HEAT AND MASS TRANSFER OF A VISCOUS FLUID FLOW IN A PIPE

By

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1 Viscous flow (Mathematics)

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A Dissertation

Submitted in the Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

IN

MATHEMATICS

Supervised by

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STARTING IN THE NAME OF "ALLAH", WHO IS THE MOST BENEFICIENT AND THE MOST MERCIFUL.

Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as confirming to the required standard

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Department of Mathematics & Statistics

Faculty of Basic & Applied Sciences

International Islamic University, Islamabad

2015

DEDICATION

After Thanking

Enormous Allah

This work is dedicated to

My Parents

Mehboob Gulistan

&

Kalsoom Akhter

"Kind words can be short and easy to speak but their echoes are truly endless."

Forwarding Sheet by Research Supervisor

The Thesis entitled ***"Heat and Mass transfer of Fractional model of a Viscous fluid flow in a Pipe"*** submitted by ***Hira Mehboob, 152-FBAS/MSMA/F13*** in partial fulfillment of MS Degree in Mathematics has been completed under my guidance and supervision. I am satisfied with the quality of her research work and allow her to submit this thesis for further process to graduate with Master of Science degree from the Department of Mathematics & Statistics, as per IIUI rules and regulations

Date 3-9-15

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I express my gratitude to all my teachers whose teaching has taken me to the level of academic excellence. I express my gratitude especially for my dedicated affectionate, kind natured and competent supervisor **Dr.Khadija Maqbool** Who was always present to help me with her knowledgeable suggestions and instructions. I feel proud to work under her supervision She always encouraged me and guided me to advance level academic standards I will never forget her support in the research work. I pray to **Almighty Allah**, give success and happiness to my supervisor **Dr.Khadija Maqbool** in her whole life I would also like to thank **Dr. Sajida Kousar** for her assistance and guidance in writing the thesis.

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DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor **Dr. Khadija Maqbool**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute

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Preface

The combined heat and mass transfer analysis under the chemical reactions have great [1-3] importance in many processes, therefore have received a considerable amount of attention in these days The heat and mass transfer occurs simultaneously in the processes such as energy transfer in a wet cooling tower, evaporation at the surface of a water body and flow in a desert cooler. The chemical reaction between a foreign mass and the fluid occur in many chemical engineering processes. The effect of chemical reaction and mass transfer on the flow past an infinite plate are studied [4-6] Also soret and dufour effect are important for intermediate molecular weight gases in coupled heat and mass transfer in binary system. Many researchers [7-10] have investigated the soret (thermo diffusion) and dufour (diffuso thermo) effect with chemical reaction but few work has been done in the axisymmetric flow

There are only few papers [11-13] in which the fractional diffusion is investigated for the axisymmetric flow In previous publications, problems for the solutions to time fractional heat conduction equation in a half-space are studied The Laplace integral transform with respect to time t, the Hankel transform with respect to the spatial coordinate r, and the sin-cos-Fourier transforms with respect to spatial coordinate z are used In this dissertation, the time fractional energy and mass concentration equations are solved by the new technique fractional VIM and HPTM [14-18] and the thesis is organized in the following fashion.

In Chapter 1 basic definitions and laws are discussed. Chapter 2 is the review of a fractional model of a viscous fluid flow in a pipe Chapter 3 is extended to solve the equation of heat and mass transfer for a fractional model of viscous fluid by HPTM and fractional VIM

Contents

Chapter 1

Basic Definitions

This Chapter comprises on the basic laws involved in chapter 2 and chapter 3

1.1 Fluid

Fluid is the material (liquid or gas) that flows

1.2 Flow

A material that goes under deformation when different forces act upon it. If the deformation is continuously increases without limit then the phenomenon is known as flow

1.3 Properties of Fluid

Properties of fluid determine the behaviours of fluid and characteristics of the fluid

1.3.1 Density

Density is the ratio between mass (m) and volume (V) of a fluid

$$\rho = \frac{m}{V} \tag{1.3.1}$$

1.3.2 Viscosity

Viscosity determines the amount of resistance of the fluid to shear stress and is denoted by μ

1.3.3 Temperature

This fluid property determines the level of heat intensity of a fluid

1.3.4 Pressure

Pressure of a fluid is the magnitude of force per unit area of the fluid In other words it is the ratio of magnitude of force on a fluid to the area of the fluid held perpendicular to the direction of the force

$$P = \frac{|F|}{A} \tag{1.3.2}$$

1.3.5 Specific Volume

Specific volume is the volume of a fluid (V) occupied per unit mass (m)

1.4 Classification of Fluids

1.4.1 Ideal Fluid

A fluid having zero or negligible viscosity is called ideal fluid The occurrence of such fluid in real world is rare

Free Convection

In free convection, the fluid motion is driven by density differences associated with temperature changes generated by heating or cooling Thus the heat transfer itself generates the flow which conveys energy away from the point at which the transfer occurs

Forced Convection

In forced convection, the fluid motion is driven by some external influence Examples are the flows of air induced by a fan, by the wind or by the motion of the vehicle etc

1.6.3 Radiation

In radiation, in the absence of intervening medium there is net heat transfer between two surfaces at different temperatures in the form of electromagnetic waves

1.7 Mass Transfer

Mass transfer is the net movement of mass from one location to another Mass transfer occurs in many processes such as absorption evaporation, drying, precipitation, membrane filtration and distillation Mass transfer is used by different scientific disciplines for different processes and mechanisms

1.8 Soret and Dufour Effect

When small light and large heavy molecules are seperated under a temperature gradient the soret effect is considered while when a chemical system is under a concentration gradient heat flux is created it is described by a dufour effect

1.9 Schmidt Number

It is non-dimensional number defined as the ratio of viscosity (momentum diffusivity) and mass diffusivity

$$Sc = \frac{\mu}{\rho D} \tag{1.9.1}$$

where D is the mass diffusivity, μ is the dynamic viscosity and ρ is the density of the fluid

1.10 Prandtl Number

The Prandtl number Pr is a dimensionless number defined as the ratio of momentum diffusivity to thermal diffusivity

$$Pr = \frac{c_p \mu}{k} \tag{1.10.1}$$

where μ is dynamic viscosity, c_p is specific heat and k is thermal conductivity

1.11 Conservation Laws

1.11.1 Law of Conservation of Mass (Continuity Equation)

In nature, mass cannot be created or destroyed. It can be stated as the rate of increase of mass in a region W equals the rate at which mass is crossing the boundary ∂W in the inward direction i.e.

$$\frac{d}{dt}\int_W \rho dV = -\int_{\partial W} \rho \mathbf{V} \cdot \mathbf{dA} \tag{1.11.1}$$

which is the integral form of law of conservation of mass, where W is a fixed subregion of region D. $\mathbf{dA}$ denotes the vector area element of ∂W. The surface integral in Eq. (1.11.1) can be changed to volume integral using the Gauss

Divergence theorem, Eq (1 11 1) becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \qquad (1\ 11\ 2)$$

is the differential form of law of conservation of mass Eq (1 11 2) is also known as the continuity equation If density of fluid is constant Eq (1 11 2) reduces to

$$\nabla \cdot \mathbf{V} = 0 \qquad (1\ 11\ 3)$$

1.11.2 Law of Conservation of Momentum (Equation of Motion)

For a viscous fluids, equation of motion in vector form is

$$\rho \frac{d\mathbf{V}}{dt} = div\mathbf{T} + \rho \mathbf{b} \qquad (1\ 11\ 4)$$

where, b is the body force, $\mathbf{V}$ is the velocity of fluid and $\mathbf{T}$ is the Cauchy stress tensor In cylinderical coordinate the equations of motion in r θ and z direction are

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z}\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial}{\partial \theta}(\tau_{r\theta}) + \frac{\partial}{\partial z}(r\tau_{rz})\right], \qquad (1\ 11\ 5)$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z}\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\tau_{\theta r}) + \frac{\partial}{\partial \theta}(\tau_{\theta\theta}) + \frac{\partial}{\partial z}(r\tau_{\theta z})\right] \qquad (1\ 11\ 6)$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\tau_{zr}) + \frac{\partial}{\partial \theta}(\tau_{z\theta}) + \frac{\partial}{\partial z}(r\tau_{zz})\right] \qquad (1\ 11\ 7)$$

where u, v and w are components of velocity in r, θ and z direction, respectively

1.11.3 Law of Conservation of Energy

It states that the temperature of a fluid element changes as it moves along with the fluid because of heat conduction and heat production by the viscous heating

Mathematically, we can write

$$\rho \frac{d\epsilon}{dt} = \mathbf{T} \cdot \mathbf{L} - \nabla \cdot \mathbf{q} + \rho r \qquad (1\ 11\ 8)$$

Definition 1.12.2 Liouville derivative is as follows

$D^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} (x-\xi)^{-\alpha} f(\xi) d\xi \qquad -\infty < x < +\infty$

Definition 1.12.3 Inversion gives the fractional derivative of f(t) as

$D^{\alpha} f(t) = L^{-1}(s^{\alpha}\overline{f}(s))$.

If $f(t) = t^{\beta}$ then we can write

$D^{\alpha} t^{\beta} = L^{-1}(\frac{\Gamma(\beta+1)}{s^{\beta-\alpha+1}}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$

1.13 Homotopy Perturbation Method

Consider the following equation

$L(u) + N(u) - f(r) = 0$

Where $f(r)$ is a known analytic function and L is a linear while N is a non-linear operator

We construct a homotopy $\nu(r, p)$ $\Omega \times [0, 1] \rightarrow R$ which satisfies

$$H(\nu, p) = L(\nu) - L(u_0) + pL(u_0) + p[N(\nu) - f(r)] = 0$$

where $p \in [0, 1]$ is an embedding parameter. According to HPM we use the embedding parameter p as a small parameter, and assume that the solution of above equation can be written as a power series in p

$$\nu = \nu_0 + \nu_1 p + \nu_2 p^2 + \ldots = \sum_{i=0}^{\infty} \nu_i p^i$$

1.14 Laplace Transform Method

The Laplace transform of a function $f(t)$ is defined as

$$L[f(t)] = \bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt \qquad (1.14.1)$$

due to the uniqueness of Laplace transform, the inverse Laplace transform L^{-1} can be defined as

$L^{-1}\hat{f}(s) = f(t)$

1.15 Homotopy Perturbation Transform method

It is the combination of Homotopy perturbation method and Laplace transform method and used to solve various types of linear and non-linear systems of partial differential equations

1.16 Variational Iteration Method

A non-linear problem is

$L(u) + N(u) = g(t)$,

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a given continuous function. Correctional functional is given as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\{L[u_n(\tau)] + N[\tilde{u}_n(\tau)] - g(\tau)\}d\tau \qquad n \geq 0 \quad (1.16.1)$$

where λ is a general Lagrange multiplier

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \qquad m \geq 1 \quad (1.16.2)$$

Here $\tilde{u}_n$ is considered as a restricted variation, which means $\delta\tilde{u}_n = 0$. Consequently, the solution may be obtained as,

$$u(t) = \lim_{n\to\infty} u_n(t)$$

1.17 Fractional Variational Iteration Method

We consider a more general FDE as given below

$$D_t^{\alpha} u + R[u] + N[u] = f(t), \qquad 0 < t,\ 0 < \alpha \leq 1 \qquad (1.17.1)$$

Correctional functional for Eq (1 17 1) is defined as follows

$$u_{n+1} = u_n + \int_0^t \lambda(t\ \tau)[D_t^{\alpha} u_n + R[u_n] + N[u_n] - f(t)]d\tau \qquad (1\ 17\ 2)$$

where,

$$\lambda(t\ \tau) = \frac{(-1)^{\alpha}(\tau - t)^{\alpha-1}}{\Gamma(\alpha)} \qquad (1\ 17\ 3)$$

By using definition 1 12 1 definition 1 12 2 and definition 1 12 3 in Eq (1 17 2) we obtain the successive approximate solutions

Chapter 2

Fractional model of a Viscous fluid flow in a Pipe

In this chapter, we review the paper of Devendra Kumar, Jagdev Singh, Sunil Kumar [19]. Here, we revised a fractional model of Navier Stokes equation formed by axisymmetric flow in a pipe. The exact solution is obtained by homotopy perturbation transform method and fractional variational iteration method. The effect of emerging parameter is shown by the help of graphs.

2.1 Problem Formulation

Here, we consider the unidirectional flow of a viscous fluid in a pipe having radius R. Initially the fluid is flowing by the parabolic velocity. The velocity is maximum at the centre of a pipe and on the boundary of the pipe fluid is at rest.

The velocity profile for the axisymmetric flow is given by

$$\mathbf{V} = (0, 0, u(r, t)) \tag{2.1.1}$$

The cauchy stress tensor for a viscous fluid is defined by

$$\mathbf{T} = -pI + \mu\mathbf{A}_1, \qquad \text{where} \qquad \mathbf{A}_1 = (grad\mathbf{V}) + (grad\mathbf{V})^T \quad (2.1.2)$$

where, p is the pressure, I is the identity tensor, μ is the viscosity and A_1 is the first Rivilin Erickson tensor.

The continuity equation is identically satisfied i.e.

$$\nabla.\mathbf{V} = 0 \quad (2.1.3)$$

The equation of motion for the fractional viscous fluid in the absence of body force is

$$\frac{\partial^\alpha \mathbf{V}}{\partial t^\alpha} + (\mathbf{V}.\nabla)\mathbf{V} = \frac{\mu}{\rho} div\mathbf{T} \quad (2.1.4)$$

Employing Eq. (2.1.1) and Eq. (2.1.2), then the momentum equation in cylindercal coordinate is given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{1}{\rho}\frac{\partial p}{\partial z}, \qquad 0 < \alpha \leq 1 \quad (2.1.5)$$

where, $0 < \alpha \leq 1$ is the fractional parameter, ν is the dynamic viscosity and ρ is the density of the fluid.

The initial and boundary conditions are defined as

$$\text{I.C.} \qquad u(r,0) = f_1(r), \qquad 0 \leq r \leq R \quad (2.1.6)$$

$$\text{B.C.'s} \qquad \frac{\partial u}{\partial r}(0,t) = 0, \qquad u(R,t) = 0 \qquad t > 0 \quad (2.1.7)$$

The non-dimensional quantities are as follows

$\xi = \frac{r}{R}$ $\qquad \overline{u} = \frac{\nu}{R^2}u$ $\qquad \overline{t} = \frac{t}{(\frac{R^2}{\nu})^{\frac{1}{\alpha}}}$

The non-dimensional form of the IBVP in the presence of constant pressure gradient P is as follows

$$\frac{\partial^\alpha \overline{u}}{\partial \overline{t}^\alpha} = \left(\frac{\partial^2 \overline{u}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \overline{u}}{\partial \xi}\right) + P \quad (2.1.8)$$

$$\overline{u}(\xi,0)=\overline{f}_1(\xi), \tag{2 1 9}$$

$$\frac{\partial\overline{u}}{\partial\xi}(0,\overline{t})=0 \qquad \overline{u}(1\ \overline{t})=0 \tag{2 1 10}$$

We drop the bar from the equation of motion, initial and boundary conditions in non-dimensionalize form for simplicity

2.1.1 Homotopy Perturbation Transform Method

Example 1. Consider the following time-fractional Navier Stokes equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial\xi}\right)+P, \tag{2 1 11}$$

subject to the initial condition

$$u(\xi,0)=1-\xi^2 \tag{2 1 12}$$

Applying Laplace transform on both sides of Eq (2 1 11) we get

$$L[u(\xi,t)]=\frac{1-\xi^2}{s}+\frac{P}{s^{\alpha+1}}+\frac{1}{s^\alpha}L\left[\frac{\partial^2 u}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial\xi}\right] \tag{2 1 13}$$

The inverse Laplace transform gives

$$u(\xi\ t)=1-\xi^2+\frac{Pt^\alpha}{\Gamma(\alpha+1)}+L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial\xi}\right]\right] \tag{2 1 14}$$

Now by applying HPM

$$u(\xi\ t)=\sum_{n=0}^{\infty}p^n u_n(\xi\ t) \tag{2 1 15}$$

Using Eq (2 1 15) in Eq (2 1 14), we have

$$\begin{aligned}\sum_{n=0}^{\infty}p^n u_n(\xi\ t) &= 1-\xi^2+\frac{Pt^\alpha}{\Gamma(\alpha+1)}+p\Bigg(L^{-1}\Bigg[\frac{1}{s^\alpha}L\Bigg[\frac{\partial^2}{\partial\xi^2}\left(\sum_{n=0}^{\infty}p^n u_n(\xi\ t)\right)\\ &+ \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\sum_{n=0}^{\infty}p^n u_n(\xi\ t)\right)\Bigg]\Bigg]\Bigg)\end{aligned} \tag{2 1 16}$$

Comparing the coefficients of powers of p

$$\begin{aligned}
p^0 \qquad u_0(\xi,t) &= 1-\xi^2+\frac{Pt^\alpha}{\Gamma(\alpha+1)}\\
p^1 \qquad u_1(\xi,t) &= L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u_0}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u_0}{\partial\xi}\right]\right]=\frac{-4t^\alpha}{\Gamma(\alpha+1)}\\
p^2 \qquad u_2(\xi,t) &= L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u_1}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u_1}{\partial\xi}\right]\right]=0\\
p^3 \qquad u_3(\xi,t) &= L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u_2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u_2}{\partial\xi}\right]\right]=0
\end{aligned} \qquad (2.1.17)$$

For $p=0$ gives initial solution and $p=1$ gives final solution for the governing equation. Hence, the solution for Eq. (2.1.5) is as follows

$$u(\xi,t)=\sum_{n=0}^{\infty}u_n(\xi,t)$$

$$u(\xi,t)=(1-\xi^2)+(P-4)\frac{t^\alpha}{\Gamma(\alpha+1)} \qquad (2.1.18)$$

which shows the exact solution for $0<\alpha\leq 1$. By substituting $\alpha=1$ in Eq. (2.1.18), we get the solution for the Navier Stokes equation as given below

$$u(\xi,t)=(1-\xi^2)+(P-4)t \qquad (2.1.19)$$

Example 2 Consider the following time-fractional Navier Stokes equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial\xi}\right) \qquad (2.1.20)$$

subject to the initial condition

$$u(\xi,0)=\xi \qquad (2.1.21)$$

Applying Laplace transform on both sides of Eq. (2.1.20), we get

$$L[u(\xi,t)]=\frac{\xi}{s}+\frac{1}{s^\alpha}L\left[\frac{\partial^2 u}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial\xi}\right] \qquad (2.1.22)$$

The inverse Laplace transform gives

$$u(\xi,t)=\xi+L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial\xi}\right]\right] \qquad (2.1.23)$$

Now, by applying HPM

$$u(\xi,t)=\sum_{n=0}^{\infty}p^n u_n(\xi,t) \tag{2.1.24}$$

Using Eq. (2.1.24) in Eq. (2.1.23), we have

$$\begin{aligned}\sum_{n=0}^{\infty}p^n u_n(\xi,t) &= \xi+p\Bigg(L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2}{\partial\xi^2}\Big(\sum_{n=0}^{\infty}p^n u_n(\xi,t)\Big)\right.\right.\\ &+ \left.\left.\frac{1}{\xi}\frac{\partial}{\partial\xi}\Big(\sum_{n=0}^{\infty}p^n u_n(\xi,t)\Big)\right]\right]\Bigg)\end{aligned} \tag{2.1.25}$$

comparing the coefficients of powers of p

$$\begin{aligned}
p^0 &\quad u_0(\xi,t) = \xi,\\
p^1 &\quad u_1(\xi,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u_0}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u_0}{\partial\xi}\right]\right]=\frac{1}{\xi}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^2 &\quad u_2(\xi,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u_1}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u_1}{\partial\xi}\right]\right]=\frac{1}{\xi^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
p^3 &\quad u_3(\xi,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u_2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial u_2}{\partial\xi}\right]\right]=\frac{9}{\xi^5}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},
\end{aligned}$$

$$p^n \quad u_n(\xi,t) = \frac{1^2\times 3^2\times\cdots\times(2n-3)^2}{\xi^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \tag{2.1.26}$$

Hence, the solution is

$$u(\xi,t)=\xi+\sum_{n=1}^{\infty}\frac{1^2\times 3^2\times\cdots\times(2n-3)^2}{\xi^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \tag{2.1.27}$$

2.1.2 Fractional Variational Iteration Method

Example 1. Consider the time-fractional Navier Stokes equation Eq. (2.1.11) subject to the initial condition (2.1.12).

The correctional functional for Eq. (2.1.11) is given as

$$u_{n+1}=u_n+\int_0^t\lambda(t,\tau)\left(\frac{\partial^\alpha u_n}{\partial\tau^\alpha}(\xi,\tau)-\frac{\partial^2 u_n}{\partial\xi^2}(\xi,\tau)-\frac{1}{\xi}\frac{\partial u_n}{\partial\xi}(\xi,\tau)-P\right)d\tau$$

$$0<\alpha\le 1 \tag{2.1.28}$$

where

$$\lambda(t,\tau) = \frac{(-1)^{\alpha}(\tau - t)^{\alpha-1}}{\Gamma(\alpha)}, \qquad 0 < \alpha \le 1 \qquad (2.1.29)$$

So, we can write Eq. (2.1.28) as follows

$$u_{n+1} = u_n - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^{\alpha} u_n}{\partial \tau^{\alpha}} - \frac{\partial^2 u_n}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial u_n}{\partial \xi} - P\right)d\tau, \qquad 0 < \alpha \le 1 \qquad (2.1.30)$$

Choose

$u_0(\xi, t) = 1 - \xi^2$.

Then, we get

$$\begin{aligned} u_1(\xi,t) &= u_0 - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^{\alpha} u_0}{\partial \tau^{\alpha}} - \frac{\partial^2 u_0}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial u_0}{\partial \xi} - P\right)d\tau \\ &= 1 - \xi^2 + (P-4)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2(\xi,t) &= u_1 - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^{\alpha} u_1}{\partial \tau^{\alpha}} - \frac{\partial^2 u_1}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial u_1}{\partial \xi} - P\right)d\tau \\ &= 1 - \xi^2 + (P-4)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ u_3(\xi,t) &= u_2 - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^{\alpha} u_2}{\partial \tau^{\alpha}} - \frac{\partial^2 u_2}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial u_2}{\partial \xi} - P\right)d\tau \\ &= 1 - \xi^2 + (P-4)\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

$$\begin{aligned} u_n(\xi,t) &= u_{n-1} - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^{\alpha} u_{n-1}}{\partial \tau^{\alpha}} - \frac{\partial^2 u_{n-1}}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial u_{n-1}}{\partial \xi} - P\right)d\tau \\ &= 1 - \xi^2 + (P-4)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \end{aligned} \qquad (2.1.31)$$

Thus, the solution is as follows

$$u(\xi,t) = \lim_{n\to\infty} u_n(\xi,t),$$

where $u_n(\xi, t)$ is given in Eq. (2.1.31)

Example 2. Consider the time-fractional Navier Stokes Eq. (2.1.20) subject to the initial condition (2.1.21)

The correctional functional for Eq. (2.1.20) is given as

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left(\frac{\partial^\alpha u_n}{\partial\tau^\alpha}(\xi,\tau) - \frac{\partial^2 u_n}{\partial\xi^2}(\xi,\tau) - \frac{1}{\xi}\frac{\partial u_n}{\partial\xi}(\xi,\tau)\right)d\tau \qquad (2.1.32)$$

$$0 < \alpha \le 1$$

where

$$\lambda(t,\tau) = \frac{(-1)^\alpha(\tau - t)^{\alpha-1}}{\Gamma(\alpha)} \qquad 0 < \alpha \le 1 \qquad (2.1.33)$$

So, we can write Eq. 2.1.32 as follows

$$u_{n+1} = u_n - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^\alpha u_n}{\partial\tau^\alpha} - \frac{\partial^2 u_n}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial u_n}{\partial\xi}\right)d\tau \qquad 0 < \alpha \le 1 \qquad (2.1.34)$$

Choose

$u_0(\xi, t) = \xi$

Then, we get

$$u_1(\xi,t) = u_0 - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^\alpha u_0}{\partial\tau^\alpha} - \frac{\partial^2 u_0}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial u_0}{\partial\xi}\right)d\tau$$

$$= \xi + \frac{1}{\xi}\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(\xi,t) = u_1 - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^\alpha u_1}{\partial\tau^\alpha} - \frac{\partial^2 u_1}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial u_1}{\partial\xi}\right)d\tau$$

$$= \xi + \frac{1}{\xi}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\xi^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3(\xi,t) = u_2 - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^\alpha u_2}{\partial\tau^\alpha} - \frac{\partial^2 u_2}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial u_2}{\partial\xi}\right)d\tau$$

$$= \xi + \frac{1}{\xi}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\xi^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{9}{\xi^5}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$u_n(\xi,t) = u_{n-1} - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^\alpha u_{n-1}}{\partial\tau^\alpha} - \frac{\partial^2 u_{n-1}}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial u_{n-1}}{\partial\xi}\right)d\tau$$

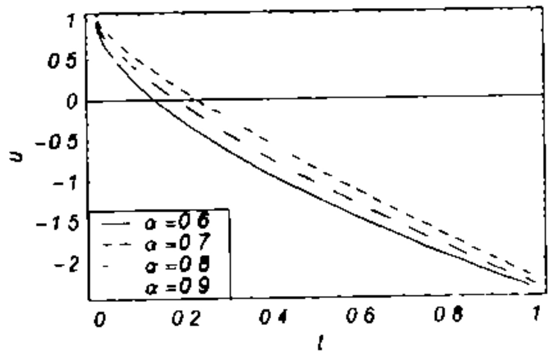
$$= \xi + \frac{1}{\xi}\frac{t^\alpha}{\Gamma(\alpha+1)} + \cdots + \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{\xi^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$n \ge 1 \qquad (2.1.35)$$

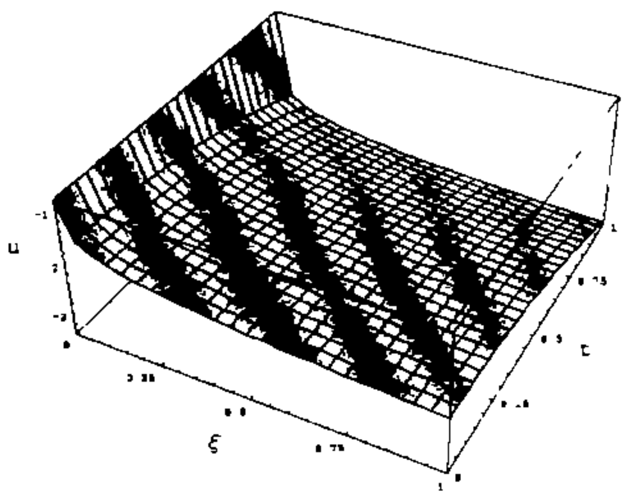
Thus the solution is as follows

$$u(\xi, t) = \lim_{n \to \infty} u_n(\xi, t)$$

where $u_n(\xi, t)$ is given in Eq. (2.1.35)



(a)



(b)

Figure 2.1. (a) Velocity profile for different values of α where $P = 1$, $\xi = 0.1$ and $0 \leq t \leq 1$. (b) 3D plot for velocity field with respect to ξ and t where $\alpha = 0.1$, $P = 1$, $0 \leq t \leq 1$ and $0 \leq \xi \leq 1$.

Fig. 2.1 (a) shows that initially magnitude of velocity profile increases and after some time magnitude of velocity profile decreases by increasing the fractional parameter and also Fig. 2.1 (b) shows the effect of velocity profile with respect to variables ξ and t

$$C(0,t) = C_0, \qquad C(R,t) = C_\infty \tag{3.1.6}$$

where D is mass diffusivity, k_T is thermal diffusion ratio T_m is mean fluid temperature k_1 is the rate of chemical reaction C_0 and C_∞ are constant. The non-dimensional quantities are given by

$$\xi = \frac{r}{R}, \qquad \overline{T} = \frac{T - T_\infty}{T_0 - T_\infty}, \qquad \overline{t} = \frac{t}{(\frac{R^2}{\nu})^{\frac{1}{\alpha}}} \qquad Pr = \frac{\mu c_p}{k_c}$$
$$\overline{C} = \frac{C - C_\infty}{C_0 - C_\infty}, \qquad Du = \frac{Dk_T(C_0 - C_\infty)}{c_p c_s \nu(T_0 - T_\infty)}, \quad \frac{1}{Sc} = \frac{D}{\nu}, \qquad \gamma = \frac{k_1 R^2}{\nu},$$
$$S_r = \frac{Dk_T(T_0 - T_\infty)}{T_m \nu(C_0 - C_\infty)} \tag{3.1.7}$$

Using the above non-dimensional quantities in Eq (3.1.1) - Eq (3.1.6) and dropping the bar, we get the following system of fractional PDE

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{Pr}\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial T}{\partial \xi}\right) + Du\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C}{\partial \xi}\right) \tag{3.1.8}$$

$$\frac{\partial^\alpha C}{\partial t^\alpha} = \frac{1}{Sc}\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C}{\partial \xi}\right) + S_r\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial T}{\partial \xi}\right) - \gamma C \tag{3.1.9}$$

$$\begin{aligned} &\text{B.C.s} \qquad T(0,t) = 1 = C(0,t) \qquad T(1,t) = 0 = C(1,t) \\ &\text{I.C.} \qquad T(\xi,0) = 1 - \xi^2 = C(\xi,0) \end{aligned} \tag{3.1.10}$$

where Pr is the prandtle number, Du is the dufour number, Sc is the schmidt number, S_r is the soret number and γ is the chemical reaction parameter.

3.1.1 Homotopy Perturbation Transform method

Applying Laplace transform on both sides of Eq. (3.1.8) and Eq. (3.1.9) respectively, we get

$$L[T(\xi,t)] = \frac{1-\xi^2}{s} + \frac{1}{s^\alpha}L\left[\frac{1}{Pr}\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial T}{\partial \xi}\right) + Du\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C}{\partial \xi}\right)\right] \tag{3.1.11}$$

$$L[C(\xi,t)] = \frac{1-\xi^2}{s} + \frac{1}{s^\alpha}L\left[\frac{1}{Sc}\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C}{\partial \xi}\right) + S_r\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial T}{\partial \xi}\right) - \gamma C\right] \tag{3.1.12}$$

Choose

$T_0(\xi, t) = 1 - \xi^2 \qquad C_0(\xi, t) = 1 - \xi^2.$

Than we obtain the following

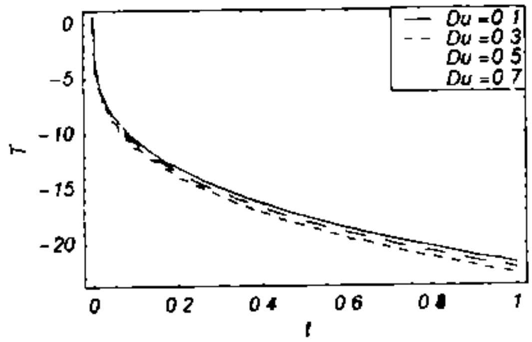
$$
\begin{aligned}
T_1(\xi, t) =& 1 - \xi^2 - \left(\frac{4}{P_r} + 4Du\right)\frac{t^\alpha}{\Gamma(\alpha+1)},\\
C_1(\xi, t) =& 1 - \xi^2 - \left(\frac{4}{Sc} + 4S_r + \gamma(1-\xi^2)\right)\frac{t^\alpha}{\Gamma(\alpha+1)}\\
T_2(\xi, t) =& 1 - \xi^2 - \left(\frac{4}{P_r} + 4Du\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + 4\gamma Du\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
C_2(\xi, t) =& 1 - \xi^2 - \left(\frac{4}{Sc} + 4S_r + \gamma(1-\xi^2)\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \gamma\left(\frac{8}{Sc} + 4S_r\right.\\
&\left. + \gamma(1-\xi^2)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
T_3(\xi, t) =& 1 - \xi^2 - \left(\frac{4}{P_r} + 4Du\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + 4\gamma Du\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&- 4\gamma^2 Du\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
C_3(\xi, t) =& 1 - \xi^2 - \left(\frac{4}{Sc} + 4S_r + \gamma(1-\xi^2)\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \gamma\left(\frac{8}{Sc} + 4S_r\right.\\
&\left. + \gamma(1-\xi^2)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \gamma^2\left(\frac{12}{Sc} + 4S_r + \gamma(1-\xi^2)\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
T_n(\xi, t) =& 1 - \xi^2 - \left(\frac{4}{P_r} + 4Du\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \cdots + \left((-1)^n 4\gamma^{n-1} Du\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)\\
& n \geq 1\\
C_n(\xi, t) =& 1 - \xi^2 + \cdots + \left((-1)^n\gamma^{n-1}\left(\frac{4n}{Sc} + 4S_r + \gamma(1-\xi^2)\right)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)\\
& n \geq 1
\end{aligned}
\tag{3.1.27}
$$

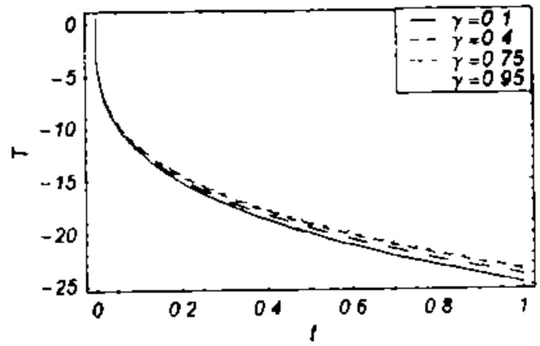
Thus the solution is as follows

$$
\begin{aligned}
T(\xi, t) &= \lim_{n\to\infty} T_n(\xi, t)\\
C(\xi, t) &= \lim_{n\to\infty} C_n(\xi, t)
\end{aligned}
\tag{3.1.28}
$$

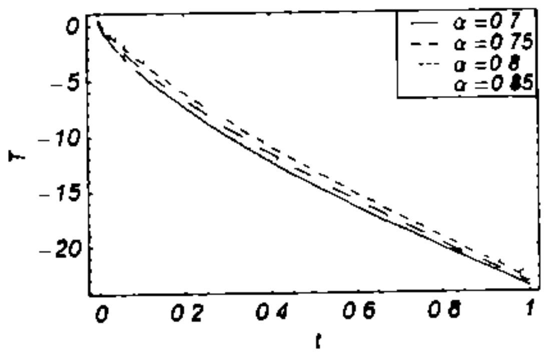
where $T_n(\xi, t)$ and $C_n(\xi, t)$ is given in above equations


Du =0 1
Du =0 3
Du =0 5
Du =0 7
T
t

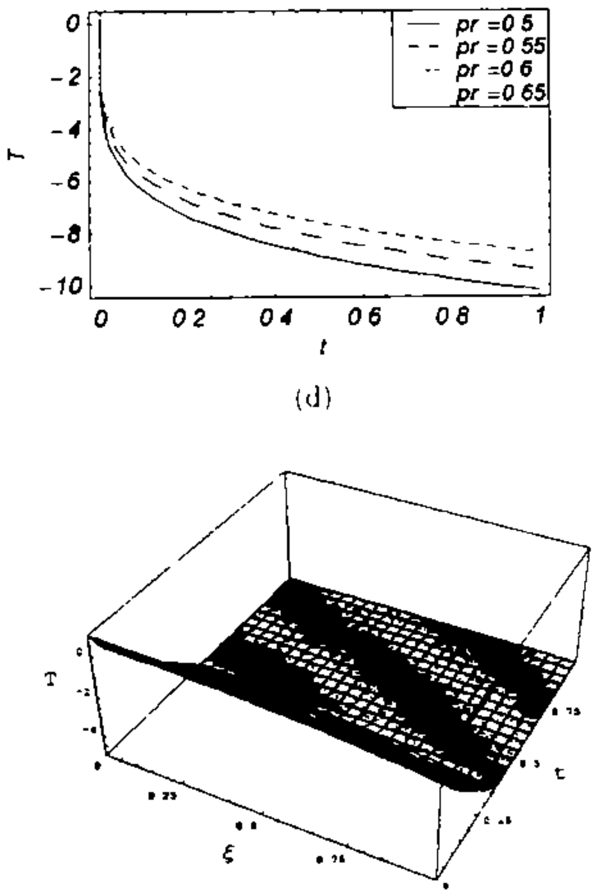
(a)


γ =0 1
γ =0 4
γ =0 75
γ =0 95
T
t

(b)


α =0 7
α =0 75
α =0 8
α =0 85
T
t

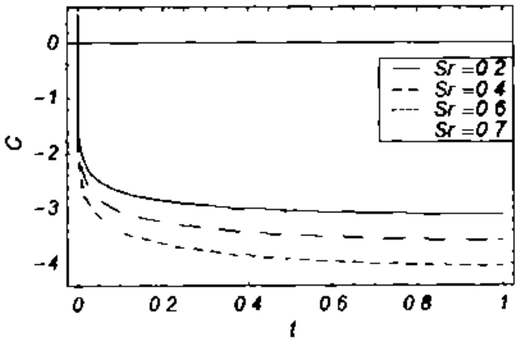
(c)



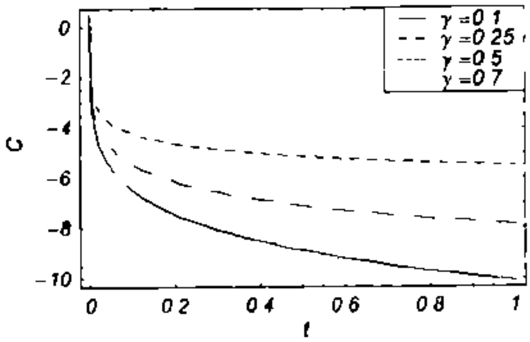
(d)

(e)

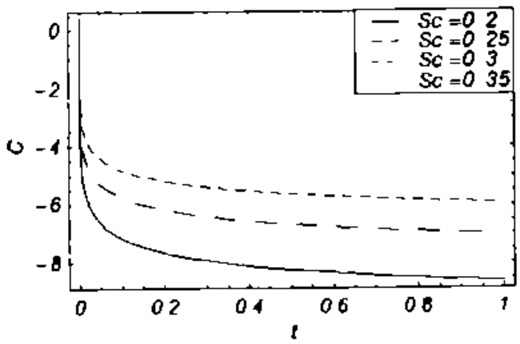
Figure 3.2 Temperature profile for different values of α, γ, Du and Pr. (a) $\alpha = 0.3$, $\gamma = 0.8$, $Pr = 0.2$, $\xi = 0.5$ and $0 \leq t \leq 1$ (b) $\alpha = 0.3$, $Du = 0.8$, $Pr = 0.2$, $\xi = 0.5$ and $0 \leq t \leq 1$ (c) $Du = 0.6$, $\gamma = 0.3$, $Pr = 0.2$, $\xi = 0.7$ and $0 \leq t \leq 1$ (d) $\alpha = 0.2$, $Du = 0.6$, $\gamma = 0.3$, $\xi = 0.7$ and $0 \leq t \leq 1$ (e) $\alpha = 0.8$, $Du = 0.6$, $\gamma = 0.1$, $Pr = 0.5$, $0 \leq t \leq 1$ and $0 \leq \xi \leq 1$


Sr =0 2
Sr =0 4
Sr =0 6
Sr =0 7
C
0
-1
-2
-3
-4
0
0 2
0 4
0 6
0 8
1
t

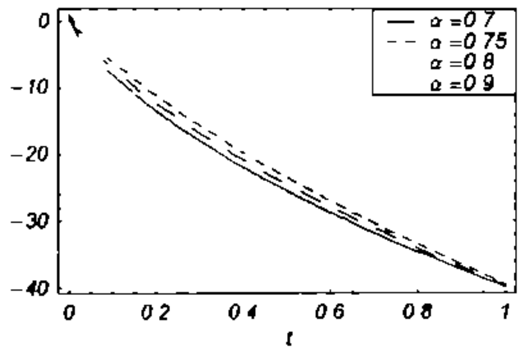
(a)


γ =0 1
γ =0 25
γ =0 5
γ =0 7
C
0
-2
-4
-6
-8
-10
0
0 2
0 4
0 6
0 8
1
t

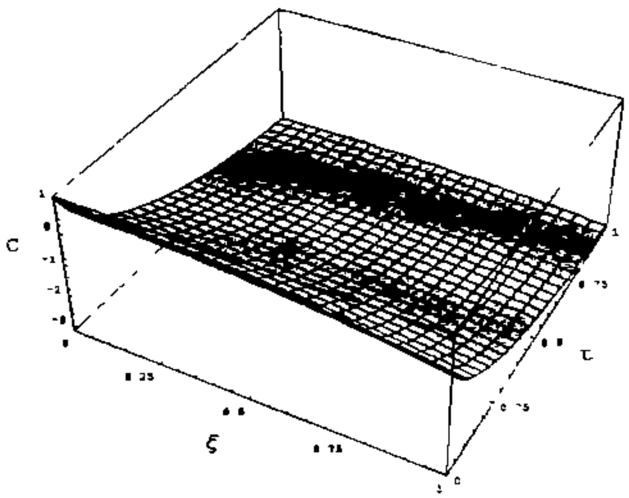
(b)


Sc =0 2
Sc =0 25
Sc =0 3
Sc =0 35
C
0
-2
-4
-6
-8
0
0 2
0 4
0 6
0 8
1
t

(c)



(d)



(e)

Figure 3 3 Concentration profile for different values of α, γ, S_r and Sc (a) $\xi = 0\ 3$, $\gamma = 0\ 8$, $Sc = 0\ 4$, $\alpha = 0\ 2$ and $0 \leq t \leq 1$, (b) $\xi = 0\ 3$ $S_r = 0\ 5$, $Sc = 0\ 4$, $\alpha = 0\ 2$ and $0 \leq t \leq 1$, (c) $\xi = 0\ 3$, $S_r = 0\ 4$, $\gamma = 0\ 6$ $\alpha = 0\ 2$ and $0 \leq t \leq 1$, (d) $\xi = 0\ 3$, $S_r = 0\ 6$, $\gamma = 0\ 1$, $Sc = 0\ 1$ and $0 \leq t \leq 1$, (e) $\alpha = 0\ 8$, $S_r = 0\ 5$, $\gamma = 0\ 9$, $Sc = 0\ 8$, $0 \leq t \leq 1$ and $0 \leq \xi \leq 1$

Fig 3 2 (a) shows that temperature profile increases by increasing the dufour effect also the temperature profile in Fig 3 2 (b) decreases by icreasing the chemical reaction parameter Fig 3 2 (c) shows that as fractional parameter increases temperature profile will decrease and in Fig 3 2 (d) temperature profile decreases by increasing prandtl number Fig 3 2 (e) shows change in temperature with respect to variables ξ and t and also graph is showing that both initial and boundary conditions are satisfied

Fig 3 3 (a) shows that concentration profile increases by increasing the soret effect also the concentration profile in Fig 3 3 (b) decreases by icreasing the chemical reaction parameter Fig 3 3 (c) shows that as schmidt number increases concentration profile will decrease and in Fig 3 3 (d) concentration profile decreases by increasing fractional parameter Fig 3 3 (e) shows change in concentration field with respect to variables ξ and t and also both initial and boundary conditions are satisfied

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