Effects of MHD and partial slip on peristaltic flow of a Jeffrey fluid in a rectangular duct

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- Peristaltic flow
- Jeffrey fluid
- Mathematics principles
- theses

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A Thesis
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE IN MATHEMATICS

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Certificate

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We accept this dissertation as conforming to the required standard.

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2015

Declaration

I hereby declare and affirm that this research work neither as a whole nor as a part has been copied out from any source. It is further declared that I have developed this research work entirely on the basis of my personal efforts. If any part of this thesis is proven to be copied out or found to be a reproduction of some other, I shall stand by the consequences.

Moreover, no portion of the work presented in this thesis has been submitted in support of an application for other degree of qualification in this or any other university or institute of learning.

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FAROOQ HUSSAIN
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Dedication

To

My Elder Sister (Late)
Her Sons
Ahmed Jahanzeb
and
Abdur Rehman

Acknowledgement

Peace and blessings upon the kindest and the loveliest of the Prophets, **Hazrat Muhammad (S.A.W.W)**. First and foremost, thanking **Allah (SWT)** The Almighty, for not only giving me the required strength to complete this piece of work in the required period of time. But, I am also extremely thankful to **Allah (SWT),** for awarding me the honor of becoming the first inhabitant of **Quetta (Balochistan)** completing his **MS (Mathematics),** form International Islamic University, Islamabad.

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Last but not the least, I dedicate this dissertation to my parents. There is no doubt, in my mind to acknowledge that they are truly, responsible for making me whatever, I am today. They provided me opportunity to embellish myself with the diadem and trinket of education. Being illiterate themselves, their dream to educate their children has finally, come true. Without their unconditional love, prayers, devotion, and active support surely no achievement in my life would have been possible.

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Preface

Over the last thirty years, or so, a very hot topic, namely non-Newtonian fluid flow came to appeared in many application in industrial processes, biology, medicine, catalytic chemistry end environmental applications [1-6]. In bio fluids the physiological system has been investigated by several researchers in order to find the treatment of diagnostic problems that arise during circulation in a human body. There are several models which have been proposed to describe such physiological fluids, however their full potential has not been exploited yet and a lot of questions remain unresolved. Among several models, non-Newtonian Jeffrey model is significant because Newtonian fluid model can be deduced from it as a special case by taking $\lambda_1 = 0$. It is also speculated that the physiological fluids such as blood exhibit Newtonian and non-Newtonian behaviors simultaneously.

Among all continuous system taking place in a human's body and becoming the main reason for one's life, peristalsis is one of them. Basically peristaltic word comes from Greek word peristaltikos which means clasping and compressing. The importance of peristalsis can be assumed by the fact that, it is an automatic series of muscular contraction and relaxation, which takes place in human's body, such as digestive tract or digestive system. Peristalsis does cause the movement of food through digestive system, chyme in the gastrointestinal tract, urine from kidney to bladder and bile from gallbladder to duodenum. Also, transport of lymph in the lymphatic vessels and vasomotion of small blood vessels like arterioles, venules and capillaries involve the peristaltic motion are the common examples of peristaltic motion. In addition peristaltic motion has been equally, playing an important and beneficial part in physiological sciences by devising mechanical and biomechanical instruments such as roller pumps and heart lung machine, which function mainly by using the principle of peristalsis. This has opened a new dimension for researchers to maneuver their equipment for obtaining better results in their respective field of interest. Different mathematical models have been employed by many authors to use peristaltic flow in Newtonian and non-Newtonian fluid [7-11].

Moreover, the effects of magnetohydrodynamics (MHD) in peristalsis are very important and have great value in medical science such as magneto therapy, hyperthermia, arterial flow, compressor, optimization of blood pump machines, magnetic wound or cancer tumor treatment causing hyperthermia, bleeding reduction during surgeries and targeted transport of drug using magnetic particles as drug carries. As MHD has based on a fundamental law of electromagnetism that is when a magnetic field and an electric current intersect in a liquid, their repulsive intersection propels the liquid in a direction perpendicular to both the field and the current. Due to this fact, MHD is also used in the study of electrically conducting fluids; examples of such fluids include electrolytes, saltwater, liquid metals and plasmas. The controlled application of low intensity and frequency pulsating fields modify the cell and tissue. Magnetic susceptible of chyme is satisfied the ions contained in the chyme or with heat generated by the magnetic field. The magnets could heat inflammations, ulceration, several diseases of uterus and bowel. A number of researchers have applied magnetic field on peristaltic mechanisms, one way or the other. An extensive literature on the said topics is now available but we can only mention a few recent interesting investigations here [12-19].

Furthermore, the fluids that exhibit slip effect have many applications, for instance, the polishing of artificial heart valves and internal cavities. In all these studies the peristaltic flow problems have been extensively investigated with no slip condition. A very less emphasis has been given to such flows in the presence of a slip/partial slip conditions. Although, the application of slip condition in the peristaltic flows has special relevance in polymers and physiology. The case of slip effects may appear for two types of fluids namely fluids having much more elastic character and rare field gases. In these fluids, slippage appears subject to large tangential traction. It is found through experiment observations that the existence of slippage is possible in the non-Newtonian fluids, polymer solution and molten polymer. Additionally, a clear layer is occasionally detected next to the wall when flow of dilute suspension of particles examined. In experimental physiology such a layer is observed when blood flow through capillary vessels is considered. The peristaltic flows subject to different flow aspects and configurations have been examined. Some pertinent studies in numerous situations can be seen from the list of references [20-31] and several therein.

Additionally, the study of non-Newtonian fluids is complicated and difficult because of the nonlinear relationship between the stress and the rate of strain occurs in real world phenomena. It is very easy to solve a linear problem but finding a solution of nonlinear problem is still very challenging task. In particular, getting an analytic solution of a nonlinear problem is often more difficult as compared to getting a numerical solution, despite the availability of high performance supercomputers. However, results obtained by numerical methods give discontinuous points of a curve when plotted. Besides that, obtaining the complete necessary understanding of a nonlinear problem is very much difficult. If a nonlinear problem has multiple solutions or contains singularity then this also adds difficulties to the numerical. Though numerical and analytic methods for solving nonlinear problems have their own limitations, at the same time they have their advantages too. Therefore, we cannot neglect either of the two approaches but usually it is pleasing to solve a nonlinear problem analytically. In applied sciences, analytical solutions of any physical model are of great importance, if available since analytical solutions not only lead to draw correct physical interpretation but are also very helpful in the validating of numerical investigations. In the present study, the resulting equations are first transformed to dimensionless partial differential equations using the appropriate variables and then analytical are obtained by the method of separation of variables.

With all abovementioned studies, one can clearly observe that no analysis for magnetohydrodynamic peristaltic flow of Jeffery fluid in the presence of partial slip conditions is reported in the available literature yet. The present work puts forward to fill this gap. The flow analysis is performed under the constraint of long wavelength approximation and low Reynolds number. An exact solution for the expression for both velocity of the fluid and pressure gradient are obtained by using the method of separation of variables. The pumping characteristics such as pressure rise and trapping phenomena are obtained numerically by using software Mathematica. The physical features of pertinent parameters have been comprehensively elaborated through graphs. The organization of thesis is in the following fashion: Chapter 1 includes some basic definition and governing equations for subsequent chapters. Chapter 2 is the review work of Nadeem and Akram [32]. Chapter 3 may be regarded as an extension of the problem considered by Nadeem and Akram [32] to serve the aforementioned purpose.

Contents

Chapter 1

Introduction

1.1 Basic definitions

Main objective of this chapter is to make a reader, familiar with the basic concepts which are largely used in fluid mechanics. Furthermore, it provides an essential aid to a reader in understanding the material presented in the subsequent chapters, as well.

1.2 Fluid mechanics

Fluid mechanics is a branch of engineering sciences, which deals with the behavior of the fluid under the conditions of both rest and motion. Moreover, one can say that fluid mechanics is the study of fluids (i.e., liquids and gases) and involves various properties of the fluid, such as velocity, pressure, density and temperature, as functions of space and time.

1.3 Branches of fluid mechanics

Based on the fluid's behavior, the fluid mechanics is divided into mainly two main branches. These are:

1.3.1 Fluid statics

This branch of fluid mechanics deals with the fluid when there is no relative motion between the fluid particles. In fluid statics, one is merely, interested in the conditions and reasons which cause the fluid to be in state of rest. It's practical applications are numerous. Some of which are Fluid Manometers, buoyancy measurements, density calculations, aircraft design and the airspeed indicator/controller.

1.3.2 Fluid kinematics

This is the branch of fluid mechanics which deals with the behavior of the fluid when it is in motion.

1.3.3 Fluid dynamics

This branch of fluid mechanics is concerned with the movement of the fluid and how different types of forces effect on it. It provides very significant methods to study the ocean current, weather pattern, evolution of stars, tectonic plates and blood circulation. It's applications includes wind turbines, rocket engine, oil pipelines and air conditioning systems.

1.4 Fluid

Fluid is defined as a substance which is capable of flowing. It has no defined shape, but it takes the shape of its container. Besides, a fluid offers very little or no resistance to the external force/stress, when applied on it. In simpler terms, one can also make it out that the fluid is a substance which offers no resistance to shear force, when applied on it. A fluid can be liquid, vapor or gas. Common examples of the daily fluid include water, petrol, diesel, air and gas etc.

1.5 Types of fluid

Physical features and nature go a long way, in understanding the different types of fluid that exist in or on the planet earth. Therefore, it is classified into four basic kinds. These are:

1.5.1 Ideal fluid (Non-viscous fluid)

An ideal fluid is defined as the fluid that has no viscosity. Moreover, it is not compressible in nature. Physically, in our universe there does not exist, such a fluid.

1.5.2 Real fluid (Viscous fluid)

A real fluid is the one which has some viscosity and is compressible in nature. Common examples of real fluid in daily lives include: Kerosene oil, Petrol, Castor oil. etc.

1.5.3 Newtonian fluid

Newtonian fluid is defined as the fluid for which the shear stress is directly and linearly proportional to the rate of strain (Also known as deformation). Mathematically, it can be represented as

$$\begin{aligned} \tau_{yx} &\,\alpha\, \tfrac{du}{dy}, \\ \tau_{yx} &= \mu \tfrac{du}{dy} \end{aligned} \tag{1.1}$$

where μ denotes the *viscosity* of the fluid, τ_{xy} is called *shear stress* acting upon the plane normal to $y - axis$. The above equation is also called Newton's law of viscosity. This implies that Newtonian fluids obey Newtonian's law of viscosity. Moreover, for Newtonian fluids, viscosity entirely depends upon the temperature and pressure of the fluid. In simple words, in Newtonian fluids the shear stress is always the function of strain rate (velocity gradient). Examples include air, water and petrol.

1.5.4 Non-Newtonian fluid

Non-Newtonian fluid is defined as the fluid for which shear stress is directly but non-linearly proportional to the rate of strain. Mathematically, it can be denoted as given

$$\tau_{yx} = \mu \left(\frac{du}{dy}\right)^n, \quad \text{For } \ n \neq 1. \tag{1.2}$$

Examples: Shampoo, blood, starch, paints and ketchup. etc.

1.6 Jeffrey fluid

Since liquids and gases are known as fluid. Jeffrey is the name of a scientist who gave a stress tensor for a non-Newtonian fluid, containing a ratio of times λ_1 (i.e., the ratio of relaxation time to retardation time) called Jeffrey parameter. It is deonted by

$$\begin{aligned} \text{Jeffrey parameter} &= \frac{\text{Relaxation time of the fluid}}{\text{Retardation time of the fluid}}, \\ \lambda_1 &= \frac{\text{Relaxation time of the fluid}}{\lambda_2}. \end{aligned}$$

1.6.1 Retardation time

It is defined as the time taken by the Jeffrey fluid to get deformed subject to shear stress. It denoted by λ_2.

1.6.2 Relaxation time

It is defined as the time taken by the Jeffrey fluid to return in to its equlibrium state, from its purturbed or deformed state.

1.7 Compressible fluid

A fluid whose density changes w.r.t. either time/space coordinates or relative to both time and space coordinates. Then this type of fluid is called a compressible fluid. All gases are compressible fluids. Mathematically, it can be expressed as

$$\nabla . \mathbf{V} \neq 0. \tag{1.3}$$

1.8 Incompressible fluid

A fluid whose density does not change w.r.t. either time/space coordinates or relative to both time and space coordinates. Then this type of fluid is called a compressible fluid. All liquids are incompressible fluids. Mathematically, it can be expressed as

$$\nabla . \mathbf{V} = 0. \tag{1.4}$$

1.9 Properties of fluid

Properties of the fluid determine how fluid can be used in engineering and technology. Moreover, these also determine the behavior of the fluid in fluid mechanics. The following are some of the most important and basic properties of the fluid which are used in this dissertation. These are:

1.9.1 Density

Density is defined as mass per unit volume of a fluid. It can also be termed as the ratio of mass of a fluid to it's volume. Mathematically, it is expressed as

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}. \tag{1.5}$$

Density is denoted by the symbol ρ. It's unit is kg/m^3. Generally, the density of a fluid decreases as the temperature of the fluid increases. Similarly, it increases as the pressure of the fluid increases. Moreover, the density of a standard liquid (i.e., water) is 1000 kg/m^3.

1.9.2 Dynamic viscosity

Second important property of the fluid is, it's viscosity. It is a measure of friction in the fluid. It can also be inferred as the measure of resistance of a fluid to the shear stress, applied on it. In general, when a fluid flows, the layers of a fluid rub against one another. In a very viscous fluid, the force of friction is so great that the layers of fluid pull against one another. Hence it hampers the fluid to flow. Simply, one can describe it as the ratio of shear stress to the rate of shear strain (deformation). Mathematically,

$$\begin{aligned} \text{Viscosity} &= \frac{\text{Shear stress}}{\text{Rate of shear strain}}, && (1.6) \\ \mu &= \frac{\text{Shear stress}}{\text{Rate of shear strain}}. && (1.7) \end{aligned}$$

It is denoted by μ pronounced as (meu). Unlike density, viscosity in liquids decreases with the increase in temperature and while in gases it increases with rise in pressure. It is also known as "*Dynamic viscosity*". Unit of the viscosity is $(N.s/m^2)$ or Pascal-second $(Pa.s)$

1.9.3 Kinematic viscosity

It is define as the ratio of dynamics viscosity μ to density of a fluid. Mathematically, it is represented as:

$$\text{Kinematic viscosity} = \frac{\text{Dynamic viscosity}}{\text{Density of the fluid}}, \tag{1.8}$$

$$\nu = \frac{\mu}{\rho}. \tag{1.9}$$

Unit of kinematic viscosity is m^2/s.

1.9.4 Pressure

Pressure is basically a type of surface forces. This is defined as the force per unit area. Infact, it is the ratio of force to an area of the fluid on which the force acts (area is normal to the direction of the force acting upon it). It is a scalar quantity. Mathematically, it can be defined as

$$\text{pressure} = \frac{\text{Force}}{\text{Area on which the force acts}},$$

$$P = \frac{\text{Force}}{\text{Area on which the force acts}}. \tag{1.10}$$

It is represented by 'P'. The fundamental S.I. unit of pressure is newton per square meter (N/m^2). This is also known as *Pascal*.

1.10 Flow

Change in the position of the fluid particles, due to shear stress or any external force, is called the flow of fluid. In simplest words it is the abilty of fluids to flow. Or a material that goes under deformation when different forces act on it. If deformation increases continuously without any limit then the phenomenon is known as *flow*.

1.11 Types of fluid flow

1.11.1 Uniform flow

A flow in which the velocity of each fluid particle is same, at every point of space or region in which fluid is moving. Mathematically, it can be expressed as

$$\frac{\partial v}{\partial s} = 0.$$

where v is the velocity of the fluid and s denotes the space.

1.11.2 Non-uniform flow

A flow in which the velocity of each fluid particle is different form the other fluid particles, at every point of space or region in which fluid is moving. Mathematically, it can be expressed as

$$\frac{\partial v}{\partial s} \neq 0.$$

1.11.3 Steady flow

A flow in which the properties of the fluid particle (i.e., velocity, pressure, density etc.) are independent of time. Mathematically, it can be expressed as

$$\frac{\partial v}{\partial t} = 0.$$

1.11.4 Unsteady fow

A flow in which the properties of the fluid particle (i.e., velocity, pressure, density etc.) are time dependent. Mathematically, it can be expressed as

$$\frac{\partial v}{\partial t} \neq 0.$$

1.11.5 Rotational flow

A flow in which the fluid particles while in the state of motion, rotate about their mass of centers. Mathematically, it can be noted as

$$\nabla \times \mathbf{V} \neq 0.$$

1.11.6 Irrotational flow

A flow in which the fluid particles while in the state of motion, do not rotate about their mass of centers. Mathematically, it can be noted as

$$\nabla \times \mathbf{V} = 0.$$

1.12 Forces in the fluid

In fluid dynamics, a moving fluid often comes under the effect of various kinds of forces, acting upon it. Which have been categorized into two main types. These forces are given as follows:

1.12.1 Body force

Body force is a force which applies on per unit mass of the fluid. This kind of force acts throughout the volume of a body. Gravitational force, Centrifugal force, Electric force and Magnetic force, are the common examples of the body force. Moreover, it is also termed as "*Long range force*" or "*Volume force*".

1.12.2 Surface force

It is define as a force, which acts upon per unit area of the fluid. Whenever, a surface force applies on the surface of any fluid it acts normally over the area. Whereas, shear stress acts tangentially over an area. Pressure, shear stresses, resistance etc. are the common examples of surface force. Mathematically, is denoted by f_s.

1.13 Dimensionless numbers

A dimensionless number is a number that has no unit associated with it. In field of *Fluid Mechanics*, in order to reach better results and conclusions, it is often preferred to ignore the dimensions of some parameters. Therefore, to meet this purpose a sort of numerical quantity/parameter is used which is known as *"Dimensionless number"* or *"Dimensionless parameter"*. Basically, this number is the ratio of a pair of forces. This can be obtained if force of inertia is divided by any one of these forces i.e., viscous force, force of gravity, pressure force, force of surface, or elastic force.

There are various dimensionless numbers in use, each depending upon its use and condition. Most commonly used non-dimensional parameters include Reynolds number, Hartmann number, Froude's number, Euler's number, Weber's number. Here is the mention of those dimensionless parameters which have been used in this dissertation.

1.13.1 Reynolds Number

It is defined as the ratio of inertial force, to viscous force of the flowing fluid. The expression for Reynolds number is obtained and denoted as:

$$\text{Reynolds number} = \frac{\text{Force of inertial}}{\text{Force of viscosity}} \tag{1.11}$$

Mathematically, it is expressed as

$$\text{Re} = \frac{\rho U L}{\mu} \tag{1.12}$$

Eq. (1.12) can also be written as

$$\text{Re} = \frac{UL}{\nu} \tag{1.13}$$

where L is the characteristic length and U is the typical velocity.

The significance of a Reynolds number is, to help one in describing the flow pattern of a fluid. It determines whether the fluid's flow is a laminar or a turbulent. For instant, a fluid is passing through a round pipe if:

(*i*). Re $\leq$ 2100, then the fluid flow is termed as Laminar flow.

(*ii*). Re $\geq$ 4000, then the fluid flow is termed as Turbulent flow.

(*iii*). 2100 $\leq$ Re $\leq$ 4000, then the fluid flow is termed as transitional flow (which means

part of the flow is laminar while some part of the flow is turbulent).

1.13.2 Hartmann number

Hartmann number (Ha) is defined as the ratio of electromagnetic force to the viscous force. It is represented by "M". Mathematically, it is defined as:

$$\begin{aligned} \text{Hartmann number} &= BL\sqrt{\frac{\sigma}{\mu}}, \\ M &= BL\sqrt{\frac{\sigma}{\mu}}. \end{aligned} \tag{1.14}$$

where

- B is the magnetic field.
- L is the characteristic length scale.
- μ is the dynamic viscosity.
- σ is the electrical conductivity.

1.14 Boundary condition

It is defined as a condition, which is required to be satisfied, by the set of differential equations, at all parts of the boundary of a region, in which the given set of differential equations is to be solved. In fluid mechanics we usually deal with the following two types of boundary conditions. These are listed below:

1.14.1 No-slip condition

If velocity of the fluid and velocity of the wall (i.e. the surface / solid boundary of the container in which the fluid is flowing or preserved) are same, then this condition is said to be *No-slip condition*. This implies that either, the fluid and wall are moving with the same velocity, or both happen to be in state of rest. Mathematically, described as

$$\begin{aligned}\text{Velocity of the fluid} &= \text{Velocity of the wall}\\ \mathbf{V}_{fluid} &= \mathbf{V}_{wall}.\end{aligned} \tag{1.15}$$

1.14.2 Partial slip condition

In fluid mechanics, partial slip condition also known as slip condition is of a great significance. For, in real life phenomena, no-slip condition rarely exists or does not hold in all the situations. In some cases, the fluids tend to slip at the stationary wall (solid boundary). This gives rise, in difference between the velocities of the fluid and the wall.

The idea of partial slip was, originally proposed by Navier. This condition states that velocity u in the $x - direction$ is directly proportional to shear stress at the wall. Mathematically, it is defined and denoted as

$$\begin{aligned}\mathbf{V}_{wall}\ \alpha\ \tau_{xy},\\ \mathbf{V}_{wall} = \alpha_1\tau_{xy},\end{aligned} \tag{1.16}$$

where α_1 denotes slip parameter. Eq. (1.16) is known as Slip-condition of the fluid at wall.

1.15 Volumetric flow rate

Volumetric flow rate is defined, as the volume (quantity) of fluid which passes through a surface/an orifice, in a given unit of time. It is given by

$$\begin{aligned}\text{Volumetric flow rate} &= \frac{\text{Volume of the fluid}}{\text{Time elapsed}},\\ Q &= \frac{\mathbf{V}}{t}.\end{aligned} \tag{1.17}$$

The volumetric flow rate is represented by the symbol Q. The S.I. unit of volumetric flow is m^3/s (cubic meters per second).

1.16 Continuity equation

Basically, continuity equation, in fluid dynamics, is the law of conservation of mass. It states that the in-coming volumetric flow rate of a fluid, at one end of the hose, is same as the out-going volumetric flow rate of the given fluid, at the other end of the hose. If Q_1 denotes the in-coming flow rate of the fluid, and Q_2 denotes the out-going flow rate, then by the condition of continuity equation, it follows:

$$\begin{aligned} \text{In-coming volumetric flow rate} &= \text{Out-going volumetric flow rate}, \\ Q_1 &= Q_2. \end{aligned} \tag{1.18}$$

However, it is believed that gases often behave as fluids. But, unlike liquids gases are not incompressible. Therefore, the continuity equation does not apply, in such cases.

In fluid dynamics the differential form of continuity equation, is given by

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\ \mathbf{V}) = 0. \tag{1.19}$$

For, incompressible fluid flow $\partial\rho/\partial t = 0$. Then the above Eq.(1.23) becomes

$$\nabla.(\rho\ \mathbf{V}) = 0. \tag{1.20}$$

This implies that, $\rho \neq 0$. Therefore

$$\nabla.\mathbf{V} = 0, \tag{1.21}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.22}$$

Eq. (1.26) is known as continuity equation in Cartesian form.

1.17 Momentum equations

Equations of motion describe the law of conservation of linear momentum. In fluid dynamics, vector differential form of momentum equations, is given by

$$\begin{aligned} \frac{D\mathbf{V}}{Dt} &= \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\mathbf{V}, \\ \Rightarrow \frac{D\mathbf{V}}{Dt} &= \mathbf{F} - \frac{1}{\rho}\boldsymbol{\nabla} p + \upsilon \boldsymbol{\nabla}^2 \mathbf{V}. \end{aligned} \tag{1.23}$$

1.18 Peristalsis

Peristalsis is one of the most important and significant systems being carried out in a human body. It is an automatic, series of muscular contraction and relaxation, which takes place in human's digestive system or digestive tract, and in some organs that connect kidney to the bladder. Peristalsis is responsible for the movement of

- Food through digestive system.
- Urine from kidney to bladder.
- Bile from gallbladder to duodenum (first opening of small intestine).

1.19 Magnetohydrodynamics (MHD)

In1942, a Swedish scientist and Nobel Prize laureate, Hannes Alfvén for the very first time introduced the use of Magnetohydrodynamics (MHD). (MHD) involves magnetic fields (magneto) and fluids (hydro) that conduct electricity and interact (dynamics). MHD technology is based on a fundamental law of electromagnetism. When a magnetic field and an electric current intersect in a liquid, their repulsive intersection propels the liquid in a direction perpendicular to both the field and the current.

1.20 Separation of variables

In the solution of differential equations, separation of variables is one of several methods, which are used to solve both an ordinary and partial differential equations. It is also known as the

Fourier method. In the solution of PDEs, separation of variables is applicable if and only if

- The given PDE is homogenous (i.e., forcing function is not present).
- Boundary conditions are homogenous.
- Domain is finite.

1.21 Tensor

Tensor is defined as an array of numbers or functions, transformed under a certain rule when the coordinates are changed. In other words scalar and vector both are special cases of a more general object known as " *Tensor*" of order n. Specification of a tensor in any coordinate system requires 3^n numbers, which are called the components of the tensor. Where $n = 1, 2, 3....$

Infact, the order or rank of a tenor is the number of suffixes used in it. For example, scalars are tensor of order 0 $(Zero)$, with $3^0 = 1$, component. Whereas, vector is a tenor of order one, with $3^1 = 3$, components. An example for the tensor of order one is velocity. However, the partial derivatives of velocity yields a tensor of order two.

Chapter 2

Peristaltic flow of a Jeffrey fluid in a rectangular duct

This chapter contains the review work of Nadeem and Akram the [32]. In present chapter, the detail calculations for the peristaltic flow of a Jeffrey fluid in a rectangular duct are given. The exact solutions of velocity and pressure gradient have been found under the assumption of long wave length and low Reynolds number. The expression for pressure rise in a rectangular duct has been evaluated numerically. The physical features of pertinent parameters are discussed through graphs.

2.1 Formulation of the problem

Let us consider the peristaltic flow of an incompressible non-Newtonian fluid (Jeffrey fluid) in a duct of rectangular cross section having the channel width $2d$ and height $2a$. Cartesian coordinates system is considered in such a way that $X - axis$ is taken along the axial direction, $Y - axis$ is taken in the lateral direction and $Z - axis$ is along the vertical direction of the rectangular duct.

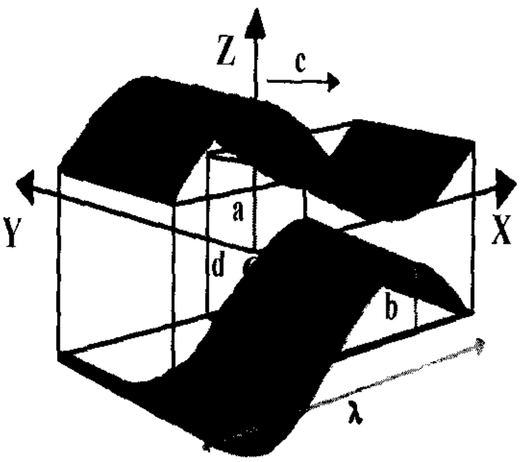

Fig. 2.1. Schematic diagram of peristaltic flow with waves propagating on horizontal walls in a rectangular duct.

The peristaltic waves on the wall are represented as

$$Z = H(X,t) = \pm a \pm b\cos\left[\frac{2\pi}{\lambda}(X - ct)\right], \tag{2.1}$$

where a and b are the amplitudes of the waves, λ is the wave length, c is the velocity of the wave propagation, t is the time and X is the direction of wave propagation. The walls parallel to XY plane remain undisturbed and are not subject to any peristaltic wave motion. It is assumed that the lateral velocity is zero as there is no change in lateral direction of the duct cross section.

If $\mathbf{V}=(U,0,W)$ be the velocity for a rectangular duct. Then the governing equations for the peristaltic flow of the Jeffrey fluid in the given rectangular duct are given as

Equation of conservation of mass:

$$\frac{\partial U}{\partial X}+\frac{\partial W}{\partial Z}=0. \tag{2.2}$$

Equation of momentum:

$$\rho\frac{D\mathbf{V}}{Dt}=-\boldsymbol{\nabla}P+\boldsymbol{\nabla}.\mathbf{S},$$

The above equation of momentum in the form of velocity components is given as:

$$\begin{aligned}
\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial X}+W\frac{\partial U}{\partial Z}\right) &= -\frac{\partial P}{\partial X}+\frac{\partial}{\partial X}S_{XX}+\frac{\partial}{\partial Y}S_{XY}+\frac{\partial}{\partial Z}S_{XZ}, && (2.3)\\
0 &= -\frac{\partial P}{\partial Y}+\frac{\partial}{\partial X}S_{YX}+\frac{\partial}{\partial Y}S_{YY}+\frac{\partial}{\partial Z}S_{YZ}, && (2.4)\\
\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial X}+W\frac{\partial W}{\partial Z}\right) &= -\frac{\partial P}{\partial Z}+\frac{\partial}{\partial X}S_{ZX}+\frac{\partial}{\partial Y}S_{ZY}+\frac{\partial}{\partial Z}S_{ZZ}. && (2.5)
\end{aligned}$$

where U and W are the velocity components, in fixed frame (X,Y). And, S denotes the shear stresses. Moreover, stress tensor for Jeffrey fluid is defined as

$$\begin{aligned}
S &= \frac{\mu}{1+\lambda_1}\left(\dot{\gamma}+\lambda_2\ddot{\gamma}\right), && (2.6)\\
\dot{\gamma} &= (\operatorname{grad}\mathbf{V})+(\operatorname{grad}\mathbf{V})^{\mathrm{T}}, &&\\
\dot{\gamma} &= (\nabla\mathbf{V})+(\boldsymbol{\nabla}\mathbf{V})^{\mathrm{T}}. && (2.7)
\end{aligned}$$

In above equation, $\lambda_1$is the ratio of relaxation to retardation times, λ_2 is the delay time, $\dot{\gamma}$ is the shear rate and dot denotes it's derivative w.r.t. time.

The boundary conditions for the problem are:

$$\begin{aligned}
(i).U(Y,Z) &= 0=U, \quad at \;\; Y=\pm d, && (2.8)\\
(ii).U(Y,Z) &= 0=U, \quad at \;\; Z=\pm H(X,t)=\pm a\pm b\cos\left[\frac{2\pi}{\lambda}(X-ct)\right]. && (2.9)
\end{aligned}$$

Now, by defining a wave frame (x,y) moving with the velocity c away from the fixed frame

(X, Y) by the following given transformation.

$$x = X - ct, y = Y, z = Z, u = U - c, w = W, p(x, z) = P(X, Z, t). \tag{2.10}$$

Selecting the following set of non-dimensional variables and parameters

$$\left.\begin{array}{c} \bar{x} = \frac{x}{\lambda}, \bar{y} = \frac{y}{d}, \bar{z} = \frac{z}{a}, \bar{u} = \frac{u}{c}, \bar{w} = \frac{w}{c\delta}, \bar{t} = \frac{ct}{\lambda}, \bar{h} = \frac{H}{a}, \bar{p} = \frac{a^2 p}{\mu c \lambda}, \mathrm{Re} = \frac{\rho a c \delta}{\mu}, \beta = \frac{a}{d}, \\ \delta = \frac{a}{\lambda}, \bar{S}_{\bar{x}\bar{x}} = \frac{a}{\mu c} S_{xx}, \bar{S}_{\bar{x}\bar{z}} = \frac{a}{\mu c} S_{xz}, \bar{S}_{\bar{x}\bar{y}} = \frac{d}{\mu c} S_{xy}, \bar{S}_{\bar{y}\bar{z}} = \frac{d}{\mu c} S_{yz}, \bar{S}_{\bar{z}\bar{z}} = \frac{\lambda}{\mu c} S_{zz}, \bar{S}_{\bar{y}\bar{y}} = \frac{\lambda}{\mu c} S_{yy}. \end{array}\right\} \tag{2.11}$$

Having used the above given transformation, non-dimensional variables and parameters. The Eqs. (2.2) - (2.6) after dropping the signs of bar, take the following form

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.12}$$

$$\mathrm{Re}\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \delta\frac{\partial}{\partial x}S_{xx} + \beta^2\frac{\partial}{\partial y}S_{xy} + \frac{\partial}{\partial z}S_{xz}, \tag{2.13}$$

$$0 = -\frac{\partial p}{\partial y} + \delta^2\frac{\partial}{\partial x}S_{yx} + \delta^2\frac{\partial}{\partial y}S_{yy} + \delta\frac{\partial}{\partial z}S_{yz}, \tag{2.14}$$

$$\mathrm{Re}\,\delta^2\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \delta^2\frac{\partial}{\partial x}S_{zx} + \delta\beta^2\frac{\partial}{\partial y}S_{zy} + \delta^2\frac{\partial}{\partial z}S_{zz} \tag{2.15}$$

where

$$\left.\begin{array}{c} S_{xx} = \frac{2\delta}{1+\lambda_1}\left(1 + \frac{\lambda_2 c\delta}{a}\left(u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\right)\frac{\partial u}{\partial x}, \\ S_{xy} = \frac{1}{1+\lambda_1}\left(1 + \frac{\lambda_2 c\delta}{a}\left(u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\right)\frac{\partial u}{\partial y}, \\ S_{xz} = \frac{1}{1+\lambda_1}\left(1 + \frac{\lambda_2 c\delta}{a}\left(u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\right)\left(\frac{\partial u}{\partial z} + \delta^2\frac{\partial w}{\partial x}\right), \\ S_{yy} = 0, \\ S_{yz} = \frac{\delta}{1+\lambda_1}\left(1 + \frac{\lambda_2 c\delta}{a}\left(u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\right)\frac{\partial w}{\partial y}, \\ S_{zz} = \frac{2}{1+\lambda_1}\left(1 + \frac{\lambda_2 c\delta}{a}\left(u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\right)\frac{\partial w}{\partial z}. \end{array}\right\} \tag{2.16}$$

Using the assumption of long wave length $\delta \leq 1$ and low Reynolds , terms of order δ and higher power are neglected. Then Eqs. (2.12) - (2.16) will reduce to the following non-homogenous, linear and second order partial differential equation:

$$\beta^2\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (1 + \lambda_1)\frac{dp}{dx}. \tag{2.17}$$

Similarly, the corresponding boundary conditions, Eqs. (2.8) - (2.9) take the form

$$u = -1, \; at \; y = \pm 1, \tag{2.18}$$

$$u = -1, \; at \; z = \pm h(x) = \pm 1 \pm \phi \cos 2\pi x. \tag{2.19}$$

where ϕ is the amplitude ratio i.e., $\phi = b/a$, such that $0 \leq \phi \leq 1$, and β is the aspect ratio.

2.2 Solution of the problem

It is noted that Eq. (2.17), is a linear, non-homogeneous and second order partial differential equation, corresponding to the boundary conditions given in Eqs. (2.18) - (2.19). In order to solve Eq. (2.17), the following transformation is taken in to consideration

$$u(y, z) = v_1(y, z) + w_1(z), \tag{2.20}$$

which is sugested by **Richard Haberman** in his book *"Elementary Applied Partial Differential Equations"*. This transformation is useful to convert non-homogenous PDEs and boundary conditions into homogenous PDEs and boundary conditions. By using Eq. (2.20) in Eq. (2.17). Which gives

$$\beta^2 \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} = 0, \tag{2.21}$$

and

$$\frac{\partial^2 w_1}{\partial z^2} = \frac{d^2 w_1}{dz^2} = (1 + \lambda_1) \frac{dp}{dx}. \tag{2.22}$$

Now, by using Eq. (2.20) in Eqs. (2.18) - (2.19). Corresponding boundary conditions take the form

$$v_1(\pm 1, z) = -1 - w_1(z), \tag{2.23}$$

and

$$v_1(y, \pm h) = 0, \tag{2.24}$$

$$w_1(\pm h) = -1. \tag{2.25}$$

Here, it can be noted that the transformation yields, the main given problem into two systems of differential equations. First one is an ODE (i.e., Eq. 2.22) whereas, the second one is PDE (i.e., Eq. 2.21), corresponding to their boundary conditions. Thus the solution of the problem is obtained by solving the Eq. (2.22) and Eq. (2.21), one-by-one respectively.

2.2.1 Solution of ordinary differential equation

By considering Eq. (2.22), with the boundary conditions Eq. (2.25) such that

$$\frac{d^2 w_1}{dz^2} = (1+\lambda_1)\frac{dp}{dx},$$

$$w_1(\pm h) = -1.$$

For the above ODE, the complementary function and particular solution are

$$w_{1_c}(z) = c_1 + c_2 z, \tag{2.26}$$

where $c_1$and c_2 are constants, which need to be determined.

$$w_{1_p}(z) = \left[\frac{z^2(1+\lambda_1)}{2}\right]\frac{dp}{dx}. \tag{2.27}$$

Thus the general solution is determined by

$$w_1(z) = c_1 + c_2\ z + \left[\frac{z^2(1+\lambda_1)}{2}\right]\frac{dp}{dx}. \tag{2.28}$$

Applying the boundary conditions Eq. (2.25), in above equation to obtain the values of c_1,and c_2 . Such that:

$$c_1 = -1 - \left[\frac{h^2(1+\lambda_1)}{2}\right]\frac{dp}{dx},$$

and

$$c_2 = 0.$$

Thus, the general solution of the given ODE is

$$w(z) = -1 - \left[\frac{(1+\lambda_1)(h^2 - z^2)}{2}\right]\frac{dp}{dx}. \tag{2.29}$$

2.2.2 Solution of partial differential equation

Considering the PDE in Eq. (2.21) with the corresponding boundary conditions Eqs. (2.23) - (2.24), such that

$$\beta^2\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} = 0,$$

$$v_1(\pm 1, z) = -1 - w_1(z),$$

$$v_1(y, \pm h) = 0.$$

Since, Eq. (2.21) is a linear, second order homogenous PDE, with homogenous boundary conditions at $z = \pm h$. Therefore, the solution of the given PDE is obtained by using method known as "*Separation of variables*".

Let us assume that

$$v(y, z) = Y(y) \times Z(z). \tag{2.30}$$

is one of the possible solutions of Eq. (2.21).

Using (2.30) in (2.21), it yields

$$\frac{Z''}{Z} = -\beta^2\frac{Y''}{Y} = -\alpha^2. \text{ (say)} \tag{2.31}$$

This implies that

$$\frac{Z''}{Z} = -\alpha^2, \tag{2.32}$$

and

$$\beta^2\frac{Y''}{Y} = \alpha^2. \tag{2.33}$$

Put Eq. (2.24) in Eq. (2.30). It gives

$$0 = Y(y) \times Z(z). \tag{2.34}$$

As $Y(y) \neq 0$. This implies that

$$Z(\pm h) = 0. \tag{2.35}$$

Therefore, there are two possible cases.

Case-I

Consider Eq. (2.32)

$$\begin{aligned}
\frac{Z''}{Z} &= -\alpha^2, \\
&\implies D^2 Z + Z\alpha^2 = 0, \\
&\implies (D^2 + \alpha^2) Z = 0, \\
&\implies Z(z) \neq 0, \\
&\implies (D^2 + \alpha^2) = 0.
\end{aligned}$$

Then

$$D = \pm (i\alpha). \tag{2.36}$$

For the values of $\alpha < 0$ and $\alpha = 0$, trivial solutions are obtained. Therefore, the only non-trivial solution is obtained, for the value of $\alpha > 0$, which is given as

$$Z(z) = c_3 \cos(\alpha z) + c_4 \sin(\alpha z), \tag{2.37}$$

where c_3 and c_4 are the constants that are determined by using $Z(\pm h) = 0$, in Eq. (2.32), such that

$$c_4 = 0.$$

and

$$c_3 \cos(\alpha h) = 0, \quad \implies c_3 \neq 0.$$

Therefore

$$\cos(\alpha h) = 0.$$

This implies that

$$\alpha_n = \frac{(2n-1)\pi}{2h}, \quad n = 1, 2, 3... \tag{2.38}$$

Thus, Eq. (2.37) reduces to

$$\begin{aligned} Z(z) &= c_3 \cos\left(\frac{\pi(2n-1)z}{2h}\right), \\ Z(z) &= c_3 \cos(\alpha_n z). \end{aligned} \tag{2.39}$$

Case-II

Now, consider Eq. (2.33)

$$\beta^2 \frac{Y''}{Y} = \alpha^2,$$

$$\Longrightarrow (D^2 - \frac{\alpha^2}{\beta^2})Y = 0. \tag{2.40}$$

This implies that $Y(y) \neq 0$. Then

$$D = \pm\left(\frac{\alpha}{\beta}\right). \tag{2.41}$$

Thus, the general solution is given as

$$Y(y) = c_5 \cosh\left(\frac{\alpha}{\beta}y\right) + c_6 \sinh\left(\frac{\alpha}{\beta}y\right). \tag{2.42}$$

Using Eq. (2.39) and Eq. (2.42) in Eq. (2.30). It is obtained

$$v_1(y,z) = \left[c_5 \cosh\left(\frac{\alpha}{\beta}y\right) + c_6 \sinh\left(\frac{\alpha}{\beta}y\right)\right] c_3 \cos(\alpha_n z), \tag{2.43}$$

or

$$v_1(y,z) = \left[A \cosh\left(\frac{\alpha}{\beta}y\right) + E \sinh\left(\frac{\alpha}{\beta}y\right)\right] \cos(\alpha_n z), \tag{2.44}$$

where $A = c_3 \times c_5$ and $E = c_3 \times c_6$.

Using the boundary conditions given in Eq. (2.23) in above Eq. (2.44). We get

$$-1 - w_1(z) = \left[A \cosh\left(\frac{\alpha}{\beta}\right) + E \sinh\left(\frac{\alpha}{\beta}\right)\right] \cos(\alpha_n z), \tag{2.45}$$

$$-1 - w_1(z) = \left[A \cosh\left(\frac{\alpha}{\beta}\right) - E \sinh\left(\frac{\alpha}{\beta}\right)\right] \cos(\alpha_n z). \tag{2.46}$$

Replacing Eq. (2.29) in above Eqs. (2.45) - (2.46). It gives

$$\frac{(1+\lambda_1)}{2}(h^2 - z^2)\frac{dp}{dx} = \left[A\cosh\left(\frac{\alpha}{\beta}\right) + E\sinh\left(\frac{\alpha}{\beta}\right)\right]\cos(\alpha_n z), \tag{2.47}$$

$$\frac{(1+\lambda_1)}{2}(h^2 - z^2)\frac{dp}{dx} = \left[A\cosh\left(\frac{\alpha}{\beta}\right) - E\sinh\left(\frac{\alpha}{\beta}\right)\right]\cos(\alpha_n z), \tag{2.48}$$

By using the orthogonal property and integrating Eq. (2.47) and Eq. (2.48), w.r.t. z, from 0 to h, respectively, it gives

$$A\cosh\left(\frac{\alpha}{\beta}\right) + E\sinh\left(\frac{\alpha}{\beta}\right) = \left(\frac{2(-1)^n(1+\lambda_1)}{h\alpha_n^3}\right)\frac{dp}{dx}, \tag{2.49}$$

$$A\cosh\left(\frac{\alpha}{\beta}\right) - E\sinh\left(\frac{\alpha}{\beta}\right) = \left(\frac{2(-1)^n(1+\lambda_1)}{h\alpha_n^3}\right)\frac{dp}{dx}. \tag{2.50}$$

Adding Eq. (2.49) and Eq. (2.50). It provides

$$A = \left[\frac{2(-1)^n(1+\lambda_1)}{h(\alpha_n)^3\cosh\left(\frac{\alpha}{\beta}\right)}\right]\frac{dp}{dx}, \quad \text{and} \quad E = 0. \tag{2.51}$$

Substituting the values of A and E, in Eq. (2.44), we have

$$v_1(y,z) = \left[\frac{2(-1)^n(1+\lambda_1)\cosh\left(\frac{\alpha}{\beta}y\right)\cos(\alpha_n z)}{h(\alpha_n)^3\left(\cosh\frac{\alpha}{\beta}\right)}\right]\frac{dp}{dx}. \tag{2.52}$$

Since the values of $v_1(y,z)$ and $w_1(z)$, are now known. Therefore, Eq. (2.20) becomes

$$u(y,z) = -1 - \frac{h^2}{2}(1+\lambda_1)\frac{dp}{dx}\left[1 - \frac{z^2}{h^2} - 4\sum_1^{\infty}\frac{(-1)^n\cos(\alpha_n z)\cosh\left(\frac{\alpha_n}{\beta}y\right)}{(\alpha_n)^3\cosh\left(\frac{\alpha_n}{\beta}\right)}\right]. \tag{2.53}$$

Now, it is assumed that

$$\alpha_n^{'} = \frac{(2n-1)\pi}{2}. \tag{2.54}$$

Then Eq. (2.38) becomes

$$\alpha_n = \frac{\alpha_n^{'}}{h(x)}. \tag{2.55}$$

Replace α_n by α_n' in Eq. (2.53), we obtained

$$u(y,z) = -1 - \frac{h^2}{2}(1+\lambda_1)\frac{dp}{dx}\left[1 - \frac{z^2}{h^2} - 4\sum_{1}^{\infty}\frac{(-1)^n \cos\left(\frac{z\alpha_n'}{h}\right)\cos\left(\frac{y\alpha_n'}{\beta h}\right)}{(\alpha_n')^3 \cosh\left(\frac{\alpha_n'}{\beta h}\right)}\right], \quad n = 1,2,3... \tag{2.56}$$

The above Eq. (2.56) shows the required velocity profile of the Jeffrey fluid peristaltically moving in the rectangular duct.

In order to find volumetric flow rate of the Jeffrey fluid in the given rectangular duct, the velocity of the fluid is integrated twice w.r.t. z and y, respectively. As shown below

$$q = \int_0^1\int_0^{h(x)} u(y,z)dzdy, \tag{2.57}$$

$$q = -h(x) - \frac{h^3(x)}{3}(1+\lambda_1)\frac{dp}{dx}\left[1 - 6\beta h(x)\sum_{n=1}^{\infty}\frac{\tanh\left(\frac{\alpha_n'}{\beta\, h(x)}\right)}{(\alpha_n')^5}\right]. \tag{2.58}$$

Average volumetric flow rate over one period $T = \lambda/c$ of the peristaltic wave is defined as

$$Q = \frac{1}{T}\int_0^T \bar{Q}dt = q + 1, \tag{2.59}$$

where

$$\bar{Q} = \int_0^1\int_0^h (u+1)dzdy = q + h. \tag{2.60}$$

Therefore, the average volumetric flow rate is given as

$$\begin{aligned} Q &= 1 + q, \\ Q &= 1 - h(x) - \frac{h^3(1+\lambda_1)}{3}\left[1 - 6\beta h\sum_{n=1}^{\infty}\frac{\tanh\left(\frac{\alpha_n'}{\beta h}\right)}{(\alpha_n')^5}\right]\frac{dp}{dx}. \end{aligned} \tag{2.61}$$

One can identify pressure from the above Eq. (2.61), such that

$$\frac{dp}{dx} = \frac{-3\left[Q - 1 + h\right]}{(1+\lambda_1)h^3\left[1 - 6\beta h \sum_{n=1}^{\infty} \frac{\tanh\left(\frac{\alpha_n'}{\beta h}\right)}{(\alpha_n')^5}\right]} \tag{2.62}$$

In order to find pressure rise of the fluid, Eq. (2.62) is numerically integrated over one wave, by using the built-in command of the mathematical software *"Mathematica"*, such that

$$\Delta p = \int_0^1 \frac{dp}{dx} dx. \tag{2.63}$$

It is interesting to note that the given rectangular duct becomes a square duct by taking $\beta = 1$.

2.3 Results and discussion

In this section, the graphical results of the problem under consideration are discussed. These graphs are obtained with the help of *"Mathematica"*.

In figures 2.2 - 2.5, the graphical behavior of pressure gradient dp/dx, for different values of aspect ratio β, amplitude ratio ϕ, Jeffrey parameter λ_1, and volumetric flow rate Q against x are sketched.

In figures 2.2 - 2.3, it is observed that for $x \in [0.2, 0.8]$, pressure gradient rises with the rise in aspect ratio β, and amplitude ratio ϕ. Moreover, for $x \in [0, 0.2]$ and $x \in [0.8, 1.0]$, pressure gradient is very small. This implies that the flow can pass without any pressure gradient. In figure 2.4, however, pressure gradient starts declining as Jeffrey parameter λ_1 increases, in region for $x \in [0.2, 0.8]$ and follows the same flow pattern in region $x \in [0, 0.2]$ and $x \in [0.8, 1.0]$, as followed by aspect ratio β. Similarly, as figure 2.5 demonstrates a different behavior of pressure gradient, altogether. As it is not only decreasing in the region, upon increase of volume flow rate Q. But, it remains positive for the smaller values of Q and negative for the higher values of Q. Figures 2.6 - 2.7, show the velocity profile U of the wave for different values of volume flow rate Q and aspect ratio β.

In figures 2.6 - 2.7, it is observed that for $z \in [-1.5, 1.5]$, the velocity profile rises, corresponding to the rise in volume flow rate and aspect ratio, respectively. In figures 2.8 - 2.11, pressure rise Δp of the fluid is depicted, for the different values of aspect ratio β, amplitude ratio ϕ, Jeffrey parameter λ_1, and volume flow rate Q.

It is inferred from the figures 2.8 - 2.9, pressure rise Δp increases for the different increasing values of β with respect to Q. Similarly, Δp rises for the Jeffrey parameter λ_1, when it is sketched w.r.t. β. Moreover, in figures 2.10 - 2.11, the values of Δp decline for rise in the numerical values of λ_1 and volumetric flow rate Q which are sketched with respect to Q and β, respectively. Figures 2.12 - 2.23, represent the stream lines for the fluid flow. The size of the boluses varies; correspond to the variation in aspect ratio β, volumetric flow rate Q and amplitude ratio respectively φ.

In figures 2.12 - 2.15, the number and size of the boluses are in increasing with the rise in the numerical value of β. This indicates the passiveness of the fluid flow through the rectangular duct.

From figures 2.16 - 2.19, it is examined that as the numerical value of volumetric flow rate rises, the boluses are getting enlarged only whereas the number of boluses remains constant. This phenomenon shows the swiftness of the fluid flow through the rectangular duct.

For the different values of the amplitude ratio, in above figures 2.20 - 2.23, the stream lines, not only increase in number, but also expand in size. Hence it can be inferred that the flow of the fluid, is finding a bite hard to pass / flow through the rectangular duct.

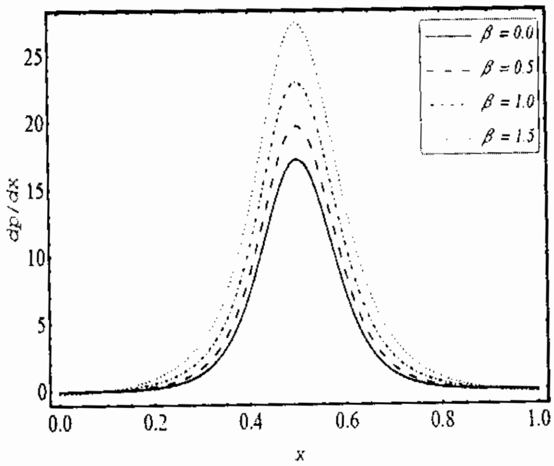

Figure. 2.2 : Variation of pressure gradient with x for different values of β at $Q = -0.5$, $\lambda_1 = 2$, $\phi = 0.6$.

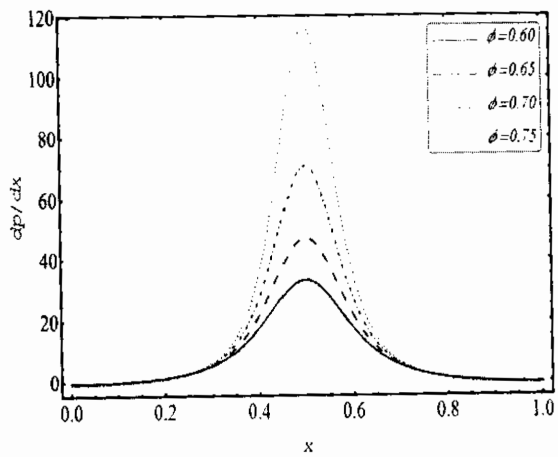

Figure. 2.3 : Variation of pressure gradient with x for different values of ϕ at $Q = -0.5$, $\lambda_1 = 2$, $\beta = 2$.

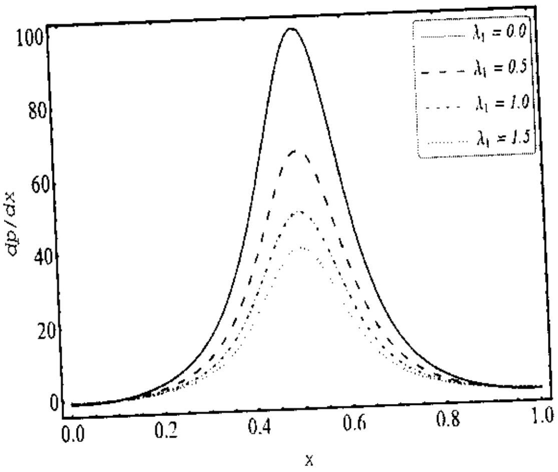

Figure. 2.4 : Variation of pressure gradient with x for different values of λ_1 at $Q = -0.5$, $\beta = 2$, $\phi = 0.6$.

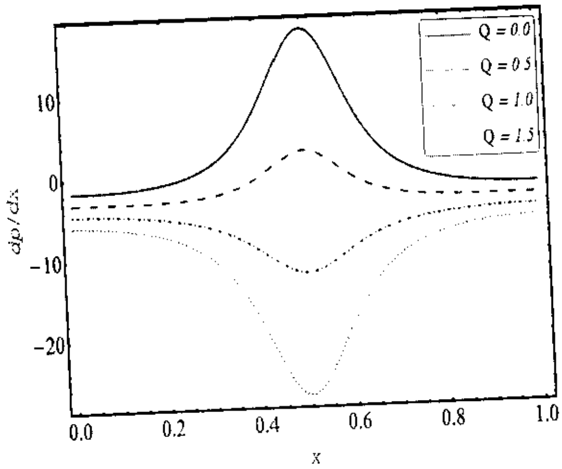

Figure. 2.5 : Variation of pressure gradient with x for different values of Q at $\beta = 2$, $\lambda_1 = 2$, $\phi = 0.6$.

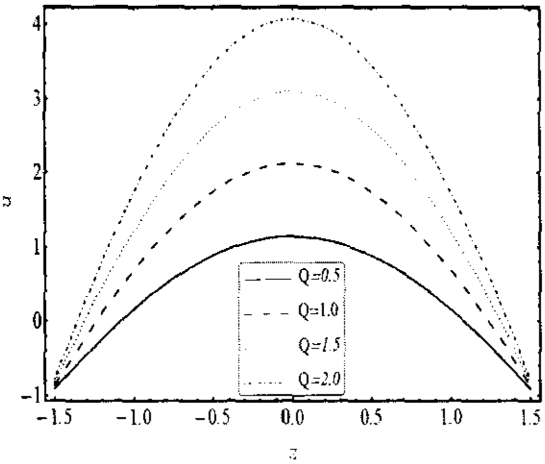

Figure. 2.6 : Velocity profile for different values of Q at $\phi = 0.6$, $y = 0.5$, $\lambda_1 = 2$, $x = 0$, $\beta = 0.5$.

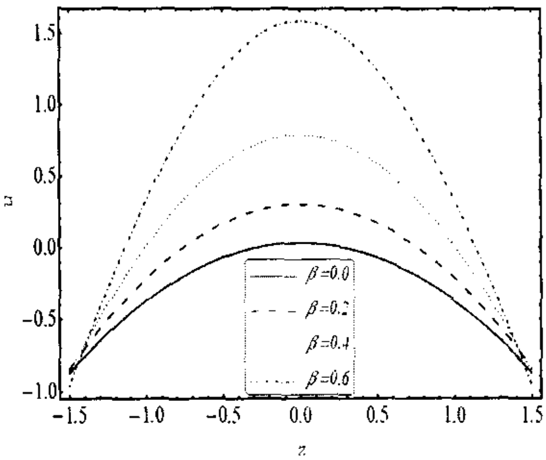

Figure. 2.7 : Velocity profile for different values of β at $\phi = 0.6$, $y = 0.5$, $\lambda_1 = 2$, $x = 0$, $Q = 0.5$.

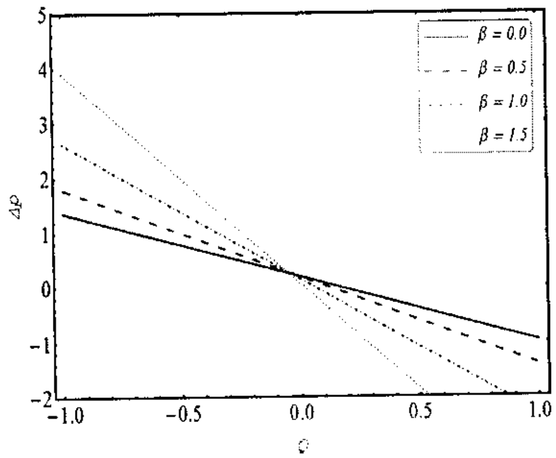

Figure. 2.8 : Variation of Δp with Q for different values of β at $\phi = 0.6$, $\lambda_1 = 2$.

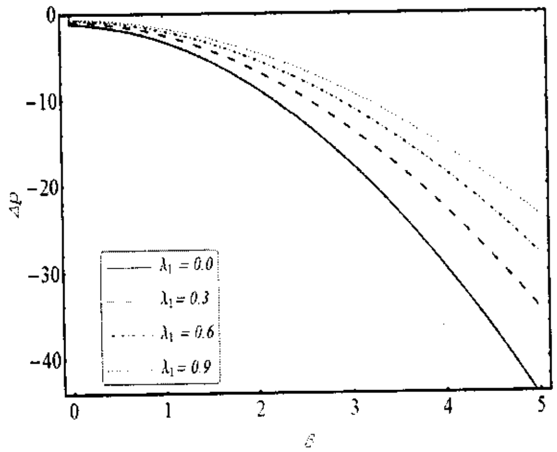

Figure. 2.9 : Variation of Δp with β for different values of λ_1 at $\phi = 0.6$, $Q = 0.5$.

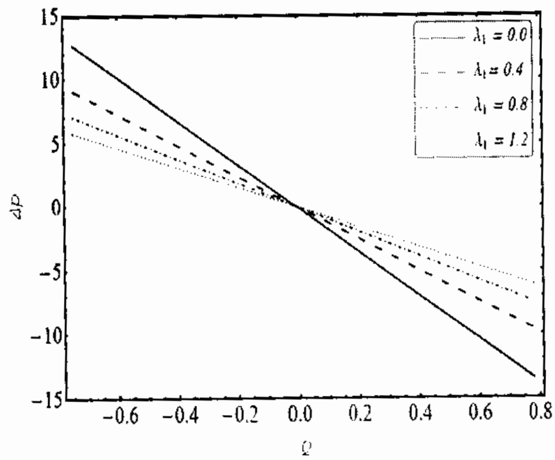

Figure. 2.10 : Variation of Δp with Q for different values of λ_1 at $\phi = 0.6$, $\beta = 2$.

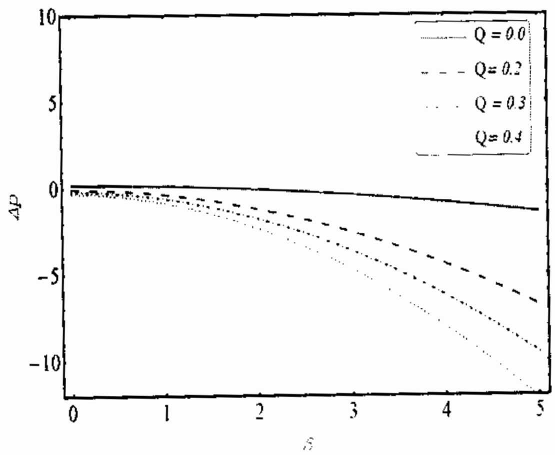

Figure. 2.11: Variation of Δp with β for different values of Q at $\phi = 0.6$, $\lambda_1 = 2$.

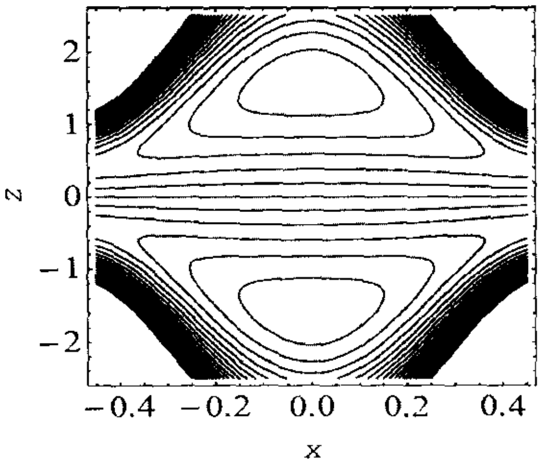

Figure. 2.12 : Stream lines for $\beta = 0$. The other parameters are $Q = 1.5$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

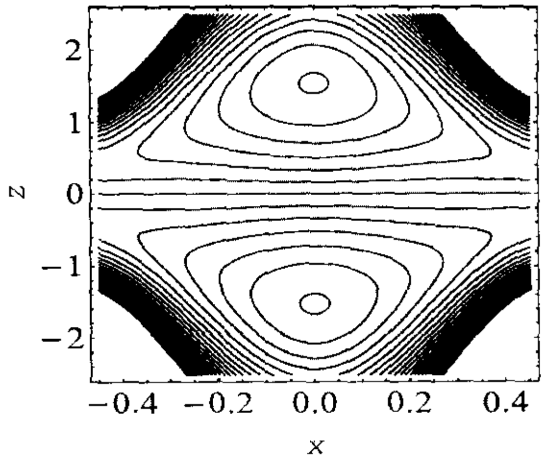

Figure. 2.13 : Stream lines for $\beta = 0.5$. The other parameters are $Q = 1.5$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

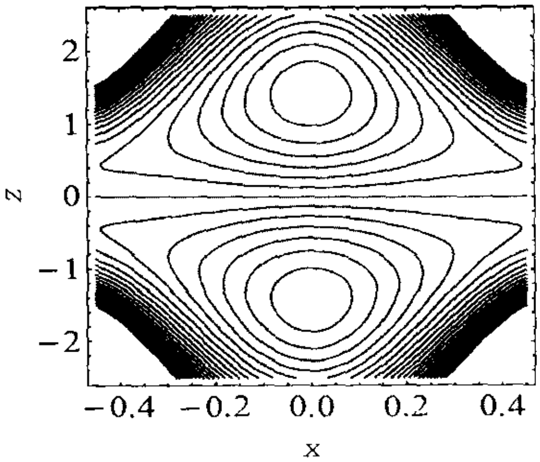

Figure. 2.14 : Stream lines for $\beta = 1$. The other parameters are $Q = 1.5$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

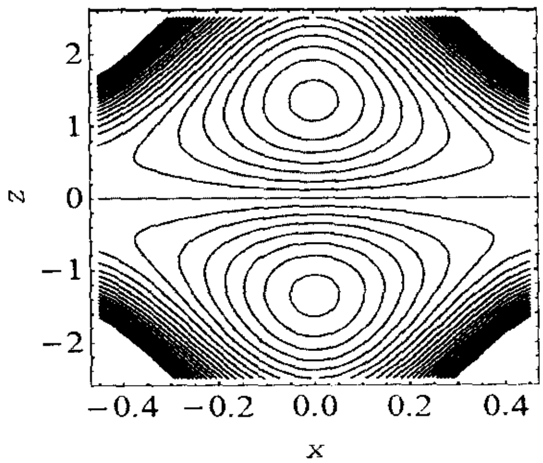

Figure. 2.15 : Stream lines for $\beta = 1.5$. The other parameters are $Q = 1.5$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

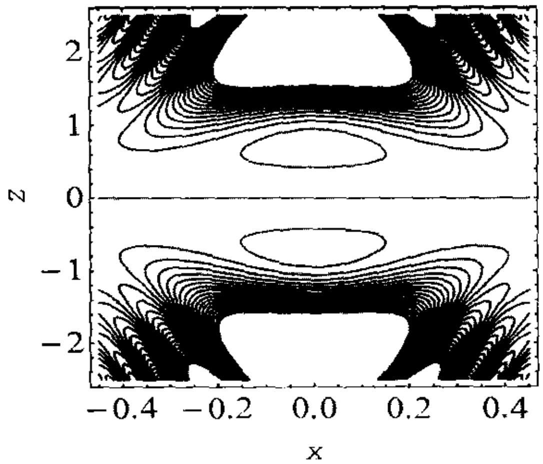

Figure. 2.16 : Stream lines for $Q = 1$. The other parameters are $\beta = 30$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

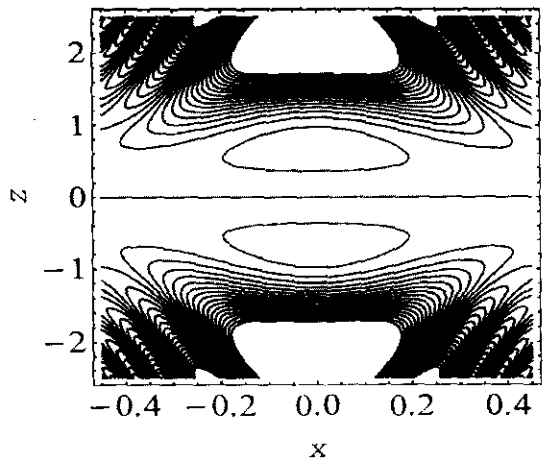

Figure. 2.17 : Stream lines for $Q = 1.5$. The other parameters are $\beta = 30$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

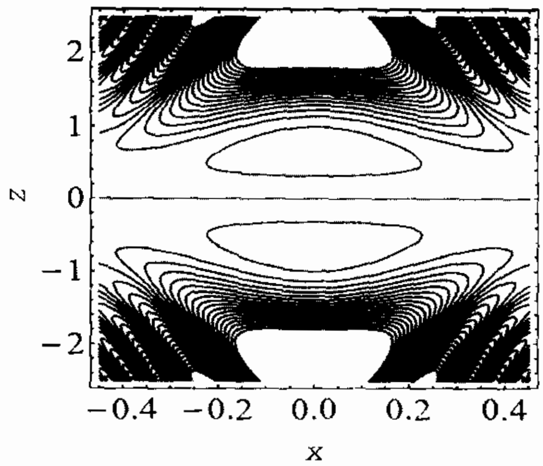

Figure. 2.18 : Stream lines for $Q = 2$ The other parameters are $\beta = 30$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

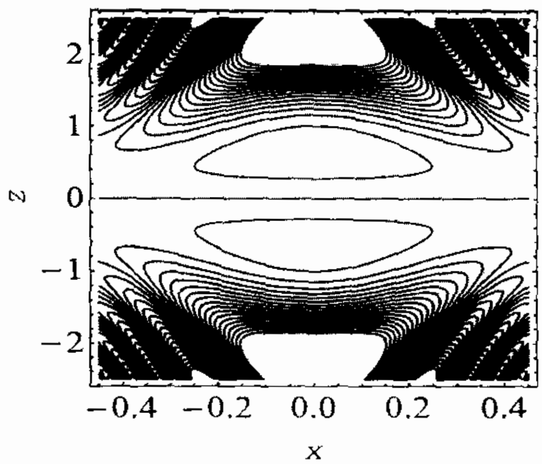

Figure. 2.19 : Stream lines for $Q = 2.5$. The other parameters are $\beta = 30$, $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 1$.

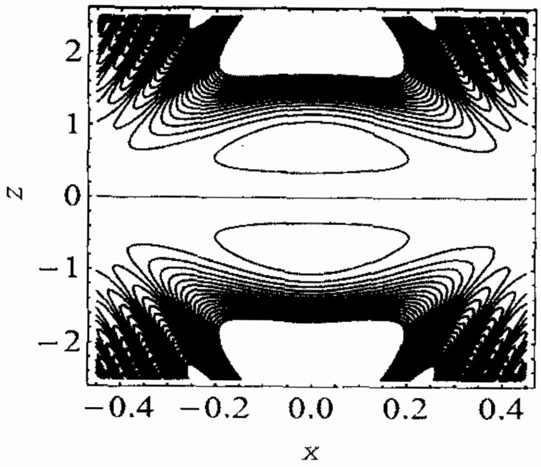

Figure. 2.20 : Stream lines for $\phi = 0.7$. The other parameters are $\beta = 30$, $y = 0.5$, $Q = 1.5$, $\lambda_1 = 1$.

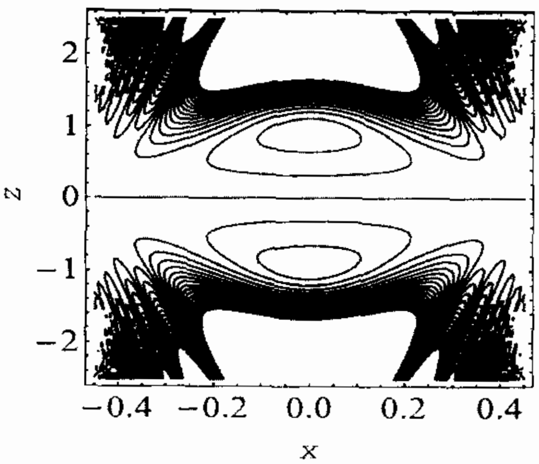

Figure. 2.21 : Stream lines for $\phi = 0.9$. The other parameters are $\beta = 30$, $y = 0.5$, $Q = 1.5$, $\lambda_1 = 1$.

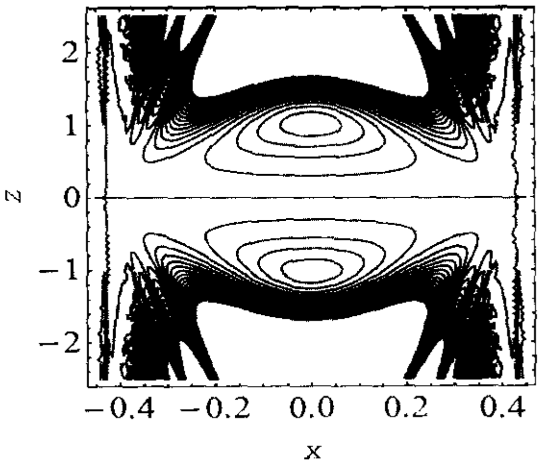

Figure. 2.22 : Stream lines for $\phi = 1.1$. The other parameters are $\beta = 30$, $y = 0.5$, $Q = 1.5$, $\lambda_1 = 1$.

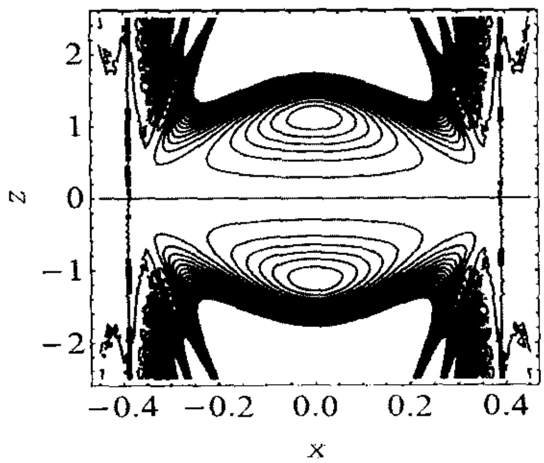

Figure. 2.23 : Stream lines for $\phi = 1.3$. The other parameters are $\beta = 30$, $y = 0.5$, $Q = 1.5$, $\lambda_1 = 1$.

Chapter 3

Effects of MHD and partial slip on the peristaltic flow of a Jeffrey fluid in a rectangular duct

The purpose of this chapter is to study the peristaltic flow of Jeffrey fluid under the simultaneous effects of magnetohydrodynamics (MHD) and partial slip conditions in a rectangular duct. The influence of sinusoidal wave is also taken into account. The analysis of mathematical model consists of continuity and the momentum equations are carried out under long wavelength and low Reynolds number assumptions. The governing equations are first reduced to the dimensionless system of partial differential equations using the appropriate variables and afterwards exact solutions are obtained by applying the method of separation of variables. The role of pertinent parameters such as Hartmann number , slip parameter, volumetric flow rate, Jeffrey parameter and the aspect ratio against the velocity profile, pressure gradient and pressure rise is illustrated graphically. The streamlines have also been presented to discuss the trapping bolus discipline against the heat transfer coefficient.

After brief introduction, Section 2 contains the formulation of the problem. Solution of the problem is given in Section 3. Section 4 is devoted for results and discussion and finally conclusion is presented in Section 5. Comparison with the existing studies is made as a limiting case of the considered problem at the end.

3.1 Mathematical formulation of the problem

Let us consider the peristaltic flow of an incompressible, Jeffrey fluid in a duct of rectangular cross section having the channel width $2d$ and height $2a$. Cartesian coordinates system is considered in such a way that $X - axis$ is taken along the axial direction, $Y - axis$ is taken along the lateral direction and $Z - axis$ is along the vertical direction of a rectangular duct.

The peristaltic waves on the wall are represented as

$$Z = H(X,t) = \pm\ a \pm b\cos\left[\frac{2\pi}{\lambda}(X - ct)\right], \tag{3.1}$$

where a and b are the amplitudes of the waves λ the wave length , c is the velocity of propagation, t is the time and X is the direction of the wave propagation. The walls parallel to XZ plane remain undisturbed and are not subject to any peristaltic wave motion. It is assumed that the lateral velocity is zero as there is no change in lateral direction of the duct cross section.

Let $\mathbf{V} = (U, 0, W)$ be the velocity for a rectangular duct. The governing equations for the flow problem are given below.

(1). Equation of conservation of mass:

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0. \tag{3.2}$$

(2). Equation of momentum:

$$\rho\frac{D\mathbf{V}}{Dt} = -\nabla P + \nabla.\mathbf{S} + \mathbf{J}\times\mathbf{B}.$$

where $\mathbf{J}$ is called the current density or Lorentz force which is the force required by a charge particle to move in magnetic and electric field. Mathematically, it is defined and denoted as

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}).$$

in which $\mathbf{B}$ is the total magnetic field such that $\mathbf{B} = \mathrm{B}_0 + \mathbf{b}$. $\mathbf{B}$ is the sum of applied magnetic field B_0 and induced magnetic field $\mathbf{b}$. Induced magnetic field is negligible compare with applied magnetic field. Moreover, the $XZ-$Walls of the rectangular duct are electrically insulated and no energy or charge/elcetricity is added or extrected from the fluid by the electric field. Therefore, this implies that threr is no electric field present in the fluid region. With the help of these assumptions the electromagnetic force $\mathbf{J} \times \mathbf{B}$ takes the following form

$$\mathbf{J} \times \mathbf{B} = -\sigma \mathrm{B}_0^2 \mathbf{V}.$$

Then, the above equation of momentum in the form of velocity components, becomes

$$\begin{aligned}
\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial X}+W\frac{\partial U}{\partial Z}\right) &= -\frac{\partial P}{\partial X}+\frac{\partial}{\partial X}S_{XX}+\frac{\partial}{\partial Y}S_{XY}+\frac{\partial}{\partial Z}S_{XZ}-\sigma B_0^2 U, && (3.3)\\
0 &= -\frac{\partial P}{\partial Y}+\frac{\partial}{\partial X}S_{YX}+\frac{\partial}{\partial Y}S_{YY}+\frac{\partial}{\partial Z}S_{YZ}, && (3.4)\\
\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial X}+W\frac{\partial W}{\partial Z}\right) &= -\frac{\partial P}{\partial Z}+\frac{\partial}{\partial X}S_{ZX}+\frac{\partial}{\partial Y}S_{ZY}+\frac{\partial}{\partial Z}S_{ZZ}-\sigma B_0^2 W. && (3.5)
\end{aligned}$$

Jeffrey stress tensor and shear stresses will remain same, which are used in previous chapter. By using the same transformation, as given in Eq. (2.10) of the previous chapter to convert the given Fixed/Lab frame in to the wave frame. In addition to the non-dimensional quantities given in Eq. (2.11) of the previous chapter, the following given non-dimensional parameters will also be used in this chapter:

$$\begin{aligned}
\beta_1 &= \frac{L}{a},\\
M &= \sqrt{\frac{\sigma}{\mu}}a\beta.
\end{aligned}$$

Neglecting the bar signs, then Eqs. (3.2) - (3.5) will be transformed into the following form:

$$\begin{aligned}
\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} &= 0, && (3.6)\\
\operatorname{Re}\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial x}+\delta\frac{\partial}{\partial x}S_{xx}+\beta^2\frac{\partial}{\partial y}S_{xy}+\frac{\partial}{\partial z}S_{xz}-M^2(u+1), && (3.7)\\
0 &= -\frac{\partial p}{\partial y}+\delta^2\frac{\partial}{\partial x}S_{yx}+\delta^2\frac{\partial}{\partial y}S_{yy}+\delta\frac{\partial}{\partial z}S_{yz}, && (3.8)\\
\operatorname{Re}\delta^2\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z}+\delta^2\frac{\partial}{\partial x}S_{zx}+\delta\beta^2\frac{\partial}{\partial y}S_{zy}+\delta^2\frac{\partial}{\partial z}S_{zz}-\delta^2M^2w. && (3.9)
\end{aligned}$$

By considering the assumption of long wave length $\delta \leq 1$ and low Reynolds, neglecting the terms of order δ and higher. Then the above Eq. (3.6)-Eq. (3.9) will reduce to the following non-homogenous, linear and second order partial differential equation:

$$\frac{\beta^2}{1+\lambda_1}\frac{\partial^2 u}{\partial y^2}+\frac{1}{1+\lambda_1}\frac{\partial^2 u}{\partial z^2}-M^2\left(u+1\right)=\frac{dp}{dx}. \tag{3.10}$$

Therefore, the corresponding slip boundary conditions at the walls are described as:

$$(i).\quad u(y,z) = -1, \quad at \quad y=\pm 1, \tag{3.11}$$

$$(ii).\quad u(y,z) = \frac{-\beta_1}{1+\lambda_1}\frac{\partial u}{\partial z}-1, \quad at \quad z=h\left(x\right)=1+\phi\cos 2\pi x, \tag{3.12}$$

$$(iii).\quad u(y,z) = \frac{\beta_1}{1+\lambda_1}\frac{\partial u}{\partial z}-1, \quad at \quad z=-h\left(x\right)=-1-\phi\cos 2\pi x. \tag{3.13}$$

3.2 Solution of the problem

Solution of the above given problem, is sought by using the following transformation in Eqs. (3.10) - (3.13).

$$u(y,z)=v_2(y,z)+w_2(y). \tag{3.14}$$

Thus Eq. (3.10) yields

$$\frac{\beta^2}{1+\lambda_1}\frac{\partial^2 v_2}{\partial y^2}+\frac{1}{1+\lambda_1}\frac{\partial^2 v_2}{\partial z^2}-M^2 v_2=0. \tag{3.15}$$

and

$$\frac{\beta^2}{1+\lambda_1}\frac{d^2 w_2}{dy^2}-M^2 w_2=M^2+\frac{dp}{dx}. \tag{3.16}$$

Moreover, the boundary conditions are transformed into the following form

$$(i).\quad w_2(y) = -1, \quad at \quad y=\pm 1, \tag{3.17}$$

$$(ii).\quad v_2(y,z) = 0, \quad at \quad y=\pm 1, \tag{3.18}$$

$$(iii).\quad v_2(y,z) = -\frac{\beta_1}{1+\lambda_1}\frac{\partial v_2}{\partial z}-1-w_2(y), \quad at \quad z=h, \tag{3.19}$$

$$(iv).\quad v_2(y,z) = \frac{\beta_1}{1+\lambda_1}\frac{\partial v_2}{\partial z}-1-w_2(y), \quad at \quad z=-h. \tag{3.20}$$

Interestingly, the transformation which is considered again yields, the main given problem into two systems of differential equations. First one is an ODE (i.e., Eq. 3.16) whereas the second one is PDE (i.e., Eq. 3.15), corresponding to their boundary conditions. Thus the solution of the problem is obtained by solving the Eq. (3.16) and Eq. (3.15), one-by-one respectively.

3.2.1 Solution of ordinary differential equation

It is observed that Eq. (3.16) is a second order linear and a non-homogenous ODE, corresponding to non-homogenous boundary conditions Eq. (3.17). Therefore, the general solution is given

$$w_2(y) = -1 - \frac{1}{M^2}\frac{dp}{dx}\left[1 - \operatorname{sech}\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\right)\cosh\left(\frac{My\sqrt{1+\lambda_1}}{\beta}\right)\right]. \tag{3.21}$$

3.2.2 Solution of partial differential equation

In order to obtain the solution of Eq. (3.15), which is a second order, linear and homogenous partial differential equation, subject to the homogenous boundary conditions Eq. (3.18). By using the method of separation of variables, it is assumed that

$$v_2(y,z) = [Y(y)] \times [Z(z)]. \tag{3.22}$$

is one the possible solutions of Eq. (3.15). Using Eq. (3.22) in Eq. (3.15), it provides

$$\frac{Y''}{Y} = \frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{M^2}{\beta^2}(1+\lambda_1) = -\alpha^2 \ \ (say), \tag{3.23}$$

$$\implies \frac{Y''}{Y} = -\alpha^2. \tag{3.24}$$

and

$$-\alpha^2 = \frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{M^2}{\beta^2}(1+\lambda_1). \tag{3.25}$$

using the boundary conditions Eq. (3.18), in Eq. (3.22). This yield

$$0 = [Y(\pm\mathbf{1})] \times [Z(z)],$$

$$\implies Z\ (z) \neq 0,$$

This implies that

$$Y\ (\pm 1) = 0. \tag{3.26}$$

Then, there are two possible cases to achieve the required solution.

Case-I

By considering Eq (3.24)

$$\begin{aligned}
\frac{Y''}{Y} &= -\alpha^2, \\
&\Longrightarrow \left[D^2 + \alpha^2\right] \times Y(y) = 0, \\
D &= \pm(i\alpha).
\end{aligned}$$

It is noted that a trivial solution is obtained for the values of $\alpha < 0$ and $\alpha = 0$. The only non-trivial solution is obtained, for the value of $\alpha > 0$. Which yields

$$Y(y) = c_3 \cos(\alpha y) + c_4 \sin(\alpha y). \tag{3.27}$$

Now applying boundary conditions given in Eq. (3.26) in Eq. (3.27). It gives

$$\begin{aligned}
c_4 &= 0, \\
\alpha_n &= \left(\frac{(2n-1)\pi}{2}\right), \quad \text{for } n = 1, 2, 3...
\end{aligned}$$

Thus Eq. (3.27) becomes

$$Y(y) = c_3 \cos\left(\frac{\pi(2n-1)y}{2}\right), \quad \text{for } n = 1, 2, 3... \tag{3.28}$$

Case-II

Similarly, Eq. (3.25), is now being considered.

$$\begin{aligned}
&\tfrac{-1}{\beta^2}\tfrac{Z''}{Z} + \tfrac{M^2}{\beta^2}(1+\lambda_1) = -\alpha^2, \\
&\Longrightarrow \tfrac{-1}{\beta^2}\tfrac{Z''}{Z} = -\alpha^2 - \tfrac{M^2}{\beta^2}(1+\lambda_1), \\
&\Longrightarrow D = \pm\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)},
\end{aligned}$$

this implies that

$$Z(z) = c_5 \cosh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) + c_6 \sinh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right). \tag{3.29}$$

Substituting the values of $Y(y)$ and $Z(z)$ in Eq. (3.22). This leads to

$$v_2(y,z) = \left.\begin{array}{c} c_3\left[c_5 \cosh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) + c_6 \sinh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right)\right] \\ \times\left[\cos\left(\frac{(2n-1)\pi y}{2}\right)\right] \end{array}\right\}. \tag{3.30}$$

For convenience, it is assumed that

$$\begin{aligned} c_3 \times c_5 &= c_7, \text{ (say)} \\ c_3 \times c_6 &= c_8. \text{ (say)} \end{aligned}$$

Therefore, Eq. (3.30) becomes

$$v_2(y,z) = \left.\begin{array}{c} \left[\begin{array}{c} c_7 \cosh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \\ +c_8 \sinh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \end{array}\right] \\ \times\left[\cos\left(\frac{(2n-1)\pi y}{2}\right)\right] \end{array}\right\}. \tag{3.31}$$

In order to calculate the values of both constants given in the above equation, use the boundary conditions Eqs. (3.19)-(3.20) in the above equation. This yields

$$\frac{-\beta_1}{1+\lambda_1}\frac{\partial v_2}{\partial z} - 1 - w_2(y) = \left.\begin{array}{c} \left[\begin{array}{c} c_7 \cosh\left(h\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \\ +c_8 \sinh\left(h\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \end{array}\right] \\ \times\left[\cos\left(\frac{(2n-1)\pi y}{2}\right)\right] \end{array}\right\}, \tag{3.32}$$

$$\frac{\beta_1}{1+\lambda_1}\frac{\partial v_2}{\partial z} - 1 - w_2(y) = \left.\begin{array}{c} \left[\begin{array}{c} c_7 \cosh\left(h\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \\ -c_8 \sinh\left(h\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \end{array}\right] \\ \times\left[\cos\left(\frac{(2n-1)\pi y}{2}\right)\right] \end{array}\right\}. \tag{3.33}$$

Subsituting the value of $w_2(y)$ in above Eqs (3.32)-(3.33) and intergrating both sides of Eqs. (3.32)-(3.33) respectivelyy, w.r.t "y" from "0" to "h". It is obtained

$$v_2(y,z) = \frac{32\frac{dp}{dx}(1+\lambda_1)^2 \cos\left(\frac{\pi y}{2}\right)\cosh\left(z\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\right)}{\begin{array}{c} 2\pi(1+\lambda_1)\left[\pi^2\beta^2+4M^2(1+\lambda_1)\right]\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}(1+\phi\cos(2\pi x))\right) \\ +\beta_1\pi\left[\pi^2\beta^2+4M^2(1+\lambda_1)\right]^{\frac{3}{2}}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}(1+\phi\cos(2\pi x))\right)\end{array}}. \tag{3.34}$$

Substituting the values of $w_2(y)$ and $v_2(y,z)$ in Eq. (3.14), it gives the desired velocity profile of the peristaltic wave, moving in the rectangular duct.

$$u(y,z) = \left\{ \begin{array}{c} \frac{-\left[M^2+\frac{dp}{dx}-\frac{dp}{dx}\cosh\left(\frac{My\sqrt{1+\lambda_1}}{\beta}\right)\right]}{M^2} \\ \\ +\frac{\left[32\frac{dp}{dx}(1+\lambda_1)^2\cos\left(\frac{\pi y}{2}\right)\cosh\left(z\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\right)\right]}{\begin{array}{c}\left[2\pi(1+\lambda_1)\left\{\pi^2\beta^2+4M^2(1+\lambda_1)\right\}\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}(1+\phi\cos(2\pi x))\right)\right] \\ +\left[\beta_1\pi\left\{\pi^2\beta^2+4M^2(1+\lambda_1)\right\}^{\frac{3}{2}}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}(1+\phi\cos(2\pi x))\right)\right]\end{array}} \end{array} \right. \tag{3.35}$$

Similarly, volumetric flow rate and average volumetric flow rate of the peristaltic wave are given by

$$q = \frac{\left[128P(1+\lambda_1)^2\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right)\right]}{\begin{array}{c}\left[\begin{array}{c}\beta_1\sqrt{\pi^2\beta^2+4M^2(1+\lambda_1)}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right) \\ +2(1+\lambda_1)\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right)\end{array}\right] \\ \times\left[\pi^2\left\{\pi^2\beta^2+4M^2(1+\lambda_1)\right\}^{\frac{3}{2}}\right]\end{array}}. \tag{3.36}$$

Average volumetric flow rate over one period $T=\lambda/c$ of the peristaltic wave is defined as

$$Q = \frac{1}{T}\int_0^T \overline{Q}dt = q+1,$$

where

$$\bar{Q} = \int_0^1 \int_0^h (u+1)dzdy = q + h.$$

Therefore, the average volumetric flow rate is given as

$$Q = 1 + \frac{\left[128P(1+\lambda_1)^2 \sinh\left(\sqrt{\frac{\pi^2\beta^2}{4} + M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right)\right]}{\left[\begin{array}{c} \beta_1\sqrt{\pi^2\beta^2 + 4M^2(1+\lambda_1)}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4} + M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right) \\ +2(1+\lambda_1)\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4} + M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right) \\ \times\left[\pi^2\left\{\pi^2\beta^2 + 4M^2(1+\lambda_1)\right\}^{\frac{3}{2}}\right] \end{array}\right]}. \tag{3.37}$$

Similarly, the pressure gradient can be obtained from the above equation, such that

$$\left.\frac{dp}{dx} = \frac{[Q+\phi\cos(2\pi x)] \times \left(\begin{array}{c} \left[2M^3\pi^2(1+\lambda_1)^{\frac{3}{2}}(\pi^2\beta^2 + 4M^2(1+\lambda_1))^{\frac{3}{2}}\right] \\ \cosh\left(\sqrt{\frac{\pi^2\beta^2}{4} + M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right) \\ +\left[\beta_1 M^3\pi^2\sqrt{1+\lambda_1}(\pi^2\beta^2 + 4M^2(1+\lambda_1))^2\right] \\ \sinh\left(\sqrt{\frac{\pi^2\beta^2}{4} + M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right) \\ -(1+\phi\cos(2\pi x))\left[M\sqrt{1+\lambda_1} - \beta\tanh(\frac{M\sqrt{1+\lambda_1}}{\beta})\right] \end{array}\right)}{128M^3(1+\lambda_1)^{\frac{5}{2}}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4} + M^2(1+\lambda_1)}\,(1+\phi\cos(2\pi x))\right)}\right\} \tag{3.38}$$

Pressure rise of the fluid can be caluculated numerically by integration of Eq. (3.38) over one wave length yields

$$\Delta p = \int_0^1 \frac{dp}{dx}dx. \tag{3.39}$$

3.3 Results and discussion

In this segment of dissertation, the relative changes in the behavior of the fluid flow, caused by the variation in the different parameters, are displayed and discussed graphically. The parameters which are given the variation for the considered problem are, aspect ratio β, amplitude ratio ϕ, Jeffrey's parameter λ_1, volumetric flow rate Q, slip parameter β_1, and the Hartmann's number M.

In figures 3.1- 3.5, the graphical behavior of pressure gradient dp/dx, for different values of aspect ratio β, amplitude ratio ϕ, Jeffrey parameter λ_1, slip parameter β_1, Hartmann's number M, and volume flow rate Q against x are sketched.

Having a glance at the above graphs. It is noticed that in figures 3.1 - 3.2, the pressure gradient dp/dx, rises with the rise in aspect ratio β and Hartmann number M whereas, in figures 3.3 - 3.5, the intensity of pressure gradient dp/dx decreases, upon the rise in slip parameter β_1, Jeffrey parameter λ_1, and volumetric flow rate Q, respectively.

Figures 3.6 - 3.10, show the velocity profile U of the peristaltic wave, for different values of volume flow rate Q and aspect ratio β, Jeffrey parameter λ_1, Hartmann number M, and slip parameter β_1.

Having witnessed the graphs presented in the figure 3.6 - 3.8, one can easily examined that the velocity profile, of the wave which propagates in the rectangular duct, is gradually speeding up with the passage of time, corresponding to the increase in numerical values of volumetric flow rate Q, Jeffrey parameter λ_1, and the aspect ratio β, respectively. Similarly, velocity profile starts making the down-turn corresponding to the increase in slip parameter β_1 and Hartmann number M, in figure 3.9 - 3.10, respectively. In figures 3.11 - 3.30, flow- path of the fluid flow has been discussed. Variation in the size and number of the boluses describes the fluid flow corresponding to the rise in the numerical values of Jeffrey parameter λ_1, aspect ratio β, Hartmann number M, slip parameter β_1, and volumetric flow rate Q, respectively.

Figures 3.11 - 3.22, indicate that boluses are reducing gradually in size, with the increase in Jeffrey parameter λ_1, aspect ratio β and Hartmann number M, respectively. Therefore, it can be concluded that fluid flow gets passive upon the variation of the above mentioned parameters. It is inferred from the graphs, depicted in the figures 3.23 - 3.30, that fluid is making its way through the considered rectangular duct, at ease. As the boluses get expanded corresponding to the increase in numerical values of slip parameter β_1 and volumetric flow rate Q, respectively. Figures 3.31 - 3.34 and Figures 3.35 - 3.38, demonstrate the behavior of the pressure rise corresponding to the variation of the numerical values of different parameters, with respect to volumetric flow rate Q and aspect ratio β, respectively.

The pressure rise Δp, in the above figures 3.31 - 3.32, as sketched with respect to volumetric flow rate Q. It is been noticed that Δp, is increasing corresponding to the rise in β and M. Whereas, for $\beta_1$and λ_1 the trend of pressure rise is quite opposite, which can be witnessed in figures 3.33 - 3.34.

Similarly, the variational trend of Δp with respect to β, has also been displayed in figures 3.35-3.36. Therefore, it is noticed that Δp declines corresponding to the rise in M and Q, respectively. Whereas in figures 3.37 - 3.38, Δp increasingly converges to -0.2 and 0.0, for the greater values of slip parameter β_1 and Jeffrey parameter λ_1, respectively.

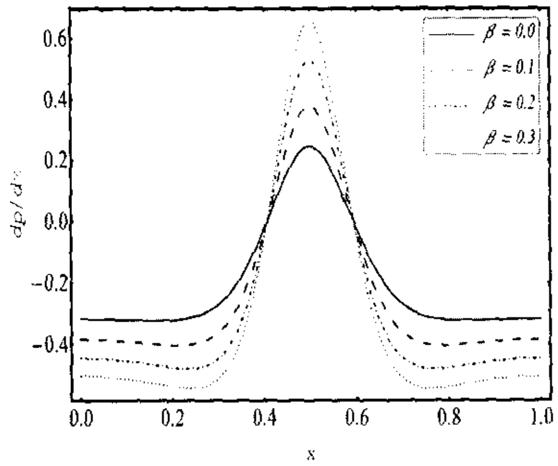

Figure. 3.1 : Variation of pressure gradient with x for different values of β at $Q = 0.5$, $\lambda_1 = 2$, $\phi = 0.6$, $M = 0.5$, $\beta_1 = 0.5$.

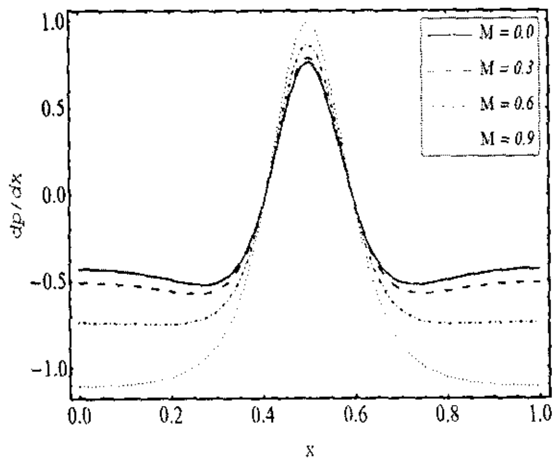

Figure. 3.2 : Variation of pressure gradient with x for different values of M at $Q = 0.5$, $\lambda_1 = 2$, $\phi = 0.6$, $\beta_1 = 0.5$, $\beta = 0.5$.

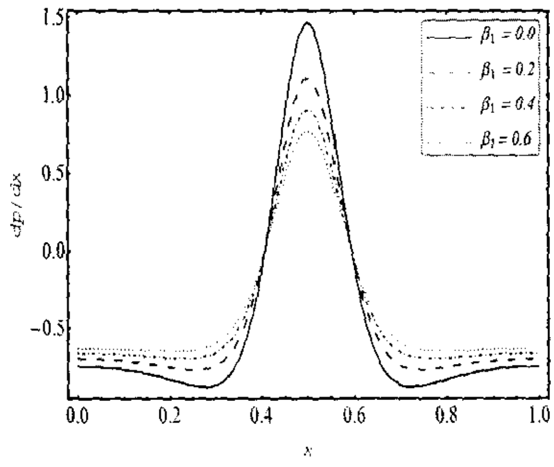

Figure. 3.3 : Variation of pressure gradient with x for different values of β_1 at $Q = 0.5$, $\lambda_1 = 2$, $\phi = 0.6$, $\beta = 0.5$, $M = 0.5$.

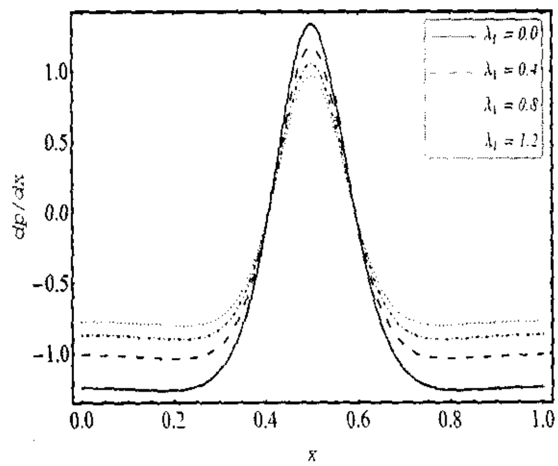

Figure. 3.4 : Variation of pressure gradient with x for different values of λ_1 at $Q = 0.5$, $\beta = 0.5$, $\phi = 0.6$, $M = 0.5$, $\beta_1 = 0.5$.

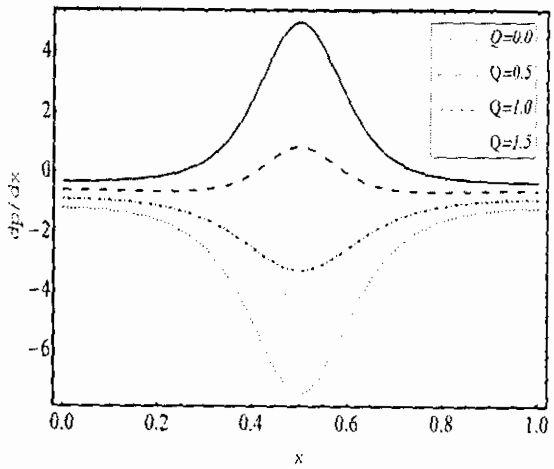

Figure. 3.5 : Variation of pressure gradient with x for different values of Q at $\beta = 0.5$, $\lambda_1 = 2$, $\phi = 0.6$, $M = 0.5$, $\beta_1 = 0.5$.

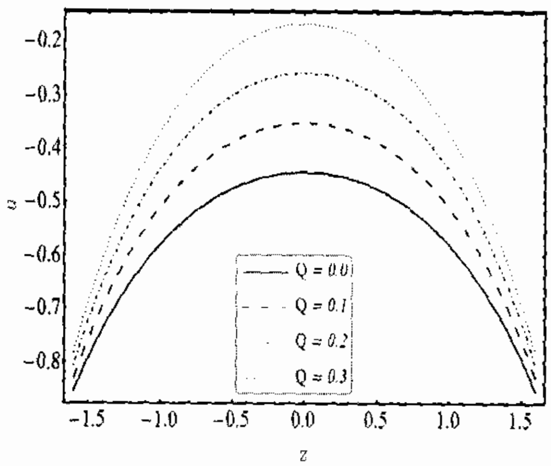

Figure. 3.6 : Velocity profile for different values of Q at $\phi = 0.6$, $x = 0$, $y = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $\beta_1 = 0.5$.

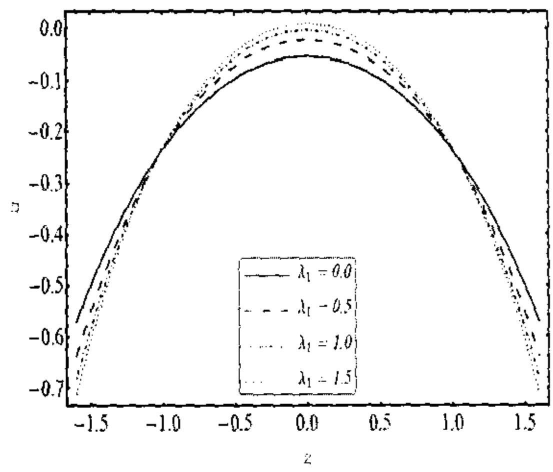

Figure. 3.7 : Velocity profile for different values of λ_1 at $\phi = 0.6$, $x = 0$, $y = 0.5$, $Q = 0.5$, $\beta = 0.5$, $M = 0.5$, $\beta_1 = 0.5$.

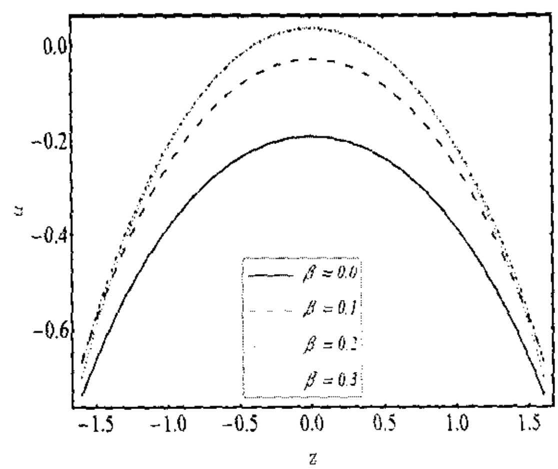

Figure. 3.8 : Velocity profile for different values of β at $\phi = 0.6$, $x = 0$, $y = 0.5$, $\lambda_1 = 2$, $Q = 0.5$, $M = 0.5$, $\beta_1 = 0.5$.

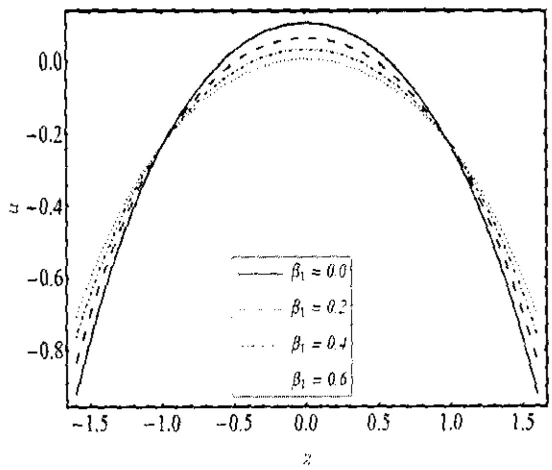

Figure. 3.9 : Velocity profile for different values of β_1 at $\phi = 0.6$, $x = 0$, $y = 0.5$, $Q = 0.5$, $M = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$.

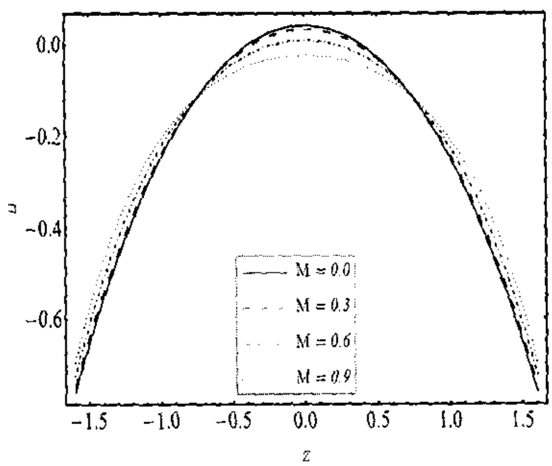

Figure. 3.10 : Velocity profile for different values of M at $\phi = 0.6$, $x = 0$, $y = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$, $Q = 0.5$, $\beta_1 = 0.5$.

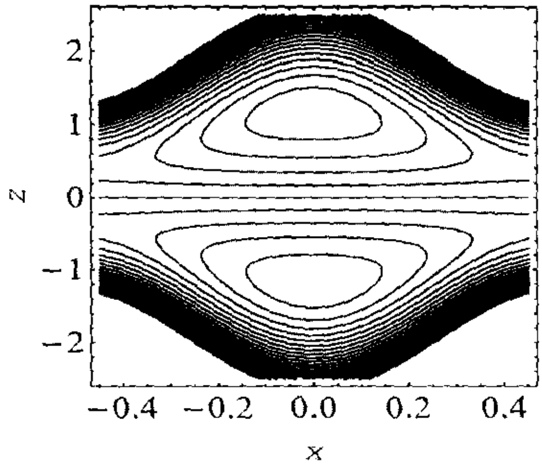

Figure. 3.11 : Stream lines for $\lambda_1 = 0$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5$, $\beta = 0.5$, $M = 0.5$, $Q = 1$.

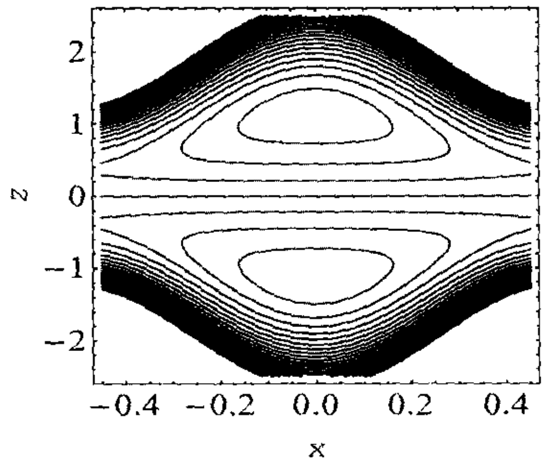

Figure. 3.12 : Stream lines for $\lambda_1 = 1$. The other parameters are $y = 0.5$, $\phi = 0.6, \beta_1 = 0.5$, $\beta = 0.5$, $M = 0.5$, $Q = 1$.

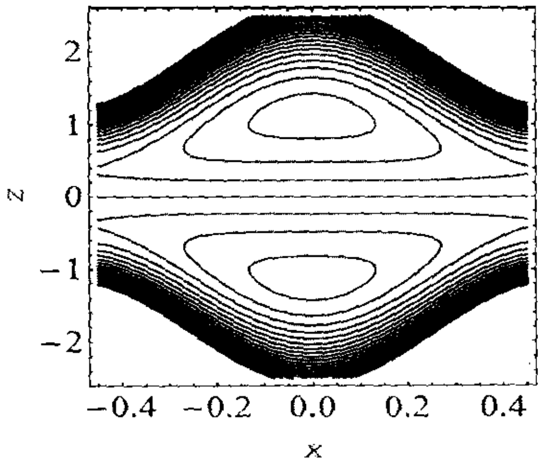

Figure. 3.13 : Stream lines for $\lambda_1 = 2$. The other parameters are $y = 0.5$, $\phi = 0.6, \beta_1 = 0.5$, $\beta = 0.5$, $M = 0.5$, $Q = 1$.

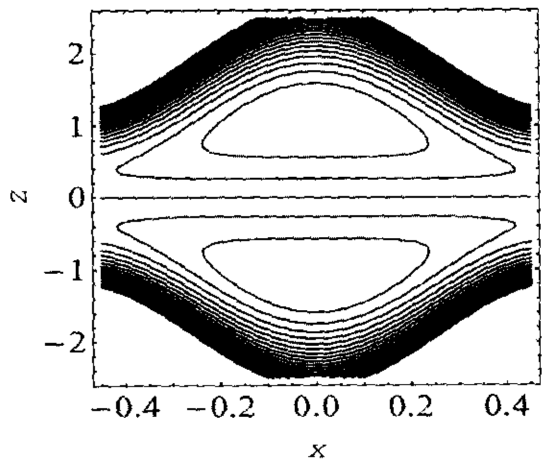

Figure. 3.14 : Stream lines for $\lambda_1 = 3$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5$, $\beta = 0.5$, $M = 0.5$, $Q = 1$.

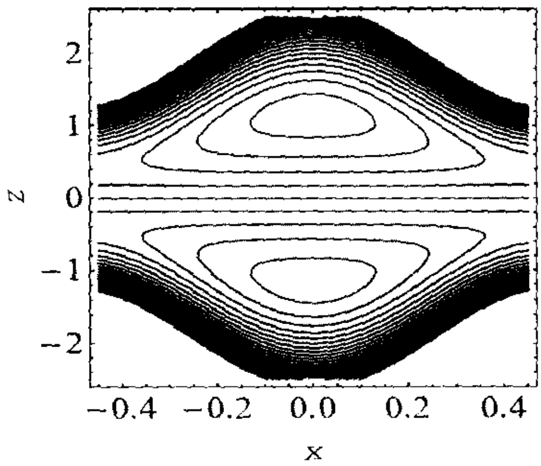

Figure. 3.15 : Stream lines for $\beta = 0.3$. The other parameters are $y = 0.5$, $\phi = 0.6, \beta_1 = 0.5$, $\lambda_1 = 2$, $M = 0.5$, $Q = 1$.

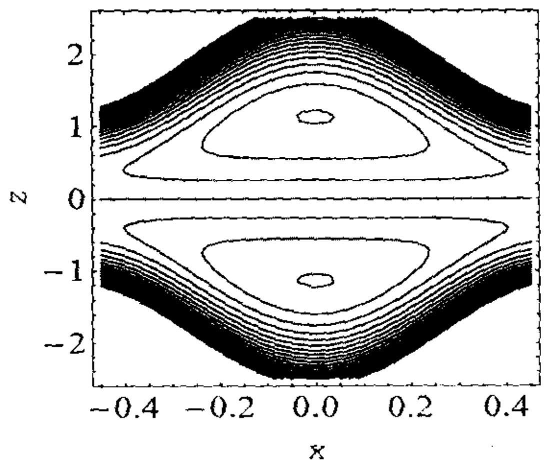

Figure. 3.16 : Stream lines for $\beta = 0.6$. The other parameters are $y = 0.5$, $\phi = 0.6, \beta_1 = 0.5$, $\lambda_1 = 2$, $M = 0.5$, $Q = 1$.

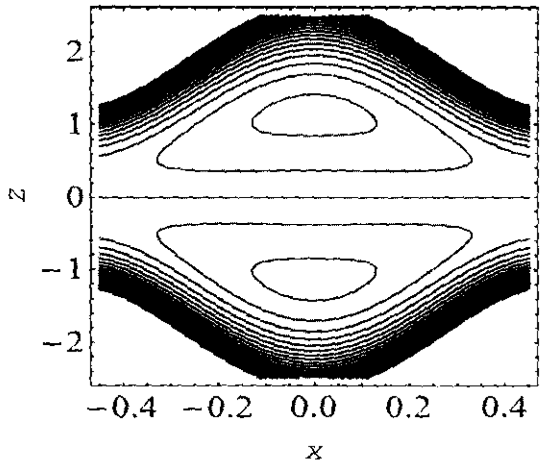

Figure. 3.17 : Stream lines for $\beta = 0.9$. The other parameters are $y = 0.5$, $\phi = 0.6, \beta_1 = 0.5$, $\lambda_1 = 2$, $M = 0.5$, $Q = 1$.

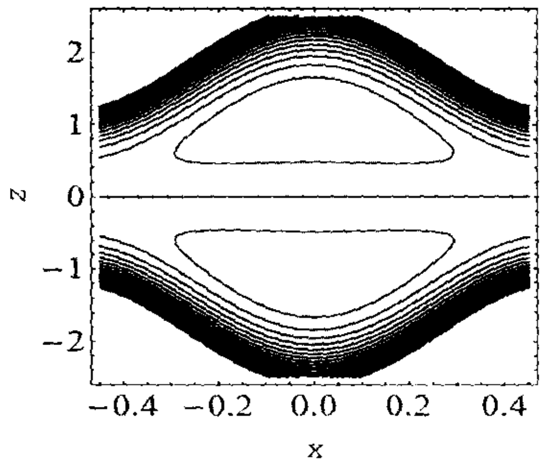

Figure. 3.18 : Stream lines for $\beta = 1.2$. The other parameters are $y = 0.5$, $\phi = 0.6, \beta_1 = 0.5$, $\lambda_1 = 2$, $M = 0.5$, $Q = 1$.

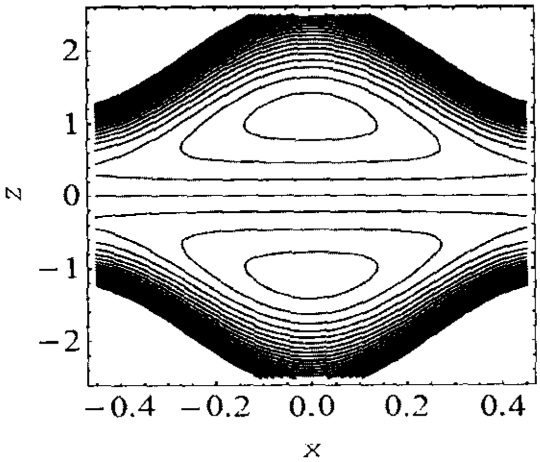

Figure. 3.19 : Stream lines for $M = 0.3$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$, $Q = 1$.

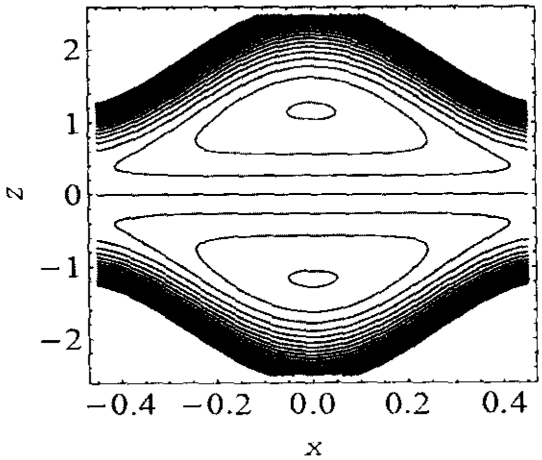

Figure. 3.20 : Stream lines for $M = 0.7$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$, $Q = 1$.

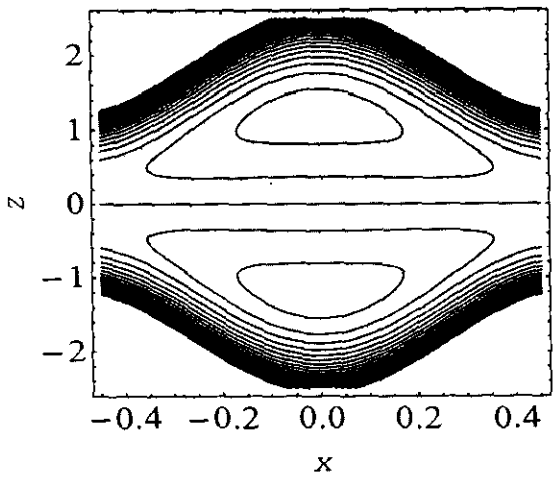

Figure. 3.21 : Stream lines for $M = 1.1$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$, $Q = 1$.

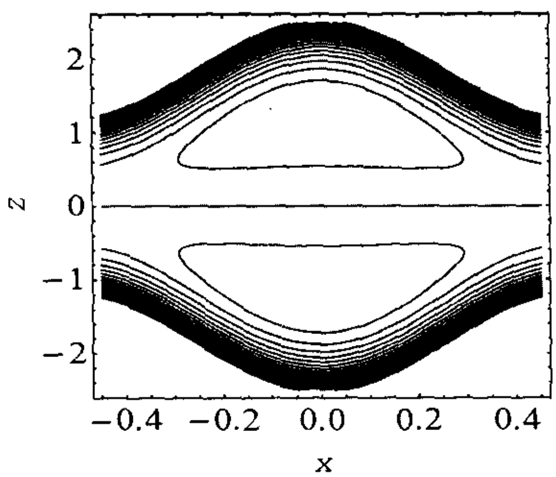

Figure. 3.22 : Stream lines for $M = 1.5$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$, $Q = 1$.

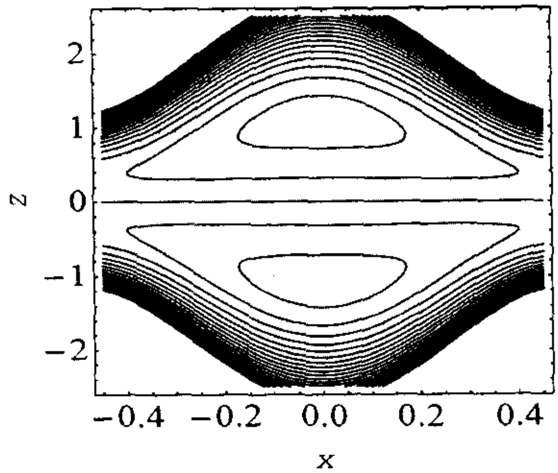

Figure. 3.23 : Stream lines for $\beta_1 = 0$ The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $Q = 1$.

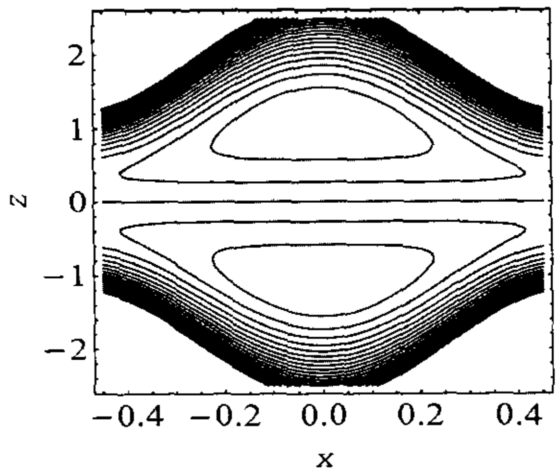

Figure. 3.24 : Stream lines for $\beta_1 = 0.3$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $Q = 1$.

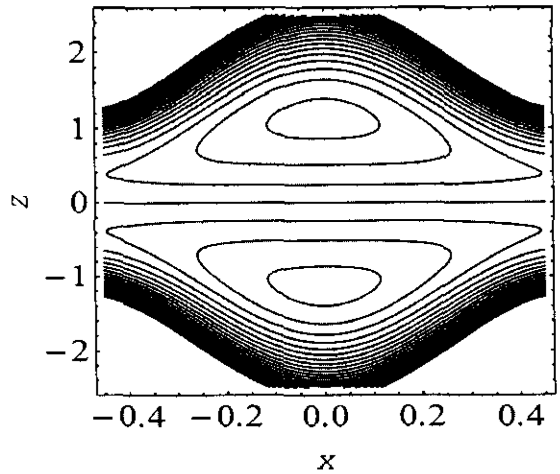

Figure. 3.25 : Stream lines for $\beta_1 = 0.6$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $Q = 1$.

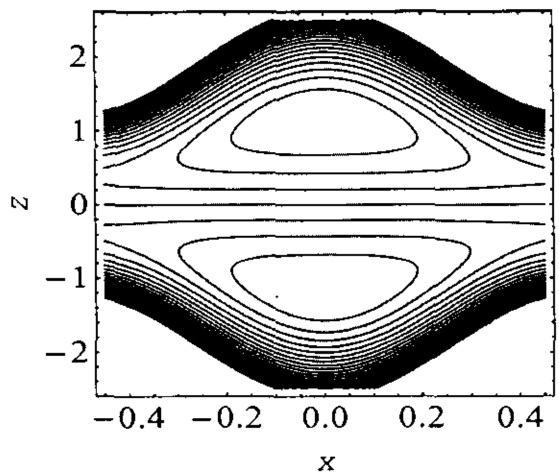

Figure. 3.26 : Stream lines for $\beta_1 = 0.9$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5, Q = 1$.

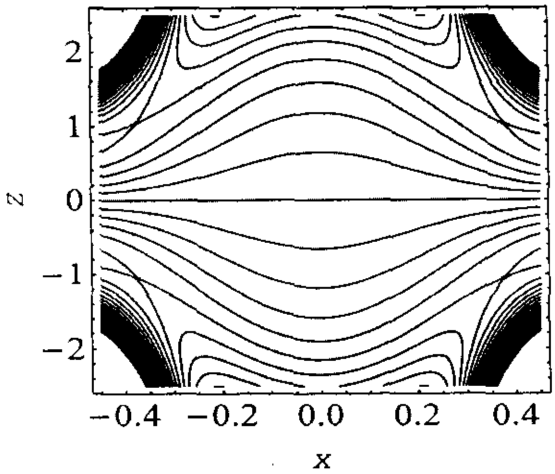

Figure. 3.27 : Stream lines for $Q = 0$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $\beta_1 = 0.5$.

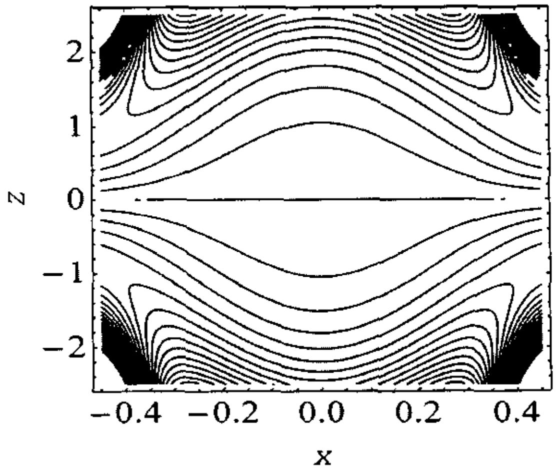

Figure. 3.28 : Stream lines for $Q = 0.3$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $\beta_1 = 0.5$.

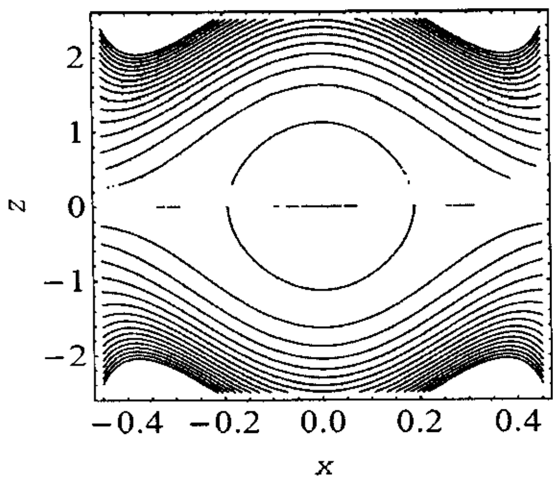

Figure. 3.29 : Stream lines for $Q = 0.6$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $\beta_1 = 0.5$.

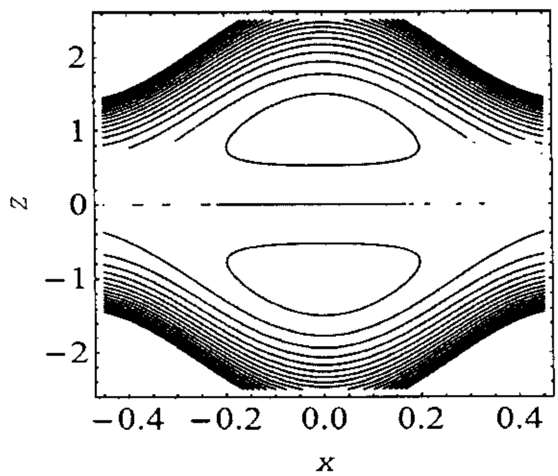

Figure. 3.30 : Stream lines for $Q = 0.9$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\lambda_1 = 2$, $\beta = 0.5$, $M = 0.5$, $\beta_1 = 0.5$.

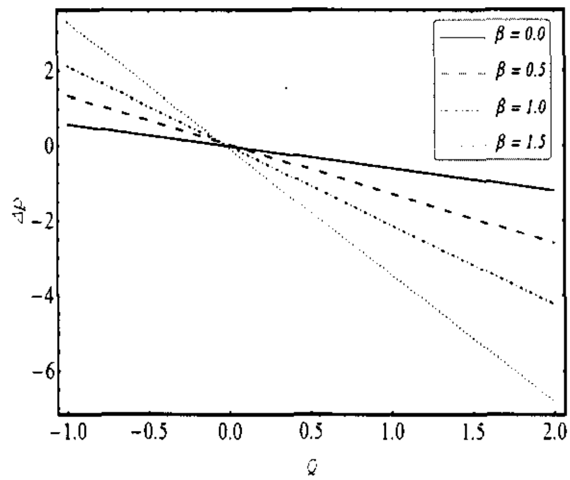

Figure. 3.31 : Variation of Δp with Q for different values of β at $\phi = 0.6$, $\lambda_1 = 2$, $M = 0.5$, $\beta_1 = 0.5$.

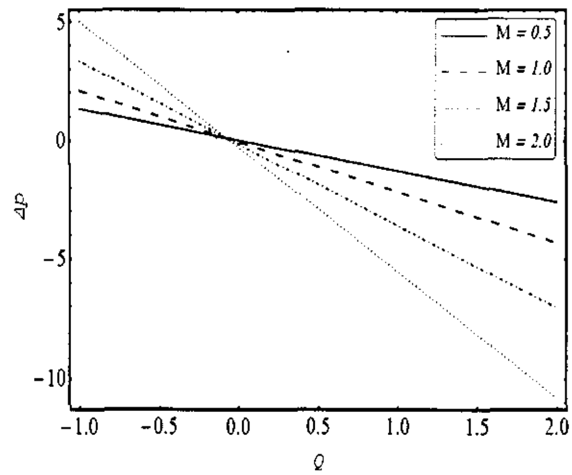

Figure. 3.32 : Variation of Δp with Q for different values of M at $\phi = 0.6$, $\lambda_1 = 2$, $\beta_1 = 0.5$, $\beta = 0.5$.

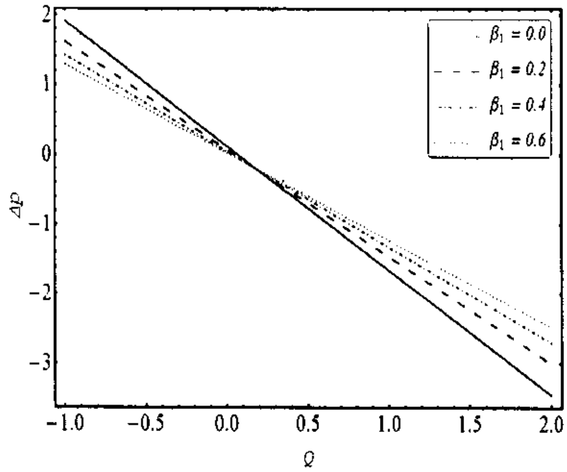

Figure. 3.33 : Variation of Δp with Q for different values of $\beta_1$at $\phi = 0.6$, $\lambda_1 = 2$, $M = 0.5$, $\beta = 0.5$.

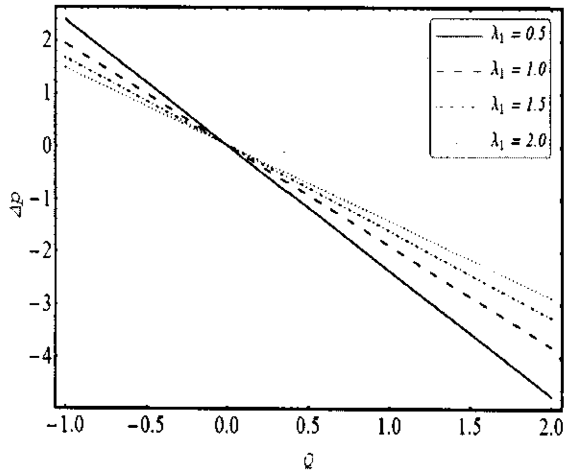

Figure. 3.34 : Variation of Δp with Q for different values of $\lambda_1$at $\phi = 0.6$, $\beta_1 = 0.5$, $M = 0.5$, $\beta = 0.5$.

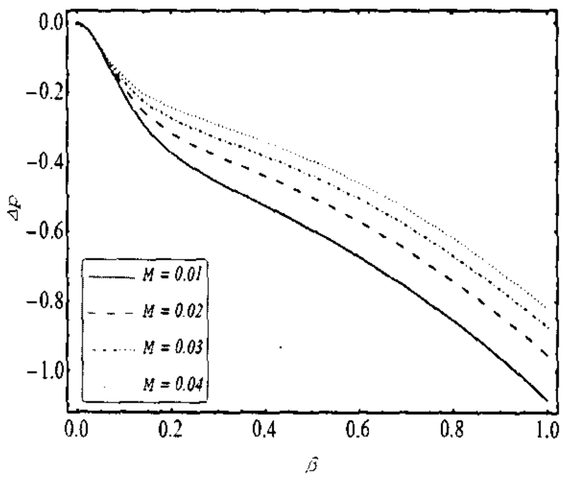

Figure. 3.35 : Variation of Δp with β for different values of M at $\phi = 0.6$, $\lambda_1 = 2$, $\beta_1 = 0.5$, $Q = 0.5$.

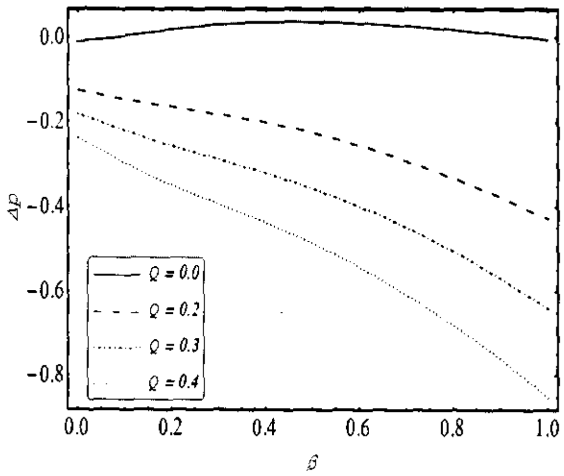

Figure. 3.36 : Variation of Δp with β for different values of Q at $\phi = 0.6$, $\lambda_1 = 2$, $M = 0.5$, $\beta_1 = 0.5$.

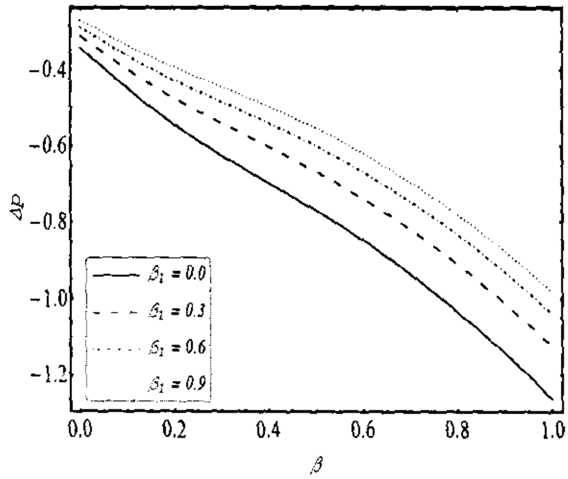

Figure. 3.37 : Variation of Δp with β for different values of β_1 at $\phi = 0.6$, $\lambda_1 = 2$, $M = 0.5$, $Q = 0.5$.

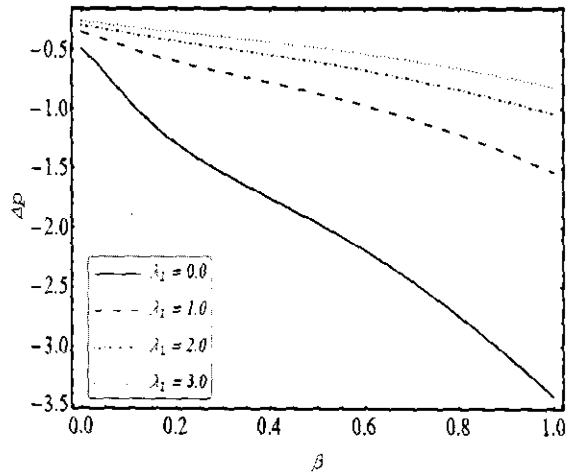

Figure. 3.38 : Variation of Δp with β for different values of λ_1 at $\phi = 0.6$, $\beta_1 = 0.5$, $M = 0.5$, $Q = 0.5$.

3.4 Conclusion

The peristaltic flow of non-Newtonian flow of Jeffrey fluid with MHD and partial slip condition has been studied under the assumptions of long wavelength and low-Reynolds number. The governing partial differential equations corresponding to their boundary condition has been solved by using the method of separation of variables. It is noted that an increase in the slip parameter decreases the pressure gradient whereas the size of trapped bolus increases by increasing the slip parameter. It is observed that velocity profile decreases by increasing the Hartmann number while quite opposite behavior is noted for the case of pressure gradient. It is noticed that the pressure gradient rises with the rise in aspect ratio and Hartmann number. Moreover, the trapped bolus increases by increasing the slip parameter. It is worth to mention that one can recover the results of [32] by taking . Also, Newtonian fluid model can be deduced from the presented model as a special case by taking The present attempt will be beneficial in many clinical applications. This analysis gives a better judgement for the speed of injection and the fluid flow characteristics within the syringe. Also, the injection can be carried out more proficiently and pain of the patient can be extenuated.

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