On Lattice Ordered Neutrosophic Soft Sets

By

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***MASTER OF SCIENCE (MS) IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS & STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY ISLAMABAD.

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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DECLARATION

I, hereby declare that this thesis neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor.

No portion of the work presented in this thesis has been submitted in support of an application for any degree or qualification of this or any other Institute of learning.

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DEDICATION

This work is dedicated

To

My Beloved Parents, my Family, my friends, Valued Teacher Dr. Muhammad Irfan Ali and Dr. Tahir Mahmood for Supporting and Encouraging me.

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First and foremost, all praise to **'ALLAH'**, Lord of the world, the Almighty, and I find no words to thank **'ALLAH'**, who created me and all the universe around us. Who is the sole creator of all the things tinier than an electron to huge galaxies. I am thankful to the Prophet Muhammad (S.A.W.) whose teachings are a blessing for the whole mankind. May Allah guide us and the whole humanity to the right path.

Words of gratitude and appreciation do not always convey the depth of one's feelings, yet I wish to record my thanks to my most respected supervisor and honorable teacher **Dr. Muhammad Irfan Ali** and **Dr. Tahir Mahmood** who encouraged me very much in time, made me realize and enjoy the hard work. I am obliged to him for his able guidance, help, immense encouragement and limitless patience, which made massive task accomplished.

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In the end I want to pay my attribute to my parents whose love and guidance always gave me a ray of hope in the darkness of desperation. I am thankful to all of my family members specially my father for their support and well wishing, enormous love, support, encouragements and constant patience. I cannot forget their prayers for me throughout my life, without them this effort would have been nothing.

WASEEM AHMAD

CONTENTS

Table for symbols and abbreviations used in this thesis.

1	**Partially ordered**	**PO**
2	**Soft set**	**SS**
3	**Soft subset**	$\subseteq$
4	**Restricted union**	$\cup_R$
5	**Restricted intersection**	$\cap_R$
6	**Extended union**	$\cup_\epsilon$
7	**Extented intersection**	$\cap_\epsilon$
8	**Lattice ordered soft set**	**LOSS**
9	**Neutrosophic set**	**NS**
10	**Neutrosophic soft set**	**NSS**
11	**Lattice ordered neutrosophic soft set**	**LONSS**

Introduction

Lotfi A. Zadeh [26] in 1965 introduced the concept of fuzzy sets. Since then the fuzzy sets and fuzzy logic have been applied in many real life problems in uncertain and ambiguous environment. After that, many theories like Theory of Rough Sets [20], Intuitionistic Fuzzy Sets [5] and Vague Set Theory [10] are all introduced. All these theories have their own difficulties and limitations. What characterizes the traditional fuzzy sets is the membership value or the grade of membership value. Sometimes it may be very difficult to assign the membership value for a fuzzy sets. Consequently the concept of interval valued fuzzy sets was proposed [25] to capture the uncertainty of grade of membership value. We must consider the truth-membership as well as the falsity-membership for proper description of an object in uncertain and ambiguous environment in some real life problems in expert system, belief system, information fusion and so on. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate for such situation. Intuitionistic fuzzy sets is appropriate for such a situation introduced by Atanassov [5]. Incomplete information considering both the truth-membership (or simply membership) and falsity-membership (or non-membership) values can only be handled by the intuitionistic fuzzy sets. It does not handle the indeterminate and inconsistent information which exists in belief system. The concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data is introduced by Smarandache [23].

In 1999, Molodtsov [19] introduced the concept of Soft Sets. Maji [16] introduced several operations on soft sets. Ali et al [1] highlighted many issues with operations defined in [16]. In 2001, Maji et al proposed the concept of Fuzzy soft sets [15] and later on applied the theories in decision making problem [17, 22]. Different algebraic structures and their applications have also been studied in soft and fuzzy soft context [1, 3, 7, 9, 12, 21, 24]. In [14] Maji introduced the concept of Neutrosophic Soft Set. Ali et al. [2] worked on lattice ordered soft set and applied it on daily life problem.

In this thesis the concept of lattice order neutrosophic soft set is introduced. Here also some basic definitions such as restricted union, restricted intersection, extended union, basic union and basic intersection of lattice order neutrosophic soft sets and some related results are discussed. An example is given to show that lattice order neutrosophic soft sets are very useful in certain decision making problems.

CHAPTER I

PRELIMINARIES

This chapter is of introductory nature including some basic definitions and results, which will help us in the subsequent chapters. In this chapter we discuss lattices, soft sets, neutrosophic sets and neutrosophic soft sets.

1.1 Lattices

In this section we discuss partial order, dictionary order, upper and lower bound, supremum, infimum and lattice. For undefined terms and notions we refer to [6].

1.1.1 Definition[6]

A set is a well-defined collection of distinct objects, considered as an object in its own right.

1.1.2 Definition[6]

For two non-empty sets X and Y, their cartesian product is denoted and defined by

$$X \times Y = \{(x, y)\,;\ x \in X,\, y \in Y\}.$$

1.1.3 Definition[6]

Let X and Y be two non-empty sets. Then any subset R of $X \times Y$ is said to be binary relation from X to Y, and we usually write it as $R : X \longrightarrow Y$.

1.1.4 Definition[6]

A binary relation $R : X \longrightarrow X$ is called relation on X.

1.1.5 Definition[6]

A set $J \neq \emptyset$ is said to be partially ordered (PO) if relation $\leq$ defined on J holds the following conditions:

(1) $\forall\, j_1 \in J,\, j_1 \leq j_1$

(2) for $j_1, j_2 \in J$, if $j_1 \leq j_2$ and $j_2 \leq j_1$ then $j_1 = j_2$

(3) for $j_1, j_2, j_3 \in J$, if $j_1 \leq j_2$ and $j_2 \leq j_3$ then $j_1 \leq j_3$.

1.1.6 Definition[6]

A PO set J is called to be tottaly ordered set if for any $j_1, j_2 \in J$, either $j_1 \leq j_2$ or $j_2 \leq j_1$.

1.1.7 Definition[6]

Let J_1 and J_2 be two PO sets. Then the dictionary order on $J_1 \times J_2$ is defined as; $(j_1, j_2) \leq_{J_1 \times J_2} (j_3, j_4)$ iff $j_1 \leq_{J_1} j_3$ and if $j_1 = j_3$ then $j_2 \leq_{J_2} j_4$. This case $\leq_{J_1 \times J_2}$is a PO on $J_1 \times J_2$. If J_1 and J_2 are totally ordered, then $\leq_{J_1 \times J_2}$ is total order on $J_1 \times J_2$.

1.1.8 Definition[6]

Let J be a PO set and $J_1 \subseteq J$. Then an element $j \in J$ is said to be

(1) lower bound of J_1 iff $\forall\, m \in J_1$, $j \leq m$.

(2) upper bound of J_1 iff $\forall\, m \in J_1$, $m \leq j$.

1.1.9 Definition[6]

Let J be a PO set and $J_1 \subseteq J$. Then

(1) a lower bound α of J_1 is said to be greatest lower bound or infimum of J_1 iff $\forall$ lower bounds j of J_1, $j \leq \alpha$. Then we write $\alpha = glb(J_1)$ or $\alpha = \inf(J_1)$.

(2) an upper bound β of J_1 is said to be least upper bound or supremum of J_1 iff for every other upper bounds j of J_1, $\beta \leq j$. Then we write $\beta = lub(J_1)$ or $\beta = \sup(J_1)$.

1.1.10 Definition[6]

Let $\leq$ be a PO on J. The pair $J = (J, \leq)$ is a lattice if $\forall\, j_1, j_2 \in J$ the set $\{j_1, j_2\}$ has supremum and an infimum in J. If $\exists\, 0$ and $1 \in J$ s.t $0 \leq j$ and $j \leq 1$, $\forall\, j \in J$. Then J is called a bounded lattice.

1.1.11 Theorem[6]

Let $J \neq \emptyset$ set. If $\wedge$ and $\vee$ are two binary operations on J, then $J = (J, \wedge, \vee)$ is a lattice iff for each $j_1, j_2, j_3 \in J$ the following hold:

(1) $j_1 \wedge j_2 = j_2 \wedge j_1$ and $j_1 \vee j_2 = j_2 \vee j_1$

(2) $(j_1 \wedge j_2) \wedge j_3 = j_1 \wedge (j_2 \wedge j_3)$ and $(j_1 \vee j_2) \vee j_3 = j_1 \vee (j_2 \vee j_3)$

(3) $j_1 \wedge j_1 = j_1$ and $j_1 \vee j_1 = j_1$

(4) $j_1 \wedge (j_1 \vee j_2) = j_1$ and $j_1 \vee (j_1 \wedge j_2) = j_1$.

1.2 Soft sets

In this section we define soft set (SS), soft subset (SSB), soft equal, soft F-subset, soft F-equal, soft M-subset, soft M-equal, soft J-subset, soft J-equal, soft L-subset, soft L-equal, injective SS, relative null SS, relative whole SS, restricted union of two SSs, restricted intersection of two SSs, extended union of two SSs, extended intersection of two SSs, basic union of two SSs, basic intersection of two SSs, restricted difference of two SSs, and complement of a SS. For undefined terms and notions we refer to [1, 8, 9, 11, 13, 16, 18, 19, 21]

1.2.1 Definition[19]

Let W be an initial universe, J be the set of all parameters with respect to W and $J_1 \subseteq J$. Then a pair (α, J_1) is called a SS over W , where α mapping given by $\alpha : J_1 \longrightarrow P(W)$.

1.2.2 Definition[16]

For two SSs (α, J_1) and (β, J_2) over W, (α, J_1) is called a SSB of (β, J_2) if

(1) $J_1 \subseteq J_2$

(2) $\alpha(j) \subseteq \beta(j)$, $\forall\ j \in J_1$.

We denote it by $(\alpha, J_1) \tilde{\subseteq} (\beta, J_2)$. And (β, J_2) is know to be a soft super set of (α, J_1).

1.2.3 Definition[16]

Any two SSs (α, J_1) and (β, J_2) over W are said to be soft equal if (α, J_1) is a SSB of (β, J_2) and (β, J_2) is a SSB of (α, J_1).

1.2.4 Example

Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ be a set.

Let $J_1 = \{j_1, j_2, j_3, j_4\}$, $J_2 = \{j_1, j_2, j_3, j_4, j_5\}$ be sets of parameters.

Consider the SSs (α, J_1) and (β, J_2) over W s.t

$(\alpha, J_1) = \{\alpha(j_1) = \{w_1, w_2\}, \alpha(j_2) = \{w_1, w_4\}, \alpha(j_3) = \{w_2, w_7\}, \alpha(j_4) = \{w_6\}\}$,

$$(\beta, J_2) = \left\{ \begin{array}{c} \beta(j_1) = \{w_1, w_2, w_4\}, \beta(j_2) = \{w_1, w_4\}, \beta(j_3) = \{w_1, w_2, w_7\}, \\ \beta(j_4) = \{w_1, w_6\}, \beta(j_5) = \{w_1, w_4\} \end{array} \right\}$$

For computer applications it is convenient to represent a soft set in tabular. Tables 1.1 and 1.2, represents the soft sets (α, J_1) and (β, J_2), respectively. If a element of W belong to $\alpha(j_i)$ we write 1, otherwise 0.

Table for SS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5	w_6	w_7
j_1	1	1	0	0	0	0	0
j_2	1	0	0	1	0	0	0
j_3	0	1	0	0	0	0	1
j_4	0	0	0	0	0	1	0

Table 1.1

Table for SS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5	w_6	w_7
j_1	1	1	0	1	0	0	0
j_2	1	0	0	1	0	0	0
j_3	1	1	0	0	0	0	1
j_4	1	0	0	0	0	1	0
j_5	1	0	0	1	0	0	0

Table 1.2

Clearly from Tables 1.1 and 1.2, we have $J_1 \subseteq J_2$ and $\alpha(j) \subseteq \beta(j)\ \forall\ j \in J_1$. Therefore $(\alpha, J_1) \subseteqq (\beta, J_2)$.

1.2.5 Definition[16]

Let (α, J_1) and (β, J_2) be two SSs over W. Then (α, J_1) is known as soft M-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_M (\beta, J_2)$, if $J_1 \subseteq J_2$ and $\alpha(j) = \beta(j)$, $\forall\ j \in J_1$. Two SSs

(α, J_1) and (β, J_2) over W are called to be soft M-equal, denoted $(\alpha, J_1) =_M (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and $(\beta, J_2) \subseteqq_M (\alpha, J_1)$.

1.2.6 Example

Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ be a set of universe and $J_1 = \{j_1, j_2, j_3, j_4\}$, $J_2 = \{j_1, j_2, j_3, j_4, j_5\}$ be sets of parameters.

Consider the SSs (α, J_1) and (β, J_2) over W as show in Tables 1.3 and 1.4, respectively.

Table for SS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5	w_6	w_7
j_1	0	1	0	1	0	1	0
j_2	1	1	0	0	1	0	0
j_3	1	1	1	0	0	1	1
j_4	1	0	0	0	1	0	1

Table 1.3

Table for SS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5	w_6	w_7
j_1	0	1	0	1	0	1	0
j_2	1	1	0	0	1	0	0
j_3	1	1	1	0	0	1	1
j_1	1	0	0	0	1	0	1
j_5	1	0	0	1	0	0	0

Table 1.4

Clearly from Tables 1.3 and 1.4, we have $J_1 \subseteq J_2$ and $\alpha(j) \subseteq \beta(j)$, $\forall\ j \in J_1$. Therefore $(\alpha, J_1) \subseteqq_M (\beta, J_2)$.

1.2.7 Definition[9]

Let (α, J_1) and (β, J_2) be two SSs over W. Then (α, J_1) is known as soft F-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_F (\beta, J_2)$, if $J_1 \subseteq J_2$ and $\alpha(j) \subseteq \beta(j) \, \forall \, j \in J_1$. Two SSs (α, J_1) and (β, J_2) over W are called to be soft F-equal, denoted $(\alpha, J_1) =_F (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_F (\beta, J_2)$ and $(\beta, J_2) \subseteqq_F (\alpha, J_1)$.

1.2.8 Definition[11]

Let (α, J_1) and (β, J_2) be two SSs over W. Then (α, J_1) is known as soft J-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_J (\beta, J_2)$, if for every $j_1 \in J_1$ there exists $j_2 \in J_2$, s.t $\alpha(j_1) = \beta(j_2)$. Two SSs (α, J_1) and (β, J_2) over W are called to be soft J-equal, denoted $(\alpha, J_1) =_J (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_J (\beta, J_2)$ and $(\beta, J_2) \subseteqq_J (\alpha, J_1)$.

1.2.9 Definition[13]

Let (α, J_1) and (β, J_2) be two SSs over W. Then (α, J_1) is known as soft L-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_L (\beta, J_2)$, if for every $j_1 \in J_1$ there exists $j_2 \in J_2$ s.t $\alpha(j_1) \subseteq \beta(j_2)$. Two SSs (α, J_1) and (β, J_2) over W are called to be soft L-equal, denoted $(\alpha, J_1) =_L (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_L (\beta, J_2)$ and $(\beta, J_2) \subseteqq_L (\alpha, J_1)$.

1.2.10 Definition[8]

Let (α, J_1) be a SS over W. Then (α, J_1) is called an injective SS over W if $\forall \, j_1, j_2 \in J_1$, $j_1 \neq j_2$ implies $\alpha(j_1) = \alpha(j_2)$

1.2.11 Definition[1]

Let initial universe be W. J be a set of parameters and $J_1 \subsetneq J$. Then

(1) (α, J_1) is known as relative null SS over W, denoted by $\emptyset_{J_1}$, if $\alpha(j) = \emptyset \, \forall \, j \in J_1$.

(2) (β, J_1) is know as relative whole SS over W, denoted by μ_{J_1},if $\beta(j) = W \, \forall \, j \in J_1$.

The relative whole SS over W w.r.t J is known as absolute SS over W and denoted by μ_J. In a similar way, the relative null SS over W w.r.t J is know as null SS over W and denoted by $\emptyset_J$.

We shall denote by $\emptyset_\emptyset$ the unique SS over W with an empty parameter set, which is called the empty SS over W. Note that $\emptyset_\emptyset$ and $\emptyset_{J_1}$ are different SSs over W and $\emptyset_\emptyset \subseteqq \emptyset_{J_1} \subseteqq (\alpha, J_1) \subseteqq \mu_{J_1} \subseteqq \mu_J$, $\forall$ SS over W (α, J_1) over W.

1.2.12 Definition[1]

Let (α, J_1) and (β, J_2) be two SSs over W s.t $J_1 \cap J_2 \neq \emptyset$. The restricted union of (α, J_1) and (β, J_2) is denoted by $(\alpha, J_1) \cup_R (\beta, J_2)$ and is defined as $(\alpha, J_1) \cup_R (\beta, J_2) = (K, C)$, where $C = J_1 \cap J_2$ and for all $j \in C$. $K(j) = \alpha(j) \cup \beta(j)$.

If $J_1 \cap J_2 = \emptyset$, then $(\alpha, J_1) \cup_R (\beta, J_2) = \emptyset_\emptyset$.

1.2.13 Definition[1]

Let (α, J_1) and (β, J_2) be two SSs over W s.t $J_1 \cap J_2 \neq \emptyset$. The restricted intersection of (α, J_1) and (β, J_2) is denoted by $(\alpha, J_1) \cap_R (\beta, J_2)$ and is defined as $(\alpha, J_1) \cap_R (\beta, J_2) = (K, J_1 \cap J_2)$ where $K(j) = \alpha(j) \cap \beta(j)$, $\forall\, j \in J_1 \cap J_2$.

If $J_1 \cap J_2 = \emptyset$, then $(\alpha, J_1) \cap_R (\beta, J_2) = \emptyset_\emptyset$.

1.2.14 Definition[1]

Extended union of two SSs (α, J_1) and (β, J_2) over W is denoted by $(\alpha, J_1) \cup_\epsilon (\beta, J_2)$ and is defined as $(\alpha, J_1) \cup_\epsilon (\beta, J_2) = (K, G)$

$$K(j) = \begin{cases} \alpha(j) & \text{if } j \in J_1 - J_2 \\ \beta(j) & \text{if } j \in J_2 - J_1 \\ \alpha(j) \cup \beta(j) & \text{if } j \in J_1 \cap J_2 \end{cases}$$

where $G = J_1 \cup J_2$ and $\forall\, j \in G$.

1.2.15 Definition[1]

Extended intersection of two SSs (α, J_1) and (β, J_2) over W is denoted by $(\alpha, J_1) \cap_\epsilon (\beta, J_2)$ and is defined as $(\alpha, J_1) \cap_\epsilon (\beta, J_2) = (K, G)$

$$K(j) = \begin{cases} \alpha(j) & \text{if } j \in J_1 - J_2 \\ \beta(j) & \text{if } j \in J_2 - J_1 \\ \alpha(j) \cap \beta(j) & \text{if } j \in J_1 \cap J_2 \end{cases}$$

where $G = J_1 \cup J_2$ and $\forall\, j \in G$.

1.2.16 Definition[18]

For the set of parameters J and $J_1, J_2 \subseteq J$, for $(j_1, j_2) \in J_1 \times J_2$, ($j_1$ and J_2) is called the conjuction parameter of (j_1, j_2), and (j_1 or J_2) is called the disjunction parameter of ordered pair (j_1, j_2). These denoted by $(j_1 \wedge j_2)$ and $(j_1 \vee j_2)$ respectively.

We denote

$J_1 \otimes J_2 = \{(j_1 \wedge j_2); (j_1, j_2) \in J_1 \times J_2\}$

also

$J_1 \oplus J_2 = \{(j_1 \vee j_2); (j_1, j_2) \in J_1 \times J_2\}$.

1.2.17 Definition[18]

Let (α, J_1) and (β, J_2) be two SSs over W. Then

(1) $(\alpha, J_1) \cap_\wedge (\beta, J_2) = (K, J_1 \otimes J_2)$ is the SS over W defined as $K(j_1 \wedge j_2) = \alpha(j_1) \cap \beta(j_2)\ \forall\ j_1 \wedge j_2 \in J_1 \otimes J_2$.

(2) $(\alpha, J_1) \cup_\vee (\beta, J_2) = (K, J_1 \oplus J_2)$ is the SS over W defined as $K(j_1 \vee j_2) = \alpha(j_1) \cup \beta(j_2)\ \forall\ j_1 \vee j_2 \in J_1 \oplus J_2$.

1.2.18 Definition[21]

Let (α, J_1) and (β, J_2) be two SSs over W. Then the basic union of (α, J_1) and (β, J_2) is denoted by $(\alpha, J_1) \vee (\beta, J_2)$ and is defined as $(K, G) = (\alpha, J_1) \vee (\beta, J_2)$, where $G = J_1 \times J_2$, and $K(j_1, j_2) = \alpha(j_1) \cup \beta(j_2),\ \forall\ (j_1, j_2) \in J_1 \times J_2$.

1.2.19 Definition[21]

Let (α, J_1) and (β, J_2) be two SSs over W. Then the basic intersection of (α, J_1) and (β, J_2) is denoted by $(\alpha, J_1) \wedge (\beta, J_2)$ and is defined as $(K, G) = (\alpha, J_1) \wedge (\beta, J_2)$, where $G = J_1 \times J_2$, and $K(j_1, j_2) = \alpha(j_1) \cap \beta(j_2),\ \forall\ (j_1, j_2) \in J_1 \times J_2$.

1.2.20 Definition[1]

Let (α, J_1) and (β, J_2) be two SSs over W, s.t $J_1 \cap J_2 \neq \emptyset$. Then restricted difference of (α, J_1) and (β, J_2) is denoted by $(\alpha, J_1) -_R (\beta, J_2)$ and is defined as $(\alpha, J_1) -_R (\beta, J_2) = (K, J_1 \cap J_2)$, where $K(j) = \alpha(j) - \beta(j),\ \forall\ j \in J_1 \cap J_2$.

If $J_1 \cap J_2 = \emptyset$, then $(\alpha, J_1) -_R (\beta, J_2) = \emptyset_\emptyset$.

1.2.21 Definition[1]

The complement of a SS (α, J_1) over W is defined by $(\alpha, J_1)^C$ and is defined by $(\alpha, J_1)^C = (\alpha^C, J_1)$ where

$\alpha^C : J_1 \longrightarrow P(W)$ is mapping given by $\alpha^C(j) = W - \alpha(j), \forall j \in J_1$.

Clearly $(\alpha, J_1)^C = \mu_{J_1} -_R (\alpha, J_1)$ and $\left((\alpha, J_1)^C\right)^C = (\alpha, J_1)$.

1.3 Neutrosophic sets

In this section we define fuzzy set, fuzzy SS and neutrosophic set (NS). For undefined term and notion we refer to [15, 23, 26].

1.3.1 Definition[26]

A fuzzy set is a pair (X, f), where X is a non-empty set and f is a function from X to $[0,1]$ i.e. $f : X \longrightarrow [0,1]$. For each $x \in X$, $f(x)$ is called the grade of membership of x in (X, f) and f is called membership function of (X, f). For $x \in X$

1) x is called not included in (X, f) if $f(x) = 0$

2) x is called fully included in (X, f) if $f(x) = 1$

3) x is called a fuzzy member if $0 < f(x) < 1$.

1.3.2 Definition

Let α and β be two fuzzy subsets of a non-empty set J, then $\alpha \subseteq \beta$ iff $\alpha(w) \leq \beta(w)$, $\forall$ $w \in W$.

1.3.3 Definition

A fuzzy subset of W is a function from W into the unit closed interval $[0,1]$. The set of all fuzzy subset of W is called the fuzzy power set of W and is denoted by $FP(W)$.

1.3.4 Definition[15]

A SS (α, J_1) over W is called a fuzzy SS over W, where α is a mapping given by $\alpha : J_1 \longrightarrow FP(W)$.

1.3.5 Definition[23]

A neutrosophic set J_1 on the universe of discourse W is defined as

$J_1 = \{\langle w, T_{J_1}(w), I_{J_1}(w), F_{J_1}(w)\rangle, w \in W\}$, where $T, I, F : W \longrightarrow [0,1]$ and $0 \leq T_{J_1}(w) + I_{J_1}(w) + F_{J_1}(w) \leq 3$.

Where T is membership function, I is indeterminacy function and F is non-membership function.

1.3.6 Definition[23]

A NS(W) J_1 is contained in another NS(W) J_2, i.e. $J_1 \subseteq J_2$ if $\forall\ w \in W$, $T_{J_1}(w) \leq T_{J_2}(w)$, $I_{J_1}(w) \leq I_{J_2}(w)$, $F_{J_2}(w) \leq F_{J_1}(w)$.

1.3.7 Definition[23]

Any two NSs(W) J_1 and J_2 are said to be NS(W) equal if $J_1 = J_2$ if $\forall\ w \in W$, $T_{J_1}(w) = T_{J_2}(w)$, $I_{J_1}(w) = I_{J_2}(w)$, $F_{J_2}(w) = F_{J_1}(w)$.

1.4 Neutrosophic soft sets and its operations

In this section we study neutrosophic soft sets and defined some new operations on neutrosophic soft sets which providing a base work for our work. For undefined term and notion we refer to [14]

1.4.1 Definition[14]

Let W, J be a universe set and set of parameters respectively. Consider $J_1 \subset J$. Let NS(W) be the set of all neutrosophis sets over W. The collection (α, J_1) is termed to be the neutrosophic soft set (NSS) over W,where α is a mapping given by $\alpha : J_1 \longrightarrow NS(W)$.

1.4.2 Example

Let W be the set of houses and J is the set of of parameters. Consider

$W = \{w_1,\ w_2,\ w_3,\ w_4,\ w_5\}$ and $J_1 \subset J$,

$J_1 = \{j_1\,(\text{beautiful}),\ j_2\,(\text{wooden}),\ j_3\,(\text{costly}),\ j_4\,(\text{moderate})\}$. Suppose that,

$$\alpha(j_1) = \left\{ \begin{array}{c} \langle w_1,\ 0.6,\ 0.9,\ 0.7\rangle,\ \langle w_2,\ 0.7,\ 0.3,\ 0.8\rangle,\ \langle w_3,\ 0.4,\ 0.6,\ 0.2\rangle \\ \langle w_4,\ 0.7,\ 0.3,\ 0.6\rangle,\ \langle w_5,\ 0.8,\ 0.1,\ 0.4\rangle \end{array} \right\}.$$

$$\alpha(j_2)=\left\{\begin{array}{c}\langle w_1,\ 0.9,\ 0.1,\ 0.5\rangle,\ \langle w_2,\ 0.3,\ 0.6,\ 0.7\rangle,\ \langle w_3,\ 0.7,\ 0.3,\ 0.8\rangle,\\ \langle w_4,\ 0.9,\ 0.8,\ 0.4\rangle,\ \langle w_5,\ 0.5,\ 0.7,\ 0.5\rangle\end{array}\right\},$$

$$\alpha(j_3)=\left\{\begin{array}{c}\langle w_1,\ 0.7,\ 0.6,\ 0.5\rangle,\ \langle w_2,\ 0.4,\ 0.2,\ 0.1\rangle,\ \langle w_3,\ 0.3,\ 0.7,\ 0.5\rangle,\\ \langle w_4,\ 0.3,\ 0.4,\ 0.8\rangle,\ \langle w_5,\ 0.4,\ 0.5,\ 0.9\rangle\end{array}\right\},$$

$$\alpha(j_4)=\left\{\begin{array}{c}\langle w_1,\ 0.7,\ 0.6,\ 0.3\rangle,\ \langle w_2,\ 0.3,\ 0.5,\ 0.6\rangle,\ \langle w_3,\ 0.7,\ 0.9,\ 0.3\rangle,\\ \langle w_4,\ 0.5,\ 0.3,\ 0.7\rangle,\ \langle w_5,\ 0.3,\ 0.9,\ 0.7\rangle\end{array}\right\}.$$

The NSS (α, J_1) over W is a parametrized family.

Thus we can view the NSS (α, J_1) as a collection of aproximation below

$$(\alpha, J_1)=\left\{\begin{array}{c}\alpha(j_1)=\left\{\begin{array}{c}\langle w_1,\ 0.6,\ 0.9,\ 0.7\rangle,\ \langle w_2,\ 0.7,\ 0.3,\ 0.8\rangle,\ \langle w_3,\ 0.4,\ 0.6,\ 0.2\rangle,\\ \langle w_4,\ 0.7,\ 0.3,\ 0.6\rangle,\ \langle w_5,\ 0.8,\ 0.1,\ 0.4\rangle\end{array}\right\},\\ \alpha(j_2)=\left\{\begin{array}{c}\langle w_1,\ 0.9,\ 0.1,\ 0.5\rangle,\ \langle w_2,\ 0.3,\ 0.6,\ 0.7\rangle,\ \langle w_3,\ 0.7,\ 0.3,\ 0.8\rangle,\\ \langle w_4,\ 0.9,\ 0.8,\ 0.4\rangle,\ \langle w_5,\ 0.5,\ 0.7,\ 0.5\rangle\end{array}\right\},\\ \alpha(j_3)=\left\{\begin{array}{c}\langle w_1,\ 0.7,\ 0.6,\ 0.5\rangle,\ \langle w_2,\ 0.4,\ 0.2,\ 0.1\rangle,\ \langle w_3,\ 0.3,\ 0.7,\ 0.5\rangle,\\ \langle w_4,\ 0.3,\ 0.4,\ 0.8\rangle,\ \langle w_5,\ 0.4,\ 0.5,\ 0.9\rangle\end{array}\right\},\\ \alpha(j_4)=\left\{\begin{array}{c}\langle w_1,\ 0.7,\ 0.6,\ 0.3\rangle,\ \langle w_2,\ 0.3,\ 0.5,\ 0.6\rangle,\ \langle w_3,\ 0.7,\ 0.9,\ 0.3\rangle,\\ \langle w_4,\ 0.5,\ 0.3,\ 0.7\rangle,\ \langle w_5,\ 0.3,\ 0.9,\ 0.7\rangle\end{array}\right\}\end{array}\right\}.$$

For the purpose NSS store in a computer, we present form of a table. Table 1.2 represents the NSS (α, J_1). In this table, the entries are c_{ik} corresponding to the house w_i and the parameter j_k, where

c_{ik} = (true-membershipvalue of w_i, indeterminacy-membership value of w_i, falsity-membership value of w_i) in $\alpha(j_k)$.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_4
j_1	(0.6, 0.9, .7)	(0.7, 0.3, 0.8)	(0.4, 0.6, 0.2)	(0.7, 0.3, 0.6)	(0.8, 0.1, 0.4)
j_2	(0.9, 0.1, 0.5)	(0.3, 0.6, 0.7)	(0.7, 0.3, 0.8)	(0.9, 0.8, 0.4)	(0.5, 0.7, 0.5)
j_3	(0.7, 0.6, 0.5)	(0.4, 0.2, 0.1)	(0.3, 0.7, 0.5)	(0.3, 0.4, 0.8)	(0.4, 0.5, 0.9)
j_4	(0.7, 0.6, 0.3)	(0.3, 0.5, 0.6)	(0.7, 0.9, 0.3)	(0.5, 0.3, 0.7)	(0.3, 0.9, 0.7)

Table 1.5

1.4.3 Definition[14]

Let (α, J_1) and (β, J_2) be a NSSs over W. Then (α, J_1) is know as neutrosophic soft subset (NS-subest) of (β, J_2) if $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) \leq T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) \leq I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) \geq F_{\beta(j)}(w)$ $\forall$ $j \in J_1, w \in W$. We denote it by $(\alpha, J_1) \subseteqq (\beta, J_2)$ or $\alpha(j) \preceq \beta(j)$, $\forall$ $j \in J_1$.

(α, J_1) is said to be neutrosophic soft super set of (β, J_2) if (β, J_2) is a NS-subset of (α, J_1). We denote it by $(\alpha, J_1) \supseteqq (\beta, J_2)$.

1.4.4 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_4, j_6\}$, $J_2 = \{j_1, j_3, j_4, j_5, j_6\}$. Consider (α, J_1) and (β, J_2) NSSs over W as shown in Tables 1.6 and 1.7, respectively.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.34, .56, .72)	(.58, .47, .65)	(.74, .36, .54)	(.37, .32, .99)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.15, .28, .64)	(.83, .37, .96)	(.18, .53, .92)
j_6	(.24, .53, .59)	(.32, .82, .93)	(.42, .66, .69)	(.59, .32, .28)	(.51, .63, .59)

Table 1.6

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.29, .57, .44)	(.63, .66, .63)	(.91, .4, .76)	(.75, .43, .29)	(.81, .47, .58)
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.66, .89, .12)	(.79, .65, .05)	(.28, .69, .12)	(.95, .58, .83)	(.57, .79, .14)
j_5	(.63, .91, .18)	(.31, .48, .68)	(.35, .49, .61)	(.59, .59, .53)	(.25, .37, .43)
j_6	(.37, .60, .46)	(.37, .97, .19)	(.59, .88, .32)	(.73, .49, .16)	(.72, .86, .54)

Table 1.7

Then clearly from Tables 1.6 and 1.7, we have $\alpha(j_1) \preceq \beta(j_1)$, $\alpha(j_4) \preceq \beta(j_4)$ and $\alpha(j_6) \preceq \beta(j_6)$ so $(\alpha, J_1) \subseteqq (\beta, J_2)$.

1.4.5 Definition[14]

Two NSSs (α, J_1) and (β, J_2) over W are called to be equal if $(\alpha, J_1) \subseteqq (\beta, J_2)$ and $(\beta, J_2) \subseteqq (\alpha, J_1)$. We denote it by $(\alpha, J_1) = (\beta, J_2)$.

1.4.6 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_4, \}$, $J_2 = \{j_1, j_2, j_4\}$. Consider (α, J_1) and (β, J_2) NSSs over W as shown in Tables 1.8 and 1.9, respectively.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)

Table 1.8

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_1	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)

Table 1.9

Then clearly from Tables 1.8 and 1.9, we have $(\alpha, J_1) \subseteqq (\beta, J_2)$ and $(\beta, J_2) \subseteqq (\alpha, J_1)$ then $(\alpha, J_1) = (\beta, J_2)$.

Now we defined some new operations on neutrosophic soft sets which providing a base for our work

1.4.7 Definition

The complement of NSS (α, J_1) over W denoted by $(\alpha, J_1)^C$ and is defined as $(\alpha, J_1)^C = (\alpha^C, J_1)$, where $\alpha^C : J_1 \longrightarrow NS(W)$ is a mapping given by $T_{\alpha^C(j)}(w) = F_{\alpha(j)}(w)$, $I_{\alpha^C(j)}(w) = I_{\alpha(j)}(w)$ and $F_{\alpha^C(j)}(w) = T_{\alpha(j)}(w)\ \forall\ j \in J_1, w \in W$.

1.4.8 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1 \subseteq J$, $J_1 = \{j_1, j_2, j_4, j_6, j_7\}$. Consider (α, J_1) NSS over W as shown in Table 1.10.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)
j_6	(.84, .93, .43)	(.32, .82, .93)	(.82, .96, .69)	(.59, .32, .28)	(.81, .63, .53)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.10

The complement of NSS (α, J_1) is (α^C, J_1) as shown in Table 1.11.

Table for the complement of NSS (α, J_1).

(α^C, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.92, .36, .15)	(.72, .56, .24)	(.65, .47, .32)	(.14, .26, .74)	(.99, .32, .17)
j_2	(.27, .54, .23)	(.76, .64, .13)	(.63, .97, .84)	(.63, .27, .56)	(.71, .84, .31)
j_4	(.34, .76, .35)	(.34, .37, .67)	(.64, .28, .85)	(.73, .37, .83)	(.92, .53, .87)
j_6	(.43, .93, .84)	(.93, .82, .32)	(.69, .96, .82)	(.28, .32, .59)	(.53, .63, .81)
j_7	(.89, .32, .54)	(.76, .41, .26)	(.51, .48, .48)	(.52, .95, .86)	(.49, .72, .48)

Table 1.11

1.4.9 Definition

A NSS (α, J_1) over W is termed to be empty or null NSS over W w.r.t the parameter J_1 if $T_{\alpha(j)}(w) = 0$, $I_{\alpha(j)}(w) = 0$ and $F_{\alpha(j)}(w) = 0$, $\forall\, j \in J_1, w \in W$. In this case the null NSS is denoted by $\emptyset_{J_1}$.

1.4.10 Definition

Let (α, J_1) and (β, J_2) be two NSSs over W such that $J_1 \cap J_2 \neq \emptyset$. Then $(\alpha, J_1) \cup_R (\beta, J_2) = (K, G)$ is said to be restricted union of two LONSSs over W, where $G = J_1 \cap J_2$. define $K(j) = \alpha(j) \cup \beta(j)$, $T_{K(j)}(w) = Max\{T_{\alpha(j)}(x), T_{\beta(j)}(w)\}$ $I_{K(j)}(w) = Max\{I_{\alpha(j)}(x), I_{\beta(j)}(w)\}$and $F_{K(j)}(w) = Min\{F_{\alpha(j)}(x), F_{\beta(j)}(w)\}$ $\forall\, j \in G, w \in W$.

1.4.11 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_4, j_6, j_7\}$, $J_2 = \{j_1, j_3, j_4, j_5, j_6\}$. Consider (α, J_1) and (β, J_2) NSSs over W as shown in Tables 1.12 and 1.13, respectively.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)
j_6	(.84, .93, .43)	(.32, .82, .93)	(.82, .96, .69)	(.59, .32, .28)	(.81, .63, .53)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.12

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.29, .73, .64)	(.63, .86, .23)	(.51, .24, .16)	(.75, .43, .29)	(.81, .42, .58)
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.16, .39, .92)	(.79, .13, .71)	(.28, .69, .12)	(.29, .58, .83)	(.57, .39, .14)
j_5	(.63, .91, .18)	(.31, .48, .68)	(.35, .49, .61)	(.59, .59, .53)	(.25, .37, .43)
j_6	(.07, .60, .46)	(.37, .59, .19)	(.59, .48, .92)	(.73, .29, .36)	(.42, .16, .54)

Table 1.13

$(\alpha, J_1) \cup_R (\beta, J_2) = (K, G)$ is a restricted union of two NSSs over W as shown in Table 1.14, where $G = J_1 \cap J_2$

Table for restricted union of two NSSs.

(K, G)	w_1	w_2	w_3	w_4	w_5
j_1	(.29, .73, .64)	(.63, .86, .23)	(.51, .47, .16)	(.76, .43, .14)	(.81, .42, .58)
j_4	(.35, .76, .34)	(.79, .37, .34)	(.85, .69, .12)	(.83, .58, .73)	(.87, .53, .14)
j_6	(.84, .93, .43)	(.37, .82, .19)	(.82, .96, .69)	(.73, .32, .28)	(.81, .63, .53)

Table 1.14

1.4.12 Definition

let (α, J_1) and (β, J_2) be two NSSs over W such that $J_1 \cap J_2 \neq \emptyset$. Then $(\alpha, J_1) \cap_R (\beta, J_2) = (K, G)$ is said to be restricted intersection of two NSSs over W, where $G = J_1 \cap J_2$ define $K(j) = \alpha(j) \cap \beta(j)$, $T_{K(j)}(w) = Min\{T_{\alpha(j)}(w), T_{\beta(j)}(w)\}$ $I_{K(j)}(w) = Min\{I_{\alpha(j)}(w), I_{\beta(j)}(w)\}$ and $F_{K(j)}(w) = Max\{F_{\alpha(j)}(w), F_{\beta(j)}(w)\}$ $\forall j \in G, w \in W$.

1.4.13 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_4, j_6, j_7\}$, $J_2 = \{j_1, j_3, j_4, j_5, j_6\}$. Consider (α, J_1) and (β, J_2) NSSs over W as shown in Tables 1.15 and 1.16, respectively.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)
j_6	(.84, .93, .43)	(.32, .82, .93)	(.82, .96, .69)	(.59, .32, .28)	(.81, .63, .53)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.15

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.29, .73, .64)	(.63, .86, .23)	(.51, .24, .16)	(.75, .43, .29)	(.81, .42, .58)
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.16, .39, .92)	(.79, .13, .71)	(.28, .69, .12)	(.29, .58, .83)	(.57, .39, .14)
j_5	(.63, .91, .18)	(.31, .48, .68)	(.35, .49, .61)	(.59, .59, .53)	(.25, .37, .43)
j_6	(.07, .60, .46)	(.37, .59, .19)	(.59, .48, .92)	(.73, .29, .36)	(.42, .16, .54)

Table 1.16

$(\alpha, J_1) \cap_R (\beta, J_2) = (K, G)$ is a restricted intersection of two NSSs over W as shown in Table 1.17, where $G = J_1 \cap J_2$

Table for restricted intersection of two NSSs.

(K, G)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .24, .65)	(.74, .26, .29)	(.17, .33, .99)
j_4	(.16, .39, .92)	(.67, .13, .71)	(.28, .28, .64)	(.29, .37, .83)	(.57, .39, .92)
j_6	(.07, .60, .46)	(.32, .59, .93)	(.59, .48, .92)	(.73, .32, .28)	(.42, .16, .54)

Table 1.17

1.4.14 Definition

let (α, J_1) and (β, J_2) be two NSSs over W. Then $(\alpha, J_1) \cup_E (\beta, J_2) = (K, G)$ is said to be extended union of two NSSs over W, where $G = J_1 \cup J_2$ define $K(j) = \alpha(j) \cup \beta(j)$,

$$T_{K(j)}(w) = \begin{cases} T_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ T_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Max\left\{T_{\alpha(j)}(w), T_{\beta(j)}(w)\right\} & \text{if } j \in J_1 \cap J_2 \end{cases}$$

$$I_{K(j)}(w)=\begin{cases} I_{\alpha(j)}(w) & \text{if } j\in J_1-J_2 \\ I_{\beta(j)}(w) & \text{if } j\in J_2-J_{12} \\ Max\left\{I_{\alpha(j)}(w),I_{\beta(j)}(w)\right\} & \text{if } j\in J_1\cap J_2 \end{cases}$$

and

$$F_{K(j)}(w)=\begin{cases} F_{\alpha(j)}(w) & \text{if } j\in J_1-J_2 \\ F_{\beta(j)}(w) & \text{if } j\in J_2-J_{12} \\ Min\left\{F_{\alpha(j)}(w),F_{\beta(j)}(w)\right\} & \text{if } j\in J_1\cap J_2 \end{cases},\ \forall\, j\in G, w\in W.$$

1.4.15 Example

let $W=\{w_1,w_2,w_3,w_4,w_5\}$ be the set of universe and $J=\{j_1,j_2,j_3,j_4,j_5,j_6,j_7\}$ be a set of parameters. Let $J_1,J_2\subseteq J$, $J_1=\{j_1,j_2,j_4,j_6,j_7\}$, $J_2=\{j_1,j_3,j_4,j_5,j_6\}$. Consider (α,J_1) and (β,J_2) NSSs over W as shown in Tables 1.18 and 1.19, respectively.

Table for NSS (α,J_1).

(α,J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)
j_6	(.84, .93, .43)	(.32, .82, .93)	(.82, .96, .69)	(.59, .32, .28)	(.81, .63, .53)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.18

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.29, .73, .64)	(.63, .86, .23)	(.51, .24, .16)	(.75, .43, .29)	(.81, .42, .58)
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.16, .39, .92)	(.79, .13, .71)	(.28, .69, .12)	(.29, .58, .83)	(.57, .39, .14)
j_5	(.63, .91, .18)	(.31, .48, .68)	(.35, .49, .61)	(.59, .59, .53)	(.25, .37, .43)
j_6	(.07, .60, .46)	(.37, .59, .19)	(.59, .48, .92)	(.73, .29, .36)	(.42, .16, .54)

Table 1.19

$(\alpha, J_1) \cup_{\in} (\beta, J_2) = (K, G)$ is a extended union of two NSSs over W as shown in Table 1.20, where $G = J_1 \cup J_2$

Table for extended union of two NSSs.

(K, G)	w_1	w_2	w_3	w_4	w_5
j_1	(.29, .73, .64)	(.63, .86, .23)	(.51, .47, .16)	(.76, .43, .14)	(.81, .42, .58)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.35, .76, .34)	(.79, .37, .34)	(.85, .69, .12)	(.83, .58, .73)	(.87, .53, .14)
j_5	(.63, .91, .18)	(.31, .48, .68)	(.35, .49, .61)	(.59, .59, .53)	(.25, .37, .43)
j_6	(.84, .93, .43)	(.37, .82, .19)	(.82, .96, .69)	(.73, .32, .28)	(.81, .63, .53)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.20

1.4.16 Definition

let (α, J_1) and (β, J_2) be two NSSs over W. Then $(\alpha, J_1) \cap_{\in} (\beta, J_2) = (K, G)$ is said to be extended intersection of two NSSs over W, where $G = J_1 \cup J_2$ define $K(j) = \alpha(j) \cup \beta(j)$,

$$T_{K(j)}(w) = \begin{cases} T_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ T_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Min\left\{T_{\alpha(j)}(w), T_{\beta(j)}(w)\right\} & \text{if } j \in J_1 \cap J_2 \end{cases}$$

$$I_{K(j)}(w) = \begin{cases} I_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ I_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Min\left\{I_{\alpha(j)}(w), I_{\beta(j)}(w)\right\} & \text{if } j \in J_1 \cap J_2 \end{cases}$$

and

$$F_{K(j)}(w) = \begin{cases} F_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ F_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Max\left\{F_{\alpha(j)}(w), F_{\beta(j)}(w)\right\} & \text{if } j \in J_1 \cap J_2 \end{cases}, \forall\, j \in G, w \in W.$$

1.4.17 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_4, j_6, j_7\}$, $J_2 = \{j_1, j_3, j_4, j_5, j_6\}$. Consider (α, J_1) and (β, J_2) NSSs over W as shown in Tables 1.21 and 1.22, respectively.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)
j_6	(.84, .93, .43)	(.32, .82, .93)	(.82, .96, .69)	(.59, .32, .28)	(.81, .63, .53)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.21

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.29, .73, .64)	(.63, .86, .23)	(.51, .24, .16)	(.75, .43, .29)	(.81, .42, .58)
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.16, .39, .92)	(.79, .13, .71)	(.28, .69, .12)	(.29, .58, .83)	(.57, .39, .14)
j_5	(.63, .91, .18)	(.31, .48, .68)	(.35, .49, .61)	(.59, .59, .53)	(.25, .37, .43)
j_6	(.07, .60, .46)	(.37, .59, .19)	(.59, .48, .92)	(.73, .29, .36)	(.42, .16, .54)

Table 1.22

$(\alpha, J_1) \cap_{\in} (\beta, J_2) = (K, G)$ is a extended intersection of two NSSs over W as shown in Table 1.23, where $G = J_1 \cup J_2$

Table for extended intersection of two NSSs.

(K, G)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .24, .65)	(.74, .26, .29)	(.17, .33, .99)
j_2	(.23, .54, .27)	(.13, .64, .76)	(.84, .97, .63)	(.56, .27, .63)	(.31, .84, .71)
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.16, .39, .92)	(.67, .13, .71)	(.28, .28, .64)	(.29, .37, .83)	(.57, .39, .92)
j_5	(.63, .91, .18)	(.31, .48, .68)	(.35, .49, .61)	(.59, .59, .53)	(.25, .37, .43)
j_6	(.07, .60, .46)	(.32, .59, .93)	(.59, .48, .92)	(.73, .32, .28)	(.42, .16, .54)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.23

1.4.18 Definition

let (α, J_1) be a NSS over W is termed to be relative whole NSS over W w.r.t the parameter J_1 if $T_{\alpha(j)}(w) = 1, I_{\alpha(j)}(w) = 1$ and $F_{\alpha(j)}(w) = 0 \ \forall \ j \in J_1, w \in W$ and it is denoted by $\cup_{J_1}$.

1.4.19 Definition

Let (α, J_1) and (β, J_2) be two NSSs over W. Then $(\alpha, J_1) \vee (\beta, J_2) = (K, G)$ is said to be basic union of two NSSs over W, where $G = J_1 \times J_2$ define $K(m,n) = \alpha(m) \cup \beta(n)$ and

$$T_{K_{(m,n)}}(w) = Max\left\{T_{\alpha(m)}(w), T_{\beta(n)}(w)\right\}$$

$$I_{K_{(m,n)}}(w) = Max\left\{I_{\alpha(m)}(w), I_{\beta(n)}(w)\right\}$$

and

$$F_{K_{(m,n)}}(w) = Min\left\{F_{\alpha(m)}(w), F_{\beta(n)}(w)\right\}, \forall (m,n) \in G, w \in W.$$

1.4.20 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_4, j_7\}$, $J_2 = \{j_3, j_4, j_6\}$. Consider (α, J_1) and (β, J_2) NSSs over W as shown in Tables 1.24 and 1.25, respectively.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.24

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.16, .39, .92)	(.79, .13, .71)	(.28, .69, .12)	(.29, .58, .83)	(.57, .39, .14)
j_6	(.07, .60, .46)	(.37, .59, .19)	(.59, .48, .92)	(.73, .29, .36)	(.42, .16, .54)

Table 1.25

$(\alpha, J_1) \vee (\beta, J_2) = (K, G)$ is a basic union of two NSSs over W as shown in Table 1.26, where $G = J_1 \times J_2$

Table for basic union of two NSSs.

(K,G)	w_1	w_2	w_3	w_4	w_5
(j_1,j_3)	(.53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.74, .97, .14)	(.58, .87, .35)
(j_1,j_4)	(.16, .39, .92)	(.79, .56, .71)	(.32, .69, .12)	(.74, .58, .14)	(.57, .39, .14)
(j_1,j_6)	(.15, .60, .92)	(.37, .59, .19)	(.59, .48, .65)	(.74, .29, .14)	(.42, .32, .54)
(j_4,j_3)	(.53, .76, .34)	(.67, .73, .34)	(.85, .48, .29)	(.83, .97, .51)	(.87, .87, .85)
(j_4,j_4)	(.35, .76, .92)	(.79, .37, .34)	(.85, .69, .12)	(.83, .58, .73)	(.87, .53, .14)
(j_4,j_6)	(.35, .76, .34)	(.67, .59, .19)	(.85, .48, .64)	(.83, .37, .36)	(.87, .53, .54)
(j_7,j_3)	(.54, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.86, .97, .51)	(.58, .87, .35)
(j_7,j_4)	(.54, .39, .89)	(.79, .41, .71)	(.48, .69, .12)	(.86, .95, .52)	(.57, .72, .14)
(j_7,j_6)	(.57, .60, .46)	(.37, .59, .19)	(.59, .48, .51)	(.86, .95, .36)	(.48, .72, .49)

Table 1.26

1.4.21 Definition

Let (α, J_1) and (β, J_2) be two NSSs over W. Then $(\alpha, J_1) \wedge (\beta, J_2) = (K, G)$ is said to be basic intersection of two NSSs over W, where $G = J_1 \times J_2$ define $K(m,n) = \alpha(m) \cap \beta(n)$ and

$$T_{K_{(m,n)}}(w) = Min\left\{T_{\alpha(m)}(w), T_{\beta(n)}(w)\right\}$$

$$I_{K_{(m,n)}}(w) = Min\left\{I_{\alpha(m)}(w), I_{\beta(n)}(w)\right\}$$

and $$F_{K_{(m,n)}}(w) = Max\left\{F_{\alpha(m)}(w), F_{\beta(n)}(w)\right\}, \forall (m,n) \in G, w \in W.$$

1.4.22 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of universe and $J = \{j_1, j_2, j_3, j_4, j_5, j_6, j_7\}$ be a set of parameters. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_4, j_7\}$, $J_2 = \{j_3, j_4, j_6\}$. Consider (α, J_1) and (β, J_2) NSSs over W as shown in Tables 1.27 and 1.28, respectively.

Table for NSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.74, .26, .14)	(.17, .32, .99)
j_4	(.35, .76, .34)	(.67, .37, .34)	(.85, .28, .64)	(.83, .37, .73)	(.87, .53, .92)
j_7	(.54, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.86, .95, .52)	(.48, .72, .49)

Table 1.27

Table for NSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_3	(53, .42, .69)	(.62, .73, .39)	(.61, .48, .29)	(.47, .97, .51)	(.58, .87, .35)
j_4	(.16, .39, .92)	(.79, .13, .71)	(.28, .69, .12)	(.29, .58, .83)	(.57, .39, .14)
j_6	(.07, .60, .46)	(.37, .59, .19)	(.59, .48, .92)	(.73, .29, .36)	(.42, .16, .54)

Table 1.28

$(\alpha, J_1) \wedge (\beta, J_2) = (K, G)$ is a basic intersection of two NSSs over W as shown in Table 1.29, where $G = J_1 \times J_2$

Table for basic union of two NSSs.

(K,G)	w_1	w_2	w_3	w_4	w_5
(j_1,j_3)	(.15, .36, .92)	(.24, .56, .72)	(.32, .47, .65)	(.47, .26, .51)	(.17, .32, .99)
(j_1,j_4)	(.15, .36, .92)	(.24, .13, .72)	(.28, .47, .65)	(.29, .26, .83)	(.17, .32, .99)
(j_1,j_6)	(.07, .36, .92)	(.24, .56, .72)	(.32, .47, .92)	(.73, .26, .36)	(.17, .16, .99)
(j_4,j_3)	(.35, .42, .69)	(.62, .37, .39)	(.61, .28, .64)	(.47, .37, .73)	(.58, .53, .92)
(j_4,j_4)	(.16, .39, .92)	(.67, .13, .71)	(.28, .28, .64)	(.29, .37, .83)	(.57, .39, .92)
(j_4,j_6)	(.07, .60, .46)	(.37, .37, .34)	(.59, .48, .92)	(.73, .29, .73)	(.42, .16, .92)
(j_7,j_3)	(.53, .32, .89)	(.26, .41, .76)	(.48, .48, .51)	(.47, .95, .52)	(.48, .72, .49)
(j_7,j_4)	(.16, .32, .92)	(.26, .13, .76)	(.28, .48, .51)	(.29, .58, .83)	(.57, .39, .49)
(j_7,j_6)	(.07, .32, .89)	(.26, .41, .76)	(.48, .48, .92)	(.73, .29, .52)	(.48, .72, .54)

Table 1.29

CHAPTER II

LATTICE ORDERED SOFT SETS

In this chapter the research paper [2] is reviewed, which contains some core material providing a base work for our work. The work in this chapter is about NSS containing some useful results and basic operations.

2.1 *Lattice (anti-lattice) ordered soft sets*

In this section we define lattice (anti-lattice) ordered soft set and lattice ordered fuzzy soft set.

2.1.1 Definition[2]

A SS (α, J_1) over W is called a lattice (anti-lattice) ordered SS over W if for the mapping $\alpha : J_1 \longrightarrow P(W)$, $j_1 \leq j_2 \Rightarrow \alpha(j_1) \subseteq \alpha(j_2)$ $(\alpha(j_2) \subseteq \alpha(j_1))$ $\forall\, j_1, j_2 \in J_1$

2.1.2 Example

Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ be a set of designs for necklaces and $J_1 = \{j_1, j_2, j_3, j_4, j_5, j_6\}$, be set of parameters to represent manufacturing materials. j_1 represents bronze, j_2 represents silver, j_3 represents gold, j_4 represents white gold, j_5 represents platinum, j_6 represents diamond. The order among the elements of set J_1 is as shown in fig. 1, $\alpha : J_1 \longrightarrow P(W)$ is a mapping representing the high cost. Therfore SS (α, J_1) showing high cost for a design in particular materrial may be considered as

$$\left\{ \begin{array}{c} \alpha(j_1) = \{w_4\}, \alpha(j_2) = \{w_4, w_5\}, \alpha(j_3) = \{w_4\}, \alpha(j_4) = \{w_2, w_4, w_5, w_7\}, \\ \alpha(j_5) = \{w_1, w_3, w_4, w_5, w_6\}, \alpha(j_6) = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\} \end{array} \right\}.$$

For computer application it is convenient to repsent a SS in a tabular form. Table 2.1. represents the soft set (α, J_1). If a design in set W has high cost for a material we write 1, otherwise 0. From Table 1, it is clear $\alpha(j_1) \subseteq \alpha(j_2) \subseteq \alpha(j_4) \subseteq \alpha(j_6)$ and $\alpha(j_1) \subseteq \alpha(j_3) \subseteq \alpha(j_5) \subseteq \alpha(j_6)$. Thus (α, J_1) is an lattice ordered SS.

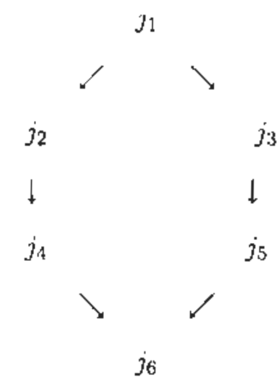

Fig. 1. Lattice of parameters

Table for LOSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5	w_6	w_7
j_1	0	0	0	1	0	0	0
j_2	0	0	0	1	1	0	0
j_3	0	0	0	1	0	0	0
j_4	0	1	0	1	1	0	1
j_5	1	0	1	1	1	1	0
j_6	1	1	1	1	1	1	1

Table 2.1

2.1.3 Example

Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be a set of six stores and $J_1 = \{j_1, j_2, j_3, j_4\}$, where

j_1 : Large stores.

j_2 : Very arge stores.

j_3 : Huge stores.

j_4 : Very huge stores.

Then clearly there is an order in the elements of parameters set J_1. This order can be described as $j_1 \leq j_2 \leq j_3 \leq j_4$. SS (α, J_1) may be represented by

$$\left\{ \begin{array}{c} \alpha(j_1) = \{w_1, w_2, w_3, w_4, w_5, w_6\}, \alpha(j_2) = \{w_2, w_3, w_4, w_5\}, \\ \alpha(j_3) = \{w_2, w_3, w_5\}, \alpha(j_4) = \{w_2\} \end{array} \right\}.$$

Tabular form for the SS (α, J_1) is given in Table 2.2.

Clearly $\alpha(j_1) \supseteq \alpha(j_2) \supseteq \alpha(j_3) \supseteq \alpha(j_4)$. Thus (α, J_1) is an anti-lattice ordered SS.

Table for anti-LOSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5	w_6
j_1	1	1	1	1	1	1
j_2	0	1	1	1	1	0
j_3	0	1	1	0	1	0
j_4	0	1	0	0	0	0

Table 2.2

2.1.4 Definition[4]

A fuzzy SS (α, J_1) over W is called a lattice ordered fuzzy SS over W if for the mapping $\alpha : J_1 \longrightarrow FP(W)$, $j_1 \leq j_2 \Longrightarrow \alpha(j_1) \leq \alpha(j_2)$, $\forall\, j_1, j_2 \in J_1$.

2.2 Properties of lattice (anti-lattice) ordered soft sets

In this section we study some basic properties of lattice (anti-lattice) ordered soft sets.

2.2.1 Proposition[2]

Restricted union of two lattice (anti-lattice) ordered soft sets (α, J_1) and (β, J_2) is a lattice (anti-lattice) ordered soft set.

2.2.2 Proposition[2]

Restricted intersection of two lattice (anti-lattice) ordered soft sets (α, J_1) and (β, J_2) is a lattice (anti-lattice) ordered soft set.

2.2.3 Example

Let $J = \{j_1, j_2, j_3, j_4, j_5\}$ with lattice ordered as shown in Fig. 2. Let $J_1 = \{j_1, j_2, j_3, j_4\}$, $J_2 = \{j_1, j_2, j_4, j_5\}$. Consider (α, J_1) and (β, J_2) is a lattice ordered soft sets over a set $W = \{w_1, w_2, w_3, w_4, w_5\}$ as shown in Tables 2.3 and 2.4, respectively.

Here $\alpha(j_1) \subseteq \alpha(j_2) \subseteq \alpha(j_4)$, $\alpha(j_1) \subseteq \alpha(j_3) \subseteq \alpha(j_4)$ and $\beta(j_1) \subseteq \beta(j_2) \subseteq \beta(j_4) \subseteq \beta(j_5)$. Then their extended intersection $(\alpha, J_1) \cap_{\in} (\beta, J_2) = (K, G)$, where $G = J_1 \cup J_2$ is given in Table 2.5.

From Table 2.5, we have $K(j_1) \subseteq K(j_2) \subseteq K(j_3)$ and $K(j_1) \subseteq K(j_4) \subseteq K(j_5)$ but $K(j_3) \nsubseteq K(j_5)$ so (K, G) is not a lattice ordered soft set. Similarly extended union $(\alpha, J_1) \cup_{\in} (\beta, J_2) = (H, G)$, where $G = J_1 \cup J_2$ is given in Table 6.

From Table 2.6, it is clear that $H(j_1) \subseteq H(j_2) \subseteq H(j_5) \subseteq H(j_4)$ but $H(j_3) \nsubseteq H(j_5)$ and $H(j_4) \nsubseteq H(j_5)$, so (H, G) is not a lattice ordered soft set.

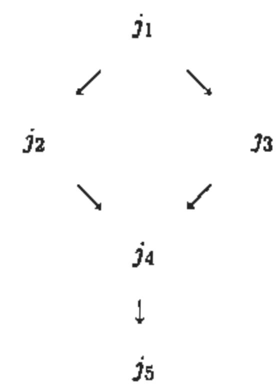

Fig. 2. Lattice of parameters

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	0	0	0	1	0
j_2	0	0	0	1	1
j_3	0	1	1	1	0
j_4	0	1	1	1	1

Table 2.3

Table for LOSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	1	0	0	0	0
j_2	1	0	0	1	0
j_4	1	0	0	1	1
j_5	1	1	0	1	1

Table 2.4

Table for extended intersection of two LOSSs.

(K, C)	w_1	w_2	w_3	w_4	w_5
j_1	0	0	0	0	0
j_2	0	0	0	1	0
j_3	0	1	1	1	0
j_4	0	0	0	1	1
j_5	1	1	0	1	1

Table 2.5

Table for extended union of two LOSSs.

(H, G)	w_1	w_2	w_3	w_4	w_5
j_1	1	0	0	1	0
j_2	1	0	0	1	1
j_3	0	1	1	1	0
j_4	1	1	1	1	1
j_5	1	1	0	1	1

Table 2.6

In above example it is seen that, in general, extended intersection of lattice ordered soft sets may not be a lattice ordered soft set also extended union of lattice ordered soft sets may not be a lattice ordered soft set. However we have the following.

2.2.4 Proposition[2]

Extended union of two lattice (anti-lattice) ordered soft sets (α, J_1) and (β, J_2) is a lattice (anti-lattice) ordered soft set if $(\alpha, J_1) \subseteqq (\beta, J_2)$ or $(\beta, J_2) \subseteqq (\alpha, J_1)$.

2.2.5 Proposition[2]

Let (α, J_1) and (β, J_2) be two lattice (anti-lattice) ordered soft sets over W. Then

1. $((\alpha, J_1) \cup_R (\beta, J_2))^C = (\alpha, J_1)^C \cap_R (\beta, J_2)^C$
2. $((\alpha, J_1) \cap_R (\beta, J_2))^C = (\alpha, J_1)^C \cup_R (\beta, J_2)^C$.

2.2.6 Proposition[2]

Basic intersection of two lattice (anti-lattice) ordered soft sets (α, J_1) and (β, J_2) is a lattice (anti-lattice) ordered soft set.

2.2.7 Proposition[2]

Basic union of two lattice (anti-lattice) ordered soft sets (α, J_1) and (β, J_2) is a lattice (anti-lattice) ordered soft set.

2.2.8 Proposition[2]

Let (α, J_1) and (β, J_2) be two lattice (anti-lattice) ordered soft sets over W. Then

1. $(\alpha, J_1) \cap_R ((\alpha, J_1) \cup_R (\beta, J_2)) = (\alpha, J_1)$
2. $(\alpha, J_1) \cup_R ((\alpha, J_1) \cap_R (\beta, J_2)) = (\alpha, J_1)$.

2.2.9 Proposition[2]

If (α, J_1) and (β, J_2) are two lattice (anti-lattice) ordered soft sets, then $(\alpha, J_1) \cap_\wedge (\beta, J_2)$ lattice (anti-lattice) ordered soft set.

2.2.10 Proposition[2]

If (α, J_1) and (β, J_2) are two lattice (anti-lattice) ordered soft sets, then $(\alpha, J_1) \cup_\vee (\beta, J_2)$ lattice (anti-lattice) ordered soft set.

CHAPTER III

LATTICE ORDERED NEUTROSOPHIC SOFT SETS

In this chapter we define lattice ordered neutrosophic soft set, lattice ordered neutrosophic soft subset, operations on lattice ordered neutrosophic soft sets and derive some results.

3.1 Lattice ordered neutrosophic soft set

In this section we define lattice ordered neutrosophic soft set and initially we start with order on set of parameters and construct an example of lattice ordered neutrosophic soft set.

3.1.1 Definition

Let $J_1 \subseteq J$, where J is a lattice. A lattice ordered neutrosophic soft set (LONSS) over W is a pair (α, J_1) s.t $\alpha : J_1 \longrightarrow NS(W)$ s.t for each $j_1 \in J_1$, $\alpha(j_1)$ is a nutrosophic soft set and for $j_1 \leq j_2$ implies that $\alpha(j_1) \preceq \alpha(j_2)$ $\forall$ $j_1, j_2 \in J_1$.

3.1.2 Example

let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of four houses and

$J_1 = \{j_1(\text{cheap}), j_2(\text{beautiful}), j_3(\text{airy}), j_4(\text{space for vehicle}\}$ be the set of parameters. The order among the elements of J_1 is shown in Fig. 3.

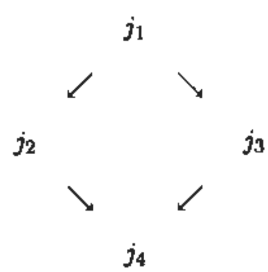

Fig. 3. Lattice of parameters

Tabular form for the LONSS (α, J_1) is given in Table 3.1.

Tahle for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)	(.4, .3, .84)
j_2	(.4, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)	(.4, .7, .79)
j_3	(.6, .7, .5)	(.6, .7, .4)	(.55, .8, .5)	(.6, .77, .2)	(.6, .76, .66)
j_4	(.7, .9, .45)	(.9, .85, .3)	(.9, .85, .2)	(.7, .8, .1)	(.8, .81, .6)

Table 3.1

Then clearly from Table 3.1, $\alpha(j_1) \preceq \alpha(j_2) \preceq \alpha(j_4)$ and $\alpha(j_1) \preceq \alpha(j_3) \preceq \alpha(j_4)$ so (α, J_1) is a LONSS over W.

3.2 Operations and results on lattice ordered neutrosophic soft sets

In this section some operations on lattice ordered neutrosophic soft sets are being studied.

3.2.1 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then (α, J_1) is said to be LONS subset of (β, J_2) if $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) \leq T_{\beta(j)}(w), I_{\alpha(j)}(w) \leq I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) \geq F_{\beta(j)}(w)$, $\forall\, j \in J_1, w \in W$. We denote $(\alpha, J_1) \subseteqq (\beta, J_2)$.

3.2.2 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then (α, J_1) is called a LONS M-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_M (\beta, J_2)$, if $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w), I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\, j \in J_1, w \in W$. Two LONSSs (α, J_1) and (β, J_2) over W are said to be LONS M-equal, denoted $(\alpha, J_1) =_M (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and $(\beta, J_2) \subseteqq_M (\alpha, J_1)$.

3.2.3 Proposition

Let (α, J_1) and (β, J_2) be two LONSSs over W.Then

1. $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ iff $J_1 \subseteq J_2$ and $\beta(j) = \alpha(j)\ \forall\, j \in J_1$.

2. $(\alpha, J_1) =_M (\beta, J_2)$ iff $J_1 = J_2$ and $\beta(j) = \alpha(j)\ \forall\ j \in J_1 = J_2$.

Proof. 1. Suppose $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ then by definition which implies that $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_1, w \in W$,then implies that $J_1 \subseteq J_2$ and $\beta(j) = \alpha(j)\ \forall\ j \in J_1$.

conversely, let $J_1 \subseteq J_2$ and $\beta(j) = \alpha(j)\ \forall\ j \in J_1$, then $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_1, w \in W$, implies that $(\alpha, J_1) \subseteqq_M (\beta, J_2)$.

2. Suppose $(\alpha, J_1) =_M (\beta, J_2)$ then by (definition 3.2.2)$(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and $(\beta, J_2) \subseteqq_M (\alpha, J_1)$, which means $J_1 \subseteq J_2$ and $J_2 \subseteq J_1$ and also $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$, $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_1, w \in W$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$, $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_2, w \in W$. implies that $J_1 = J_2$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$, $F_{\alpha(j)}(w) = F_{\beta(j)}(w)$, $\forall\ j \in J_1 = J_2$. This implies that $J_1 = J_2$ and $\beta(j) = \alpha(j)\ \forall\ j \in J_1 = J_2$.

Conversely, let $J_1 = J_2$ and $\beta(j) = \alpha(j)\ \forall\ j \in J_1 = J_2$. Then $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$, $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_1 = J_2$ which implies that $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and $(\beta, J_2) \subseteqq_M (\alpha, J_1)$ then $(\alpha, J_1) =_M (\beta, J_2)$. ■

3.2.4 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then (α, J_1) is called a LONS F-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_F (\beta, J_2)$, if $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) \leq T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) \leq I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) \geq F_{\beta(j)}(w)\ \forall\ j \in J_1, w \in W$. Two LONSSs (α, J_1) and (β, J_2) over W are said to be LONS F-equal, denoted $(\alpha, J_1) =_F (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_F (\beta, J_2)$ and $(\beta, J_2) \subseteqq_F (\alpha, J_1)$.

3.2.5 Note

It is easy to see that for two LONSSs (α, J_1) and (β, J_2) over W,if (α, J_1) is a LONS M-subset of (β, J_2) then (α, J_1) is also LONS F-subset of (β, J_2). However, the converse may not true as illustrated by the following example.

3.2.6 Example

let $W = \{w_1, w_2, w_3, w_4\}$ be the set of cars and

$J = \{j_1(\text{cheap}), j_2(\text{good looking}), j_3(\text{good condition}), j_4(\text{speed})\}$ be a set of parameters with lattice order. This order can be described as $j_1 \leq j_2 \leq j_3 \leq j_4$. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_3\}$, $J_2 = \{j_1, j_2, j_3, j_4\}$. Consider (α, J_1), (β, J_2) and (K, J_2) LONSSs over W as shown in Tables 3.2, 3.3 and 3.4, respectively.

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)	(.32, .3, .8)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)	(.4, .45, .59)
j_3	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)	(.7, .6, .5)

Table 3.2

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)	(.32, .3, .8)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)	(.4, .45, .59)
j_3	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)	(.7, .6, .5)
j_4	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .2)	(.75, .8, .37)

Table 3.3

Table for LONSS (K, J_2).

(K, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.1, .3, .9)	(.2, .55, .6)	(.4, .45, .7)	(.5, .64, .4)	(.4, .5, .7)
j_2	(.2, .5, .6)	(.4, .6, .3)	(.5, .75, .6)	(.6, .75, .27)	(.5, .55, .5)
j_3	(.5, .8, .4)	(.7, .8, .2)	(.7, .85, .3)	(.7, .8, .1)	(.75, .69, .34)
j_4	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .3)	(.8, .75, .25)

Table 3.4

that $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and $(\beta, J_2) \subseteqq_M (\alpha, J_1)$, then $(\alpha, J_1) =_M (\beta, J_2)$. ∎

3.2.8 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then (α, J_1) is called a LONS J-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_J (\beta, J_2)$, if for every $m \in J_1$ there exists $n \in J_2$ s.t $\alpha(m) \preceq \beta(n)$ then $T_{\alpha(m)}(w) \leq T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) \leq I_{\beta(n)}(w)$ and $F_{\beta(n)}(w) \leq F_{\alpha(m)}(w)$ $\forall\, w \in W$. Two LONSSs (α, J_1) and (β, J_2) over W are said to be LONS J-equal, denoted $(\alpha, J_1) =_J (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_J (\beta, J_2)$ and $(\beta, J_2) \subseteqq_J (\alpha, J_1)$.

3.2.9 Example

let $W = \{w_1, w_2, w_3, w_4\}$ be the set of cars and

$J = \{j_1(\text{cheap}), j_2(\text{good looking}), j_3(\text{good condition}), j_4(\text{speed})\}$ be a set of parameters. This order can be described as $j_1 \leq j_2 \leq j_3 \leq j_4$. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2\}$, $J_2 = \{j_2, j_3, j_4\}$. Consider (α, J_1) and (β, J_2) as LONSSs over W as shown in Tables 3.5 and 3.6, respectively.

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4
j_1	(.5, .8, .7)	(.4, .5, .7)	(.7, .9, .75)	(.7, .6, .7)
j_2	(.9, .9, .7)	(.8, .6, .6)	(.8, .9, .6)	(.9, .7, .3)

Table 3.5

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)
j_3	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)
j_4	(.9, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .2)

Table 3.6

$J = \{j_1(\text{cheap}), j_2(\text{good looking}), j_3(\text{good condition}), j_4(\text{speed})\}$ be a set of parameters with lattice order. This order can be described as $j_1 \le j_2 \le j_3 \le j_4$. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_3\}$, $J_2 = \{j_1, j_2, j_3, j_4\}$. Consider (α, J_1), (β, J_2) and (K, J_2) LONSSs over W as shown in Tables 3.2, 3.3 and 3.4, respectively.

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)	(.32, .3, .8)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)	(.4, .45, .59)
j_3	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)	(.7, .6, .5)

Table 3.2

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)	(.32, .3, .8)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)	(.4, .45, .59)
j_3	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)	(.7, .6, .5)
j_4	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .2)	(.75, .8, .37)

Table 3.3

Table for LONSS (K, J_2).

(K, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.1, .3, .9)	(.2, .55, .6)	(.4, .45, .7)	(.5, .64, .4)	(.4, .5, .7)
j_2	(.2, .5, .6)	(.4, .6, .3)	(.5, .75, .6)	(.6, .75, .27)	(.5, .55, .5)
j_3	(.5, .8, .4)	(.7, .8, .2)	(.7, .85, .3)	(.7, .8, .1)	(.75, .69, .34)
j_4	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .3)	(.8, .75, .25)

Table 3.4

From Tables 3.2 and 3.3, we have $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and from Tables 3.2 and 3.4, we have $(\alpha, J_1) \subseteqq_F (K, J_2)$ but (α, J_1) is not a LONS M-subset of (K, J_2).

3.2.7 Proposition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then the following conditions are equivalent;

1. $(\alpha, J_1) =_M (\beta, J_2)$
2. $(\alpha, J_1) =_F (\beta, J_2)$
3. $J_1 = J_2$ and $(\alpha, J_1) = (\beta, J_2)$

Proof. (1) $\Longrightarrow$ (2)

suppose $(\alpha, J_1) =_M (\beta, J_2)$ then $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and $(\beta, J_2) \subseteqq_M (\alpha, J_1)$. For $(\alpha, J_1) \subseteqq_M (\beta, J_2)$, implies that $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_1, w \in W$. implies that $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) \leq T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) \leq I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) \geq F_{\beta(j)}(w)\ \forall\ j \in J_1, w \in W$ this implies that $(\alpha, J_1) \subseteqq_F (\beta, J_2)$.

Now take $(\beta, J_2) \subseteqq_M (\alpha, J_1)$ implies that that $J_2 \subseteq J_1$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_2, w \in W$, implies that $J_2 \subseteq J_1$ and $T_{\beta(j)}(w) \leq T_{\alpha(j)}(w)$, $I_{\beta(j)}(w) \leq I_{\alpha(j)}(w)$ and $F_{\alpha(j)}(w) \leq F_{\beta(j)}(w)\ \forall\ j \in J_2, w \in W$ implies that $(\beta, J_2) \subseteqq_F (\alpha, J_1)$. From both result $(\alpha, J_1) \subseteqq_F (\beta, J_2)$ and $(\beta, J_2) \subseteqq_F (\alpha, J_1)$ implies that $(\alpha, J_1) =_F (\beta, J_2)$.

(2) $\Longrightarrow$ (3)

suppose $(\alpha, J_1) =_F (\beta, J_2)$ then $(\alpha, J_1) \subseteqq_F (\beta, J_2)$ and $(\beta, J_2) \subseteqq_F (\alpha, J_1)$. From $(\alpha, J_1) \subseteqq_F (\beta, J_2)$ implies that $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) \leq T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) \leq I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) \geq F_{\beta(j)}(w)\ \forall\ j \in J_1, w \in W$. Now take $(\beta, J_2) \subseteqq_F (\alpha, J_1)$ implies that $J_2 \subseteq J_1$ and $T_{\beta(j)}(w) \leq T_{\alpha(j)}(w)$, $I_{\beta(j)}(w) \leq I_{\alpha(j)}(w)$ and $F_{\alpha(j)}(w) \leq F_{\beta(j)}(w)\ \forall\ j \in J_2, w \in W$.

combine both result $J_1 = J_2$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_2 = J_1, w \in W$, implies that $J_1 = J_2$ and $(\alpha, J_1) = (\beta, J_2)$.

(3) $\Longrightarrow$ (1)

suppose $J_1 = J_2$ and $(\alpha, J_1) = (\beta, J_2)$ implies that $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)\ \forall\ j \in J_2 = J_1, w \in W$ then $J_1 \subseteq J_2$ and $J_2 \subseteq J_1$ implies that $(\alpha, J_1) \subseteqq_M (\beta, J_2)$ and $(\beta, J_2) \subseteqq_M (\alpha, J_1)$, then $(\alpha, J_1) =_M (\beta, J_2)$. ∎

3.2.8 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then (α, J_1) is called a LONS J-subset of (β, J_2), denoted $(\alpha, J_1) \subseteqq_J (\beta, J_2)$, if for every $m \in J_1$ there exists $n \in J_2$ s.t $\alpha(m) \preceq \beta(n)$ then $T_{\alpha(m)}(w) \leq T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) \leq I_{\beta(n)}(w)$ and $F_{\beta(n)}(w) \leq F_{\alpha(m)}(w)$ $\forall\, w \in W$. Two LONSSs (α, J_1) and (β, J_2) over W are said to be LONS J-equal, denoted $(\alpha, J_1) =_J (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_J (\beta, J_2)$ and $(\beta, J_2) \subseteqq_J (\alpha, J_1)$.

3.2.9 Example

let $W = \{w_1, w_2, w_3, w_4\}$ be the set of cars and

$J = \{j_1(\text{cheap}), j_2(\text{good looking}), j_3(\text{good condition}), j_4(\text{speed})\}$ be a set of parameters. This order can be described as $j_1 \leq j_2 \leq j_3 \leq j_4$. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2\}$, $J_2 = \{j_2, j_3, j_4\}$. Consider (α, J_1) and (β, J_2) as LONSSs over W as shown in Tables 3.5 and 3.6, respectively.

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4
j_1	(.5, .8, .7)	(.4, .5, .7)	(.7, .9, .75)	(.7, .6, .7)
j_2	(.9, .9, .7)	(.8, .6, .6)	(.8, .9, .6)	(.9, .7, .3)

Table 3.5

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)
j_3	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)
j_4	(.9, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .2)

Table 3.6

Then it is easy to see that $(\alpha, J_1) \subseteqq_J (\beta, J_2)$, since $\alpha(j_1) \preceq \beta(j_4)$ and $\alpha(j_2) \preceq \beta(j_4)$. On the other hand, the parameter set J_1 is not a subset of J_2, hence $(\alpha, J_1) \nsubseteqq_F (\beta, J_2)$ and $(\beta, J_2) \nsubseteqq_J (\alpha, J_1)$ then $(\alpha, J_1) \neq_J (\beta, J_2)$.

3.2.10 Proposition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then the following conditions are equivalent;

1. $(\alpha, J_1) \subseteqq_J (\beta, J_2)$
2. There exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $\alpha(m) \preceq \beta(\phi(m))\ \forall\ m \in J_1$.
3. There exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $(\alpha, J_1) \subseteqq_F (\beta o\phi, J_1)$.

Proof. $(1) \Longrightarrow (2)$

Suppose $(\alpha, J_1) \subseteqq_J (\beta, J_2)$. For every $m \in J_1$ by (definition 3.2.8) there exists $n \in J_2$ s.t $T_{\alpha(m)}(w) \leq T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) \leq I_{\beta(n)}(w)$ and $F_{\beta(n)}(w) \leq F_{\alpha(m)}(w)\ \ \forall\ w \in W$, then $\alpha(m) \preceq \beta(n)$. Then the Axiom of choice ensures that find a mapping $\phi : J_1 \longrightarrow J_2$ s.t $\phi(m) = n$, whence $\alpha(m) \preceq \beta(n) = \beta(\phi(m))$ as required.

$(2) \Longrightarrow (3)$

Suppose that there exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $\alpha(m) \preceq \beta(\phi(m))\ \forall\ m \in J_1$. Let define a LONSS over W $(\beta o\phi, J_1)$ over W, where $\beta o\phi(m) = \beta(\phi(m))$ then definition of LONS F-subset, (α, J_1) is a LONS F-subset of $(\beta o\phi, J_1)$.

$(3) \Longrightarrow (1)$

Assume that there exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $(\alpha, J_1) \subseteqq_F (\beta o\phi, J_1)$. Then for every $m \in J_1$, $\alpha(m) \preceq \beta o\phi(m) = \beta(\phi(m))$. That is, there exists $n = \phi(m)$ s.t $\alpha(m) \preceq \beta(n) = \beta(\phi(m))$, hence by definition of LONS J-subset, (α, J_1) is a LONS J-subset of (β, J_2). ■

3.2.11 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then (α, J_1) is called a LONS L-subset of (β, J_2). denoted $(\alpha, J_1) \subseteqq_L (\beta, J_2)$, if for every $m \in J_1$ there exists $n \in J_2$, s.t $T_{\alpha(m)}(w) = T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) = I_{\beta(n)}(w)$ and $F_{\alpha(m)}(w) = F_{\beta(n)}(w)$ then $\alpha(m) = \beta(n)\ \ \forall\ w \in W$.

Two LONSSs (α, J_1) and (β, J_2) over W are said to be LONS L-equal, denoted $(\alpha, J_1) =_L (\beta, J_2)$, if $(\alpha, J_1) \subseteqq_L (\beta, J_2)$ and $(\beta, J_2) \subseteqq_L (\alpha, J_1)$.

3.2.12 Example

let $W = \{w_1, w_2, w_3, w_4\}$ be the set of cars and

$J = \{j_1(\text{cheap}), j_2(\text{good looking}), j_3(\text{good condition}), j_4(\text{speed})\}$ be a set of parameters. This order can be described as shown in Fig. 4. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2\}$, $J_2 = \{j_1, j_3, j_4\}$. Consider (α, J_1) and (β, J_2) LONSSs over W as shown in Tables 3.7 and 3.8, respectively.

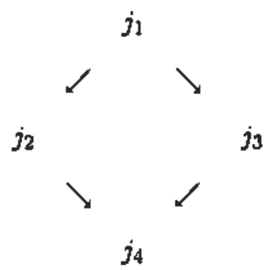

Fig. 4. Lattice of parameters

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5
j_1	(.5, .8, .7)	(.4, .4, .7)	(.7, .5, .75)	(.3, .6, .7)	(.5, .3, .8)
j_2	(.9, .9, .3)	(.8, .7, .6)	(.8, .9, .6)	(.9, .8, .3)	(.8, .9, .3)

Table 3.7

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5
j_1	(.5, .8, .7)	(.4, .4, .7)	(.7, .5, .75)	(.3, .6, .7)	(.5, .3, .8)
j_3	(.7, .8, .5)	(.6, .5, .7)	(.7, .8, .7)	(.6, .77, .5)	(.7, .5, .6)
j_4	(.9, .9, .3)	(.8, .7, .6)	(.8, .9, .6)	(.9, .8, .3)	(.8, .9, .3)

Table 3.8

Then it is easy to see that $(\alpha, J_1) \subseteqq_L (\beta, J_2)$, since $\alpha(j_1) = \beta(j_1)$ and $\alpha(j_2) = \beta(j_4)$.

3.2.13 Proposition

For two LONSSs (α, J_1) and (β, J_2) over W, $(\alpha, J_1) \subseteqq_M (\beta, J_2) \Longrightarrow (\alpha, J_1) \subseteqq_L (\beta, J_2) \Longrightarrow (\alpha, J_1) \subseteqq_J (\beta, J_2)$.

Proof. Suppose that $(\alpha, J_1) \subseteqq_M (\beta, J_2)$, then $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) = T_{\beta(j)}(w)$, $I_{\alpha(j)}(w) = I_{\beta(j)}(w)$ and $F_{\alpha(j)}(w) = F_{\beta(j)}(w)$ $\forall\ j \in J_1, w \in W$. For every $m \in J_1$ implies that $m = n \in J_2$ by definition of LONS M-subset, $T_{\alpha(m)}(w) = T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) = I_{\beta(n)}(w)$ and $F_{\alpha(m)}(w) = F_{\beta(n)}(w)$ $\forall\ w \in W$, implies that $\alpha(m) = \beta(n)$ $\forall\ x \in W$ implies that $(\alpha, J_1) \subseteqq_L (\beta, J_2)$.

Now suppose that $(\alpha, J_1) \subseteqq_L (\beta, J_2)$ then for every $m \in J_1$ there exists $n \in J_2$ s.t $T_{\alpha(m)}(w) = T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) = I_{\beta(n)}(w)$ and $F_{\beta(n)}(w) = F_{\alpha(m)}(w)$ $\forall\ w \in W$. Implies that for every $m \in J_1$ there exists $n \in J_2$ s.t $T_{\alpha(m)}(w) \leq T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) \leq I_{\beta(n)}(w)$ and $F_{\beta(n)}(w) \leq F_{\alpha(m)}(w)$ $\forall\ w \in W$. which implies that for every $m \in J_1$ there exists $n \in J_2$ s.t $\alpha(m) \preceq \beta(n)$ $\forall\ w \in W$, then $(\alpha, J_1) \subseteqq_J (\beta, J_2)$. ■

3.2.14 Proposition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then the following conditions are equivalent:

1. $(\alpha, J_1) \subseteqq_L (\beta, J_2)$.
2. There exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $\alpha(m) = \beta(\phi(m))$, $\forall\ m \in J_1$.
3. There exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $(\alpha, J_1) = (\beta o \phi, J_1)$.
4. The set $\{\alpha(m) : m \in J_1\}$ is a subset of $\{\beta(n) : n \in J_2\}$.

Proof. (1) $\Longrightarrow$ (2)

Suppose $(\alpha, J_1) \subseteqq_L (\beta, J_2)$. For every $m \in J_1$ by (definition 3.2.11) there exists $n \in J_2$, s.t $T_{\alpha(m)}(w) = T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) = I_{\beta(n)}(w)$ and $F_{\beta(n)}(w) = F_{\alpha(m)}(w)$ $\forall\ w \in W$, then $\alpha(m) = \beta(n)$. Then the Axiom of choice ensures that find a mapping $\phi : J_1 \longrightarrow J_2$ s.t $\phi(m) = n$, whence $\alpha(m) = \beta(n) = \beta(\phi(m))$, $\forall\ m \in J_1$, as required.

(2) $\Longrightarrow$ (3)

Suppose that there exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $\alpha(m) = \beta(\phi(m))\ \forall\ m \in J_1$. Let define a LONSS $(\beta o\phi, J_1)$ over W, where $\beta o\phi(m) = \beta(\phi(m))$ then definition of LONS equal, $(\alpha, J_1) \equiv (\beta o\phi, J_1)$.

$(3) \Longrightarrow (4)$

Assume that there exists a mapping $\phi : J_1 \longrightarrow J_2$ s.t $(\alpha, J_1) \equiv (\beta o\phi, J_1)$. Then for every $m \in J_1$, $\alpha(m) = \beta o\phi(m) = \beta(\phi(m))$. That is, there exists $n = \phi(m)$ s.t $\alpha(m) = \beta(n) = \beta(\phi(m))$, then $\{\alpha(m) : m \in J_1\}$ is a subset of $\{\beta(n) : n \in J_2\}$

$(4) \Longrightarrow (1)$

Assume $\{\alpha(m) : m \in J_1\}$ is a subset of $\{\beta(n) : n \in J_2\}$, then for every $m \in J_1$ there exists $n \in J_2$ s.t $\alpha(m) = \beta(n)$, implies that $(\alpha, J_1) \subseteqq_L (\beta, J_2)$. ■

3.2.15 Proposition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) =_L (\beta, J_2)$ iff $\{\alpha(m) : m \in J_1\} = \{\beta(n) : n \in J_2\}$.

Proof. Assume that $(\alpha, J_1) =_L (\beta, J_2)$. Then $(\alpha, J_1) \subseteqq_L (\beta, J_2)$ and $(\beta, J_2) \subseteqq_L (\alpha, J_1)$. Let $(\alpha, J_1) \subseteqq_L (\beta, J_2)$, for every $m \in J_1$ there exists $n \in J_2$ s.t $T_{\alpha(m)}(w) = T_{\beta(n)}(w)$, $I_{\alpha(m)}(w) = I_{\beta(n)}(w)$ and $F_{\beta(n)}(w) = F_{\alpha(m)}(w)\ \ \forall\ w \in W$, then $\alpha(m) = \beta(n)$. Which implies that $\{\alpha(m) : m \in J_1\}$ is a subset of $\{\beta(n) : n \in J_2\}$.

Now let $(\beta, J_2) \subseteqq_L (\alpha, J_1)$, then for every $n \in J_2$ there exists $m \in J_1$ s.t $T_{\beta(n)}(w) = T_{\alpha(m)}(w)$, $I_{\beta(n)}(w) = I_{\alpha(m)}(w)$ and $F_{\alpha(m)}(w) = F_{\beta(n)}(w)\ \forall\ w \in W$, then $\beta(n) = \alpha(m)$. Which implies that $\{\beta(n) : n \in J_2\}$ is a subset of $\{\alpha(m) : m \in J_1\}$. From both result implies that $\{\alpha(m) : m \in J_1\} = \{\beta(n) : n \in J_2\}$.

Conversely, assume that $\{\alpha(m) : m \in J_1\} = \{\beta(n) : n \in J_2\}$, which implies that

$\{\alpha(m) : m \in J_1\}$ is a subset of $\{\beta(n) : n \in J_2\}$ and also $\{\beta(n) : n \in J_2\}$ is a subset of $\{\alpha(m) : m \in J_1\}$. Let $\{\alpha(m) : m \in J_1\}$ is a subset of $\{\beta(n) : n \in J_2\}$, implies that for every $m \in J_1$ there exists $n \in J_2$ s.t $\alpha(m) = \beta(n)$, which implies that $(\alpha, J_1) \subseteqq_L (\beta, J_2)$. Now let $\{\beta(n) : n \in J_2\}$ is a subset of $\{\alpha(m) : m \in J_1\}$,implies that for every $n \in J_2$ there exists $m \in J_1$ s.t $\beta(n) = \alpha(m)$, which implies that $(\beta, J_2) \subseteqq_L (\alpha, J_1)$. From both result implies that $(\alpha, J_1) =_L (\beta, J_2)$. ■

3.2.16 Definition

Let (α, J_1) be a LONSS over W. Then (α, J_1) is called an injective LONSS over W if $\forall\, m_1$, $m_2 \in J_1$, $m_1 \neq m_2$ implies $\alpha(m_1) = \alpha(m_2)$.

3.2.17 Theorem

let (α, J_1) and (β, J_2) be two injective LONSSs over W. Then $(\alpha, J_1) =_L (\beta, J_2)$ iff there exists a bijective mapping $\phi : J_1 \longrightarrow J_2$ s.t $\alpha(m) = \beta(\phi(m))\ \forall\, m \in J_1$.

Proof. First assume that (α, J_1) and (β, J_2) are two injective LONSSs over W, s.t $(\alpha, J_1) =_L (\beta, J_2)$. For every $m \in J_1$, there exists $n \in J_2$ s.t $\alpha(m) = \beta(n)$. In addition, find the parameter n is indeed unique, since (β, J_2) is an injective LONSS over W. Thus we obtain a mapping $\phi : J_1 \longrightarrow J_2$ given by $\phi(m) = \beta(\alpha(m)) = n\ \forall\, m \in J_1$. Now it is easy to verify that $\beta(\phi(m)) = \beta(n) = \alpha(m)\ \forall\, m \in J_1$. For two different parameters m_1, $m_2 \in J_1$, since (α, J_1) is injective then $\alpha(m_1) = \alpha(m_2)$. it follows that $\phi(m_1) = \beta(\alpha(m_1)) \neq \beta(\alpha(m_2)) = \phi(m_1)$, whence ϕ is an injective mapping. Furthermore, also show that ϕ is surjective. In fact, for every $n \in J_2$, there exists $m \in J_1$ s.t $\alpha(m) = \beta(n)$. But note that $\alpha(m) = \beta(\phi(m))$ as shown above. This yields to $\phi(m) = n$ since (β, J_2) are injective LONSS over W. Thus ϕ is an bijective mapping as required.

Conversely, suppose that there exists a bijective mapping $\phi : J_1 \longrightarrow J_2$ s.t $\alpha(m) = \beta(\phi(m))\ \forall\, m \in J_1$.By proposition (3.2.14), $(\alpha, J_1) \subseteq_L (\beta, J_2)$. Then it remains to show that $(\beta, J_2) \subseteq_L (\alpha, J_1)$ also holds. Since $\phi : J_1 \longrightarrow J_2$ is a mapping, consider its inverse mapping $\phi^{-1} : J_2 \longrightarrow J_1$. For every $n \in J_2$, it is clear that $\beta(n) =\beta\left(\phi\left(\phi^{-1}(n)\right)\right)$. That is, $\phi^{-1} : J_2 \longrightarrow J_1$ is a mapping s.t $\phi : J_1 \longrightarrow J_2$ s.t $\beta(n) = \alpha\left(\phi^{-1}(n)\right)\ \forall\, n \in J_2$. Again by Proposition (3.2.14), deduce that $(\beta, J_2) \subseteq_L (\alpha, J_1)$, completing the proof. ∎

3.2.18 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W such that $J_1 \cap J_2 \neq \emptyset$. Then $(\alpha, J_1) \cup_R (\beta, J_2) = (K, G)$ is said to be restricted union of two LONSSs over W, where $G = J_1 \cap J_2$ define $K(j) = \alpha(j) \cup \beta(j)$, $T_{K(j)}(w) = Max\left\{T_{\alpha(j)}(x), T_{\beta(j)}(w)\right\}$ $I_{K(j)}(w) = Max\left\{I_{\alpha(j)}(x), I_{\beta(j)}(w)\right\}$ and $F_{K(j)}(w) = Min\left\{F_{\alpha(j)}(x), F_{\beta(j)}(w)\right\}\ \forall\, j \in G, w \in W$.

3.2.19 Proposition

Restricted union of two LONSSs (α, J_1) and (β, J_2) is a LONSS.

Proof. Let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \cup_R (\beta, J_2) = (K, G)$, $\alpha(c) \cup \beta(c) = K(c)$ where $G = J_1 \cap J_2$. If $J_1 \cap J_2 = \emptyset$, then the required result hold trivially. If $J_1 \cap J_2 \neq \emptyset$, since $J_1, J_2 \subseteq J$ so J_1 and J_2 inherit the partial order from J. therefore for any $m_{1 \leq_{J_1}} m_2$ we have $\alpha(m_1) \preceq \alpha(m_2)$ $\forall$ $m_1, m_2 \in J_1$. also for any $n_{1 \leq_{J_2}} n_2$ we have $\beta(n_1) \preceq \beta(n_2)$ $\forall$ $n_1, n_2 \in J_2$. Therefore for any $c_1, c_2 \in G$ and $c_1 \leq_G c_2$ implies $c_1, c_2 \in J_1 \cap J_2$ implies $c_1, c_2 \in J_1$ and $c_1, c_2 \in J_2$ implies that $\alpha(c_1) \preceq \alpha(c_2)$ and $\beta(c_1) \preceq \beta(c_2)$ whenever $c_1 \leq_{J_1} c_2$ and $c_1 \leq_{J_2} c_2$

implies that

$$\begin{aligned} T_{\alpha(c_1)}(w) &\leq T_{\alpha(c_2)}(w), \; T_{\beta(c_1)}(w) \leq T_{\beta(c_2)}(w) \\ I_{\alpha(c_1)}(w) &\leq I_{\alpha(c_2)}(w), \; I_{\beta(c_1)}(w) \leq I_{\beta(c_2)}(w) \\ F_{\alpha(c_2)}(w) &\leq F_{\alpha(c_1)}(w), \; F_{\beta(c_2)}(w) \leq F_{\beta(c_1)}(w) \end{aligned} \tag{1}$$

implies that

$$\begin{aligned} Max\left\{T_{\alpha(c_1)}(w), T_{\beta(c_1)}(w)\right\} &\leq Max\left\{T_{\alpha(c_2)}(w), T_{\beta(c_2)}(w)\right\} \\ Max\left\{I_{\alpha(c_1)}(w), I_{\beta(c_1)}(w)\right\} &\leq Max\left\{I_{\alpha(c_2)}(w), I_{\beta(c_2)}(w)\right\} \\ Min\left\{F_{\alpha(c_2)}(w), F_{\beta(c_2)}(w)\right\} &\leq Min\left\{F_{\alpha(c_1)}(w), F_{\beta(c_1)}(w)\right\} \end{aligned}$$

implies that

$$\begin{aligned} T_{\alpha(c_1)\cup\beta(c_1)}(w) &\leq T_{\alpha(c_2)\cup\beta(c_2)}(w) \\ I_{\alpha(c_1)\cup\beta(c_1)}(w) &\leq I_{\alpha(c_2)\cup\beta(c_2)}(w) \\ F_{\alpha(c_2)\cup\beta(c_2)}(w) &\leq F_{\alpha(c_1)\cup\beta(c_1)}(w) \end{aligned}$$

implies that

$$\begin{aligned} T_{(\alpha\cup\beta)(c_1)}(w) &\leq T_{(\alpha\cup\beta)(c_2)}(w) \\ I_{(\alpha\cup\beta)(c_1)}(w) &\leq I_{(\alpha\cup\beta)(c_2)}(w) \\ F_{(\alpha\cup\beta)(c_2)}(w) &\leq F_{(\alpha\cup\beta)(c_1)}(w) \end{aligned}$$

implies that

$$\begin{aligned} T_{K(c_1)}(w) &\leq T_{K(c_2)}(w) \\ I_{K(c_1)}(w) &\leq I_{K(c_2)}(w) \\ F_{K(c_2)}(w) &\leq F_{K(c_1)}(w) \end{aligned}$$

implies that $K(c_1) \preceq K(c_2)$ for $c_1 \leq_G c_1$.

Thus $(\alpha, J_1) \cup_R (\beta, J_2)$ is a LONSS over W. ■

3.2.20 Example

let $W = \{w_1, w_2, w_3, w_4\}$ be the set of cars and

$J = \{j_1(\text{cheap}), j_2(\text{good looking}), j_3(\text{good condition}), j_4(\text{speed}), j_5(\text{well assembled})\}$ be a set of parameters. This order can be described as shown in Fig. 5. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_4\}$, $J_2 = \{j_2, j_4, j_5\}$. Consider (α, J_1) and (β, J_2) LONSSs over W as shown in Tables 3.9 and 3.10, respectively.

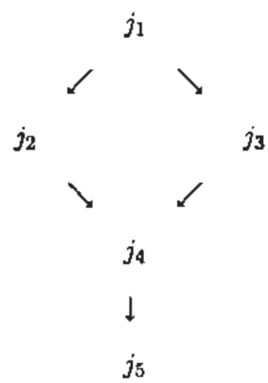

Fig. 5. Lattice of parameters

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)
j_4	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)

Table 3.9

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4
j_2	(.2, .1, .8)	(.4, .6, .3)	(.1, .4, .6)	(.5, .3, .9)
j_4	(.4, .5, .4)	(.7, .8, .2)	(.4, .6, .4)	(.7, .7, .7)
j_5	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .3)

Table 3.10

$(\alpha, J_1) \cup_R (\beta, J_2) = (K, G)$ is a restricted union of two LONSS over W.

Table for restricted union of two LONSS.

(K, G)	w_1	w_2	w_3	w_4
j_2	(.2, .4, .7)	(.4, .6, .3)	(.4, .7, .6)	(.56, .7, .3)
j_4	(.4, .7, .4)	(.7, .8, .2)	(.55, .8, .4)	(.7, .77, .2)

Table 3.11

clearly from Table 3.11, $K(j_2) \preceq K(j_4)$ for $j_2 \leq j_3$, then (K, G) LONSS over W.

3.2.21 Definition

let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \cup_{\in} (\beta, J_2) = (K, G)$ is said to be extended union of two LONSSs over W, where $G = J_1 \cup J_2$ define $K(j) = \alpha(j) \cup \beta(j)$,

$$T_{K(j)}(w) = \begin{cases} T_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ T_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Max\{T_{\alpha(j)}(w), T_{\beta(j)}(w)\} & \text{if } j \in J_1 \cap J_2 \end{cases}$$

$$I_{K(j)}(w) = \begin{cases} I_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ I_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Max\{I_{\alpha(j)}(w), I_{\beta(j)}(w)\} & \text{if } j \in J_1 \cap J_2 \end{cases}$$

and

$$F_{K(j)}(w) = \begin{cases} F_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ F_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Min\{F_{\alpha(j)}(w), F_{\beta(j)}(w)\} & \text{if } j \in J_1 \cap J_2 \end{cases}, \forall\, j \in G, w \in W.$$

3.2.22 Proposition

Extended union of two LONSSs (α, J_1) and (β, J_2) is a LONSS if $(\alpha, J_1) \subseteqq (\beta, J_2)$ or $(\beta, J_2) \subseteqq (\alpha, J_1)$.

Proof. Let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \cup_{\in} (\beta, J_2) = (K, G)$, $\alpha(c) \cup \beta(c) = K(c)$ where $G = J_1 \cup J_2$.

$$T_{K(j)}(w) = \begin{cases} T_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ T_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Max\{T_{\alpha(j)}(w), T_{\beta(j)}(w)\} & \text{if } j \in J_1 \cap J_2 \end{cases}$$

$$I_{K(j)}(w) = \begin{cases} I_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ I_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Max\{I_{\alpha(j)}(w), I_{\beta(j)}(w)\} & \text{if } j \in J_1 \cap J_2 \end{cases}$$

and $$F_{K(j)}(w) = \begin{cases} F_{\alpha(j)}(w) & \text{if } j \in J_1 - J_2 \\ F_{\beta(j)}(w) & \text{if } j \in J_2 - J_{12} \\ Min\{F_{\alpha(j)}(w), F_{\beta(j)}(w)\} & \text{if } j \in J_1 \cap J_2 \end{cases}, \forall\, j \in G, w \in W.$$

Suppose (α, J_1) is a LONS subset of (β, J_2), then $J_1 \subseteq J_2$ and $T_{\alpha(j)}(w) \leq T_{\beta(j)}(w)$,

$I_{\alpha(j)}(w) \leq I_{\beta(j)}(w)$ and $F_{\beta(j)}(w) \leq F_{\alpha(j)}(w)\ \forall\ j \in J_1, w \in W$. Since $J_1, J_2 \subseteq E$ so J_1 and J_2 inherit the partial order from J. Therefore for any $m_{1 \leq_{J_1}} m_2$ we have $\alpha(m_1) \preceq \alpha(m_2)\ \forall$ $m_1, m_2 \in J_1$. also for any $n_{1 \leq_{J_2}} n_2$ we have $\beta(n_1) \preceq \beta(n_2)\ \forall\ n_1, n_2 \in J_2$. Therefore for any $c_1, c_2 \in G$ and $c_1 \leq_G c_2$ implies $c_1, c_2 \in J_1 \cup J_2$ implies $c_1, c_2 \in J_1 \cap J_2$ or $c_1, c_2 \in J_2$ and $c_1, c_2 \notin J_1$ because $J_1 \subseteq J_2$.

Suppose $c_1, c_2 \in J_1 \cap J_2$ implies that $c_1, c_2 \in J_1$ and $c_1, c_2 \in J_2$, implies that $\alpha(c_1) \preceq \alpha(c_2)$ and $\beta(c_1) \preceq \beta(c_2)$ whenever $c_1 \leq_{J_1} c_2$ and $c_1 \leq_{J_2} c_2$,implies that

$$\begin{aligned} T_{\alpha(c_1)}(w) &\leq T_{\alpha(c_2)}(w),\ T_{\beta(c_1)}(w) \leq T_{\beta(c_2)}(w) \qquad (2) \\ I_{\alpha(c_1)}(w) &\leq I_{\alpha(c_2)}(w),\ I_{\beta(c_1)}(w) \leq I_{\beta(c_2)}(w) \\ F_{\alpha(c_2)}(w) &\leq F_{\alpha(c_1)}(w),\ F_{\beta(c_2)}(w) \leq F_{\beta(c_1)}(w) \end{aligned}$$

implies that

$$\begin{aligned} Max\left\{T_{\alpha(c_1)}(w), T_{\beta(c_1)}(w)\right\} &\leq Max\left\{T_{\alpha(c_2)}(w), T_{\beta(c_2)}(w)\right\} \\ Max\left\{I_{\alpha(c_1)}(w), I_{\beta(c_1)}(w)\right\} &\leq Max\left\{I_{\alpha(c_2)}(w), I_{\beta(c_2)}(w)\right\} \\ Min\left\{F_{\alpha(c_2)}(w), F_{\beta(c_2)}(w)\right\} &\leq Min\left\{F_{\alpha(c_1)}(w), F_{\beta(c_1)}(w)\right\} \end{aligned}$$

implies that

$$\begin{aligned} T_{\alpha(c_1)\cup\beta(c_1)}(w) &\leq T_{\alpha(c_2)\cup\beta(c_2)}(w) \\ I_{\alpha(c_1)\cup\beta(c_1)}(w) &\leq I_{\alpha(c_2)\cup\beta(c_2)}(w) \\ F_{\alpha(c_2)\cup\beta(c_2)}(w) &\leq F_{\alpha(c_1)\cup\beta(c_1)}(w) \end{aligned}$$

implies that

$$\begin{aligned} T_{(\alpha\cup\beta)(c_1)}(w) &\leq T_{(\alpha\cup\beta)(c_2)}(w) \\ I_{(\alpha\cup\beta)(c_1)}(w) &\leq I_{(\alpha\cup\beta)(c_2)}(w) \\ F_{(\alpha\cup\beta)(c_2)}(w) &\leq F_{(\alpha\cup\beta)(c_1)}(w) \end{aligned}$$

implies that

$$\begin{aligned} T_{K(c_1)}(w) &\leq T_{K(c_2)}(w) \\ I_{Kc_1}(w) &\leq I_{K(c_2)}(w) \\ F_{K(c_2)}(w) &\leq F_{K(c_1)}(w) \end{aligned}$$

implies that $K(c_1) \preceq K(c_2)$ for $c_1 \leq_G c_1$.

Thus $(\alpha, J_1) \cup_{\in} (\beta, J_2)$ is a LONSS over W if $c_1, c_2 \in J_1 \cap J_2$.

Now suppose for any $c_1, c_2 \in J_2$ and $c_1, c_2 \notin J_1$ and $c_1 \leq_{J_2} c_2$,

implies that $\beta(c_1) \preceq \beta(c_2)$ whenever $c_1 \leq_{J_2} c_2$.

implies this is also a LONSS over W.

Hence $(\alpha, J_1) \cup_{\in} (\beta, J_2)$ is a LONSS over W for both cases.

Similarly we can prove for (β, J_2) is a LONS subset of (α, J_1).

Thus extended union of two LONSSs over W is a LONSS over W if one of them is a LONS subset of other. ■

3.2.23 Example

$J = \{j_1, j_2, j_3, j_4, j_5\}$ with lattice order as shown in Fig. 6. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_4, j_5\}$, $J_2 = \{j_1, j_2, j_3, j_4, j_5\}$. Consider (α, J_1) and (β, J_2) LONSSs over $W = \{w_1, w_2, w_3, w_4\}$ as shown in Tables 3.12 and 3.13, respectively.

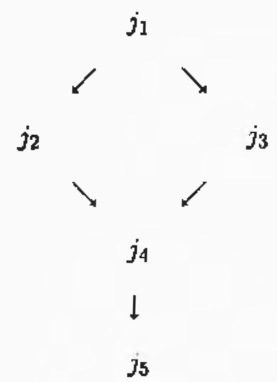

Fig. 6. Lattice of parameters

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4
j_1	$(.1, .2, .9)$	$(.3, .6, .5)$	$(.3, .7, .7)$	$(.7, .6, .8)$
j_4	$(.2, .4, .7)$	$(.4, .8, .4)$	$(.6, .9, .7)$	$(.8, .7, .6)$
j_5	$(.9, .8, .3)$	$(.5, .9, .3)$	$(.6, 1, .4)$	$(.9, .8, .3)$

Table 3.12

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4
j_1	$(.2, .3, .9)$	$(.4, .6, .4)$	$(.3, .8, .6)$	$(.8, .7, .7)$
j_2	$(.3, .4, .8)$	$(.4, .7, .3)$	$(.4, .8, .5)$	$(.8, .7, .6)$
j_3	$(.4, .6, .2)$	$(.5, .8, .3)$	$(.6, .9, .4)$	$(.85, .77, .3)$
j_4	$(.9, .9, .1)$	$(.7, .9, .2)$	$(.6, 1, .3)$	$(.9, .9, .2)$
j_5	$(1, 1, .1)$	$(.9, 1, .1)$	$(.66, 1, .1)$	$(1, .9, .1)$

Table 3.13

Then clearly from Tables 3.12 and 3.13, (α, J_1) is a LONS subset of (β, J_2). Now $(\alpha, J_1) \cup_R (\beta, J_2) = (K, G)$ is a extended union of two LONSSs over W where $G = J_1 \cup J_2$.

Table for extended union of two LONSSs.

(K, G)	w_1	w_2	w_3	w_4
j_1	$(.2, .3, .9)$	$(.4, .6, .4)$	$(.3, .8, .6)$	$(.8, .7, .7)$
j_2	$(.3, .4, .8)$	$(.4, .7, .3)$	$(.4, .8, .5)$	$(.8, .7, .6)$
j_3	$(.4, .6, .2)$	$(.5, .8, .3)$	$(.6, .9, .4)$	$(.85, .77, .3)$
j_4	$(.9, .9, .1)$	$(.7, .9, .2)$	$(.6, 1, .3)$	$(.9, .9, .2)$
j_5	$(1, 1, .1)$	$(.9, 1, .1)$	$(.66, 1, .1)$	$(1, .9, .1)$

Table 3.14

Clearly from Table 3.14, we have $K(j_1) \preceq K(j_2) \preceq K(j_3) \preceq K(j_4) \preceq K(j_5)$, so (K, G) is a LONSS over W.

3.2.24 Definition

let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \cap_R (\beta, J_2) = (K, G)$ is said to be restricted intersection of two LONSSs over W, where $G = J_1 \cap J_2 \neq \emptyset$, define $K(j) = \alpha(j) \cap \beta(j)$, $T_{K(j)}(w) = Min\{T_{\alpha(j)}(w), T_{\beta(j)}(w)\}$ $I_{K(j)}(w) = Min\{I_{\alpha(j)}(w), I_{\beta(j)}(w)\}$ and $F_{K(j)}(w) = Max\{F_{\alpha(j)}(w), F_{\beta(j)}(w)\}$ $\forall\, j \in G, w \in W$.

3.2.25 Proposition

Restricted intersection of two LONSSs (α, J_1) and (β, J_2) is a LONSS.

Proof. Let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \cap_R (\beta, J_2) = (K, G)$, $\alpha(c) \cap \beta(c) = K(c)$ where $G = J_1 \cap J_2$. If $J_1 \cap J_2 = \emptyset$ then the required result hold trivially. If $J_1 \cap J_2 \neq \emptyset$, since $J_1, J_2 \subseteq J$ so J_1 and J_2 inherit the partial order from E. therefore for any $m_{1 \leq_{J_1}} m_2$ we have $\alpha(m_1) \preceq \alpha(m_2)$ $\forall\, m_1, m_2 \in J_1$. also for any $n_{1 \leq_{J_2}} n_2$ we have $\beta(n_1) \preceq \beta(n_2)$ $\forall\, n_1, n_2 \in J_2$. Therefore for any $c_1, c_2 \in G$ and $c_1 \leq_G c_2$ implies $c_1, c_2 \in J_1 \cap J_2$ implies $c_1, c_2 \in J_1$ and $c_1, c_2 \in J_2$ implies that $\alpha(c_1) \preceq \alpha(c_2)$ and $\beta(c_1) \preceq \beta(c_2)$ whenever $c_1 \leq_{J_1} c_2$ and $c_1 \leq_{J_2} c_2$

implies that

$$\begin{aligned} T_{\alpha(c_1)}(w) &\leq T_{\alpha(c_2)}(w),\ T_{\beta(c_1)}(w) \leq T_{\beta(c_2)}(w) \\ I_{\alpha(c_1)}(w) &\leq I_{\alpha(c_2)}(w),\ I_{\beta(c_1)}(w) \leq I_{\beta(c_2)}(w) \\ F_{\alpha(c_2)}(w) &\leq F_{\alpha(c_1)}(w),\ F_{\beta(c_2)}(w) \leq F_{\beta(c_1)}(w) \end{aligned} \tag{3}$$

implies that

$$\begin{aligned} Min\{T_{\alpha(c_1)}(w), T_{\beta(c_1)}(w)\} &\leq Min\{T_{\alpha(c_2)}(w), T_{\beta(c_2)}(w)\} \\ Min\{I_{\alpha(c_1)}(w), I_{\beta(c_1)}(w)\} &\leq Min\{I_{\alpha(c_2)}(w), I_{\beta(c_2)}(w)\} \\ Max\{F_{\alpha(c_2)}(w), F_{\beta(c_2)}(w)\} &\leq Ma\bar{u}\{F_{\alpha(c_1)}(w), F_{\beta(c_1)}(w)\} \end{aligned}$$

implies that

$$\begin{aligned} T_{\alpha(c_1)\cap\beta(c_1)}(w) &\leq T_{\alpha(c_2)\cap\beta(c_2)}(w) \\ I_{\alpha(c_1)\cap\beta(c_1)}(w) &\leq I_{\alpha(c_2)\cap\beta(c_2)}(w) \\ F_{\alpha(c_2)\cap\beta(c_2)}(w) &\leq F_{\alpha(c_1)\cap\beta(c_1)}(w) \end{aligned}$$

implies that

$$\begin{aligned} T_{(\alpha\cap\beta)(c_1)}(w) &\leq T_{(\alpha\cap\beta)(c_2)}(w) \\ I_{(\alpha\cap\beta)(c_1)}(w) &\leq I_{(\alpha\cap\beta)(c_2)}(w) \\ F_{(\alpha\cap\beta)(c_2)}(w) &\leq F_{(\alpha\cap\beta)(c_1)}(w) \end{aligned}$$

implies that

$$\begin{aligned} T_{K(c_1)}(w) &\leq T_{K(c_2)}(w) \\ I_{Kc_1}(w) &\leq I_{K(c_2)}(w) \\ F_{K(c_2)}(w) &\leq F_{K(c_1)}(w) \end{aligned}$$

implies that $K(c_1) \preceq K(c_2)$ for $c_1 \leq_G c_1$.

Thus $(\alpha, J_1) \cap_R (\beta. J_2)$ is a LONSS over W. ■

3.2.26 Example

$J = \{j_1, j_2, j_3, j_4\}$ with lattice order as shown in Fig. 7. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_4\}$, $J_2 = \{j_1, j_3, j_4\}$. Consider (α, J_1) and (β, J_2) LONSSs over $W = \{w_1, w_2, w_3, w_4\}$ as shown in Tables 3.15 and 3.16, respectively.

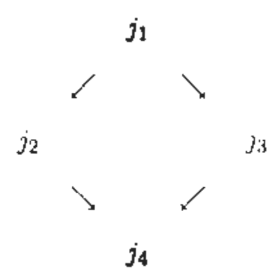

Fig. 7. Lattice of parameters

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4
j_1	$(.1, .3, .9)$	$(.2, .5, .7)$	$(.3, .4, .75)$	$(.5, .6, .4)$
j_2	$(.2, .4, .7)$	$(.3, .6, .6)$	$(.4, .7, .6)$	$(.56, .7, .3)$
j_4	$(.3, .7, .5)$	$(.6, .7, .4)$	$(.55, .8, .4)$	$(.6, .77, .2)$

Table 3.15

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4
j_1	$(.2, .1, .8)$	$(.4, .6, .3)$	$(.1, .4, .6)$	$(.5, .3, .9)$
j_3	$(.4, .5, .4)$	$(.7, .8, .2)$	$(.4, .6, .4)$	$(.7, .7, .7)$
j_4	$(.6, .9, .2)$	$(.9, 1, .1)$	$(.8, .9, .2)$	$(.9, .8, .3)$

Table 3.16

Now $(\alpha, J_1) \cap_R (\beta, J_2) = (K, G)$ is a restricted intersection of two LONSSs over W where

$$G = J_1 \cap J_2.$$

Table for restricted intersection of two LONSSs.

(K, G)	w_1	w_2	w_3	w_4
j_1	$(.1, .1, .9)$	$(.2, .5, .7)$	$(.1, .4, .75)$	$(.5, .3, .9)$
j_4	$(.3, .7, .5)$	$(.6, .7, .4)$	$(.55, .8, .4)$	$(.6, .77, .3)$

Table 3.17

Then clearly from Table 3.17, we have $K(j_1) \preceq K(j_4)$, so (K, G) is a restricted intersection of two LONSSs over W is a LONSS over W.

3.2.27 Proposition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \cap_R (\beta, J_2)$ is largest LONSS over W which is contained in (α, J_1) and (β, J_2).

Proof. Let $(\alpha, J_1) \cap_R (\beta, J_2) = (K, G)$ where $G = J_1 \cap J_2 \neq \emptyset$ then $T_{K(c)}(w) = Min\{T_{\alpha(m)}(w), T_{\beta(n)}(w)\}$, $I_{K(c)}(w) = Min\{I_{\alpha(m)}(w), I_{\beta(n)}(w)\}$ and

$F_{K(c)}(w) = Max\{F_{\alpha(m)}(w), F_{\beta(n)}(w)\}\ \forall\ c \in G, w \in W.$

then $T_{K(c)}(w) \leq T_{\alpha(m)}(w)$, $T_{K(c)}(w) \leq T_{\beta(n)}(w)$ and $I_{K(c)}(w) \leq I_{\alpha(m)}(w)$, $I_{K(c)}(w) \leq I_{\beta(n)}(w)$ and $F_{\alpha(m)}(w) \leq F_{K(c)}(w)$, $F_{\beta(n)}(w) \leq F_{K(c)}(w)$.

implies that $(K, G) \subseteq (\alpha, J_1)$ and $(K, G) \subseteq (\beta, J_2)$

implies that $(K, G) \subseteq (\alpha, J_1) \cap_R (\beta, J_2)$.

let (L, D) be another LONSS over W contained in both (α, J_1) and (β, J_2).

then $(L, D) \subseteq (\alpha, J_1) \cap_R (\beta, J_2)$ where $D = J_1 \cap J_2 \neq \emptyset$

this implies $(L, D) \subseteq (K, G)$, $D = J_1 \cap J_2 = G$, so $(L, D) \subseteq (K, G)$, $\forall\ c \in G$.

Therefore (K, G) is largest LONSS over W. ■

3.2.28 Definition

Let (α, J_1) be a LONSS over W is termed to be relative whole LONSS over W w.r.t the parameters J_1 if $T_{\alpha(j)}(w) = 1, I_{\alpha(j)}(w) = 1$ and $F_{\alpha(j)}(w) = 0$, $\forall\ j \in J_1, w \in W$ and it is denoted by $\cup_{J_1}$.

3.2.29 Definition

The complement of a LONSS over W (α, J_1) denoted by $(\alpha, J_1)^C$ and is defined as $(\alpha, J_1)^C = (\alpha^C, J_1)$, where $\alpha^C : J_1 \longrightarrow NS(W)$ is a mapping given by $T_{\alpha^C(j)}(w) = 1 - T_\alpha(w)$.

$I_{\alpha^C(j)}(w) = 1 - I_\alpha(w)$ and $F_{\alpha^C(j)}(w) = 1 - F_\alpha(w)$, $\forall\ j \in J$ and $w \in W$.

3.2.30 Proposition

Complement of LONSS over W is an anti-LONSS over W.

Proof. Let (α, J_1) be a LONSS over W. Then $m_{1 \leq_{J_1}} m_2$ implies $\alpha(m_1) \preceq \alpha(m_2)$

implies $T_{\alpha(m_1)}(w) \leq T_{\alpha(m_2)}(w)$

$I_{\alpha(m_1)}(w) \leq I_{\alpha(m_2)}(w)$

$F_{\alpha(m_2)}(w) \leq F_{\alpha(m_1)}(w)$

implies that $-T_{\alpha(m_1)}(w) \geq -T_{\alpha(m_2)}(w)$

$-I_{\alpha(m_1)}(w) \geq -I_{\alpha(m_2)}(w)$

$-F_{\alpha(m_2)}(w) \geq -F_{\alpha(m_1)}(w)$

implies that $1 - T_{\alpha(m_1)}(w) \geq 1 - T_{\alpha(m_2)}(w)$

$1 - I_{\alpha(m_1)}(w) \geq 1 - I_{\alpha(m_2)}(w)$

$1 - F_{\alpha(m_2)}(w) \geq 1 - F_{\alpha(m_1)}(w)$

implies that $T_{\alpha^C(m_1)}(w) \geq T_{\alpha^C(m_2)}(w)$

$I_{\alpha^C(m_1)}(w) \geq I_{\alpha^C(m_2)}(w)$

$F_{\alpha^C(m_2)}(w) \geq F_{\alpha^C(m_1)}(w)$

implies that $(\alpha(m_2))^C \preceq (\alpha(m_1))^C$.

This show that complement of LONSS over W is an anti-LONSS over W. ■

3.2.31 Proposition

Let (α, J_1) be LONSS over W then $\left((\alpha, J_1)^C\right)^C = (\alpha, J_1)$.

Proof. Let (α, J_1) be a LONSS over W. The complement of (α, J_1) is $T_{\alpha^C(j)}(w) = 1 - T_{\alpha(j)}(w), I_{\alpha^C(j)}(w) = 1 - I_{\alpha(j)}(w)$ and $F_{\alpha^C(j)}(w) = 1 - F_{\alpha(j)}(w)$. Now the complement of $(\alpha, J_1)^C$ is $T_{(\alpha^C)^C(j)}(w) = 1 - T_{\alpha^C(j)}(w)$, $I_{(\alpha^C)^C(j)}(w) = 1 - I_{\alpha^C(j)}(w)$ and $F_{(\alpha^C)^C(j)}(w) = 1 - F_{\alpha^C(j)}(w)$ implies that $T_{(\alpha)^C(j)}(w) = 1 - \left(1 - T_{\alpha(j)}(w)\right)$, $I_{(\alpha)^C(j)}(w) = 1 - \left(1 - I_{\alpha(j)}(w)\right)$ and $F_{(\alpha)^C(j)}(w) = 1 - \left(1 - F_{\alpha(j)}(w)\right)$ which is implies that $T_{(\alpha^C)^C(j)}(w) = T_{\alpha(j)}(w)$, $I_{(\alpha^C)^C(j)}(w) = I_{\alpha(j)}(w)$ and $F_{(\alpha^C)^C(j)}(w) = F_{\alpha(j)}(w)$. then $\left((\alpha, J_1)^C\right)^C = (\alpha, J_1)$. ■

3.2.32 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \vee (\beta, J_2) = (K, G)$ is said to be basic union of two LONSSs over W, where $G = J_1 \times J_2$ define $K(m, n) = \alpha(m) \cup \beta(n)$ and

$$T_{K_{(m,n)}}(w) = Max\left\{T_{\alpha(m)}(w), T_{\beta(n)}(w)\right\}$$

$$I_{K_{(m,n)}}(w) = Max\left\{I_{\alpha(m)}(w), I_{\beta(n)}(w)\right\}$$

and $F_{K_{(m,n)}}(w) = Min\left\{F_{\alpha(m)}(w), F_{\beta(n)}(w)\right\} \forall (m, n) \in G, w \in W.$

3.2.33 Proposition

Basic union of two LONSSs over W is a LONSS over W.

Proof. Let (α, J_1) and (β, J_2) be two LONSSs over W. Let $(\alpha, J_1) \vee (\beta, J_2) = (K, G)$ where $G = J_1 \times J_2$. since $J_1, J_2 \subseteq J$, so both J_1 and J_2 inherit a partial order from J. Therefore $m_{1 \leq_{J_1}} m_2$ then $\alpha(m_1) \preceq \alpha(m_2)$ $\forall$ $m_1, m_2 \in J_1$ also for any $n_{1 \leq_{J_2}} n_2$ then $\beta(n_1) \preceq \beta(n_2)$ $\forall$ $n_1, n_2 \in J_2$.

Therefore for any $(m_1, n_1), (m_2, n_2) \in G$ and $\leq$ is the partial order on G which is induced by partial orders on J_1 and J_2. The order on $J_1 \times J_2$ is a $(m_1, n_1) \leq (m_2, n_2)$ iff $m_{1 \leq_{J_1}} m_2$ and $n_{1 \leq_{J_2}} n_2$. If $(m_1, n_1) \leq (m_2, n_2)$ then $\alpha(m_1) \preceq \alpha(m_2)$ and $\beta(n_1) \preceq \beta(n_2)$.

Implies that

$T_{\alpha(m_1)}(w) \leq T_{\alpha(m_2)}(w)$, $T_{\beta(n_1)}(w) \leq T_{\beta(n_2)}(w)$

$I_{\alpha(m_1)}(w) \leq I_{\alpha(m_2)}(w)$, $I_{\beta(n_1)}(w) \leq I_{\beta(n_2)}(w)$

$F_{\alpha(m_2)}(w) \leq F_{\alpha(m_1)}(w)$, $F_{\beta(n_2)}(w) \leq F_{\beta(n_1)}(w)$

then

$T_{\alpha(m_1)}(w) \vee T_{\beta(n_1)}(w) \leq T_{\alpha(m_2)}(w) \vee T_{\beta(n_2)}(w)$

$I_{\alpha(m_1)}(w) \vee I_{\beta(n_1)}(w) \leq I_{\alpha(m_2)}(w) \vee I_{\beta(n_2)}(w)$

$F_{\alpha(m_2)}(w) \vee F_{\beta(n_2)}(w) \leq F_{\alpha(m_1)}(w) \vee F_{\beta(n_1)}(w)$

implies

$T_{(\alpha(m_1) \vee \beta(n_1))}(w) \leq T_{(\alpha(m_2) \vee \beta(n_2))}(w)$

$I_{(\alpha(m_1) \vee \beta(n_1))}(w) \leq I_{(\alpha(m_2) \vee \beta(n_2))}(w)$

$F_{(\alpha(m_2) \vee \beta(n_2))}(w) \leq F_{(\alpha(m_1) \vee \beta(n_1))}(w)$

implies

$T_{K(m_1, n_1)}(w) \leq T_{K(m_2, n_2)}(w)$

$I_{K(m_1, n_1)}(w) \leq I_{K(m_2, n_2)}(w)$

$F_{K(m_2, n_2)}(w) \leq F_{K(m_1, n_1)}(w)$

which is implies that $K(m_1, n_1) \preceq K(m_2, n_2)$ $\forall$ $(m_1, n_1) \leq (m_2, n_2)$.

Thus $(\alpha, J_1) \vee (\beta, J_2)$ is a LONSS over W. ■

3.2.34 Example

let $J = \{j_1, j_2, j_3, j_4\}$ be a set of parameters with lattice order. This order can be described as $j_1 \leq j_2 \leq j_3 \leq j_4$. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_3\}$, $J_2 = \{j_2, j_3, j_4\}$. Consider (α, J_1) and (β, J_2) LONSSs over $W = \{w_1, w_2, w_3, w_4\}$ as shown in Tables 3.18 and 3.19, respectively.

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4
j_1	$(.1, .3, .9)$	$(.2, .5, .7)$	$(.3, .4, .75)$	$(.5, .6, .4)$
j_2	$(.2, .4, .7)$	$(.3, .6, .6)$	$(.4, .7, .6)$	$(.56, .7, .3)$
j_3	$(.3, .7, .5)$	$(.6, .7, .4)$	$(.55, .8, .4)$	$(.6, .77, .2)$

Table 3.18

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4
j_2	$(.2, .1, .8)$	$(.4, .6, .3)$	$(.1, .4, .6)$	$(.5, .3, .9)$
j_3	$(.4, .5, .4)$	$(.7, .8, .2)$	$(.4, .6, .4)$	$(.7, .7, .7)$
j_4	$(.6, .9, .2)$	$(.9, 1, .1)$	$(.8, .9, .2)$	$(.9, .8, .3)$

Table 3.19

Let $(\alpha, J_1) \vee (\beta, J_2) = (K, G)$ is a basic union of two LONSS over W, where $G = J_1 \times J_2$. Since $J_1, J_2 \subseteq J$, so both J_1 and J_2 inherit a partial order from J. Therefore for any $(m_1, n_1), (m_2, n_2) \in G$ and $\leq$ is the partial order on G, which is induced by partial orders on J_1 and J_2. Define order on $J_1 \times J_2$ is a $(m_1, n_1) \leq (m_2, n_2)$ iff $m_1 \leq_{J_1} m_2$ and $n_1 \leq_{J_2} n_2$. So we have

$G = \{(j_1, j_2), (j_1, j_3), (j_1, j_4), (j_2, j_2), (j_2, j_3), (j_2, j_4), (j_3, j_2), (j_3, j_3), (j_3, j_4)\}$ with lattice order.

Table for basic union of two LONSSs.

(K,G)	w_1	w_2	w_3	w_4
(j_1,j_2)	(.2, .3, .8)	(.4, .6, .3)	(.3, .4, .6)	(.5, .3, .4)
(j_1,j_3)	(.4, .5, .4)	(.7, .8, .2)	(.4, .6, .4)	(.7, .7, .4)
(j_1,j_4)	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .3)
(j_2,j_2)	(.2, .4, .7)	(.4, .6, .3)	(.4, .7, .6)	(.56, .7, .3)
(j_2,j_3)	(.4, .5, .4)	(.7, .8, .2)	(.4, .7, .4)	(.7, .7, .3)
(j_2,j_4)	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .3)
(j_3,j_2)	(.3, .7, .5)	(.6, .7, .3)	(.55, .8, .4)	(.6, .77, .2)
(j_3,j_3)	(.4, .7, .4)	(.7, .8, .2)	(.55, .8, .4)	(.7, .77, .2)
(j_3,j_4)	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .2)

Table 3.20

Then clearly from Table 3.20, (K,G) is a LONSS over W.

3.2.35 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then $(\alpha, J_1) \wedge (\beta, J_2) = (K, G)$ is said to be basic intersection of two LONSSs over W, where $G = J_1 \times J_2$ define $K(m,n) = \alpha(m) \cap \beta(n)$ and

$$T_{K_{(m,n)}}(w) = Min\left\{T_{\alpha(m)}(w), T_{\beta(n)}(w)\right\}$$
$$I_{K_{(m,n)}}(w) = Min\left\{I_{\alpha(m)}(w), I_{\beta(n)}(w)\right\}$$
$$\text{and } F_{K_{(m,n)}}(w) = Max\left\{F_{\alpha(m)}(w), F_{\beta(n)}(w)\right\} \; \forall \; (m,n) \in G, w \in W.$$

3.2.36 Proposition

Basic intersection of two LONSSs over W is a LONSS over W.

Proof. Let (α, J_1) and (β, J_2) be two LONSSs over W. Consider $(\alpha, J_1) \wedge (\beta, J_2) = (K, G)$ where $G = J_1 \times J_2$. since $J_1, J_2 \subseteq J$, so both J_1 and J_2 inherit a partial order from J. Therefore $m_1 \leq_{J_1} m_2$ then $\alpha(m_1) \preceq \alpha(m_2) \; \forall \; m_1, m_2 \in J_1$ also for any $n_1 \leq_{J_2} n_2$ then $\beta(n_1) \preceq \beta(n_2) \; \forall \; n_1, n_2 \in J_2$.

Therefore for any $(m_1, n_1), (m_2, n_2) \in G$ and $\leq$ is the partial order on G which is induced by partial orders on J_1 and J_2. The order on $J_1 \times J_2$ is a $(m_1, n_1) \leq (m_2, n_2)$ iff $m_{1 \leq_{J_1}} m_2$ and $n_{1 \leq_{J_2}} n_2$. If $(m_1, n_1) \leq (m_2, n_2)$ then $\alpha(m_1) \preceq \alpha(m_2)$ and $\beta(n_1) \preceq \beta(n_2)$.

Implies that

$T_{\alpha(m_1)}(w) \leq T_{\alpha(m_2)}(w)$, $T_{\beta(n_1)}(w) \leq T_{\beta(n_2)}(w)$

$I_{\alpha(m_1)}(w) \leq I_{\alpha(m_2)}(w)$, $I_{\beta(n_1)}(w) \leq I_{\beta(n_2)}(w)$

$F_{\alpha(m_2)}(w) \leq F_{\alpha(m_1)}(w)$, $F_{\beta(n_2)}(w) \leq F_{\beta(n_1)}(w)$

then

$T_{\alpha(m_1)}(w) \wedge T_{\beta(n_1)}(w) \leq T_{\alpha(m_2)}(w) \wedge T_{\beta(n_2)}(w)$

$I_{\alpha(m_1)}(w) \wedge I_{\beta(n_1)}(w) \leq I_{\alpha(m_2)}(w) \wedge I_{\beta(n_2)}(w)$

$F_{\alpha(m_2)}(w) \wedge F_{\beta(n_2)}(w) \leq F_{\alpha(m_1)}(w) \wedge F_{\beta(n_1)}(w)$

implies

$T_{(\alpha(m_1) \wedge \beta(n_1))}(w) \leq T_{(\alpha(m_2) \wedge \beta(n_2))}(w)$

$I_{(\alpha(m_1) \wedge \beta(n_1))}(w) \leq I_{(\alpha(m_2) \wedge \beta(n_2))}(w)$

$F_{(\alpha(m_2) \wedge \beta(n_2))}(w) \leq F_{(\alpha(m_1) \wedge \beta(n_1))}(w)$

implies

$T_{K(m_1,n_1)}(w) \leq T_{K(m_2,n_2)}(w)$

$I_{K(m_1,n_1)}(w) \leq I_{K(m_2,n_2)}(w)$

$F_{K(m_2,n_2)}(w) \leq F_{K(m_1,n_1)}(w)$

which implies that $K(m_1, n_1) \preceq K(m_2, n_2)\ \forall\ (m_1, n_1) \leq (m_2, n_2)$.

Thus $(\alpha, J_1) \wedge (\beta, J_2)$ is a LONSS over W. ∎

3.2.37 Example

Let $J = \{j_1, j_2, j_3, j_4\}$ be a set of parameters with lattice order. Tbis order can be described as $j_1 \leq j_2 \leq j_3 \leq j_4$. Let $J_1, J_2 \subseteq J$, $J_1 = \{j_1, j_2, j_3\}$, $J_2 = \{j_2, j_3, j_4\}$. Consider (α, J_1) and

(β, J_2) LONSSs over $W = \{w_1, w_2, w_3, w_4\}$, as shown in Tables 3.18 and 3.19, respectively.

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4
j_1	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .75)	(.5, .6, .4)
j_2	(.2, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.56, .7, .3)
j_3	(.3, .7, .5)	(.6, .7, .4)	(.55, .8, .4)	(.6, .77, .2)

Table 3.21

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4
j_2	(.2, .1, .8)	(.4, .6, .3)	(.1, .4, .6)	(.5, .3, .9)
j_3	(.4, .5, .4)	(.7, .8, .2)	(.4, .6, .4)	(.7, .7, .7)
j_4	(.6, .9, .2)	(.9, 1, .1)	(.8, .9, .2)	(.9, .8, .3)

Table 3.22

Let $(\alpha, J_1) \wedge (\beta, J_2) = (K, G)$ is a basic intersection of two LONSS over W, where $G = J_1 \times J_2$. Since $J_1, J_2 \subseteq J$, so both J_1 and J_2 inherit a partial order from J. Therefore for any $(m_1, n_1), (m_2, n_2) \in G$ and $\leq$ is the partial order on G, which is induced by partial orders on J_1 and J_2. Define order on $J_1 \times J_2$ is a $(m_1, n_1) \leq (m_2, n_2)$ iff $m_1 \leq_{J_1} m_2$ and $n_1 \leq_{J_2} n_2$. So we have

$G = \{(j_1, j_2), (j_1, j_3), (j_1, j_4), (j_2, j_2), (j_2, j_3), (j_2, j_4), (j_3, j_2), (j_3, j_3), (j_3, j_4)\}$ with lattice order.

Table for basic intersection of two LONSSs.

(K,G)	w_1	w_2	w_3	w_4
(j_1,j_2)	(.1,.1,.9)	(.2,.5,.7)	(.1,.4,.75)	(.5,.3,.9)
(j_1,j_3)	(.1,.3,.9)	(.2,.5,.7)	(.3,.4,.75)	(.5,.6,.7)
(j_1,j_4)	(.1,.3,.9)	(.2,.5,.7)	(.3,.4,.75)	(.5,.6,.4)
(j_2,j_2)	(.2,.1,.8)	(.3,.6,.6)	(.1,.4,.6)	(.5,.3,.9)
(j_2,j_3)	(.2,.4,.7)	(.3,.6,.8)	(.4,.6,.6)	(.56,.7,.7)
(j_2,j_4)	(.2,.4,.7)	(.3,6,.6)	(.4,.7,.6)	(.56,.7,.3)
(j_3,j_2)	(.2,.1,.8)	(.4,.6,.4)	(.1,.4,.6)	(.5,.3,.9)
(j_3,j_3)	(.3,.5,.4)	(.6,.7,.4)	(.4,.6,.4)	(.6,.7,.7)
(j_3,j_4)	(.3,.7,.5)	(.6,.7,.4)	(.55,.8,.4)	(.6,.77,.3)

Table 3.23

Then clearly from Table 3.23, (K,G) is a LONSS over W.

3.2.38 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then define

$(\alpha, J_1) \cap_\wedge (\beta, J_2) = (K, J_1 \otimes J_2)$ as $K(m \wedge n) = \alpha(m) \cap \beta(n)\ \forall\ m \wedge n \in J_1 \otimes J_2$.

3.2.39 Definition

Let (α, J_1) and (β, J_2) be two LONSSs over W. Then define

$(\alpha, J_1) \cup_\vee (\beta, J_2) = (K, J_1 \oplus J_2)$ as $K(m \vee n) = \alpha(m) \cup \beta(n)\ \forall\ m \wedge n \in J_1 \oplus J_2$.

3.2.40 Proposition

If (α, J_1) and (β, J_2) are LONSSs over W. Then $(\alpha, J_1) \cap_\wedge (\beta, J_2)$ is LONSS over W .

Proof. If (α, J_1) and (β, J_2) are LONSSs over W (i.e $\forall\ m_1, m_2 \in J_1$) $\alpha(m_1) \preceq \alpha(m_2)$ whenever $m_1 \leq m_2$ and $\forall\ n_1, n_2 \in J_2$ $\beta(n_1) \preceq \beta(n_2)$ whenever $n_1 \leq n_2$.

we have to prove that $(\alpha, J_1) \cap_\wedge (\beta, J_2)$ is LONSS over W .

As $(\alpha, J_1) \cap_\wedge (\beta, J_2) = (K, J_1 \otimes J_2)$

defined as $K(m_1 \wedge n_1) = \alpha(m_1) \cap \beta(n_1)\ \forall\ (m_1 \wedge n_2) \in J_1 \otimes J_2$

and $K(m_2 \wedge n_2) = \alpha(m_2) \cap \beta(n_2)$

$\alpha(m_1) \preceq \alpha(m_2)$ and $\beta(n_1) \preceq \beta(n_2)$

$T_{\alpha(m_1)}(w) \leq T_{\alpha(m_2)}(w)$, $T_{\beta(n_1)}(w) \leq T_{\beta(n_2)}(w)\ \forall\ w \in W$

$I_{\alpha(m_1)}(w) \leq I_{\alpha(m_2)}(w)$, $I_{\beta(n_1)}(w) \leq I_{\beta(n_2)}(w)\ \forall\ w \in W$

$F_{\alpha(m_2)}(w) \leq F_{\alpha(m_1)}(w)$, $F_{\beta(n_2)}(w) \leq F_{\beta(n_1)}(w)\ \forall\ w \in W$

implies

$T_{\alpha(m_1)}(w) \cap T_{\beta(n_1)}(w) \leq T_{\alpha(m_2)}(w) \cap T_{\beta(n_2)}(w)$

$I_{\alpha(m_1)}(w) \cap I_{\beta(n_1)}(w) \leq I_{\alpha(m_2)}(w) \cap I_{\beta(n_2)}(w)$

$F_{\alpha(m_2)}(w) \cap F_{\beta(n_2)}(w) \leq F_{\alpha(m_1)}(w) \cap F_{\beta(n_1)}(w)$

implies

$(T_{\alpha(m_1)} \cap T_{\beta(n_1)})(w) \leq (T_{\alpha(m_2)} \cap T_{\beta(n_2)})(w)$

$(I_{\alpha(m_1)} \cap I_{\beta(n_1)})(w) \leq (I_{\alpha(m_2)} \cap I_{\beta(n_2)})(w)$

$(F_{\alpha(m_2)} \cap F_{\beta(n_2)})(w) \leq (F_{\alpha(m_1)} \cap F_{\beta(n_1)})(w)$

implies

$(T_{\alpha(m_1)\cap\beta(n_1)})(w) \leq (T_{\alpha(m_2)\cap\beta(n_2)})(w)$

$(I_{\alpha(m_1)\cap\beta(n_1)})(w) \leq (I_{\alpha(m_2)\cap\beta(n_2)})(w)$

$(F_{\alpha(m_2)\cap\beta(n_2)})(w) \leq (F_{\alpha(m_1)\cap\beta(n_1)})(w)$

implies

$T_{K(m_1 \cap n_1)}(w) \leq T_{K(m_2 \cap n_2)}(w)$

$I_{K(m_1 \cap n_1)}(w) \leq I_{K(m_2 \cap n_2)}(w)$

$F_{K(m_2 \cap n_2)}(w) \leq F_{K(m_1 \cap n_1)}(w)$

then

$K(m_1 \wedge n_1) \preceq K(m_2 \wedge n_2)\ \forall\ m_1 \leq m_2, n_1 \leq n_2, m_1 \wedge n_1 \leq m_2 \wedge n_2$ for $m_1 \wedge n_1, m_2 \wedge n_2 \in J_1 \otimes J_2$

thus $(\alpha, J_1) \cap_\wedge (\beta, J_2)$ is LONSS over W . ■

3.2.41 Proposition

If (α, J_1) and (β, J_2) are LONSSs over W. Then $(\alpha, J_1) \cup_\vee (\beta, J_2)$ is LONSS over W .

Proof. If (α, J_1) and (β, J_2) are LONSSs over W (i.e $\forall\ m_1, m_2 \in J_1)$ $\alpha(m_1) \preceq \alpha(m_2)$ whenever $m_1 \leq m_2$ and $\forall\ n_1, n_2 \in J_2 \quad \beta(n_1) \preceq \beta(n_2)$ whenever $n_1 \leq n_2$.

we have to prove that $(\alpha, J_1) \cup_\vee (\beta, J_2)$ is LONSS over W .

As $(\alpha, J_1) \cup_\vee (\beta, J_2) = (K, J_1 \oplus J_2)$

defined as $K(m_1 \vee n_1) = \alpha(m_1) \cup \beta(n_1)\ \forall\ (m_1 \vee n_2) \in J_1 \otimes J_2$

and $K(m_2 \vee n_2) = \alpha(m_2) \cup \beta(n_2)$

$\alpha(m_1) \preceq \alpha(m_2)$ and $\beta(n_1) \preceq \beta(n_2)$

$T_{\alpha(m_1)}(w) \leq T_{\alpha(m_2)}(w)$, $T_{\beta(n_1)}(w) \leq T_{\beta(n_2)}(w)\ \forall\ w \in W$

$I_{\alpha(m_1)}(w) \leq I_{\alpha(m_2)}(w)$, $I_{\beta(n_1)}(w) \leq I_{\beta(n_2)}(w)\ \forall\ w \in W$

$F_{\alpha(m_2)}(w) \leq F_{\alpha(m_1)}(w)$, $F_{\beta(n_2)}(w) \leq F_{\beta(n_1)}(w)\ \forall\ w \in W$

implies

$T_{\alpha(m_1)}(w) \cup T_{\beta(n_1)}(w) \leq T_{\alpha(m_2)}(w) \cup T_{\beta(n_2)}(w)$

$I_{\alpha(m_1)}(w) \cup I_{\beta(n_1)}(w) \leq I_{\alpha(m_2)}(w) \cup I_{\beta(n_2)}(w)$

$F_{\alpha(m_2)}(w) \cup F_{\beta(n_2)}(w) \leq F_{\alpha(m_1)}(w) \cup F_{\beta(n_1)}(w)$

implies

$(T_{\alpha(m_1)} \cup T_{\beta(n_1)})(w) \leq (T_{\alpha(m_2)} \cup T_{\beta(n_2)})(w)$

$(I_{\alpha(m_1)} \cup I_{\beta(n_1)})(w) \leq (I_{\alpha(m_2)} \cup I_{\beta(n_2)})(w)$

$(F_{\alpha(m_2)} \cup \alpha)(w) \leq (F_{\alpha(m_1)} \cup F_{\beta(n_1)})(w)$

implies

$(T_{\alpha(m_1) \cup \beta(n_1)})(w) \leq (T_{\alpha(m_2) \cup \beta(n_2)})(w)$

$(I_{\alpha(m_1) \cup \beta(n_1)})(w) \leq (I_{\alpha(m_2) \cup \beta(n_2)})(w)$

$(F_{\alpha(m_2) \cup \beta(n_2)})(w) \leq (F_{\alpha(m_1) \cup \beta(n_1)})(w)$

implies

$T_{K(m_1 \cup n_1)}(w) \leq T_{K(m_2 \cup n_2)}(w)$

$I_{K(m_1 \cup n_1)}(w) \leq I_{K(m_2 \cup n_2)}(w)$

$F_{K(m_2 \cup n_2)}(w) \leq F_{K(m_1 \cup n_1)}(w)$

then

$K(m_1 \vee n_1) \preceq K(m_2 \vee n_2)\ \forall\ m_1 \leq m_2, n_1 \leq n_2, m_1 \vee n_1 \leq m_2 \vee n_2$ for $m_1 \vee n_1, m_2 \vee n_2 \in J_1 \oplus J_2$

thus $(\alpha, J_1) \cup_V (\beta, J_2)$ is LONSS over W . ■

3.2.42 Proposition

Let (α, J_1) be a LONSS over W. Then

1. $(\alpha, J_1) \cap_R (\alpha, J_1) = (\alpha, J_1)$
2. $(\alpha, J_1) \cup_R (\alpha, J_1) = (\alpha, J_1)$
3. $(\alpha, J_1) \cap_R \emptyset_{J_1} = \emptyset_{J_1}$
4. $(\alpha, J_1) \cup_R \emptyset_{J_1} = (\alpha, J_1)$

Proof. 1. Let $(\alpha, J_1) \cap_R (\alpha, J_1) = (\beta, G)$ where $G = J_1 \cap J_1 = J_1$ then by the definition of restricted intersection.

$T_{\beta(j)}(w) = Min\left\{T_{\alpha(j)}(w), T_{\alpha(j)}(w)\right\}$

$I_{\beta(j)}(w) = Min\left\{I_{\alpha(j)}(w), I_{\alpha(j)}(w)\right\}$

and $F_{\beta(j)}(w) = Max\left\{F_{\alpha(j)}(w), F_{\alpha(j)}(w)\right\} \ \forall\ j \in G = J_1, w \in W$

implies that $T_{\beta(j)}(w) = T_{\alpha(j)}(w)$

$I_{\beta(j)}(w) = I_{\alpha(j)}(w)$

$F_{\beta(j)}(w) = F_{\alpha(j)}(w)$

implies that $(\beta, G) = (\alpha, J_1)$ implies that $(\alpha, J_1) \cap_R (\alpha, J_1) = (\alpha, J_1)$.

2. Let $(\alpha, J_1) \cup_R (\alpha, J_1) = (\beta, G)$ where $G = J_1 \cap J_1 = J_1$ then by the definition of restricted union

$T_{\beta(j)}(w) = Max\left\{T_{\alpha(j)}(w), T_{\alpha(j)}(w)\right\}$

$I_{\beta(j)}(w) = Max\left\{I_{\alpha(j)}(w), I_{\alpha(j)}(w)\right\}$

and $F_{\beta(j)}(w) = Min\left\{F_{\alpha(j)}(w), F_{\alpha(j)}(w)\right\} \ \forall\ j \in G = J_1, w \in W$.

implies that $T_{\beta(j)}(w) = T_{\alpha(j)}(w)$

$I_{\beta(j)}(w) = I_{\alpha(j)}(w)$

$F_{\beta(j)}(w) = F_{\alpha(j)}(w)$

implies that $(\beta, G) = (\alpha, J_1)$ implies that $(\alpha, J_1) \cup_R (\alpha, J_1) = (\alpha, J_1)$.

3. Let $(\alpha, J_1) \cap_R (\emptyset, J_1) = (\beta, G)$ where $G = J_1 \cap J_1 = J_1$ then by the definition of restricted intersection

then $T_{\beta(j)}(w) = Min\left\{T_{\alpha(j)}(w), T_{\emptyset_{(j)}}(w)\right\}$

$I_{\beta(j)}(w) = Min\left\{I_{\alpha(j)}(w), I_{\emptyset_{(j)}}(w)\right\}$

and $F_{\beta(j)}(w) = Max\left\{F_{\alpha(j)}(w), F_{\emptyset_{(j)}}(w)\right\} \; \forall\, j \in G = J_1, w \in W.$

implies that $T_{\beta(j)}(w) = Min\{T_{\alpha(j)}(w), 0\}$

$I_{\beta(j)}(w) = Min\{I_{\alpha(j)}(w), 0\}$

and $F_{\beta(j)}(w) = Max\{F_{\alpha(j)}(w), 1\}$

implies that $T_{\beta(j)}(w) = 0$

$I_{\beta(j)}(w) = 0$

$F_{\beta(j)}(w) = 1$

implies that $T_{\beta(j)}(w) = T_{\emptyset_{(j)}}(w)$

$I_{\beta(j)}(w) = I_{\emptyset_{(j)}}(w)$

$F_{\beta(j)}(w) = F_{\emptyset_{(j)}}(w)$

implies that $(\beta, G) = (\emptyset, J_1)$ implies that $(\alpha, J_1) \cap_R (\emptyset, J_1) = (\emptyset, J_1)$

implies that $(\alpha, J_1) \cap_R \emptyset_{J_1} = \emptyset_{J_1}$.

4. Let $(\alpha, J_1) \cup_R (\emptyset, J_1) = (\beta, G)$ where $G = J_1 \cap J_1 = J_1$ then by the definition of restricted union then

$T_{\beta(j)}(w) = Max\left\{T_{\alpha(j)}(w), T_{\emptyset_{(j)}}(w)\right\}$

$I_{\beta(j)}(w) = Max\left\{I_{\alpha(j)}(w), I_{\emptyset_{(j)}}(w)\right\}$

and $F_{\beta(j)}(w) = Min\left\{F_{\alpha(j)}(w), F_{\emptyset_{(j)}}(w)\right\} \; \forall\, j \in G = J_1, w \in W.$

implies that $T_{\beta(j)}(w) = Max\{T_{\alpha(j)}(w), 0\}$

$I_{\beta(j)}(w) = Max\{I_{\alpha(j)}(w), 0\}$

and $F_{\beta(j)}(w) = Min\{F_{\alpha(j)}(w), 1\}$

implies that $T_{\beta(j)}(w) = T_{\alpha(j)}(w)$

$I_{\beta(j)}(w) = I_{\alpha(j)}(w)$

$F_{\beta(j)}(w) = F_{\alpha(j)}(w)$

implies that $(\beta, G) = (\alpha, J_1)$ implies that $(\alpha, J_1) \cup_R (\emptyset, J_1) = (\alpha, J_1)$

implies that $(\alpha, J_1) \cup_R \emptyset_{J_1} = (\alpha, J_1)$. ■

3.3 An application of LONSS in a decision making problem

Lattice ordered neutrosophic soft sets are used for many daily life decision making problems. A specific order on set of parameters is decided and objectives are placed according to the importance of order. In this example, decision will be made for a problem on the basis of different opinions.

3.3.1 Example

The problem is that to select the suitable car from a set of six cars having different colors, sizes, and designs. Two experts were sent to point out the suitable and best option of car. Both have different opinions for car.

Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be the set of cars and

$$J = \left\{ \begin{array}{c} j_1\,(\text{stylish})\,, j_2\,(\text{capacity})\,, j_3\,(\text{average milage})\,, j_4\,(\text{comfortable})\,, j_5\,(\text{reliable})\,, \\ j_6\,(\text{well assembled})\,, j_7\,(\text{invincible}) \end{array} \right\}$$

be the set of parameters with lattice order as shown in Fig. 8.

As there are two experts to choose a car so one member will consider $J_1 = \{j_2, j_3, j_4, j_6\}$ set of parameters and other follow $J_2 = \{j_3, j_6, j_7\}$ set of parameters, where $J_1, J_2 \subseteq J$. And obviously order on J_1 and J_2 is same as on J. Let (α, J_1) and (β, J_2) be LONSSs as shown in Tables 3.24 and 3.25 determined on J_1 and J_2 respectively to choose suitable car.

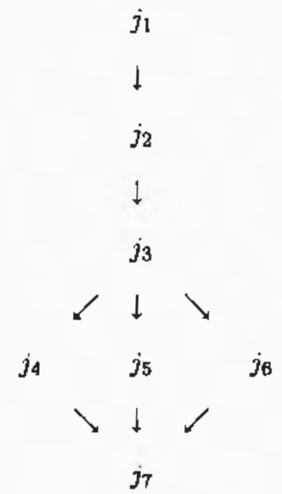

Fig. 8. Lattice of parameters

The combination of both opinions will help to select suitable choice. In order to do so, basic union of LONSSs is taken which is shown in Table 3.26. By definition of basis union (3.2.32), the basic union of two LONSSs (α, J_1) and (β, J_2) are denoted by $(\alpha, J_1) \vee (\beta, J_2) = (K, G)$ where $G = J_1 \times J_2$ and there is dictionary order on G, define in (1.1.7) and $G = \{(j_2, j_3), (j_2, j_6), (j_2, j_7), (j_3, j_3), (j_3, j_6), (j_3, j_7), (j_4, j_3), (j_4, j_6), (j_4, j_7), (j_6, j_3), (j_6, j_6), (j_6, j_7)\}$. We know from proposition (3.2.33) that (K, G) is LONSS.

Table for LONSS (α, J_1).

(α, J_1)	w_1	w_2	w_3	w_4	w_5	w_6
j_2	(.1, .3, .9)	(.2, .5, .7)	(.3, .4, .8)	(.5, .6, .4)	(.3, .5, .8)	(.4, .2, .5)
j_3	(.3, .4, .7)	(.3, .6, .6)	(.4, .7, .6)	(.5, .7, .3)	(.4, .7, .4)	(.6, .4, .3)
j_4	(.7, .7, .5)	(.6, .7, .4)	(.5, .8, .4)	(.6, .7, .2)	(.7, .8, .2)	(.6, .7, .2)
j_6	(.8, .8, .3)	(.7, .9, .1)	(.9, .8, .2)	(.9, .8, .1)	(.9, .9, .1)	(.8, .9, .2)

Table 3.24

Table for LONSS (β, J_2).

(β, J_2)	w_1	w_2	w_3	w_4	w_5	w_6
j_3	(.2, .1, .8)	(.4, .6, .3)	(.1, .4, .6)	(.5, .3, .9)	(.7, .4, .8)	(.4, .1, .7)
j_6	(.4, .5, .4)	(.7, .8, .2)	(.4, .6, .4)	(.7, .7, .7)	(.7, .8, .8)	(.7, .3, .4)
j_7	(.6, .9, .2)	(.9, .8, .1)	(.8, .9, .2)	(.9, .8, .3)	(.9, .9, .6)	(.9, .5, .1)

Table 3.25

Table for basic union of two LONSSs.

(K,G)	w_1	w_2	w_3	w_4	w_5	w_6
(j_2,j_3)	(.2,.3,.8)	(.4,.6,.3)	(.3,.4,.6)	(.5,.6,.4)	(.7,.5,.8)	(.4,.2,.5)
(j_2,j_6)	(.4,.5,.4)	(.7,.8,.2)	(.4,.6,.4)	(.7,.7,.4)	(.7,.8,.8)	(.7,.3,.4)
(j_2,j_7)	(.6,.9,.2)	(.9,.8,.1)	(.8,.9,.2)	(.9,.8,.3)	(.9,.9,.6)	(.9,.5,.1)
(j_3,j_3)	(.3,.4,.7)	(.4,.6,.3)	(.4,.7,.6)	(.5,.7,.3)	(.7,.7,.4)	(.6,.4,.3)
(j_3,j_6)	(.4,.5,.4)	(.7,.8,.2)	(.4,.7,.4)	(.7,.7,.3)	(.7,.8,.4)	(.7,.4,.3)
(j_3,j_7)	(.6,.9,.2)	(.9,.8,.1)	(.8,.9,.2)	(.9,.8,.3)	(.9,.9,.4)	(.9,.5,.1)
(j_4,j_3)	(.7,.7,.5)	(.6,.7,.3)	(.5,.8,.4)	(.6,.7,.2)	(.7,.8,.2)	(.6,.7,.2)
(j_4,j_6)	(.7,.7,.5)	(.7,.8,.4)	(.5,.8,.4)	(.7,.7,.2)	(.7,.8,.2)	(.7,.7,.2)
(j_4,j_7)	(.7,.9,.2)	(.9,.8,.1)	(.8,.9,.2)	(.9,.8,.2)	(.9,.9,.2)	(.9,.7,.1)
(j_6,j_3)	(.8,.8,.3)	(.7,.9,.1)	(.9,.8,.2)	(.9,.8,.1)	(.9,.9,.1)	(.8,.9,.2)
(j_6,j_6)	(.8,.8,.3)	(.7,.9,.1)	(.9,.8,.2)	(.9,.8,.1)	(.9,.9,.1)	(.8,.9,.2)
(j_6,j_7)	(.8,.9,.2)	(.9,.9,.1)	(.9,.9,.2)	(.9,.8,.1)	(.9,.9,.1)	(.9,.9,.1)

Table 3.26

Now define

$C_i = \sum \left[T_{\alpha(j)}(w_i) + I_{\alpha(j)}(w_i) - F_{\alpha(j)}(w_i)\right]$, for all $j \in G$, where $i = 1,2,3,4,5,6$

Next we compute the score for each w_i as shown in Table 3.27,

	w_1	w_2	w_3	w_4	w_5	w_6
score (C_i)	10.6	15.6	12.8	15.1	15.3	13.3

Table 3.27

The Selction of car will be judged by maximum grades in Table 3.27. Thus the ranking of car is given by Table 3.28.

Car	Score in Table 3.27	Rank
w_2	15.6	1*st*
w_5	15.3	2*nd*
w_4	15.1	3rd
w_6	13.3	4th
w_3	12.8	5th
w_1	10.6	6th

Table 3.28

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