Thin film flows of non-Newtonian fluid: Exact solution

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1. Non-Newtonian fluids
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We accept this dissertation as conforming to the required standard.

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DEDICATED TO

My grateful Dada (late), my parents, my faimly and my Special friend Ayinos, whose prayers and invocations always made me successful and confident.

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Foremost, I am always grateful to Almighty ALLAH, who made human being, the best creation of all the living species and made them understand to write with pen. He provided me the boldness and capability to achieve this task. I offer countless Darood and Salaams to my beloved Holy Prophet Hazrat Muhammad (PBUH), for whom this universe has been manifested. ALLAH has shown His existence and oness by sending him as a messenger of Islam and born me as a Muslim.

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PREFACE

A layer of liquid on a solid substrate with a free surface is called a thin film if its thickness is small compared to all relevant length scales parallel to the substrate. Thin film flows have gained considerable attention by various researchers over the years. This is because of their numerous applications in engineering and industry. For example thin film flows are involved in coating of electronic components, extrusion process involving polymers, oil behavior in gears and cold rolling of steel. The various agents causing flow of a thin film are gravity, thermal and centrifugal force, surface shear etc.

Dynamics, morphology and stability of thin film flow of various fluids are now well understood and lot of literature is available on these issues (see for example [1] and refs. therein).

It is now an established fact that most of fluids in technological and industrial applications do not exhibit Newtonian behavior. Therefore, lot of efforts has been made to study the flows of non-Newtonian fluids. The governing equations of non-Newtonian are highly nonlinear and have higher order when compared with Navier-stokes equations. Due to nonlinearity the superposition principle is not applicable and hence the possibility of getting an exact solution narrows down. The inadequacy of Navier-stokes equations to describe rheologically complex fluids such as polymer solution has lead to the development of non-Newtonian models. Due to complex nature of the fluids exists in industry and technology it is not possible to describe all the properties by a single constitutive relationship. Therefore, several constitutive relationships have been proposed for non-Newtonian fluids in the literature. The study of thin film flow of non-Newtonian fluids under various configurations is an active area of research. Mention may be made to the works of Siddiqui et. al [2-4], Hayat and Sajid [5,6], Sajid et .al [7,8] and Asghar et. al [9]. Amongst several non-Newtonian models there is a class known as generalized Newtonian fluid models. These models are quite popular in predicting the behavior of non-Newtonian fluids because of their simplicity [10]. Among generalized Newtonian fluid models the Carreau, Ellis, power law and Eyring-Powell models have been widely used for study of visco elastic fluids [11]. Keeping these facts in mind we investigate thin film flows of an Eyring-Powell non-Newtonian fluid in this dissertation.

The dissertation consists of three chapters. Basic concept and definition are presented in chapter 1 to facilitate the readers. In chapter 2 three thin film flow problems for a micro-polar fluid are re-investigated. This chapter is infact a review of work conducted by Sajid et al. [8]. Thin film flows of an Eyring-Powell fluid are discussed in chapter 3. Exact, numerical and perturbation solutions are developed for each case. Graphical results are presented and discussed in detail.

Contents

Chapter 1

Basic definitions and equations

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The aim of this chapter is to provide the readers with some basic concepts and governing equations of fluid motion. The contents of this chapter are based on the refs. [12, 13].

1.1 Deformation

It is a relative change in position or length of the fluid particles.

1.2 Flow

A material goes under deformation when certain forces act upon it. If the deformation exceed continuously with out limit, then the phenomena is known as flow.

1.3 Fluid

A fluid is a substance that continually deforms (flows) under an applied shear (tangential) stress.

1.4 Fluid Mechanics

It is the branch of engineering and physics that examines the nature and properties of the fluid, both at motion and at rest.

1.5 Internal flow system

Internal flow is a flow for which the fluid is confined by a surface. Hence the boundary layer is unable to develop without eventually being constrained. Example includes flow in a pipe.

1.6 External flow system

External flow is such a flow for which the boundary layers develop freely, without constraints imposed by adjacent surfaces. It can be defined as the flow of a fluid around a body that is completely submerged in it. Examples include fluid motion over a flat plate (inclined or parallel to the free stream velocity) and flow over curved surfaces such as a sphere, cylinder, airfoil, or turbine blade, air flowing around an airplane and water flowing around the submarines are the examples of this case.

1.7 Characteristics of fluid

1.7.1 Pressure

Pressure is an effect which occurs when a force is applied on a surface. Pressure is the amount of force acting on a unit area. Mathematically:

$$p = \frac{F}{A}, \tag{1.1}$$

where F is the magnitude of force acting in the direction perpendicular to the surface of the fluid and A is the area of the surface of the fluid.

1.7.2 Specific volume

The specific volume denoted by V_s is defined as the volume occupied by a unit mass of the fluid. Accordingly

$$V_s = \frac{1}{\rho}. \tag{1.2}$$

1.7.3 Density

The density of the fluid is the mass of unit volume of the fluid at given temperature and pressure. If the density of the fluid varies then the density at the point is given as

$$\rho = \lim_{\delta V \to \delta V'} \left(\frac{\delta m}{\delta V'} \right), \tag{1.3}$$

where $\delta V'$ is the small volume over which the substance can be considered as a continuum.

1.7.4 Specific gravity

The specific gravity of a liquid (gas) is the ratio of the weight of the liquid (gas) to the weight of an equal volume of water (air) at a standard temperature. The standard temperature of water is taken as 4^oC while that of air is taken as 0^oC. Thus

$$\text{Specific gravity of a liquid} = \frac{\text{density of liquid}}{\text{density of water}}.$$

1.7.5 Viscosity

The measure of resistance to the motion of the fluid is called viscosity. It is also known as absolute or dynamic viscosity. Mathematically, viscosity is the ratio of shear stress to the rate of shear strain.

$$\mu = \frac{\text{Shear stress}}{\text{Rate of deformation}}. \tag{1.4}$$

1.7.6 Kinematic viscosity

Kinematic viscosity is the ratio of absolute viscosity μ to the density ρ. It is denoted by ν and given by

$$\nu = \frac{\mu}{\rho}. \tag{1.5}$$

1.7.7 Shear thinning effect

Shear thinning is an effect where viscosity decreases with increasing rate of shear stress. Materials that execute shear thinning are called pseudoplastic. There are certain complex solutions such as lava, ketchup, whipped cream, blood, paint and nail polish, which describe such effects.

1.7.8 Shear thickening effect

A shear thickening effect is one in which viscosity of a fluid increases with the rate of shear stress. Fluids which describe such effects are termed as dilatant. Mixture of cornstarch and water can easily be seen to perform this effect.

1.8 Types of Flow

1.8.1 Uniform flow

It is a flow in which the velocities of fluid particles are same at each layer.

1.8.2 Non-uniform flow

It is a flow in which the velocities of fluid particles are different at different layers.

1.8.3 Steady flow

It is a flow in which fluid quantity flowing per second is uniform. Mathematically, it is defined as

$$\frac{\partial \eta}{\partial t} = 0, \tag{1.6}$$

where η represent any fluid property and t is the time.

1.8.4 Unsteady flow

It is a flow in which fluid quantity flowing per second is not uniform i.e.,

$$\frac{\partial \eta}{\partial t} \neq 0. \tag{1.7}$$

1.8.5 Laminar flow

A flow in which every particle of fluid have a separate, definite path and that every particle do not intersect its path itself. Laminar flow is also known as stream line flow.

1.8.6 Trubulant flow

A flow in which every particle of fluid do not have a definite path and that every particle can intersect its path itself.

1.8.7 Compressible flow

A flow in which change in density occurs during the flow is called compressible flow.

1.8.8 Incompressible flow

A flow in which change in density does not occur during the flow is called incompressible flow.

1.8.9 Rotational flow

Flow of a fluid in which the curl of the fluid velocity is not zero, so that each minute particle of fluid rotates about its own axis. Mathematically, for rotational flow

$$\nabla \times \mathbf{V} \neq 0, \tag{1.8}$$

where $\mathbf{V}$ is fluid velocity.

1.8.10 Irrotational flow

Flow of a fluid in which the curl of the fluid velocity is zero every where , so that each minute particle of fluid do not rotates about its own axis. Mathematically, for irrotational flow

$$\nabla \times \mathbf{V} = 0. \tag{1.9}$$

1.8.11 Couette flow

It is a flow between two plates, in which one plate remains at rest and the other one is moving with uniform velocity.

1.8.12 Poiseuille flow

A flow between two plates produced by a constant pressure gradient in the direction of flow is called Poiseuille flow.

1.8.13 One dimensional flow

A flow for which the velocity field depends only on one space variable is called a one–dimensional flow.

1.8.14 Two dimensional flow

A flow for which the velocity field depends upon two space variables is called a two-dimensional flow.

1.8.15 Three dimensional flow

A flow for which the velocity field have three space variables is called a three-dimensional flow.

1.9 Classification of fluids

There are two main types of fluids.

(a) Ideal fluids

(b) Real fluids

1.9.1 Ideal fluids

A fluid with zero viscosity is called ideal fluid. i.e. a fluid in which there is no friction. All ideal fluids are incompressible. Mineral water is an example of an ideal fluid.

1.9.2 Real fluids

It is a fluid for which viscosity is not equal to zero, i.e.,

$$\mu \neq 0. \tag{1.10}$$

Real fluids are also known as viscous fluids. Real fluids are further divided in to three main classes.

1.10 Newtonian fluids

The fluids which satisfy the Newton's law of viscosity are called Newtonian fluids. In Newtonian fluids shear stress is directly and linearly proportional to the rate of deformation, i.e.,

$$\tau = \mu \left(\frac{du}{dy} \right), \tag{1.11}$$

where τ is shear stress and du/dy is the shear rate. Water, air, syrup and gassolin are some of the examples of Newtonian fluids.

1.11 Non-Newtonian fluids

The fluids which do not satisfy the Newton's law of viscosity are called non-Newtonian fluids. Such fluids obey the power law model. In which shear stress is directly but non-linearly proportional to the rate of deformation, i.e.,

$$\tau \propto \left(\frac{du}{dy} \right)^n, \quad n \neq 1, \tag{1.12}$$

$$\tau = k \left(\frac{du}{dy} \right)^n, \tag{1.13}$$

where n is the flow behavior index and k is the consistency index. Shampoo, gel, soap and blood etc are the examples of Non-Newtonian fluids. The above equation can be rewritten in the form

$$\tau = k \left| \frac{du}{dy} \right|^{n-1} \frac{du}{dy} = \eta^* \frac{du}{dy}. \tag{1.14}$$

The coefficient $\eta^* = k \, |du/dy|^{n-1}$ is called as the apparent viscosity. Non-Newtonian fluids are commonly classified in to following main types.

1.11.1 Time independent Non-Newtonian fluids

Such fluids where apparent viscosity does not depend upon time are known time-independent non-Newtonian fluids. These are further sub divided in the following types.

Psudoplastic fluids

The fluids for which apparent viscosity decreases with increasing rate of deformation are known as psudoplastic fluids. Polymer solutions and paper pulp in water are some examples of the fluids.

Dilatent fluids

The fluids for which apparent viscosity increases with increasing deformation rate. Suspension of starch and of sand are examples of dilatant fluids.

Bingham plastic fluids

The fluids that act as a solid untill a minimum stress is exceeded and later on shows a linear relationship between stress and the rate of deformation are turmed as Bingham plastic fluids. Toothpaste and drilling muds include in examples of such fluids.

1.11.2 Time dependent non-Newtonian fluids

The fluids in which the behavior of apparent viscosity is dependent upon time are classified as time dependent non-Newtonian fluids. Following the main categories of these fluids.

Thixotropic fluids

Such fluids show a decrease in η^* under a constant applied shear stress. An example of such a fluid is yougrut.

Rheopatic fluids

Such fluids show an increase in η^* with time under a constant applied shear stress. An example of such a fluid is blood.

1.11.3 Viscoelastic fluids

After deformation when the applied stress is released, some fluids partially come to their original shape or position. Such fluids are called viscoelastic fluids. Examples of such fluids are nylon, flour dough etc.

1.12 Thin film flow

A typical thin film flow consists of an expense of fluids partially bounded by a solid substrate with a free surface where the fluid is exposed to another fluid (usually a gas and most often air in application). A simple and obvious example of thin film flow is the flow of a thin rain drop down a window pane under the action of gravity.

1.13 Micropolar fluid

A Micropolar fluid is the fluid with internal structures in which coupling between the spin of each particle and the microscopic velocity field is taken in to account. These fluids can support stress moments and body moments and are influenced by the spin inertia.

1.14 Exact solution

In mathematics and especially physics an exact solution to a problem that summarize the whole mathematics or physics of the problem without using an approximation.

1.15 Governing Equations

1.15.1 Equation of continuity

The mathematical relation of conservation of mass for fluids is known as equation of continuity. It has the following form

$$\frac{\partial \rho}{\partial t} + \mathbf{\nabla} . (\rho \mathbf{V}) = 0, \tag{1.15}$$

and for an incomprssible fluid it reduces to

$$\nabla.\mathbf{V} = 0. \tag{1.16}$$

1.15.2 Equation of motion

The motion of fluid is governed by law of conservation of momentum. The application of this law to an arbitrary control volume in flowing fluid yield the following equation commonly known an equation of motion.

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + \operatorname{div}\mathbf{T} + \rho\mathbf{b}. \tag{1.17}$$

In above equation $\mathbf{T}$ is cauchy stress tensor and $\mathbf{b}$ is body force per unit mass.

Chapter 2

Thin film flows of a micropolar fluid

2.1 Introduction

The aim of this chapter is to provide exact analytic solutions for the thin film flow problems of a micropolar fluid in three different geometries:

1. Flow down an inclined plane

2. Flow on a moving belt

3. Flow down a vertical cylinder

Graphs have been drawn for different values of the material parameters and discussed in detail. The material of this chapter is based on the work by Sajid et al [8].

2.2 Governing equations

The behavior of an incompressible micropolar fluid in absence of body couples is governed by the following set of equations.

$$\nabla . \mathbf{V} = 0, \tag{2.1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + (\mu + k)\nabla^2 \mathbf{V} + k\nabla \times \mathbf{\Omega} + \rho g, \tag{2.2}$$

$$\rho j \frac{D\mathbf{\Omega}}{Dt} = (\alpha + \beta + \gamma)\nabla(\nabla . \mathbf{\Omega}) - \gamma \nabla \times (\nabla \times \mathbf{\Omega}) + k\nabla \times \mathbf{V} - 2k\mathbf{\Omega}, \tag{2.3}$$

where k, α, β and γ are the material constants, j represents the gyration parameter of the fluid, t is the time and $\mathbf{\Omega}$ denotes the micro-rotation vector.

We now proceed to obtain the reduced form of governing equation for two-dimensional steady flow of a micropolar fluid in absence of pressure gradient. For this we assume

$$\mathbf{V} = [u, v, 0] \quad \text{and} \quad \mathbf{\Omega} = [0, 0, N] \tag{2.4}$$

Thus

$$\nabla \times \mathbf{\Omega} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & N \end{vmatrix} = \frac{\partial N}{\partial y}\mathbf{i} - \frac{\partial N}{\partial x}\mathbf{j} + 0\mathbf{k}, \tag{2.5}$$

$$\nabla \times (\nabla \times \mathbf{\Omega}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} - \left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right)\mathbf{k}, \tag{2.6}$$

and

$$\nabla \times \mathbf{V} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u & v & 0 \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}. \tag{2.7}$$

$$\nabla^2 \mathbf{V} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\mathbf{V}. \tag{2.8}$$

Utilizing above results in Eqs. (2.1) – (2.3) we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.9}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (\nu + \frac{k}{\rho})\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] - \frac{k}{\rho}\frac{\partial N}{\partial y} + g_x, \tag{2.10}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (\nu + \frac{k}{\rho})\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - \frac{k}{\rho}\frac{\partial N}{\partial x} + g_y, \tag{2.11}$$

$$u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y} = \frac{\gamma}{\rho j}\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right) - \frac{k}{\rho j}\left(2N - \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \tag{2.12}$$

Following ref. [14] we assumed γ to be

$$\gamma = \left(\mu + \frac{k}{2}\right) j, \tag{2.13}$$

where j is the reference length. The micro-rotation N at the walls is related to the shear stress at the wall by the relation

$$N_w = -n\tau_w, \tag{2.14}$$

where N_w and τ_w are micro-rotation and shear stress at the wall and n is a constant $0 \leq n \leq 1$.

2.3 Thin film flow down an inclined plane

We consider the thin film flow of an incompressible micropolar fluid down an inclined plane due to gravity only. It is assumed that surface tension effects are negligible. For this case the velocity $\mathbf{V}$ and micro-rotation $\mathbf{\Omega}$ are of the form

$$\mathbf{V} = [u(y), 0, 0] \quad \text{and} \quad \mathbf{\Omega} = [0, 0, N(y)] \tag{2.15}$$

Using Eq. (2.15) we find that Eqs. (2.9) and (2.11) are satisfied identically and Eqs. (2.10) and (2.12) reduce to

$$\left(\nu + \frac{k}{\rho}\right) \frac{d^2u}{dy^2} + \frac{k}{\rho}\frac{dN}{dy} + g_1 \sin\alpha = 0, \tag{2.16}$$

$$\left(\nu + \frac{k}{2\rho}\right) \frac{d^2N}{dy^2} - \frac{k}{\rho j}\left(2N + \frac{du}{dy}\right) = 0. \tag{2.17}$$

The boundary conditions of the problem under consideration are

$$u = 0, \quad N = -n\frac{du}{dy} \quad \text{at} \quad y = 0 \quad \text{and} \quad \frac{du}{dy} = N = 0 \quad \text{at} \quad y = \delta, \tag{2.18}$$

where $g_1 \sin\alpha$ is the component of gravity along the plane and δ is the film thickness which is assumed uniform. In order to non-dimensionalize the above equations and boundary conditions, we use the following dimensionless quantities

$$\bar{y} = \frac{y}{\delta}, \quad \bar{u} = \frac{\delta}{\nu}u, \quad \bar{N} = \frac{\delta^2}{\nu}N, \quad K = \frac{k}{\mu}, \quad m_1 = \frac{\delta^3 g_1 \sin\alpha}{\nu^2}, \tag{2.19}$$

where j is equal to δ^2.

With the help of (2.19) Eqs. (2.16) – (2.18) take the following form

$$(1+K)\,u'' + KN' + m_1 = 0, \tag{2.20}$$

$$\left(1+\frac{K}{2}\right)N'' - K\left(2N+u'\right) = 0, \tag{2.21}$$

$$u(0) = u'(1) = N(1) = 0, \qquad N(0) = -nu'(0). \tag{2.22}$$

In above equations prime indicate differentiation with respect to y and bars have been removed for simplicity. Integrating Eq. (2.20) and then utilization of Eq. (2.22) yields

$$(1+K)\,u' + KN + m_1(y-1) = 0. \tag{2.23}$$

From Eq. (2.23) we obtain

$$u' = -\frac{KN}{1+K} - \frac{m_1(y-1)}{1+K}. \tag{2.24}$$

Subtituting for u' in Eq. (2.21), we have

$$\left(1+\frac{K}{2}\right)N'' - \frac{K(K+2)}{K+1}N + \frac{m_1K}{K+1}(y-1) = 0. \tag{2.25}$$

Also Eq. (2.23) implies that at the wall

$$(1+K)\,u'(0) + KN(0) - m_1 = 0. \tag{2.26}$$

In view of Eq. (2.22) and Eq. (2.26) we can write

$$N(0) = \frac{m_1n}{K(n-1)-1}, \qquad N(1) = 0. \tag{2.27}$$

Solving Eq. (2.25) subject to boundary condition (2.27) gives

$$N(y) = \frac{m_1 e^{-\sqrt{\frac{2k}{k+1}}y}\left\{\coth\left(\sqrt{\frac{2k}{k+1}}\right)-1\right\}}{2(K+2)\{K(n-1)-1\}}.$$
$$\times\left[(K+1)(2n-1)\left\{e^{2\sqrt{\frac{2k}{k+1}}}-e^{2\sqrt{\frac{2k}{k+1}}y}\right\}+(y-1)\{K(n-1)-1\}e^{\sqrt{\frac{2k}{k+1}}y}\left\{e^{2\sqrt{\frac{2k}{k+1}}}-1\right\}\right]. \tag{2.28}$$

Using Eq. (2.28) in to Eq. (2.20) and then solving the resulting equation for u we get

$$u(y) = \frac{m_1\sqrt{K+1}e^{-\sqrt{\frac{2k}{k+1}}y}\left\{\coth\left(\sqrt{\frac{2k}{k+1}}\right)-1\right\}}{4\sqrt{K}(K+2)\{K(n-1)-1\}}$$
$$\times\left[K\sqrt{2}(2n-1)\left\{\left(e^{2\sqrt{\frac{2k}{k+1}}}-e^{\sqrt{\frac{2k}{k+1}}y}+e^{2\sqrt{\frac{2k}{k+1}}y}\right)-e^{\sqrt{\frac{2k}{k+1}}(y+2)}\right\}\right.$$
$$\left.-2e^{\sqrt{\frac{2k}{k+1}}y}\left(e^{2\sqrt{\frac{2k}{k+1}}}-1\right)\sqrt{\frac{K}{K+!}}\{K(n-1)-1\}(y-2)y\right]. \tag{2.29}$$

2.4 Thin film flow on a moving belt

Let us consider a container filled with micropolar fluid. A wide moving belt pass through the container in vertical direction with velocity U_0. The belt picks up a thin film of thickness δ. The fluid drains down due to gravity. Let us assume this flow to be steady laminar flow with uniform film thickness. Let x-axis be taken in the direction perpendicular to the belt. In view of above assumptions the appropriate form of the velocity and micro-rotation field is

$$\mathbf{V} = [0, v(x), 0] \quad \text{and} \quad \mathbf{\Omega} = [0, N(x), 0] \tag{2.30}$$

For the velocity and micro-rotation fields defined by Eq. (2.30), Eqs. (2.9) and (2.10) identically satisfied and Eqs. (2.11) and (2.12) give

$$\left(\nu+\frac{k}{\rho}\right)\frac{d^2v}{dx^2}+\frac{k}{\rho}\frac{dN}{dx}-g_1=0, \tag{2.31}$$

$$\left(\nu+\frac{k}{2\rho}\right)\frac{d^2N}{dx^2}-\frac{k}{\rho j}\left(2N-\frac{dv}{dx}\right)=0. \tag{2.32}$$

The relevant boundary conditions are

$$v = U_0, \quad N = -n\frac{dv}{dx} \quad \text{at} \quad x = 0 \quad \text{and} \quad \frac{dv}{dx} = N = 0 \quad \text{at} \quad x = \delta. \tag{2.33}$$

Introducing the dimensionless quantities as

$$\bar{x} = \frac{x}{\delta}, \quad \bar{v} = \frac{v}{U_0}, \quad \bar{N} = \frac{\delta}{U_0}N, \quad m_2 = \frac{\rho g_1 \delta^2}{\mu U_0}, \tag{2.34}$$

Eqs. (2.31) – (2.33) become

$$(1+K)\,v'' + KN' - m_2 = 0, \tag{2.35}$$

$$\left(1+\frac{K}{2}\right)N'' - K(2N - v') = 0, \tag{2.36}$$

$$v(0) = v'(1) = N(1) = 0, \qquad N(0) = -nv'(0), \tag{2.37}$$

where prime indicate differentiation with respect to x. Without going in to detail we find, after employing the procedure as in the previous case the following are the expressions for N and v.

$$\begin{aligned} N(x) \;&=\; \frac{m_2 e^{-\sqrt{\frac{2k}{k+1}}x}\left\{\coth\left(\sqrt{\frac{2k}{k+1}}\right) - 1\right\}}{2(K+2)\{K(n-1)-1\}} \\ &\times \left[\begin{array}{c} (K+1)(2n-1)\left\{e^{2\sqrt{\frac{2k}{k+1}}} - e^{2\sqrt{\frac{2k}{k+1}}x}\right\} \\ +(x-1)\left\{K(n-1) - 1\}e^{\sqrt{\frac{2k}{k+1}}x}\{e^{2\sqrt{\frac{2k}{k+1}}} - 1\right\} \end{array}\right]. \end{aligned} \tag{2.38}$$

$$\begin{aligned} v(x) = \frac{\coth\left(\sqrt{\frac{2k}{k+1}}\right) - 1}{2(K+2)\{K(n+1)+1\}} &\times \left[\left(e^{2\sqrt{\frac{2k}{k+1}}} - 1\right)\{K + 2 + m_2(x-2)x\}\right. \\ &\times \{K(n+1)+1\} - 2m_2\sqrt{2K(K+1)}(2n+1)e^{\sqrt{\frac{2k}{k+1}}} \\ &\times \left.\sinh\left\{\sqrt{\frac{K}{2(K+1)}}(x-2)\right\}\sinh\left\{\sqrt{\frac{K}{2(K+1)}}x\right\}\right]. \end{aligned} \tag{2.39}$$

2.5 Thin film flow down a vertical cylinder

In this section we discuss the thin flow of the micropolar fluid falling on the outer surface of a vertical cylinder of radius R. For such a flow the velocity $\mathbf{V}$ and micro-rotation $\boldsymbol{\Omega}$ are given by

$$\mathbf{V} = [0, 0, w(r)] \quad \text{and} \quad \boldsymbol{\Omega} = [0, 0, N(r)] . \tag{240}$$

A cylindrical coordinate system is a natural choice for this particular flow. Neglecting the pressure gradient the governing equations for two dimensional flow of an incompressible micropolar fluid in cylindrical coordinates are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{2.41}$$

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = (\mu + k)\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right\} - k\frac{\partial N}{\partial z}, \tag{2.42}$$

$$\rho\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = (\mu + k)\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 w}{\partial z^2}\right\} + k\left(\frac{\partial N}{\partial z} + \frac{N}{r}\right) + \rho g_1, \tag{2.43}$$

$$\rho j\left(u\frac{\partial N}{\partial r} + w\frac{\partial N}{\partial z}\right) = \gamma\left\{\frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} - \frac{N}{r^2} + \frac{\partial^2 N}{\partial z^2}\right\} - k\left(2N + \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right). \tag{2.44}$$

With the help of Eq. (2.40) we observe that Eqs. (2.41) and (2.42) are identically satisfied and Eqs. (1.43) and (1.44) take the form

$$(\mu + k)\left\{\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right\} + k\left(\frac{dN}{dr} + \frac{N}{r}\right) + \rho g_1 = 0, \tag{2.45}$$

$$\left(\mu + \frac{k}{2}\right) j\left\{\frac{d^2 N}{dr^2} + \frac{1}{r}\frac{dN}{dr} - \frac{N}{r^2}\right\} - k\left(2N + \frac{dw}{dr}\right) = 0. \tag{2.46}$$

We have the following boundary conditions for the problem under investigation

$$w(R) = w'(R + \delta) = N(R + \delta) = 0, \quad \text{and} \quad N(R) = -nw'(R). \tag{2.47}$$

Now we define the following dimensionless variable

$$\eta = \frac{r}{R}, \quad f = \frac{R}{\nu}w, \quad g = \frac{R^2}{\nu}N, \quad K = \frac{k}{\mu} \quad \text{and} \quad m_3 = \frac{g_1}{\nu^2}R^3. \tag{2.48}$$

Upon making use of (2.48), Eqs. (2.45) – (2.47) can be casted in the following form

$$(1+K)\left\{\eta f''+f'\right\}+K\left(\eta g'+g\right)+m_3\eta=0, \tag{2.49}$$

$$\left(1+\frac{K}{2}\right)\left\{\eta^2 g''+\eta g'-g\right\}-K\eta^2\left\{2g+f'\right\}=0, \tag{2.50}$$

$$f(1)=f'(d)=g(d)=0, \quad \text{and} \quad g(1)=-nf'(1), \tag{2.51}$$

where primes denote the differentiation with respect to η. Integrating Eq. (2.49) with respect to η and using (2.51) we have

$$(1+K)\,\eta f'+K\eta g+\frac{m_3}{2}\left(\eta^2-d^2\right)=0. \tag{2.52}$$

Eliminating f' between Eqs. (2.50) and (2.52) one obtains

$$\eta^2 g''+\eta g'-\left(1+L_1\eta^2\right)g=L_3\eta-L_2\eta^3, \tag{2.53}$$

where

$$L_1=\frac{2K}{K+1}, \quad L_2=\frac{Km_3}{(K+1)(K+2)}, \quad \text{and} \quad L_3=L_2d^2. \tag{2.54}$$

Eq. (2.52) at $\eta=1$ become

$$(1+K)\,f'(1)+Kg(1)+\frac{m_3}{2}\left(1^2-d^2\right)=0. \tag{2.55}$$

Using boundry condition from Eq. (2.51) results in

$$f'(1)=\frac{m_3\left(1-d^2\right)}{2\left\{K(n-1)-1\right\}}. \tag{2.56}$$

Thus

$$g(1)=-nf'(1)=\frac{-nm_3\left(1-d^2\right)}{2\left\{K(n-1)-1\right\}}=\frac{m_3 n\left(d^2-1\right)}{2\left\{K(n-1)-1\right\}}, \tag{2.57}$$

or

$$g(1)=A, \quad \text{and} \quad g(d)=0, \tag{2.58}$$

in which

$$A = \frac{m_3 n\left(d^2-1\right)}{2\left\{K(n-1)-1\right\}}. \tag{2.59}$$

Solution of Eq. (2.53) subject to boundary conditions (2.58) is given by

$$\begin{aligned}
g(\eta) = {} & C_1 I_1\left(\sqrt{L_1}\eta\right) + C_2 K_1\left(\sqrt{L_1}\eta\right) + I_1\left(\sqrt{L_1}\eta\right)\left(-\frac{L_3 K_0\left(\sqrt{L_1}\eta\right)}{\sqrt{L_1}}\right) \\
& + K_1\left(\sqrt{L_1}\eta\right)\left(\frac{L_3 - L_3 I_0\left(\sqrt{L_1}\eta\right) + L_2\eta^2 I_2\left(\sqrt{L_1}\eta\right)}{\sqrt{L_1}}\right) \\
& - \frac{1}{4}L_2\eta^3 G_{1,3}^{2,1}\left(\frac{\sqrt{L_1}\eta}{2},\frac{1}{2}\Bigg|_{-\frac{1}{2},\frac{1}{2},-\frac{3}{2}}^{-\frac{1}{2}}\right),
\end{aligned} \tag{2.60}$$

where I_0 and I_1 are modified Bessel functions of first type of order zero and one respectively, K_0 and K_1 are second kind of order zero and one, respectively and $G_{1,3}^{2,1}$ is the Meiger$G$ function. The constants C_1 and C_2 appearing in above solution have the following values

$$\begin{aligned}
C_1 = {} & \frac{1}{4\sqrt{L_1}\left\{I_1\left(\sqrt{L_1}d\right)K_1\left(\sqrt{L_1}\right) - I_1\left(\sqrt{L_1}\right)K_1\left(\sqrt{L_1}d\right)\right\}} \\
& \times\Big[4L_3 I_1\left(\sqrt{L_1}d\right)K_0\left(\sqrt{L_1}d\right)K_1\left(\sqrt{L_1}\right) - 4\Big\{A\sqrt{L_1} + L_3 I_1\left(\sqrt{L_1}\right)K_0\left(\sqrt{L_1}\right) \\
& + L_3 I_0\left(\sqrt{L_1}\right)K_1\left(\sqrt{L_1}\right) - L_3 I_0\left(\sqrt{L_1}d\right)K_1\left(\sqrt{L_1}\right) - L_2 I_2\left(\sqrt{L_1}\right)K_1\left(\sqrt{L_1}\right) \\
& + d^2 L_2 I_2\left(\sqrt{L_1}d\right)K_1\left(\sqrt{L_1}\right)\Big\}K_1\left(\sqrt{L_1}d\right) + d^3\sqrt{L_1}L_2 I_1\left(\sqrt{L_1}d\right)K_1\left(\sqrt{L_1}\right) \\
& \times G_{1,3}^{2,1}\left(\frac{\sqrt{L_1}d}{2},\frac{1}{2}\Bigg|_{-\frac{1}{2},\frac{1}{2},-\frac{3}{2}}^{-\frac{1}{2}}\right) - 2L_2 I_1\left(\sqrt{L_1}\right)I_1\left(\sqrt{L_1}d\right)G_{1,3}^{2,1}\left(\frac{\sqrt{L_1}}{2},\frac{1}{2}\Bigg|_{0,1,-1}^{0}\right)\Big],
\end{aligned} \tag{2.61}$$

$$C_2 = \frac{1}{4\sqrt{L_1}\left\{I_1\left(\sqrt{L_1}d\right)K_1\left(\sqrt{L_1}\right) - I_1\left(\sqrt{L_1}\right)K_1\left(\sqrt{L_1}d\right)\right\}}$$
$$\times\left[4\left\{L_3 - L_3 I_0\left(\sqrt{L_1}d\right) + d^2 L_2 I_2\left(\sqrt{L_1}d\right)\right\} I_1\left(\sqrt{L_1}\right)\right.$$
$$+ 4\left\{A\sqrt{L_1} + L_3 I_1\left(\sqrt{L_1}\right)K_0\left(\sqrt{L_1}\right) - L_3 I_1\left(\sqrt{L_1}\right)K_0\left(\sqrt{L_1}d\right)\right.$$
$$+ L_3 I_0\left(\sqrt{L_1}\right)K_1\left(\sqrt{L_1}\right) - L_2 I_2\left(\sqrt{L_1}\right)K_1\left(\sqrt{L_1}\right)$$
$$\left. - L_3 K_1\left(\sqrt{L_1}\right)\right\} I_1\left(\sqrt{L_1}d\right) - d^3\sqrt{L_1}L_2 I_1\left(\sqrt{L_1}\right) I_1\left(\sqrt{L_1}d\right)$$
$$\left.\times G_{1,3}^{2,1}\left(\frac{\sqrt{L_1}d}{2}, \frac{1}{2}\middle|_{-\frac{1}{2},\frac{1}{2},-\frac{3}{2}}^{-\frac{1}{2}}\right) - 2L_2 I_1\left(\sqrt{L_1}\right) I_1\left(\sqrt{L_1}d\right) G_{1,3}^{2,1}\left(\frac{\sqrt{L_1}}{2}, \frac{1}{2}\middle|_{0,1,-1}^{0}\right)\right]. \quad (2.62)$$

Utilizing the solution for $g(\eta)$ in Eq. (2.52) and solving the resulting equation for $f(\eta)$ while boundry condition $f(1) = 0$ completes the solution for this case.

2.6 Results and discussion

We have plotted velocity and micro-rotation for several values of parameters of interest in the three considered problems, i.e. (i) flow down an inclined plane, (ii) flow on a moving belt and (iii) flow down a vertical cylinder. Figs. 2.1, 2.5 and 2.9 show the variation of velocity u and micro-rotation N for different values of micropolar fluid parameter K for strong concentration of microelements ($i.e.$ $n = 0$), respectively for three considered problems. It is noted that magnitude of both velocity and micro-rotation decreases by increasing K. In order to see the influence of K on u and N in three considered problems, Figs. 2.2, 2.6 and 2.10 have been plotted when $n = 1/2$. Here as in previous case, the magnitudes of velocity and micro-rotation decreases by increasing K. However, the magnitudes of velocity and micro-rotation are greater for $n = 1/2$ when compared with the case $n = 0$. We have also prepared some figures just to see the effect of film thickness δ on the velocity and micro-rotation. Figs. 2.3 and 2.4 correspond to the case of flow down an inclined plane for $n = 0$ and $n = 1/2$, respectively. While Figs. 2.7 and 2.8 correspond to the case of flow on a moving belt when $n = 0$ and $n = 1/2$, respectively. It is seen from these figures that an increase in the film thickness increases the magnitudes of velocity and micro-rotation. Further more, this increase is enhanced when the value of n increases from zero to 1/2.

Case (1)

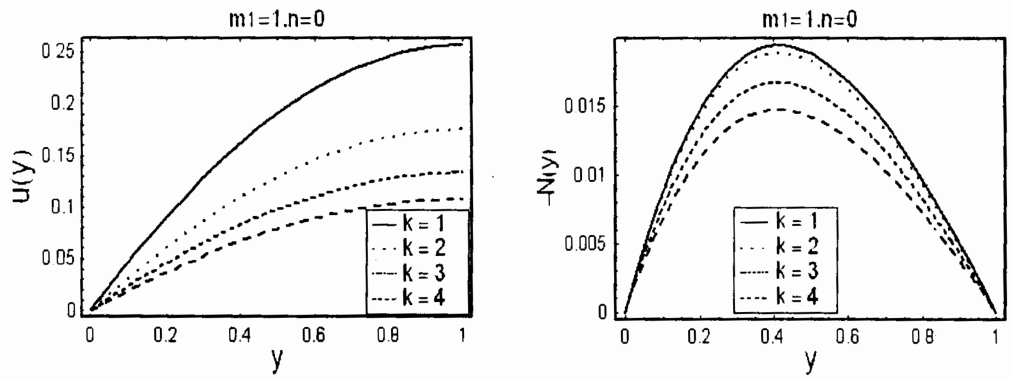

Fig. 2.1: Influence of parameter K on the velocity and micro-ration for $n = 0$.

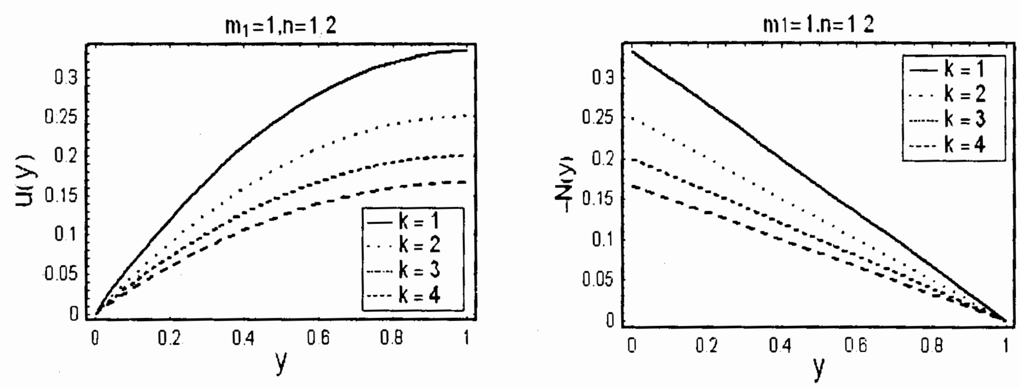

Fig. 2.2: Influence of parameter K on the velocity and micro-ration for $n = 1/2$.

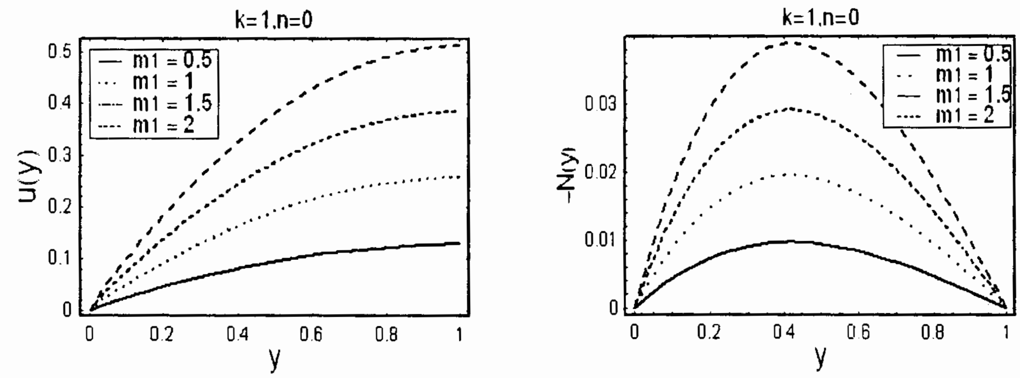

Fig. 2.3: Influence of parameter m_1 on the velocity and micro-ration for $n = 0$.

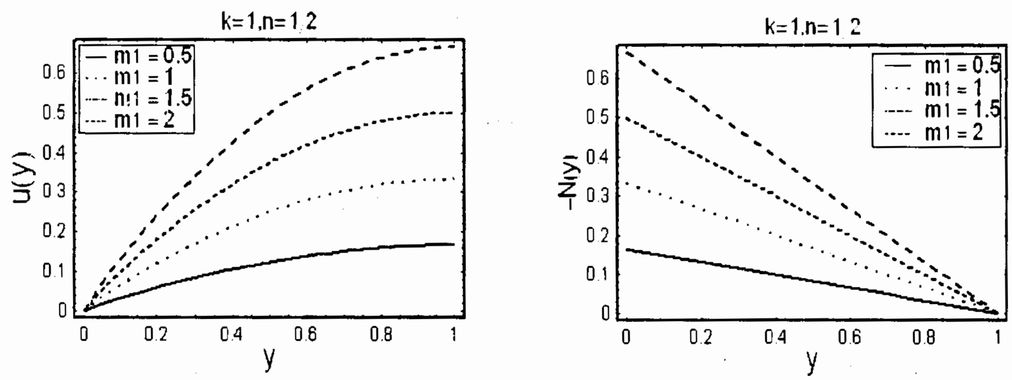

Fig. 2.4: Influence of parameter m_1 on the velocity and micro-ration for $n = 1/2$.

Case (2)

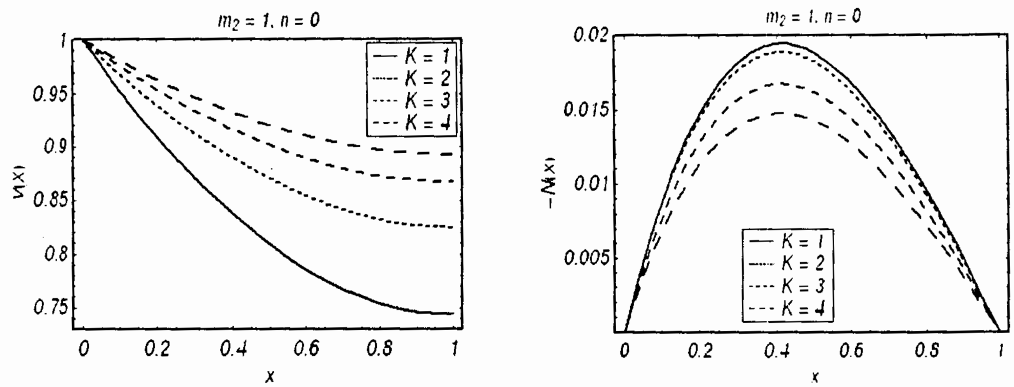

Fig. 2.5: Influence of parameter K on the velocity and micro-ration for $n = 0$.

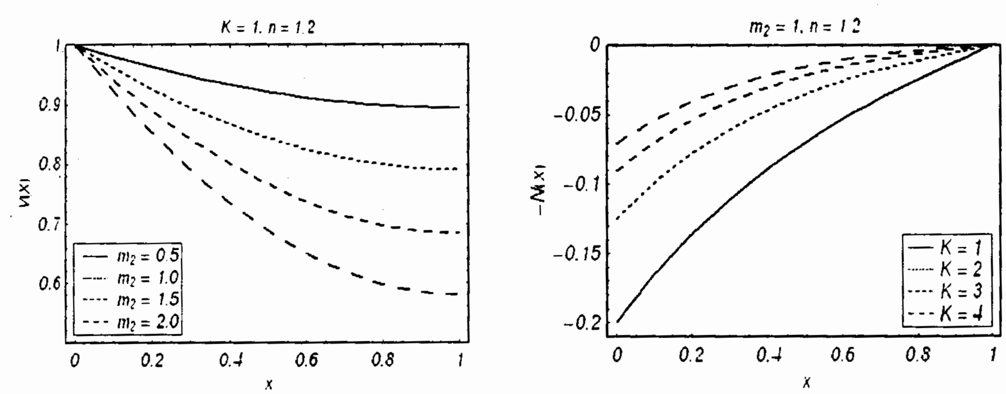

Fig. 2.6: Influence of parameter K on the velocity and micro-ration for $n = 1/2$.

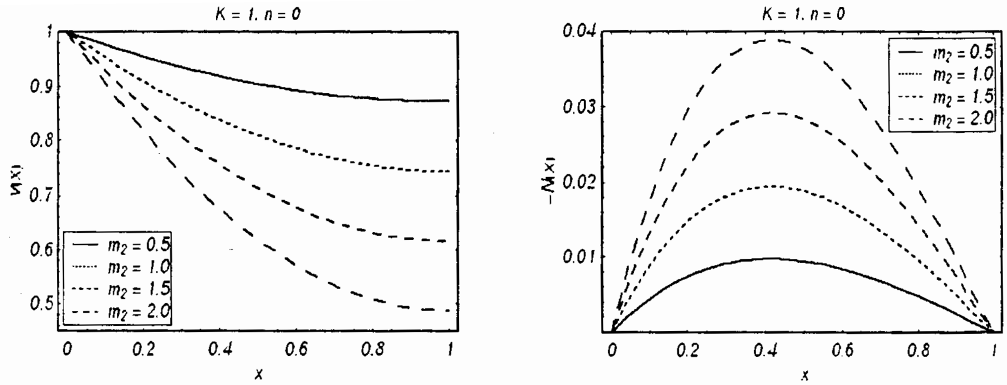

Fig. 2.7: Influence of parameter m_2 on the velocity and micro-ration for $n = 0$.

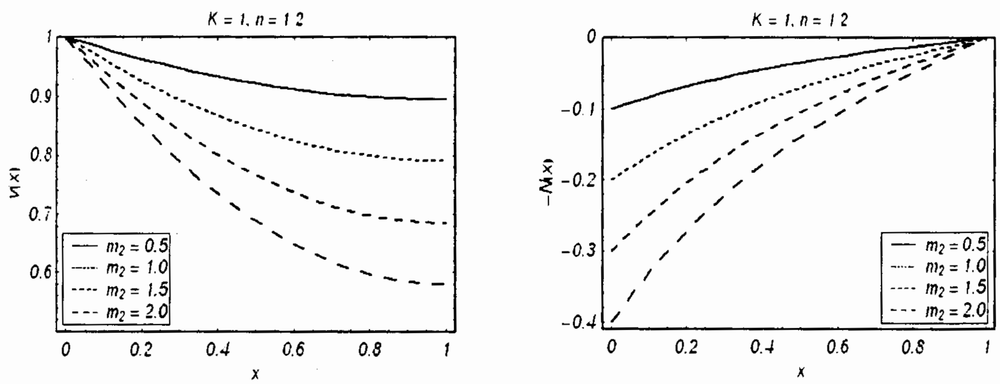

Fig. 2.8: Influence of parameter m_2 on the velocity and micro-ration for $n = 1/2$.

Case (3)

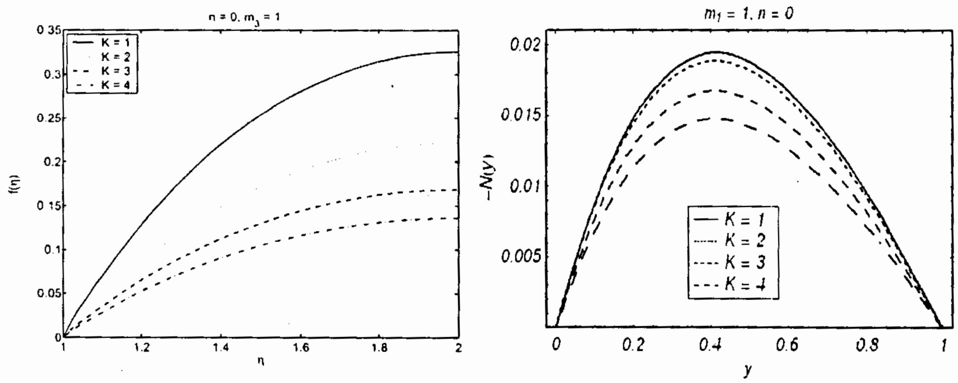

Fig. 2.9: Influence of parameter K on the velocity and micro-ration for $n = 0$.

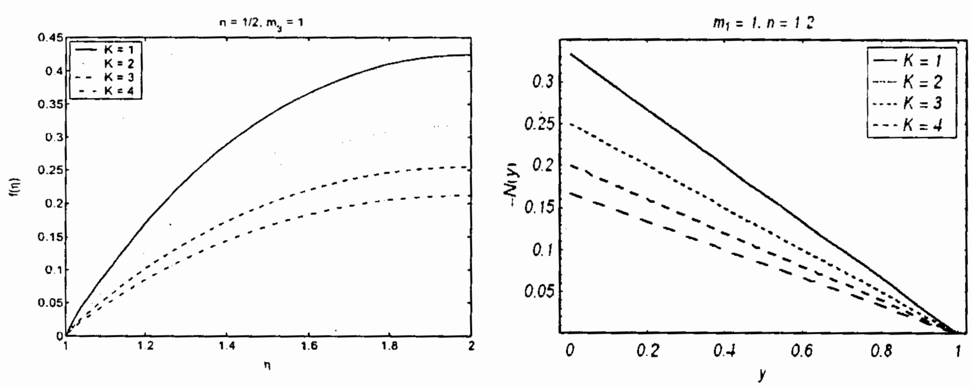

Fig. 2.10: Influence of parameter K on the velocity and micro-ration for $n = 1/2$.

Chapter 3

Thin film flows of an Eyring-Powell fluid

This chapter is devoted to the study of thin film flows of an Eyring-Powell fluid. All the three cases considered in previous chapter are investigated. The governing equation is nonlinear in each case. In first two cases exact, perturbation and numerical solutions are provided. However, in the last case only exact and perturbation solutions are obtained. A comparison of the numerical and perturbation solution is also presented. Finally, influence of emerging parameters on the velocity profile is depicted with the help of graphs.

3.1 Thin film flow down an inclined plane

For an incompressible flow the equation of continuity and momentum are given by Eq. (1.16) and (1.17). For an Eyring-Powell fluid $\mathbf{T}$ is given by the following relation [15].

$$\mathbf{T} = \mu\nabla\mathbf{V} + \frac{1}{\beta}sinh^{-1}\left(\frac{1}{c^*}\nabla\mathbf{V}\right), \tag{3.1}$$

where β and c^* are material constants. In index notation Eq. (3.1) reads

$$T_{ij} = \mu\frac{\partial u_i}{\partial x_j} + \frac{1}{\beta}sinh^{-1}\left(\frac{1}{c^*}\frac{\partial u_i}{\partial x_j}\right), \tag{3.2}$$

For subsequent analysis we shall take the first and second order approximation of the $sinh^{-1}$ function:

$$sinh^{-1}\left(\frac{1}{c^*}\nabla\mathbf{V}\right) \cong \frac{1}{c^*}\nabla\mathbf{V} - \frac{1}{6}\left(\frac{1}{c^*}\nabla\mathbf{V}\right)^3, \quad \left|\frac{1}{c^*}\nabla\mathbf{V}\right| << 1. \tag{3.3}$$

We assume the same form of the velocity field as defined in Eq. (2.15). Substituting of this velocity field results in the following form of **T:**

$$\mathbf{T} = \begin{pmatrix} 0 & \mu\frac{du}{dy} + \frac{1}{\beta}\left[\frac{1}{c}\frac{du}{dy} - \frac{1}{6}\left(\frac{1}{c}\frac{du}{dy}\right)^3\right] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

Further we note that by using Eq. (2.4), Eq. (1.16) is satisfied identically and Eq. (1.17) in scalar form reduce to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{\partial}{\partial_x}T_{xx} + \frac{\partial}{\partial_y}T_{xy} + \frac{\partial}{\partial_z}T_{xz}\right) + g_1 sin\alpha. \tag{3.5}$$

Neglecting the pressure gradient and utilizing Eq. (3.4), Eq. (3.5) take the form

$$\left(\nu + \frac{1}{\rho\beta c}\right)\frac{d^2u}{dy^2} - \frac{1}{2\rho\beta c^3}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} + g_1 sin\alpha = 0. \tag{3.6}$$

The appropriate boundary conditions of the problem are as defined in Eq. (2.18). Using the dimensionless variable given by (2.19), we have the following form non-dimensional governing boundary value problem.

$$(1+M)\frac{d^2u}{dy^2} - A\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy} + m_1 = 0, \tag{3.7}$$

$$u = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad \frac{du}{dy} = 0 \quad \text{at} \quad y = 1, \tag{3.8}$$

where $M = 1/\mu\beta c$ and $A = \nu^2/2\mu\beta c^3\delta^4 = M\nu^2/2c^2\delta^4$ are the non-dimensional parameters associated with Eyring-Powell fluid and over bars are removed for simplicity.

3.2 Solution of the problem

Eq. (3.7) can be rewritten in the following form

$$(1+M)\frac{d}{dy}\left(\frac{du}{dy}\right)-\frac{A}{3}\frac{d}{dy}\left(\frac{du}{dy}\right)^3+m_1=0. \tag{3.9}$$

Integrating above equation once yields

$$(1+M)\left(\frac{du}{dy}\right)-\frac{A}{3}\left(\frac{du}{dy}\right)^3+m_1y=c_1. \tag{3.10}$$

In view of second boundary condition of Eq. (3.8), we find that $c_1 = m_1$. Thus we have

$$(1+M)\left(\frac{du}{dy}\right)-\frac{A}{3}\left(\frac{du}{dy}\right)^3+m_1(y-1)=0. \tag{3.11}$$

The above equation is a cubic equation in du/dy. It can be solved by Mathematica for its real roots. After finding the real roots Eq. (3.11) can be integrated numerically with the remaining condition $u(0) = 0$. In this way an exact solution of the boundary value problem consisting of Eq. (3.7) and (3.8) can be obtained. The details can be seen in [16]. We have also obtained the perturbation and numerical solution of boundary value problem given by Eq. (3.7) and (3.8). By expanding u in a power series of small parameter A, one can write

$$u=u_0(y)+Au_1(y)+A^2u_2(y). \tag{3.12}$$

Substituting Eq. (3.12) in to governing equation and boundary conditions and equating like powers of A we obtain the following differential systems:

Zeroth order system

$$(1+M)\frac{d^2u_0}{dy^2}+m_1=0,\;\; u_0(0)=0,\;\; \frac{du_0(1)}{dy}=0. \tag{3.13}$$

1st order system

$$(1+M)\frac{d^2u_1}{dy^2}-\left(\frac{du_0}{dy}\right)^2\frac{d^2u_0}{dy^2}=0,\;\; u_1(0)=0,\;\; \frac{du_1(1)}{dy}=0. \tag{3.14}$$

2nd order system

$$(1+M)\frac{d^2u_2}{dy^2}-\left(\frac{du_0}{dy}\right)^2\frac{d^2u_1}{dy^2}-2\left(\frac{d}{dy}(u_0u_1)\right)\frac{d^2u_0}{dy^2}=0,\quad u_2(0)=0,\quad \frac{du_2(1)}{dy}=0. \quad (3.15)$$

These equations are solved with their respective boundary conditions to get the final form of $u(y)$ as

$$u=-\frac{m_1}{2(1+M)}y^2+\frac{m_1}{(1+M)}y-A\frac{m_1^3}{12(1+M)^4}y^4$$
$$+A\frac{m_1^3}{3(1+M)^4}y-A^2\frac{m_1^5}{84(1+M)^7}y^7+A^2\frac{m_1^5}{12(1+M)^7}y. \quad (3.16)$$

To solve the boundary value problem numerically, we have made use of shooting method with Rung-Kutta algorithm. In the next section the results obtained by numerical method and perturbation method are compared for different values of emerging parameters.

3.3 Results and discussion

Fig. (3.1) is prepared to make a comparison between perturbation and numerical solutions. It is observed from this figure that both the solutions are in good agreement for small values of A. However, the perturnation solution diverges form numerical solution as the value of A increases. It is also important to mention that a good agreement between the solution is highly dependent on the values of M and m_1. In fact for a fixed value of A perturbation solution diverges form numerical solution by decreasing (increasing) the value of M (m_1). The influence of A on the film velocity u is illustrated with the help of Fig. (3.2). This figure shows that the film velocity is an increasing function of A. We have plotted Figs. (3.3) and (3.4) to see the variation of u with y for different values of M and m_1 respectively. These figures depict that film velocity increases for large values of m_1. However, its magnitude decreases by increasing M.

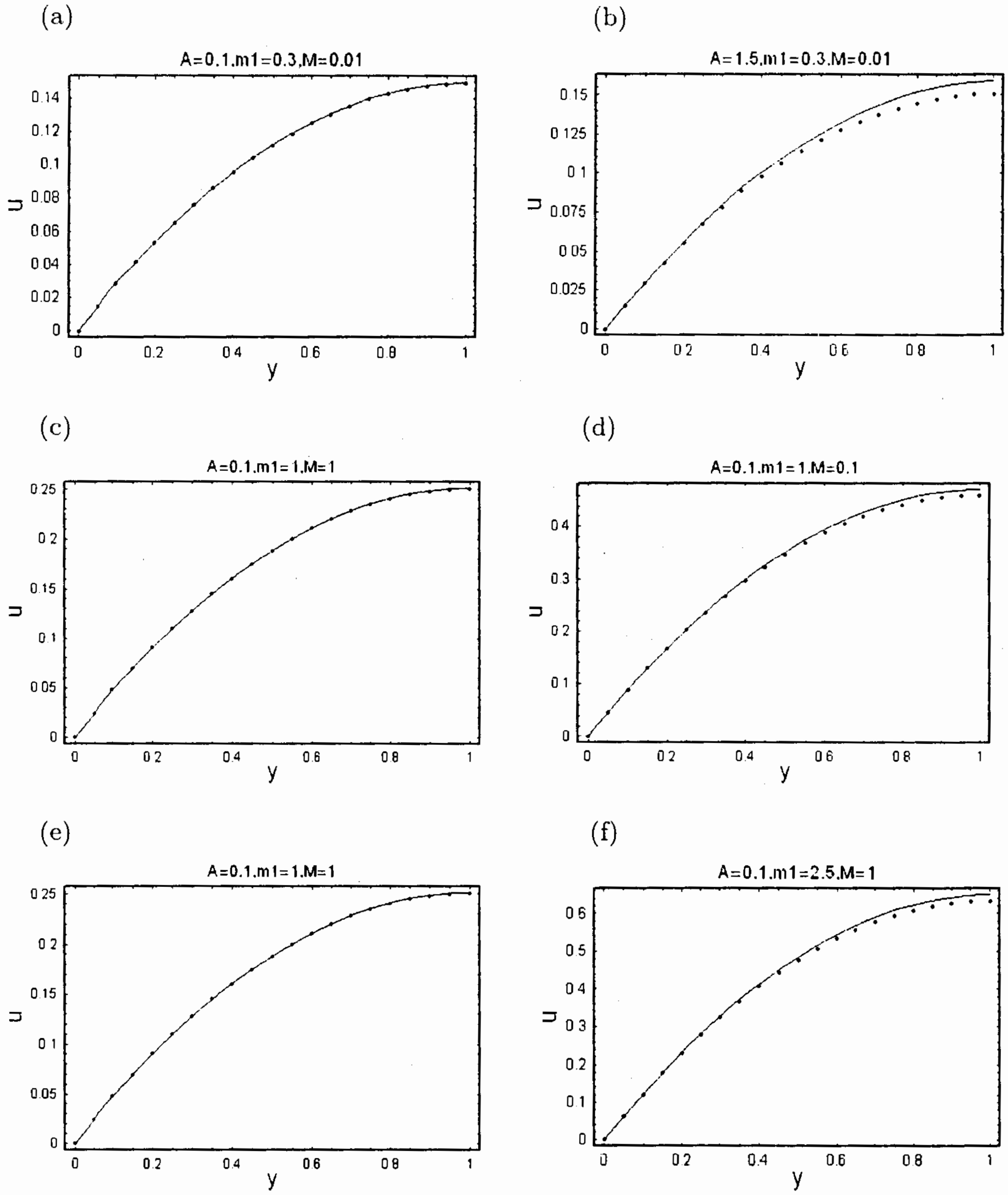

Fig. (3.1): Comparison of perturbation and numerical solutions for different values of A, M and m_1. Solid line indicates the perturbation solution while dotted line indicates the numerical solution.

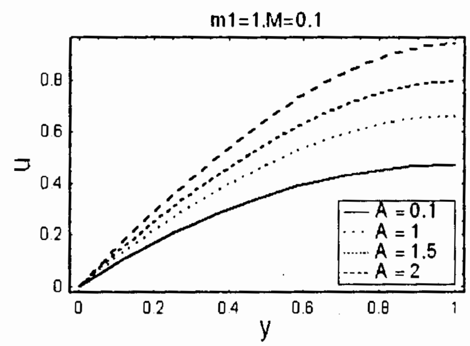

Fig. (3.2): Influence of A on u for $m_1 = 1$ and $M = 0.1$.

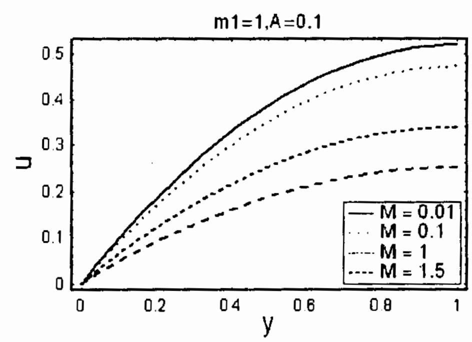

Fig. (3.3): Influence of M on u for, $m_1 = 1$ and $A = 0.1$.

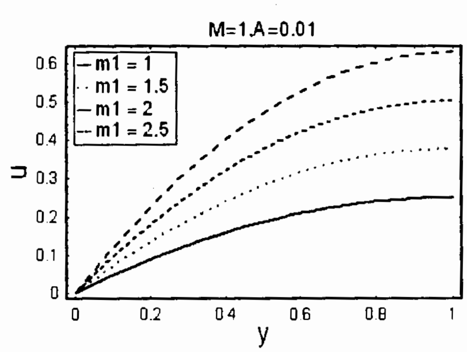

Fig. (3.4): Influence of $m1$ on u for, $M = 1$ and $A = 0.01$.

3.4 Thin film flow on a moving belt

For this case Eq. (2.30) in the appropriate form of the velocity field. Thus the equation of motion (1.17) yields

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial_x}T_{xx} + g_1. \tag{3.17}$$

From Eq. (3.2) we find that

$$T_{xx} = \mu\frac{dv}{dx} + \frac{1}{\beta}sinh^{-1}\left(\frac{1}{c}\frac{dv}{dx}\right), \tag{3.18}$$

or

$$T_{xx} = \mu\frac{dv}{dx} + \frac{1}{\beta c}\frac{dv}{dx} - \frac{1}{6\beta c^3}\left(\frac{dv}{dx}\right)^3. \tag{3.19}$$

With the help of Eq. (3.19) and neglecting pressure gradient we get

$$\left(\nu + \frac{1}{\rho\beta c}\right)\frac{d^2v}{dx^2} - \frac{1}{2\rho\beta c^3}\left(\frac{dv}{dx}\right)^2\frac{d^2v}{dx^2} + g_1 = 0. \tag{3.20}$$

Equation (3.20) is subject to the boundary conditions (2.33).

$$v = U_0 \quad \text{at} \quad x = 0 \quad \text{and} \quad \frac{dv}{dx} = 0 \quad \text{at} \quad x = \delta \tag{3.21}$$

In view of dimensionless variables defined by (2.34), Eq. (3.20) and boundary conditions (3.21) after dropping the bars can be casted in the following dimensionless form:

$$(1 + M)\frac{d^2v}{dx^2} - A_1\left(\frac{dv}{dx}\right)^2\frac{d^2v}{dx^2} + m_2 = 0, \tag{3.22}$$

$$v = 1 \quad \text{at} \quad x = 0 \quad \text{and} \quad \frac{dv}{dx} = 0 \quad \text{at} \quad x = 1, \tag{3.23}$$

where $A_1 = U_0^2/2\mu\beta c^3\delta^2 = MU_0^2/2c^2\delta^2$.

In a similar manner as described in previous section, an exact solution of Eq. (3.22) satisfying the boundary condition (3.23) is obtained. The results of perturbation solution are summarized below.

$$v_0 = -\frac{m_2}{2(1 + M)}x^2 + \frac{m_2}{(1 + M)}x + 1, \tag{3.24}$$

$$v_1 = -\frac{1}{12m_2}\left[-\frac{m_2}{1+M}x + \frac{m_2}{1+M}\right]^4 + \frac{1}{12m_2}\left(\frac{m_2}{1+M}\right)^4, \tag{3.25}$$

$$\begin{aligned} v_2 \;=\; & \frac{1}{1260\,(1+M)^7}\left(210m_2^4x + 210Mm_2^4x + 357m_2^5x - 420m_2^4x^2 - 420Mm_2^4x^2 - 630m_2^5x^2\right. \\ & +420m_2^4x^3 + 420Mm_2^4x^3 + 560m_2^5x^3 - 210m_2^4x^4 - 210Mm_2^4x^4 - 210m_2^5x^4 + 42m_2^4x^5 \\ & \left.+42Mm_2^4x^5 - 42m_2^5x^5 + 63m_2^5x^6 - 15m_2^5x^7\right) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} v = & -\frac{m_2}{2(1+M)}x^2 + \frac{m_2}{(1+M)}x + 1 - \frac{A_1}{12m_2}\left[-\frac{m_2}{1+M}x + \frac{m_2}{1+M}\right]^4 + \frac{A_1}{12m_2}\left(\frac{m_2}{1+M}\right) \\ & \frac{A_1^2}{1260\,(1+M)^7}\left(210m_2^4x + 210Mm_2^4x + 357m_2^5x - 420m_2^4x^2 - 420Mm_2^4x^2 - 630m_2^5x^2\right. \\ & + 420m_2^4x^3 + 420Mm_2^4x^3 + 560m_2^5x^3 - 210m_2^4x^4 - 210Mm_2^4x^4 - 210m_2^5x^4 + 42m_2^4x^5 \\ & \left.+ 42Mm_2^4x^5 - 42m_2^5x^5 + 63m_2^5x^6 - 15m_2^5x^7\right). \end{aligned} \tag{3.27}$$

Similarly as in the previous section the boundary value problem consisting of Eq. (3.22) and boundary conditions (3.23) is integrated numerically using shooting method with Runge-Kutta algorithm. A comparison of both numerical and perturbation solution is presented in the next section. Further effect of various emerging parameter of film velocity $v(y)$ are also illustrated in this section with the help of graphs.

3.5 Graphical results

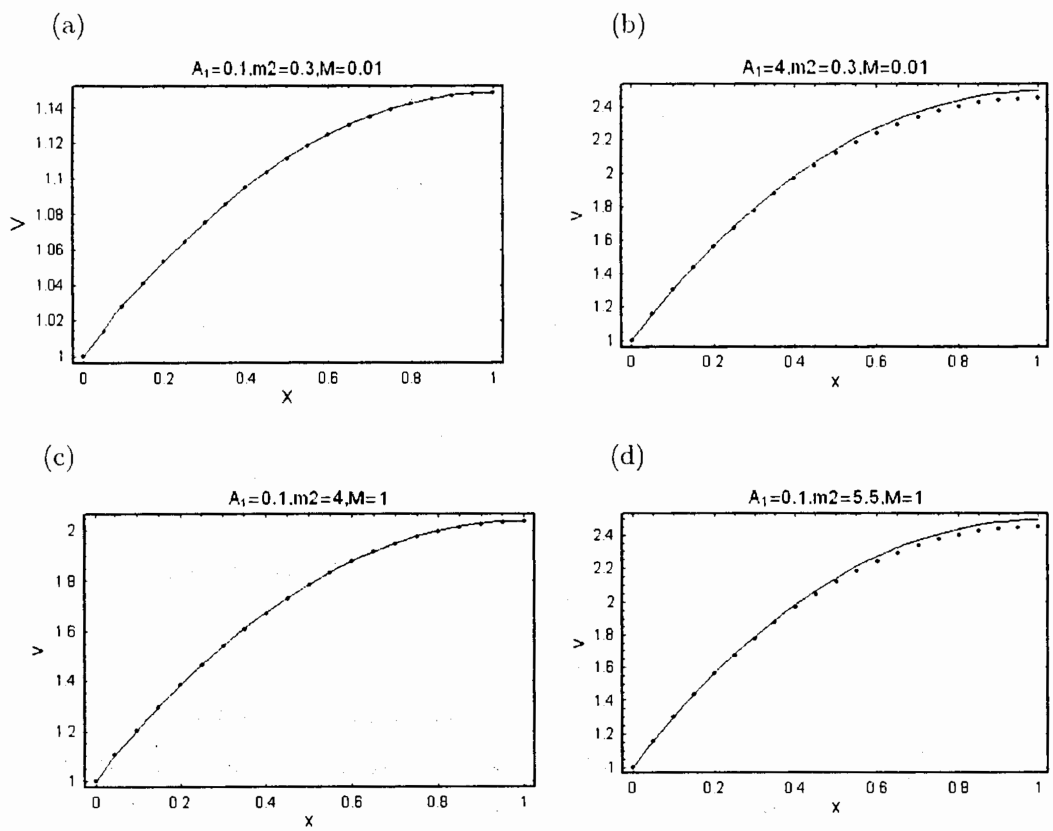

Fig. (3.5): Plots showing comparison between perturbation and numerical solutions. Solid line indicates the perturbation solution while dotted line indicates the numerical solution of the problem.

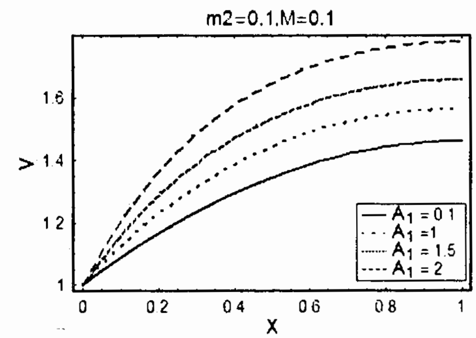

Fig. (3.6): Influence of A_1 on v for $m_2 = 1$ and $M = 0.1$.

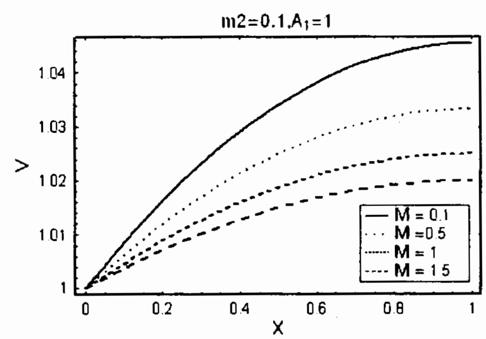

Fig. (3.7):. Influence of M on v for $m_2 = 0.1$ and $A_1 = 1$.

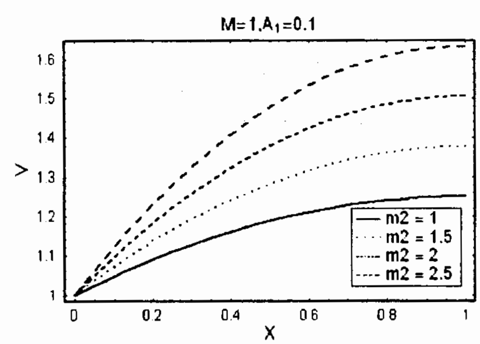

Fig. (3.8): Influence of $m2$ on v for $M = 1$ and $A_1 = 0.1$.

A comparison of perturbation and numerical solutions is presented in Fig. (3.5). It is observed form panel (a) that both the solution are in good agreement for small values of A_1.

However it is interesting to note that perturbation solution diverges form numerical solution for very large values of A_1 (panel (b)). Moreover, for a fixed value of A_1 and M perturbation solution starts diverging form numerical solution after a large value of m_2. For instance, if we choose $A_1 = 0.1$ and $M = 1$ the perturbation solution is identical with numerical solution up to $m_2 = 4$. Thus it can be concluded that for thin film flow on a moving belt the perturbation solution shows a significant deviation from the numerical solution for very large values of the non-Newtonian parameter. However, this is not the case with the previous problem of thin film flow down an inclined plane. Figs. (3.6) – (3.8) present the variation of film velocity $v(x)$ for different values of A_1, M and m_2. These figures elucidate that the film velocity increases by increasing A_1 and m_2, while it decreases when large values of M are taken into account.

3.6 Thin film flow down a vertical cylinder

For this case equation of motion is given by

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{r\rho}\frac{\partial}{\partial_r}\left(rT_{rz}\right) + g_1, \tag{3.28}$$

where we have made use of Eq. (2.40) and

$$T_{rz} = \left(\mu + \frac{1}{\beta c}\right)\frac{dw}{dr} - \frac{1}{6\beta c^3}\left(\frac{dw}{dr}\right)^3. \tag{3.29}$$

Substitution of Eq. (3.29) into Eq. (3.28) and neglecting the pressure gradient we have

$$0 = \frac{1}{r}\left[\left(\frac{\mu}{\rho} + \frac{1}{\rho\beta c}\right)\left(\frac{dw}{dr} + r\frac{d^2w}{dr^2}\right) - \frac{1}{6\rho\beta c^3}\left(\frac{dw}{dr}\right)^3 - \frac{3r}{6\rho\beta c^3}\left(\frac{dw}{dr}\right)^2\frac{d^2w}{dr^2}\right] + g_1. \tag{3.30}$$

The relevant boundary conditions are

$$w(R) = w'(R+\delta) = 0. \tag{3.31}$$

After making use of (2.48) the governing equation (3.30) and boundary conditions (3.31) have the following dimensionless forms

$$(1+M)\left(\frac{df}{d\eta}+\eta\frac{d^2f}{d\eta^2}\right)-A_2\left\{\left(\frac{df}{d\eta}\right)^3-3\eta\left(\frac{df}{d\eta}\right)^2\frac{d^2f}{d\eta^2}\right\}+m_3\eta=0, \tag{3.32}$$

$$f(1)=0, \qquad f'(d)=0, \tag{3.33}$$

where $A_2 = M\nu^2/6R^4c^2$ and $m_3 = R^3g_1/\nu^2$.

Eq. (3.32) can be written as

$$(1+M)\frac{d}{d\eta}\left(\eta\frac{df}{d\eta}\right)-A_2\frac{d}{d\eta}\left(\eta\left(\frac{df}{d\eta}\right)^3\right)+m_3\frac{d}{d\eta}\frac{\eta^2}{2}=0. \tag{3.34}$$

Integrating above equation and utilizing the second boundary condition in (3.33) results in

$$(1+M)\eta\frac{df}{d\eta}-A_2\eta\left(\frac{df}{d\eta}\right)^3+\frac{m_3}{2}\left(\eta^2-d^2\right)=0. \tag{3.35}$$

Note that Eq. (3.35) is a cubic equation in $df/d\eta$. Solving it for its real root and numerically integrating it with remaining boundary condition using Mathematica we have obtained an exact solution. To obtain perturbation solution we assume

$$f=f_0+A_2f_1+A_2^2f_2+o(A_2^3), \tag{3.36}$$

and get the following systems.

Zeroth order system

$$(1+M)\left(\frac{df_0}{d\eta}+\eta\frac{d^2f_0}{d\eta^2}\right)+m_3\eta=0 \quad f_0(1)=0 \quad \text{and} \quad \frac{df_0(d)}{d\eta}=0, \tag{3.37}$$

Ist order system

$$(1+M)\left(\frac{df_1}{d\eta}+\eta\frac{d^2f_1}{d\eta^2}\right)-\left(\frac{df_0}{d\eta}\right)+3\eta\left(\frac{df_0}{d\eta}\right)^2\frac{d^2f_0}{d\eta^2}=0 \quad f_1(1)=0 \quad \text{and} \quad \frac{df_1(d)}{d\eta}=0, \tag{3.38}$$

2nd order system

$$(1+M)\left(\frac{df_2}{d\eta}+\eta\frac{d^2f_2}{d\eta^2}\right)-3\left(\frac{df_0}{d\eta}\right)^2\left(\frac{df_1}{d\eta}\right)+6\eta\frac{d^2f_0}{d\eta^2}\frac{df_0}{d\eta}\frac{df_1}{d\eta}+3\eta\frac{d^2f_1}{d\eta^2}\left(\frac{df_0}{d\eta}\right)^2=0, \quad (3.39)$$

with

$$f_2(1)=0 \text{ and } \frac{df_2(d)}{d\eta}=0. \quad (3.40)$$

The above systems have following solutions

$$f_0=-\frac{m_3\eta^2}{4(1+M)}+\frac{m_3d^2}{2(1+M)}\ln\eta-\frac{m_3}{4(1+M)}, \quad (3.41)$$

$$f_1=\frac{1}{64(1+M)^4\eta^2}\left(8d^6m_3^3-m_3^3\eta^2-8d^6m_3^3\eta^2+m_3^3\eta^6+12d^4m_3^3\eta^2\ln\eta\right. \quad (3.42)$$
$$\left.+48d^4m_3^3\eta^2\ln d\ln\eta-24d^4m_3^3\eta^2\ln\eta^2\right),$$

$$f_2=\frac{1}{78(1+M)^7\eta^4}\left(-54d^{10}m_3^5+72d^8m_3^5\eta^2+4m_3^5\eta^4-9d^2m_3^5\eta^4-108d^4m_3^5\eta^4-72d^8m_3^5\eta^4\right.$$
$$+54d^{10}m_3^5\eta^4+108d^4m_3^5\eta^6+9d^2m_3^5\eta^8-4m_3^5\eta^{10}+432d^8m_3^5\eta^2\ln d-432d^8m_3^5\eta^4\ln d$$
$$-432d^8m_3^5\eta^2\ln\eta+132d^6m_3^5\eta^4\ln\eta+1584d^6m_3^5\eta^4\ln\eta\ln d+864d^6m_3^5\eta^4\ln\eta\ln d^2$$
$$\left.-792d^6m_3^5\eta^4\ln\eta^2-864d^6m_3^5\eta^4\ln\eta^2\ln d+288d^6m_3^5\eta^4\ln\eta^3\right), \quad (3.43)$$

upon making use of above results in (3.32) we have

$$f=-\frac{m_3\eta^2}{4(1+M)}+\frac{m_3d^2}{2(1+M)}\ln\eta-\frac{m_3}{4(1+M)}+\frac{A}{64(1+M)^4\eta^2}\left(8d^6(m_3)^3-m_3^3\eta^2-8d^6m_3^3\eta^2\right.$$
$$\left.+m_3^3\eta^6+12d^4m_3^3\eta^2\ln\eta+48d^4m_3^3\eta^2\ln d\ln\eta-24d^4m_3^3\eta^2L\ln\eta^2\right)$$
$$\frac{A^2}{78(1+M)^7\eta^4}\left(-54d^{10}m_3^5+72d^8m_3^5\eta^2+4m_3^5\eta^4-9d^2m_3^5\eta^4-108d^4m_3^5\eta^4-72d^8m_3^5\eta^4\right.$$
$$+54d^{10}m_3^5\eta^4+108d^4m_3^5\eta^6+9d^2m_3^5\eta^8-4m_3^5\eta^{10}+432d^8m_3^5\eta^2\ln d-432d^8m_3^5\eta^4\ln d$$
$$-432d^8m_3^5\eta^2\ln\eta+132d^6m_3^5\eta^4\ln\eta+1584d^6m_3^5\eta^4\ln\eta\ln d+864d^6m_3^5\eta^4\ln\eta\ln d^2$$
$$\left.-792d^6m_3^5\eta^4\ln\eta^2-864d^6m_3^5\eta^4\ln\eta^2\ln d+288d^6m_3^5\eta^4\ln\eta^3\right). \quad (3.44)$$

The validity of perturbation results is discussed by comparing it with exact solution in the next section.

3.7 Numerical results and discussion

Fig. (3.9) is sketched to see how the non-Newtonian parameter A_2 effects the perturbation solution. As expected the perturbation solution matches the numerical solution of small values of A_2 and deviates form it as the values of A_2 exceeds unity. The effects of A_2, M and m_3 on film velocity f can be analyzed through Figs. (3.10) – (3.12). Figs. (3.10) and (3.12) show that the magnitude of film velocity increases for increasing values of parameters A_2 and m_3. However, an increase in M causes a decrease in the velocity of film.

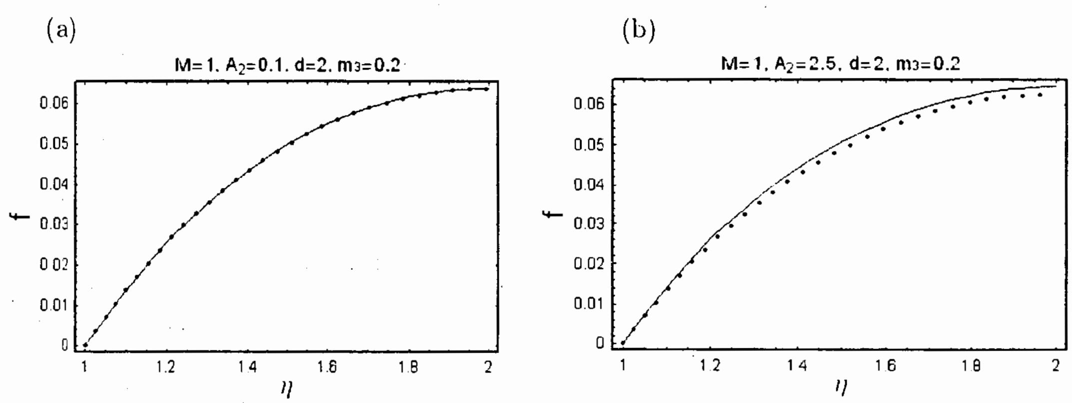

Fig. (3.9): Graphs showing comparison between perturbation and exact solutions. The solid line shows the exact solution while the dotted line represents the perturbation solution.

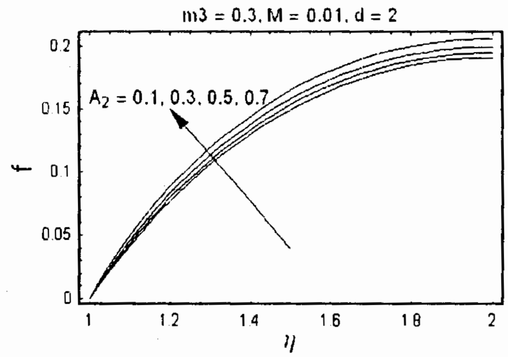

Fig. (3.10): Influence of A_2 on f for $m3 = 0.3$, $M = 0.01$ and $d = 2$.

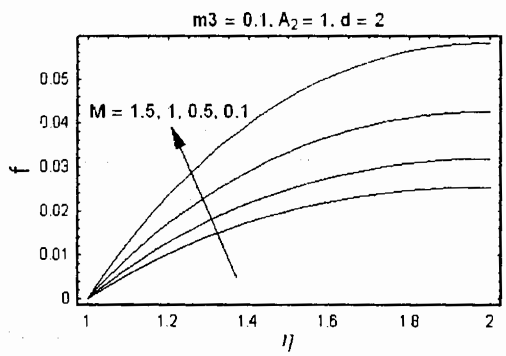

Fig. (3.11): Influence of M on f for $m3 = 0.1$, $A_2 = 1$ and $d = 2$.

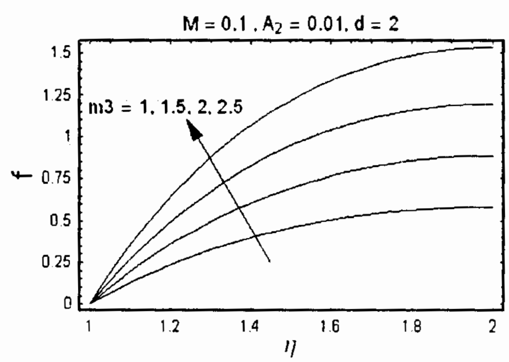

Fig. (3.12): Influence of $m3$ on f for $M = 0.1$, $A_2 = 0.01$ and $d = 2$.

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