\sqrt{\pi}}$$

$$+\frac{13440\eta c_{15}}{\sqrt{\pi}}-\frac{3072\eta c_{14}}{\sqrt{\pi}}-\frac{43008\eta c_{16}}{\sqrt{\pi}}+\frac{32c_{12}}{\sqrt{\pi}}+\frac{640c_{14}}{\sqrt{\pi}}+\frac{3360c_{16}}{\sqrt{\pi}}-\frac{192c_{13}}{\sqrt{\pi}}$$

$$-\frac{1600c_{15}}{\sqrt{\pi}}+\frac{3670016\eta^6 c_{18}}{\sqrt{\pi}}+\frac{688128\eta^5 c_{17}}{\sqrt{\pi}}-\frac{11010048\eta^5 c_{18}}{\sqrt{\pi}}+\frac{122880\eta^4 c_{16}}{\sqrt{\pi}}$$
$$+\frac{12779520\eta^4 c_{18}}{\sqrt{\pi}}-\frac{1720320\eta^4 c_{17}}{\sqrt{\pi}}+\frac{1576960\eta^3 c_{17}}{\sqrt{\pi}}+\frac{26357760\eta^3 c_{19}}{\sqrt{\pi}}$$

$$-\frac{7208960\eta^3 c_{18}}{\sqrt{\pi}}+\frac{2027520\eta^2 c_{18}}{\sqrt{\pi}}-\frac{645120\eta^2 c_{17}}{\sqrt{\pi}}-\frac{5474304\eta^2 c_{19}}{\sqrt{\pi}}+\frac{112896\eta c_{17}}{\sqrt{\pi}}$$
$$+\frac{532224\eta c_{19}}{\sqrt{\pi}}-\frac{258048\eta c_{18}}{\sqrt{\pi}}+\frac{10752c_{18}}{\sqrt{\pi}}-\frac{6272c_{17}}{\sqrt{\pi}}-\frac{17280c_{19}}{\sqrt{\pi}}+\frac{94371840\eta^8 c_{110}}{\sqrt{\pi}}$$

$$+\frac{188743368\eta^7 c_{19}}{\sqrt{\pi}}-\frac{377487360\eta^7 c_{110}}{\sqrt{\pi}}+\frac{623902720\eta^6 c_{110}}{\sqrt{\pi}}-\frac{66060288\eta^6 c_{19}}{\sqrt{\pi}}$$
$$+\frac{92897280\eta^5 c_{19}}{\sqrt{\pi}}-\frac{550502400\eta^5 c_{110}}{\sqrt{\pi}}-\frac{67092480\eta^4 c_{19}}{\sqrt{\pi}}+\frac{279552000\eta^4 c_{110}}{\sqrt{\pi}}$$
$$-\frac{82001920\eta^3 c_{110}}{\sqrt{\pi}}+\frac{13178880\eta^2 c_{110}}{\sqrt{\pi}}-\frac{1013760\eta c_{110}}{\sqrt{\pi}}+\frac{26400c_{110}}{\sqrt{\pi}}$$

$$+\frac{461373440\eta^9 c_{111}}{\sqrt{\pi}}-\frac{2076180480\eta^8 c_{111}}{\sqrt{\pi}}+\frac{3944742912\eta^7 c_{111}}{\sqrt{\pi}}-\frac{4117757952\eta^6 c_{111}}{\sqrt{\pi}}$$

$$+\frac{2573598720\eta^5 c_{111}}{\sqrt{\pi}}-\frac{984023040\eta^4 c_{111}}{\sqrt{\pi}}+\frac{225505280\eta^3 c_{111}}{\sqrt{\pi}}-\frac{28993536\eta^2 c_{111}}{\sqrt{\pi}}$$

$$+\frac{1812096\eta c_{111}}{\sqrt{\pi}}-\frac{38720 c_{111}}{\sqrt{\pi}}$$

$$=\frac{96\eta^2 e_{13}}{\sqrt{\pi}}-\frac{64\eta^3 e_{13}}{\sqrt{\pi}}-\frac{16\eta^2 e_{12}}{\sqrt{\pi}}+\frac{16\eta e_{12}}{\sqrt{\pi}}-\frac{4\eta e_{11}}{\sqrt{\pi}}-\frac{36\eta e_{13}}{\sqrt{\pi}}-\sqrt{\frac{2}{\pi}}e_{10}+\frac{2e_{11}}{\sqrt{\pi}}+\frac{2e_{13}}{\sqrt{\pi}}$$

$$-\frac{2e_{12}}{\sqrt{\pi}}-\frac{4096\eta^6 e_{16}}{\sqrt{\pi}}+\frac{12288\eta^5 e_{16}}{\sqrt{\pi}}-\frac{1024\eta^5 e_{15}}{\sqrt{\pi}}+\frac{2560\eta^4 e_{15}}{\sqrt{\pi}}-\frac{256\eta^4 e_{14}}{\sqrt{\pi}}+\frac{64\eta e_{14}}{\sqrt{\pi}}$$

$$-\frac{13824\eta^4 e_{16}}{\sqrt{\pi}}+\frac{512\eta^3 e_{14}}{\sqrt{\pi}}+\frac{7168\eta^3 e_{16}}{\sqrt{\pi}}-\frac{2240\eta^3 e_{15}}{\sqrt{\pi}}+\frac{800\eta^2 e_{15}}{\sqrt{\pi}}-\frac{320\eta^2 e_{14}}{\sqrt{\pi}}-\frac{1680\eta^2 e_{16}}{\sqrt{\pi}}$$

$$+\frac{144\eta e_{16}}{\sqrt{\pi}}-\frac{100\eta e_{15}}{\sqrt{\pi}}+\frac{2e_{15}}{\sqrt{\pi}}-\frac{2e_{14}}{\sqrt{\pi}}-\frac{2e_{16}}{\sqrt{\pi}}-\frac{65536\eta^8 e_{18}}{\sqrt{\pi}}+\frac{262144\eta^7 e_{18}}{\sqrt{\pi}}-\frac{16384\eta^7 e_{17}}{\sqrt{\pi}}$$

$$+\frac{57344\eta^6 e_{17}}{\sqrt{\pi}}-\frac{425984\eta^6 e_{18}}{\sqrt{\pi}}+\frac{360448\eta^5 e_{18}}{\sqrt{\pi}}-\frac{78848\eta^5 e_{17}}{\sqrt{\pi}}+\frac{53760\eta^4 e_{17}}{\sqrt{\pi}}-\frac{168960\eta^4 e_{18}}{\sqrt{\pi}}$$

$$+\frac{43008\eta^3 e_{18}}{\sqrt{\pi}}-\frac{18816\eta^3 e_{17}}{\sqrt{\pi}}+\frac{3136\eta^2 e_{17}}{\sqrt{\pi}}-\frac{5376\eta^2 e_{18}}{\sqrt{\pi}}+\frac{256\eta e_{18}}{\sqrt{\pi}}-\frac{196\eta e_{17}}{\sqrt{\pi}}+\frac{2e_{17}}{\sqrt{\pi}}$$

$$-\frac{2e_{18}}{\sqrt{\pi}}-\frac{262144\eta^9 e_{19}}{\sqrt{\pi}}+\frac{1179648\eta^8 e_{19}}{\sqrt{\pi}}-\frac{2211840\eta^7 e_{19}}{\sqrt{\pi}}+\frac{2236416\eta^6 e_{19}}{\sqrt{\pi}}-\frac{1317888\eta^5 e_{19}}{\sqrt{\pi}}$$

$$+\frac{456192\eta^4 e_{19}}{\sqrt{\pi}}-\frac{88704\eta^3 e_{19}}{\sqrt{\pi}}+\frac{8640\eta^2 e_{19}}{\sqrt{\pi}}-\frac{324\eta e_{19}}{\sqrt{\pi}}+\frac{2e_{19}}{\sqrt{\pi}}-\frac{1048576\eta^{10} e_{110}}{\sqrt{\pi}}$$

$$+\frac{5242880\eta^9 e_{110}}{\sqrt{\pi}}-\frac{11141120\eta^8 e_{110}}{\sqrt{\pi}}+\frac{13107200\eta^7 e_{110}}{\sqrt{\pi}}-\frac{9318400\eta^6 e_{110}}{\sqrt{\pi}}+\frac{4100096\eta^5 e_{110}}{\sqrt{\pi}}$$

$$-\frac{1098240\eta^4 e_{110}}{\sqrt{\pi}}+\frac{168960\eta^3 e_{110}}{\sqrt{\pi}}-\frac{13200\eta^2 e_{110}}{\sqrt{\pi}}+\frac{400\eta e_{110}}{\sqrt{\pi}}-\frac{2e_{110}}{\sqrt{\pi}}-\frac{4194304\eta^{11} e_{111}}{\sqrt{\pi}}$$

$$+\frac{23068672\eta^{10} e_{111}}{\sqrt{\pi}}-\frac{54788096\eta^9 e_{111}}{\sqrt{\pi}}+\frac{73531392\eta^8 e_{111}}{\sqrt{\pi}}-\frac{61276160\eta^7 e_{111}}{\sqrt{\pi}}$$

$$+\frac{32800768\eta^6 e_{111}}{\sqrt{\pi}}-\frac{11275264\eta^5 e_{111}}{\sqrt{\pi}}+\frac{2416128\eta^4 e_{111}}{\sqrt{\pi}}-\frac{302016\eta^3 e_{111}}{\sqrt{\pi}}$$
$$+\frac{19360\eta^2 e_{111}}{\sqrt{\pi}}-\frac{484\eta e_{111}}{\sqrt{\pi}}+\frac{2e_{111}}{\sqrt{\pi}}, \quad (3\ 38)$$

$$\frac{20480\eta^3 e_{15}}{\sqrt{\pi}}+\frac{3072\eta^2 e_{14}}{\sqrt{\pi}}-\frac{30720\eta^2 e_{15}}{\sqrt{\pi}}+\frac{384\eta e_{13}}{\sqrt{\pi}}+\frac{13440\eta e_{15}}{\sqrt{\pi}}-\frac{3072\eta e_{14}}{\sqrt{\pi}}$$

$$+\frac{32e_{12}}{\sqrt{\pi}}+\frac{640e_{14}}{\sqrt{\pi}}-\frac{192e_{13}}{\sqrt{\pi}}-\frac{1600e_{15}}{\sqrt{\pi}}+\quad+\frac{461373440\eta^9 e_{111}}{\sqrt{\pi}}-\frac{2076180480\eta^8 e_{111}}{\sqrt{\pi}}$$
$$+\frac{3944742912\eta^7 e_{111}}{\sqrt{\pi}}-\frac{4117757952\eta^6 e_{111}}{\sqrt{\pi}}+\frac{2573598720\eta^5 e_{111}}{\sqrt{\pi}}$$

$$=-\sqrt{\frac{2}{\pi}}f_{10}+\frac{2f_{11}}{\sqrt{\pi}}+\frac{2f_{13}}{\sqrt{\pi}}-\frac{2f_{12}}{\sqrt{\pi}}-\frac{64\eta^3 f_{13}}{\sqrt{\pi}}+\frac{96\eta^2 f_{13}}{\sqrt{\pi}}-\frac{16\eta^2 f_{12}}{\sqrt{\pi}}+\frac{16\eta f_{12}}{\sqrt{\pi}}-\frac{4\eta f_{11}}{\sqrt{\pi}}$$
$$-\frac{36\eta f_{13}}{\sqrt{\pi}}+\quad-\frac{4194304\eta^{11} f_{111}}{\sqrt{\pi}}+\frac{23068672\eta^{10} f_{111}}{\sqrt{\pi}}$$

$$-\frac{54788096\eta^9 f_{111}}{\sqrt{\pi}}++\frac{73531392\eta^8 f_{111}}{\sqrt{\pi}}, \quad (3\ 39)$$

$$\frac{3072\eta^2 f_{14}}{\sqrt{\pi}}+\frac{384\eta f_{13}}{\sqrt{\pi}}-\frac{3072\eta f_{14}}{\sqrt{\pi}}+\frac{32f_{12}}{\sqrt{\pi}}+\frac{640f_{14}}{\sqrt{\pi}}-\frac{192f_{13}}{\sqrt{\pi}}+\quad+\frac{461373440\eta^9 f_{111}}{\sqrt{\pi}}$$

$$-\frac{2076180480\eta^8 f_{111}}{\sqrt{\pi}}+\frac{3944742912\eta^7 f_{111}}{\sqrt{\pi}}$$

$$=e^{-\eta}\left(\sqrt{\frac{2}{\pi}}c_{10}-\frac{2c_{11}}{\sqrt{\pi}}+\frac{4\eta c_{11}}{\sqrt{\pi}}+\frac{2c_{12}}{\sqrt{\pi}}-\frac{16\eta c_{12}}{\sqrt{\pi}}+\quad+\frac{4194304\eta^{11}c_{111}}{\sqrt{\pi}}\right.$$

$$\left.-\frac{23068672\eta^{10}c_{111}}{\sqrt{\pi}}++\frac{54788096\eta^9 c_{111}}{\sqrt{\pi}}\right)^2, \quad (3\ 40)$$

$$\sqrt{\frac{2}{\pi}}c_{10}+\frac{2c_{12}}{\sqrt{\pi}}+\frac{2c_{14}}{\sqrt{\pi}}+\frac{2c_{16}}{\sqrt{\pi}}+\frac{2c_{18}}{\sqrt{\pi}}+\frac{2c_{110}}{\sqrt{\pi}}-\frac{2c_{11}}{\sqrt{\pi}}$$
$$-\frac{2c_{13}}{\sqrt{\pi}}-\frac{2c_{15}}{\sqrt{\pi}}-\frac{2c_{17}}{\sqrt{\pi}}-\frac{2c_{19}}{\sqrt{\pi}}-\frac{2c_{111}}{\sqrt{\pi}}=1, \quad (3\ 41)$$

$$\sqrt{\frac{2}{\pi}}c_{10}+\frac{2c_{11}}{\sqrt{\pi}}+\frac{2c_{12}}{\sqrt{\pi}}+\frac{2c_{13}}{\sqrt{\pi}}+\frac{2c_{14}}{\sqrt{\pi}}+\frac{2c_{15}}{\sqrt{\pi}}+\frac{2c_{16}}{\sqrt{\pi}}$$
$$+\frac{2c_{17}}{\sqrt{\pi}}+\frac{2c_{18}}{\sqrt{\pi}}+\frac{2c_{19}}{\sqrt{\pi}}+\frac{2c_{110}}{\sqrt{\pi}}+\frac{2c_{111}}{\sqrt{\pi}}=e, \quad (3\ 42)$$

$$\sqrt{\frac{2}{\pi}}e_{10}+\frac{2e_{12}}{\sqrt{\pi}}+\frac{2e_{14}}{\sqrt{\pi}}+\frac{2e_{16}}{\sqrt{\pi}}+\frac{2e_{18}}{\sqrt{\pi}}-\frac{2e_{11}}{\sqrt{\pi}}-\frac{2e_{13}}{\sqrt{\pi}}$$
$$-\frac{2e_{15}}{\sqrt{\pi}}-\frac{2e_{17}}{\sqrt{\pi}}-\frac{2e_{19}}{\sqrt{\pi}}+\frac{2e_{110}}{\sqrt{\pi}}-\frac{2e_{111}}{\sqrt{\pi}}=1, \quad (3\ 43)$$

$$\sqrt{\frac{2}{\pi}}e_{10}+\frac{2e_{11}}{\sqrt{\pi}}+\frac{2e_{12}}{\sqrt{\pi}}+\frac{2e_{13}}{\sqrt{\pi}}+\frac{2e_{14}}{\sqrt{\pi}}+\frac{2e_{15}}{\sqrt{\pi}}+\frac{2e_{16}}{\sqrt{\pi}}$$
$$+\frac{2e_{17}}{\sqrt{\pi}}+\frac{2e_{18}}{\sqrt{\pi}}+\frac{2e_{19}}{\sqrt{\pi}}+\frac{2e_{110}}{\sqrt{\pi}}+\frac{2e_{111}}{\sqrt{\pi}}=e, \quad (3\ 44)$$

$$\sqrt{\frac{2}{\pi}}f_{10}+\frac{2f_{12}}{\sqrt{\pi}}+\frac{2f_{14}}{\sqrt{\pi}}+\frac{2f_{16}}{\sqrt{\pi}}+\frac{2f_{18}}{\sqrt{\pi}}-\frac{2f_{11}}{\sqrt{\pi}}-\frac{2f_{13}}{\sqrt{\pi}}$$
$$-\frac{2f_{15}}{\sqrt{\pi}}-\frac{2f_{17}}{\sqrt{\pi}}-\frac{2f_{19}}{\sqrt{\pi}}+\frac{2f_{110}}{\sqrt{\pi}}-\frac{2f_{111}}{\sqrt{\pi}}=1, \quad (3\ 45)$$

$$\sqrt{\frac{2}{\pi}}f_{10}+\frac{2f_{11}}{\sqrt{\pi}}+\frac{2f_{12}}{\sqrt{\pi}}+\frac{2f_{13}}{\sqrt{\pi}}+\frac{2f_{14}}{\sqrt{\pi}}+\frac{2f_{15}}{\sqrt{\pi}}+\frac{2f_{16}}{\sqrt{\pi}}$$
$$+\frac{2f_{17}}{\sqrt{\pi}}+\frac{2f_{18}}{\sqrt{\pi}}+\frac{2f_{19}}{\sqrt{\pi}}+\frac{2f_{110}}{\sqrt{\pi}}+\frac{2f_{111}}{\sqrt{\pi}}=e, \quad (3\ 46)$$

Where last six equations (3 41)-(3 46) are the consequence of boundary conditions in (3 34) Now collocating the first three equations (3 38)-(3 40) at ten collocation points $\eta_i \in [0,1]$, we get thirty simultaneous algebraic equations These thirty equations along with Eqs (3 41)-(3 46) are solved simultaneously for thirty six unknowns $c_{10}, \ldots, c_{111}$, $e_{10}, \ldots, e_{111}$ and $f_{10}, \ldots, f_{111}$ The final solution is given by

$$y(\eta) = 1 + \eta + 0\ 5\eta^2 + 0\ 166667\eta^3 + 0\ 0416667\eta^4 + 0\ 00833334\eta^5$$

$$+ 0\ 00138887\eta^6 + 0\ 000198456\eta^7 + 0\ 000024732\eta^8 + 2\ 82825\times10^{-6}\eta^9$$

$$+ 2\ 28856\times10^{-7}\eta^{10} + 4\ 15155\times10^{-8}\eta^{11} \qquad (3\ 47)$$

A comparison of above solutıon wıth that of exact solution ıs gıven ın **Table 3.2** The solutıon obtaıned by HPM [35] ıs also included ın the **Table 3.2** Clearly, the CWS ıs superıor over the HPM solutıon ın terms of the absolute error

Table 3.2: Comparıson of HPM and Chebyshev wavelets solutıon

η	Chebyshev wavelets solutıon, $k = 1$, $M = 12$	Exact Solutıon e^{η}	HPM Solutıon $O(x^{13})$	Absolute Error $\lvert y(\eta) - e^{\eta}\rvert$
0 1	1 1051709180756477	1 1051709180756477	1 105294273	0
0 2	1 221402758160169	1 2214027581601699	1 221638169	$8\ 88178\times10^{-16}$
0 3	1 3498588075760032	1 3498588075760032	1 350184525	0
0 4	1 4918246976412706	1 4918246976412703	1 492210231	$2\ 22045\times10^{-16}$
0 5	1 648721270700127	1 6487212707001282	1 649129880	$1\ 11022\times10^{-15}$
0 6	1 8221188003905093	1.8221188003905089	1 822510698	$4\ 44089\times10^{-16}$
0 7	2 013752707470476	2 0137527074704766	2 014088799	$4\ 44089\times10^{-16}$
0 8	2 2255409284924665	2 225540928492468	2 225786815	$1\ 33227\times10^{-16}$
0 9	2 45960311115695	2 45960311115695	2 459733002	0

Conclusion:

Chebyshev wavelets method ıs applıed for the solutıon of five nonlınear boundary value problems It ıs observed that ın each case Chebyshev wavelets solutıon ıs ın excellent agreement wıth the correspondıng numerical solution In the last example, it is observed that Chebyshev wavelet method is superior over Homotopy perturbation method in terms of the absolute error As far as second order nonlinear boundary value problems are concerned, the Chebyshev wavelet method produce excellent results However, its applicability to higher order nonlinear boundary value problems in bounded domain is to be tested The efforts in this regard are underway and will be communicated intimately

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