Fixed Points and Coincidence Points of Multivalued Monotone Operators

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Fixed point theory

Coupled fixed point theory

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A Thesis
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In
MATHEMATICS

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE

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We accept this dissertation as confirming to the required standard.

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2017

DEDICATION

This thesis is most respectfully dedicated to my parents, teachers and friends.

For their love, support and encouragement.

DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor **Dr. Maliha Rashid**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

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ACKNOWLEDGEMENT

All praises to The Allah Almighty who has created this world of knowledge for us. He is The Gracious, The Merciful. He bestowed man with intellectual power and understanding, and gave him spiritual insight, enabling him to discover his "Self" know his Creator through His wonders, and conquer nature. Next to all His Messenger Hazrat Muhammad (SAW) Who is an eternal torch of guidance and knowledge for whole mankind.

I am deeply thankful to my research supervisor, **Dr. Maliha Rashid**, whose teaching methodology, continuous guidance, feedback, advice and encouragements have been truly exceptional. I would like to say special thanks to **Dr.Nayyar Mehmood** for his guidance, keen interest, valuable attitude throughout my research work. It would not be justified if I don't mention the support of my fellows. Sincere thanks to my friends who have played a vital role in the completion of this work are **Ammarah safdar** and **Iqra Batool**. I would like to thanks my parents and friends especially **Hina Azmat, Hira Masood, Hira Afzal, Aroosa Nawaz** and **Izat Zahra** for giving me encouragement, love, prayers and confidence.

ABSTRACT

It is well known that the mixed monotone operator equation is important for the applications point of view because of a quite extensive class of integro-differential equations as well as the boundary value problems in nonlinear analysis which are related to the solvability of this kind of equation. Multivalued monotone operators were introduced in 1984, by Nishniannidze [19].Nguyen [15] showed the existence of fixed points for multivalued increasing operators. Mixed monotone operators were introduced by Guo and Lakshmikantham [13] in 1987. Bhaskar and Lakshmikantham [7] established the notion of coupled fixed point and proved some coupled fixed point results in a partially ordered space. Huang and Fang [9] generalized the notion of mixed monotone operators to multivalued mappings and also provided an application for a class of integral inclusions. In all the above mentioned results the existence of a lower or an upper solution to the operator inclusion was necessary.

Feng and Wang [10] relaxed the condition of existence of a lower or an upper solution by using the characteristics of reproducing cones in a partially ordered Banach space and discussed the existence and uniqueness of fixed point in a partially ordered set based on the characterization in the context of reproducing cones. Feng and Wang [11] established some fixed point theorems for multivalued monotone and mixed monotone operators on the basis of characterizations of reproducing cones and also compare some results by removing the requirement of the existence of lower or upper solution. They also established some coupled fixed point theorems for single-valued and multivalued mixed monotone operators. In the final section, as an application of these results, the solvability of fraction integral inclusion was discussed.

In this thesis the fixed point and the coupled fixed point results given by Feng and Wang in [11] are extended by improving the contractive condition in the view of contractions used by Beg and Azam [5], Lakshmikantham [7] and Azam and Mehmood [2] in coupled.

Chapter 01 is introductory and is related with some basic concepts and results that will be useful in the upcoming chapters.

In Chapter 02 we include some relations on the subsets of a partially ordered set, fixed point results for multivalued operators having monotone and mixed monotone property, coupled fixed point results for single valued as well as for multivalued mappings taken from the Feng and Wang [11], Lakshmikantham [7] and Azam and Mehmood [2].

In Chapter 03 we will prove fixed point theorem for multivalued operator by generalizing contractive condition used in Beg and Azam [5]. This result will generalize theorems of Feng and Wang[11]. We will also prove some coupled fixed point results for mixed monotone operators which are generalizations of [2 ,7 ,11] .

Contents

Chapter 1

Preliminaries

This chapter is introductory and is related with some basic concepts and results that will be useful in the upcoming chapters.

1.1 Basic Definitions

In the following section we will discuss fixed point, coincidence point, complete metric space, Banach space, contraction, cone, multivalued monotone operators.

1.1.1 Fixed Point and Coincidence Point

1. A point of a set that is invariant under any transformation is called a *fixed point.*
2. A point $x \in \Omega$ is called *common fixed point* of the pair (S, T), where $S, T : \Omega \to \Omega$ if $Sx = Tx = x$.
3. A point $x \in \Omega$ is said to be *coincidence point* of the pair (S, T) where $S, T : \Omega \to \Omega$ if $Sx = Tx$.
4. A point $y \in \Omega$ is said to be *point of coincidence* of the pair (S, T) where $S, T : \Omega \to \Omega$ if $y = Sx = Tx$ for some $x \in \Omega$.

1.1.2 Complete Metric Space

Consider (Ω, δ) be a metric space. A sequence$\{x_r\}$

1. Converges towards x if for every $\varepsilon > 0$ there is a natural number r_0 such that $\delta(x_r, x) < \varepsilon$ for all $r \geq r_0$.

2. *Cauchy sequence* if for every $m \geq r_\varepsilon, r \geq r_\varepsilon, \delta(x_m, x_r) < \varepsilon$ for all $\varepsilon > 0$ and $r_\varepsilon \in \mathbb{N}$.

3. A metric space (Ω, δ) is *complete* if each Cauchy sequence in Ω converges (to a point in Ω).

4. A complete normed space is a *Banach space*.

1.2 Contraction Principles

For a metric space (Ω, δ), a map $T : \Omega \to \Omega$ is a *Banach contraction* [4] on Ω if there exists $\alpha \in [0, 1)$ for which

$$\delta(Tx, Ty) \leq \alpha\delta(x, y) \text{ for each } x, y \in \Omega. \tag{1.1}$$

Theorem: For a complete metric space (Ω, δ), if $T : \Omega \to \Omega$ satisfies (1.1) then T has a unique fixed point in Ω.

For a metric space (Ω, δ), a map $T : \Omega \to CB(\Omega)$ is a *Nadler contraction* [17] on Ω if there exists $\alpha \in [0, 1)$ for which

$$H(Tx, Ty) \leq \alpha\delta(x, y) \text{ for all } x, y \in \Omega.$$

1.3 Cone

For a complete normed space $\Im$, $N \subset \Im$ is said to be a *cone* [14] if it satisfies the conditions given as:

1. N is nonempty closed and $N \neq \{\theta\}$.

2. $\alpha x + \beta y$ belongs to N, whenever x, y are in N and α, β in $\mathbb{R}(\alpha, \beta \geq 0)$.

3. $N \cap (-N) = \{\theta\}$.

Given a cone $N \subset \Im$, a partial ordering $\leq$ with respect to N is defined as follows:

$$x \leq y \text{ iff } y - x \in N (x < y \text{ implies } x \leq y, \text{ where } x \neq y).$$

For $x, y \in N$, $x \ll y$ represents $y - x \in I(N)$, where $I(N)$ represents the interior of N.

A cone N is

1. *normal*, if $\|x\| \leq l\|y\|$ for all $x, y \in \Im$, whenever $\theta \leq x \leq y$ and $l > 0$ is a real constant.

2. *reproducing* [10] if $x = e - f(e, f \in N)$, for each $x \in \Im$. The elements e and f are not necessarily unique.

1.3.1 Lemma

For a complete normed space $\Im$ and a cone $N \subset \Im$, the equivalence of the following conditions is achieved [10]:

1. N is reproducing.

2. for each $x, y \in \Im$ a lower bound exists;

3. for every $x, y \in \Im$ an upper bound exists;

4. for all $x \in \Im$, there exist $e \geq 0$ in such a way $x \leq e$;

5. for each $x \in \Im$, there exist $e \leq 0$ in such a way $x \geq e$.

1.3.2 Cone Metric Space

For a set $\Omega \neq \phi$. A vector-valued function $\delta : \Omega \times \Omega \to \Im$ become a *cone metric* [14] if for each $x, y, z \in \Omega$ it satisfies:

1. $\delta(x, y) \geq \theta$.

2. $\delta(x, y) = \theta$ iff $x = y$.

3. $\delta(x, y) = \delta(y, x)$.

4. $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$.

Then (Ω, δ) become a *cone metric space.*

For a *cone metric space* (Ω, δ), $x \in \Omega$ and consider $\{x_r\}$ be a sequence in Ω then

1. $\{x_r\}$ converges towards x if for all $r \geq r_0, \delta(x_r, x) \ll c$ for each $c \in \Im$ where $\theta \ll c$ with $r_0 \in \mathbb{N}$ in such a way we have $\lim_{r \to \infty} x_r = x$.

2. $\{x_r\}$ be a Cauchy sequence if for all $r, m \geq r_0, \delta(x_r, x_m) \ll c$ for each $c \in \Im$ where $\theta \ll c$ with $r_0 \in \mathbb{N}$.

3. If the Cauchy convergence implies the convergence for every sequence in Ω, then (Ω, δ) is *complete.*

1.4 Multivalued Monotone Operators

In this section we will discuss increasing, decreasing, and fixed point of multivalued operator. Consider $(\Omega, \leq)$ be a partial ordered complete metric space, where the partial order $\leq$ is induced by cone N and $\prec$ is a partial order on 2^{Ω} [11].

1. A multivalued operator $T : \Omega \to 2^{\Omega}\backslash\{\phi\}$, if for each $w, z \in \Omega, w \leq z$ implies $T(w) \prec T(z)$ then T is increasing.

2. A multivalued operator $T : \Omega \to 2^{\Omega}\backslash\{\phi\}$, if for every $w, z \in \Omega, w \leq z$ implies $T(z) \prec T(w)$ then T is decreasing.

Let $T : \Omega \to 2^{\Omega}\backslash\{\phi\}$ be a multivalued operator, $w \in \Omega$ becomes a *fixed point* of T, if $w \in T(w)$. A point $w \in \Omega$ is said to be a *coincidence point* of a pair of multivalued mapping (T, S) if $Tw \cap Sw \neq \phi$ and if $w \in Tw \cap Sw$, then w is a *common fixed point* of the pair (T, S).

Chapter 2

Fixed Points and Coupled Fixed Points of Single-valued and Multivalued Mappings

In this chapter we include some relations on the subsets of a partially ordered set, fixed point results for multivalued operators having monotone and mixed monotone property, coupled fixed point results for single-valued as well as for multivalued mappings taken from the articles [2, 11, 7].

2.1 Fixed Points for Multivalued Monotone Operators

Feng and Wang [10] established some fixed point theorems in partial ordered Banach spaces. Then in [11] Feng and Wang proved some fixed point results for multivalued monotone and mixed monotone operators on the basis of characterizations of reproducing cones. In this section we will include the fixed point theorems presented in [11].

2.1.1 Relations on Subsets of a Partially Ordered Set

We consider two nonempty subsets G and H of $(\Omega, \leq)$, G and H have relations which are define as follows [2]:

1. $G \prec_1 H$ if for each $g \in G$, there exists $h \in H$ in such a way $g \leq h$;

2. $G \prec_2 H$ if for each $h \in H$, there exists $g \in G$ in such a way $g \leq h$;

3. $G \prec_3 H$ if $G \prec_1 H$ and $G \prec_2 H$;

4. $G \prec_4 H$ if for each $g \in G$, there exists $h \in H$ in such a way $g \asymp h$ (read as ;"a is compareble with b");

5. $G \prec_5 H$ if for each $h \in H$, there exists $g \in G$ in such a way $g \asymp h$.

2.1.2 Theorem

Suppose $T : \Omega \to 2^{\Omega}/(\phi)$, fulfil the following conditions [11]:

1. For all $w \in \Omega, T(w)$ is a nonempty and closed in Ω.

2. A linear operator $\Gamma : \Omega \to \Omega$ with spectral radius $\gamma(\Gamma) < 1, \Gamma(N) \subset N$ in such a way that for every $w, m \in \Omega, w \leq m$ we have:

(i) For every h in $T(w)$ there exists i in $T(m)$ for which

$$\theta \preceq i - h \preceq \Gamma(m - w).$$

(ii) For every i in $T(m)$ there exists h in $T(w)$ for which

$$\theta \preceq i - h \preceq \Gamma(m - w).$$

Then a fixed point of T exists in Ω.

Remark

1. In Theorem 2.1.2, the assumptions $2(i)$ and $2(ii)$ implies that $T(w) \prec T(m)$ for $w \leq m$, that is T is a multivalued increasing operator.

2. Theorem 2.1.2 does not implies the uniqueness of a fixed point.

3. If T is single-valued, condition 1 of Theorem 2.12 is satisfied and the condition 2 is as follow: "A linear operator $\Gamma : \Omega \to \Omega$ with spectral radius $\gamma(\Gamma) < 1, \Gamma(N) \subset N$ in such a way for every $w, m \in \Omega$, $w \leq m$ we have

$$T(m) - T(w) \preceq \Gamma(m - w)."$$

In this case we have the existence of fixed point of T which is unique.

Remark

In the absence of any one of the conditions of Theorem 2.1.2 the existence of a fixed point is not possible.

2.1.3 Theorem

Suppose $T : \Omega \to 2^{\Omega}/(\phi)$, fulfil the following conditions [11]:

1. For some $w \in \Omega, T(w)$ is a nonempty and closed subset of Ω.

2. A positive constant $\alpha \in (0,1)$ exists in such a way for every $w, m \in \Omega, w \leq m$ we have

(i) For every h in $T(w)$ there exists i in $T(m)$ for which

$$-\alpha(m-w) \preceq i-h \preceq \theta.$$

(ii) For every i in $T(m)$ there exists h in $T(w)$ for which

$$-\alpha(m-w) \preceq i-h \preceq \theta.$$

Then a fixed point of T in Ω.

2.2 Coupled Fixed Point for Single-valued Mappings

In the upcoming section, we will consider some results of coupled fixed point for single-valued mappings taken from [7, 11].

2.2.1 Coupled Fixed Point

The mapping $\vartheta : \Omega \times \Omega \to \Omega$ has a *coupled fixed point* $(u, v) \in \Omega \times \Omega$ if $\vartheta(u, v) = u, \vartheta(v, u) = v$ [7].

2.2.2 Mixed Monotone Property

Consider a partial ordered set $(\Omega, \leq)$ and the mapping $\vartheta : \Omega \times \Omega \to \Omega$ has *mixed monotone* (MM) property if $\vartheta(u, v)$ is monotonically nondecreasing in first component

$$u_1, u_2 \in \Omega, u_1 \leq u_2 \Rightarrow \vartheta(u_1, v) \leq \vartheta(u_2, v) \text{ for any } u, v \in \Omega,$$

and monotonically nonincreasing in second component [7].

$$v_1, v_2 \in \Omega, v_1 \leq v_2 \Rightarrow \vartheta(u, v_1) \geq \vartheta(u, v_2) \text{ for any } u, v \in \Omega.$$

Coupled fixed point results taken from [7].

2.2.3 Theorem

Consider a continuous mapping $\vartheta : \Omega \times \Omega \to \Omega$ having the MM-property on Ω. Assume
1. for some $\lambda \in [0, 1)$,

$$\delta(\vartheta(x, m), \vartheta(h, i)) \leq \frac{\lambda}{2}[\delta(x, h) + \delta(m, h)], \text{ forall } x \geq h, m \leq i,$$

and
2. if there exist $x_0, m_0 \in \Omega$ in such a way that

$$x_0 \leq \vartheta(x_0, m_0) \text{ and } m_0 \geq \vartheta(m_0, x_0).$$

Then x, $m \in \Omega$ exist such as

$$x = \vartheta(x, m) \text{ and } m = \vartheta(m, x).$$

2.2.4 Some Properties

The properties of product space $\Omega \times \Omega$ equipped with partial order are as follows:
1. For every $(h, i) \in \Omega \times \Omega$ a lower bound or an upper bound of (h, i) exist.
2. (see [18]) that condition (1) is similar to:
"For every (h, i) , (h', i') $\in \Omega \times \Omega$, there exists an element $(h^*, i^*) \in \Omega \times \Omega$ which can be compared to both (h, i) and (h', i) ."

2.2.5 Theorem

Consider a continuous mapping $\vartheta : \Omega \times \Omega \to \Omega$ having the MM property on Ω. Aussume

1. for some $\lambda \in [0, 1)$,

$$\delta(\vartheta(x, m), \vartheta(h, i)) \leq \frac{\lambda}{2}[\delta(x, h) + \delta(m, h)], \text{ for all } x \geq h, m \leq i,$$

and

2. If there exist $x_0, m_0 \in \Omega$ in such a way

$$x_0 \leq \vartheta(x_0, m_0) \text{ and } m_0 \geq \vartheta(m_0, x_0)$$

Then $x, m \in \Omega$ exist such as

$$x = \vartheta(x, m) \text{ and } m = \vartheta(m, x).$$

3. In addition if consider property 2 of (2.2.4) then coupled fixed point of ϑ will be unique.

2.2.6 Theorem

Consider a continuous mapping $\vartheta : \Omega \times \Omega \to \Omega$ having the MM property on Ω. Aussume

1. for some $\lambda \in [0, 1)$,

$$\delta(\vartheta(x, m), \vartheta(h, i)) \leq \frac{\lambda}{2}[\delta(x, h) + \delta(m, h)], \text{ for all } x \geq h, m \leq i,$$

and

2. If there exist $x_0, m_0 \in \Omega$ in such a way

$$x_0 \leq \vartheta(x_0, m_0) \text{ and } m_0 \geq \vartheta(m_0, x_0)$$

Then $x, m \in \Omega$ exist such as

$$x = \vartheta(x, m) \text{ and } m = \vartheta(m, x).$$

3. In addition, if we consider that every pair of elements of Ω has an upper bound or a lower bound in Ω. Then $x = m$.

2.2.7 Theorem

Consider a continuous mapping $\vartheta : \Omega \times \Omega \to \Omega$ having the MM property on Ω. Aussume
1. for some $\lambda \in [0, 1)$,

$$\delta(\vartheta(x, m), \vartheta(h, i)) \leq \frac{\lambda}{2}[\delta(x, h) + \delta(m, h)], \text{ for all } x \geq h, m \leq i,$$

and
2. If there exist $x_0, m_0 \in \Omega$ in such a way

$$x_0 \leq \vartheta(x_0, m_0) \text{ and } m_0 \geq \vartheta(m_0, x_0)$$

Then $x, m \in \Omega$ exist such as

$$x = \vartheta(x, m) \text{ and } m = \vartheta(m, x).$$

3. In addition, if we consider that x_0, m_0 in Ω are comparable. Then $x = m$.

Coupled fixed point theorem taken from [11].

2.2.8 Theorem

Suppose a mixed monotone operator $T : \Omega \times \Omega \to \Omega$. Consider that two linear operators $\Gamma, S : \Omega \to \Omega$ with $\|\Gamma\| + \|S\| < 1, \Gamma(N) \subset N, S(N) \subset N$ in such a way, for any

$x_1, x_2, m_1, m_2 \in \Omega, x_1 \leq x_2, m_2 \leq m_1$

$$T(x_2, m_2) - T(x_1, m_1) \leq \Gamma(x_2 - x_1) + S(m_1 - m_2).$$

Then a unique coupled fixed point $(\bar{x}, \bar{m})$ of T in $\Omega \times \Omega$. For every $(x, m) \in \Omega \times \Omega$.
$\lim_{r\to\infty} T^r(x, m) = \bar{x}, \lim_{r\to\infty} T^r(m, x) = \bar{m}$.

2.3 Coupled Fixed Point of Multivalued Monotone Operator

In this section we will, consider some results of coupled fixed point for multivalued mappings taken from [2, 11].

Assume $T : \Omega \times \Omega \to 2^{\Omega} \backslash \{\phi\}$ be a multivalued operator. Then

1. T has a *coupled fixed point* $(x, m) \in \Omega \times \Omega$ of, if $x \in T(x, m)$ and $m \in T(m, x)$ [11].
2. T is called *mixed monotone*, if for all $x_1, x_2, m_1, m_2 \in \Omega, x_1 \leq x_2, m_2 \leq m_1$ implies $T(x_1, m_1) \prec T(x_2, m_2)$ [11].

Coupled fixed point theorem for multivalued operator taken from [11].

2.3.1 Theorem

Suppose $T : \Omega \times \Omega \to 2^{\Omega}$ is a multivalued mixed monotone operator, fulfil the following conditions:

1. For every $(x, m) \in \Omega \times \Omega$, $T(x, m) \neq \phi$ and closed in Ω.
2. Two linear operators $\Gamma, S : \Omega \to \Omega$ exist with $\|\Gamma\| + \|S\| < 1, \Gamma(N) \subset N, S(N) \subset N$ in such a way, for every $x_1, x_2, m_1, m_2 \in \Omega, x_1 \leq x_2, m_2 \leq m_1$ we have:

(*i*) For every $h \in T(x_1, m_1)$, there exist $i \in T(x_2, m_2)$

$$0 \leq i - h \leq \Gamma(x_2 - x_1) + S(m_1 - m_2).$$

(*ii*) For every $i \in T(x_2, m_2)$, there exist $h \in T(x_1, m_1)$,

$$0 \leq i - h \leq \Gamma(x_2 - x_1) + S(m_1 - m_2).$$

Then a coupled fixed point of T in $\Omega \times \Omega$.

We define

$$\sigma(U,V) = \cap_{u\in U, v\in V} s(u-v) \text{ for } U,V \in C(\Omega).$$

2.3.3 Lemma

Consider a cone metric space (Ω,δ) with a cone N. If $q \in \sigma(U,V)$ then $\delta(u,v) \preceq q$ for all $u \in U, v \in V$ [2].

Coupled fixed point theorem for multivalued operator taken from [2].

2.3.4 Theorem

Consider (Ω,δ) be a complete cone metric space endowed with a partial order " $\leq$ " on Ω. Assume $\vartheta : \Omega \times \Omega \to C(\Omega)$ be a multivalued mapping having CCM property on Ω. Consider that

1. Ω has limit comparison property,
2. there exist a $0 \leq k < 1$ such that

$$\frac{k}{2}[\delta(x,h) + \delta(m,i)] \in \sigma(\vartheta(x,m), \vartheta(h,i)),$$

for every $x \asymp h, m \asymp i$, and

3. if there exist $x_0, m_0 \in \Omega$ in such a way that $\{x_0\} \leq_4 \vartheta(x_0,m_0)$ and $\vartheta(m_0,x_0) \leq_5 \{m_0\}$.

If Ω has limit comparison property then there exist $\bar{x}, \overline{m} \in \Omega$ such that $\bar{x} \in \vartheta(\bar{x},\overline{m})$ and $\overline{m} \in \vartheta(\overline{m},\bar{x})$.

$x_1, x_2, m_1, m_2 \in \Omega, x_1 \leq x_2, m_2 \leq m_1$

$$T(x_2, m_2) - T(x_1, m_1) \leq \Gamma(x_2 - x_1) + S(m_1 - m_2).$$

Then a unique coupled fixed point $(\bar{x}, \bar{m})$ of T in $\Omega \times \Omega$. For every $(x, m) \in \Omega \times \Omega$.
$\lim_{r\to\infty} T^r(x, m) = \bar{x}, \lim_{r\to\infty} T^r(m, x) = \bar{m}$.

2.3 Coupled Fixed Point of Multivalued Monotone Operator

In this section we will, consider some results of coupled fixed point for multivalued mappings taken from [2, 11].

Assume $T : \Omega \times \Omega \to 2^{\Omega} \backslash \{\phi\}$ be a multivalued operator. Then

1. T has a *coupled fixed point* $(x, m) \in \Omega \times \Omega$ of, if $x \in T(x, m)$ and $m \in T(m, x)$ [11].
2. T is called *mixed monotone*, if for all $x_1, x_2, m_1, m_2 \in \Omega, x_1 \leq x_2, m_2 \leq m_1$ implies $T(x_1, m_1) \prec T(x_2, m_2)$ [11].

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2.3.1 Theorem

Suppose $T : \Omega \times \Omega \to 2^{\Omega}$ is a multivalued mixed monotone operator, fulfil the following conditions:

1. For every $(x, m) \in \Omega \times \Omega$, $T(x, m) \neq \phi$ and closed in Ω.
2. Two linear operators $\Gamma, S : \Omega \to \Omega$ exist with $\|\Gamma\| + \|S\| < 1, \Gamma(N) \subset N, S(N) \subset N$ in such a way, for every $x_1, x_2, m_1, m_2 \in \Omega, x_1 \leq x_2, m_2 \leq m_1$ we have:

(i) For every $h \in T(x_1, m_1)$, there exist $i \in T(x_2, m_2)$

$$0 \leq i - h \leq \Gamma(x_2 - x_1) + S(m_1 - m_2).$$

(ii) For every $i \in T(x_2, m_2)$, there exist $h \in T(x_1, m_1)$,

$$0 \leq i - h \leq \Gamma(x_2 - x_1) + S(m_1 - m_2).$$

Then a coupled fixed point of T in $\Omega \times \Omega$.

Multivalued results

In the following, we list some properties in ordered cone metric space and for partially ordered sets [2].

1. An ordered cone metric space is said to have *limit comparision* property if for every non decreasing sequence $\{x_r\}$ in Ω with $x_r \to x$, we have $x_r \asymp x$, for all r.

2. An ordered cone metric space is said to have a *subsequential limit comparision* property if for every non decreasing sequence $\{x_r\}$ in Ω with $x_r \to x$, there exists a subsequence $\{x_{r_k}\}$ of $\{x_r\}$ such that $x_{r_k} \asymp x$, for all r.

3. Consider a partially ordered set $(\Omega, \leq)$ and $\vartheta : \Omega \times \Omega \to 2^{\Omega}$ be a set valued mapping . ϑ has *comparable combined monotone* (CCM) property if for any $x, m \in \Omega$,

$$x_1, x_2, m_1, m_2 \in \Omega, x_1 \asymp x_2 \text{ and } m_1 \asymp m_2 \Rightarrow \vartheta(x_1, m_1) \leq_4 \vartheta(x_2, m_2).$$

4. Consider a partially ordered set $(\Omega, \leq)$ and $\vartheta : \Omega \times \Omega \to 2^{\Omega}$ be a set valued mapping . ϑ has *combined monotone* (CM) property if for any $x, m \in \Omega$,

$$x_1, x_2, m_1, m_2 \in \Omega, x_1 \leq x_2 \text{ and } m_1 \geq m_2 \Rightarrow \vartheta(x_1, m_1) \leq_1 \vartheta(x_2, m_2).$$

2.3.2 Remark

Combined monotone property is equivalent to mixed monotone property in multivalued mappings [2].

Let $C(\Omega)$ the family of nonempty closed subsets of Ω, let $p \in \Im$ [8].

$$s(p) = \{q \in \Im : p \preceq q\} \text{ for } q \in \Im.$$

According to [2] for $U, V \in C(\Omega)$,

$$\sigma(U, V) = \cap_{u \in U, v \in V} s(\delta(u, v)) \text{ for } U, V \in C(\Omega).$$

We define

$$\sigma(U,V) = \cap_{u\in U, v\in V} s(u-v) \text{ for } U,V \in C(\Omega).$$

2.3.3 Lemma

Consider a cone metric space (Ω,δ) with a cone N. If $q \in \sigma(U,V)$ then $\delta(u,v) \preceq q$ for all $u \in U, v \in V$ [2].

Coupled fixed point theorem for multivalued operator taken from [2].

2.3.4 Theorem

Consider (Ω,δ) be a complete cone metric space endowed with a partial order " $\leq$ " on Ω. Assume $\vartheta : \Omega \times \Omega \to C(\Omega)$ be a multivalued mapping having CCM property on Ω. Consider that

1. Ω has limit comparison property,

2. there exist a $0 \leq k < 1$ such that

$$\frac{k}{2}[\delta(x,h) + \delta(m,i)] \in \sigma(\vartheta(x,m), \vartheta(h,i)),$$

for every $x \asymp h, m \asymp i$, and

3. if there exist $x_0, m_0 \in \Omega$ in such a way that $\{x_0\} \leq_4 \vartheta(x_0,m_0)$ and $\vartheta(m_0,x_0) \leq_5 \{m_0\}$.

If Ω has limit comparison property then there exist $\bar{x}, \overline{m} \in \Omega$ such that $\bar{x} \in \vartheta(\bar{x},\overline{m})$ and $\overline{m} \in \vartheta(\overline{m},\bar{x})$.

Chapter 3

Fixed Points and Coincidence Points of Multivalued Monotone Operators

In this chapter, we generalize the fixed points and coupled fixed points results already presented in [2, 7, 11].

3.1 Fixed Point of Multivalued Monotone Operators

In this section we will prove fixed point theorem for multivalued operator by considering generalized contractive condition used in [5]. This result will generalize Theorem 3.1 of [11].

3.1.1 Theorem

Suppose $T : \Omega \to 2^{\Omega}/(\phi)$, fulfil the following assumptions:

1. For every $w \in \Omega, T(w) \neq \phi$ and closed in Ω.

2. A linear operator $\Gamma : \Omega \to \Omega$ with spectral radius $\gamma(\Gamma) < 1, \Gamma(N) \subset N$, exists in such a way that for every $w, m \in \Omega$, $w \leq m$ we have:

(i) For every $h \in T(w)$, there exists $i \in T(m)$ for which

$$\theta \preceq i - h \preceq \rho \in \left\{ \begin{array}{c} \Gamma(m-w), \Gamma(h-w), \Gamma(i-m), \frac{\Gamma(i-w)+\Gamma(h-m)}{2}, \\ \frac{\Gamma(h-w)+\Gamma(i-m)}{2} \end{array} \right\},$$

(ii) For every $i \in T(m)$, there exists $h \in T(w)$ for which

$$\theta \preceq i - h \preceq \rho \in \left\{ \begin{array}{c} \Gamma(m-w), \Gamma(h-w), \Gamma(i-m), \frac{\Gamma(i-w)+\Gamma(h-m)}{2}, \\ \frac{\Gamma(h-w)+\Gamma(i-m)}{2} \end{array} \right\}.$$

Then a fixed point of T in Ω.

Proof:

Case 1: If $\rho = \Gamma(m-w)$. Already proved by [11].

Case 2: If $\rho = \Gamma(h-w)$. There exists $w_0 \in \Omega$, such that $\{w_0\} \prec_1 T(w_0)$. In fact

(i) if $\{\theta\} \prec_1 T(\theta)$, then $w_0 = \theta$;

(ii) if $\{\theta\} \prec_1 T(\theta)$ is not satisfied .

Consider $m_0 \in T(\theta)$. As N is reproducing, by Lemma 1.3.1, there is $i \in (-N)$ in such a way that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theorem the equation $(I - \Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$ by condition 2(ii), there exists $h_0 \in T(w_0)$ in such a way that

$$\theta \preceq m_0 - h_0 \preceq \Gamma(h_0 - w_0),$$

that is

$$m_0 - \Gamma(h_0) - \Gamma(t_0) \preceq h_0.$$

Since $i \preceq m_0$,

$$h_0 \succeq i - \Gamma(t_0) - \Gamma(h_0).$$

Since t_0 is the solution so we have

$$h_0 + \Gamma(h_0) \succeq w_0,$$

as $h_0 \preceq m_0 \in -N$ therefore $h_0 \in -N$ and so

$$h_0 \succeq h_0 + \Gamma(h_0) \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.

Assuming $w_1 = h_0$ and because of condition $2(i)$, there exists $w_2 \in T(w_1)$ in such a way that

$$\theta \preceq w_2 - w_1 \leq \Gamma(w_1 - w_0).$$

Due to the fact that N is normal, there exists $l_1 > 0$, such that

$$\|w_2 - w_1\| \leq l_1\|\Gamma\|\|w_1 - w_0\|.$$

Now for $w_2 \in T(w_1)$ by condition $2(i)$, there exists $w_3 \in T(w_2)$ in such a way that

$$\theta \preceq w_3 - w_2 \preceq \Gamma(w_2 - w_1),$$

that is

$$\begin{aligned} \|w_3 - w_2\| &\leq l_2\|\Gamma\|\|w_2 - w_1\|, \text{ where } l_2 > 0, \\ &\leq l_2 l_1\|\Gamma\|^2\|w_1 - w_0\|. \end{aligned}$$

Continuing in this way, we have

$$\|w_{r+1} - w_r\| \leq l_r \ldots l_2 l_1\|\Gamma\|^r\|w_1 - w_0\|,$$

by considering $l_r \ldots l_2 l_1 = l$, we obtain

$$\|w_{r+1} - w_r\| \leq l\|\Gamma\|^r\|w_1 - w_0\|.$$

Since $lim_{r\to\infty}(\|\Gamma^r\|)^{1/r} = q < 1$, we have $\|\Gamma^r\| \leq q^r$ for some $q \in (0,1)$ and for all sufficiently large r

$$\|w_{r+1} - w_r\| \leq lq^r\|w_1 - w_0\|.$$

UPThis implies $\{w_r\}$ is fundamental. As Ω is complete, a unique element $w^* \in \Omega$ exists in such a way that $w_r \to w^*$. Since $\{w_r\}$ is an increasing sequence such that $w_{r+1} \in T(w_r)$, therefore

$w_r \leq w^*$ for all r. By condition 2(i), there exists $m_r \in T(w^*)$ such that

$$\theta \preceq m_r - w_{r+1} \preceq \Gamma(w_r - w_{r+1}).$$

Due to the fact that N is normal, there exists $l > 0$ such that

$$\|m_r - w_{r+1}\| \leq l\|\Gamma\|\|w_r - w_{r+1}\|,$$

which implies $lim_{r\to\infty} m_r = w^*$. As we know that $T(w^*)$ is closed, we have $w^* \in T(w^*)$.

Case 3: If $\rho = \Gamma(i - m)$. There exists $w_0 \in \Omega$, such that $\{w_0\} \prec_1 T(w_0)$. In fact

(i) if $\{\theta\} \prec_1 T(\theta)$, then $w_0 = \theta$;

(ii) if $\{\theta\} \prec_1 T(\theta)$ is not satisfied.

Consider $m_0 \in T(\theta)$. As N is reproducing , by Lemma 1.3.1, there is $i \in (-N)$ such that $i \preceq m_0$. Because of given condition $r(\Gamma) < 1$, by Banach's contraction theorem.the equation $(I - \Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$, by condition 2(ii), there exists $h_0 \in T(w_0)$ in such a way that

$$\theta \preceq m_0 - h_0 \preceq \Gamma(m_0 - \theta),$$

that is

$$h_0 \succeq i - \Gamma(m_0).$$

Since t_0 is the solution so we have,

$$h_0 \succeq i - \Gamma(m_0) + \Gamma(t_0) - \Gamma(t_0) \succeq w_0 + \Gamma(t_0 - m_0).$$

As $t_0, -m_0 \in N$, therefore

$$h_0 \succeq w_0 + \Gamma(t_0 - m_0) \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.

Assuming $w_1 = h_0$ and using condition 2(i), there exists $w_2 \in T(w_1)$ in such a way that

$$\theta \preceq w_2 - w_1 \preceq \Gamma(w_2 - w_1).$$

Due to the fact that N is normal, there exists $l_1 > 0$, such that

$$\|w_2 - w_1\| \leq l_1\|\Gamma\|\|w_2 - w_1\|,$$

that is

$$(1 - l_1\|\Gamma\|)\,\|w_2 - w_1\| \leq 0,$$

which gives $w_1 = w_2$. Hence $w_1 \in T(w_1)$.

Case 4: If $\rho = \frac{\Gamma(i-w)+\Gamma(h-m)}{2}$. There exists $w_0 \in \Omega$, such that $\{w_0\} \prec_1 T(w_0)$. In fact

(i). if $\{\theta\} \prec_1 T(\theta)$, then $w_0 = \theta$;

(ii). if $\{\theta\} \prec_1 T(\theta)$ is not satisfied.

Consider $m_0 \in T(\theta)$. As N is reproducing , by Lemma 1.3.1, there is $i \in (-N)$ in such a way that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theorem$(I - \Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$ by condition $2(ii)$, there exists $h_0 \in T(w_0)$ in such a way that

$$\theta \preceq m_0 - h_0 \preceq \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2},$$

which implies

$$\begin{aligned}
h_0 &\succeq m_0 - \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2} \\
&\succeq i - \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2} \\
&\succeq i - \Gamma(t_0) + \Gamma(t_0) - \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2} \\
&\succeq w_0 + \frac{\Gamma(-w_0)}{2} + \frac{\Gamma(-m_0)}{2} + \frac{\Gamma(-h_0)}{2}.
\end{aligned}$$

Since $w_0, m_0, h_0 \in -N$, therefore

$$h_0 \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.

Assuming $w_1 = h_0$ and using condition 2(i), there exists $w_2 \in T(w_1)$ in such a way that

$$\theta \preceq w_2 - w_1 \preceq \frac{\Gamma(w_2 - w_0) + \Gamma(w_1 - w_1)}{2} \preceq \frac{\Gamma(w_2 - w_1) + \Gamma(w_1 - w_0)}{2}.$$

Due to the fact that N is normal, there exists $l_1 > 0$, such that

$$\|w_2 - w_1\| \le \frac{l_1\|\Gamma\|(\|w_2 - w_1\| + \|w_1 - w_0\|)}{2},$$

which gives

$$\left(1 - \frac{l_1\|\Gamma\|}{2}\right)\|w_2 - w_1\| \le \frac{l_1\|\Gamma\|}{2}\|w_1 - w_0\|.$$

As $2 - l_1\|\Gamma\| \neq 0$, so

$$\|w_2 - w_1\| \le \frac{l_1\|\Gamma\|}{2 - l_1\|\Gamma\|}\|w_1 - w_0\|.$$

Let $a_1 = \frac{l_1}{2 - l_1\|\Gamma\|} < 1$

$$\|w_2 - w_1\| \le a_1\|\Gamma\|\|w_1 - w_0\|.$$

Continuing in this manner we have

$$\|w_{r+1} - w_r\| \le a_r ... a_1\|\Gamma\|^r\|w_1 - w_0\|,$$

that is

$$\|w_{r+1} - w_r\| \le a\|\Gamma\|^r\|w_1 - w_0\|, \text{ where } a = a_r ... a_1.$$

Since $\lim_{r\to\infty}(\|\Gamma^r\|)^{1/r} = q < 1$, we have$\|\Gamma^r\| < q^r$ for some $q \in (0,1)$ and for all sufficiently large r

$$\|w_{r+1} - w_r\| \le aq^r\|w_1 - w_0\|.$$

This implies $\{w_r\}$ is fundamental. As Ω is complete, a unique element $w^* \in \Omega$ exists, in such a way that $w_r \to w^*$. Since $\{w_r\}$ is an increasing sequence such that $w_{r+1} \in T(w_r)$, therefore $w_r \le w^*$ for all r. By condition 2(i), there exist $m_r \in T(w^*)$ in such a way that

$$\theta \preceq m_r - w_{r+1} \preceq \frac{\Gamma(m_r - w_r) + \Gamma(w_{r+1} - w^*)}{2}.$$

Due to the fact that N is normal, there exists $l > 0$ such that

$$\|m_r - w_{r+1}\| \le \frac{l\|\Gamma\|}{2}\|(m_r - w_r + w_{r+1} - w^*)\|.$$

which implies $lim_{r\to\infty} m_r = w^*$. As we know that $T(w^*)$ is closed, we have $w^* \in T(w^*)$.

Case 5: If $\rho = \frac{\Gamma(h-w)+\Gamma(i-m)}{2}$. There exists $w_0 \in \Omega$, such that $\{w_0\} \prec_1 T(w_0)$. In fact

(i) if $\{\theta\} \prec_1 T(\theta)$, then $w_0 = \theta$;

(ii) if $\{\theta\} \prec_1 T(\theta)$ is not satisfied.

Let $m_0 \in T(\theta)$. Since N is reproducing , by Lemma 1.3.1, there is $i \in (-N)$ such that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theorem the equation $(I-\Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$, by condition $2(ii)$, there exists $h_0 \in T(w_0)$ in such a way that

$$\theta \preceq m_0 - h_0 \preceq \frac{\Gamma(h_0 - w_0) + \Gamma(m_0)}{2},$$

which implies

$$\begin{aligned} h_0 &\succeq m_0 - \frac{\Gamma(h_0 - w_0) + \Gamma(m_0)}{2} \\ &\succeq i - \Gamma(t_0) + \Gamma(t_0) - \frac{\Gamma(h_0 - w_0) + \Gamma(m_0)}{2} \\ &\succeq w_0 + \Gamma(t_0) - \frac{\Gamma(h_0 - w_0) + \Gamma(m_0)}{2} \\ &\succeq w_0 - \frac{\Gamma(w_0)}{2} - \frac{\Gamma(h_0)}{2} - \frac{\Gamma(m_0)}{2}. \end{aligned}$$

Since $w_0, m_0, h_0 \in -N$, therefore

$$h_0 \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.
considering $w_1 = h_0$ and using condition $2(i)$, there exists $w_2 \in T(w_1)$ in such a way that

$$\theta \preceq w_2 - w_1 \preceq \frac{\Gamma(w_2 - w_1) + \Gamma(w_1 - w_0)}{2}.$$

By using the same arguments as in case 4, we observe that $\{w_r\}$ is fundamental. As Ω is complete, a unique element $w^* \in \Omega$ exists, in such a way that $w_r \to w^*$. Since $\{w_r\}$ is an increasing sequence such that $w_{r+1} \in T(w_r)$, therefore $w_r \leq w^*$ for all r. By condition $2(i)$, there exist $m_r \in T(w^*)$ such that

$$\theta \preceq m_r - w_{r+1} \preceq \frac{\Gamma(w_{r+1} - w_r) + \Gamma(m_r - w^*)}{2}.$$

Due to the fact that N is normal, there exists $l > 0$ such that

$$\|m_r - w_{r+1}\| \leq \frac{l\|\Gamma\|}{2}\|(w_{r+1} - w_r + m_r - w^*)\|.$$

which implies $lim_{r\to\infty} m_r = w^*$. As we know that $T(w^*)$ is closed, we have $w^* \in T(w^*)$.

3.2 Coincidence Point of Multivalued Monotone Operator

We will prove coincidence point theorem for a pair of multivalued operators.

3.2.1 Theorem

Suppose $T, S : \Omega \to 2^{\Omega}/\{\phi\}$ fulfil the following conditions:

1. For every $w \in \Omega$, $T(w)$ and $S(w)$ are nonempty and closed subsets of Ω.

2. $\{\theta\} \preceq_1 S(\theta)$ is not satisfied.

3. A linear operator $\Gamma : \Omega \to \Omega$ with spectral radius $\gamma(\Gamma) < 1, \Gamma(N) \subset N$ in such a way for every $w, m \in \Omega, w \preceq m$ we have:

(i) For every $h \in T(w)$, there exists $i \in S(m)$ or

(ii) For every $h \in S(w)$, there exists $i \in T(m)$ or

(iii) For every $i \in S(m)$, there exists $h \in T(w)$ in such a way that

$$\theta \preceq i - h \preceq \rho \in \left\{ \begin{array}{c} \Gamma(m-w), \Gamma(h-w), \Gamma(i-m), \frac{\Gamma(i-w)+\Gamma(h-m)}{2}, \\ \frac{\Gamma(h-w)+\Gamma(i-m)}{2} \end{array} \right\}.$$

Then T and S have a common fixed point in Ω.

Proof:

Case 1: If $\rho = \Gamma(m - w)$. Since $\{\theta\} \prec_1 S(\theta)$ is not satisfied.

Let $m_0 \in S(\theta)$. As N is reproducing, by Lemma 1.3.1, there is $i \in (-N)$ such that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theorem the equation $(I - \Gamma)w =$

$-i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$, by condition $3(iii)$ for $m_0 \in S(\theta)$, there exists $h_0 \in T(w_0)$ in such a way that

$$\theta \preceq m_0 - h_0 \preceq \Gamma(\theta - w_0),$$

that is

$$\theta \preceq m_0 - h_0 \preceq \Gamma(-w_0)$$

or

$$m_0 - \Gamma(t_0) \preceq m_0 - \Gamma(-w_0) \preceq h_0.$$

As $i \preceq m_0$,

$$i - \Gamma(t_0) \preceq h_0. \tag{3.1}$$

As t_0 is the solution therefore

$$(I - \Gamma)t_0 = -i,$$

which gives

$$i - \Gamma(t_0) = -t_0. \tag{3.2}$$

By using equations 3.1 and 3.2, we have

$$h_0 \succeq -t_o = w_0,$$

which implies that $\{w_0\} \prec_1 T(w_0)$
considering $w_1 = h_0$ and using condition $3(i)$, there exists $w_2 \in S(w_1)$ in such a way that

$$w_2 - w_1 \preceq \Gamma(w_1 - w_0).$$

Due to the fact that N is normal, there exists $l_1 > 0$, such that

$$\|w_2 - w_1\| \leq l_1 \|\Gamma\| \|w_1 - w_0\|, \text{ where } l_1 > 0.$$

Now for $w_2 \in S(w_1)$ by condition $3(ii)$, there exists $w_3 \in T(w_2)$ in such a way that

$$w_3 - w_2 \preceq \Gamma(w_2 - w_1),$$

that is

$$\begin{aligned} \|w_3 - w_2\| &\leq l_2\|\Gamma\|\|w_2 - w_1\|, \text{ where } l_2 > 0, \\ &\leq l_2 l_1 \|\Gamma\|^2 \|w_1 - w_0\|. \end{aligned}$$

Continuing in this way, we have

$$\|w_{r+1} - w_r\| \leq l_r ... l_2 l_1 \|\Gamma\|^r \|w_1 - w_0\|.$$

that is

$$\|w_{r+1} - w_r\| \leq l\|\Gamma\|^r \|w_1 - w_0\|, \text{ where } l_r ... l_2 l_1 = l.$$

As $lim_{r\to\infty}(\|\Gamma^r\|)^{1/r} = q < 1$, we have $\|\Gamma^r\| \leq q^r$ for some $q \in (0,1)$ and for all sufficiently large r

$$\|w_{r+1} - w_r\| \leq l\|\Gamma^r\|\|w_1 - w_0\| \leq lq^r\|w_1 - w_0\|.$$

This implies $\{w_r\}$ is fundamental. As Ω is complete, a uniqe element $w^* \in \Omega$ exists, in such a way that $w_r \to w^*$. As $\{w_r\}$ is an increasing sequence, $w_r \preceq w^*$, for $r = 0, 1, 2, 3, ...$ Now there exist two subsequences $\{w_{2r+1}\}$ and $\{w_{2r+2}\}$, where $w_{2r+1} \in T(w_{2r})$ and $w_{2r+2} \in S(w_{2r+1})$, such that $w_{2r} \preceq w^*$ and $w_{2r+1} \preceq w^*$, for $r = 0, 1, 2, 3, ...$ So by given conditions $3(i)$ (and $3(ii)$), there exist $m_r \in S(w^*)$ (and $t_r \in T(w^*)$) in such a way that

$$\theta \preceq m_r - w_{2r+1} \preceq \Gamma(w^* - w_{2r}) \ \ (\text{and } \theta \preceq t_r - w_{2r+2} \preceq \Gamma(w^* - w_{2r+1})).$$

Due to the fact that N is normal, there exist $l_1, l_2 > 0$, in such a way that

$$\|m_r - w_{2r+1}\| \leq l_1\|\Gamma\|\|w^* - w_{2r}\| \ \ (\text{and } \|t_r - w_{2r}\| \leq l_2\|\Gamma\|\|w^* - w_{2r+1}\|)$$

which implies $lim_{r\to\infty} m_r = w^* = lim_{r\to\infty} t_r$. As we know that $T(w^*)$ and $S(w^*)$ are closed, so we have $w^* \in T(w^*) \cap S(w^*)$.

Case 2: If $\rho = \Gamma(h - w)$. As $\{\theta\} \prec_1 S(\theta)$ is not satisfied.

Let $m_0 \in S(\theta)$. As N is reproducing, by Lemma 1.3.1, there is $i \in (-N)$ such that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theoremthe equation $(I - \Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$, by condition $3(iii)$ for $m_0 \in S(\theta)$, there exists $h_0 \in T(w_0)$ such that

$$\theta \preceq m_0 - h_0 \preceq \Gamma(h_0 - w_0),$$

that is,

$$-h_0 \preceq -m_0 + \Gamma(h_0 - w_0),$$

which implies

$$i - \Gamma(t_0) \preceq m_0 - \Gamma(t_0) \preceq h_0 + \Gamma(h_0).$$

As t_0 is the solution so we have

$$h_0 + \Gamma(h_0) \succeq w_0.$$

As $h_0 \preceq m_0 \in -N$ therefore $h_0 \in -N$ and so

$$h_0 \succeq h_0 + \Gamma(h_0) \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.

consider $w_1 = h_0$ and using conditions $3(i)$, there exists $w_2 \in S(w_1)$ insuch a way that

$$\theta \preceq w_2 - w_1 \preceq \Gamma(w_1 - w_0).$$

By condition $3(ii)$, there exists $w_3 \in T(w_2)$ in such a way that

$$\theta \preceq w_3 - w_2 \preceq \Gamma(w_2 - w_1).$$

Now applying the same procedure as in Case 1 we have a fundamental sequence $\{w_r\}$ in Ω. As $\{w_r\}$ is an increasing sequence, $w_r \preceq w^*$, for $r = 0, 1, 2, 3, \ldots$. Now there exists two subsequences $\{w_{2r+1}\}$ and $\{w_{2r+2}\}$, where $w_{2r+1} \in T(w_{2r})$ and $w_{2r+2} \in S(w_{2r+1})$, such that $w_{2r} \preceq w^*$

and $w_{2r+1} \preceq w^*$, for $r = 0, 1, 2, 3, ...$ So by given conditions $3(i)$ (and $3(ii)$), there exist $m_r \in S(w^*)$ (and $t_r \in T(w^*)$) such that

$$\theta \preceq m_r - w_{2r+1} \preceq \Gamma(w_{2r+1} - w_{2r}) \ \text{(and } \theta \preceq t_r - w_{2r+2} \preceq \Gamma(w_{2r+2} - w_{2r+1})\text{)}.$$

By using triangular inequality and the definition of normality we deduce

$$lim_{r \to \infty} m_r = w^* = lim_{r \to \infty} t_r.$$

As we know that $T(w^*)$ and $S(w^*)$ are closed, so we have $w^* \in T(w^*) \cap S(w^*)$.

Case 3 : If $\rho = \Gamma(i - m)$. As $\{\theta\} \prec_1 S(\theta)$ is not satisfied.

Let $m_0 \in S(\theta)$. As N is reproducing , by Lemma 1.3.1, there is $i \in (-N)$ such that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theorem the equation $(I - \Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$, by condition $3(iii)$, for $m_0 \in S(\theta)$, there exists $h_0 \in T(w_0)$ such that

$$\theta \preceq m_0 - h_0 \preceq \Gamma(m_0 - \theta),$$

that is

$$h_0 \succeq i - \Gamma(m_0).$$

As t_0 is the solution so we have,

$$h_0 \succeq i - \Gamma(m_0) + \Gamma(t_0) - \Gamma(t_0) \succeq w_0 + \Gamma(t_0 - m_0).$$

As $t_0, -m_0 \in N$, therefore

$$h_0 \succeq w_0 + \Gamma(t_0 - m_0) \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.

Assuming $w_1 = h_0$ and using condition $3(i)$, there exists $w_2 \in S(w_1)$ such that

$$\theta \preceq w_2 - w_1 \preceq \Gamma(w_2 - w_1).$$

Due to the fact that N is normal, there exists $l_1 > 0$, such that

$$\|w_2 - w_1\| \leq l_1\|\Gamma\|\|w_2 - w_1\|,$$

that is

$$(1 - l_1\|\Gamma\|)\,\|w_2 - w_1\| \leq 0$$

which gives $w_1 = w_2$. By condition $3(ii)$, for $w_2 \in S(w_2)$ there exists $w_3 \in T(w_2)$ such that

$$\theta \preceq w_3 - w_2 \preceq \Gamma(w_3 - w_2)$$

that is $w_2 = w_3$. Hence $w_2 \in S(w_2) \cap T(w_2)$.

Case 4: If $\rho = \frac{\Gamma(i-w)+\Gamma(h-m)}{2}$. As $\{\theta\} \prec_1 S(\theta)$ is not satisfied.

Let $m_0 \in S(\theta)$. As N is reproducing , by Lemma 1.3.1, there is $i \in (-N)$ such that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theorem the equation $(I - \Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$, by condition $3(iii)$, for $m_0 \in S(\theta)$, there exists $h_0 \in T(w_0)$ in such a way that

$$0 \preceq m_0 - h_0 \preceq \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2},$$

which implies

$$\begin{aligned} h_0 &\succeq m_0 - \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2} \\ &\succeq i - \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2} \\ &\succeq i - \Gamma(t_0) + \Gamma(t_0) - \frac{\Gamma(m_0 - w_0) + \Gamma(h_0)}{2} \\ &\succeq w_0 + \frac{\Gamma(-w_0)}{2} + \frac{\Gamma(-m_0)}{2} + \frac{\Gamma(-h_0)}{2}. \end{aligned}$$

As $w_0, m_0, h_0 \in -N$, therefore

$$h_0 \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.

Assuming $w_1 = h_0$ and using condition $3(i)$, there exists $w_2 \in S(w_1)$ such that

$$0 \preceq w_2 - w_1 \preceq \frac{\Gamma(w_2 - w_0) + \Gamma(w_1 - w_1)}{2} \preceq \frac{\Gamma(w_2 - w_1) + \Gamma(w_1 - w_0)}{2}.$$

Due to the fact that N is normal, there exists $l_1 > 0$, such that

$$\|w_2 - w_1\| \leq \frac{l_1\|\Gamma\|(\|w_2 - w_1\| + \|w_1 - w_0\|)}{2},$$

which gives

$$\left(1 - \frac{l_1\|\Gamma\|}{2}\right)\|w_2 - w_1\| \leq \frac{l_1\|\Gamma\|}{2}\|w_1 - w_0\|.$$

As $2 - l_1\|\Gamma\| \neq 0$, so

$$\|w_2 - w_1\| \leq \frac{l_1\|\Gamma\|}{2 - l_1\|\Gamma\|}\|w_1 - w_0\|.$$

Let $a_1 = \frac{l_1}{2 - l_1\|\Gamma\|} < 1$

$$\|w_2 - w_1\| \leq a_1\|\Gamma\|\|w_1 - w_0\|.$$

Continuing in this manner we have

$$\|w_{r+1} - w_r\| \leq a_r \ldots a_1\|\Gamma\|^r\|w_1 - w_0\|,$$

that is

$$\|w_{r+1} - w_r\| \leq a\|\Gamma\|^r\|w_1 - w_0\|, \text{ where } a = a_r \ldots a_1.$$

As $\lim_{r\to\infty}(\|\Gamma^r\|)^{1/r} = q < 1$, we have$\|\Gamma^r\| < q^r$ for some $q \in (0,1)$ and for all sufficiently large r

$$\|w_{r+1} - w_r\| \leq aq^r\|w_1 - w_0\|.$$

This implies $\{w_r\}$ is fundamental. As $\{w_r\}$ is an increasing sequence, $w_r \preceq w^*$, for $r = 0, 1, 2, 3, \ldots$. Now there exist two subsequences $\{w_{2r+1}\}$ and $\{w_{2r+2}\}$, where $w_{2r+1} \in T(w_{2r})$ and $w_{2r+2} \in S(w_{2r+1})$, such that $w_{2r} \preceq w^*$ and $w_{2r+1} \preceq w^*$, for $r = 0, 1, 2, 3, \ldots$. So by given conditions $3(i)$ (and $3(ii)$), there exist $m_r \in S(w^*)$ (and $t_r \in T(w^*)$) such that

$$\theta \preceq m_r - w_{2r+1} \preceq \frac{\Gamma(m_r - w_{2r}) + \Gamma(w_{2r+1} - w^*)}{2}$$

and

$$\theta \preceq t_r - w_{2r+2} \preceq \frac{\Gamma(t_r - w_{2r+1}) + \Gamma(w_{2r+2} - w^*)}{2}.$$

By using the definition of normality we deduce $lim_{r\to\infty} m_r = w^* = lim_{r\to\infty} t_r$. Noting that $T(w^*)$ and $S(w^*)$ are closed, so we have $w^* \in T(w^*) \cap S(w^*)$.

Case 5: If $\rho = \frac{\Gamma(h-w)+\Gamma(i-m)}{2}$. As $\{\theta\} \prec_1 S(\theta)$ is not satisfied.

Let $m_0 \in S(\theta)$. As N is reproducing , by Lemma 1.3.1, there is $i \in (-N)$ such that $i \preceq m_0$. Because of given condition $\gamma(\Gamma) < 1$, by Banach's contraction theorem the equation $(I - \Gamma)w = -i$ has a unique solution $t_0 \in N$. Let $w_0 = -t_0$, by condition $3(iii)$ for $m_0 \in S(\theta)$, there exists $h_0 \in T(w_0)$ in such a way that

$$0 \preceq m_0 - h_0 \preceq \frac{\Gamma(h_0 - w_0) + \Gamma(m_0 - \theta)}{2},$$

which implies

$$\begin{aligned} h_0 &\succeq m_0 - \frac{\Gamma(h_0 - w_0) + \Gamma(m_0)}{2} \\ &\succeq i - \frac{\Gamma(h_0 - w_0) + \Gamma(m_0)}{2} \\ &\succeq i - \frac{\Gamma(-w_0)}{2} + \frac{\Gamma(-m_0)}{2} + \frac{\Gamma(-h_0)}{2} \\ &\succeq i - \Gamma(t_0) + \Gamma(t_0) - \frac{\Gamma(t_0)}{2} + \frac{\Gamma(-m_0)}{2} + \frac{\Gamma(-h_0)}{2} \\ &\succeq w_0 + \frac{\Gamma(-w_0)}{2} + \frac{\Gamma(-m_0)}{2} + \frac{\Gamma(-h_0)}{2}. \end{aligned}$$

As $w_0, m_0, h_0 \in -N$, therefore

$$h_0 \succeq w_0,$$

which implies $\{w_0\} \prec_1 T(w_0)$.

Consider $w_1 = h_0$ and using condition $3(i)$, there exists $w_2 \in S(w_1)$ in such a way that

$$\theta \preceq w_2 - w_1 \preceq \frac{\Gamma(w_1 - w_0) + \Gamma(w_2 - w_1)}{2}.$$

By using the same arguments as in case 4, we observe that $\{w_r\}$ is fundamental. As $\{w_r\}$ is an increasing sequence, $w_r \preceq w^*$, for $r = 0, 1, 2, 3, ...$ Now there exists two subsequences

$\{w_{2r+1}\}$ and $\{w_{2r+2}\}$, where $w_{2r+1} \in T(w_{2r})$ and $w_{2r+2} \in S(w_{2r+1})$, such that $w_{2r} \preceq w^*$ and $w_{2r+1} \preceq w^*$, for $r = 0, 1, 2, 3, \ldots$. So by given condition $3(i)$ (and $3(ii)$), there exist $m_r \in S(w^*)$ (and $t_r \in T(w^*)$) in such a way that

$$\theta \preceq m_r - w_{2r+1} \preceq \frac{\Gamma(w_{2r+1} - w_{2r}) + \Gamma(m_r - w^*)}{2},$$

and

$$\theta \preceq t_r - w_{2r+2} \preceq \frac{\Gamma(w_{2r+2} - w_{2r+1}) + \Gamma(t_r - w^*)}{2}.$$

By using the definition of normality we deduce $lim_{r \to \infty} m_r = w^* = lim_{r \to \infty} t_r$. As we know that $T(w^*)$ and $S(w^*)$ are closed, so we have $w^* \in T(w^*) \cap S(w^*)$.

3.3 Coupled Fixed Point Theorems for Operators having MM Property

In this section we will prove some coupled fixed point theorem for a mixed monotone operator, which are geralizations of [7, 11].

3.3.1 Theorem

Suppose $\vartheta : \Omega \times \Omega \longrightarrow \Omega$ is a mixed monotone operator. Consider a linear operator $\Gamma : \Omega \to \Omega$ with $\|\Gamma\| < 1, \Gamma(N) \subset N$ and there exists $k \in [0, 1)$ in such a way

$$\vartheta(x, m) - \vartheta(h, i) \preceq \frac{k}{2}\left(\Gamma(x - h) + \Gamma(i - m)\right),$$

for every $x, m, h, i \in \Omega$ with $h \preceq x, m \preceq i$. If $x_0, m_0 \in \Omega$ exist in such a way that

$$x_0 \preceq \vartheta(x_0, m_0) \text{ and } m_0 \succeq \vartheta(m_0, x_0).$$

Then there exist $x, m \in \Omega$ in such a way

$$x = \vartheta\left(x, m\right) \text{ and } m = \vartheta\left(m, x\right).$$

Proof: As $x_0 \preceq \vartheta(x_0, m_0) = x_1$ (say), $m_0 \succeq \vartheta(m_0, x_0) = m_1$ (say).
Letting $x_2 = \vartheta(x_1, m_1)$ and $m_2 = \vartheta(m_1, x_1)$.
Due to mixed monotone property of ϑ,

$$\begin{aligned} x_2 &= \vartheta(\vartheta(x_0, m_0), \vartheta(m_0, x_0)) = \vartheta(x_1, m_1) \succeq x_1, \\ m_2 &= \vartheta(\vartheta(m_0, x_0), \vartheta(x_0, m_0)) = \vartheta(m_1, x_1) \preceq m_1. \end{aligned}$$

Further for $r = 1, 2, 3 ...$,

$$\begin{aligned} x_{r+1} &= \vartheta^{r+1}(x_0, m_0) = \vartheta(\vartheta^r(x_0, m_0), \vartheta^r(m_0, x_0)) = \vartheta(x_r, m_r) \succeq x_r, \\ m_{r+1} &= \vartheta^{r+1}(m_0, x_0) = \vartheta(\vartheta^r(m_0, x_0), \vartheta^r(x_0, m_0)) = \vartheta(m_r, x_r) \preceq m_r. \end{aligned}$$

Then by using given condition,

$$\begin{aligned} x_2 - x_1 &= \vartheta(x_1, m_1) - \vartheta(x_0, m_0) \preceq \frac{k}{2}\left(\Gamma(x_1 - x_0) + \Gamma(m_0 - m_1)\right), \\ m_1 - m_2 &= \vartheta(m_0, x_0) - \vartheta(m_1, x_1) \preceq \frac{k}{2}\left(\Gamma(m_0 - m_1) + \Gamma(x_1 - x_0)\right), \end{aligned}$$

which gives from the normality of cone N,

$$\begin{aligned} \|x_2 - x_1\| &\leq \frac{a_1 k}{2}\|\Gamma\|\left(\|x_1 - x_0\| + \|m_0 - m_1\|\right), \\ \|m_1 - m_2\| &\leq \frac{b_1 k}{2}\|\Gamma\|\left(\|m_0 - m_1\| + \|x_1 - x_0\|\right). \end{aligned}$$

$$\begin{aligned} x_3 - x_2 &= \vartheta(x_2, m_2) - \vartheta(x_1, m_1) \preceq \frac{k}{2}\left(\Gamma(x_2 - x_1) + \Gamma(m_1 - m_2)\right), \\ m_2 - m_3 &= \vartheta(m_1, x_1) - \vartheta(m_2, x_2) \preceq \frac{k}{2}\left(\Gamma(m_1 - m_2) + \Gamma(x_2 - x_1)\right), \end{aligned}$$

which gives from the normality of the cone N,

$$\begin{aligned}
\|x_3 - x_2\| &\leq \frac{a_2 k}{2} \|\Gamma\| \left(\|x_2 - x_1\| + \|m_1 - m_2\|\right), \\
&\leq \frac{a_2 k}{2} \|\Gamma\| \left(\begin{array}{c} \frac{a_1 k}{2} \|\Gamma\| \left(\|x_1 - x_0\| + \|m_0 - m_1\|\right) \\ + \frac{b_1 k}{2} \|\Gamma\| \left(\|m_0 - m_1\| + \|x_1 - x_0\|\right) \end{array} \right) \\
&\leq \|\Gamma\|^2 \frac{k^2}{2^2} (a_1 + b_1)\, a_2 (\|x_1 - x_0\| + \|m_0 - m_1\|), \\
\|m_2 - m_3\| &\leq \frac{b_2 k}{2} \|\Gamma\| \left(\|m_1 - m_2\| + \|x_2 - x_1\|\right) \\
&\leq \frac{b_2 k}{2} \|\Gamma\| \left(\begin{array}{c} \frac{b_1 k}{2} \|\Gamma\| \left(\|m_0 - m_1\| + \|x_1 - x_0\|\right) \\ + \frac{a_1 k}{2} \|\Gamma\| \left(\|x_1 - x_0\| + \|m_0 - m_1\|\right) \end{array} \right) \\
&\leq \|\Gamma\|^2 \frac{k^2}{2^2} (a_1 + b_1)\, b_2 (\|x_1 - x_0\| + \|m_0 - m_1\|).
\end{aligned}$$

$$\begin{aligned}
x_4 - x_3 &= \vartheta(x_3, m_3) - \vartheta(x_2, m_2) \preceq \frac{k}{2} \left(\Gamma(x_3 - x_2) + \Gamma(m_2 - m_3)\right), \\
m_3 - m_4 &= \vartheta(m_2, x_2) - \vartheta(m_3, x_3) \preceq \frac{k}{2} \left(\Gamma(m_2 - m_3) + \Gamma(x_3 - x_2)\right),
\end{aligned}$$

which gives from the normality of the cone N,

$$\begin{aligned}
\|x_4 - x_3\| &\leq \frac{a_3 k}{2} \|\Gamma\| \left(\|x_3 - x_2\| + \|m_2 - m_3\|\right), \\
&\leq \frac{a_3 k}{2} \|\Gamma\| \left(\begin{array}{c} \|\Gamma\|^2 \frac{k^2}{2^2} (a_1 + b_1)\, a_2 (\|x_1 - x_0\| + \|m_0 - m_1\|) + \\ \|\Gamma\|^2 \frac{k^2}{2^2} (a_1 + b_1)\, b_2 (\|x_1 - x_0\| + \|m_0 - m_1\|) \end{array} \right) \\
&\leq \|\Gamma\|^3 \frac{k^3}{2^3} (a_1 + b_1)(a_2 + b_2)\, a_3 (\|x_1 - x_0\| + \|m_0 - m_1\|), \\
\|m_3 - m_4\| &\leq \frac{b_3 k}{2} \|\Gamma\| \left(\|m_2 - m_3\| + \|x_3 - x_2\|\right) \\
&\leq \frac{b_3 k}{2} \|\Gamma\| \left(\begin{array}{c} \|\Gamma\|^2 \frac{k^2}{2^2} (a_1 + b_1)\, b_2 (\|x_1 - x_0\| + \|m_0 - m_1\|) + \\ \|\Gamma\|^2 \frac{k^2}{2^2} (a_1 + b_1)\, a_2 (\|x_1 - x_0\| + \|m_0 - m_1\|) \end{array} \right) \\
&\leq \|\Gamma\|^3 \frac{k^3}{2^3} (a_1 + b_1)(a_2 + b_2)\, b_3 (\|x_1 - x_0\| + \|m_0 - m_1\|).
\end{aligned}$$

Continuing in this way, we have

$$\begin{aligned}\|x_{r+1}-x_r\| &\leq \|\Gamma\|^r \frac{k^r}{2^r}(a_1+b_1)(a_2+b_2)\dots(a_{r-1}+b_{r-1})\,a_r\times \\ &\quad (\|x_1-x_0\|+\|m_0-m_1\|), \\ \|m_r-m_{r+1}\| &\leq \|\Gamma\|^r \frac{k^r}{2^r}(a_1+b_1)(a_2+b_2)\dots(a_{r-1}+b_{r-1})\,b_r\times \\ &\quad (\|x_1-x_0\|+\|m_0-m_1\|),\end{aligned}$$

where a_i and b_i are positive, for all $i = 1, 2, \dots, r$. This implies $\{x_r\}$ and $\{m_r\}$ are fundamental. Letting $r \to \infty$ we have $\|x_{r+1}-x_r\| \to 0$ and $\|m_r - m_{r+1}\| \to 0$. As Ω is complete so there exist $x, m \in \Omega$, in such a way

$$x_r = \lim_{r\to\infty} \vartheta^r(x_0, m_0) \to x \text{ and } m_r = \lim_{r\to\infty} \vartheta^r(m_0, x_0) \to m. \tag{3.3}$$

Finally, we claim that $\vartheta(x,m) = x$ and $\vartheta(m,x) = m$.
Since (3.3) holds, so for $\eta_1, \eta_2 > 0$, there exist r_0, m_0 such that, for $r \geq r_0, m \geq m_0$

$$\|\vartheta^r(x_0,m_0) - x\| < \eta_1 \text{ and } \|\vartheta^m(m_0,x_0) - m\| < \eta_2.$$

Now, for $r \geq \max\{r_0, m_0\}$,

$$\begin{aligned}\|\vartheta(x,m)-x\| &= \left\|\vartheta(x,m) - \vartheta^{r+1}(x_0,m_0) + \vartheta^{r+1}(x_0,m_0) - x\right\| \\ &\leq \|\vartheta(x,m) - \vartheta(x_r,m_r)\| + \left\|\vartheta^{r+1}(x_0,m_0) - x\right\| \\ &< k_3\left\|\frac{k}{2}\left(\Gamma(x_r-x)+\Gamma(m-m_r)\right)\right\| + \left\|\vartheta^{r+1}(x_0,m_0)-x\right\| \\ &\leq \frac{kk_3}{2}\|\Gamma\|\left\{\|x_r-x\|+\|m-m_r\|\right\} + \left\|\vartheta^{r+1}(x_0,m_0)-x\right\|.\end{aligned}$$

Letting $r \to \infty$, we have

$$\|\vartheta(x,m) - x\| = 0,$$

which implies $\vartheta(x,m) = x$. Similarly, we have $\vartheta(m,x) = m$.

If the product space $\Omega \times \Omega$ endowed with the partial order, we can prove the uniqueness of coupled fixed point by using the property 1 and 2 of (2.2.4).

3.3.2 Theorem

Suppose $\vartheta : \Omega \times \Omega \longrightarrow \Omega$ is a mixed monotone operator. Consider that a linear operator $\Gamma : \Omega \to \Omega$ with $\|\Gamma\| < 1, \Gamma(N) \subset N$ and there exists $k \in [0,1)$ in such a way

$$\vartheta(x,m) - \vartheta(h,i) \preceq \frac{k}{2}\left(\Gamma(x-h) + \Gamma(i-m)\right),$$

for every $x, m, h, i \in \Omega$ with $h \preceq x, m \preceq i$. If there exist $x_0, m_0 \in \Omega$ such that

$$x_0 \preceq \vartheta(x_0, m_0) \text{ and } m_0 \succeq \vartheta(m_0, x_0).$$

Then there exist $x, m \in \Omega$ such that

$$x = \vartheta\left(x, m\right) \text{ and } m = \vartheta\left(m, x\right).$$

In addition if we consider condition 2 of (2.2.4), we have coupled fixed point of ϑ is unique.

Proof: Suppose $\left(x^{'}, m^{'}\right) \in \Omega \times \Omega$ is another coupled fixed point of ϑ, then we show that $\left\|(x,m) - \left(x^{'}, m^{'}\right)\right\| = 0$ where

$$\lim_{r\to\infty} \vartheta^r(x_0, m_0) = x \text{ and } \lim_{t\to\infty} \vartheta^t(m_0, x_0) = m.$$

We prove this result by considering two cases:

Case 1: If (x,m) and $\left(x^{'}, m^{'}\right)$ are comparable with respect to the ordering in $\Omega \times \Omega$, then for every $r = 0, 1, 2, 3...,$ $(x,m) = (\vartheta^r(x,m), \vartheta^r(m,x))$ and $\left(x^{'}, m^{'}\right) = \left(\vartheta^r(x^{'}, m^{'}), \vartheta^r(m^{'}, x^{'})\right)$ are also comparable. So, we have

$$
\begin{aligned}
\left\|(x,m)-\left(x',m'\right)\right\| &= \left\|(\vartheta^r(x,m),\vartheta^r(m,x))-\left(\vartheta^r(x',m'),\vartheta^r(m',x')\right)\right\| \\
&= \left\|\left(\vartheta^r(x,m)-\vartheta^r(x',m'),\vartheta^r(m,x)-\vartheta^r(m',x')\right)\right\| \\
&= \left\|\vartheta^r(x,m)-\vartheta^r(x',m')\right\|+\left\|\vartheta^r(m,x)-\vartheta^r(m',x')\right\| \\
&= \left\|\begin{array}{c}\vartheta\left(\vartheta^{r-1}(x,m),\vartheta^{r-1}(m,x)\right)\\ -\vartheta\left(\vartheta^{r-1}(x',m'),\vartheta^{r-1}(m',x')\right)\end{array}\right\| \\
&\quad+\left\|\begin{array}{c}\vartheta\left(\vartheta^{r-1}(m,x),\vartheta^{r-1}(x,m)\right)\\ -\vartheta\left(\vartheta^{r-1}(m',x'),\vartheta^{r-1}(x',m')\right)\end{array}\right\|.
\end{aligned}
$$

By using the normality of the cone N, there exist normal constants $k_1, k_1' > 0$ such that

$$
\begin{aligned}
\left\|(x,m)-\left(x',m'\right)\right\| &\leq k_1\left\|\frac{k}{2}\left(\begin{array}{c}\Gamma(\vartheta^{r-1}(x,m)-\vartheta^{r-1}(x',m'))\\ +\Gamma(\vartheta^{r-1}(m',x')-\vartheta^{r-1}(m,x))\end{array}\right)\right\| \\
&\quad+k_1'\left\|\frac{k}{2}\left(\begin{array}{c}\Gamma(\vartheta^{r-1}(m,x)-\vartheta^{r-1}(m',x'))\\ +\Gamma(\vartheta^{r-1}(x',m')-\vartheta^{r-1}(x,m))\end{array}\right)\right\| \\
&\leq l_1 k\left\|\Gamma\right\|\left(\begin{array}{c}\left\|\vartheta^{r-1}(x,m)-\vartheta^{r-1}(x',m')\right\|\\ +\left\|\vartheta^{r-1}(m,x)-\vartheta^{r-1}(m',x')\right\|\end{array}\right),
\end{aligned}
$$

where $l_1 = \max\{k_1, k_1'\}$. Continuing in the similar manner we have

$$
\begin{aligned}
\left\|(x,m)-\left(x',m'\right)\right\| &\leq l_{r-1}k^{r-1}\left\|\Gamma\right\|^{r-1}\left(\begin{array}{c}\left\|\vartheta(x,m)-\vartheta(x',m')\right\|\\ +\left\|\vartheta(m,x)-\vartheta(m',x')\right\|\end{array}\right) \\
&= l_{r-1}k^{r-1}\left\|\Gamma\right\|^{r-1}\left(\left\|x-x'\right\|+\left\|m-m'\right\|\right).
\end{aligned}
$$

Letting $r \to \infty$, we have $(x,m) = \left(x',m'\right)$.

Case 2: If (x,m) and $\left(x',m'\right)$ are not comparable with respect to the ordering in $\Omega\times\Omega$, then there exists an element $(x^*,m^*)\in\Omega\times\Omega$ which is comparable to both (x,m) and $\left(x',m'\right)$.

So we have,

$$\begin{aligned}\left\|(x,m)-\left(x',m'\right)\right\| &= \left\|x-x'\right\|+\left\|m-m'\right\| \\ &\leq \|x-x^*\|+\left\|x^*-x'\right\|+\|m-m^*\|+\left\|m^*-m'\right\| \\ &= \|\vartheta^r(x,m)-\vartheta^r(x^*,m^*)\|+\left\|\vartheta^r(x^*,m^*)-\vartheta^r(x',m')\right\| \\ &\quad +\|\vartheta^r(m,x)-\vartheta^r(m^*,x^*)\|+\left\|\vartheta^r(m^*,x^*)-\vartheta^r(m',x')\right\|.\end{aligned}$$

By using the normality of the cone N, there exist normal constants $k_1, k_1', k_1', k_1' > 0$ such that

$$\begin{aligned}\left\|(x,m)-\left(x',m'\right)\right\| &\leq k_1\left\|\frac{k}{2}\begin{pmatrix}\Gamma(\vartheta^{r-1}(x,m)-\vartheta^{r-1}(x^*,m^*)) \\ +\Gamma\left(\vartheta^{r-1}(m^*,x^*)-\vartheta^{r-1}(m,x)\right)\end{pmatrix}\right\| \\ &\quad +k_1'\left\|\frac{k}{2}\begin{pmatrix}\Gamma(\vartheta^{r-1}(x^*,m^*)-\vartheta^{r-1}(x',m')) \\ +\Gamma(\vartheta^{r-1}(m',x')-\vartheta^{r-1}(m^*,x^*))\end{pmatrix}\right\| \\ &\quad +k_1'\left\|\frac{k}{2}\begin{pmatrix}\Gamma(\vartheta^{r-1}(m,x)-\vartheta^{r-1}(m^*,x^*)) \\ +\Gamma(\vartheta^{r-1}(x^*,m^*)-\vartheta^{r-1}(x,m))\end{pmatrix}\right\| \\ &\quad +k_1'\left\|\frac{k}{2}\begin{pmatrix}\Gamma(\vartheta^{r-1}(m^*,x^*)-\vartheta^{r-1}(m',x')) \\ +\Gamma(\vartheta^{r-1}(x',m')-\vartheta^{r-1}(x^*,m^*))\end{pmatrix}\right\| \\ &\leq l_1 k\,\|\Gamma\|\begin{pmatrix}\left\|\vartheta^{r-1}(x,m)-\vartheta^{r-1}(x^*,m^*)\right\| \\ +\left\|\vartheta^{r-1}(m^*,x^*)-\vartheta^{r-1}(m,x)\right\| \\ +\left\|\vartheta^{r-1}(x^*,m^*)-\vartheta^{r-1}(x',m')\right\| \\ +\left\|\vartheta^{r-1}(m',x')-\vartheta^{r-1}(m^*,x^*)\right\|\end{pmatrix},\end{aligned}$$

where $l_1 = \max\{k_1, k_1', k_1', k_1'\}$. Continuing in the similar manner we have

$$\begin{aligned}\left\|(x,m)-\left(x',m'\right)\right\| &\leq l_{r-1}k^{r-1}\|\Gamma\|^{r-1}\begin{pmatrix}\|\vartheta(x,m)-\vartheta(x^*,m^*)\| \\ +\|\vartheta(m^*,x^*)-\vartheta(m,x)\| \\ +\left\|\vartheta(x^*,m^*)-\vartheta(x',m')\right\| \\ +\left\|\vartheta(m',x')-\vartheta(m^*,x^*)\right\|\end{pmatrix} \\ &= l_{r-1}k^{r-1}\|\Gamma\|^{r-1}\begin{pmatrix}\|x-x^*\|+\|m^*-m\| \\ +\left\|x^*-x'\right\|+\left\|m'-m^*\right\|\end{pmatrix}.\end{aligned}$$

Letting $r \to \infty$, we have $(x, m) = (\acute{x}, \acute{m})$.

3.3.3 Theorem

Suppose $\vartheta : \Omega \times \Omega \longrightarrow \Omega$ is a mixed monotone operator. Consider that a linear operator $\Gamma : \Omega \to \Omega$ with $\|\Gamma\| < 1, \Gamma(N) \subset N$ and there exists $k \in [0, 1)$ in such a way

$$\vartheta(x, m) - \vartheta(h, i) \preceq \frac{k}{2}\left(\Gamma(x - h) + \Gamma(i - m)\right),$$

for any $x, m, h, i \in \Omega$ with $h \preceq x, m \preceq i$. If there exist $x_0, m_0 \in \Omega$ such that

$$x_0 \preceq \vartheta(x_0, m_0) \text{ and } m_0 \succeq \vartheta(m_0, x_0).$$

Then there exist $x, m \in \Omega$ in such a way

$$x = \vartheta(x, m) \text{ and } m = \vartheta(m, x).$$

In addition, if we consider that every pair of elements has either a lower bound or an upper bound. we obtain $x = m$.

Proof:

Case 1. If x is comparable to m then $x = \vartheta(x, m)$ is comparable to $m = \vartheta(m, x)$ and we have

$$\begin{aligned}\|x - m\| &= \|\vartheta^r(x, m) - \vartheta^r(m, x)\| \\ &= \|\vartheta(\vartheta^{r-1}(x, m), \vartheta^{r-1}(m, x)) - \vartheta(\vartheta^{r-1}(m, x), \vartheta^{r-1}(x, m))\|.\end{aligned}$$

By using the normality of the cone N, there exists a normal constant $k_1 > 0$ such that

$$\begin{aligned}\|x - m\| &\leq k_1\|\frac{k}{2}[\Gamma(\vartheta^{r-1}(x, m) - \vartheta^{r-1}(m, x)) + \Gamma(\vartheta^{r-1}(x, m) - \vartheta^{r-1}(m, x))]\| \\ &\leq kk_1\|\Gamma\|\left\|\vartheta^{r-1}(x, m) - \vartheta^{r-1}(m, x)\right\|.\end{aligned}$$

Continuing in this manner we have

$$\|x - m\| \leq k^{r-1}k_1...k_{r-1}\|\Gamma\|^{r-1}\|\vartheta(x, m) - \vartheta(m, x)\|.$$

Letting $r \to \infty$, we have $x = m$.

Case 2. If x and m are not comparable then there exists an upper bound or lower bound of x and m that is, there exists a $z \in \Omega$ comparable to both x and m. Suppose that $x \preceq z,\ m \preceq z$ holds, then we have

$$\begin{aligned} \vartheta(x,m) &\preceq \vartheta(z,m) \text{ and } \vartheta(x,m) \succeq \vartheta(x,z), \\ \vartheta(m,x) &\preceq \vartheta(z,x) \text{ and } \vartheta(m,x) \succeq \vartheta(m,z). \end{aligned}$$

By using the mixed monotone property of ϑ, we have

$$\begin{aligned} (i)\ \vartheta^2(x,m) &= \vartheta(\vartheta(x,m),\vartheta(m,x)) \preceq \vartheta(\vartheta(z,m),\vartheta(m,z)) = \vartheta^2(z,m). \\ (ii)\ \vartheta^2(m,x) &= \vartheta(\vartheta(m,x),\vartheta(x,m)) \preceq \vartheta(\vartheta(z,x),\vartheta(x,z)) = \vartheta^2(z,x). \\ (iii)\ \vartheta^2(x,m) &= \vartheta(\vartheta(x,m),\vartheta(m,x)) \succeq \vartheta(\vartheta(x,z),\vartheta(z,x)) = \vartheta^2(x,z). \\ (ii)\ \vartheta^2(m,x) &= \vartheta(\vartheta(m,x),\vartheta(x,m)) \succeq \vartheta(\vartheta(m,z),\vartheta(z,m)) = \vartheta^2(m,z). \end{aligned}$$

Similar relations can be shown to hold for any $r > 2$. Now consider

$$\begin{aligned} \|x-m\| &= \|\vartheta^r(x,m) - \vartheta^r(z,x) + \vartheta^r(z,x) - \vartheta^r(m,x)\| \\ &= \|\vartheta(\vartheta^{r-1}(x,m),\vartheta^{r-1}(m,x)) - \vartheta(\vartheta^{r-1}(z,x),\vartheta^{r-1}(x,z)) \\ &\quad + \vartheta(\vartheta^{r-1}(z,x),\vartheta^{r-1}(x,z)) - \vartheta(\vartheta^{r-1}(m,x),\vartheta^{r-1}(x,m))\| \\ &\leq \|\vartheta(\vartheta^{r-1}(x,m),\vartheta^{r-1}(m,x)) - \vartheta(\vartheta^{r-1}(z,x),\vartheta^{r-1}(x,z))\| \\ &\quad + \|\vartheta(\vartheta^{r-1}(z,x),\vartheta^{r-1}(x,z)) - \vartheta(\vartheta^{r-1}(m,x),\vartheta^{r-1}(x,m))\|. \end{aligned}$$

Due to the normality of N there exist $k_1, k_1' > 0$, such that

$$\begin{aligned} \|x-m\| &\leq k_1\|\frac{k}{2}[\Gamma(\vartheta^{r-1}(x,m) - \vartheta^{r-1}(z,x)) + \Gamma(\vartheta^{r-1}(x,z) - \vartheta^{r-1}(m,x))]\| \\ &\quad + k_1'\|\frac{k}{2}[\Gamma(\vartheta^{r-1}(z,x) - \vartheta^{r-1}(m,x)) + \Gamma(\vartheta^{r-1}(x,m) - \vartheta^{r-1}(x,z))]\| \\ &\leq l_1\frac{k}{2}\|\Gamma\|\left\{\left\|\vartheta^{r-1}(x,m) - \vartheta^{r-1}(z,x)\right\| + \left\|\vartheta^{r-1}(x,z) - \vartheta^{r-1}(m,x)\right\| + \|\vartheta^{r-1}(z,x) - \vartheta^{r-1}(m,x)\| - \right. \end{aligned}$$

where $l_1 = \max\{k_1, k_1'\}$. Continuing in this manner we have

$$\begin{aligned}\|x - m\| &\leq l_{r-1}(\frac{k}{2})^{r-1}\|\Gamma\|^{r-1}\{\|\vartheta(x,m) - \vartheta(z,x)\| + \|\vartheta(x,z) - \vartheta^r(m,x)\| \\ &\quad +\|\vartheta(z,x) - \vartheta(m,x)\| + \|(\vartheta(x,m) - \vartheta(x,z)\|\}.\end{aligned}$$

Letting $r \to \infty$, we have $x = m$.

3.3.4 Theorem

Suppose $\vartheta : \Omega \times \Omega \longrightarrow \Omega$ is a mixed monotone operator. Consider that a linear operator $\Gamma : \Omega \to \Omega$ with $\|\Gamma\| < 1, \Gamma(N) \subset N$ and there exists $k \in [0,1)$ such that

$$\vartheta(x,m) - \vartheta(h,i) \preceq \frac{k}{2}\left(\Gamma(x-h) + \Gamma(i-m)\right),$$

for any $x, m, h, i \in \Omega$ with $h \preceq x, m \preceq i$. If there exist $x_0,\ m_0 \in \Omega$ such that

$$x_0 \preceq \vartheta(x_0, m_0) \text{ and } m_0 \succeq \vartheta(m_0, x_0).$$

Then there exist $x, m \in \Omega$ such that

$$x = \vartheta\left(x, m\right) \text{ and } m = \vartheta\left(m, x\right).$$

In addion, if we consider that x_0 and m_0 in Ω are comparable then $x = m$.

Proof: Recall that $x_0 \in \Omega$ is such that $x_0 \leq \vartheta(x_0, m_0)$. Now if $x_0 \preceq m_0$ we claim that, for all $r \in \mathbb{N}, x_r \preceq m_r$.

Indeed by the mixed monotone property of ϑ

$$x_1 = \vartheta(x_0, m_0) \preceq \vartheta(m_0, x_0) = m_1.$$

Assume that $x_r \preceq m_r$, for some r. Now consider

$$\begin{aligned}x_{r+1} &= \vartheta^{r+1}(x_0, m_0) = \vartheta(\vartheta^r(x_0, m_0), \vartheta^r(m_0, x_0)) \\ &= \vartheta(x_r, m_r) \preceq \vartheta(m_r, x_r) = m_{r+1}.\end{aligned}$$

Hence for all $r \in \mathbb{N}, x_r \preceq m_r$. Now

$$\begin{aligned}
\|x-m\| &= \|\vartheta^r(x,m)-\vartheta^r(x_0,m_0)+\vartheta^r(x_0,m_0)-\vartheta^r(m,x)\| \\
&\leq \|\vartheta^r(x,m)-\vartheta^r(x_0,m_0)\|+\|\vartheta^r(x_0,m_0)-\vartheta^r(m,x)\| \\
&= \|\vartheta(\vartheta^{r-1}(x,m),\vartheta^{r-1}(m,x))-\vartheta(\vartheta^{r-1}(x_0,m_0),\vartheta^{r-1}(m_0,x_0))\| \\
&\quad +\|\vartheta(\vartheta^{r-1}(x_0,m_0),\vartheta^{r-1}(m_{0,}x_0))-\vartheta(\vartheta^{r-1}(m,x),\vartheta^{r-1}(x,m))\|.
\end{aligned}$$

Due to the normality of N there exist $k_1, k_1^{'} > 0$ such that

$$\begin{aligned}
\|x-m\| &\leq k_1\left\|\frac{k}{2}\begin{bmatrix} \Gamma(\vartheta^{r-1}(x,m)-\vartheta^{r-1}(x_0,m_0)) \\ +\Gamma(\vartheta^{r-1}(m_0,x_0)-\vartheta^{r-1}(m,x)) \end{bmatrix}\right\| \\
&\quad +k_1^{'}\left\|\frac{k}{2}\begin{bmatrix} \Gamma(\vartheta^{r-1}(x_0,m_0)-\vartheta^{r-1}(m,x)) \\ +\Gamma(\vartheta^{r-1}(x,m)-\vartheta^{r-1}(m_{0,}x_0)) \end{bmatrix}\right\| \\
&\leq l_1\frac{k}{2}\|\Gamma\|\begin{pmatrix} \|\vartheta^{r-1}(x,m)-\vartheta^{r-1}(x_0,m_0)\| \\ +\|\vartheta^{r-1}(m_0,x_0)-\vartheta^{r-1}(m,x)\| \\ +\|\vartheta^{r-1}(x_0,m_0)-\vartheta^{r-1}(m,x)\| \\ +\|(\vartheta^{r-1}(x,m)-\vartheta^{r-1}(m_{0,}x_0)\| \end{pmatrix},
\end{aligned}$$

where $l_1 = \max\{k_1, k_1^{'}\}$. Continuing in this manner we have

$$\begin{aligned}
\|x-m\| &\leq l_{r-1}(\frac{k}{2})^{r-1}\|\Gamma\|^{r-1}\{\|\vartheta(x,m)-\vartheta(x_0,m_0)\|+\|\vartheta(m_0,x_0)-\vartheta(m,x)\| \\
&\quad +\|\vartheta^{r-1}(x_0,m_0)-\vartheta^{r-1}(m,x)\|+\|(\vartheta^{r-1}(x,m)-\vartheta^{r-1}(m_{0,}x_0)\|\}.
\end{aligned}$$

Letting $r \to \infty$, we have $x = m$. Similarly, if $x_0 \geq m_0$, it can be shown that $x_r \succeq m_r$ for all $r \in \mathbb{N}$ and $x = m$.

3.3.5 Theorem

Suppose $\vartheta : \Omega \times \Omega \to \Omega$ is a mixed monotone operator. Consider that two commuting and nondecreasing linear operators $\Gamma, S : \Omega \to \Omega$ with $\|\Gamma\| + \|S\| < 1, \Gamma(N) \subset N, S(N) \subset N$ in such a way for any $x, m, h, i \in \Omega$ with $x \preceq h, i \preceq m$

$$\vartheta(x,m) - \vartheta(h,i) \preceq \Gamma(h-x) + S(m-i).$$

If there exist $x_0, m_0 \in \Omega$ in such a way

$$x_0 \preceq \vartheta(x_0, m_0) \text{ and } m_0 \succeq \vartheta(m_0, x_0).$$

Then $x, m \in \Omega$ exist in such a way that

$$x = \vartheta(x,m) \text{ and } m = \vartheta(m,x).$$

Proof: Since $x_0 \preceq \vartheta(x_0, m_0) = x_1$ (say), $m_0 \succeq \vartheta(m_0, x_0) = m_1$ (say).
Letting $x_2 = \vartheta(x_1, m_1)$ and $m_2 = \vartheta(m_1, x_1)$.
Because of MM-property of ϑ,

$$\begin{aligned} x_2 &= \vartheta(\vartheta(x_0,m_0), \vartheta(m_0,x_0)) = \vartheta(x_1,m_1) \succeq x_1, \\ m_2 &= \vartheta(\vartheta(m_0,x_0), \vartheta(x_0,m_0)) = \vartheta(m_1,x_1) \preceq m_1. \end{aligned}$$

Further for $r = 1, 2, 3...,$

$$\begin{aligned} x_{r+1} &= \vartheta^{r+1}(x_0,m_0) = \vartheta(\vartheta^r(x_0,m_0), \vartheta^r(m_0,x_0)) = \vartheta(x_r,m_r) \succeq x_r, \\ m_{r+1} &= \vartheta^{r+1}(m_0,x_0) = \vartheta(\vartheta^r(m_0,x_0), \vartheta^r(x_0,m_0)) = \vartheta(m_r,x_r) \preceq m_r. \end{aligned}$$

Then by using given condition,

$$\begin{aligned} x_2 - x_1 &= \vartheta(x_1,m_1) - \vartheta(x_0,m_0) \preceq \Gamma(x_1 - x_0) + S(m_0 - m_1), \\ m_1 - m_2 &= \vartheta(m_0,x_0) - \vartheta(m_1,x_1) \preceq \Gamma(m_0 - m_1) + S(x_1 - x_0), \end{aligned}$$

$$
\begin{aligned}
x_3 - x_2 &\preceq \Gamma(x_2 - x_1) + S(m_1 - m_2) \\
&\preceq \Gamma(\Gamma(x_1 - x_0) + S(m_0 - m_1)) + S(\Gamma(m_0 - m_1) + S(x_1 - x_0)) \\
&\preceq \Gamma^2(x_1 - x_0) + 2\Gamma S(m_0 - m_1) + S^2(x_1 - x_0),
\end{aligned}
$$

$$
\begin{aligned}
m_2 - m_3 &\preceq \Gamma(m_1 - m_2) + S(x_2 - x_1) \\
&\preceq \Gamma\left(\Gamma(m_0 - m_1) + S(x_1 - x_0)\right) + S\left(\Gamma(x_1 - x_0) + S(m_0 - m_1)\right) \\
&\preceq \Gamma^2(m_0 - m_1) + 2\Gamma S(x_1 - x_0) + S^2(m_0 - m_1),
\end{aligned}
$$

$$
\begin{aligned}
x_4 - x_3 &\preceq \Gamma(x_3 - x_2) + S(m_2 - m_3) \\
&\preceq \Gamma(\Gamma^2(x_1 - x_0) + 2\Gamma S(m_0 - m_1) + S^2(x_1 - x_0)) \\
&\quad + S(\Gamma^2(m_0 - m_1) + 2\Gamma S(x_1 - x_0) + S^2(m_0 - m_1)) \\
&\preceq \Gamma^3(x_1 - x_0) + 3\Gamma^2 S(m_0 - m_1) + 3\Gamma S^2(x_1 - x_0) + S^3(m_0 - m_1),
\end{aligned}
$$

$$
\begin{aligned}
m_3 - m_4 &\preceq \Gamma(m_2 - m_3) + S(x_3 - x_2) \\
&\preceq \Gamma(\Gamma^2(m_0 - m_1) + 2\Gamma S(x_1 - x_0) + S^2(m_0 - m_1)) \\
&\quad + S(\Gamma^2(x_1 - x_0) + 2\Gamma S(m_0 - m_1) + S^2(x_1 - x_0)) \\
&\preceq \Gamma^3(m_0 - m_1) + 3\Gamma^2 S(x_1 - x_0) + 3\Gamma S^2(m_0 - m_1) + S^3(x_1 - x_0).
\end{aligned}
$$

Continuing in this way we have

$$
\begin{aligned}
x_{r+1} - x_r &\preceq \binom{r}{0}\Gamma^r(x_1 - x_0) + \binom{r}{1}\Gamma^{r-1}S(m_0 - m_1) + \binom{r}{2}\Gamma^{r-2}S^2(x_1 - x_0) \\
&\quad + \ldots.. + \binom{r}{r-1}\Gamma S^{r-1}(x_1 - x_0) + \binom{r}{r}S^r(m_0 - m_1),
\end{aligned}
$$

$$
\begin{aligned}
m_r - m_{r+1} &\preceq \binom{r}{0}\Gamma^r(m_0 - m_1) + \binom{r}{1}\Gamma^{r-1}S(x_1 - x_0) + \binom{r}{2}\Gamma^{r-2}S^2(m_0 - m_1) \\
&\quad + \ldots.. + \binom{r}{r-1}\Gamma S^{r-1}(m_0 - m_1) + \binom{r}{r}S^r(x_1 - x_0).
\end{aligned}
$$

Due to the normality of cone N, there exist constants $l_1, l_2 > 0$, such that

$$\begin{aligned} \|x_{r+1} - x_r\| &\leq l_1 \left\| \begin{array}{c} \binom{r}{0}\Gamma^r(x_1 - x_0) + \binom{r}{1}\Gamma^{r-1}S(m_0 - m_1) \\ +\binom{r}{2}\Gamma^{r-2}S^2(x_1 - x_0) + ... + \binom{r}{r-1}\Gamma S^{r-1}(x_1 - x_0) \\ +\binom{r}{r}S^r(m_0 - m_1) \end{array} \right\| \\ &\leq l_1 \left\{ \begin{array}{c} |\binom{r}{0}| \|\Gamma\|^r \|x_1 - x_0\| + |\binom{r}{1}| \|\Gamma\|^{r-1} \|S\| \|m_0 - m_1\| \\ +|\binom{r}{2}| \|\Gamma\|^{r-2} \|S\|^2 \|x_1 - x_0\| + ... \\ +|\binom{r}{r-1}| \|\Gamma\| \|S\|^{r-1} \|x_1 - x_0\| + |\binom{r}{r}| \|S\|^r \|m_0 - m_1\| \end{array} \right\}, \end{aligned}$$

$$\begin{aligned} \|m_r - m_{r+1}\| &\leq l_2 \left\| \begin{array}{c} \binom{r}{0}\Gamma^r(m_0 - m_1) + \binom{r}{1}\Gamma^{r-1}S(x_1 - x_0) \\ +\binom{r}{2}\Gamma^{r-2}S^2(m_0 - m_1) + ... + \binom{r}{r-1}\Gamma S^{r-1}(m_0 - m_1) \\ +\binom{r}{r}S^r(x_1 - x_0) \end{array} \right\| \\ &\leq l_2 \left\{ \begin{array}{c} |\binom{r}{0}| \|\Gamma\|^r \|m_0 - m_1\| + |\binom{r}{1}| \|\Gamma\|^{r-1} \|S\| \|x_1 - x_0\| \\ +|\binom{r}{2}| \|\Gamma\|^{r-2} \|S\|^2 \|m_0 - m_1\| + ... \\ +|\binom{r}{r-1}| \|\Gamma\| \|S\|^{r-1} \|m_0 - m_1\| + |\binom{r}{r}| \|S\|^r \|x_1 - x_0\| \end{array} \right\}. \end{aligned}$$

Letting $r \to \infty$ then $\|x_{r+1} - x_r\| \to 0$ and $\|m_r - m_{r+1}\| \to 0$. As Ω is complete so there exist $x, m \in \Omega$, such that

$$x_r = \lim_{r \to \infty} \vartheta^r(x_0, m_0) \to x \text{ and } m_r = \lim_{r \to \infty} \vartheta^r(m_0, x_0) \to m. \tag{3.4}$$

We finally claim that $\vartheta(x, m) = x$ and $\vartheta(m, x) = m$.

Since (3.4) holds, so for $\eta_1, \eta_2 > 0$, there exist r_0, t_0 such that, for $r \geq r_0, t \geq t_0$

$$\|\vartheta^r(x_0, m_0) - x\| < \eta_1 \text{ and } \|\vartheta^t(m_0, x_0) - m\| < \eta_2.$$

Now, for $r \geq \max\{r_0, t_0\}$,

$$\begin{aligned}
\|\vartheta(x,m) - x\| &= \left\|\vartheta(x,m) - \vartheta^{r+1}(x_0,m_0) + \vartheta^{r+1}(x_0,m_0) - x\right\| \\
&\leq \|\vartheta(x,m) - \vartheta(x_r,m_r)\| + \left\|\vartheta^{r+1}(x_0,m_0) - x\right\| \\
&\leq l_3 \|\Gamma(x_r - x) + S(m - m_r)\| + \left\|\vartheta^{r+1}(x_0,m_0) - x\right\| \\
&\leq l_3 \{\|\Gamma\| \|x_r - x\| + \|S\| \|m - m_r\|\} + \left\|\vartheta^{r+1}(x_0,m_0) - x\right\|.
\end{aligned}$$

Letting $r \to \infty$, we have

$$\|\vartheta(x,m) - x\| = 0,$$

we have that $\vartheta(x,m) = x$. Similarly, we have prove that $\vartheta(m,x) = m$.

3.4 Coupled Fixed Point Theorem for a Multivalued Operator having CCM Property

In this section we will prove the generalization of [2].

3.4.1 Theorem

Consider Ω be a complete normed space and N be a normal and reproducing cone in Ω and partial order " $\leq$ " is induced by the cone N. Let $\vartheta : \Omega \times \Omega \to C(\Omega)$ be a multivalued mapping having CCM property on Ω.Two linear operators $\Gamma, S : \Omega \to \Omega$ exists. Assume that there exists a $k \in [0,1)$ such that for all $x \asymp h, m \asymp i$

$$\frac{k}{2}[\Gamma(x-h) + S(m-i)] \in \sigma(\vartheta(x,m), \vartheta(h,i)) = \cap_{z\in\vartheta(x,m), w\in\vartheta(h,i)} s(z-w),$$

and

$$\frac{k}{2}[\Gamma(m-i) + S(x-h)] \in \sigma(\vartheta(m,x), \vartheta(i,h)) = \cap_{m\in\vartheta(m,x), r\in\vartheta(i,h)} s(m-r).$$

If there exist $x_0, m_0 \in \Omega$ such that $\{x_0\} \leq_4 \vartheta(x_0,m_0), \vartheta(m_0,x_0) \leq_5 \{m_0\}$ and Ω has limit comparison property then $\overline{x}, \overline{m} \in \Omega$, exist in such a way that

$$\overline{x} \in \vartheta(\overline{x}, \overline{m}) \text{ and } \overline{m} \in \vartheta(\overline{m}, \overline{x}).$$

Proof: Since $\{x_0\} \leq_4 \vartheta(x_0, m_0)$ and $\vartheta(m_0, x_0) \leq_5 \{m_0\}$, then there exist some $x_1 \in \vartheta(x_0, m_0)$ and $m_1 \in \vartheta(m_0, x_0)$ such that $x_0 \asymp x_1$ and $m_0 \asymp m_1$,so by given conditions we have

$$\frac{k}{2}[\Gamma(x_0 - x_1) + S(m_0 - m_1)] \in \sigma(\vartheta(x_0, m_0), \vartheta(x_1, m_1))$$

and

$$\frac{k}{2}[\Gamma(m_0 - m_1) + S(x_0 - x_1)] \in \sigma(\vartheta(m_0, x_0), \vartheta(m_1, x_1)).$$

Also by using CCM property we have

$$\vartheta(x_0, m_0) \leq_4 \vartheta(x_1, m_1) \text{ and } \vartheta(m_0, x_0) \leq_4 \vartheta(m_1, x_1),$$

then there exist $x_2 \in \vartheta(x_1, m_1)$ and $m_2 \in \vartheta(m_1, x_1)$ such that $x_1 \asymp x_2$ and $m_1 \asymp m_2$. By using Lemma we have

$$\frac{k}{2}[\Gamma(x_0 - x_1) + S(m_0 - m_1)] \in s(x_1 - x_2),$$

which implies

$$x_1 - x_2 \preceq \frac{k}{2}[\Gamma(x_0 - x_1) + S(m_0 - m_1)],$$

and also

$$\frac{k}{2}[\Gamma(m_0 - m_1) + S(x_0 - x_1)] \in s(m_1 - m_2),$$

which gives

$$m_1 - m_2 \preceq \frac{k}{2}[\Gamma(m_0 - m_1) + S(x_0 - x_1)].$$

Due to the normality of the cone N we have

$$\begin{aligned} \|x_1 - x_2\| &\leq a_1 \frac{k}{2} \left(\|\Gamma\| \, \|x_0 - x_1\| + \|S\| \, \|m_0 - m_1\| \right), \\ \|m_1 - m_2\| &\leq b_1 \frac{k}{2} \left(\|\Gamma\| \, \|m_0 - m_1\| + \|S\| \, \|x_0 - x_1\| \right). \end{aligned}$$

If $l_1 = \max\{a_1, b_1\}$ then

$$\begin{aligned} \|x_1 - x_2\| &\leq l_1 \frac{k}{2} \left(\|\Gamma\| \|x_0 - x_1\| + \|S\| \|m_0 - m_1\| \right), \\ \|m_1 - m_2\| &\leq l_1 \frac{k}{2} \left(\|\Gamma\| \|m_0 - m_1\| + \|S\| \|x_0 - x_1\| \right). \end{aligned}$$

As $x_1 \asymp x_2$ and $m_1 \asymp m_2$ so again by using CCM property we have

$$\vartheta(x_1, m_1) \leq_4 \vartheta(x_2, m_2) \text{ and } \vartheta(m_1, x_1) \leq_4 \vartheta(m_2, x_2),$$

then there exist $x_3 \in \vartheta(x_2, m_2)$ and $m_3 \in \vartheta(m_2, x_2)$ such that $x_2 \asymp x_3$ and $m_2 \asymp m_3$. Again by using Lemma we have

$$\begin{aligned} x_2 - x_3 &\preceq \frac{k}{2}[\Gamma(x_1 - x_2) + S(m_1 - m_2)], \\ m_2 - m_3 &\preceq \frac{k}{2}[\Gamma(m_1 - m_2) + S(x_1 - x_2)]. \end{aligned}$$

Again by the normality of the cone N we have

$$\begin{aligned} \|x_2 - x_3\| &\leq a_2 \frac{k}{2} \left(\|\Gamma\| \|x_1 - x_2\| + \|S\| \|m_1 - m_2\| \right) \\ &\leq a_2 \frac{k}{2} \begin{pmatrix} \|\Gamma\| \left(l_1 \frac{k}{2} \left(\|\Gamma\| \|x_0 - x_1\| + \|S\| \|m_0 - m_1\| \right) \right) \\ + \|S\| \left(l_1 \frac{k}{2} \left(\|\Gamma\| \|m_0 - m_1\| + \|S\| \|x_0 - x_1\| \right) \right) \end{pmatrix} \\ &\leq \frac{k^2}{2^2} l_2^2 \left(\left(\|\Gamma\|^2 + \|S\|^2 \right) \|x_0 - x_1\| + 2 \|\Gamma\| \|S\| \|m_0 - m_1\| \right) \end{aligned}$$

and

$$\begin{aligned} \|m_2 - m_3\| &\leq b_2 \frac{k}{2} \left(\|\Gamma\| \|m_1 - m_2\| + \|S\| \|x_1 - x_2\| \right) \\ &\leq b_2 \frac{k}{2} \begin{pmatrix} \|\Gamma\| \left(l_1 \frac{k}{2} \left(\|\Gamma\| \|m_0 - m_1\| + \|S\| \|x_0 - x_1\| \right) \right) \\ + \|S\| \left(l_1 \frac{k}{2} \left(\|\Gamma\| \|x_0 - x_1\| + \|S\| \|m_0 - m_1\| \right) \right) \end{pmatrix} \\ &\leq \frac{k^2}{2^2} l_2^2 \left(\left(\|\Gamma\|^2 + \|S\|^2 \right) \|m_0 - m_1\| + 2 \|\Gamma\| \|S\| \|x_0 - x_1\| \right), \end{aligned}$$

where $l_2 = \max\{a_2, b_2, l_1\}$. As $x_2 \asymp x_3$ and $m_2 \asymp m_3$ so again by using CCM property we have

$$\vartheta(x_2, m_2) \leq_4 \vartheta(x_3, m_3) \text{ and } \vartheta(m_2, x_2) \leq_4 \vartheta(m_3, x_3),$$

then there exist $x_4 \in \vartheta(x_3, m_3)$ and $m_4 \in \vartheta(m_3, x_3)$ such that $x_3 \asymp x_4$ and $m_3 \asymp m_4$. Again by using Lemma we have

$$\begin{aligned} x_3 - x_4 &\preceq \frac{k}{2}[\Gamma(x_2 - x_3) + S(m_2 - m_3)], \\ m_3 - m_4 &\preceq \frac{k}{2}[\Gamma(m_2 - m_3) + S(x_2 - x_3)]. \end{aligned}$$

Again by the normality of the cone N we have

$$\begin{aligned} \|x_3 - x_4\| &\leq a_3 \frac{k}{2} \left(\|\Gamma\| \|x_2 - x_3\| + \|S\| \|m_2 - m_3\| \right) \\ &\leq a_3 \frac{k}{2} \begin{pmatrix} \|\Gamma\| \left(\frac{k^2}{2^2} l_2^2 \begin{pmatrix} \left(\|\Gamma\|^2 + \|S\|^2 \right) \|x_0 - x_1\| \\ +2 \|\Gamma\| \|S\| \|m_0 - m_1\| \end{pmatrix} \right) \\ + \|S\| \left(\frac{k^2}{2^2} l_2^2 \begin{pmatrix} \left(\|\Gamma\|^2 + \|S\|^2 \right) \|m_0 - m_1\| \\ +2 \|\Gamma\| \|S\| \|x_0 - x_1\| \end{pmatrix} \right) \end{pmatrix} \\ &\leq \frac{k^3}{2^3} l_3^3 \begin{pmatrix} \left(\|\Gamma\|^3 + 3 \|\Gamma\| \|S\|^2 \right) \|x_0 - x_1\| \\ + \left(\|S\|^3 + 3 \|\Gamma\|^2 \|S\| \right) \|m_0 - m_1\| \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \|m_3 - m_4\| &\leq b_3 \frac{k}{2} \left(\|\Gamma\| \|m_2 - m_3\| + \|S\| \|x_2 - x_3\| \right) \\ &\leq b_3 \frac{k}{2} \begin{pmatrix} \|\Gamma\| \left(\frac{k^2}{2^2} l_2^2 \begin{pmatrix} \left(\|\Gamma\|^2 + \|S\|^2 \right) \|m_0 - m_1\| \\ +2 \|\Gamma\| \|S\| \|x_0 - x_1\| \end{pmatrix} \right) \\ + \|S\| \left(\frac{k^2}{2^2} l_2^2 \begin{pmatrix} \left(\|\Gamma\|^2 + \|S\|^2 \right) \|x_0 - x_1\| \\ +2 \|\Gamma\| \|S\| \|m_0 - m_1\| \end{pmatrix} \right) \end{pmatrix} \\ &\leq \frac{k^3}{2^3} l_3^3 \begin{pmatrix} \left(\|\Gamma\|^3 + 3 \|\Gamma\| \|S\|^2 \right) \|m_0 - m_1\| \\ + \left(\|S\|^3 + 3 \|\Gamma\|^2 \|S\| \right) \|x_0 - x_1\| \end{pmatrix}, \end{aligned}$$

where $l_3 = \max\{a_3, b_3, l_2\}$. Continuing in this manner we have

$$\begin{aligned}
\|x_r - x_{r+1}\| &\leq \frac{k^r}{2^r} l_r^r \left(A \|x_0 - x_1\| + B \|m_0 - m_1\|\right), \\
\|m_r - m_{r+1}\| &\leq \frac{k^r}{2^r} l_r^r \left(C \|m_0 - m_1\| + D \|x_0 - x_1\|\right),
\end{aligned}$$

where A, B, C and D are the combinations of powers of $\|\Gamma\|$ and $\|S\|$. This implies $\{x_r\}$ and $\{m_r\}$ are fundamental. Applying limit $r \to \infty$, as $\|\Gamma\| < 1, \|S\| < 1$ and $k^r \to 0$, we have $\|x_{r+1} - x_r\| \to 0$ and $\|m_r - m_{r+1}\| \to 0$. As Ω is complete so there exist $\overline{x}, \overline{m} \in \Omega$, such that $\lim_{r\to\infty} x_r = \bar{x}$ and $\lim_{r\to\infty} m_r = \overline{m}$. Hence for every $\epsilon > 0$, there exist natural numbers k_1 and k_2 such that

$$\|x_r - \bar{x}\| \leq \frac{\epsilon}{3}, \text{ for all } r \geq k_1 \text{ and } \|m_r - \overline{m}\| \leq \frac{\epsilon}{3}, \text{ for all } r \geq k_2.$$

Now we prove that $\bar{x} \in \vartheta(\bar{x}, \overline{m})$ and $\overline{m} \in \vartheta(\overline{m}, \bar{x})$. By limit comparision property of Ω we have $x_r \asymp \bar{x}$ and $m_r \asymp \overline{m}$, for all r so we have

$$\begin{aligned}
\frac{k}{2}[\Gamma(x_r - \bar{x}) + S(m_r - \overline{m})] &\in \sigma(\vartheta(x_r, m_r), \vartheta(\bar{x}, \overline{m})), \\
\frac{k}{2}[\Gamma(m_r - \overline{m}) + S(x_r - \bar{x})] &\in \sigma(\vartheta(m_r, x_r), \vartheta(\overline{m}, \bar{x})).
\end{aligned}$$

Then there exists a sequence i_r in $\vartheta(\bar{x}, \overline{m})$ such that

$$x_{r+1} - i_r \preceq \frac{k}{2}[\Gamma(x_r - \bar{x}) + S(m_r - \overline{m})],$$

and also there exists a sequence h_r in $\vartheta(\overline{m}, \bar{x})$ such that

$$m_{r+1} - h_r \preceq \frac{k}{2}[\Gamma(m_r - \overline{m}) + S(x_r - \bar{x})].$$

Now consider

$$\begin{aligned}
\bar{x} - i_r &= (x_{r+1} - i_r) + (\bar{x} - x_{r+1}) \\
&\preceq \frac{k}{2}[\Gamma(x_r - \bar{x}) + S(m_r - \overline{m})] + (\bar{x} - x_{r+1}).
\end{aligned}$$

Due to the normality of the cone N a normal constant $c > 0$ exist in such a way that

$$\|\bar{x} - i_r\| \leq c\left(\frac{k}{2}\left(\|\Gamma\|\,\|x_r - \bar{x}\| + \|S\|\,\|m_r - \overline{m}\|\right) + \|\bar{x} - x_{r+1}\|\right).$$

Applying limit $r \to \infty$, we have $\|\bar{x} - i_r\| \to 0$, which implies $i_r \to \bar{x}$. Since $\vartheta(\bar{x}, \overline{m})$ is closed so $\bar{x} \in \vartheta(\bar{x}, \overline{m})$. Similarly $h_r \to \overline{m}$ and $\vartheta(\overline{m}, \bar{x})$ is closed so $\overline{m} \in \vartheta(\overline{m}, \bar{x})$.

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