Effects of variable viscosity in a third grade fluid with porous medium

By

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1- Boundary value problems

Certificate

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We accept this dissertation as conforming to the required standard.

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Acknowledgements

To begin with the name of Almighty ***ALLAH,*** Who inculcated the consecration upon me to fulfill the requirement for this dissertation. I offer my humblest Darood to the Holy Prophet ***Muhammad*** (Peace be upon Him) who is forever a torch of guidance for humanity.

I express my gratitude to all my teachers whose teachings have brought me to this stage of academic zenith, in particular, I wish to express my profound gratitude to my kind natured, eminent, affectionate, candid and devoted supervisors, *Dr. Rahmat Ellahi*, who aided me with many inspirational discussions. His many valuable comments and suggestions was most welcome and instructive and greatly improved the clarity of this document. I would never have been able to do it up to the standard without his help. I am placing my earnest thanks to *Dr. Rahmat Ellahi.*

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Finally, it would have been impossible for me to complete this work without a great support and understanding of my family. I am placing regards to them.

Date: 3rd January 2008 **Saeed Afzal**

Preface

During the past several years, interest in non-Newtonian fluids has increased due to their technological applications. Many materials such as, polymer solutions or metals, drilling mud, clastomers, certain oils and gases and many other emulsions are classified as non-Newtonian fluids. Due to complexity of fluids, there are many models of non-Newtonian fluids. However, to describe several non standard features, such as normal stress effects, rod climbing, sheer thinning and sheer thickening, Rivlin-Erickson fluids [1] of differential types have acquired a special status. Several interesting studies [2-7] are available in the literature which deal with the equations of motions on non-Newtonian fluids of second and third grades in various geometrical configurations. Moreover study of heat transfer plays an important role during the handling and processing of non-Newtonian fluids. The understanding of heat transfer in boundary layer flows of non-Newtonian fluids is of importance in many engineering applications such as the design of the thrust bearings and radial diffusers, transpiration cooling, drag reduction, thermal recovery of oil etc.

All the above mentioned studies have lacked the effects of variable viscosity and heat transfer on the flow. Very little work has been reported which deals with such effects. Massoudi and Christie [8] have discussed the effects of variable viscosity and viscous dissipation on he flow of third grade fluid in a pipe through the numerical solution.

In this study, the analytical solutions are obtained and observations are found to be in agreement with the numerical solution of the problem given by Massoudi and Christie [8] for temperature dependent viscosity. For temperature dependent viscosity, the governing equations are complicated system of coupled equations in velocity and temperature distributions. We use homotopy analysis method (HAM) [9] to obtain analytical solution. The HAM has already been applied successfully to discuss many flow problems [10-32].

In chapter one, the basic definitions of fluids and homotopy are given. Chapter two is devoted to the study of influence of variable viscosity and viscous dissipation on non-Newtonian flow. An analytical solution for the flow of a third grade fluid in a pipe using HAM is studied.

In chapter three, we find an analytic solution for the flow consider in [9] with effect of porous medium by using modified Darcy's law. The governing equations are modeled and analytical solution using HAM is studied. The fluid is assumed to be in a compressible. Expression for velocity and temperature profiles are constructed analytically and explained with the help of graphs.

Contents

Chapter 1

Preliminaries

In this chapter, some basic definitions and concepts of various types of fluids are presented. Basic equations which govern the flow are given. The basic idea of Homotopy Analysis Method (HAM) is also explained in this chapter.

1.1 Fluid

A fluid is a substance that deforms continuously under the application of shearing, i. e., tangentail stress, no matter how small the shearing stress is. According to physical forms in which matter exists, fluid comprises the liquid and gas phases.

1.2 Flow

A material undergoes a deformation when different forces act upon it. If the deformation continuously increases without limit, the phenomena is known as flow.

1.3 Deformation

It is a relative change in position or length of the fluid particles.

1.4 Pressure

The magnitude of force exerted perpendicularly on unit area of an object is called pressure.

Let A is the surface area of a fluid and F is the magnitude of force acting normal to the surface, then the pressure p^* due to the force on unit area of this surface is

$$p^* = \frac{F}{A}. \tag{1.1}$$

1.5 Density

It is defined as the mass per unit volume. Mathematically one can write

$$\rho = \frac{m}{V}. \tag{1.2}$$

Here ρ is the density, m is the mass and V is the volume.

1.6 Viscosity

Although a fluid offers no resistance to change of shape, it does inhibit resistance to the rate of change of shape. The property of producing this resistance is called the viscosity. It is the property which gives rise to tangential (shearing) stresses whenever relative motion occurs within a fluid.

In mathematical form we have

$$\mu = \frac{\text{shear stress}}{\text{rate of shear strain}}, \tag{1.3}$$

where μ is the coefficient of viscosity.

1.7 Kinematic viscosity

It is a ratio of dynamic viscosity to density. Mathematically

$$\nu = \frac{\mu}{\rho}, \tag{1.4}$$

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in which ν is a kinematic viscosity.

1.8 Classification of fluids

1.8.1 Ideal fluids

The fluids which offer no resistance to flow i. e., for which viscosity is negligible are termed as ideal fluid.

1.8.2 Real fluids

These are the fluids for which dynamic viscosity is non-zero. Real fluids are further classified into two classes on the basis of Newton's law of viscosity. According to this law

"shear stress is directly and linearly proportional to the rate of deformation".

For one dimensional flow we can write as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.5}$$

where τ_{yx} is the shear stress and du/dy is a deformation rate.

1.8.3 Newtonian fluids

The fluids which obey Newton's law of viscosity are called Newtonian fluids. Water, gasoline and mercury are some examples of Newtonian fluids.

1.8.4 Non-Newtonian fluids

The fluids for which "shear stress is directly proportional to the rate of deformation but in a non-linear manner". Mathematically

$$\tau_{yx} = k \left(\frac{du}{dy}\right)^n, \qquad n \neq 1 \tag{1.6}$$

or

$$\tau_{yx} = \eta \left(\frac{du}{dy}\right), \tag{1.7}$$

where

$$\eta = \left(\frac{du}{dy}\right)^{n-1} \tag{1.8}$$

is the apparent viscosity. Examples of non-Newtonian fluids are tooth paste, ketchup, gel, shampoo, blood and soaps.

1.9 Types of flows

1.9.1 Uniform flow

It is a flow in which the velocities of fluid particles are equal at all sections of the flow domain.

1.9.2 Non-uniform flow

It is a flow in which the velocities of fluid particles are not the same at all sections of the flow domain.

1.9.3 Steady flow

A flow for which fluid properties do not depend upon time. For such flow

$$\frac{\partial \mathbf{V}}{\partial t} = 0, \tag{1.9}$$

where $\mathbf{V}$ is the fluid velocity and t is the time.

1.9.4 Unsteady flow

A flow for which fluid velocity depends upon time i.e.,

$$\frac{\partial \mathbf{V}}{\partial t} \neq 0. \tag{1.10}$$

1.9.5 Compressible flow

It is a flow which possesses a variable density. All the gases are treated as compressible flows.

1.9.6 Incompressible flow

It is a flow of constant density. All the liquids are treated as incompressible flows.

1.9.7 Porosity

The measurement of openings in buildings, which allow air to enter during a period of high wind,or a measure of the amount of voids (pores) in a material.

1.9.8 Porous medium

A porous medium is a continuous solid phase with intervening void or gas pockets. Natural porous media include soil,sand , mineral salts, sponge, wood and others .Synthetic porous media include paper , cloth filters , chemical reaction catalysts and membranes.

1.9.9 Prandtl number

It is the ration of the product of dynamic viscosity and specific heat with thermal conductivity k, i. e .,

$$P_r = \frac{\upsilon \rho c_p}{k} \tag{1.11}$$

The Prandtl number serves as a direct measure for the ratio of the thickness of the layers in forced flow.

1.9.10 Eckert number

The ratio between square of the velocity and the two temperature difference is called Eckert number

$$E_c = \frac{v_0^2}{c_p(T_0 - T_\infty)} \tag{1.12}$$

1.10 Homotopy analysis method

In topology two functions are said to be homotopic if one function can transform continuously into the other.

Definition: Formally a homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function

$$H : X \times [0,1] \to Y, \tag{1.13}$$

from the product of the space X with the unit interval $[0,1]$ to Y such that for all point x in X and

$$H(x,0) = f(x)\ , \qquad\qquad H(x,1) = g(x). \tag{1.14}$$

If we consider second coordinate as time t then at time $t = 0$ we have a function f and at time 1 we have a function g.

In fluid mechanics sometimes governing differential equations become non-linear, which is very difficult to deal with. So we have no choice but to solve these numerically. Fortunately we have some method in which we can solve non-linear differential equations by HAM.

The zeroth order deformation is defined as

$$(1-p)\,\mathcal{L}[u(y\ ;p) - u_0] = pH\hbar R_m, \tag{1.15}$$

in which $p \in [0,1]$ is the embedding parameter, $\hbar$ is auxiliary non-zero parameter and R_m is non-linear differential operator. For $p = 0$ and $p = 1$ we have

$$u(y,0) = u_0\ ,\ \ u(y,1) = u(y), \tag{1.16}$$

where u_0 is the initial guess and $u(y)$ is a solution of a given differential equation. By Taylor's theorem

$$u(y,p) = u_0(y) + \sum_{m=1}^{\infty} u_m(y)p^m, \tag{1.17}$$

where

$$u_m(y) = \frac{1}{m!}\frac{\partial^m u(y;p)}{\partial p^m} \qquad at \quad p = 0. \tag{1.18}$$

Note that convergence is depending upon auxiliary parameter $\hbar$. Thus we can write

$$u(y) = u_0(y) + \sum_{m=1}^{\infty} u_m(y), \tag{1.19}$$

which is a solution of given differential equation.

The advantages of HAM are

1. it is valid even if a given non-linear problem does not contain any small/large parameters at all

2. it can provide us with a convenient way to control the convergence of approximation series and adjust convergence regions where necessary

3. it can be employed to efficiently approximate a nonlinear problem by choosing different sets of base functions.

Chapter 2

The influence of variable viscosity and viscous dissipation on the non-Newtonian flow: An analytic solution

2.1 Introduction

This chapter is the review work of Hayat et al. [33]. In this chapter we seek the effect of constant and variable viscosity on the velocity and temperature profile. The resultant equation is non-linear and solved by homotopy analysis method (HAM). Different graphs has been drawn for different parameters.

2.2 Mathematical formulation

Let us consider the incompressible third grade fluid. We choose cylinder in $z - axis$. For the problem under consideration the velocity field is given by

$$\mathbf{V} = [0, 0, v(r)], \tag{2.1}$$

In the absence of body forces, the equations governing the flow of an incompressible fluid are

$$\nabla.\mathbf{V} = 0, \tag{2.2}$$

$$\rho\frac{dv}{dt} = div\mathbf{T}, \tag{2.3}$$

where d/dt is the material derivative and $\mathbf{T}$ is the Cauchy stress tensor which for third grade fluid is

$$\mathbf{T} = -p_1\mathbf{I} + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2(A_1A_2 + A_2A_1) + \beta_3(trA_1^2)A_1 \tag{2.4}$$

where p_1 is hydrostatic pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamics viscosity and $\alpha_i(i = 1,2)$ and $\beta_j(j = 1,2)$ are material constants. The first three Rivlin-Ericksen tensors $(A_1 - A_3)$ are defined through the following expressions

$$A_1 = \nabla V + (\nabla V)^t, \tag{2.5}$$

$$A_n = \frac{dA_{n-1}}{dt} + A_{n-1}\mathbf{L} + \mathbf{L}^t A_{n-1}, \qquad n > 1, \tag{2.6}$$

$$\mathbf{L} = \nabla V = grad\mathbf{V}, \tag{2.7}$$

in which ∇ is the gradient operator and V is the velocity field. For thermodynamic third grade fluid the coefficients μ, α_1, α_2, β_1, β_2,and β_3 satisfy the following conditions

$$\mu \geq 0, \alpha_1 \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \beta_1 = \beta_2 = 0, \beta_3 \geq 0. \tag{2.8}$$

$$A_1 = \nabla V + (\nabla V)^t = \begin{bmatrix} 0 & 0 & \frac{dv}{dr} \\ 0 & 0 & 0 \\ \frac{dv}{dr} & 0 & 0 \end{bmatrix}, \tag{2.9}$$

$$A_2 = \begin{bmatrix} 2\left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.10}$$

$$A_1^2 = \begin{bmatrix} \left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{dv}{dr}\right)^2 \end{bmatrix}, \tag{2.11}$$

$$tr(A_1^2)A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\left(\frac{dv}{dr}\right)^3 & 0 & 0 \end{bmatrix}, \tag{2.12}$$

$$\tau_{rr} = p_1 + (2\alpha_1 + \alpha_2)\left(\frac{dv}{dr}\right)^2, \tau_{rz} = \mu\frac{dv}{dr} + 2\beta_3\left(\frac{dv}{dr}\right)^3, \tau_{r\theta} = 0 = \tau_{r\theta}, \tag{2.13}$$

$$\tau_{\theta\theta} = -p_1, \tau_{\theta z} = 0 = \tau_{\theta z}, \tau_{zz} = -p_1 + \alpha_2\left(\frac{dv}{dr}\right)^2, \tag{2.14}$$

Equation of motion in cylindrical form is

$$\frac{1}{r}\frac{d(r\tau_{rz})}{dr} + \frac{1}{r}\frac{d\tau_{\theta z}}{d\theta} + \frac{d\tau_{zz}}{dz} = -\frac{\partial\hat{p}}{\partial z}, \tag{2.15}$$

where

$$\hat{p} = p_1 + \alpha_2\left(\frac{dv}{dr}\right)^2, \tag{2.16}$$

In dimensionless form the equations for flow of a thermodynamic third grade in a pipe with heat transfer was derived

$$\frac{1}{r}\frac{d}{dr}\left[r\mu(r)(\frac{dv}{dr})\right] + \frac{\Lambda}{r}\frac{d}{dr}(r(\frac{dv}{dr})^3) = c, \tag{2.17}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}(\frac{d\theta}{dr}) + \Gamma(\frac{dv}{dr})^2(\mu(r) + \Lambda(\frac{dv}{dr})^2) = 0, \tag{2.18}$$

$$v(1) = \theta(1) = 0, \qquad \frac{dv(0)}{dr} = \frac{d\theta(0)}{dr} = 0. \tag{2.19}$$

In the above equations the dimensionless parameters are

$$c_1 = \frac{\partial\hat{p}}{\partial z}, \quad c = \frac{c_1 R^2}{v_0\mu_0}, \quad \Lambda = \frac{2\beta_3 v_0^2}{\mu_0 R^2}, \qquad \Gamma = \frac{\mu_0 v_0^2}{k(\theta_1 - \theta_0)}. \tag{2.20}$$

And r $(0 < r < 1)$, v, θ,and μ, are the dimensionless radius , velocity , temperature and viscosity respectively . These terms are related to the dimensional ones as follows :

$$v = \frac{v^*}{v_0}, \ r = \frac{r^*}{R}, \mu = \frac{\mu^*}{\mu_0}, \theta = \frac{\theta^* - \theta_0}{\theta_1 - \theta_0}, \tag{2.21}$$

in which R, v_0, μ_0, θ_0, θ^* and θ_1 are the radius, reference velocity, reference viscosity, reference temperature, pipe and fluid temperatures respectively. Also, c_1 is the axial pressure drop, Λ is the dimensionless third grade parameter and Γ is related to the Prandtl and Eckert numbers.

2.3 Analytic solution

Now, we discuss the solutions of Equations (2.17) and (2.18) in two cases :

Case 1 : For the constant viscosity

If $\mu = 1$, after simplification equation (2.17) becomes

$$\frac{d^2v}{d\,r^2} + \frac{1}{r}\frac{dv}{dr} + 3\Lambda(\frac{dv}{dr})^2\frac{d^2v}{d\,r^2} + \frac{\Lambda}{r}(\frac{dv}{dr})^3 = c. \tag{2.22}$$

For an initial guess, we mostly take linear part of equation (2.16) as linear operator

$$\mathcal{L} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}, \tag{2.23}$$

which satisfies

$$\mathcal{L}[C_1 + C_2 \ln r] = 0, \tag{2.24}$$

and initial guess is given by

$$v_0(r) = \frac{c}{4}(r^2 - 1), \tag{2.25}$$

$$(1-p)\mathcal{L}[\overline{v}(r,p) - v_0(r)] = p\hbar[\frac{1}{r}\frac{d}{dr}(r(\frac{d\overline{v}}{dr})) + \frac{\Lambda}{r}\frac{d}{dr}(r(\frac{d\overline{v}}{dr})^3) - c], \tag{2.26}$$

$$(1-p)L[\overline{v}(r,p) - v_0(r)] = p\hbar[\frac{1}{r}\frac{d}{dr}(r(\frac{d\overline{v}}{dr})) + \frac{\Lambda}{r}\frac{d}{dr}(r(\frac{d\overline{v}}{dr})^3) - c], \tag{2.27}$$

$$\overline{v}(1) = \overline{\theta}(1) = 0, \frac{d\overline{v}(0)}{dr} = \frac{d\overline{\theta}(0)}{dr} = 0. \tag{2.28}$$

$$\overline{v}(r;0) = v_0, \quad \overline{v}(r;1) = v. \tag{2.29}$$

As p increases from 0 to 1, $\overline{v}(r;p)$ varies from the initial guess v_0 to the exact solution $v(r)$. One can write by Taylor's theorem

$$\overline{v}(r;p) = v_0 + \sum_{m=1}^{\infty} v_m(r)p^m, \tag{2.30}$$

$$v_m(r) = \frac{1}{m!}\frac{\partial^m \overline{v}(r;p)}{\partial p^m}, \tag{2.31}$$

$$v(r) = v_0 + \sum_{m=1}^{\infty} v_m(r), \tag{2.32}$$

mth order deformation equation is given by

$$\mathcal{L}[v_m - \chi_m v_{m-1}] = \hbar R_m(r) \tag{2.34}$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{2.35}$$

$$v_m(0) = v_m(\infty) = 0 \tag{2.36}$$

and

$$R_m(r) = \left(v''_{m-1} + \frac{1}{r}v'_{m-1} + \Lambda \sum_{k=0 m-1-k}^{m-1} v'_{m-1-k} \sum_{l=0 k-l}^{k} v'_{k-l}(\frac{1}{r}v'_l + 3v''_l) \right). \tag{2.37}$$

$$\begin{aligned} \theta(r) \quad = \quad & C_3(1-r^4) + C_4(1-r^8) + C_5(1-r^{12}) + C_6(1-r^{16}) \\ & + C_7(1-r^{20}) + C_8(1-r^{24}) + C_9(1-r^{28}) + C_{10}(1-r^{32}) \\ & + C_{11}(1-r^{36}) + C_{12}(1-r^{40}) + C_{13}(1-r^{44}) + C_{14}(1-r^{48}) \\ & + C_{15}(1-r^{52}) + C_{16}(1-r^{56}), \end{aligned} \tag{2.38}$$

where the coefficients $C_3 - C_{16}$ are given in Appendix A.

Case 2 For the variable viscosity

Let $\mu = r$, the equation (2.18) becomes

$$\frac{1}{r}\frac{d}{dr}(r^2\frac{dv}{dr}) + \frac{\Lambda}{r}\frac{d}{dr}(r(\frac{dv}{dr})^3) = c, \tag{2.39}$$

$$\frac{d}{dr}(r^2\frac{dv}{dr}) + \Lambda\frac{d}{dr}(r(\frac{dv}{dr})^3) = cr, \tag{2.40}$$

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + \frac{\Lambda}{r^2}[3r(\frac{dv}{dr})^2\frac{d^2v}{dr^2} + (\frac{dv}{dr})^3] = \frac{c}{r}, \tag{2.41}$$

$$\mathcal{L}_1 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}, \tag{2.42}$$

satisfies

$$\mathcal{L}_1[C_{17} + 2C_{18}\ln r] = 0 \tag{2.43}$$

$$(1-p)\mathcal{L}_1[v(r,p) - v_0(r)] = p\hbar\left[\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + \frac{3\Lambda}{r}(\frac{dv}{dr})^2\frac{d^2v}{dr^2} + \frac{\Lambda}{r^2}(\frac{dv}{dr})^3 - \frac{c}{r}\right] \tag{2.44}$$

and mth order deformation is given by

$$\mathcal{L}_1[v_m - \chi_m v_{m-1}] = \hbar R^*_m(r) \tag{2.45}$$

and

$$R^*_m(y) = \left(rv''_{m-1} + 2v'_{m-1} + \Lambda\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}(\frac{1}{r}v'_l + 3v''_l)\right) - (c - \chi_m) \tag{2.46}$$

For finding the solution of temperature we use the Mathematica to solve the Cauchy-Euler equation

$$\begin{aligned}\theta(r) \;=\; & [C_{19}(1-r^4) + C_{20}(1-r^5) + C_{21}(1-r^6) + C_{22}(1-r^7) \\ & + C_{23}(1-r^8) + C_{24}(1-r^9) + C_{25}(1-r^{10}) + C_{26}(1-r^{11}) \\ & + C_{27}(1-r^{12}) + C_{28}(1-r^{13}) + C_{29}(1-r^{14}) + C_{30}(1-r^{15}) \\ & + C_{31}(1-r^{16}) + C_{32}(1-r^{17}) + C_{33}(1-r^{18}) + C_{34}(1-r^{19}) \\ & + C_{35}(1-r^{20}) + C_{36}(1-r^{21}) + C_{37}(1-r^{22})],\end{aligned} \tag{2.47}$$

where the coefficients $C_{19} - C_{37}$ are given in Appendix A.

2.4 Graphs

In this section we drew the graphs of velocity and temperature profile for both constant and variable viscosity.

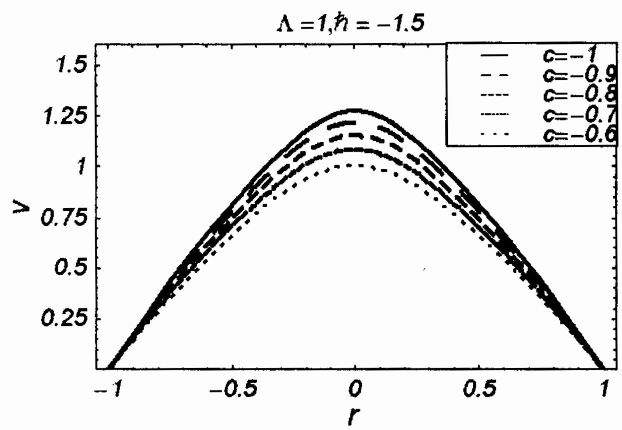

Fig. 2.1 Influence of c on the constant velocity.

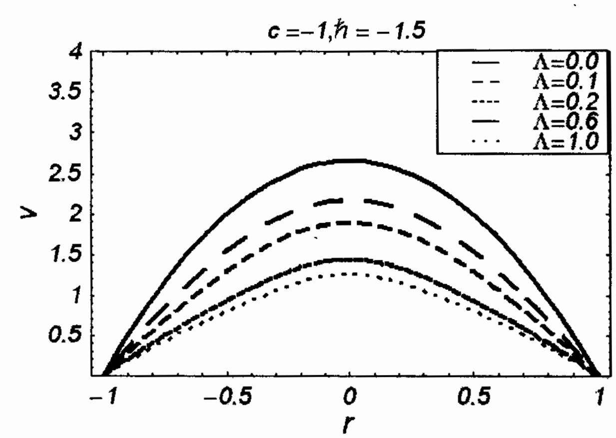

Fig. 2.2 Influence of Λ on the constant velocity.

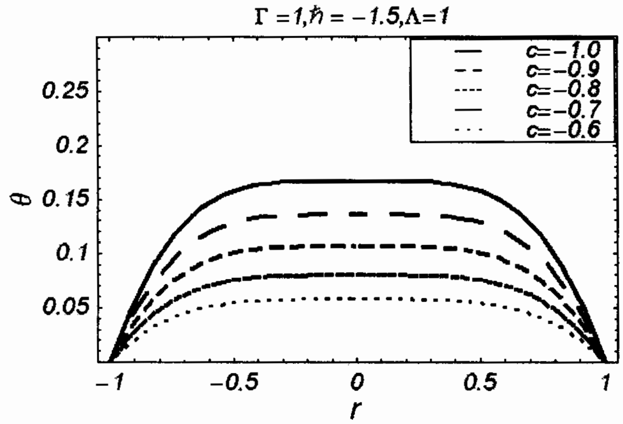

Fig. 2.3 Influence of c on the constant temperature.

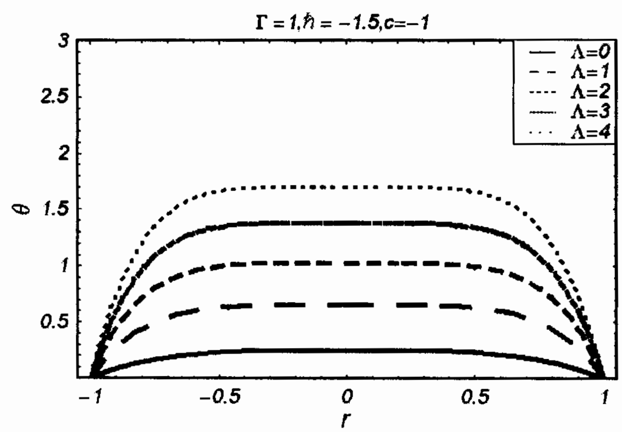

Fig. 2.4 Influence of Λ on the constant temperature.

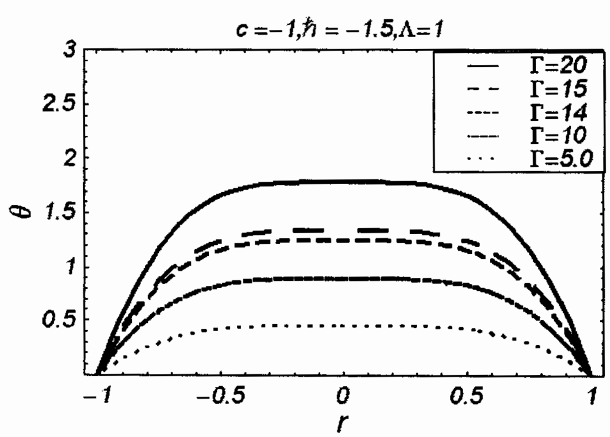

Fig. 2.5 Influence of Γ on the constant temperature.

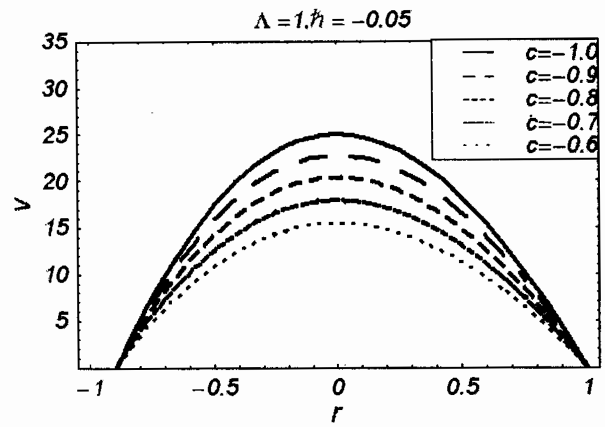

Fig. 2.6 Influence of c on the variable velocity.

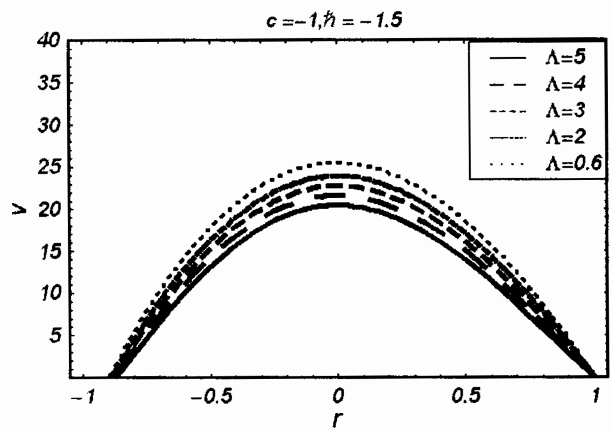

Fig. 2.7 Influence of Λ on the variable velocity.

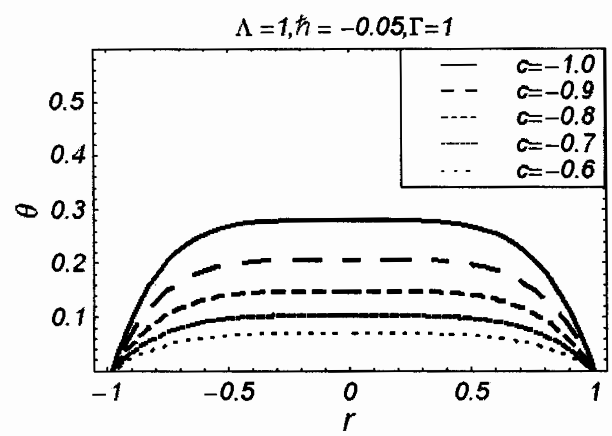

Fig. 2.8 Influence of c on the variable temperature.

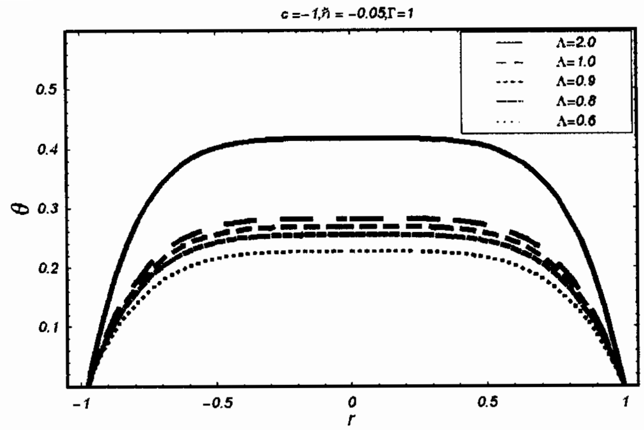

Fig. 2.9 Influence of Λ on the variable temperature.

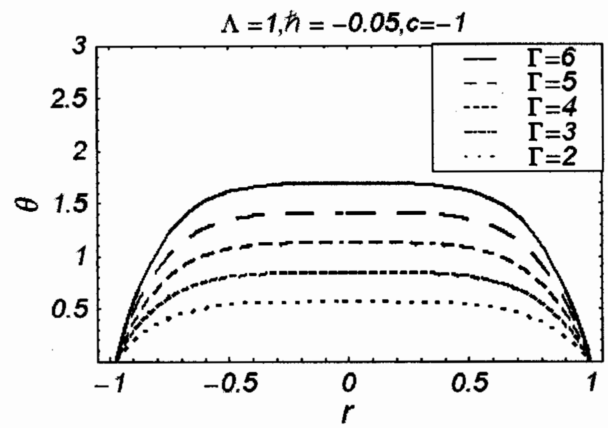

Fig. 2.10 Influence of Γ on the variable temperature.

2.5 Discussions

Here, the solution for the velocity and temperature distributions are plotted against the pipe radius. The velocity and temperature distributions for constant viscosity are displayed in Figures 2.1 to 2.4 and for space dependent viscosity in Figures 2.5 to 2.9. In these Figures, the variation of the emerging parameters Λ, c and Γ are taken into account.

In Figure 2.1, the effect of pressure gradient c is depicted when $h = -1.5$ and $\Lambda = 1$. It can be seen that the maximum velocity of the fluid increases at the center of the pipe with the more negative values of c. Also, the effect of c on θ (in Figure 2.3) is similar to that of velocity.

The effect of third grade parameter Λ on the velocity and temperature distributions are shown in Figures 2.2 and 2.4, respectively when $h = -1.5$ and $c = -1$. As expected, an increase in Λ results in a decrease in velocity and increases for temperature. However, the profiles are more flatter in case of temperature distribution than the velocity distribution.

In order to illustrate the effect of Γ on the temperature distribution, we prepared Figure 2.5. It is found that θ increases with the increase of Γ and hence the thermal boundary layer thickness decreases.

Until now, we discussed the results of the velocity and temperature for constant viscosity model. Now, we turn our attention to a discussion of the results of the velocity and temperature distributions for space dependent viscosity. For this case, the velocity and temperature distributions are plotted against the pipe radius in Figures 2.6 to 2.10, respectively for various values of c, Λ and Γ. From these figures, it is noted that the observations about c, Λ and Γ are similar to that of constant viscosity model.

Chapter 3

Effects of variable viscosity in a third grade with porous medium

3.1 Introduction

In this Chapter we extend the work of chapter [33]. and introduce the porous medium term. Analytic solution has been obtained by homotopy analysis method (HAM). Different graphs has been drawn for different pertinent parameters.

3.2 Problem formulation

Let us consider an incompressible third grade fluid in a pipe. The fluid saturates the porous medium. The velocity filed is given by

$$\mathbf{V} = [0{,}0, v(r)], \tag{3.1}$$

and the governing equations are

$$\nabla.\mathbf{V} = 0, \tag{3.2}$$

$$\rho\frac{d\mathbf{V}}{dt} = div\mathbf{T}, \tag{3.3}$$

$$\rho c_p\frac{d\theta}{dt} = \mathbf{T}\cdot\mathbf{I}-\nabla^2\theta, \tag{3.4}$$

where d/dt is the material derivative, ρ is the specific heat, θ is the temperature and the Cauchy stress $\mathbf{T}$ is

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \qquad (3.5)$$

where p_1 is hydrostatic pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity and $\alpha_i(i = 1, 2)$ and $\beta_j(j = 1-3)$ are material constants. Note that radiation effects in the energy equation are neglected. The first three Rivlin-Ericksen tensors $(A_1 - A_3)$ are defined through the following expressions

$$A_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^t, \qquad (3.6)$$

$$A_n = \frac{dA_{n-1}}{dt} + A_{n-1}\mathbf{L} + \mathbf{L}^t A_{n-1}, \qquad n > 1, \qquad (3.7)$$

$$\mathbf{L} = \nabla\mathbf{V} = grad\mathbf{V}, \qquad (3.8)$$

in which ∇ is the gradient operator and $\mathbf{V}$ is the velocity field. For thermodynamic third grade fluid the coefficients μ, α_1, α_2, β_1, β_2,and β_3 satisfy the following conditions [34]

$$\mu \geq 0, \alpha_1 \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \beta_1 = \beta_2 = 0, \beta_3 \geq 0. \qquad (3.9)$$

Keeping in mind the constitutive equation, the following law in an Oldroyd-B fluid is proposed [35]:

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu}{k}\phi\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V} \qquad (3.10)$$

and the Darcy's resistance $\mathbf{R}$ is

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\mathbf{R} = -\frac{\mu}{k}\phi\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \qquad (3.11)$$

where λ and λ_r are the relaxation and retardation times, ϕ is porosity and k is the permeability.

Having in mind the above equation the $z-$components of $\mathbf{R}$ for steady flow of a third grade fluid is

$$R_z = -\frac{\phi}{k}\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right]v. \qquad (3.12)$$

Invoking Eq. (1) the continuity equation is identically satisfied and momentum and energy equations reduce to

$$\frac{1}{r}\frac{d}{dr}(r\mu\frac{dv}{dr}) + \frac{2\beta_3}{r}\frac{d}{dr}(r(\frac{dv}{dr})^3) - \frac{\phi}{k}\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right] v = -\frac{\partial\hat{p}}{\partial z}, \tag{3.13}$$

$$\mu(\frac{dv}{dr})^2 + 2\beta_3\left(\frac{dv}{dr}\right)^4 + k[\frac{1}{r}\frac{d}{dr}(r\frac{d\theta}{dr})] = 0. \tag{3.14}$$

where k is the thermal conductivity and $\hat{p}$ is

$$\hat{p} = p_1 - \alpha_2\left(\frac{dv}{dr}\right)^2. \tag{3.15}$$

The boundary conditions are

$$v(1) = \theta(1) = 0, \qquad \frac{dv(0)}{dr} = \frac{d\theta(0)}{dr} = 0. \tag{3.16}$$

Equations (3.13), (3.14) and (3.16) in dimensionless variables become

$$\frac{d\mu}{dr}\frac{dv}{dr} + \frac{\mu}{r}\frac{dv}{dr} + \mu\frac{d^2v}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 + 3\Lambda\frac{d^2v}{dr^2}\left(\frac{dv}{dr}\right)^2 - P\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right] v = c, \tag{3.17}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2\left(\mu(r) + \Lambda\left(\frac{dv}{dr}\right)^2\right) = 0, \tag{3.18}$$

$$v(1) = \theta(1) = 0, \qquad \frac{dv(0)}{dr} = \frac{d\theta(0)}{dr} = 0, \tag{3.19}$$

where

$$\begin{aligned} \Lambda &= \frac{2\beta_3 V_0^2}{R_0^2 v_0}, \qquad c = \frac{\frac{\partial p_1}{\partial z}R^2}{\mu_0 V_0}, \quad \frac{\phi}{kR^2} = P, \qquad r = \frac{\overline{r}}{\overline{R}}, \quad v = \frac{\overline{v}}{\overline{v_0}}, \quad \mu = \frac{\overline{\mu}}{\mu_0} \\ \overline{\theta} &= \frac{\theta - \theta_0}{\theta_1 - \theta_0}, \qquad \Gamma = \frac{\mu_0^2 v_0}{k(\theta_1 - \theta_0)}, \end{aligned} \tag{3.20}$$

where R, v_0, μ_0, θ_0, θ and θ_1 are the radius, reference velocity, reference viscosity, reference temperature and pipe fluid temperatures respectively. Here c is the pressure drop, P is porous medium parameter and Γ is related to the Prandtl and Eckert numbers and bars have been suppressed for simplicity.

3. Series solutions

In this section, Eqs. (3.17) and (3.18) with boundary conditions (3.19) will be solved by using HAM.

Case- I Reynold model

For Reynold model

$$\mu = e^{-B\theta}, \tag{3.21}$$

For HAM solutions we select

$$v_0(r) = \frac{c}{4}(r^2-1), \quad \theta_0(r) = \frac{c}{2}(r^2-1) \tag{3.22}$$

as the initial approximations of v and θ. Further we choose the following auxiliary linear operators

$$\mathcal{L}_2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \tag{3.23}$$

such that

$$\mathcal{L}_2(C_1 + C_2 \ln r) = 0, \tag{3.24}$$

where C_1 and C_2 are arbitrary constants.

The zeroth – order problems are

$$(1-p)\mathcal{L}_2[v^*(r,p) - v_0(r)] = p\hbar\mathcal{N}_1[v^*(r,p), \theta^*(r,p)], \tag{3.25}$$

$$(1-p)\mathcal{L}_2[\theta^*(r,p) - \theta_0(r)] = p\hbar\mathcal{N}_2[v^*(r,p), \theta^*(r,p)], \tag{3.26}$$

$$v^*(1,p) = 0, \quad \theta^*(1,p) = 0, \quad \left.\frac{\partial v^*(r,p)}{\partial r}\right|_{r=0} = 0, \quad \left.\frac{\partial \theta^*(r,p)}{\partial r}\right|_{r=0} = 0. \tag{3.27}$$

After the Taylor's series expansion of Eq. (3.21) and retrieving only first two components, we get

$$\mu \approx 1 - \theta B \tag{3.27a}$$

In view of Eq. (3.17) and 3.27a we can write

$$\mathcal{N}_1\left[v^*(r,p),\theta^*(r,p)\right]= \begin{array}{c}\left[\frac{1}{r}\frac{dv}{dr}+\frac{d^2v}{dr^2}-\frac{B}{r}\frac{dv}{dr}\theta-B\frac{d^2v}{dr^2}\theta+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3+3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}\right.\\ \left.+BPv\theta-Pv-P\Lambda\left(\frac{dv}{dr}\right)^2v-c\right],\end{array} \tag{3.28}$$

$$\mathcal{N}_2\left[v^*(r,p),\theta^*(r,p)\right]= \begin{array}{c}\left[\frac{1}{r}\frac{d\theta^*}{dr}+\frac{d^2\theta^*}{dr^2}-\Gamma B\left(\frac{dv^*}{dr}\right)^2\theta+\Gamma\left(\frac{dv}{dr}\right)^2\right.\\ \left.+\Lambda\Gamma\left(\frac{dv}{dr}\right)^4\right]\end{array} \tag{3.29}$$

For $p=0$ and $p=1$, we have

$$v^*(r;0)=v_0(r),\ \theta(r;0)=\theta_0(y)\ \text{ and } v^*(r;1)=v(r),\ \theta^*(r;1)=\theta(r). \tag{3.30}$$

When p increases from 0 to 1, $v^*(r,p)$, $\theta^*(r,p)$ varies from $v_0(r)$, $\theta_0(r)$ to $v(r)$ and $\theta(r)$ respectively. By Taylor's theorem and Eq. (3.30) one can write

$$v^*(r,p)=v_0(r)+\sum_{m=1}^{\infty}v_m(r)\ p^m,\ \theta^*(r,p)=\theta_0(r)+\sum_{m=1}^{\infty}\theta_m(r)\ p^m, \tag{3.31}$$

$$v_m(r)=\frac{1}{m!}\frac{\partial^m v^*(r,p)}{\partial p^m}\bigg|_{p=0},\ \theta_m(r)=\frac{1}{m!}\frac{\partial^m\theta^*(r,p)}{\partial p^m}\bigg|_{p=0}. \tag{3.32}$$

Clearly, the convergence of the series (3.31) depends upon $\hbar$. We choose $\hbar$ in such a way that the series (3.31) is convergent at $p=1$, then due to Eq. (3.30) we have

$$v(r)=v_0(r)+\sum_{m=1}^{\infty}v_m(r), \qquad \theta(r)=\theta_0(r)+\sum_{m=1}^{\infty}\theta_m(r). \tag{3.33}$$

The mth order deformation problems are

$$\mathcal{L}_1\left[v_m(r)-\chi_m v_{m-1}(r)\right]=\hbar\mathcal{R}1_m(r), \tag{3.34}$$

$$\mathcal{L}_2\left[\theta_m(r)-\chi_m\theta_{m-1}(r)\right]=\hbar\mathcal{R}2_m(r), \tag{3.35}$$

$$v_m(1)=0,\quad v_m'(0)=0,\quad \theta_m(1)=0,\quad \theta_m'(0)=0, \tag{3.36}$$

$$\mathcal{R}1_m(r) = \left[\begin{array}{c} \frac{1}{r}\frac{dv_{m-1}}{dr} + \frac{d^2v_{m-1}}{dr^2} - B\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{d\theta_k}{dr} - \frac{B}{r}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\theta_k \\ -B\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr^2}\theta_k + \end{array}\right.$$

$$\frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-l}}{dr}\frac{dv_l}{dr} + 3\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{d^2v_l}{dr^2} \tag{3.37}$$

$$\left. -P\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}v_l - (1-\chi_m)c + BP\sum_{k=0}^{m-1}v_{m-1-k}\theta_k - Pv_{m-1}\right],$$

$$\mathcal{R}2_m(r) = \left[\frac{1}{r}\frac{\partial\theta_{m-1}}{\partial r} + \frac{\partial^2\theta_{m-1}}{\partial r^2} + \Gamma\sum_{k=0}^{m-1}\frac{\partial v_{m-1-k}}{\partial r}\frac{\partial v_k}{\partial r}\right.$$

$$-\Gamma B\sum_{k=0}^{m-1}\frac{\partial v_{m-1-k}}{\partial r}\frac{\partial v_{k-l}}{\partial r}\theta_l$$

$$\left. +\Lambda\Gamma\sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{f=0}^{l}\frac{\partial v_{m-1-k}}{\partial r}\frac{\partial v_{k-l}}{\partial r}\frac{\partial v_{l-f}}{\partial r}\frac{\partial v_f}{\partial r}\right]. \tag{3.38}$$

Case II Vogel model

Here

$$\mu = \mu_0 e^{\left(\frac{A}{B+\theta}-\theta\right)}. \tag{3.39}$$

The problems at the zeroth order are

$$(1-p)\mathcal{L}_2\left[v^*(r,p) - v_0(r)\right] = p\hbar\mathcal{N}_3\left[v^*(r,p), \theta^*(r,p)\right], \tag{3.40}$$

$$(1-p)\mathcal{L}_2\left[v^*(r,p) - v_0(r)\right] = p\hbar\mathcal{N}_4\left[v^*(r,p), \theta^*(r,p)\right], \tag{3.41}$$

$$v^*(1,p) = 0, \quad \theta^*(1,p) = 0, \quad \left.\frac{\partial v^*(r,p)}{\partial r}\right|_{r=0} = 0, \quad \left.\frac{\partial\theta^*(r,p)}{\partial r}\right|_{r=0} = 0, \tag{3.42}$$

By Taylors series, Eq. (3.39) can be approximated [36] as

$$\mu = \frac{c}{S}\left(1 - \frac{\theta A}{B^2}\right) \qquad \text{where} \qquad S = \frac{c}{\mu_o e^{\left(\frac{A}{B}-w_0\right)}} \tag{3.42a}$$

In view of Eq. (3.17) and 3.42a we can write

$$\mathcal{N}_3\left[v^*(r,p),\theta^*(r,p)\right]=\left[\begin{array}{c}\frac{c}{rS}\frac{dv^*}{dr}-\frac{Ac}{rSB^2}\frac{dv^*}{dr}\theta+\frac{c}{S}\frac{d^2v^*}{dr^2}-\frac{cA}{SB^2}\frac{d^2v^*}{dr^2}\theta+\frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3\\+3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2}-P\Lambda\left(\frac{dv^*}{dr}\right)^2v-\frac{Pvc}{S}-\frac{PAc}{sB^2}v^*\theta^*-c\end{array}\right]. \qquad (3.43)$$

$$\mathcal{N}_4\left[v^*(r,p),\theta^*(r,p)\right] = \left[\frac{1}{r}\frac{d\theta^*}{dr}+\frac{d^2\theta^*}{dr^2}-\frac{\Gamma c}{S}\left(\frac{dv^*}{dr}\right)^2+\frac{\Gamma cA}{SB^2}\theta\left(\frac{dv^*}{dr}\right)^2 +\Lambda\Gamma\left(\frac{dv^*}{dr}\right)^4\right]. \qquad (3.44)$$

The mth order deformation problems are

$$\mathcal{L}_3\left[v_m(r)-\chi_m v_{m-1}(r)\right]=\hbar\mathcal{R}3_m(r), \qquad (3.45)$$

$$\mathcal{L}_3\left[v_m(r)-\chi_m v_{m-1}(r)\right]=\hbar\mathcal{R}4_m(r), \qquad (3.46)$$

$$v_m(1)=0,\quad v_m'(0)=0,\quad \theta_m(1)=0,\quad \theta_m'(0)=0, \qquad (3.47)$$

$$\begin{aligned}\mathcal{R}3_m(r) = & \left[\frac{-AC}{B^2S}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{d\theta_k}{dr}+\frac{c}{rs}\frac{dv_{m-1}}{dr}-\frac{Ac}{rsB^2}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\theta_k+\frac{c}{s}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{dv_k}{dr}-\right.\\ & \frac{cA}{sB^2}\frac{d^2v_{m-1}}{dr^2}\theta+\frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{dv_l}{dr}+\\ & 3\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{d^2v_l}{dr^2}-P\Lambda\sum_{k=0}^{m-1}\frac{d^2v_{m-1-k}}{dr^2}\frac{dv_k}{dr}-\frac{Pc}{s}\frac{dv_{m-1}}{dr}-\\ & \left.\frac{PAc}{sB^2}\sum_{k=0}^{m-1}\sum_{l=0}^{k}v_{m-1-k}\theta_k-c(1-\chi_m)\right],\end{aligned} \qquad (3.48)$$

$$\mathcal{R}4_m(r)=\left[\begin{array}{c}\frac{1}{r}\frac{\partial\theta_{m-1}}{\partial r}+\frac{\partial^2\theta_{m-1}}{\partial r^2}+\Gamma\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{dv_k}{dr}-\\ \Gamma B\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\theta_l\\ +\Lambda\Gamma\sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p_2=0}^{l}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{dv_{l-p_2}}{dr}\frac{dv_{p_2}}{dr}\end{array}\right]. \qquad (3.49)$$

The above equation can be solved by Mathematica. The mth solution is given by

$$V_m = \sum_{n=0}^{6m+2} a_{m,n} r^n, \qquad \theta_m = \sum_{n=0}^{6m+2} b_{m,n} r^n, \qquad m \geq 0, \tag{3.50}$$

$$V_m = \sum_{n=0}^{6m+2} \overline{a}_{m,n} r^n, \qquad \theta_m = \sum_{n=0}^{6m+2} \overline{b}_{m,n} r^n, \qquad m \geq 0. \tag{3.51}$$

The related coefficients are

$$I1m, n = \sum_{\max\{0,n-6k-2\}}^{\min\{n,6m-6k-4\}} e_{m-1-k,n_1} h_{k,n-n_1}, \tag{3.52}$$

$$I2m, n = \sum_{n_3=\max\{0,n-6k-2\}}^{\min\{n,6m-6k-4\}} e1_{m-1-k,n_3} a_{k,n-n_3}, \tag{3.53}$$

$$I3m, n = \sum_{n_5=\max\{0,n-6k-2\}}^{\min\{n,6m-6k-4\}} f_{m-1-k,n_5} a_{k,n-n_5}, \tag{3.54}$$

$$I4m, n = \sum_{n_7=0}^{6m-6k-4} \sum_{n_8=\max\{0,n-6k-2\}}^{\min\{n,6m-6k-4\}} e_{m-1-k,n_7} g_{k-l,n_8} a_{l,n-n_7-n_8}, \tag{3.55}$$

$$I5m, n = \sum_{n_{10}=0}^{6m-6k-4} \sum_{n_{11}=\max\{0,n-6k-2\}}^{\min\{n,6m-6k-4\}} e_{m-1-k,n_{10}} g_{k-l,n_{11}} i_{l,n-n_{11}-n_{10}}, \tag{3.56}$$

$$I6m, n = \sum_{n_{13}=\max\{0,n-6k-2\}}^{\min\{n,6m-6k-4\}} i_{m-1-k,n_{13}} b_{k,n-n_{13}}, \tag{3.57}$$

$$I7m, n = \sum_{n_{17}=0}^{6l+2} \sum_{n_{16}=\max\{0,n-n_{17}-6m+6k+4\}}^{\min\{n-n_{17},6k-6l+2\}} e_{m-1-k,n-n_{17}-n_{16}} g_{k-l,n_{16}} a_{l,n_{17}}, \tag{3.58}$$

$$c1m, n = c1_{m-1,n+1}, \tag{3.59}$$

$$I8m, n = I2_{m,n+1}, \tag{3.60}$$

$$\overline{I}m, n = \sum_{n_{21}=\max\{0,n-6m+6k+4\}}^{\min\{n,6k+2\}} \overline{c1}_{m-1-k,n_{20}} \overline{d}_{k,n_{21}}, \tag{3.61}$$

$$I9m, n = I4_{m,n+1}, \tag{3.62}$$

$$\overline{I2}m,n=\sum_{n_{23}=\max\{0,n-6m+6k+4\}}^{\min\{n,6k+2\}}\overline{c1}_{m-1-k,n-n_{23}}\overline{b}_{k,n_{23}},\tag{3.63}$$

$$\overline{I3}m,n=\sum_{n_{23}=\max\{0,n-6m+6k+4\}}^{\min\{n,6k+2\}}\overline{e}_{m-1-k,n-n_{23}}\overline{b}_{k,n_{23}},\tag{3.64}$$

$$\overline{I4}m,n=\sum_{n_{26}=\max\{0,n-n_{27}-6m+6k+4\}}^{\min\{n-n_{27},6k-6l+2\}}\sum_{n_{27}=0}^{6l+2}\overline{c1}_{l,n_{27}}\overline{c}_{m-1-k,n-n_{26}-n_{27}}\overline{c}_{l,n_{27}},\tag{3.65}$$

$$\overline{I5}m,n=\sum_{n_{28}=\max\{0,n-n_{29}-6m+6k+4\}}^{\min\{n-n_{29},6k-6l+2\}}\sum_{n_{29}=0}^{6l+2}\overline{e}_{m,n_{27}}\overline{c}_{m-1-k,n-n_{28}-n_{29}}\overline{c}_{k-l,n_{26}},\tag{3.66}$$

$$\overline{I6}m,n=\sum_{n_{30}=\max\{0,n-6m+6k+4\}}^{\min\{n,6k+2\}}\overline{e}_{m,n-n_{30}}\overline{a}_{k,n_{30}},\tag{3.67}$$

$$\overline{I7}m,n=\sum_{n_{31}=\max\{0,n-6m+6k+4\}}^{\min\{n,6k+2\}}\overline{a}_{m-1-k,n_{30}}\overline{b}_{k,n-n_{30}},\tag{3.68}$$

$$\overline{\overline{I1}}m,n=\sum_{n_{41}=\max\{0,n-n_{42}-6m+6k+4\}}^{\min\{n-n_{42},6k-6l+2\}}\sum_{n_{42}=0}^{6l+2}\overline{\overline{e}}_{m-1-k,n-n_{41}-n_{42}}\overline{\overline{e}}_{k-l,n_{41}}\overline{\overline{b}}_{l,n_{42}},\tag{3.69}$$

$$\overline{\overline{I2}}m,n=\sum_{n_{43}=\max\{0,n-6k-4\}}^{\min\{n,6m-6k-4\}}\overline{\overline{e}}_{m-1-k,n-n_{43}}\overline{\overline{e}}_{k,n_{43}},\tag{3.70}$$

$$\overline{\overline{I3}}m,n=\sum_{n_{46}=\max\{0,n-n_{47}-n_{48}-6m+6k+4\}}^{\min\{n-n_{47}-n_{48},6k-6p+2\}}\sum_{n_{47}=0}^{6(p-l)+2}\sum_{n_{48}=0}^{6l+2}\overline{\overline{e}}_{m-1-k,n-n_{46}-n_{47}-n_{48}}\overline{\overline{e}}_{k-l,n_{46}}\overline{\overline{e}}_{l-p,n_{47}}\overline{\overline{e}}_{p,n_{48}},\tag{3.71}$$

$$\overline{\overline{\overline{I1}}}m,n=\sum_{n_{49}=\max\{0,n-n_{50}-6m+6k+4\}}^{\min\{n-n_{50},6k-6l+2\}}\sum_{n_{50}=0}^{6l+2}\overline{\overline{\overline{e}}}_{m-1-k,n-n_{49}-n_{50}}\overline{\overline{\overline{e}}}_{k-l,n_{49}}\overline{\overline{\overline{b}}}_{l,n_{50}},\tag{3.72}$$

$$\overline{\overline{\overline{I2}}}m,n=\sum_{n_{51}=\max\{0,n-6k-4\}}^{\min\{n,6m-6k-4\}}\overline{\overline{\overline{e}}}_{m-1-k,n-n_{51}}\overline{\overline{\overline{e}}}_{k,n_{51}},\tag{3.73}$$

$$\overline{\overline{\overline{I3}}}m,n=\sum_{n_{52}=\max\{0,n-n_{53}-n_{54}-6m+6k+4\}}^{\min\{n-n_{53}-n_{54},6k-6p+2\}}\sum_{n_{53}=0}^{6(p-l)+2}\sum_{n_{54}=0}^{6l+2}\overline{\overline{\overline{e}}}_{m-1-k,n-n_{52}-n_{53}-n_{54}}\overline{\overline{\overline{e}}}_{k-l,n_{52}}\overline{\overline{\overline{e}}}_{l-p,n_{53}}\overline{\overline{\overline{e}}}_{p,n_{54}},\tag{3.74}$$

and

$$\Gamma_{m,n} = \big[\chi_{6m-2-n}c1m,n + k_{m-1,n} - B\chi_{6m-n}I1m,n - B\chi_{6m-n}I8m,n - B\chi_{6m-n}I3m,n$$
$$+\Lambda\chi_{6m+2-n}I9m,n + 3\Lambda\chi_{6m+2-n}I5m,n + pB\chi_{6m+2-n}I6m,n - p\chi_{6m-2-n}a_{m-1,n}$$
$$-p\Lambda\chi_{6m+2-n}I7m,n - c(1-\chi_m)\big], \tag{3.75}$$

$$\overline{\Gamma}_{m,n} = \left[\frac{-AC}{B^2S}\chi_{6m-n}\overline{I}m,n + \frac{C}{S}\chi_{6m-n-2}\overline{c}1_{m-1,n} - \frac{AC}{SB^2}\chi_{6m-n}\overline{I}1m,n + \frac{C}{S}\chi_{6m-n-2}\overline{e}_{m-1,n}\right.$$
$$\frac{AC}{SB^2}\chi_{6m-n}\overline{I}3m,n + \Lambda\chi_{6m+2-n}\overline{\overline{I}}4m,n + 3\Lambda\chi_{6m-n+2}\overline{I}5m,n - p\Lambda\chi_{6m-n}\overline{I}6m,n -$$
$$\left.\frac{pC}{S}\chi_{6m-n-2}\overline{c}_{m-1,n} - \frac{pAC}{SB^2}\chi_{6m-n}\overline{I}7_{m,n}\right] \tag{3.76}$$

$$\overline{\overline{\Gamma}}_{m,n} = \chi_{6m-2-n}\overline{\overline{c1}}_{m-1,n} + \chi_{6m-2-n}\overline{\overline{d}}_{m-1,n} - B\Gamma\chi_{6m+2-n}\overline{\overline{I1}}m,n + \Gamma\chi_{6m+2-n}\overline{\overline{I2}}m,n + \Lambda\Gamma\overline{\overline{I3}}m,n \tag{3.77}$$

$$\overline{\overline{\overline{\Gamma}}}_{m,n} = \chi_{6m-2-n}\overline{\overline{\overline{c1}}}_{m-1,n} + \chi_{6m-2-n}\overline{\overline{\overline{d}}}_{m-1,n} - B\Gamma\chi_{6m+2-n}\overline{\overline{\overline{I1}}}m,n + \Gamma\chi_{6m+2-n}\overline{\overline{\overline{I2}}}m,n + \Lambda\Gamma\overline{\overline{\overline{I3}}}m,n \tag{3.78}$$

$$a_{m,0} = \chi_m\chi_{6m-n}a_{m-1,0} - \sum_{n=0}^{6m+2}\frac{\Gamma_{m,n}}{(n+2)^2} - \frac{C}{4}, \tag{3.79}$$

$$a_{m,1} = \chi_m\chi_{6m-3}a_{m-1,1}, \tag{3.80}$$

$$a_{m,2} = \chi_m\chi_{6m-4}a_{m-1,n} + \frac{\Gamma_{m,0}}{4} + \frac{C}{4}, \tag{3.81}$$

$$a_{m,n} = \chi_m\chi_{6m-n-2}a_{m-1,n} + \sum_{n=2}^{6m+2}\frac{\Gamma_{m,n}}{n^2}, \tag{3.82}$$

$$\overline{a}_{m,0} = \chi_m\chi_{6m-n}\overline{a}_{m-1,0} - \sum_{n=0}^{6m+2}\frac{\overline{\Gamma}_{m,n}}{(n+2)^2} - \frac{C}{4}, \tag{3.83}$$

$$\overline{a}_{m,2} = \chi_m\chi_{6m-4}\overline{a}_{m-1,n} + \frac{\overline{\Gamma}_{m,0}}{4} + \frac{C}{4}, \tag{3.84}$$

$$\overline{a}_{m,n} = \chi_m\chi_{6m-n-2}\overline{a}_{m-1,n} + \sum_{n=2}^{6m+2}\frac{\overline{\Gamma}_{m,n}}{n^2}, \tag{3.85}$$

$$\overline{\overline{a}}_{m,0} = \chi_m\chi_{6m-n}\overline{a}_{m-1,0} - \sum_{n=0}^{6m+2}\frac{\overline{\Gamma}_{m,n}}{(n+2)^2} - \frac{C}{4}, \tag{3.86}$$

$$\overline{\overline{a}}_{m,2} = \chi_m \chi_{6m-4} \overline{\overline{a}}_{m-1,n} + \frac{\overline{\overline{\Gamma}}_{m,0}}{4} + \frac{C}{4}, \tag{3.87}$$

$$\overline{\overline{a}}_{m,n} = \chi_m \chi_{6m-n-2} \overline{\overline{a}}_{m-1,n} + \sum_{n=2}^{6m+2} \frac{\overline{\overline{\Gamma}}_{m,n}}{n^2}, \tag{3.88}$$

$$\overline{\overline{\overline{a}}}_{m,0} = \chi_m \chi_{6m-n} \overline{\overline{\overline{a}}}_{m-1,0} - \sum_{n=0}^{6m+2} \frac{\overline{\overline{\overline{\Gamma}}}_{m,n}}{(n+2)^2} - \frac{C}{4}, \tag{3.89}$$

$$\overline{\overline{\overline{a}}}_{m,2} = \chi_m \chi_{6m-4} \overline{\overline{\overline{a}}}_{m-1,n} + \frac{\overline{\overline{\overline{\Gamma}}}_{m,0}}{4} + \frac{C}{4}, \tag{3.90}$$

$$\overline{\overline{\overline{a}}}_{m,n} = \chi_m \chi_{6m-n-2} \overline{\overline{\overline{a}}}_{m-1,n} + \sum_{n=2}^{6m+2} \frac{\overline{\overline{\overline{\Gamma}}}_{m,n}}{n^2}, \tag{3.91}$$

$$a_{m-1,n+1} = c1_{m-1,n}, \tag{3.92}$$

$$d_{m-1,n} = (n+1)c1_{m-1,n}, \tag{3.93}$$

$$e_{m-1-k,n} = (n+1)a_{m-1-k,n+1}, \tag{3.94}$$

$$f_{m-1-k,n} = (n+1)e_{m-1-k,n+1}, \tag{3.95}$$

$$g_{k-l,n} = (n+1)a_{k-l,n+1}, \tag{3.96}$$

$$h_{k-l,n} = (n+1)b_{k-l,n+1}, \tag{3.97}$$

$$\overline{c1} = (n+1)\overline{a}_{m,n+1}, \tag{3.98}$$

$$\overline{d}_{m-1,n} = (n+1)\overline{b}_{m-1,n}, \tag{3.99}$$

$$\overline{e}_{m,n} = (n+1)\overline{c1}, \tag{3.100}$$

$$\overline{\overline{c1}} = (n+1)\overline{\overline{a}}_{m,n+1}, \tag{3.101}$$

$$\overline{\overline{d}}_{m-1,n} = (n+1)\overline{\overline{b}}_{m-1,n} \tag{3.102}$$

$$\overline{\overline{e}}_{m,n} = (n+1)\overline{\overline{c1}}, \tag{3.103}$$

With the recurrence formulas, we can calculate all the coefficients $a_{m,n}$, $b_{m,n}$, $\overline{a}_{m,n}$, $\overline{b}_{m,n}$, $\overline{\overline{a}}_{m,n}$,

$\bar{\bar{b}}_{m,n}$, $\bar{\bar{\bar{a}}}_{m,n}$, $\bar{\bar{\bar{b}}}_{m,n}$ using only

$$\begin{aligned} \bar{\bar{\bar{a}}}_{0,0} &= \bar{\bar{a}}_{0,0} = \bar{a}_{0,0} = a_{0,0} = \frac{C}{4}r^2, \quad \bar{\bar{\bar{a}}}_{0,1} = \bar{\bar{a}}_{0,1} = \bar{a}_{0,1} = a_{0,1} = -\frac{C}{4}, \\ b_{0,0} &= \bar{b}_{0,0} = \bar{\bar{b}}_{0,0} = \bar{\bar{\bar{b}}}_{0,0} = \frac{C}{2}r^2, \quad b_{0,1} = \bar{b}_{0,1} = \bar{\bar{b}}_{0,1} = \bar{\bar{\bar{b}}}_{0,1} = -\frac{C}{2}, \end{aligned} \tag{3.104}$$

given by the initial guess approximations for the solutions of V and θ for Reynold and Vogel models in Eq. (3.12). The corresponding Mth-order approximations of are

$$\sum_{m=0}^{M} v_m(r) = a_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} a_{m,n} r^n, \tag{3.105}$$

$$\sum_{m=0}^{M} \theta_m(r) = b_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} b_{m,n} r^n, \tag{3.106}$$

$$\sum_{m=0}^{M} v_m(r) = \bar{a}_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} \bar{a}_{m,n} r^n, \tag{3.107}$$

$$\sum_{m=0}^{M} \theta_m(r) = \bar{b}_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} \bar{b}_{m,n} r^n, \tag{3.108}$$

and the explicit analytic solutions are

$$v(r) = \lim_{M->\infty} \left(a_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} a_{m,n} r^n \right), \tag{3.109}$$

$$\theta(r) = \lim_{M->\infty} \left(b_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} b_{m,n} r^n \right), \tag{3.110}$$

$$\theta(r) = \lim_{M->\infty} \left(\bar{b}_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} \bar{b}_{m,n} r^n \right), \tag{3.111}$$

$$v(r) = \lim_{M->\infty} \left(\bar{a}_{0,0} + \sum_{m=n-1}^{2M} \sum_{n=0}^{6m+2} \bar{a}_{m,n} r^n \right). \tag{3.112}$$

3.3 Convergence of the analytic solution

The explicit, analytic expression contain Eqs. (3.37) and (3.38) the auxiliary parameter $\hbar$. As pointed out by Liao [37], the convergence region and rate of approximations given by the HAM are strongly dependent upon the auxiliary parameter. In Fig. 3.1 and 3.2 the $\hbar$−curves are plotted to see the range of admissible values for the parameter $\hbar$. It is clear from figures 3.1 and 3.2 that the range for the admissible values for $\hbar$ is $-1.5 \leq \hbar \leq 0$. The solution given in Eqs. (3.37) and (3.38) converges in the whole region of r, when $\hbar$ is in the neighborhood of -1.0. It is also observed that the interval for the values of $\hbar$ converges to value -1.0 as the rest of the parameter keeping fixed.

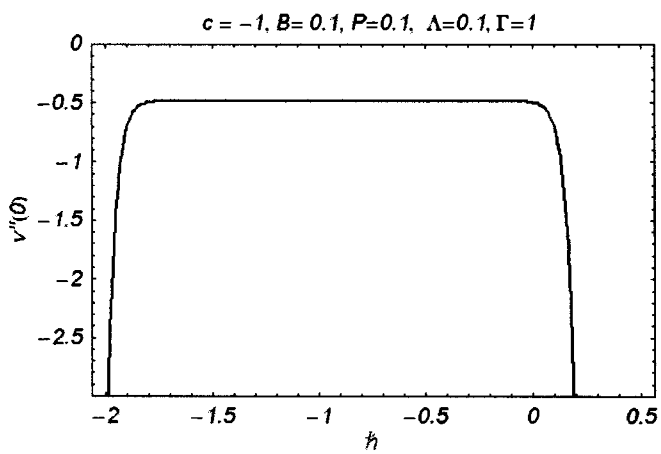

Fig. 3.1 $\hbar$−curve for Reynold model in 30th order approximation.

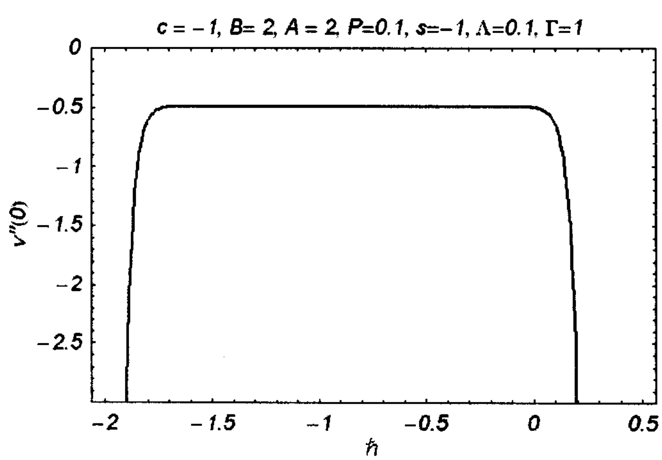

Fig. 3.2 $\hbar$−curve for Vogel model in 30^{th} order approximation.

Case-I: The graphs for Reynold model have been drawn Below.

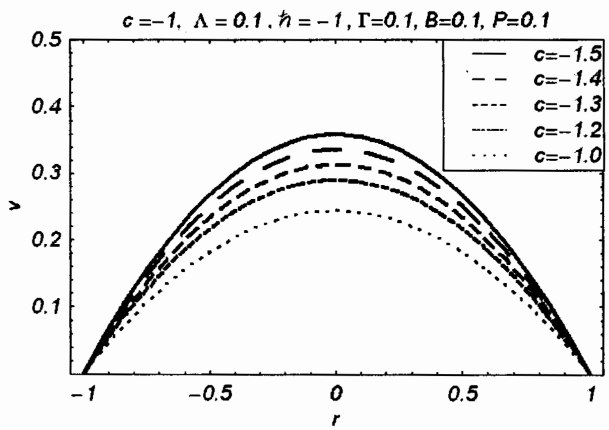

Fig.3.3 Influence of c on the velocity

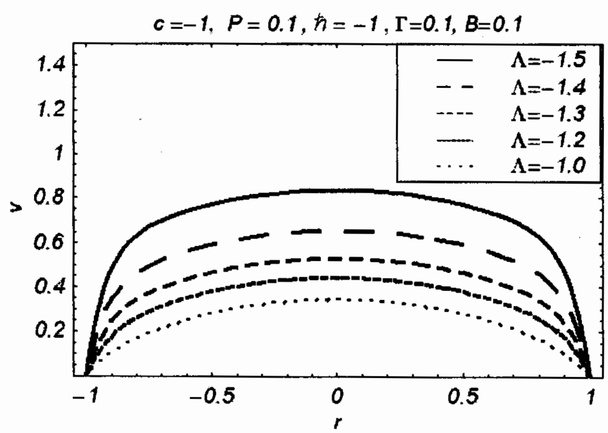

Fig. 3.4 Influence of Λ on the velocity

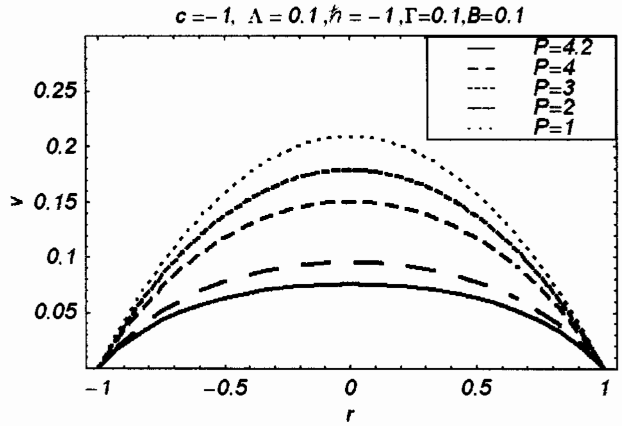

Fig. 3.6 Influence of P on the velocity profile.

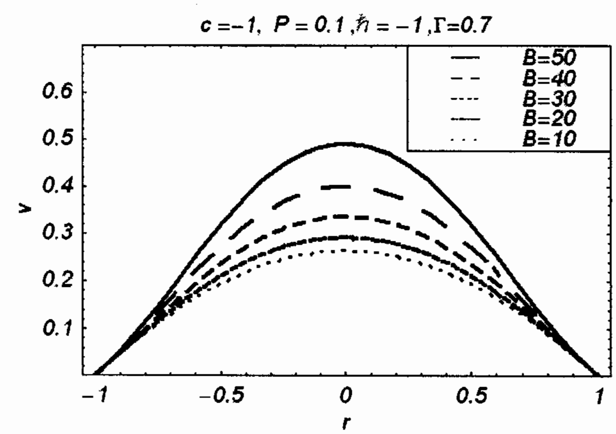

Fig. 3.7 Influence of B on the velocity.

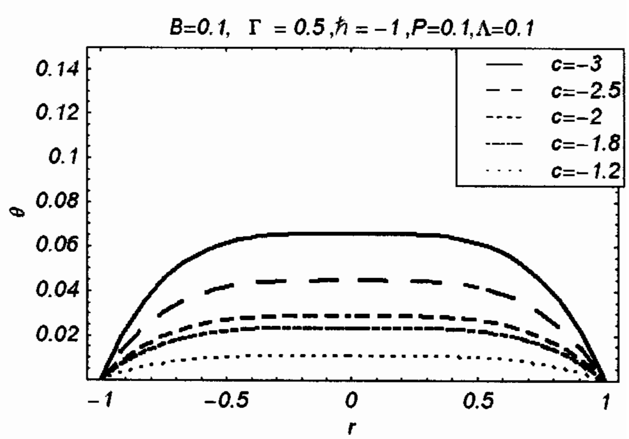

Fig. 3.8 Influence of c on the temparature

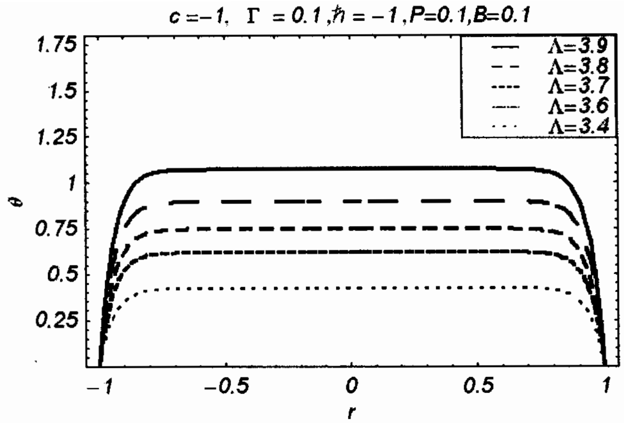

Fig. 3.9 Influence of Λ on temparature

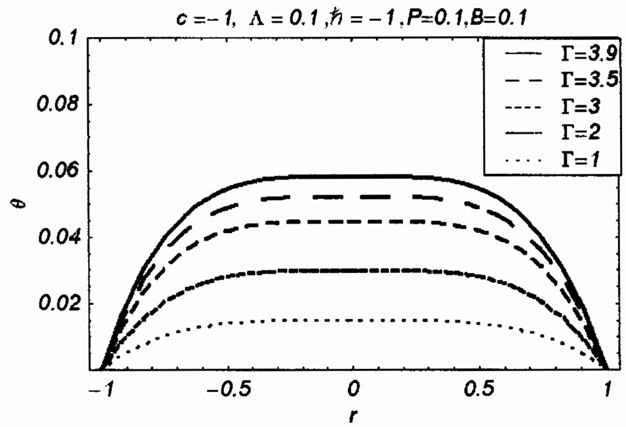

Fig. 3.10 Influence of Γ on the temparature

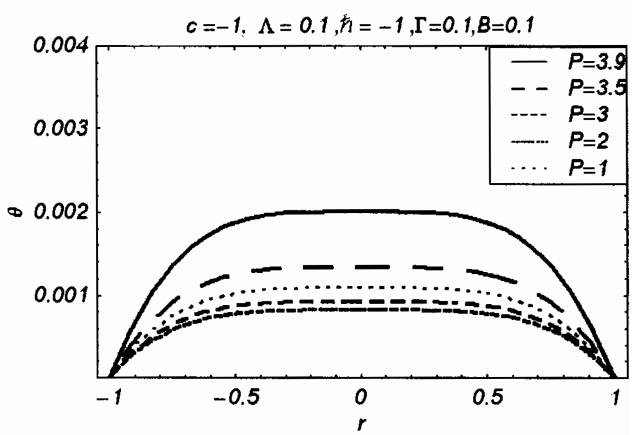

Fig. 3.11 Influence of P on the temparature

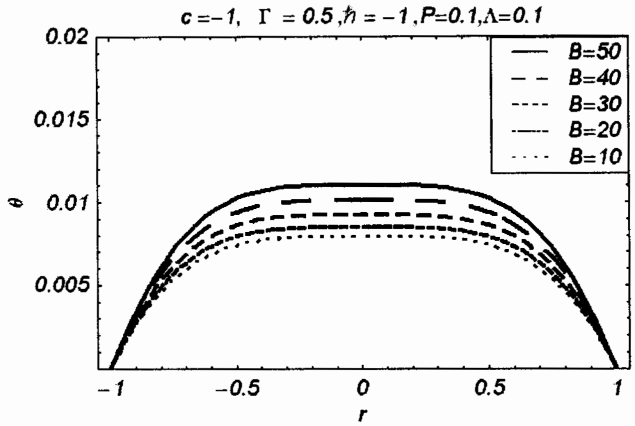

Fig. 3.12 Influence of B on the temparature

The graphs for Vogel model have been drawn Below.

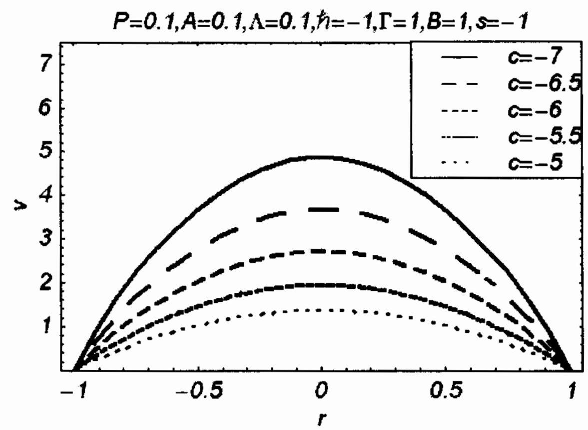

Fig. 3.13 Influence of c on the velocity

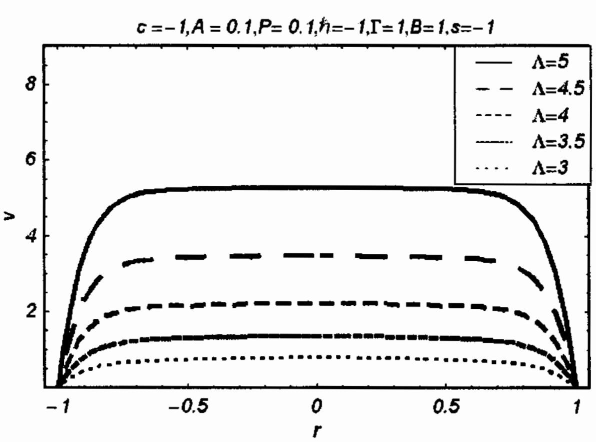

Fig. 3.14 Influence of Λ on the velocity

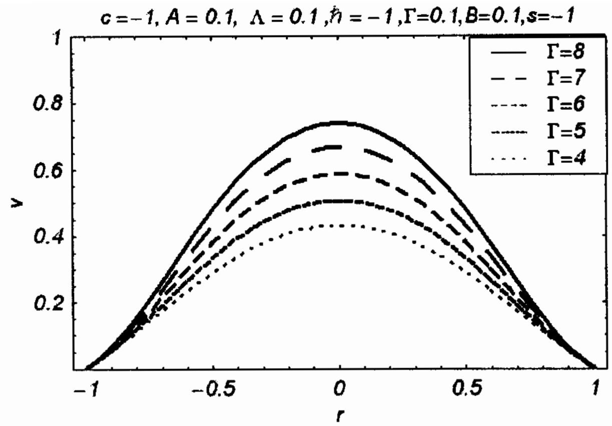

Fig. 3.15 Influence of Γ on the velocity

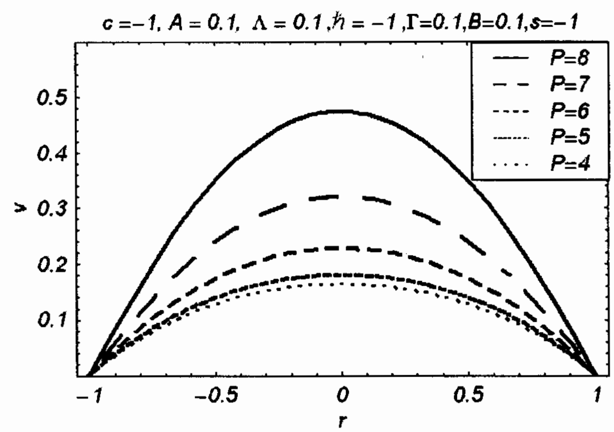

Fig. 3.16 Influence of P on the velocity

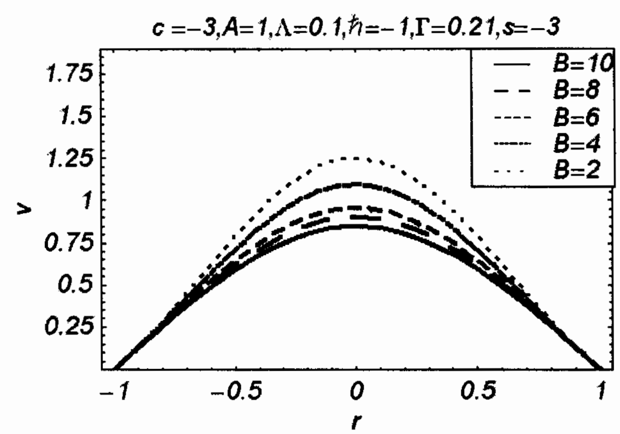

Fig. 3.17 Influence of B on the velocity

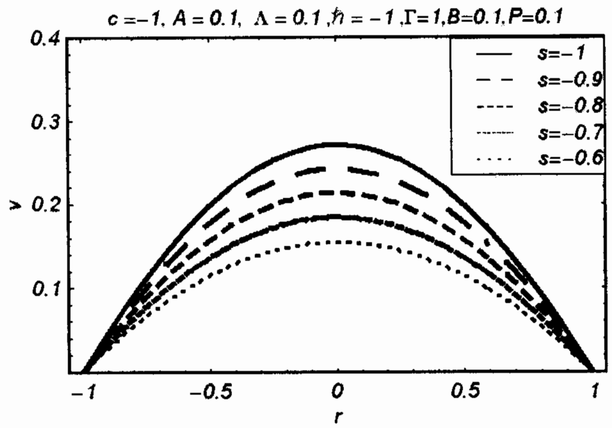

Fig. 3.18 Influence of s on the velocity

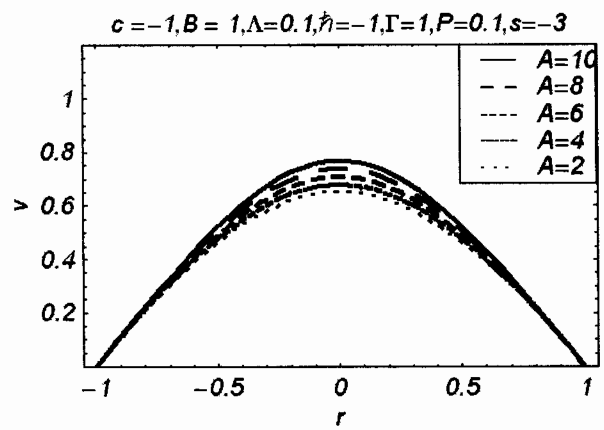

Fig. 3.19 Influence of A on the velocity

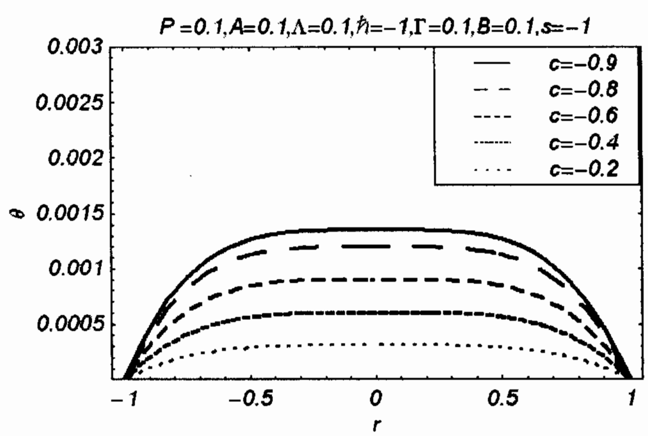

Fig. 3.20 Influence of c on the temparature

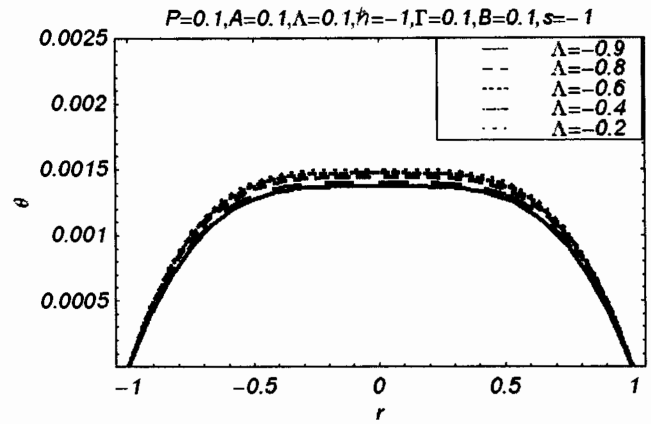

Fig. 3.21 Influence of Λ on the temparature

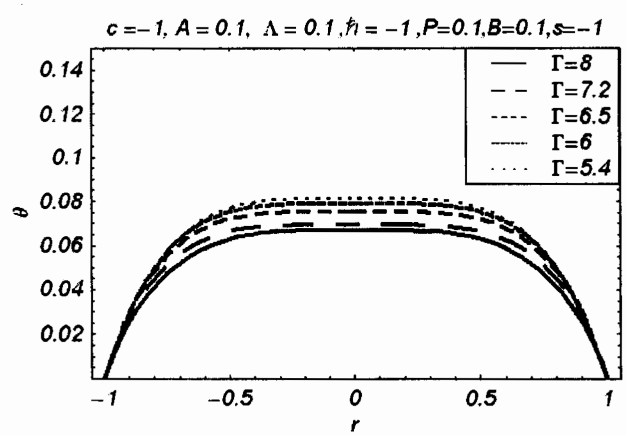

Fig. 3.22 Influence of Γ on the temparature

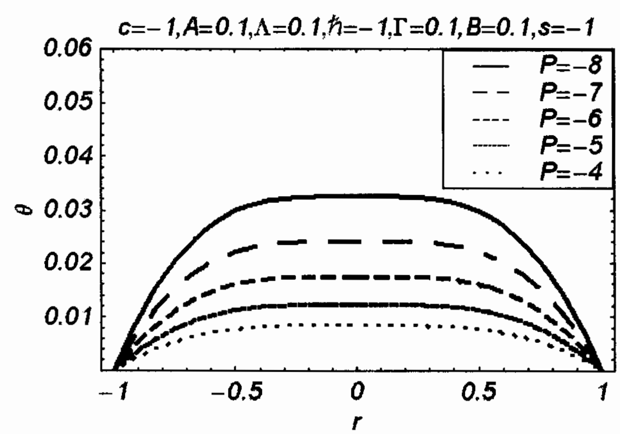

Fig. 3.23 Influence of P on the temperature.

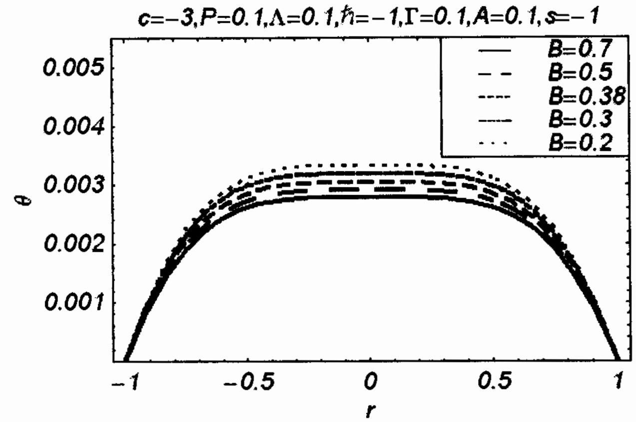

Fig. 3.24 Influence of B on the temperature.

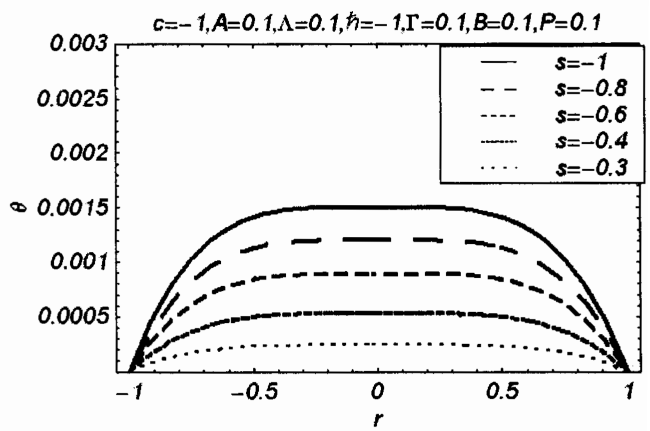

Fig. 3.25 Influence of s on the temparature

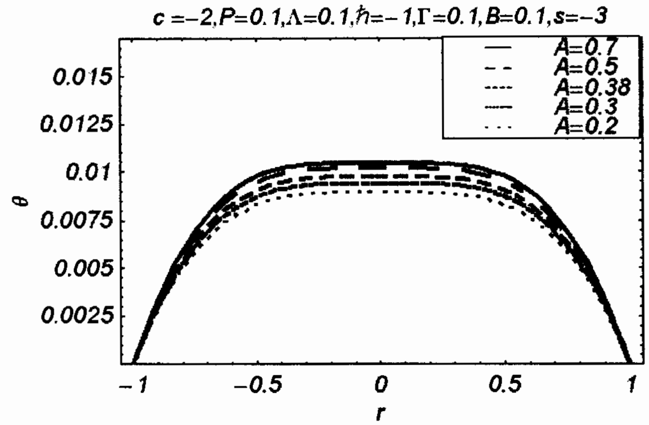

Fig. 3.26 Influence of A on the temparature

3.4 Discussion

Here, the solution for the velocity and temperature distributions are plotted against the pipe radius. The velocity and temperature distributions for Reynold model are displayed in Figures 3.3 to 3.12 and for Vogel model Figures 3.13 to 3.26 are plotted. In these Figures, the variation of the emerging parameters c, Λ, Γ, P and B are taken into account.

In Figure 3.3, the effect of c is depicted when $\hbar = -1$. It can be seen that the maximum velocity of the fluid increases at the centre of the pipe with the more negative values of c. Also, the effect of c on θ (in Figure 3.8) is similar to that of velocity. The effect of third grade parameter Λ on the velocity and temperature distributions are shown in Figures 3.4 and 3.9 respectively. As expected an increase in Λ results in a decrease in velocity and increase for temperature. However, the profiles are more flatter in case of temperature distributions as compare to the velocity distributions .The effect of Γ on the velocity and temperature distributions are shown in Figures 3.5 and 3.10 respectively. It is found that velocity and temperature distribution increase with the increase in Γ. The effect of P on the velocity and temperature distributions are shown in Figures 3.6 and 3.11 respectively.We observe that with the increase in P the velocity and temperature increases. The effect of B on the velocity and temperature distributions are shown in Figures 3.7 and 3.12, respectively when $\hbar = -1$. We observe that with the increase in B the velocity decreases but the temperature increases.

Up to yet we discussed the results of the velocity and temperature for Reynold Model.

Now, we turn our discussion of the results of the velocity and temperature distributions for Vogel Model. For this case, the velocity and temperature distributions are plotted against the pipe radius in Figures 3.13 to 3.26, respectively for various values of c, Λ, Γ, P and B. From these Figures, it is noted that the observations are similar to that of Reynold Model .

In this chapter an analytic technique for nonlinear problems in general, namely the HAM given by Liao [9] is employed to obtain the analytical solution for the pipe flow of a third grade fluid with variable viscosity. The expressions for velocity and temperature distributions are given into two cases. The important parameters in this study are c, Λ, Γ, P and B. The validity of this analytic solutions is confirmed by the graphical results.

To the best of our knowledge, such kind of analytic solutions has never been reported. The explicit analytic solutions might be fruitful and new contribution in literature.

Appendix A

The related coefficients are given by

$$C_3 = \frac{\Gamma}{4} c^2 (7 + \hbar(4 + \hbar))^2,$$

$$\begin{aligned} C_4 \;=\; & \frac{\Gamma}{1048576} (131072 c^4 \hbar (3 + \hbar)(6 + \hbar)(7 + \hbar(4 + \hbar))\Lambda + 3211264 c^4 (7 + \hbar(4 + \hbar))^2 \Lambda \\ & + 917504 c^4 \hbar (4 + \hbar)(7 + \hbar(4 + \hbar))^2 \Lambda + 65536 c^4 \hbar^2 (4 + \hbar)^2 (7 + \hbar(4 + \hbar))^2 \Lambda), \end{aligned}$$

$$\begin{aligned} C_5 \;=\; & \frac{\Gamma}{1048576} (16384 c^4 \hbar^2 (3 + \hbar)^2 (6 + \hbar)^2 \Lambda^2 + 196608 c^6 \hbar^2 (4 + \hbar)(7 + \hbar(4 + \hbar))\Lambda^2 \\ & + 1605632 c^6 \hbar (3 + \hbar)(6 + \hbar)(7 + \hbar(4 + \hbar))\Lambda^2 + \\ & 458752 c^6 \hbar^2 (3 + \hbar)(4 + \hbar)(6 + \hbar)(7 + \hbar(4 + \hbar))\Lambda^2 + \\ & 32768 c^6 \hbar^3 (3 + \hbar)(4 + \hbar)^2 (6 + \hbar)(7 + \hbar(4 + \hbar))\Lambda^2 \\ & + 229376 c^6 \hbar (3 + \hbar)(6 + \hbar)(7 + \hbar(4 + \hbar))\Lambda^2 + \\ & 32768 c^6 \hbar^2 (3 + \hbar)(4 + \hbar)(6 + \hbar)(7 + \hbar(4 + \hbar))^2 \Lambda^2), \end{aligned}$$

$$\begin{aligned} C_6 \;=\; & \frac{\Gamma}{1048576} (49152 c^8 \hbar^3 (3 + \hbar)(4 + \hbar)(6 + \hbar)\Lambda^3 + 57344 c^8 \hbar \,(3 + \hbar)^2 (4 + \hbar)\,(6 + \hbar)^2 \Lambda^3 \\ & + 4096 c^8 \hbar^4 (3 + \hbar)^2 (4 + \hbar)^2 (6 + \hbar)^2 \Lambda^3 + 98304 c^8 \hbar^3 (7 + \hbar\,(4 + \hbar)) \\ & + 2408448 c^8 \hbar^2 (4 + \hbar)\,(7 + \hbar\,(4 + \hbar))\Lambda^3 + 688128 c^8 \hbar^3 (4 + \hbar)^2 (7 + \hbar\,(4 + \hbar))\Lambda^3 \\ & + 49152 c^8 \hbar^4 (4 + \hbar)^3 (7 + \hbar\,(4 + \hbar))\Lambda^3 + 114688 c^8 \hbar^2 (3 + \hbar)^2 (6 + \hbar)^2 (7 + \hbar\,(4 + \hbar))\Lambda^3 \\ & + 16384 c^8 \hbar^3 (3 + \hbar)^2 (4 + \hbar)\,(6 + \hbar)^2 (7 + \hbar\,(4 + \hbar))\Lambda^3 + 344064 c^8 \hbar^2 (4 + \hbar)\,(7 + \hbar\,(4 + \hbar))\Lambda^3 \\ & + 49152 c^8 \hbar^3 (4 + \hbar)^2 (7 + \hbar\,(4 + \hbar))^2 \Lambda^3 + 4096 c^8 \hbar^2 (3 + \hbar)^2 (6 + \hbar)^2 (7 + \hbar\,(4 + \hbar))^2 \Lambda^3), \end{aligned}$$

$$C_7 = \frac{\Gamma}{1048576}(36864c^{10}\hbar^4(4+\hbar)^2\Lambda^4 + 24576c^{10}\hbar^4(3+\hbar)(6+\hbar)\Lambda^4$$
$$+602112c^{10}\hbar^3(3+\hbar)(4+\hbar)(6+\hbar)\Lambda^4 + 172032c^{10}\hbar^4(3+\hbar)(4+\hbar)^2(6+\hbar)\Lambda^4$$
$$+12288c^{10}\hbar^5(3+\hbar)(4+\hbar)^3(6+\hbar)\Lambda^4 + 14336c^{10}\hbar^3(3+\hbar)^3(6+\hbar)^3\Lambda^4$$
$$+2048c^{10}\hbar^4(3+\hbar)^3(4+\hbar)(6+\hbar)^3\Lambda^4 + 1204224c^{10}\hbar^3(7+\hbar(4+\hbar))\Lambda^4$$
$$+344064c^{10}\hbar^4(4+\hbar)(7+\hbar(4+\hbar))\Lambda^4 + 24576c^{10}\hbar^5(4+\hbar)^2(7+\hbar(4+\hbar))\Lambda^4$$
$$+344064c^{10}\hbar^3(3+\hbar)(4+\hbar)(6+\hbar)(7+\hbar(4+\hbar))\Lambda^4$$
$$+49152c^{10}\hbar^4(3+\hbar)(4+\hbar)^2(6+\hbar)(7+\hbar(4+\hbar))\Lambda^4$$
$$+2048c^{10}\hbar^3(3+\hbar)^3(6+\hbar)^3(7+\hbar(4+\hbar))\Lambda^4 + 172032c^{10}\hbar^3(7+\hbar(4+\hbar))^2\Lambda^4$$
$$+24576c^{10}\hbar^4(4+\hbar)(7+\hbar(4+\hbar))^2\Lambda^4 +$$
$$12288c^{10}\hbar^3(3+\hbar)(4+\hbar)(6+\hbar)(7+\hbar(4+\hbar))^2\Lambda^4),$$

$$C_8 = \frac{\Gamma}{1048576}(36864c^{12}\hbar^5(4+\hbar)\Lambda^5 + 451584c^{12}\hbar^4(4+\hbar)^2\Lambda^5$$
$$+129024c^{12}\hbar^5(4+\hbar)^3\Lambda^5 + 9216c^{12}\hbar^6(4+\hbar)^4\Lambda^5$$
$$+301056c^{12}\hbar^4(3+\hbar)(6+\hbar)\Lambda^5 + 86016c^{12}\hbar^5(3+\hbar)(4+\hbar)(6+\hbar)\Lambda^5$$
$$+6144c^{12}\hbar^6(3+\hbar)(4+\hbar)^2(6+\hbar)\Lambda^5 + 64512c^{12}\hbar^4(3+\hbar)^2(4+\hbar)(6+\hbar)^2\Lambda^5$$
$$+9216c^{12}\hbar^5(3+\hbar)^2(4+\hbar)^2(6+\hbar)^2\Lambda^5 + 256c^{12}\hbar^4(3+\hbar)^4(6+\hbar)^4\Lambda^5 +$$
$$258048c^{12}\hbar^4(4+\hbar)^2(7+\hbar(4+\hbar))\Lambda^5 + 36864c^{12}\hbar^5(4+\hbar)^3(7+\hbar(4+\hbar))\Lambda^5 +$$
$$172032c^{12}\hbar^4(3+\hbar)(6+\hbar)(4+\hbar)^2(7+\hbar(4+\hbar))\Lambda^5 +$$
$$24576c^{12}\hbar^5(3+\hbar)(4+\hbar)(6+\hbar)(7+\hbar(4+\hbar))\Lambda^5 +$$
$$9216c^{12}\hbar^4(3+\hbar)^2(4+\hbar)(6+\hbar)^2(7+\hbar(4+\hbar))\Lambda^5 +$$
$$9216c^{12}\hbar^4(4+\hbar)^2(7+\hbar(4+\hbar))^2\Lambda^5 +$$
$$6144c^{12}\hbar^4(3+\hbar)(6+\hbar)(7+\hbar(4+\hbar))\Lambda^5),$$

$$
\begin{aligned}
C_9 &= \frac{\Gamma}{1048576}(9216c^{14}\hbar^6\Lambda^6 + 451584c^{14}\hbar^5(4+\hbar)\Lambda^6 \\
&+129024c^{14}\hbar^6(4+\hbar)^2\Lambda^6 + 9216c^{14}\hbar^7(4+\hbar)^3\Lambda^6 \\
&+96768c^{14}\hbar^5(3+\hbar)(4+\hbar)^2(6+\hbar)\Lambda^6 + 13824c^{14}\hbar^6(3+\hbar)(4+\hbar)^3(6+\hbar)\Lambda^6 \\
&+32256c^{14}\hbar^5(3+\hbar)^2(6+\hbar)^2\Lambda^6 + 4608c^{14}\hbar^6(3+\hbar)^2(4+\hbar)(6+\hbar)^2\Lambda^6 \\
&+1536c^{14}\hbar^5(3+\hbar)^3(4+\hbar)(6+\hbar)^3\Lambda^6 + 258048c^{14}\hbar^5(4+\hbar)(7+\hbar(4+\hbar))\Lambda^6 + \\
&36864c^{14}\hbar^6(4+\hbar)^2(7+\hbar(4+\hbar))\Lambda^6 + 13824c^{14}\hbar^5(3+\hbar)(4+\hbar)^2(6+\hbar)(7+\hbar(4+\hbar))\Lambda^6 \\
&+4608c^{14}\hbar^5(3+\hbar)^2(6+\hbar)^2(7+\hbar(4+\hbar))\Lambda^6 + 9216c^{14}\hbar^5(4+\hbar)((7+\hbar(4+\hbar))\Lambda^6),
\end{aligned}
$$

$$
\begin{aligned}
C_{10} &= \frac{\Gamma}{1048576}(112896c^{16}\hbar^6\Lambda^7 + 32256c^{16}\hbar^7(4+\hbar)\Lambda^7 \\
&+2304c^{16}\hbar^8(4+\hbar)^2\Lambda^7 + 48384c^{16}\hbar^6(4+\hbar)^3\Lambda^7 + \\
&6912c^{16}\hbar^7(4+\hbar)^4\Lambda^6 + 96768c^{16}\hbar^6(3+\hbar)(4+\hbar)(6+\hbar)\Lambda^7 \\
&+13824c^{16}\hbar^7(3+\hbar)(6+\hbar)(4+\hbar)^2\Lambda^7 + 3456c^{16}\hbar^6(3+\hbar)^2(4+\hbar)^2(6+\hbar)^2\Lambda^7 \\
&+768c^{16}\hbar^6(3+\hbar)^3(6+\hbar)^3\Lambda^7 + 64512c^{16}\hbar^6(7+\hbar(4+\hbar))\Lambda^7 + \\
&9216c^{16}\hbar^7(4+\hbar)(7+\hbar(4+\hbar))\Lambda^7 + 6912c^{16}\hbar^6(4+\hbar)^3(7+\hbar(4+\hbar))\Lambda^7 + \\
&13824c^{16}\hbar^6(4+\hbar)(3+\hbar)(6+\hbar)(7+\hbar(4+\hbar))\Lambda^7 + 2304c^{16}\hbar^6(7+\hbar(4+\hbar))^2\Lambda^7),
\end{aligned}
$$

$$
\begin{aligned}
C_{11} &= \frac{\Gamma}{1048576}(72576c^{18}\hbar^7(4+\hbar)^2\Lambda^8 + 10368c^{18}\hbar^8(4+\hbar)^3\Lambda^8 \\
&+24192c^{18}\hbar^7(3+\hbar)(6+\hbar)\Lambda^8 + 3456c^{18}\hbar^8(3+\hbar)(4+\hbar)(6+\hbar)\Lambda^8 + \\
&3456c^{18}\hbar^7(3+\hbar)(4+\hbar)^3(6+\hbar)\Lambda^8 + 3456c^{18}\hbar^7(3+\hbar)^2(4+\hbar)(6+\hbar)^2\Lambda^8 + \\
&10368c^{18}\hbar^7(4+\hbar)^2(6+\hbar)^2\Lambda^8(7+\hbar(4+\hbar)) + \\
&3456c^{18}\hbar^7(3+\hbar)(6+\hbar)\Lambda^8(7+\hbar(4+\hbar)),
\end{aligned}
$$

$$C_{12} = \frac{\Gamma}{1048576}(36288c^{20}\hbar^8(4+\hbar)\Lambda^9 + 5184c^{20}\hbar^9(4+\hbar)^2\Lambda^9 +$$
$$+1296c^{20}\hbar^8(4+\hbar)^4\Lambda^9 + 5184c^{20}\hbar^8(3+\hbar)(4+\hbar)^2(6+\hbar)\Lambda^9 +$$
$$864c^{20}\hbar^8(3+\hbar)^2(6+\hbar)^2\Lambda^9 + 5184c^{20}\hbar^8(4+\hbar)\Lambda^9(7+\hbar 4+\hbar),$$

$$C_{13} = \frac{\Gamma}{1048576}(6048c^{22}\hbar^9\Lambda^{10} + 864c^{22}\hbar^{10}(4+\hbar)\Lambda^{10} +$$
$$2592c^{22}\hbar^9(4+\hbar)^3\Lambda^{10} + 2596c^{18}\hbar^7(3+\hbar)(6+\hbar)\Lambda^{10}(7+\hbar(4+\hbar)) +$$
$$864c^{22}\hbar^9(7+\hbar(4+\hbar)\Lambda^{10},$$

$$C_{14} = \frac{\Gamma}{1048576}(1944c^{24}\hbar^{10}(4+\hbar)^2\Lambda^{11} + 432c^{24}\hbar^{10}(6+\hbar)\Lambda^{11})$$

$$C_{15} = \frac{\Gamma}{131072}(81c^{26}\hbar^{11}(4+\hbar)\Lambda^{12},$$

$$C_{16} = \frac{\Gamma}{131072}(81c^{28}\hbar^{12}\Lambda^{13}),$$

$$C_{19} = \frac{-c^2\Gamma}{691201211741302764902400000}(-97200170401120701314400000$$
$$+86400151467662845612800000\hbar - 19200033659480632358400000\hbar^2),$$

$$C_{20} = \frac{-c^2\Gamma}{691201211741302764902400000}(-691201211741302764902400000\hbar +$$
$$583681023248211223695360000\hbar^2 - 122880215420676047093760000\hbar^3),$$

$$C_{21} = \frac{-c^2\Gamma}{691201211741302764902400000}(-277333819525831356288000000\hbar^2 +$$
$$170666965862050065408000000\hbar^3 - 21333370732756258176000000\hbar^4 -$$
$$97200170401120701314400000c^2\Lambda + 162986952398257812464640000c^2\Lambda\hbar -$$
$$104443870160947119280384000c^2\Lambda\hbar^2 + 31288943741375324800000c^2\Lambda\hbar^3 -$$
$$37925992413788903424000000c^2\Lambda\hbar^4),$$

$$
\begin{aligned}
C_{22} \;=\; & \frac{-c^2\Gamma}{6912012117413027649024000000}(-58775613243308058240000000\hbar^3 + \\
& 23510245297323223296000000\hbar^4 - 158694155756931757248000000c^2\Lambda\hbar + \\
& 266284004236089441331880000 0c^2\Lambda\hbar^2 - 1723388055276374946201600000c^2\Lambda\hbar^3 + \\
& 505034898979535907840000000c^2\Lambda\hbar^4 - 557279888529143070720000000c^2\Lambda\hbar^5),
\end{aligned}
$$

$$
\begin{aligned}
C_{23} \;=\; & \frac{-c^2\Gamma}{6912012117413027649024000000}(-6750011833411159813500000\hbar^4 - \\
& 137700241401587660195400000 0c^2\Lambda\hbar^2 + 2166470464694937881079600000c^2\Lambda\hbar^3 - \\
& 1283868917410687562354400000c^2\Lambda\hbar^4 + 336000589040911066272000000c^2\Lambda\hbar^5 - \\
& 32000056099134387264000000c^2\Lambda\hbar^6 - 248400435469530681136800000c^4\Lambda^2\hbar + \\
& 486866038707543065931360000c^4\Lambda^2\hbar^2 - 357689515952546595417600000c^4\Lambda^2\hbar^3 + \\
& 116474278264534515124800000c^4\Lambda^2\hbar^4 - 142222471551708387840000000c^4\Lambda^2\hbar^5),
\end{aligned}
$$

$$
\begin{aligned}
C_{24} \;=\; & \frac{-c^2\Gamma}{6912012117413027649024000000}(-77629765721974161696000000 0c^2\Lambda\hbar^3 + \\
& 1101091642252272698227200000c^2\Lambda\hbar^4 - 565729386839017809408000000c^2\Lambda\hbar^5 + \\
& 120098975976998194176000000c^2\Lambda\hbar^6 + 8427998314175311872000000c^2\Lambda\hbar^7 - \\
& 457245246038742466905600000c^4\Lambda^2\hbar^2 + 803766493669685360855040000c^4\Lambda^2\hbar^3 - \\
& 530296677259839102412800000c^4\Lambda^2\hbar^4 + 155262457832251856486400000c^4\Lambda^2\hbar^5 - \\
& 16855996628350623744000000c^4\Lambda^2\hbar^6),
\end{aligned}
$$

$$C_{25} = \frac{-c^2\Gamma}{6912012117413027649024000000}(-3076538726797611882206400000c^2\Lambda\hbar^4 +$$
$$3739739889452172058252800000c^2\Lambda\hbar^5 + 15488027151981043435776000000c^2\Lambda\hbar^6 +$$
$$238933752206870091571200000c^2\Lambda\hbar^7 - 853334829310250327040000c^2\Lambda\hbar^8 -$$
$$4023687053905157854575360000c^4\Lambda^2\hbar^3 + 6268879730681676457405056000c^4\Lambda^2\hbar^3c^4\Lambda^2\hbar^4 -$$
$$3543046211296159357870080000c^4\Lambda^2\hbar^5 + 840819251813699988910080000c^4\Lambda^2\hbar^6 -$$
$$682667863448200261632000000c^4\Lambda^2\hbar^7 - 3630086363885804891228160000c^6\Lambda^3\hbar^2 +$$
$$7145889070631392553472000000c^6\Lambda^3\hbar^3 - 5358247418207452590581760000c^6\Lambda^3\hbar^4 +$$
$$1802748839402099209420800000c^6\Lambda^3\hbar^5 - 227555954482733420544000000c^6\Lambda^3\hbar^6),$$

$$C_{26} = \frac{-c^2\Gamma}{6912012117413027649024000000}(-866117220865000977600000000c^2\Lambda\hbar^5 +$$
$$848927108084474240640000000c^2\Lambda\hbar^6 - 253884742604702576640000000c^2\Lambda\hbar^7 +$$
$$21157061883725214720000000c^2\Lambda\hbar^8 - 2197386160154180993088000000c^4\Lambda^2\hbar^4 +$$
$$2900633184258726938112000000c^4\Lambda^2\hbar^5 - 1318084955356080877056000000c^4\Lambda^2\hbar^6 +$$
$$226929078575067488256000000c^4\Lambda^2\hbar^7 - 9403138614988984320000000c^4\Lambda^2\hbar^8 -$$
$$672488965897474686105600000c^6\Lambda^3\hbar^3 + 1154211757143360352819200000\Lambda^3\hbar^4c^6 -$$
$$724867059965911936819200000c^6\Lambda^3\hbar^5 + 195271845237937907712000000c^6\Lambda^3\hbar^6 -$$
$$18806277229977968640000000c^6\Lambda^3\hbar^7),$$

$$
\begin{aligned}
C_{27} = {} & \frac{-c^2\Gamma}{691201211741302764902400000}(-1700002980266514323400000 0c^2\Lambda\hbar^6 + \\
& 1200002103717539522400000 0c^2\Lambda\hbar^7 - 2000003506195899204000000c^2\Lambda\hbar^8 - \\
& 7945528332557190014640000 0c^4\Lambda^2\hbar^5 + 8423224643255173511760000 0c^4\Lambda^2\hbar^6 - \\
& 2753585074209465175680000 0c^4\Lambda^2\hbar^7 + 2581074483716189260800000 0c^4\Lambda^2\hbar^8 - \\
& 5718333755920154172804000 0c^6\Lambda^3\hbar^4 + 8222631699004118139744000 0c^6\Lambda^3\hbar^5 - \\
& 4126690362051746064576000 0c^6\Lambda^3\hbar^6 + 8006598398466546278400000c^6\Lambda^3\hbar^7 - \\
& 3950624209769677440000 00c^6\Lambda^3\hbar^8 - 3604959223356348329472000c^8\Lambda^4\hbar^3 + \\
& 6775473342255292426752000c^8\Lambda^4\hbar^4 - 4750698771955625448960000c^8\Lambda^4\hbar^5 + \\
& 1483678869891278860800000c^8\Lambda^4\hbar^6 - 1755832982119856640000 00c^8\Lambda^4\hbar^7),
\end{aligned}
$$

$$
\begin{aligned}
C_{28} = {} & \frac{-c^2\Gamma}{691201211741302764902400000}(-2130181249202732880000000c^2\Lambda\hbar^7 \\
& +8520724996810931520000000c^2\Lambda\hbar^8 - 1917794289097038179520000 0c^4\Lambda^2\hbar^6 + \\
& 1459253051305694346240000 0c^4\Lambda^2\hbar^7 - 2676138813813211084800000c^4\Lambda^2\hbar^8 - \\
& 2840470755054861253248000 0c^6\Lambda^3\hbar^5 + 3264356369230917827328000 0c^6\Lambda^3\hbar^6 - \\
& 1171516700576623306752000 0c^6\Lambda^3\hbar^7 + 1249238803373322854400000c^6\Lambda^3\hbar^8 - \\
& 6384701880022618994688000c^8\Lambda^4\hbar^4 + 9995168254479888420864000c^8\Lambda^4\hbar^5 - \\
& 5561356816574232791040000c^8\Lambda^4\hbar^6 + 1226797387743861964800000c^8\Lambda^4\hbar^7 - \\
& -7480471876486963200000 0c^8\Lambda^4\hbar^8),
\end{aligned}
$$

$$
\begin{aligned}
C_{29} \quad = \quad & \frac{-c^2\Gamma}{6912012117413027649024000000}(-137755343539003261500000c^2\Lambda\hbar^8 - \\
& 286531114561126783920000c^4\Lambda^2\hbar^7 + 1240818501803170118400000c^4\Lambda^2\hbar^8 - \\
& 879489574078603048704000c^6\Lambda^3\hbar^6 + 7225933554620358589440000c^6\Lambda^3\hbar^7 - \\
& 146463345703420327296000c^6\Lambda^3\hbar^8 - 4919634390941768893824000c^8\Lambda^4\hbar^5 + \\
& 6125812816340545840512000c^8\Lambda^4\hbar^6 - 24193945160564996044800000c^8\Lambda^4\hbar^7 + \\
& 296699940652090060800000c^8\Lambda^4\hbar^8 - 2581274619478625900544000c^{10}\Lambda^5\hbar^4 + \\
& 4370264311015638097920000c^{10}\Lambda^5\hbar^5 - 2678899464148581273600000c^{10}\Lambda^5\hbar^6 + \\
& 6736665319153735680000c^{10}\Lambda^5\hbar^7 - 537499892485670400000c^{10}\Lambda^5\hbar^8),
\end{aligned}
$$

$$
\begin{aligned}
C_{30} \quad = \quad & \frac{-c^2\Gamma}{6912012117413027649024000000}(-216889269116355291456000c^4\Lambda^2\hbar^8 - \\
& 161306620234369109660160\!0c^6\Lambda^3\hbar^7 + 7570449485707973889024\!00c^6\Lambda^3\hbar^8 - \\
& 2088278106918251781012480c^8\Lambda^4\hbar^6 + 18524761104429716174438\!40c^8\Lambda^4\hbar^7 \\
& -4158748945916951113728\!00c^8\Lambda^4\hbar^8 - 411873341109533773037568c^{10}\Lambda^5\hbar^5 + \\
& 552880851384376594759680c^{10}\Lambda^5\hbar^6 - 239917018676857211289600c^{10}\Lambda^5\hbar^7 + \\
& 34024141342411444224000c^{10}\Lambda^5\hbar^8),
\end{aligned}
$$

$$
\begin{aligned}
C_{31} \quad = \quad & \frac{-c^2\Gamma}{6912012117413027649024000000}(-1458683112765446624306\!25c^6\Lambda^3\hbar^8 - \\
& 4905220076225190306585\!00c^8\Lambda^4\hbar^7 + 24920848443984375682800\!0c^8\Lambda^4\hbar^8 - \\
& 26994435374408384642760\!0c^{10}\Lambda^5\hbar^6 + 25782121101567959528400\!0c^{10}\Lambda^5\hbar^7 - \\
& 63930153227887436490000\!0c^{10}\Lambda^5\hbar^8 - 13351700827245734553600c^{12}\Lambda^6\hbar^5 + \\
& 19177598015820211968000c^{12}\Lambda^6\hbar^6 - 9071803740952592640000c^{12}\Lambda^6\hbar^7 \\
& 1463194151766547200000c^{12}\Lambda^6\hbar^8),
\end{aligned}
$$

$$C_{32} = \frac{-c^2\Gamma}{6912012117413027649024000000}(-544886920275656999904000c^8\Lambda^4\hbar^8 -$$
$$8674675859134553748480 0c^{10}\Lambda^5\hbar^7 + 47501931514621199616000c^{10}\Lambda^5\hbar^8 -$$
$$-1801285271716723752960 0c^{12}\Lambda^6\hbar^6 + 18413496765354442752000c^{12}\Lambda^6\hbar^7$$
$$-4990048982633594880000c^{12}\Lambda^6\hbar^8),$$

$$C_{33} = \frac{-c^2\Gamma}{6912012117413027649024000000}(-1232015522251605851760 0c^{10}\Lambda^5\hbar^8 -$$
$$8911185567014367436800c^{12}\Lambda^8\hbar^7 + 5218846235469554688000c^{12}\Lambda^8\hbar^8 -$$
$$4857349635523280896 00c^{14}\Lambda^7\hbar^6 + 5275259829834874880 00c^{14}\Lambda^7\hbar^7 -$$
$$1541472028198502400 00c^{14}\Lambda^7\hbar^8),$$

$$C_{34} = \frac{-c^2\Gamma}{6912012117413027649024000000}(-145703789785392943104c^{14}\Lambda^7\hbar^8 -$$
$$113452341275409776 64c^{16}\Lambda^7\hbar^7 + 73990657353528115 20c^{16}\Lambda^7\hbar^8)$$

$$C_{35} = \frac{3904}{3954917491875}c^{18}\Lambda^8\hbar^8\Gamma,$$
$$C_{36} = \frac{64c^{20}\Lambda^9\hbar^8}{325540827562}\Gamma.$$

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