Some Fixed Point Results in Dualistaic Partail Metric Spaces

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1. Metric spaces
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بسم الله الرحمن الرحيم

In the name of almighty ***ALLAH***,
the most beneficent, the most merciful

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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2016

Dedicated to…

My Parents, Teachers and Friends

DECLARTION

I hereby declare that this thesis, neither as a whole nor as a part there of, has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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ZAIN UL ABIDIN

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Preface

Fixed point theorems deals with the assurance that a mapping T on a set X has one or more fixed points, i.e. the functional equation $x = Tx$ has one or more solutions. A large variety of the problems of analysis and applied mathematics relate to finding solutions of nonlinear functional equations which can be formulated in terms of finding the fixed solutions of nonlinear mappings. In fact, fixed point theorems are extremely substantial tools for proving the existence and uniqueness of the solutions to various mathematical models (differential, integral and partial differential equations and variational inequalities etc.) exhibiting phenomena arising in broad spectrum of fields, such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, financial analysis, epidemics, biomedical research and flow of fluids etc. Thus the fixed point theory started as purely analytical theory. This field of mathematics can be divided into three major areas: Metric fixed point theory, Topological fixed point theory and Discrete fixed point theory. Classical and major results in these areas: Brouwer's fixed point theorem, Banach's fixed point theorem and Tarski's fixed point theorem.

The Banach fixed point theorem is commonly known as Banach contraction principle, which states that if X is a complete metric space and T is a single-valued contraction self mapping on X, then T has a unique fixed point in X. This theorems certainly plays an important and fundamental role in the field of fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. Subsequently many authors generalized the Banach fixed point theorems in different way.

Ran and Reuring [23] proved an analogue of Banach's fixed point theorem in metric space endowed with an order and gave applications to matrix equations. In this way, they weakened the usual contractive condition. Subsequently, Nieto [20] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equations with periodic boundary conditions. Samet and Vetro [10] generalized these results in ordered metric spaces and introduced the concept of

$\alpha - \psi$ contractive type mappings and established fixed point theorems for such mappings in complete metric spaces.

Matthews [17] introduced the concept of partial metric space (in which self distance may not be zero) as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. O'Neill [19] introduced the concept of dualistic partial metric, which is more general than partial metric (which allows negative values also) and established a robust relationship between dualistic partial metric and quasi metric. Oltra and Valero [21] presented a Banach fixed point theorem on complete dualistic partial metric spaces, Valero [27] generalized the main theorem of [21] using nonlinear contractive condition instead of Banach contractive condition.

In our thesis, we establish an order relation on quasi dualistic partial metric spaces. Then using this order relation, we prove fixed point theorems for singlevalued mappings. Same results are then proved for multivalued mappings in dualistic partial metric space. Instead of monotone mappings, the notion of dominating mappings is used. Our work improves/generalizes various well known primary and conventional results. In support of our results we will present some examples which proves that our results are applicable. Moreover we have discussed the application of our fixed point results to show the existence and uniquenes of solution appearing in dynamic programing. Some important corollaries are developed as a generalization of our theorems.

Let us recall some mathematical basic and results to make our thesis self sufficient. Throughout these thesis the letters $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent set of positive real numbers, set of real numbers and set of natural numbers respectively.

Chapter 1, is essentially an introduction, where we fix notations and terminologies to be used. It is a survey aimed at recalling some basic definitions and facts. Moreover, some of the recent results about fixed point existence are also presented in this chapter.

Chapter 2, concerned with the study of fixed points resuls of dominating mappings in dualistic partial metric spaces.

Chapter 3, is devoted to the study of dualistic partial metric spaces and some corresponding results for a mapping satisfying generalized contraction. An application of our fixed point results to show the existence of solution has been discussed.

Chapter 1

Preliminaries

The aim of this chapter is to present basic concepts and to explain the terminologies used through out this dissertation. Some previously known results are given without proof. Section 1.1 is concerned with the introduction of dualistic partial metric space and some other basic definitions. Section 1.2, deals with some classical fixed point results including partial spaces and dualistic partial metric spaces.

1.1 Basic Concepts

Definition 1.1.1 [4,16] Let (X, d) be a metric space. A point $x \in X$ is said to be a fixed point of mapping $T : X \to X$ if $x = Tx$.

Definition 1.1.2 [5] Define a sequence $\{x_n\}$ in X by a simple iterative method such that $x_n = Tx_{n-1}$, where $n \in \{0, 1, 2...\}$. This particular sequence is known as Picard iterative sequence and its convergence plays a very important role in proving an existence of a fixed point of a mapping T.

Definition 1.1.3 [16] A pair (X, d) where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ is said to be metric space such that for all $x, y, z \in X$ if it satisfyies the following properties:

(M1) $d(x, y) \geq 0$ where d is real-valued, finite and nonnegative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$ (Symmetry).

(M4) $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality).

A metric space is a pair (X, d) such that X is a nonempty set and d is a metric on X.

Example1.1.4 [16] This is the set of all real numbers, taken with the usual metric defined by

$$d(x, y) = |x - y| \text{ for all } x, y \in X .$$

Definition 1.1.5 [4,13,14,18,27] A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,

(ii) $p(x, x) \leq p(x, y)$,

(iii) $p(x, y) = p(y, x)$,

(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Example 1.1.6 [13] A partial metric space is the pair (R^+, p) where

$$p(x, y) = \max\{x, y\} \text{ for all } x, y \in R^+.$$

Then(R^+, p) is a partial metric space.

Definition 1.1.7 Let X be a non empty set. Then (X, d) is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.

Definition 1.1.8 [5,26] Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 1.1.9 [1,4,5,26] Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is said to be

(i) a dominated map if $fx \preceq x$ for each x in X,

(ii) a dominating map if $x \preceq fx$ for each x in X.

Definition 1.1.10 Let $T : N \to N$ be defined by $T(x) = x^k$ where k is positive integer, $T(x) = 2^x$ and $T(x) = x! \; \forall \; x \in N$ are dominating mappings with respect to usual order defined on N. Dominating mappings frequently occurs in agriculture and industry.

Example 1.1.11 [5] Let $X = [0, 1]$ be endowed with the usual ordering and $F : X \to X$ be defined by $Fx = x^n$ for some $n \in N$. Since $Fx = x^n \leq x$ for all $x \in X$. Therefore F is dominated mapping.

Definition 1.1.12 [1,4] Let f and g be self-mappings on a set X. If $fx = gx = w$ for some x in X, then w is called a coincidence point of f and g. Further more, if $fgx = gfx$ whenever x is a coincidence point of f and g, then f and g are called weakly compatible mappings.

Definition 1.1.13 [19] Let X be a non emty set and a function $D : X \times X \to R$ is said to be dualistic partial metric space if it satisfying following properties for all $x, y, z \in X$:

(D1) $D(x,x) = D(y,y) = D(x,y) \Leftrightarrow x = y$

(D2) $D(x,x) \leq D(x,y)$

(D3) $D(x,y) = D(y,x)$

(D4) $D(x,y) \leq D(x,z) + D(z,y) - D(z,z)$.

A dualistic partial metric space is a pair (X, D) such that X is a nonempty set and D is a dualistic partial metric on X.

Definition 1.1.14 [18,21] Let X be a non-empty set and the function $q : X \times X \to R^+$ is said to be quasi metric if it satisfies following properties $\forall x, y, z \in X$

(i) $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$

(ii) $q(x,z) \leq q(x,y) + q(y,z)$.

A pair (X, d_q) is called quasi metric space.

Example 1.1.15 Let $D : X \times X \to \mathbb{R}$ by $D(x,y) = x \vee y = \sup\{x,y\}$. Now if $X = R$ then D is dualistic partial metric space but not partial on X for $x = -4$ and $y = -8$ then

$x \vee y = -4 = D(x, y)$.

If (X, D) is a dualistic partial metric space, then the function $d_D : X \times X \longrightarrow R^+$ defined by

$$d_D(x, y) = D(x, y) - D(x, x)$$

is a quasi metric on X such that $\tau(D) = \tau(d_D) \ \forall \ x, y \in X$.

Remark 1.1.16 It is obvious that every partial metric is dualistic partial metric but converse is not true. To support this comment, define $D_\vee : R \times R \longrightarrow R$ by

$$D_\vee(x, y) = x \vee y = \sup\{x, y\} \ \ \forall \ x, y \in \mathbb{R}.$$

It is clear that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, because $D_\vee(-1, -2) = -1 \notin R^+$. However, the restriction of $D_\vee$ to R^+, $D_\vee|_{\mathbb{R}^+}$, is a partial metric.

Example 1.1.17 If (X, d) is a metric space and $c \in R$ is arbitrary constant, then

$$D(x, y) = d(x, y) + c.$$

defines a dualistic partial metric on X and the corresponding metric is $d^s_D(x, y) = d(x, y)$.

Example 1.1.18 Let $X = R$ and define the function $D : X \times X \to R$ by

$$D(x, y) = x + y - xy \;\; \forall \; x \leq y \wedge 1.$$

Then (X, D) is dualistic partial metric space.

Definition 1.1.19 [19] Let (X, D) be a dualistic partial metric space, then

(1) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, D) converges to a point $x \in X$ if and only if

$$D(x, x) = \lim_{n \to \infty} D(x, x_n).$$

(2) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, D) is called a Cauchy sequence if $\lim_{n,m \to \infty} D(x_n, x_m)$ exists and is finite.

(3) A dualistic partial metric space (X, D) is said to be complete if every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges, with respect to $\tau(D)$, to a point $x \in X$ such that

$$D(x, x) = \lim_{n,m \to \infty} D(x_n, x_m).$$

Definition 1.1.20 [26] Let (X, q) be a quasi partial metric space. A sequence $\{x_n\}$ in X is said to be 0-Cauchy sequence if $\lim_{n \to \infty} q(x_n, x_m) = 0$ and (X, q) is said to be 0-complete if every 0-Cauchy sequence converges in to a point $x \in X$ such that $q(x, x) = 0$.

Definition 1.1.21 [11] Let A and B be two nonempty subsets of an ordered set X, the relations between A and B is defined as follows:

If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.

Example 1.1.22 Let $X = R$, $A = [\frac{1}{3}, 1]$, $B = [-1, 1]$, $\preceq$ be usual order on X, then $A \prec_1 B$. The relation $\prec_1$, is reflexive and transitive, but are not antisymmetric. For instance, let $X = R$, $A = [-1, 3]$, $B = [-1, 1] \cup [2, 3]$, $\preceq$ be usual order on X, then $A \prec_1 B$ and $B \prec_1 A$, but $A \neq B$. Hence, $\prec_1$ is not partial orders on 2^X.

Remark 1.1.23 [11] The relation $\prec_1$, is reflexive and transitive, but are not antisymmetric. For instance, let $X = R$, $A = [-1, 3]$, $B = [-1, 1] \cup [2, 3]$, $\preceq$ be usual order on X, then $A \prec_1 B$ and $B \prec_1 A$, but $A \neq B$. Hence, $\prec_1$ is not partial orders on 2^X.

Definition 1.1.24 Let X be a nonempty set. Then, $(X, \preceq, D)$ is called an ordered dualistic partial metric space if:

(a) (X, D) is a dualistic partial metric space,

(b) $(X, \preceq)$ is a partially ordered set,

(c) $D(x, x) \leq D(y, y)$ whenever $x \preceq y$.

Definition 1.1.25 [11] A multivalued mapping $T : X \to 2^X$ is called order closed if for monotone sequences $\{u_n\}$, $\{v_n\}$ in X, $u_n \to u_0$, $v_n \to v_0$ and $v_n \in T(u_n)$ imply $v_0 \in T(u_0)$.

Definition 1.1.26 Let $(X, \preceq, D)$ be an ordered dualistic partial metric space. A multivalued mapping $T : X \to 2^X$ is called D-order closed if for monotone sequences, $\{u_n\}$,

$\{v_n\} \subseteq X$, $\lim_{n\to\infty} D(u_n, u_0) = D(u_0, u_0)$, $\lim_{n\to\infty} D(v_n, v_0) = D(v_0, v_0)$ and $v_n \in T(u_n)$ imply $v_0 \in T(u_0)$.

Remark 1.1.27 [14] Let (X, p) be a complete partial metric space. Therefore

(a) If $p(x, y) = 0$, then $x = y$

(b) If $x \neq y$, then $p(x, y) > 0$.

Definition 1.1.28 [5] Let (X, p) be a partial metric space. A mapping $T : X \longrightarrow X$ is said to be Banach contraction mapping or simply contraction if $k \in [0, 1[$ such that

$$p(T(x), T(y)) \leq kp(x, y), \qquad \forall\, x, y \in X$$

and Kannan contraction mapping

$$p(Tx, Ty) \leq k\ [p\ (x, Tx) + p(y, Ty)]$$

Definition 1.1.29 [2] A mapping $F : X \to X$ is said to be a weakly contractive if

$$D(Fx, Fy) \leq D(x, y) - \phi(d(x, y)),$$

for all $x, y \in X$.

Definition 1.1.30 [18] Let be T a mapping of a complete dualistic partial metric X into itself , then $T : X \to X$ is called a partial metric contraction mapping if there exists a constant L, $0 \leq L < 1$, such that

$$p(Tx, Ty) \leq Lp(x, y)$$

for all $x, y \in X$.

Definition 1.1.31 [16] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be-Cauchy (or fundamental) if for every $\in > 0$ there is an $N = N(\in)$ such that for every $m, n > N$.

The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Definition 1.1.32 [16] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$, x is called the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or, simply $x_n \to x$. We say that $\{x_n\}$ converges to x or has the limit x. If $\{x_n\}$ is not convergent, it is said to be divergent.

Definition 1.1.33 **[12,20,27]** A function $f : \mathbb{R} \to \mathbb{R}$ is monotone non decreasing (or monotone or increasing) if $x \geq y$ implies $f(x) \geq f(y)$.

Definition 1.1.34 [11] If $\{x_n\} \subset X$ satisfies $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ or $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$, then $\{x_n\}$ is called a monotone sequence.

1.2 Classical Fixed Point Results in Partial Metric Spaces

Fixed point theorems are very important tools for providing evidence of existence and uniqueness of solutions to various mathematical models. For last four decades, the literature flourished with results which discover fixed points of nonself and self nonlinear mappings in a metric space. The Banach contraction theorem played a fundamental role in the development of fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer.

Matthews [17] introduced the concept of partial metric space (in which self distance may not be zero) as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces.This section presents some previously well known results from literature in partial metric spaces.

Theorem 1.2.1 [13,Theorem 2.1] Let $(X, \leq)$ be a partially ordered set and p be partial metric on X such that (X, p) is a complete partial metric space. Let $F : X \to X$ be a weackly contractive and F is continouse non decreasing mapping such that

$$p(Fx,Fy) \leq \frac{\alpha p(x,Fx)p(y,Fy)}{p(x,y)} + \beta p(x,y)$$

$x \geq y,\ x, y \in X$ for all ,$x \neq y$ with $\alpha + \beta \prec 1$ if there exists $x_0 \in x$ with $x_0 \leq Fx_0$. Then F has a fixed point.

Theorem 1.2.2 [14, Theorem 2.2] Suppose that (X, p) is a complete partial metric space and T, S are self mappings on X. If there exists an $\gamma \in [0, 1)$ such that $p(Tx, Ty) \leq \gamma M(x, y)$ for any $x, y \in X$, where

$$M(x,y) = \max\{p(Tx,x), p(Sy,y), p(x,y) , \frac{1}{2}[p(Tx,y) + p(Sy,x)]\},$$

then there exists $z \in X$ such that $Tz = Sz = z$.

Theorem 1.2.4 [5, Theorem 2] Let $(X, \preceq, p)$ be a complete ordered partial metric space and $S, T : X \to X$ be two dominated mappings. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following condition holds for x,$y \in X$,

$$p(Sx; Ty) \leq (x, Sx) + p(y, Ty)$$

for all x, y in ∇. Then there exists a point x^* such that $p(x^*, x^*) = 0$. Also if, for a nonincreasing sequence $\{x_n\}$ in X $x_n \to u$ implies that $u \preceq x_n$, then $x^* = Sx^* = Tx^*$. Moreover, x^* is unique, if for any two points x, y in X there exists a point $z_0 \in X$ such that

$z_0 \preceq x$ and $z_0 \preceq y$.

Theorem 1.2.5 [5, Theorem 1] Let $(X, \preceq, p)$ be a complete ordered partial metric space and $S, T : X \to X$ be two dominated mappings. Suppose that there exists $t \in [0,$

$\frac{1}{2}$) such that following condition holds for $x,\ y \in X$,

$$p(Sx, Ty) \leq t(x, Sx) + p(y, Ty)$$

for all (x, y) in $B(x_0,\ r) \times B(x_0,\ r) \cap \nabla$ and

$$p(x_0\ , Sx_0) \leq (1 - \lambda)[r + p(x_0\ , Sx_0)]$$

where $\lambda = \frac{t}{1-t}$. Then there exists a point x^* such that $p(x^*, x^*) = 0$. Also if, for a nonincreasing sequence $\{x_n\}$ in $B(x_0,\ r)$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then $x^* = Sx^* = Tx^*$. Moreover, x^* is unique, if for any two points x, y in $B(x_0,\ r)$ such that $z_0 \preceq x,\ z_0 \preceq y$ and

$$p(x_0, Sx_0) + p(z, Tz) \leq (x_0, z) + p(Sx_0, Tz)$$

where $\lambda = \frac{t}{1-t}$ for all $z \in B(x_0,\ r)$ such that $z_0 \preceq Sx_0$.

Theorem 1.2.6 [26, Theorem 10] Let (X, q) be a 0-complete partial metric space, $S : X \to X$ a map, and x_0 an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with $q(Sx, Sy) \leq kq(x, y)$ for all elements $x,\ y$ in $\overline{B(x_0,\ r)}$

$$q(x_0, Sx_0) \leq (1 - k)[r + p(x_0\ , x_0)].$$

where $\lambda = \frac{t}{1-t}$. Then there exists a unique point x^* in $\overline{B(x_0,\ r)}$ such that $x^* = Sx^*$.

Further $q(x^*, x^*) = 0$.

Theorem 1.2.7 [**26,Theorem 13**] Let $(X, \preceq, d)$ be a 0-complete ordered quasi partial metric space, $S : X \to X$ a dominated map, and x_0 an arbitrary point in X be a 0-complete ordered quasi partial metric space, $S : X \to X$ a dominated map, and x_0 an arbitrary point in X. Suppose that there exists $b \in [0, 1/2)$ such that

$$q(Sx, Sy) \leq b[q(x, Sx) + q(y, Sy)]$$

for all comparable elements x, y in $\overline{B(x_0, r)}$. And

$$q(x_0, Sx_0) \leq (1 - k)[r + q(x_0, x_0)],$$

where $k = \frac{b}{(1-b)}$. If for a nonincreasing sequence $\{x_n\}$ in $\overline{Bq(x_0, Sx_0)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$ and $q(x^*, x^*)$ Moreover, x^* is unique, if for any two points x, y in $\overline{B(x_0, r)}$. There exists a point $z \in \overline{B(x_0, r)}$ such that $z \preceq x$ and $z \preceq y$ and

$$q(x_0, Sx_0) + q(z, Sz) \leq q(x_0, z) + q(Sx_0, Sz)$$

for all $z \preceq Sx_0$.

1.3 Some Fixed Point Results in Dualistic Partial Metric Spaces

O'Neill [19] introduced the concept of dualistic partial metric, which is more general than partial metric (which allows negative values also), established a robust relationship between dualistic partial metric and quasi metric. Oltra and Valero [21] presented a Banach fixed point theorem on complete dualistic partial metric spaces.In this section presents some previously well known results from literature in dualistic partial metric spaces.

Lemma 1.3.1 [27, Lemma 2.1]

(1) Let (X, p) is a dualistic partial metric (X, p) is complete if and only if the metric space (X, d_p^s) is complete.

(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$, with respect to $\tau(d_p^s)$ if and only if

$$\lim_{n\to\infty} p(x, x_n) = p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

(3) If $\lim_{n\to\infty} x_n = v$ such that $p(v, v) = 0$ then

$$\lim_{n\to\infty} p(x_n, y) = p(v, y)$$

for every $y \in X$.

Theorem 1.3.2 [21,Theorem 2.3] Let f be a mapping of a complete dualistic partial metric space (X, p) into itself such that there exists $0 \leq c < 1$, satisfying

$$|p(f(x), f(y))| \leq c|p(x, y)|$$

for all $x, y \in X$. Then f has a unique fixed point $x^* \in X$.

Lemma 1.3.3 [21, Lemma 2.1] Let (X, p) is dualistic partial metric space, then the function $d_p : X \times X \to R^+$ defined by

$$d_p(x, y) = p(x, y) - p(x, x)$$

is a quasi metric on X such that $\top(p) = \top(d_p)$ for all $x, y \in X$.

Proposition 1.3.4 [27, Proposition 2.7] Let (X, p) be a complete dualistic partial metric space, $x_0 \in X$ and $r > 0$. Suppose that $f : B_p(x_0, r) \to X$ is a contraction with contracion constant c such that

$$|p(f(x_0), x_0)| < (1 - c)r - 2|p(x_0, x_0)| - |p(f(x_0), f(x_0))|.$$

for all $x, y \in B_p(x_0, r)$ then f has a unique fixed point in $B_p(x_0, r)$.

Theorem 1.3.5 [27, Theorem 3.2] Let f be a mapping of a complete dualistic

(X, p) into itself such that

$$|p(f(x), f(y))| \leq \Phi(|p(x, y)|)$$

for all $x, y \in X$, where $\Phi : [0, +\infty) \to [0, +\infty)$ is any monotone non decreasing function with $\lim_{n \to \infty} \Phi^n(t) = 0$ for any fixed $t > 0$.

Then f has a unique fixed point.

Lemma 1.3.6 [27, Lemma 3.4] Let (X, p) be a dualistic partial metric space and $Y \subseteq X$. Then

$$\delta\,(d_p)^s\,(Y) \leq 4\delta_p(Y).$$

Lemma 1.3.7 [27, Lemma 3.5] Let (X, p) be a dualistic partial metric space and let $\varphi : X \to R^+$ be an arbitrary non negative function. Assume that $\inf\{\varphi(x) + \varphi(y) : |p(x, y)| + |p(x, x)| + |p(y, y)| \geq a\} = u\,(a) > 0$ for all $a > 0$.

Then each sequence $(x_n)_{n \in N}$ for which $\lim_{n \to \infty} \varphi\,(x_n) = 0$ converges with respect to $\top\,(d_p)^s$ to the same point of X.

Theorem 1.3.8 [27,Theorem 3.6] Let (X, p) be a dualistic partial metric space (X, p) and let $f : X \to X$ be a continous mapping from $(X, (d_p)^s)$ to $(X, (d_p)^s)$ such that the function

$$\varphi\,(x) = d_p\,(x, f\,(x)) \text{ and } \psi\,(x) = d_p\,(f\,(x)\,, x)$$

satisfying the following conditions:

(1) $\inf\{\varphi(x) + +\varphi(y) + \psi(x) + \psi(y) : |p(x,y)| + |p(x,x)| + |p(y,y)| \geq \in\} = \mu(a) > 0$ for all $a > 0$.

(2) $\inf_{x\in X} (\varphi(x) + \psi(x)) = 0$.

Then f has a unique fixed point.

Lemma 1.3.9 [21, Lemma 2.2] If (X,p) is a dualistic partial metric space (X,p) is complete if and only if the metric space $(X, (d_p)^s)$ is complete. Furthermore $\lim_{n\to\infty} (d_p)^s (a, x_n) = 0$ if and only if

$$p(a,a) = \lim_{n\to\infty} p(a,x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Theorem 1.3.10 [18,Theorem 12] If X is a complete dualistic partial metric space, and $T : X \to X$, is such that T^γ is contraction for some integer $r > 0$, then T^γ has a unique fixed point.

Corollary 1.3.11 [18, Corollary 15] Let (X,p) be a dualistic partial metric space and let $T : X \to X$ be a contraction on closed ball

$$B = B(x_0, r) = \{x : p(x, x_0) \leq r\}.$$

Moreover, assume that

$$p(x_0, Tx_0) < (1 - L)r.$$

Then, prior error estimate is the following estimate.

$$p(x_m, u) \leq L^n r$$

and posterior estimate $p(x_m, u) \leq Lr$.

Chapter 2

Fixed Point Results in Dualistic Partial Metric Spaces:

2.1 Fixed Point Results of Dominating mapping in Dualistic Partial Metric Spaces:

In the development of non-linear analysis, fixed point theory plays a very important role. Also, it has been widely used in different branches of engineering and sciences. Metric fixed point theory is an essential part of mathematical analysis because of its applications in different areas like variational and linear inequalities, improvement, and approximation theory. The fixed point theorem in metric spaces plays a significant role to construct methods to solve the problems in mathematics and sciences.

Though metric fixed point theory is vast field of study and is capable of solving many equations. To overcome the problem of measurable functions with respect to a measure and their convergence, O'Neill [19] introduced the concept of dualistic partial metric spaces by extending the range R^+ to R. Oltra and Valero [27] established Banach fixed point theorem for complete dualistic partial metric spaces. In this section we present a fixed point theorem for dominating mappings in an ordered dualistic partial metric spaces.

We begin with the following lemma.

Lemma 2.1.1 Let (X, D) be a dualistic partial metric space and $\varphi : X \to R$ be a mapping. Define the relation $\preceq$ on X as follows:

$$x \preceq y \quad \Leftrightarrow D(x,y) - D(x,x) \leq \varphi(x) - \varphi(y).$$

Then $\preceq$ is an order on X, named the order induced by φ.

Proof As $0 \leq 0$ this implies

$$D(x,x) - D(x,x) \leq \varphi(x) - \varphi(x) \;\Rightarrow\; x \preceq x.$$

So $\preceq$ is reflexive.

Now if $x \preceq y$ and $y \preceq x$, we will prove that $x = y$ for this

$$\text{Since } x \preceq y \quad \Leftrightarrow D(x,y) - D(x,x) \leq \varphi(x) - \varphi(y). \tag{2.1.1}$$

$$\text{And } y \preceq x \quad \Leftrightarrow D(y,x) - D(y,y) \leq \varphi(y) - \varphi(x). \tag{2.1.2}$$

Adding equations (2.1.1) and (2.1.2) we get

$$D(x,y) - D(x,x) + D(y,x) - D(y,y) \leq 0.$$

This gives

$$d_D(x,y) + d_D(y,x) \leq 0.$$

since $d_D(x,y)$ and $d_D(y,x)$ are non-negative, therefore

$$d_D(x,y) = d_D(y,x) = 0 \text{ entails } x = y.$$

Thus $\preceq$ is anti-symmetric

Lastly, if $x \preceq y$ and $y \preceq z$, we show that $x \preceq z$. For this

$$\text{since } x \preceq y \Leftrightarrow D(x,y) - D(x,x) \leq \varphi(x) - \varphi(y). \tag{2.1.3}$$

$$\text{and } y \preceq z \Leftrightarrow D(y,z) - D(y,y) \leq \varphi(y) - \varphi(z). \tag{2.1.4}$$

Adding equations (2.1.3) and (2.1.4) we obtain

$$D(x,y) - D(x,x) + D(y,z) - D(y,y) \leq \varphi(x) - \varphi(z)$$

Which implies

$$d_D(x,y) + d_D(y,z) \leq \varphi(x) - \varphi(z).$$

By triangular inequality

$$d_D(x,z) \leq d_D(x,y) + d_D(y,z).$$

Consequently, we have

$$d_D(x,z) \leq d_D(x,y) + d_D(y,z) \leq \varphi(x) - \varphi(z),$$

that is

$$D(x,z) - D(x,x) \leq \varphi(x) - \varphi(z) \;\Rightarrow x \preceq z.$$

Therefore $\preceq$ is transitive. Hence $\preceq$ is a partial order on X.

It can be observed from lemma (2.1.1) that φ is decreasing function, with respect to usual order " $\leq$ "

2.2 Fixed Point Results For single-valued mapping in Dualistic Partial Metric Space

Theorem 2.2.1 Let $(X, \preceq)$ be a partially ordered set and (X, D) be a 0-complete dualistic partial metric space. Suppose that $\varphi : X \to \mathbb{R}$ is a bounded below function. If $T : X \to X$ is a

(1) $\tau(D)$-continuous map.

(2) dominating map.

Then T has a fixed point.

Proof Let $x_0 \in X$ be an initial point and $x_n = T(x_{n-1})$ for all $n \geq 1$, if there exists a positive integer r such that $x_{r+1} = x_r$ then $x_r = T(x_r)$, so we are done. Suppose that $x_n \neq x_{n+1}\ \forall\ \ n \in \mathbb{N}$. As T is dominating mapping, so $x_0 \preceq T(x_0) = x_1$, so we have $x_0 \preceq x_1$, and $x_1 \preceq T(x_1)$ that is $x_1 \preceq x_2$, and $x_2 \preceq T(x_2)$ implies $x_2 \preceq x_3$ continuing in the similar way we get;

$$x_0 \leq x_1 \leq x_2 \leq x_3 \leq \ldots, \leq x_n \leq x_{n+1} \leq x_{n+2} \leq \ldots$$

Now by definition of φ as defined in order we deduce that

$$\varphi(x_0) \geq \varphi(x_1) \geq \varphi(x_2) \geq \varphi(x_3) \geq \ldots \geq \varphi(x_n) \geq \cdots . \tag{2.2.1}$$

Since φ is bounded below, so from (2.2.1) we infer that $\{\varphi(x_n)\}_{n=1}^{\infty}$ is monotone bounded sequence and hence convergent sequence and $\{\varphi(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore, for $\epsilon > 0$ there exist n_0 such that for

$$n > m > n_0 \qquad |\varphi(x_n) - \varphi(x_m)| < \epsilon.$$

Since $x_n \preceq x_m$, we have

$$x_n \preceq x_m \quad \Leftrightarrow \ D(x_n, x_m) - D(x_n, x_n) \leq \varphi(x_n) - \varphi(x_m).$$

Which implies, $D(x_n, x_m) - D(x_n, x_n) \leq |\varphi(x_n) - \varphi(x_m)| < \epsilon$.

In consequence, $d_D(x_n, x_m) < \epsilon$. Since

$$d^s_D(x, y) = \max\{d_D(x, y), d_D(y, x)\},$$

therefore

$$d^s_D(x_n, x_m) < \epsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence in complete metric space (X, d^s_D). So there exist $v \in X$ such that

$$\lim_{n\to\infty} d^s_D(x_n, v) = 0.$$

By Lemma (1.2.1), we get

$$D(v, v) = \lim_{n\to\infty} D(x_n, v) = \lim_{n,m\to\infty} D(x_n, x_m).$$

As

$$\lim_{n,m\to\infty} d_D(x_n, x_m) = 0.$$

This implies

$$\lim_{n,m\to\infty} D(x_n, x_m) = \lim_{n\to\infty} D(x_n, x_n).$$

Since (X, D) is 0-complete dualistic partial metric space, so

$$\lim_{n\to\infty} D(x_n, x_m) = 0.$$

This implies that

$$\lim_{n\to\infty} D(x_n, v) = 0.$$

This shows that $\{x_n\}$ is 0-Cauchy sequence in (X, D) which converges to v. Now since T is continuous, therefore

$$v = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^{n+1}(x_0) = T(\lim_{n\to\infty} T^n(x_0)) = T(v).$$

Hence $v = T(v)$ that is v is fixed point of T.

If we assume that $\varphi(X)$ is compact in $\mathbb{R}$ instead of boundedness of $\varphi(X)$ in Theorem (2.2.1), we can have the following theorem.

Theorem 2.2.2 Let (X, D) be a 0-complete dualistic partial metric space, $\varphi : X \to \mathbb{R}$ be a function such that $\varphi(X)$ is compact and $\preceq$ be an order induced by φ, and $T : X \to X$ is a $\tau(D)$-continuous, dominating mapping. Then T has a fixed point in (X, D).

Example 2.2.3 Let $\varphi(x) = 1 + \frac{1}{x^2}$ for all $x \in R - \{0\}$, then $\varphi(x) = 1 + \frac{1}{x^2} > 1$ so it is bounded below. Let $D_\vee(x, y) = x \vee y \ \forall \ x, y \in R$ and $\preceq$ be an order as defined in Lemma

(2.1.1). Therefore

$$x \preceq y \Leftrightarrow D_{\vee}(x,y) - D_{\vee}(x,x) \leq \varphi(x) - \varphi(y).$$

This implies either

$$x \preceq y \Leftrightarrow 0 \leq \frac{1}{x^2} - \frac{1}{y^2}.$$

or

$$x \preceq y \Leftrightarrow y - x \leq \frac{1}{x^2} - \frac{1}{y^2}.$$

Let the mapping $T : X \to X$ is defined by

$$T(x) = \begin{cases} x^2 - 1 & \text{if } x \in (-\infty, -1); \\ x & \text{if } x \in [-1, \infty). \end{cases}$$

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T is dominating and if $T(x) = x$ then the result is obvious

and if $T(x) = x^2 - 1$, then

$$T(x) \preceq T(y) \Leftrightarrow T(x) \vee T(y) \leq T(x) + \frac{1}{(T(x))^2} - \frac{1}{(T(y))^2}.$$

this implies either

$$T(x) \preceq T(y) \Leftrightarrow y^2 \leq x^2 + \frac{1}{(x^2-1)^2} - \frac{1}{(y^2-1)^2} \quad \text{if} \quad T(x) \vee T(y) = T(y).$$

or

$$T(x) \preceq T(y) \;\Leftrightarrow\; 0 \leq \frac{1}{(x^2-1)^2} - \frac{1}{(y^2-1)^2} \quad \text{if} \quad T(x) \vee T(y) = T(x).$$

So in both cases we have

$$T(x) \preceq T(y) \;\Leftrightarrow x \preceq y.$$

So all the conditions of theorem (2.2.1) are satisfied and T has a fixed point.

Theorem 2.2.4 Let (X, D) be a 0-complete dualistic partial metric space, $\varphi : X \to \mathbb{R}$ be a bounded below function and $\preceq$ be an order induced by φ. Suppose that $T : X \to 2^X$ is a D-order closed and satisfies following property

$$\forall\, x, y \in X,\; x \preceq y \;\Rightarrow\; T(x) \prec_1 T(y) \tag{2.2.2}$$

Then T has a fixed point in X.

Proof Since $T(x)$ is non-empty set and $\{x_0\} \prec_1 T(x_0)$ for some $x_0 \in X$. We can choose $x_1 \in T(x_0)$ such that $x_0 \preceq x_1$, by equation (2.2.2), we get $T(x_0) \prec_1 T(x_1)$. For every $x_1 \in T(x_0)$, there is $x_2 \in T(x_1)$ such that $x_1 \preceq x_2$ which implies $T(x_1) \prec_1 T(x_2)$. Again for every $x_2 \in T(x_1)$, there exist $x_3 \in T(x_2)$ such that $x_2 \preceq x_3$ and this implies that $T(x_2) \prec_1 T(x_3)$. Continuing in a similar way we get a monotone sequence

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \ldots \preceq x_n \preceq \ldots.$$

Now by definition of φ as defined in order we deduce that

$$\varphi(x_0) \geq \varphi(x_1) \geq \varphi(x_2) \geq \varphi(x_3) \geq ... \geq \varphi(x_n) \geq ....$$

Since φ is bounded above, therefore $\{\varphi(x_n)\}_{n=1}^{\infty}$ is monotone bounded sequence and hence convergent sequence. Thus $\{\varphi(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, so for $\epsilon > 0$ there exist n_0 such that for $n > m > n_0$,

$$|\varphi(x_n) - \varphi(x_m)| < \epsilon.$$

On the other hand since $x_n \preceq x_m$, so we have from order defined in Lemma (2.1.1)

$$x_n \preceq x_m \Leftrightarrow D(x_n, x_m) - D(x_n, x_n) \leq \varphi(x_n) - \varphi(x_m).$$

This implies

$$D(x_n, x_m) - D(x_n, x_n) \leq |\varphi(x_n) - \varphi(x_m)| < \epsilon.$$

Therefore

$$d_D(x_n, x_m) < \epsilon.$$

Since

$$d_D^s(x, y) = \max\{d_D(x, y), d_D(y, x)\},$$

$$\text{So, } d_D^s(x_n, x_m) < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in complete metric space (X, d_D^s). So there exist $v \in X$ such that

$$\lim_{n\to\infty} d_D^s(x_n, v) = 0.$$

By Lemma (1.2.1), we obtain

$$D(v, v) = \lim_{n\to\infty} D(x_n, v) = \lim_{n,m\to\infty} D(x_n, x_m). \tag{2.2.3}$$

Since (X, D) is 0-complete so

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0,$$

therefore from (2.2.3) we deduce that $\lim_{n\to\infty} D(x_n, v) = 0$. Now as T is D-order closed and $x_{n+1} \in T(x_n)$, this implies that $v \in T(v)$ which completes the proof.

Example 2.2.5 Let $X = R^2$ and define multivalued mapping T by

$$T(p,q) = \begin{cases} \{(0,0), (3,4)\} & \text{if } pq \geq 0; \\ \{(\frac{pq}{p^5+q^5}, \frac{pq}{p^5+q^5}), (1+\frac{pq}{p^5+q^5}, 1+\frac{pq}{p^5+q^5})\} & \text{if } pq < 0. \end{cases}$$

Then T is ordered closed and $\forall\ (p,q), (u,v) \in R^2$

$$(p,q) \preceq (u,v) \Longleftrightarrow T(p,q) \prec_1 T(u,v).$$

Further $\{x_0\} \prec_1 T(x_0)$. Hence T satisfies all the conditions of Theorem (2.2.4) and it has a fixed point.

Chapter 3

Fixed Point Results of Generalizesd Contraction on Dualistic Partial Metric Space.

Valero [27] generalized the main theorem of [21] using nonlinear contractive condition instead of Banach contractive condition.

We introduce the notion of generalized contraction on dualistic partial metric spaces. We discuss an application of our fixed point results to show the existence of solution.

Theorem 3.1 Let (X, D) be a complete dualistic partial metric space and $T : X \to X$ be a mapping satisfying

$$\varphi(|D(T(x), T(y))|) \leq \varphi(M(x, y)) - \psi(M(x, y)) \text{ for all } x, y \in X, \tag{3.1}$$

where

$$M(x, y) = \max\left\{|D(x, y)|, \left|\frac{D(y, T(y))(1 + D(x, T(x))}{1 + D(x, y)}\right|\right\}$$

and

$\varphi : [0,\infty) \to [0,\infty)$ is a continuous and monotone non-decreasing function with $\varphi(t) = 0$ if and only if $t = 0$ and $\psi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof Let x_0 be an initial point of X and let us define Picard iterative sequence $\{x_n\}$ by

$$x_n = T(x_{n-1}) \text{ for all } n \in \mathbb{N}.$$

If there exists a positive integer i such that $x_i = x_{i+1}$, then $x_i = x_{i+1} = T(x_i)$, so x_i is a fixed point of T. In this case proof is complete. On the other hand if $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then from contractive condition (3.1) we have for $x_n, x_{n+1} \in X$

$$\varphi(|D(T(x_{n-1}), T(x_n))|) \leq \varphi(M(x_{n-1}, x_n)) - \psi(M(x_{n-1}, x_n)).$$

That is

$$\varphi(|D(x_n, x_{n+1})|) \leq \varphi(M(x_{n-1}, x_n)) - \psi(M(x_{n-1}, x_n)). \tag{3.2}$$

Where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{|D(x_{n-1}, x_n)|, \left|\frac{D(x_n, x_{n+1})(1 + D(x_{n-1}, x_n)}{1 + D(x_{n-1}, x_n)}\right|\right\}. \\ &= \max\{|D(x_{n-1}, x_n)|, |D(x_n, x_{n+1})|\}. \end{aligned}$$

If

$$|D(x_{n-1}, x_n)| \leq |D(x_n, x_{n+1})|,$$

then

$$M(x_{n-1}, x_n) = |D(x_n, x_{n+1})|$$

and therefore (3.2) implies,

$$\begin{aligned} \varphi(|D(x_n, x_{n+1})|) &\leq \varphi(|D(x_n, x_{n+1})|) - \psi(|D(x_n, x_{n+1})|). \\ &< \varphi(|D(x_n, x_{n+1})|). \end{aligned}$$

Which is a contradiction due to the fact $|D(x_n, x_{n+1})| > 0$. Hence

$$M(x_{n-1}, x_n) = |D(x_{n-1}, x_n)|.$$

So in this case (3.2) gives,

$$\varphi(|D(x_n, x_{n+1})|) \quad < \quad \varphi(|D(x_{n-1}, x_n)|).$$

which implies $|D(x_n, x_{n+1})| \leq |D(x_{n-1}, x_n)|$.

Thus, $\{|D(x_n, x_{n+1})|\}_{n\in\mathbf{N}}$ is a nonincreasing sequence of positive real numbers. There exists a number $L \geq 0$ such that

$$\lim_{n\to\infty} |D(x_n, x_{n+1})| = L.$$

We claim that $L = 0$. On contrary suppose that $L > 0$ and taking upper limit of

$$\varphi(|D(x_n, x_{n+1})|) \leq \varphi(|D(x_{n-1}, x_n)|) - \psi(|D(x_{n-1}, x_n)|).$$

we get

$$\begin{aligned} \varphi(L) &\leq \varphi(L) - \lim_{n\to\infty} \inf \psi(|D(x_{n-1}, x_n)|). \\ \varphi(L) &\leq \varphi(L) - \psi(L) < \varphi(L). \end{aligned}$$

Which is a contradiction, so $L = 0$ and hence

$$\lim_{n\to\infty} |D(x_n, x_{n+1})| = 0 \text{ implies } \lim_{n\to\infty} D(x_n, x_{n+1}) = 0 \tag{3.3}$$

Now to find $\lim_{n\to\infty} |D(x_n, x_n)|$, we use (3.1) again,

$$\varphi(|D(T(x_{n-1}), T(x_{n-1}))|) \leq \varphi(M(x_{n-1}, x_{n-1})) - \psi(M(x_{n-1}, x_{n-1})).$$

That is

$$\varphi(|D(x_n, x_n)|) \leq \varphi(M(x_{n-1}, x_{n-1})) - \psi(M(x_{n-1}, x_{n-1})). \tag{3.4}$$

Where

$$M(x_{n-1}, x_{n-1})) = \max\left\{|D(x_{n-1}, x_{n-1})|, \left|\frac{D(x_n, x_{n-1})(1 + D(x_{n-1}, x_n)}{1 + D(x_{n-1}, x_{n-1})}\right|\right\}$$

If

$$M(x_{n-1}, x_{n-1}) = \left|\frac{D(x_n, x_{n-1})(1 + D(x_{n-1}, x_n)}{1 + D(x_{n-1}, x_{n-1})}\right|$$

, then taking upper limit on (3.4) and using (3.3), we obtain,

$$\lim_{n\to\infty} \varphi(|D(x_n, x_n)|) \leq 0.$$

This implies that

$$\lim_{n\to\infty} \varphi(|D(x_n, x_n)|) = 0.$$

By continuity of φ, we have $\lim_{n\to\infty} |D(x_n, x_n)| = 0$. Similarly if

$$M(x_{n-1}, x_{n-1}) = |D(x_{n-1}, x_{n-1})|,$$

then

$$\varphi(|D(x_n, x_n)|) \leq \varphi(|D(x_{n-1}, x_{n-1})|) - \psi(|D(x_{n-1}, x_{n-1})|). \tag{3.5}$$

$$\varphi(|D(x_n, x_n)|) \;<\; \varphi(|D(x_{n-1}, x_{n-1})|).$$

$$\text{it implies } |D(x_n, x_n)| \;\leq\; |D(x_{n-1}, x_{n-1})|.$$

Thus $\{|D(x_n, x_n)|\}_{n\in\mathbf{N}}$ is a nonincreasing sequence of positive real numbers and arguing like above, we get

$$\lim_{n\to\infty} D(x_n, x_n) = 0. \tag{3.6}$$

Since

$$d_D(x_n, x_{n+1}) = D(x_n, x_{n+1}) - D(x_n, x_n)$$

, so using (3.6), we get

$$\lim_{n\to\infty} d_D(x_n, x_{n+1}) = 0. \tag{3.7}$$

Now we show that $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). For this we have to show that $\lim_{n,m\to\infty} d_D^s(x_n, x_m) = 0$. That is

$$\lim_{n,m\to\infty} d_D(x_n, x_m) = 0 = \lim_{n,m\to\infty} d_D(x_m, x_n).$$

Suppose on contrary that $\lim_{n,m\to\infty} d_D(x_n, x_m) \neq 0$. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{n_k}\}, \{x_{m_k}\}$ of $\{x_n\}$ such that n_k is smallest index for which

$$n_k > m_k \text{ and } d_D(x_{n_k}, x_{m_k}) \geq \epsilon. \tag{3.8}$$

This means that

$$d_D(x_{n_k-1}, x_{m_k}) < \epsilon. \tag{3.9}$$

Now using (3.8) and (3.9), we have

$$\begin{aligned}\epsilon \leq d_D(x_{n_k}, x_{m_k}) &\leq d_D(x_{n_k}, x_{n_k-1}) + d_D(x_{n_k-1}, x_{m_k}).\\ &\leq d_D(x_{n_k}, x_{n_k-1}) + \epsilon.\end{aligned}$$

Taking $\lim_{k\to\infty}$ and using (3.7), we obtain

$$\lim_{k\to\infty} d_D(x_{n_k}, x_{m_k}) = \epsilon. \tag{3.10}$$

Due to triangular inequality, we get

$$\begin{aligned}d_D(x_{n_k}, x_{m_k}) &\leq d_D(x_{n_k}, x_{n_k-1}) + d_D(x_{n_k-1}, x_{m_k}).\\ &\leq d_D(x_{n_k}, x_{n_k-1}) + d_D(x_{n_k-1}, x_{m_k-1}) + d_D(x_{m_k-1}, x_{m_k}).\end{aligned}$$

But then

$$\begin{aligned}d_D(x_{n_k-1}, x_{m_k-1}) &\leq d_D(x_{n_k-1}, x_{n_k}) + d_D(x_{n_k}, x_{m_k-1}).\\ &\leq d_D(x_{n_k-1}, x_{n_k}) + d_D(x_{n_k}, x_{m_k}) + d_D(x_{m_k}, x_{m_k-1})\end{aligned}$$

Taking $\lim_{k\to\infty}$ in above expressions and using (3.7), (3.10) we obtain,

$$\lim_{k\to\infty} d_D(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{3.11}$$

Following (3.1) for $x_{n_k} \neq x_{m_k}$, we have

$$\varphi(|D(T(x_{n_k-1}), T(x_{m_k-1}))|) \leq \varphi(M(x_{n_k-1}, x_{m_k-1})) - \psi(M(x_{n_k-1}, x_{m_k-1})).$$

That is

$$\varphi(|D(x_{n_k}, x_{m_k})|) \leq \varphi(M(x_{n_k-1}, x_{m_k-1})) - \psi(M(x_{n_k-1}, x_{m_k-1})). \tag{3.12}$$

Where

$$M(x_{n_k-1}, x_{m_k-1}) = \max\left\{|D(x_{n_k-1}, x_{m_k-1})|, \left|\frac{D(x_{m_k-1}, x_{m_k})(1 + D(x_{n_k-1}, x_{n_k}))}{1 + D(x_{n_k-1}, x_{n_k-1})}\right|\right\}$$

By using (3.10) and (3.11), we deduce that

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{3.13}$$

Now applying upper limit on (3.12) and using (3.10), (3.11) along with properties of φ, ψ we get

$$\varphi(\epsilon) \leq \varphi(\epsilon) - \lim_{k\to\infty} \inf \psi(M(x_{n_k-1}, x_{m_k-1})).$$

That is $\varphi(\epsilon) < \varphi(\epsilon)$, a contradiction and therefore

$$\lim_{n,m\to\infty} d_D(x_n, x_m) = 0.$$

Similarly we can prove that $\lim_{n,m\to\infty} d_D(x_m, x_n) = 0$.

Hence $\lim_{n,m\to\infty} d_D^s(x_n, x_m) = 0$ which entails that $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). Since $(X. D)$ is a complete dualistic partial metric space, so by Lemma (1.2.1) (X, d_D^s) is also a complete metric space. Thus, there exists v in X such that $\lim_{n\to\infty} d_D^s(x_n, v) = 0$, again from Lemma (1.2.1), we get

$$\lim_{n\to\infty} d_D^s(x_n, v) = 0 \iff \lim_{n\to\infty} D(v, x_n) = D(v, v) = \lim_{n,m\to\infty} D(x_n, x_m). \quad (3.14)$$

Since $\lim_{n,m\to\infty} d_D(x_n, x_m) = 0$, thus, $\lim_{n,m\to\infty} D(x_n, x_m) = 0$. From (3.14)

$$D(v, v) = 0 = \lim_{n\to\infty} D(v, x_n).$$

Now we prove that v is a fixed point of T. On contrary suppose that $v \neq T(v)$, then using (3.1) and Lemma (1.2.1) we have

$$\varphi(|D(x_n, T(v)|) = \varphi(|D(T(x_{n-1}), T(v))|) \leq \varphi(M(x_{n-1}, v)) - \psi(M(x_{n-1}, v)).$$

That is

$$\varphi(|D(x_n, T(v))|) \leq \varphi(M(x_{n-1}, v)) - \psi(M(x_{n-1}, v)).$$

Letting $n \to \infty$ and using properties of φ, ψ we get $\varphi(D(v,T(v))) < \varphi(D(v,T(v)))$. Which is a contradiction as $D(v,T(v)) \geq 0$. Hence $v = T(v)$ that is v is a fixed point of T. Finally, we shall prove the uniqueness. Suppose that ω is another fixed point of T such that $v \neq \omega$, then from (3.1),we have

$$\varphi(|D(v,\omega)|) \leq \varphi(M(v,\omega)) - \psi(M(v,\omega)).$$

which implies that

$$\varphi(|D(v,\omega)|) < \varphi(|D(v,\omega)|).$$

A contradiction, hence $v = \omega$. So T has a unique fixed point in X.

In Theorem (3.1), if we take $\varphi(t) = t$ and $\psi(t) = (1-h)t$ where $h \in [0,1[$ and $t \geq 0$. Then we have following result

Corollary 3.2 Let (X, D) be a complete dualistic partial metric space and $T : X \to X$ be a mapping satisfying

$$|D(T(x),T(y))| \leq h \max\left\{|D(x,y)|, \left|\frac{D(y,T(y))(1+D(x,T(x))}{1+D(x,y)}\right|\right\} \text{ for all } x,y \in X, \tag{3.15}$$

Then T has a unique fixed point.

For if $D(x,y) \in \mathbb{R}_0^+$ for all $x,y \in X$, then $D(x,y) = p(x,y)$. The partial metric version of Corollary (3.2) can be obtained as follows:

Corollary 3.3 Let (X,p) be a complete partial metric space and $T : X \to X$ be a mapping satisfying

$$p(T(x),T(y)) \leq h \max\left\{p(x,y), \frac{p(y,T(y))(1+p(x,T(x))}{1+p(x,y)}\right\} \text{ for all } x,y \in X,$$

Then T has a unique fixed point.

Now if

$$\max\left\{|D(x,y)|, \left|\frac{D(y,T(y))(1+D(x,T(x))}{1+D(x,y)}\right|\right\} = |D(x,y)|,$$

then the result obtained by Valero [21] can be viewed as a special case of Corollary (3.2).

Corollary 3.4 Let (X, D) be a complete dualistic partial metric space and let $T : X \to X$ be a self-mapping such that there exists $\alpha \in [0,1[$ satisfying

$$|D(T(x),T(y))| \leq \alpha|D(x,y)|,$$

for all $x,y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, $D(x^*,x^*) = 0$ and the Picard iterative sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_D^s)$, for every $x \in X$.

Remark 3.5 As every dualistic partial metric D is an extension of partial metric p, therefore, Theorem (3.1) is an extension of Theorem (3.2). Consequently, Corollary (3.2) generalizes Corollary (3.3).

There arises the following natural question:

Whether the contractive condition in the statement of Corollary (3.2) can be replaced by the contractive condition in Corollary (3.3). Following example will give the negative answer to this question.

Example 3.6 Consider the complete dualistic partial metric $(\mathbb{R}, D_\vee)$. Define the self-mapping $T_0 : \mathbb{R} \to \mathbb{R}$ by

$$T_0(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}.$$

It is easy to check that fixed point free mapping T_0 does not satisfy the contractive condition in the statement of Corollary (3.1). Indeed,

$$1 = |D_\vee(-1,-1)| = |D_\vee(T_0(0), T_0(0))|) > hM(0,0).$$

Where

$$M(0,0) = \max\left\{|D_\vee(0,0)|, \left|\frac{D_\vee(0,T_0(0))(1+D_\vee(0,T_0(0)))}{1+D_\vee(0,0)}\right|\right\}.$$

Nevertheless, the contractive condition in the statement of Corollary (3.3) holds true. Indeed,

$$-1 = p_\vee(-1,-1) = p_\vee(T_0(0), T_0(0))|) \leq hM(0,0).$$

Where

$$M(0,0) = \max\left\{p_V(0,0), \frac{p_V(0,T_0(0))(1+p_V(0,T_0(0)))}{1+p_V(0,0)}\right\} = 0.$$

Example 3.7 Let $(\mathbb{R}, D_V)$ be a complete dualistic partial metric space. Define the self-mapping $T_1 : \mathbb{R} \to \mathbb{R}$ by

$$T_1(x) = \begin{cases} 0 & \text{if } x = 0 \\ -1 & \text{if } x \geq 2 \end{cases}.$$

The mapping T_1 has a unique fixed point $x = 0$. It is easy to check that T_1 satisfies the contractive condition in the statement of Corollary (3.1). Indeed, for all $x \geq y \geq 2$ and $h > \frac{1}{2}$

$$\begin{aligned} |D_V(T_1(x), T_1(y))| &\leq h \max\left\{|D_V(x,y)|, \left|\frac{D_V(y,T_1(y))(1+D_V(x,T_1(x))}{1+D_V(x,y)}\right|\right\} \\ 1 = |D_V(-1,-1)| &\leq hx \end{aligned}$$

holds. Also note that for $x = 0 = y$ the contractive condition in the statement of Corollary (3.2) trivially holds.

Application to Functional Equations

As an application of our fixed point result [Corollary 3.2], we present the study about the existence and uniqueness of the solution of functional equations. We introduce some notations for the sake of convenience.

$$\begin{aligned}
S &= \text{State space.} \\
W &= \text{Decision space.} \\
B(S) &= \text{Space of bounded functions.} \\
c_n &= \text{Sequence of real numbers such that } \lim_{n \to \infty} |c_n| = 0. \\
g &: S \times W \to \mathbb{R}. \\
F_n &: S \times W \times \mathbb{R} \to \mathbb{R} \text{ where } n = 0, 1, 2, 3, \cdots . \\
\phi &: S \times W \to S.
\end{aligned}$$

In the following we shall prove the existence and uniqueness of solution of functional equation appearing in dynamic programming. (for example see [8])

$$u(x) = \sup_{y \in W}\{g(x, y) + F_n(x, y, u(\phi(x, y)))\} \;\; \forall x \in S \tag{3.16}$$

We observe that the spaces $(B(S), \|.\|_\infty)$ is a Banach space and distance function in $B(S)$ is defined by

$$d_\infty(u, v) = \sup_{x \in S} |u(x) - v(x)| \;\; \forall\, u, v \in B(S)$$

where as for dualistic partial metric space distant function is given by

$$D_\infty(u,v) = d_\infty(u,v) + c_n, \ \ \forall\ u,v \in B(S).$$

In calculations following two lemmas will be helpful.

Lemma 3.7 Let $G, H : S \to R$ be two bounded functions then,

$$|\sup_{x\in S} G(x) - \sup_{x\in S} H(x)| \leq \sup_{x\in S} |G(x) - H(x)|.$$

Lemma 3.8 Let

(1) g, F be bounded functions.

(2) $\exists\ k > 0$ such that $\forall\ t, r \in R$, $x \in S$ and $y \in W$.

$$|F_n(x,y,t) - F_n(x,y,r)| \leq k|t - r|.$$

Then the operator $R : B(S) \to B(S)$ defined by

$$(Ru)(x) = \sup_{y\in W}\{g(x,y) + F_n(x,y,u(\phi(x,y)))\}$$

is well define.

Theorem 3.9 Let all the conditions of lemma (3.8) be satisfied and for $n \to \infty$

$$|F_n(x,y,u) - F_n(x,y,v)| \leq h \max\left\{|D_\infty(u,v)|, \left|\frac{D_\infty(v,Rv)(1 + D_\infty(u,Ru)}{1 + D_\infty(u,v)}\right|\right\} \quad (3.17)$$

Then the functional equation (3.17) has unique solution.

Proof Let $R : B(S) \to B(S)$ be an operator as defined in lemma (3.8). We shall show that R satisfies contractive condition (3.17). Indeed by lemma (3.7), for all $u, v \in B(S)$.

$$
\begin{aligned}
|(Ru)(x) - (Rv)(x)| &= |\sup_{y \in W}\{g(x,y) + F_n(x,y,u(\phi(x,y)))\} - \sup_{y \in W}\{g(x,y) + F_n(x,y,v(\phi(x,y)))\}| \\
&\leq \sup_{y \in W} |g(x,y) + F_n(x,y,u(\phi(x,y))) - g(x,y) - F_n(x,y,v(\phi(x,y)))| \\
&\leq \sup_{y \in W} |F_n(x,y,u(\phi(x,y))) - F_n(x,y,v(\phi(x,y)))|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|D_\infty(Ru, Rv)| &= |\sup_{x \in S} |(Ru)(x) - (Rv)(x)| + c_n| \\
&\leq \sup_{x \in S} |(Ru)(x) - (Rv)(x)| + |c_n| \\
&\leq \sup_{y \in W} |F_n(x,y,u(\phi(x,y))) - F_n(x,y,v(\phi(x,y)))| + |c_n|
\end{aligned}
$$

When $n \to \infty$ then by (3.17) we obtain,

$$
|D_\infty(Ru, Rv)| \leq h \max\left\{|D_\infty(u,v)|, \left|\frac{D_\infty(v, Rv)(1 + D_\infty(u, Ru))}{1 + D_\infty(u, v)}\right|\right\} + \lim_{n \to \infty} |c_n|.
$$

Finally, definition of c_n gives

$$
|D_\infty(Ru, Rv)| \leq h \max\left\{|D_\infty(u,v)|, \left|\frac{D_\infty(v, Rv)(1 + D_\infty(u, Ru))}{1 + D_\infty(u, v)}\right|\right\}.
$$

Hence, R satisfies all the conditions of Corollary (3.2). Thus there exists a unique solution of (3.17) $u_0 \in B(S)$ such that $Ru_0 = u_0$.

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