Generalized Bi-Quasi-Variational Inequalities for Quasi-Pseudo-Monotone Type I Operators in Non-Compact Settings.

By

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2008

variation inequalities (Mathematics)
Critical point theory (Mathematical analysis)

Starting in the name of 'Allah'

The Lord of the Worlds....

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A Dissertation
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We accept this thesis as conforming to the required standard.

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Declaration

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1. Dr. M.S.R. Chowdhury ______________________
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dedicated To

my mother and father
& my family
who waited
patiently for me
to complete my studies.
The wait is over.

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ABSTRACT

In this dissertation we shall prove some existence results of solutions for a new class of generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators in non-compact settings in locally convex Hausdorff topological vector spaces.

In obtaining these results on generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators in non-compact settings, we shall use the concept of escaping sequences, introduced by Border [**2**], and apply Chowdhury and Tan's result on generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators on compact sets [11].

Symbols and Abbreviations

2^X .... the family of all non-empty subsets of X.

$\Im(X)$ .... the family of all non-empty finite subsets of X.

Φ ....either the real field **R** or the complex field **C**.

C .... the set all complex numbers.

R .... the real line.

N .... the set of all natural numbers.

ϕ ....the empty set.

E^* ....the dual space of E.

G(T) .... the graph of the mapping T.

KKM Theorem .... Knaster-Kuratowski-Mazurkiewicz Theorem.

CONTENTS

CHAPTER 1

Introduction

In this dissertation we shall review and obtain some results on generalized bi-quasi-variational inequalities in non-compact settings. Thus we shall begin this chapter by defining the generalized bi-quasi-variational inequalities. For this we need to introduce some notations which will be used throughout this thesis.

Given a non-empty set X, we shall denote by 2^X the class or family of all non-empty subsets of X, and by $\Im(X)$ the family of all non-empty finite subsets of X. Moreover Φ will denote either the real field **R** or the complex field **C.**

Definition 1.1. Suppose X, Y and Z are vector spaces and B maps $X \times Y$ into Z. Associate to each $x \in X$ and to each $y \in Y$ the mappings $B_x : Y \to Z$ and $B^y : X \to Z$ by defining $B_x(y) = B(x,y) = B^y(x)$. B is said to be bilinear if every B^x and every B^y is linear.

We shall present some examples of bilinear mappings:

Examples 1.1

- Matrix multiplication is a bilinear map: *M*(*m*, *n*) × *M*(*n*, *p*) → *M*(*m*, *p*).
- If a vector space *V* over the real numbers **R** carries an inner product, then the inner product is a bilinear map *V* × *V* → **R**.
- In general, for a vector space *V* over a field *F*, a bilinear form on *V* is the same as a bilinear map *V* × *V* → *F*.

- If V is a vector space with dual space V^*, then the application operator, $b(f, v) = f(v)$ is a bilinear map from $V^* \times V$ to the base field.
- Let V and W be vector spaces over the same base field F. If f is a member of V^* and g a member of W^*, then $b(v, w) = f(v)g(w)$ defines a bilinear map $V \times W \to F$.
- The cross product in $\mathbf{R}^3$ is a bilinear map $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$.
- Let $B : V \times W \to X$ be a bilinear map, and $L : U \to W$ be a linear operator, then $(v, u) \to B(v, Lu)$ is a bilinear map on $V \times U$.

The generalized bi-quasi-variational inequality problem was first introduced by Shih and Tan [**14**] in 1989. The following is the definition due to Shih and Tan in [**14**].

Definition 1.2. Let E and F be vector spaces over Φ, let $<,>: F \times E \to \Phi$ be a bilinear functional, and X be a non empty subset of E. If $S : X \to 2^X$ and $M, T : X \to 2^F$, the generalized bi-quasi variational inequality (GBQVI) problem for the triple (S, M, T) is to find $\hat{y} \in X$ satisfying the properties

(i) $\hat{y} \in S(\hat{y})$ and

(ii) $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle \le 0$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$.

When T is single-valued, a generalized bi-quasi variational inequality problem reduces to a bi-quasi variational inequality problem. Note that the generalized bi-quasi variational inequality problem include the following generally known variational type inequality problems:

Suppose E is a topological vector space, $F = E^*$, the vector space of all continuous linear functionals on E and $\langle , \rangle$ is the usual duality pairing between E^* and E. Then:

(i) If $T \equiv 0$, a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a generalized quasi-variational inequality (GQVI) problem. Chan and Pang

[5] first studied GQVI problems in finite dimensional case and Shih and Tan [15] studied them in infinite dimensional case.

(ii) If $T \equiv 0$ and M is single-valued, a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a quasi-variational inequality problem which was introduced by Bensoussan and Lions [1].

(iii) If $S(x) \equiv X$ and $M \equiv 0$, a generalized bi-quasi-variational inequality problem becomes a generalized variational inequality problem which was studied by Browder [4] and Yen [16] among many others.

The following definition of generalized bi-quasi–variational inequality problem due to Chowdhury and Tan in [8] is a slight modification of **Definition 1.2**.

Definition 1.3. Let E and F be vector spaces over Φ, let $<,>: F \times E \to \Phi$ be a bilinear functional, and X be a non-empty subset of E. If $S: X \to 2^X$ and $M, T: X \to 2^F$, then the generalized bi-quasi variational inequality (GBQVI) problems for the triple (S, M, T) is:

(i) to find a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and $\mathrm{Re}\langle f - \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$

or

(ii) to find a point $\hat{y} \in X$, a point $\hat{w} \in T(\hat{y})$ and a point $\hat{f} \in M(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and $\mathrm{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$.

Our main result will be obtained on generalized bi-quasi-variational inequalities using Chowdhury and Tan's following definition of quasi-pseudo-monotone type I operators given in [11]:

Definition 1.4. Let E be a topological vector space, X be a non-empty subset of E and F be a topological vector space over Φ. Let $\langle , \rangle : F \times E \to \Phi$ be a bilinear functional. Suppose we have the following three maps:

(i) $h : X \to$ **R**.

(ii) $M : X \to 2^F$ and

(iii) $T : X \to 2^F$.

Then T is said to be an (1) *h*–quasi–pseudo-monotone type I operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with

$$\limsup_\alpha \left[\inf_{f\in M(y)} \inf_{u\in T(y_\alpha)} \mathrm{Re}\langle f-u, y_\alpha - y\rangle + h(y_\alpha) - h(y) \right] \le 0$$

we have

$$\limsup_\alpha \left[\inf_{f\in M(x)} \inf_{u\in T(y_\alpha)} \mathrm{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x) \right]$$

$$\ge \inf_{f\in M(x)} \inf_{w\in T(y)} \mathrm{Re}\langle f-w, y-x\rangle + h(y) - h(x) \quad \text{for all } x \in X;$$

(2) a quasi-pseudo-monotone type I operator if T is an *h*-quasi–pseudo-monotone type I operator with $h \equiv 0$.

Note that when $M \equiv 0$, and T is replaced by $-T$, and $F = E^*$, an *h*-quasi–pseudo–monotone type I operator is reduced to the following *h*-pseudo-monotone operator (respectively, *h*-demi-monotone operator) defined in [**6**].

Definition 1.5. Let E be a Topological vector space, X be a non-empty subset of E, and $T : X \to 2^{E^*}$. If $h : X \to$ **R**, then T is said to be an *h*-pseudo-monotone (respectively, *h*-demi-monotone) operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (respectively, weakly to y) with

$$\limsup_\alpha \left[\inf_{u\in T(y_\alpha)} \mathrm{Re}\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y) \right] \le 0$$

we have

$$\limsup_{\alpha}\left[\inf_{u\in T(y_\alpha)} \mathrm{Re}\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]$$

$$\geq \inf_{w\in T(y)} \mathrm{Re}\langle \mathrm{w}, \mathrm{y} - \mathrm{x}\rangle + \mathrm{h(y)} - \mathrm{h(x)} \text{ for all } x \in X;$$

T is said to be pseudo-monotone (respectively, demi-monotone) if T is *h*-pseudo-monotone (respectively, *h*-demi-monotone) with $h \equiv 0$. This definition is slightly more general than the definition of *h*-pseudo-monotone operator given in **[7]**.

Later, these operators were re-named as pseudo-monotone type I operators in [9]. The pseudo-monotone type I operators are set-valued generalization of the classical (single-valued) pseudo-monotone operators with slight variations. The classical definition of a single-valued pseudo-monotone operator was introduced by Brézis, Nirenberg and stampacchia in **[3]**.

We observe that the definition of quasi-pseudo-monotone type I operators given in **Definition 1.4** above is a generalization of pseudo-monotone type I operators. In this dissertation we shall obtain some general theorems on solutions for a new class of generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators defined in non compact settings in topological vector spaces.

CHAPTER 2

Preliminary Concepts, and Results on Quasi Pseudo-Monotone Type I Operators

2.1 Preliminary and Basic Definitions, and Examples

We shall begin with some basic definitions.

Definition 2.1.1. Let E be a vector space over a field Φ where Φ is the field of real or complex numbers. A norm on E is a function $\|\cdot\|: E \to \mathbf{R}$ satisfying the following conditions:

N1: $\|x\| \geq 0$ for all $x \in E$ and $\|x\| = 0$ if and only if $x = 0$.

N2: $\|ax\| = |a|\|x\|$ for all $a \in \Phi$ and $x \in E$.

N3: $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

Then $(E, \|\cdot\|)$ is called a normed space.

Definition 2.1.2. A sequence $\{x_n\}$ in a normed space N is said to be a Cauchy sequence in N if for every $\varepsilon > 0$ there exist a natural number $n_0 \in N$ such that $m, n > n_0 \Rightarrow \|x_m - x_n\| < \varepsilon$.

Definition 2.1.3. A complete metric space is a metric space in which every Cauchy sequence is convergent.

Definition 2.1.4. A complete normed space is called a Banach space.

Definition 2.1.5. A subset X of a vector space E is said to be convex if for all $x,\ y \in X$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in X$.

Definition 2.1.6. Let X be a subset of a vector space E. For any elements $x_1, x_2 \cdots x_n$ of X, the linear combination $\sum_{i=1}^{n} \alpha_i x_i$, with $\sum_{i=1}^{n} \alpha_i = 1$, and $\alpha_i \geq 0$, for $i = 1,2,\cdots,n$ is called a convex combination.

Definition 2.1.7. Let X be subset of a vector space E. For each subset X of E, there is a unique smallest convex set containing X, namely the intersection of all convex subsets containing X. We shall call this intersection the convex hull of X which will be denoted by $co(X)$.

Definition 2.1.8. Suppose that $(X, \Im)$ is a topological space. Then a collection Ω of subsets of X is to be a cover for X if $X = \bigcup_{G \in \Omega} G$.

Definition 2.1.9. If every set of a cover Ω is in $\Im$ then Ω is called an open cover for X.

Definition 2.1.10. A finite sub collection Ω_1 of Ω is said to be a finite sub-cover for X if $X = \bigcup_{G \in \Omega_1} G$.

Definition 2.1.11. A topological space $(X, \Im)$ is said to be compact if every open cover of X contains a finite sub-cover.

Definition 2.1.12. Let $(X, \Im)$ be a topological space and $x \in X$. Then a sub-collection β of $\Im$ is said to be a neighborhood base or simply a base at x, if for any $U \in \Im$ with $x \in U$, there is a $B \in \beta$ such that $x \in B \subseteq U$.

Definition 2.1.13. A topological space $(X, \Im)$ is said to be a Hausdorff space if for any two distinct points a,b in X there exist open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \phi$.

Definition 2.1.14. A topological space $(X, \Im)$ is said to be a regular space if for any closed set A and any point x not in A, there are open sets U and V such that $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

Definition 2.1.15. A topological space $(X, \Im)$ is said to be a normal space if for any two disjoint closed subsets A and B of X there are open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

Definition 2.1.16. (**Partially Ordered Set**) A partially ordered set, consists of a set D and a binary relation "$\leq$" on D which satisfies the following properties:

(1) $a \leq a$ for all $a \in D$ (reflexive property);

(2) if $a \leq b$ and $b \leq a$, then $a = b$ for all $a,b \in D$ (anti-symmetric property);

(3) if $a \leq b$ and $b \leq c$, then $a \leq c$ for all $a,b,c \in D$ (transitive property).

Definition 2.1.17. (Directed Set) A **directed set** is a **partially ordered set** $(D,\leq)$ such that whenever $a,b \in D$ there is an $x \in D$ such that $a \leq x$ and $b \leq x$ (finite upper bound property).

Example 2.1.1. If N is a set of natural numbers then $(N, \leq)$ is a directed set where "$\leq$" is the usual less then or equal to relation.

Example 2.1.2. If **R** is the set of real numbers then $(\mathbf{R}, \leq)$ is a directed set where "$\leq$" is the usual less then or equal to relation.

Definition 2.1.18. A function $f:D\to(X,\Im)$, from a directed set $(D,\leq)$ in to a topological space $(X,\Im)$ is called a net in $(X,\Im)$. A point $f(\alpha)\in X$ is usually denoted by x_α and a net f itself is denoted by $(x_\alpha)_{\alpha\in D}$ or simply by (x_α) if the index set is understood.

Definition 2.1.19. Let E be a vector space. Then a subset X of E is called a subspace of E if X is itself a vector space under the operations of addition and scalar multiplication inherited from E.

Note that in the above definition, X becomes a subspace of E, if X is closed under the operations of addition and scalar multiplication inherited from E. In particular, X will be a subspace of E, if we can show that $\alpha u+\beta v\in X$ for all vectors $u,v\in X$ and all scalars $\alpha,\beta\in\Phi$.

Definition 2.1.20. If X is a subset of vector space E, then X is said to be convex if

$$tX+(1-t)X\subset X \text{ for all } t \text{ with } 0\leq t\leq 1;$$

$$\text{or } \quad tx+(1-t)y\in X \text{ for all } x,y\in X \text{ and all } t \text{ with } 0\leq t\leq 1.$$

Definition 2.1.21. A subset B of a vector space X is said to be balanced if $\alpha B\subset B$ for every $\alpha\in\Phi$ with $|\alpha|\leq 1$.

Definition 2.1.22. Let E be a vector space and τ be a topology on E. The (E,τ) is said to be a topological vector space if the vector space operations, i.e., addition and scalar multiplications, are continuous with respect to τ.

Definition 2.1.23. A subset X of a topological vector space E is said to be bounded if to every neighborhood V of 0 in E corresponds a number $s>0$ such that $X \subset tV$ for every $t > s$.

Definition 2.1.24. Let X and Y be subsets of a vector space E such that $co(X) \subset Y$. Then $T: X \to 2^Y$ is called a KKM-map if for each $A \in \Im(X)$, $\mathrm{co}(A) \subset \bigcup_{x \in A} T(x)$ where $\Im(\mathrm{X})$ is the family of all non-empty finite subsets of X. Note that if T is a KKM-map, then $x \in \mathrm{T(x)}$ for all $\mathrm{x} \in \mathrm{X}$.

Definition 2.1.25. If X and Y are topological spaces and $T: X \to 2^Y$, then the graph of T is defined to be the set $G(T) := \{(x, y) \in X \times Y \mid y \in T(x)\}$.

Definition 2.1.26 (Closed Graph) If X and Y are sets and f maps X into Y, the graph of f is the set of all points $(x, f(x))$ in the cartesian product $X \times Y$. If X and Y are topological spaces, if $X \times Y$ is given the usual product topology (the smallest topology that contains all sets $U \times V$ with U and V open in X and Y, respectively), and if $f: X \to Y$ is continuous and Y is Hausdorff, then the graph G of f is closed.

2.2 Preliminary Results on Quasi-Pseudo-Monotone Type I Operators

Recall that throughout this dissertation, Φ will denote either the real field **R** or the complex field **C.** Let E be a topological vector space over Φ, F be a vector space over Φ and $\langle , \rangle : F \times E \to \Phi$ be a bilinear functional. For each $x_0 \in E$, each non-empty subset A of E and each $\varepsilon > 0$, let

$$W(x_0;\varepsilon) := \{y \in F : |< y, x_0 >| < \varepsilon\}$$

$$\text{and} \quad U(A;\varepsilon) := \left\{y \in F : \sup_{x \in A} |< y, x >| < \varepsilon\right\}.$$

Let $\sigma < F, E >$ be the (weak) topology on F generated by the family $\{W(x;\varepsilon); x \in E \text{ and } \varepsilon > 0\}$ as a sub-base for the neighborhood system at 0 and $\delta < F, E >$ be the (strong) topology on F generated by the family $\{U(A,\varepsilon) : A$ is a non-empty bounded subset of E and $\varepsilon > 0\}$ as a base for a neighborhood system at 0. We note then that F, when equipped with the (weak) topology $\sigma < F, E >$ or the (strong) topology $\delta < F, E >$, becomes a locally convex topological vector space which is not necessarily Hausdorff. But if the bilinear functional $<,> : F \times E \to \Phi$ separates points in F, i.e., for each $y \in F$ with $y \neq 0$, there exist $x \in E$ such that $< y, x > \neq 0$, then F also becomes Hausdorff.

Furthermore, for a net $\{y_\alpha\}_{\alpha \in \Gamma}$ in F and for $y \in F$, (i) $y_\alpha \to y$ in $\sigma\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$ and (ii) $y_\alpha \to y$ in $\delta\langle F, E\rangle$ if and only if

$\langle y_{\alpha}, x\rangle \to \langle y, x\rangle$ uniformly for $x \in A$ for each non-empty bounded subset A of E.

Let X be a non-empty subset of E, then X is a cone in E if X is convex and $\lambda X \subset X$ for all $\lambda \geq 0$. If X is a cone in E and $\langle , \rangle : F \times E \to \Phi$ is a bilinear functional, then $\hat{X} = \{w \in F : \mathrm{Re}\langle w, x\rangle \geq 0$ for all $x \in X\}$ is also a cone in F, called the dual cone of X (with respect to the bilinear functional $\langle , \rangle$).

Definition 2.2.1. (Linear mapping) Let X and Y be vector spaces over the same scalar field Φ. A linear mapping, $T : X \to Y$, is a function such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all x and y in X and all scalars α and β.

Proposition 2.2.1. Let E be a topological vector space and $T : E \to 2^E$ be a set-valued linear mapping. Then T is always single-valued.

Proof: We have $T(0) = 0T(0) = \{0\}$. Then for any vector $z \in E$, we have

$$\{0\} = T(z - z) = T(z) - T(z)$$

by the linearity of T. But $T(z) - T(z) = \{x - y \mid x, y \in T(z)\} = \{0\}$. Thus $x - y = 0$ for all $x, y \in T(z)$. Hence, $x = y$ for all $x, y \in T(z)$. Consequently, $T(z)$ is single-valued, say $T(z) = x$, for some $x \in E$.

Definition 2.2.2. Let X be a convex set in a topological vector space E. Then $f : X \to$ **R** is called

(i) lower semi-continuous $\Leftrightarrow$ for all $\lambda \in$ **R**, $\{x \in X \mid f(x) \leq \lambda\}$ is closed in X;

(ii) upper semi-continuous $\Leftrightarrow$ $-f$ is lower semi-continuous, i.e., for all $\lambda \in$ **R**, $\{x \in X \mid f(x) \geq \lambda\}$ is closed in X.

Definition 2.2.3. Let X and Y be topological spaces and $T: X \to 2^Y$. Then T is said to be **upper (respectively, lower) semi-continuous** at $x_0 \in X$ if for each open set G in Y with $T(x_0) \subset G$ (respectively, $T(x_0) \cap G \neq \phi$,) there exists an open neighborhood U of x_0 in X such that $T(x) \subset G$ (respectively, $T(x) \cap G \neq \phi$) for all $x \in U$. Moreover, T is said to be **continuous** at the point $x_0 \in X$ if T is **both upper semi-continuous and lower semi-continuous** at $x_0 \in X$. And T is said to be **continuous on** X if T is **continuous at each point** x_0 **of** X.

Definition 2.2.4. Let X be a convex set in a vector space E. Then $f: X \to$ R is:

(i) convex if and only if for all $x, y \in X$ and for all $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

(ii) concave if and only if for all $x, y \in X$ and for all $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

(iii) quasi-concave if and only if for all $\lambda \in$ **R**

$$\{x \in X \mid f(x) > \lambda\} \text{ is convex.}$$

The following definition was given by K. C. Border in **[2]**:

Definition 2.2.5. Let X be a topological space such that $X = \cup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact subsets of X. Then a sequence $\{x_n\}_{n=1}^{\infty}$ is said to be escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ if for each $n \in$

N, there exists $m \in \mathbf{N}$ such that $x_k \notin C_n$ for all $k \geq m$.

In obtaining the results on generalized bi-quasi-variational inequalities (GBQVI) for quasi-pseudo-monotone type I operates in non-compact settings, we shall use the concept of escaping sequences introduced by Border [**2**] with the application of Chowdhury and Tan's result [**Theorem 2.2.2 below**] on generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators on non compact sets.

We shall first state the following result of M. S. R. Chowdhury and K. K. Tan in [**11, Theorem 3.1**]:

Theorem 2.2.1. let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E and F a Hausdorff topological vector space over Φ. Let $\langle , \rangle : \mathrm{F} \times \mathrm{E} \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that

(a) $S : \mathrm{X} \to 2^{\mathrm{X}}$, is upper semi-continuous such that each $S(x)$ is closed and convex ;

(b) $h : \mathrm{X} \to \mathrm{R}$ is convex and continuous ;

(c) $T : \mathrm{X} \to 2^{\mathrm{F}}$ is an *h*-quasi-pseudo-monotone type I operator and is upper semi-continuous such that each $T(x)$ is compact and convex and $T(x)$ is strongly bounded;

(d) $M : X \to 2^{F}$ is a linear map in X (and is therefore single-valued for each $x \in X$);

(e) the set

$$\Sigma = \left\{ y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re} < M(x) - w, y - x > + h(y) - h(x)] > 0 \right\}$$

is open in X.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re} < M(\hat{y}) - \hat{w}, \hat{y} - x > \leq h(x) - h(\hat{y})$, for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex and if $T \equiv 0$, the continuity assumption on $\langle\ ,\ \rangle$ can be weakened to the assumption that for each $f \in F$, the map $x \mapsto < f, x >$ is continuous on X.

Applying the above **Theorem 2.2.1**, Chowdhury and Tan obtained the following result in [**11, Theorem 3.2**]:

Theorem 2.2.2. Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E and F be a vector space over Φ. Let $\langle\ ,\ \rangle : \mathrm{F} \times \mathrm{E} \to \Phi$ be a bilinear functional such that $\langle\ ,\ \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto < f, x >$ is continuous on X. Equip F with the strong topology $\delta < F, E >$. Suppose that

(a) $S : \mathrm{X} \to 2^{\mathrm{X}}$, is a continuous map such that each $S(x)$ is closed and convex;

(b) $h : \mathrm{X} \to \mathbf{R}$ is convex and continuous;

(c) $T : X \to 2^F$ is an h-quasi-pseudo-monotone type I operator and is upper semi-continuous such that each $T(x)$ is strongly $(\delta < F, E >)$-compact and convex;

(d) $M : X \to 2^F$ is a continuous linear map in X and for each $y \in \Sigma$, where

$$\Sigma = \left\{ y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re} < M(x) - w, y - x > + h(y) - h(x)] > 0 \right\},$$

$\inf_{w \in T(y)} \operatorname{Re} < M(x) - w, y - x > + h(y) - h(x) > 0$ for some point x in $S(y)$.

Then there exist a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re} < M(\hat{y}) - \hat{w}, \hat{y} - x > \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

For completeness we shall include the proof here as outlined in **[11]**:

Proof: As $\langle \, , \, \rangle : F \times E \to \Phi$ is a bilinear functional such that for each $f \in F$, the map $x \mapsto < f, x >$ is continuous on X and as F is equipped with strong topology $\delta < F, E >$, the bilinear functional, $\langle \, , \, \rangle$ is continuous on compact subsets of $F \times X$. Thus by **Theorem 2.2.1**, it suffices to show that the set

$$\Sigma = \left\{ y \in X : \sup_{x \in S(y)} \left[\inf_{w \in T(y)} \operatorname{Re} < M(x) - w, y - x > + h(y) - h(x) \right] > 0 \right\}$$

is an *open* in X. Indeed, let $y_0 \in \Sigma$; then by the last part of the **hypothesis (d)**, M is a continuous linear map on X and

$$\inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, y_0 - x_0 > + h(y_0) - h(x_0) > 0$$

for some point x_0 in $S(y_0)$. Let

$$\alpha := \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, y_0 - x_0 > + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$W := \left\{ w \in F : \sup_{z_1, z_2 \in X} \left| < w, z_1 - z_2 > \right| < \alpha/6 \right\}.$$

Then W is an open neighborhood of 0 in F so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in F. Since T is upper semi–continuous at y_0, there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Let $U_2 := M(x_0) + W$, then U_2 is an open neighborhood of $M(x_0)$ in F. Since M is continuous at x_0, there exist an open neighborhood V_1 of x_0 in X such that $M(x) \in U_2$ for all $x \in V_1$.

As the map $x \mapsto \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, x_0 - x > + h(x_0) - h(x)$ is continuous at x_0, there exist an open neighborhood V_2 of x_0 in X such that

$$\left| \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, x_0 - x > + h(x_0) - h(x) \right| < \alpha/6$$

for all $x \in V_2$. Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X.

since $x_0 \in V_0 \cap S(y_0) \neq \phi$ and S is lower semi-continuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_0 \neq \phi$ for all $y \in N_2$.

Since the map $y \mapsto \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, y - y_0 > + h(y) - h(y_0)$ is continuous at y_0, there exists an open neighborhood N_3 of y_0 in X such that

$$\left| \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, y - y_0 > + h(y) - h(y_0) \right| < \alpha/6$$

for all $y \in N_3$.

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

(i) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$;

(ii) $S(y_1) \cap V_0 \neq \phi$ as $y_1 \in N_2$; so we can choose any $x_1 \in S(y_1) \cap V_0$;

(iii) $\left| \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, y_1 - y_0 > + h(y_1) - h(y_0) \right| < \alpha/6$ as $y_1 \in N_3$;

(iv) $M(x_1) \in U_2 = M(x_0) + W$ as $x_1 \in V_1$;

(v) $\left| \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, x_0 - x_1 > + h(x_0) - h(x_1) \right| < \alpha/6$ as $x_1 \in V_2$.

It follows that

$$\inf_{w \in T(y_1)} \mathrm{Re} < M(x_1) - w,\ y_1 - x_1 > + h(y_1) - h(x_1)$$

$$\geq \inf_{f \in W} \inf_{[w \in T(y_0) + W]} \mathrm{Re} < (M(x_0) + f) - w,\ y_1 - x_1 > + h(y_1) - h(x_1)$$

(by (i) and (iv)),

$$\geq \inf_{w \in T(y_0)} \mathrm{Re} < M(x_0) - w, y_1 - x_1 > + h(y_1) - h(x_1)$$

$$+\inf_{f\in W}\inf_{w\in W}\mathrm{Re}< f-w, y_1 - x_1 >$$

$$\geq \inf_{w\in T(y_0)}\mathrm{Re}< M(x_0) - w,\ y_1 - y_0 > + h(y_1) - h(y_0)$$

$$+ \inf_{w\in T(y_0)}\mathrm{Re}< M(x_0) - w,\ y_0 - x_0 > + h(y_0) - h(x_0)$$

$$+ \inf_{w\in T(y_0)}\mathrm{Re}< M(x_0) - w, x_0 - x_1 > + h(x_0) - h(x_1)$$

$$+ \inf_{f\in W}\mathrm{Re}< f, y_1 - x_1 > + \inf_{w\in W}\mathrm{Re}< -w, y_1 - x_1 >$$

$$\geq -\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{3} > 0 \qquad \text{(by (iii) and (v)).}$$

Therefore

$$\sup_{x\in S(y_1)}\left[\inf_{w\in T(y_1)}\mathrm{Re}< M(x) - w,\ y_1 - x > + h(y_1) - h(x)\right] > 0$$

as $x_1 \in S(y_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$. Thus N_0 is an open neighborhood of y_0 which is contained in Σ. Hence, Σ is open in X. Consequently, the conclusion of **Theorem 2.2.2** follows from **Theorem 2.2.1.**

CHAPTER 3

Generalized Bi-Quasi-Variational Inequalities for Quasi-Pseudo-Monotone Type I Operators in Non-Compact Settings

In this chapter we shall present our main result of existence theorem on non-compact generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type I operators. In obtaining this result we shall mainly use the concept of escaping sequences given in **Definition 2.2.5** and apply **Theorem 2.2.2.**

We shall now present our main result:

Theorem 3.1. let E be a locally convex Hausdorff topological vector space over Φ, X a non-empty (convex) subset of E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and let F be a vector space over Φ. Let $\langle \, , \, \rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle \, , \, \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X. Equip F with the strong topology $\delta \langle F, E \rangle$. Suppose that

(1) $S : X \to 2^X$ Is a continuous map such that

(a) for each $x \in X$, $S(x)$ is a closed and convex subset of X and

(b) for each $n \in N$, $S(x) \subset C_n$ for all $x \in C_n$;

(2) $h : X \to \mathbf{R}$ is convex and continuous;

(3) $T: X \to 2^F$ is an h-quasi-pseudo-monotone type I operator and is upper semi-continuous such that each $T(x)$ is $\delta < F, E >$-compact and convex;

(4) $M : X \to 2^F$ is a continuous linear map in X and for each $y \in \Sigma$, where

$$\Sigma = \left\{ y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re} < M(x) - w, y - x > + h(y) - h(x)] > 0 \right\},$$

$\inf_{w \in T(y)} \operatorname{Re} < M(x) - w, y - x > + h(y) - h(x) > 0$ for some point x in $S(y)$;

(5) for each sequence $\{y_n\}_{n=1}^{\infty}$ in X, with $y_n \in C_n$ for each $n \in N$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, either there exists $n_0 \in N$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in N$ and $x_{n_0} \in S(y_{n_0})$ such that

$$\min_{w \in T(y_{n_0})} \operatorname{Re} < M(y_{n_0}) - w,\ y_{n_0} - x_{n_0} > + h(y_{n_0}) - h(x_{n_0}) > 0 \qquad (*)$$

holds.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re} < M(\hat{y}) - \hat{w}, \hat{y} - x > \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Proof. Let us fix an arbitrary $n \in N$. We note that C_n is a non-empty compact and convex subset of E. Let us define $S_n : C_n \to 2^{C_n}$, $h_n : C_n \to \mathbf{R}$ and $M_n, T_n : C_n \to 2^F$ by $S_n(x) = S(x)$, $h_n(x) = h(x)$, $M_n(x) = M(x)$, and $T_n(x) = T(x)$

respectively for each $x \in C_n$; i.e., $S_n = S|_{C_n}$, $h_n = h|_{C_n}$, $M_n = M|_{C_n}$, and $T_n = T|_{C_n}$ respectively. Then by **Theorem 2.2.2**, there exists a point $\hat{y}_n \in C_n$ such that

$(i)'$ $\hat{y}_n \in S_n(\hat{y}_n)$ and

$(ii)'$ there exists a point $\hat{w}_n \in T(\hat{y}_n) = T_n(\hat{y}_n)$ with

$$\mathrm{Re} < M_n(\hat{y}_n) - \hat{w}_n, \hat{y}_n - x > \leq h(x) - h(\hat{y}_n)$$

for all $x \in S_n(\hat{y}_n)$.

Note that $\{\hat{y}_n\}_{n=1}^{\infty}$ is a sequence in $X = \bigcup_{n=1}^{\infty} C_n$ with $\hat{y}_n \in C_n$ for each $n \in N$.

Case 1: $\{\hat{y}_n\}_{n=1}^{\infty}$ is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$. Then by hypothesis (5), there exists $n_0 \in N$ such that $\hat{y}_{n_0} \notin S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$, which contradicts $(i)'$ or there exist $n_0 \in N$ and $x_{n_0} \in S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$ such that

$$\min_{w \in T(\hat{y}_{n_0})} \mathrm{Re} < M(\hat{y}_{n_0}) - w, \hat{y}_{n_0} - x_{n_0} > + h(\hat{y}_{n_0}) - h(x_{n_0}) > 0,$$

which contradicts $(ii)'$.

Case 2: $\{\hat{y}_n\}_{n=1}^{\infty}$ is not escaping from X relative to $\{C_n\}_{n=1}^{\infty}$. Then there exist $n_1 \in N$ and a subsequence $\{\hat{y}_{n_j}\}_{j=1}^{\infty}$ of $\{\hat{y}_n\}_{n=1}^{\infty}$ such that $\hat{y}_{n_j} \in C_{n_1}$ for all $j = 1,2,3,\cdots$. Since C_{n_1} is compact, there exist a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_{n_j}\}_{j=1}^{\infty}$ and $\hat{y} \in C_{n_1} \subset X$ such that $\hat{z}_\alpha \to \hat{y}$.

For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \to \infty$. Then according to our choice of $\hat{y}_{n_\alpha}$ in C_{n_α}, we have

$(i)''$ $\hat{y}_{n_\alpha} \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$, and

$(ii)''$ there exist a point $\hat{w}_{n_\alpha} \in T_{n_\alpha}(\hat{y}_{n_\alpha}) = T(\hat{y}_{n_\alpha})$ with

$$\mathrm{Re}\langle \mathrm{M}(\hat{y}_{n_\alpha}) - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x\rangle + h(\hat{y}_{n_\alpha}) - h(x) \le 0$$

for all $x \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$. Since $\mathrm{n}_\alpha \to \infty$, there exists $\alpha_0 \in \Gamma$ such that $n_\alpha \ge n_1$ for all $\alpha \ge \alpha_0$. Thus $C_{n_1} \subset C_{n_\alpha}$, for all $\alpha \ge \alpha_0$. From $(i)''$ above we have $(\hat{y}_{n_\alpha}, \hat{y}_{n_\alpha}) \in G(S)$ for all $\alpha \in \Gamma$. Since S is upper semi-continuous with closed values, $G(S)$ is closed in $X \times X$; it follows that $\hat{y} \in S(\hat{y})$.

Moreover, since $\{M(\hat{y}_{n_\alpha})\}_{\alpha \ge \alpha_0}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha \ge \alpha_0}$ are nets in the compact sets $\bigcup_{x \in C_{n_1}} M(x) = M(C_{n_1})$ (since M is a continuous single-valued function) and $\bigcup_{x \in C_{n_1}} T(x)$ respectively, without loss of generality, we may assume that the nets $\{M(\hat{y}_{n_\alpha})\}_{\alpha \in \Gamma}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha \in \Gamma}$ converges to $M(\hat{y})$ and some point $\hat{w} \in \bigcup_{x \in C_{n_1}} T(x)$ respectively. Note that M has a closed graph. Also, since T has a closed graph on C_{n_1}, $\hat{w} \in T(\hat{y})$.

Let $x \in S(\hat{y})$ be arbitrarily fixed. Let $\mathrm{n}_2 \ge n_1$ be such that $\mathrm{x} \in C_{n_2}$. Since S is lower semi-continuous at $\hat{y}$, without loss of generality we may assume that for each $\alpha \in \Gamma$, there is an $x_{n_\alpha} \in S(\hat{y}_{n_\alpha})$ such that $\mathrm{x}_{\mathrm{n}\alpha} \to x$. By $(\mathrm{ii})''$ we have $\mathrm{Re}\langle \mathrm{M}(\hat{y}_{n_\alpha}) - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha}) \le 0$ for all $\alpha \in \Gamma$. Note that $\mathrm{M}(\hat{y}_{n_\alpha}) - \hat{w}_{n_\alpha} \to M(\hat{y}) - \hat{w}$ in $\delta\langle \mathrm{F}, \mathrm{E}\rangle$ and $\{\hat{\mathrm{y}}_{\mathrm{n}_\alpha} - x_{n_\alpha}\}_{\alpha \in \Gamma}$ is a net in the compact (and hence bounded) set $C_{n_2} - \bigcup_{y \in C_{n_2}} S(y)$. Thus for each $\varepsilon > 0$, there exist

$\alpha_1 \geq \alpha_0$ such that $\left|\mathrm{Re}\langle \mathrm{M}(\hat{\mathrm{y}}_{\mathrm{n}_\alpha}) - \hat{w}_{n_\alpha} - (M(\hat{y}) - \hat{w}), \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle\right| < \varepsilon/2$ for all $\alpha \geq \alpha_1$.

Since $\langle \mathrm{M}(\hat{\mathrm{y}}) - \hat{\mathrm{w}}, \hat{\mathrm{y}}_{\mathrm{n}_\alpha} - x_{n_\alpha}\rangle \to \langle M(\hat{y}) - \hat{w}, \hat{y} - x\rangle$, there exists $\alpha_2 \geq \alpha_1$ such that

$$\left|\mathrm{Re}\langle \mathrm{M}(\hat{\mathrm{y}}) - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle - \mathrm{Re}\,\langle \mathrm{M}(\hat{\mathrm{y}}) - \hat{\mathrm{w}}, \hat{\mathrm{y}} - \mathrm{x}\rangle\right| < \varepsilon/2 \text{ for all } \alpha \geq \alpha_2.$$

Thus for all $\alpha \geq \alpha_2$,

$$\left|\mathrm{Re}\,\langle \mathrm{M}(\hat{\mathrm{y}}_{\mathrm{n}_\alpha}) - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle - \mathrm{Re}\,\langle \mathrm{M}(\hat{\mathrm{y}}) - \hat{\mathrm{w}}, \hat{\mathrm{y}} - \mathrm{x}\rangle\right|$$

$$\leq \left|\mathrm{Re}\,\langle \mathrm{M}(\hat{\mathrm{y}}_{\mathrm{n}_\alpha}) - \hat{w}_{n_\alpha} - (M(\hat{y}) - \hat{w}), \hat{\mathrm{y}}_{\mathrm{n}_\alpha} - \mathrm{x}_{\mathrm{n}_\alpha}\rangle\right|$$

$$+ \left|\mathrm{Re}\langle \mathrm{M}(\hat{\mathrm{y}}) - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} - (\hat{\mathrm{y}} - \mathrm{x})\rangle\right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus,

$$\lim_\alpha \mathrm{Re}\langle M(\hat{y}_{n_\alpha}) - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle = \mathrm{Re}\langle \mathrm{M}(\hat{\mathrm{y}}) - \hat{\mathrm{w}}, \hat{\mathrm{y}} - \mathrm{x}\rangle.$$

Since h is continuous, we have

$$\mathrm{Re}\langle M(\hat{y}) - \hat{w}, \hat{y} - x\rangle + \mathrm{h}(\hat{\mathrm{y}}) - \mathrm{h}(\mathrm{x})$$

$$= \lim_\alpha \left[\ \mathrm{Re}\left\langle M(\hat{y}_{n_\alpha}) - \hat{w}_{n\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\right\rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha})\ \right]$$

$$\leq 0.$$

This completes the proof.

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