Inertial and Leakage Effects on Second Grade Fluid Flow near a Corner

By

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Department of Mathematics & Statistics
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A Thesis
Submitted in the Partial Fulfillment of the
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In
MATHEMATICS

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<u>Certificate</u>

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We accept this thesis as conforming to the required standard

1.__________________ 2. ____________________

(External Examiner) (Internal Examiner)

3.______________________ 4. ____________________

Dr. Khadija Maqbool
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Department of Mathematics & Statistics
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2025

Thesis Certificate

The thesis entitled ***"Inertial and Leakage Effects on Second Grade Fluid Flow near a Corner"*** submitted by ***Saba Ahmed***, *933-FOS/MSMA/F23* in partial fulfillment of MS Degree in Mathematics has been completed under my guidance and supervision. I am satisfied with the quality of her research work and allow her to submit this thesis for further process to graduate with Master of Science degree from the Department of Mathematics & Statistics, as per IIUI rules and regulations.

Date………………………….

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Assistant Professor

Department of Mathematics

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Islamabad.

DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely based on my efforts under the supervision of my supervisor **Dr. Khadija Maqbool**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

Saba Ahmed

MS (Mathematics)

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Dedication

To my parents,

The reason of what I become today.

Thanks for your great support and continuous care.

To my brother,

I am really grateful to you.

You have been my inspiration and my strength.

Acknowledgments

All recommendations to Almighty Allah, who has disclosed upon me endurance and resoluteness for the accomplishment of this assignment. I offer my humblest and sincerest words of thanks to his Holy Prophet (P.B.U.H) who is forever a torch of guidance and knowledge for me.
Needless to say, about the distinguished and outstanding supervisor **Dr. Khadija Maqbol** who has helped the most important and pivotal role during my academic journey. Her pains-taking supervision and challenging queries were absolutely necessary to my research work. She has been much more than a supervisor, so again are my sincere thanks for the guidance, inspiration and unabated vigor to her.

I would like to thanks my beloved father **Mr. Ali Ahmed,** my mother **Waheeda Parveen** and my brother **Mr. Muhammad Shahbaz Ahmed** without whom I would not have reached the pinnacle of success. They always kept me motivated on my core task. Their encouragement, guidance, and sustenance since my childhood enabled me to unfold my hidden abilities.

Lastly, I offer my regards and blessings to all of those who supported me in any aspect during the complrtion of the thesis.

Saba Ahmed

Contents

Preface

The study of non-Newtonian fluids continues to engage scientists due to their complex flow characteristics and importance across various engineering and geophysical applications. Second-grade fluids, in particular, incorporate both viscous and elastic responses, making them well-suited for modeling viscoelastic behaviors beyond the reach of Newtonian models.

Corner flow problems are known for their complexity, primarily due to the sharp boundary angles that create stress singularities and secondary flow patterns. The addition of inertia and leakage at corner, increases this complexity and calls for more refined analytical and numerical method.

Different researchers like Taylor [1], Hancock et al. [2] and Riedler et al. [3] have examined scraping flow problems for creeping and inertial flows of a Newtonian fluid between two intersecting planes. Their study showed that the flow was generated due to the movement of one plate near a corner and regular perturbation technique was used to observe the leakage effect on flow and expressions for normal and tangential stresses to the plate. Mansutti et al. [4] have configured the non-inertial flow of a shear thinning fluid between intersecting planes and used multi-parameter continuation technique to analyze both converging and diverging flow. Huang et al. [5] calculated the flow of Oldroyd-B fluid between two intersecting plate, one of which was fixed and the other one was moving and observed the flow pattern with the help of streamlines. Hills et al. [6] examined the corner flow induced by the rotation of plate with fixed angle and determined the flow in three different geometries. Mahmood [7] has compared the inertial and non-inertial flow by regular perturbation technique. Chaffin et al. [8] analyzed Taylor's paint scraping problem for Carreau fluid and reveals the perturbation results for dynamics of the fluid near and far from the corner.

In this research, steady and non-creeping flow of second grade fluid with leakage at corner is described which has not been addressed by a single author. The mathematical model represents the set of non-linear differential equations which are solved by recursive approach introduced by Langlois $[10-12]$. This study presents the effects of angle of the scraper with the flow and emerging parameters on velocity profile, stresses and pressure distribution near a corner. Our results subsume Mahmood et al. [7] results as a special case of Newtonian fluid and comparison between inertial and non-inertial flow behavior is also shown through graphical results. Recently, researchers have studied the different flow problems near a corner with its applications $[13-18]$.

Keeping view of the literature survey, this research is organized in following three chapters. Chapter one includes the preliminaries, chapter two presents the mathematical modelling of second grade fluid for creeping flow near a corner and chapter three examines the inertial and leakage effects when leakage is present at the apex of two intersecting planes.

Chapter 1

Preliminaries

This chapter includes the basic concepts and definitions of different types of flows, fluid properties, and laws of fluid mechanics.

1.1 Fluid

The fluid is a type of matter which is continuously deformable, and which spontaneously tends to adopt its shape to its container by occupying all of the space made available to it.

1.2 Fluid Mechanics

Fluid mechanics is a branch of engineering that explores the characteristics and behavior of fluids and forces acting on them. It is divided into two main sub disciplines fluid statics and fluid dynamics.

1.3 Types of Flows

Flows can be categorized based on various characteristics:

1.3.1 Laminar vs Turbulent Flow

Laminar flow refers to a type of fluid flow where all fluid particles follow a certain path and move smoothly without crossing each other. On the other hand, turbulent flow is characterized by fluid particles that do not follow a specific path and the fluid flows in an irregular pattern.

1.3.2 Steady vs Unsteady Flow

In steady flow, fluid properties such as velocity, pressure, and density remain constant over time and do not change with respect to time i.e. $S \neq S(t)$; while in case of unsteady flow fluid properties change with respect to time i.e.$S = S(t)$.

1.3.3 Compressible vs Incompressible Flow

Compressible flow have a notable change in density with varying pressure, temperature, and space components. On contrary incompressible flow have a density that does not change with respect to space, pressure and temperature.

1.3.4 Creeping vs Non-Creeping Flow

In creeping flow the viscous forces are dominant over the inertial forces, it is also known as Stokes flow, but in non-creeping flow inertial forces have significant contribution in the flow field.

1.4 Flow Properties

1.4.1 Density

The density ρ is defined as mass per unit volume at a specific pressure and temperature. Mathematically, it is defined as

$$\rho = \frac{m}{V}, \tag{1.1}$$

where V denotes the volume and m is mass of fluid.

1.4.2 Pressure

Pressure is the magnitude of force per unit area and mathematically, it is represented as follows:

$$p = \frac{|\mathbf{F}|}{A}. \tag{1.2}$$

1.4.3 Shear stress

A force per unit area that tends to create deformation in fluid flow is known as shear stress. It is denoted by τ and mathematically, defined as

$$\tau = \frac{\mathbf{F}}{A}. \tag{1.3}$$

1.5 Inertial and Non Inertial Flow

The term "inertial flow" refers to a fluid's motion in which internal friction causes viscous forces to oppose the fluid's motion, yet inertial forces, which are connected to the fluid's mass and acceleration, are important. On the other hand, "non-inertial flow" happens when viscous forces predominate and the fluid moves steadily and smoothly.

1.6 Newtonian Fluids

Newtonian fluids are those which obey Newton's law of viscosity. Most common fluids such as water, air, gasoline, glycerine and syrup are Newtonian under normal conditions. Mathematically, Newtonian fluid hold the following relation:

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}. \tag{1.4}$$

The Cauchy stress tensor $\mathbf{T}$ satisfy the following relation for Newtonian fluids

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1, \tag{1.5}$$

$$\mathbf{A}_1 = \operatorname{grad} V + (\operatorname{grad} V)^T, \tag{1.6}$$

where p is the hydrostatic pressure, $\mathbf{I}$ is the unit tensor, μ is the dynamic viscosity of the fluid, V is the velocity of the fluid.and $\mathbf{A}_1$ is the first Rivlin-Erickson tensor.

1.7 Non-Newtonian Fluids

A fluid in which the viscosity varies with the applied strain rate is called a non-Newtonian fluid. As a result, the non-Newtonian fluids may not have well-defined viscosity. Mathematically, non-Newtonian fluid hold following relation:

$$\tau_{xy} = \eta \frac{\partial u}{\partial y}, \tag{1.7}$$

where $\eta = \left(\frac{\partial u}{\partial y}\right)^{n-1}$ and $n \neq 1$.

There are different types of non-Newtonian fluid.

1.7.1 Dilatant (Shear thickening)

In such type of fluid viscosity is an increasing function of shear stress.

1.7.2 Pseudoplastic (Shear thinning)

In such type of fluid viscosity is constant at low shear rate but decreases at intermediate shear rate.

1.7.3 Thixotropic

It has time dependent shear thinning property. e.g. gels, clays, and cytoplasm.

1.7.4 Rheopectic

This type of fluid has increasing viscosity with stress over time.

1.7.5 Viscoelastic fluids

This type of fluid formed by viscous and elastic component. In other words it is a mixture of polymer and solvent. e.g. second grade, FENEP fluid model and Maxwell fluid model.

1.7.6 Visco-inelastic fluids

It is also known as generalized Newtonian fluids, such type of fluid has shear rate dependent viscosity. e.g. Carreau, Ellis, Power law and Williamson fluid model.

1.8 Second Grade Fluid

A second-grade fluid is a type of non-Newtonian fluid that exhibits elasticity and common stress effects. This theoretical model is used in fluid mechanics to analyse materials whose behavior deviates slightly from that of Newtonian fluids (like water or air), particularly when memory-effect or slow flows are present.

In contrast to Newtonian fluids, the stress in a second-grade fluid is dependent on both acceleration (second derivative of velocity) and strain rate (first derivative of velocity) and satisfy the following Cauchy stress tensor

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1.8}$$

and

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{1.9}$$

in above equation μ is the viscosity coefficient and α_1, α_2 are the second grade fluid parameters, $\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivilain Erickson tensor defined as follows:

$$\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T, \tag{1.10}$$

$$\mathbf{A}_2 = (\mathbf{V} \cdot \nabla) \mathbf{A}_1 + \mathbf{A}_1 (\nabla \mathbf{V}) + (\nabla \mathbf{V})^T \mathbf{A}_1. \tag{1.11}$$

The second order fluids are fluids for which the Cauchy stress tensor T is expanded as a power series in the rate-of-deformation tensor and its derivatives, truncated after second-order terms."The second-order fluid" is a second-order asymptotic approximation about the state of the rest of a viscoelastic fluid.

It is possible to think of second-grade fluid models as simplified or idealized versions of more intricate polymer fluid models. They provide a more straightforward mathematical representation of the elastic and normal stress behavior of polymers. Utilized frequently in theoretical and scholarly research to understand viscoelastic effects without having to solve difficult constitutive equations.

1.8.1 Polymer Fluid

Polymer fluid demonstrates viscoelasticity, shear thickening or shear thinning and normal stress differences. The behavior of polymer fluids is often modeled with more complex constitutive models like Oldroyd-B, Maxwell, Giesekus, and FENE-P.

1.9 Types of boundary condition

1.9.1 Slip boundary condition

In slip boundary conditions, there is relative velocity between the surface and fluid. For example ice skating, the skates slide over the snow with very little friction. This is because a thin layer of water forms between the skates and the surface, mathematically it is defined as follows

$$\frac{\partial V}{\partial y} = 0. \tag{1.12}$$

1.9.2 No-slip boundary condition

In no slip boundary conditions the velocity of fluid is supposed to be velocity of surface. For example when you drive, your tyres grip the road, as tyres are designed to grip the road and avoid slipping effect.

For a flat wall located at $y = 0$, and fluid velocity V satisfy the following condition:

$$V = V_{\mathrm{wall}} \qquad \text{on the boundary} \tag{1.13}$$

If the wall is stationary:

$$V = 0. \tag{1.14}$$

1.9.3 Partial-slip boundary condition

In partial slip, both slip and no slip condition occurs between fluid and wall. For example blood flow in blood vessel because walls of blood vessels are not perfectly smooth. Mathematically it is represented as:

$$V = \lambda \frac{\partial V}{\partial y}, \quad \text{at } y = 0. \tag{1.15}$$

1.10 Geometrical arrangement of fluid flow

Numerous geometrical arrangements of fluid flow are possible, and each has unique physical properties and uses. The following list of fluid mechanics geometries is frequently used, particularly for theoretical and experimental analysis. Flow between parallel plates, flow in a circular pipe or tube, annular flow (between two coaxial cylinders), corner or wedge flow, flow over a flat plate, flow past a cylinder or sphere, channel flow (rectangular duct), jet and plume flows, and open channel flow.

1.10.1 Corner Flow

The term "Corner Flow" describes the flow of fluids caused by capillary forces in wedge-shaped structures, including corners or gradients. This phenomenon is important for a number of applications, such as spacecraft fluid management, CO_2 sequestration, and oil recovery.

1.11 Coordinate systems and their types

A coordinate system is a mathematical framework that uses coordinates to represent a point's location in space. In fluid mechanics, physics, and engineering, it aids in the definition of geometry, motion, and fields (such as pressure or velocity).The main purpose of coordinate system are used to specify position in (1D, 2D, or 3D).

1.11.1 Main types of coordinate systems

Cartesian Coordinate System (Rectangular)

This system is used for rectangular or flat geometry and its coordinates define distances along three perpendicular axes x, y, and z. e.g. a box-shaped room, flow between two flat plates.

Cylindrical Coordinate System

This system is used for cylindrical symmetry and its coordinates radial distance, angle around the axis, and distance from the centre axis to the boundary. e.g. whirling machinery and flow inside a pipe.

Spherical Coordinate System

This system is used for spherical symmetry and its coordinates define angle about the horizontal plane, angle from the vertical axis, and distance from the origin to the boundary. e.g. radiation patterns, gravitational fields, and flow around a sphere.

1.12 Non-dimensional Parameters

1.12.1 Reynold's Number

The Reynold's number is a dimensionless quantity that is the ratio of inertial to the viscous forces. It is used in fluid mechanics to predict flow patterns in different fluid flow situations. It helps to determine whether the flow is laminar, turbulent, or in transition. Mathematically, it is defined as

$$\mathrm{Re} = \frac{\rho \mathbf{V} L}{\mu}. \tag{1.16}$$

Where ρ denotes density, μ is dynamic viscosity, V represents velocity of fluid and L is characteristics length.

The $\mathrm{Re} < 2300$ represents the laminar flow and $2300 < \mathrm{Re} < 4000$ indicates the transition from laminar to turbulent flow but $\mathrm{Re} > 4000$ predict the turbulent flow.

1.13 Basic Laws of Fluid

1.13.1 Principle of Conservation of Mass

This law states that mass cannot be created or destroyed during flow. It remains constant with respect to all physical changes. Mathematically, it can be defined as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\left(\rho \cdot \mathbf{V}\right) = 0, \tag{1.17}$$

where ρ is the density of fluid, t is the time, $\boldsymbol{\nabla}$ is the divergence, $\mathbf{V}$ is the velocity vector. For any incompressible flow, above equation becomes

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0. \tag{1.18}$$

1.13.2 Principle of Conservation of Momentum

The principle states that the total momentum of an isolated system remains constant within some domain if no external forces act on it. Mathematically, it can be defined as

$$\rho\left(\frac{\partial}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\right)\mathbf{V} + \boldsymbol{\nabla} p - \rho \mathbf{f} = \boldsymbol{\nabla} \tau. \tag{1.19}$$

where ρ is the density of fluid, t is the time, $\boldsymbol{\nabla}$ is the divergence, $\mathbf{V}$ is the velocity vector, p is the pressure, $\mathbf{f}$ is the body force term, and $\boldsymbol{\tau}$ is stress tensor.

1.14 Methodology

The technique which is going to be used in this research is Recursive Langloi's technique, which was introduced by W.E.Langlois in 1963. This method will help us to make the non-linear system into linear system. After linearizing, we will use inverse method to convert the linear PDE's into set of ODE's. In this approach, one can linearize velocity profie, shear stress and pressure with the help of small dimensionless number ε. In order to obtain the $1st, 2nd$ and $3rd$

order solutions for velocity profile, shear stress and pressure, we consider following series i.e.

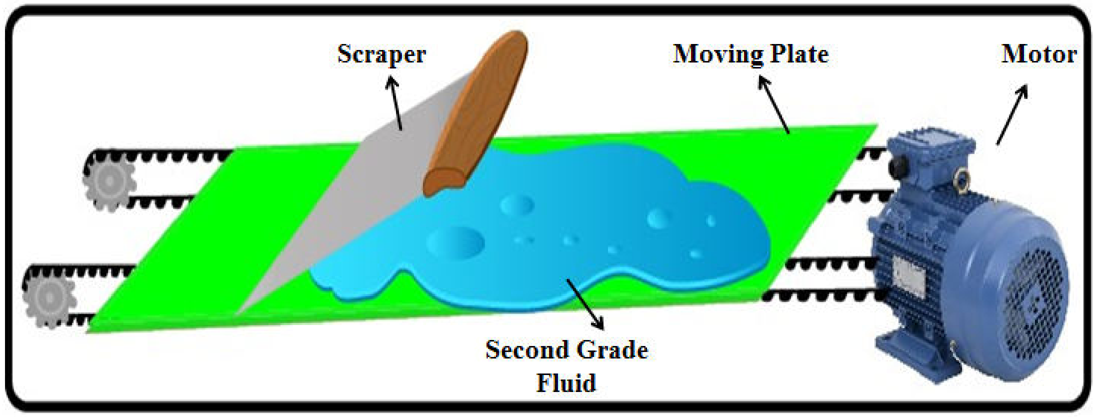
$$u = \sum_{i=1}^{\infty} \varepsilon^i u^{(i)}, \ v = \sum_{i=1}^{\infty} \varepsilon^i v^{(i)}, \ p = p^{(0)} + \sum_{i=1}^{\infty} \varepsilon^i p^{(i)}. \tag{1.20}$$

Chapter 2

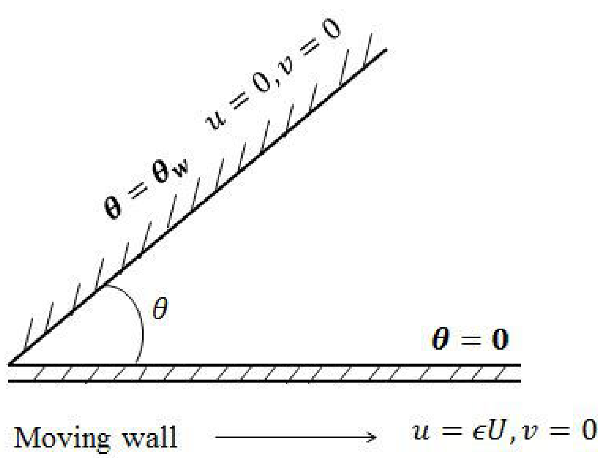
Creeping flow of second grade fluid near a corner

This chapter is the extension of review work of Mehmood et.al.[9] about a creeping flow of second grade fluid near a corner. His study investigated the steady state incompressible flow near a corner for Newtonian fluid. This chapter extend the idea of Mehmood et.al. for second grade fluid model. The creeping flow of second grade fluid model near a corner presents a system of non-linear partial differential equation. The non-linear system is solved by Langloi's recursive approach and inverse method to obtain the approximate results for stream function, velocity field, pressure distribution, and shear stress. The results of mathematical expression can be visualized by the graphs that are plotted by the Software Mathematica.

2.1 Mathematical Modeling



(a)



(b)

Fig.2:(a) The flat plate corner model. (b) Corner flow geometry.

The creeping flow of second grade fluid over a moving plate making an angle θ with a scraper is modeled by the corner flow. The corner flow is sketched in figure 2 (b), the flow is assumed incompressible, steady state, and two dimensional therefore, following governing equations are used.

$$\nabla.\mathbf{V} = 0, \tag{2.1}$$

$$\rho\left(\nabla.\mathbf{V}\right)\mathbf{V} = \operatorname{div}\mathbf{T}, \tag{2.2}$$

where Cauchy stress tensor $\mathbf{T}$ for second grade fluid Ref.[19] is given as follows.

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{2.3}$$

and

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{2.4}$$

In above equation μ is the viscosity coefficient and α_1, α_2 are the second grade fluid parameters. $\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivilin Erickson tensor defined as follows:

$$\mathbf{A}_1 = \nabla\mathbf{V} + \left(\nabla\mathbf{V}\right)^T, \tag{2.5}$$

$$\mathbf{A}_2 = \left(\mathbf{V}\cdot\nabla\right)\mathbf{A}_1 + \mathbf{A}_1\left(\nabla\mathbf{V}\right) + \left(\nabla\mathbf{V}\right)^T\mathbf{A}_1, \tag{2.6}$$

where,

$$\mathbf{V} = \left(u\left(r,\theta\right), v\left(r,\theta\right), 0\right). \tag{2.7}$$

After using Eq.(2.3) – (2.7), equation of continuity and momentum take the following form

$$\frac{1}{r}\frac{\partial\left(ru\right)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} = 0, \tag{2.8}$$

$r-$component of momentum equation is

$$0 = -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(rS_{rr}\right) + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial\theta} - \frac{S_{\theta\theta}}{r}, \tag{2.9}$$

$\theta-$component of momentum equation is

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2S_{r\theta}\right) + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta}. \tag{2.10}$$

The no slip boundary condition is imposed at the moving boundary $(\theta = 0)$ and is given as follows

$$u = \varepsilon U, v = 0, \quad \text{at} \quad \theta = 0, \tag{2.11}$$

where U is speed of moving plate.

The flow is stationary near the scraper $(\theta = \theta_w)$ and satisfy the following expressions.

$$u = 0, v = 0, \quad \text{at} \quad \theta = \theta_w. \tag{2.12}$$

2.2 Methodology

We will employee the Recursive technique $[10 - 12]$, which will linearize the velocity profile, pressure, stream function, and shear stress using a small dimensionless number ε. This is achieved by assuming a series solutions in the following form

$$\mathbf{V}(r,\theta) = \varepsilon \mathbf{V}^{(1)} + \varepsilon^2 \mathbf{V}^{(2)} + ..., \tag{2.13}$$

$$p(r,\theta) = const + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + ..., \tag{2.14}$$

$$\psi(r,\theta) = \varepsilon \psi^{(1)} + \varepsilon^2 \psi^{(2)} + ..., \tag{2.15}$$

$$\mathbf{S}(r,\theta) = \varepsilon \mathbf{S}^{(1)} + \varepsilon^2 \mathbf{S}^{(2)} .... \tag{2.16}$$

This leads to dynamical equations and boundary conditions for $\left(\mathbf{V}^{(i)}, p^{(i)}, \psi^{(i)}, \mathbf{S}^{(i)}\right)$ where $i = 1, 2, 3$ so that $(\mathbf{V}, p, \psi, \mathbf{S})$, as given by Eqs.$(2.13) - (2.16)$,provides a solution to the equations of motion (with appropriate boundary conditions) for Rivilin-Erickson fluid, retaining terms up to third order in ε, neglecting all higher-order terms. At each stage of the analysis, the dynamical equation of system is linear, with the results from the previous stage used explicitly in the analysis.

Making use of Eq$(2.13) - (2.16)$, in Eq$(2.8) - (2.10)$, one can get the first and second-order problem given as follows:

2.2.1 First Order Problem and its Solution

The first order problem represent the creeping flow of Newtonian fluid as this system does not involve second grade parameters.

$$\frac{1}{r}\frac{\partial\left(ru^{(1)}\right)}{\partial r}+\frac{1}{r}\frac{\partial v^{(1)}}{\partial\theta}=0, \tag{2.17}$$

$$0=-\frac{\partial p^{(1)}}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(rS_{rr}^{(1)}\right)+\frac{1}{r}\frac{\partial S_{r\theta}^{(1)}}{\partial\theta}-\frac{S_{\theta\theta}^{(1)}}{r}, \tag{2.18}$$

$$0=-\frac{1}{r}\frac{\partial p^{(1)}}{\partial\theta}+\frac{1}{r^{2}}\frac{\partial}{\partial r}\left(r^{2}S_{r\theta}^{(1)}\right)+\frac{1}{r}\frac{\partial S_{\theta\theta}^{(1)}}{\partial\theta}. \tag{2.19}$$

where, stress tensor are given as follows

$$S_{rr}^{(1)}=2\mu\frac{\partial u^{(1)}}{\partial r}, \tag{2.20}$$

$$S_{r\theta}^{(1)}=S_{\theta r}^{(1)}=\mu\left(\frac{\partial v^{(1)}}{\partial r}+\frac{1}{r}\frac{\partial u^{(1)}}{\partial\theta}-\frac{v^{(1)}}{r}\right), \tag{2.21}$$

$$S_{\theta\theta}^{(1)}=-2\mu\frac{\partial u^{(1)}}{\partial r}. \tag{2.22}$$

To reduce the number of unknown from above equation we introduce stream function $\psi^{(1)}\left(r,\theta\right)$ in the following form

$$u^{(1)}=\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta},\qquad v^{(1)}=-\frac{\partial\psi^{(1)}}{\partial r}. \tag{2.23}$$

Using above relation in Eq.(2.20) – (2.22), and incorporating the Eq.(2.18) – (2.19) one can get the following form of equations

$$0=-\frac{\partial p^{(1)}}{\partial r}+\frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^{2}\psi^{(1)}, \tag{2.24}$$

$$0=-\frac{1}{r}\frac{\partial p^{(1)}}{\partial\theta}-\mu\frac{\partial}{\partial r}\nabla^{2}\psi^{(1)}. \tag{2.25}$$

After eliminating the pressure gradient from above equation by cross differentiation one can get the following expression.

$$\nabla^4\psi^{(1)} = 0. \tag{2.26}$$

B.C's in stream function are given as follows

$$\psi^{(1)} = 0, \qquad \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta} = U \quad \text{at} \quad \theta = 0, \tag{2.27}$$

$$\psi^{(1)} = 0, \qquad \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta} = 0 \quad \text{at} \quad \theta = \theta_w. \tag{2.28}$$

The following solution is assumed as suggested by the Krutchkoff, R. G.[20]

$$\psi^{(1)} = Urf_1(\theta). \tag{2.29}$$

Using Eq.(2.29) in Eq.(2.26) – (2.28), one can get the following BVP

$$f_1^{(iv)}(\theta) + 2f_1''(\theta) + f_1(\theta) = 0, \tag{2.30}$$

with B.C's

$$f_1'(0) = 1, \qquad f_1(0) = 0, \tag{2.31}$$

$$f_1'(\theta_w) = 0, \qquad f_1(\theta_w) = 0. \tag{2.32}$$

The solution of above problem is expressed as follows

$$f_1(\theta) = B\sin\theta + C\theta\cos\theta + D\theta\sin\theta. \tag{2.33}$$

where B, C, D are defined in Appendix.

After using Eq.(2.29) in stream function, one can get following radial and axial components of velocity

$$u^{(1)} = \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta} = Uf_1'(\theta), \tag{2.34}$$

$$v^{(1)} = -\frac{\partial\psi^{(1)}}{\partial r} = -Uf_1(\theta). \tag{2.35}$$

To find the pressure gradient, we require the momentum equation in the following form

$$\frac{\partial p^{(1)}}{\partial r} = \frac{\mu}{r}\frac{\partial}{\partial \theta}\nabla^2\psi^{(1)}, \tag{2.36}$$

$$\frac{\partial p^{(1)}}{\partial \theta} = -\mu r\frac{\partial}{\partial r}\nabla^2\psi^{(1)}. \tag{2.37}$$

Integrating Eq.(2.36) with respect to r and then differentiating the resulting expression with respect to θ, after comparing the expression with Eq.(2.37), one can get following expression of pressure for the first order

$$p^{(1)}(r,\theta) = p_0 - \frac{\mu}{r}U\left(f_1' + f_1'''\right). \tag{2.38}$$

where p_0 is constant.

Finally, we obtain tangential and normal stresses in the following form

$$T_{r\theta}^{(1)} = \frac{\mu}{r}U\left(f_1 + f_1''\right), \tag{2.39}$$

$$T_{rr}^{(1)} = T_{\theta\theta}^{(1)} = -p^{(1)}. \tag{2.40}$$

It is noted that pressure, normal and tangential stresses has singularity at $r = 0$ that can be removed by introducing the different boundary conditions.

2.2.2 Second Order Problem and its Solution

Second order problem represents the creeping flow of second grade fluid also it involves the parameter of second grade fluid.

The momentum equation for the second order takes the following form

$$0 = -\frac{\partial p^{(2)}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(rS_{rr}^{(2)}\right) + \frac{1}{r}\frac{\partial S_{r\theta}^{(2)}}{\partial \theta} - \frac{S_{\theta\theta}^{(2)}}{r}, \tag{2.41}$$

$$0 = -\frac{1}{r}\frac{\partial p^{(2)}}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 S_{r\theta}^{(2)}\right) + \frac{1}{r}\frac{\partial S_{\theta\theta}^{(2)}}{\partial \theta}, \tag{2.42}$$

where stress tensor components of the second order are

$$S_{rr}^{(2)} = 2\mu\frac{\partial u^{(2)}}{\partial r} + \frac{2\alpha_1 U^2}{r^2}\left(f_1 + f_1''\right)^2 + \frac{\alpha_2 U^2}{r^2}\left(f_1 + f_1''\right)^2, \tag{2.43}$$

$$S_{r\theta}^{(2)} = S_{\theta r}^{(2)} = \mu\left(\frac{\partial v^{(2)}}{\partial r} + \frac{1}{r}\frac{\partial u^{(2)}}{\partial \theta} - \frac{v^{(2)}}{r}\right) - \frac{\alpha_1 U^2}{r^2}\left(f_1^{'}f_1^{''} + 2f_1 f_1^{'} + f_1 f_1^{'''}\right), \tag{2.44}$$

$$S_{\theta\theta}^{(2)} = -2\mu\frac{\partial u^{(2)}}{\partial r} + \frac{\alpha_2 U^2}{r^2}\left((f_1)^2 + \left(f_1^{''}\right)^2\right). \tag{2.45}$$

To reduce the unknown function, introduce the following stream function $\psi^{(2)}(r,\theta)$

$$u^{(2)} = \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta}, \qquad v^{(2)} = -\frac{\partial \psi^{(2)}}{\partial r}. \tag{2.46}$$

Using stress tensor components and stream function for second order, Eq.(2.41) – (2.42) take the following form

$$\begin{aligned}\frac{\partial p^{(2)}}{\partial r} \;=\;& \frac{\mu}{r}\frac{\partial}{\partial \theta}\nabla^2\psi^{(2)} - \frac{\alpha_1 U^2}{r^3}\left(2(f_1)^2 + 3\left(f_1^{''}\right)^2 + 6f_1 f_1^{''} + 2\left(f_1^{'}\right)^2 + 2f_1^{'}f_1^{'''} + f_1 f_1^{iv}\right)\\ & - \frac{\alpha_2 U^2}{r^3}\left(2(f_1)^2 + 2\left(f_1^{''}\right)^2 + 2f_1 f_1^{''}\right),\end{aligned} \tag{2.47}$$

$$\frac{1}{r}\frac{\partial p^{(2)}}{\partial \theta} = -\mu\frac{\partial}{\partial r}\nabla^2\psi^{(2)} + \frac{\alpha_2 U^2}{r^3}\left(2f_1 f_1^{'} + 2f_1^{''}f_1^{'''}\right). \tag{2.48}$$

After eliminating pressure gradient, one can get following system of BVP

$$\mu\nabla^4\psi^{(2)} = \frac{1}{r^4}\left[\left(E_1 + E_3\theta + E_5\theta^2\right)\sin 2\theta + \left(E_2 + E_4\theta + E_6\theta^2\right)\cos 2\theta\right], \tag{2.49}$$

and B.C's in stream function take the following form

$$\psi^{(2)} = 0, \quad \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \tag{2.50}$$

$$\psi^{(2)} = 0, \quad \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta} = 0 \quad \text{at} \quad \theta = \theta_w. \tag{2.51}$$

Where constants E_j, $j = 1...8.$ are defined in Appendix.

To find the second order stream function following form of function is assumed

$$\psi^{(2)} = \frac{f_2(\theta)}{\mu}. \tag{2.52}$$

Using Eq.(2.52), in Eq.(2.49) −Eq.(2.51) ,following BVP can be obtained

$$f_2^{iv} + 4f_2^{''} = \left(E_1 + E_3\theta + E_5\theta^2\right)\sin 2\theta + \left(E_2 + E_4\theta + E_6\theta^2\right)\cos 2\theta + E_7\theta + E_8, \tag{2.53}$$

with B.C's

$$f_2 = 0, \qquad f_2^{'} = 0 \qquad \text{at} \qquad \theta = 0, \theta = \theta_w. \tag{2.54}$$

The solution of above BVP is mentioned as follows:

$$\begin{aligned} f_2(\theta) &= U_1 + U_2\theta + \left(U_3 + \gamma_1\theta + \gamma_3\theta^2 + \gamma_5\theta^3\right)\cos 2\theta + \left(U_4 + \gamma_2\theta + \gamma_4\theta^2\right. \\ &\quad \left. + \gamma_6\theta^3\right)\sin 2\theta + \gamma_7\theta^3 + \gamma_8\theta^4. \end{aligned} \tag{2.55}$$

where, γ_i , $E_i$and U_j are defined in Appendix. Here, $i = 1...8$, $j = 1...4$.

Now, radial and axial velocity with help of:

$$u^{(2)} = \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta} = \frac{1}{\mu r} f_2^{'}, \tag{2.56}$$

$$v^{(2)} = -\frac{\partial \psi^{(2)}}{\partial r} = 0. \tag{2.57}$$

To calculate expression of second order pressure, integrating Eq.(2.41) with respect to r and then differentiating the resulting expression with respect to θ, after comparing the expression with Eq.(2.42), one can get the expression of pressure for the second order as follows

$$p^{(2)}(r,\theta) = -\frac{1}{2r^2} f_2^{'''} + \frac{U^2\alpha_2}{2r^2}\left((f_1)^2 + \left(f_1^{''}\right)^2\right). \tag{2.58}$$

Finally, we obtain tangential and normal stresses as follows

$$T_{r\theta}^{(2)} = \frac{1}{r^2} f_2^{''} - \frac{\alpha_1 U^2}{r^2}\left(f_1^{'} f_1^{''} + 2 f_1 f_1^{'} + f_1 f_1^{'''}\right), \tag{2.59}$$

$$T_{rr}^{(2)} = -p^{(2)} - \frac{2}{r^2} f_2^{'} + \frac{(2\alpha_1 + \alpha_2)U^2}{r^2}\left(f_1 + f_1^{''}\right)^2, \tag{2.60}$$

$$T_{\theta\theta}^{(2)} = -p^{(2)} + \frac{2}{r^2} f_2' + \frac{\alpha_2 U^2}{r^2} \left((f_1)^2 + \left(f_1'' \right)^2 \right). \tag{2.61}$$

2.2.3 Third Order Problem and its Solution

The momentum equation for the third order takes the following form

$$0 = -\frac{\partial p^{(3)}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(r S_{rr}^{(3)} \right) + \frac{1}{r}\frac{\partial S_{r\theta}^{(3)}}{\partial \theta} - \frac{S_{\theta\theta}^{(3)}}{r}, \tag{2.62}$$

$$0 = -\frac{1}{r}\frac{\partial p^{(3)}}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 S_{r\theta}^{(3)} \right) + \frac{1}{r}\frac{\partial S_{\theta\theta}^{(3)}}{\partial \theta}, \tag{2.63}$$

where stress tensor components of the third order are

$$S_{rr}^{(3)} = 2\mu \frac{\partial u^{(3)}}{\partial r} + \frac{2\alpha_1 U}{\mu r^3}\left(3 f_1 f_2'' + 2 f_1'' f_2'' \right) + \frac{2\alpha_2 U}{\mu r^3}\left(f_1 f_2'' + f_1'' f_2'' \right), \tag{2.64}$$

$$S_{r\theta}^{(3)} = S_{\theta r}^{(3)} = \mu \left(\frac{\partial v^{(3)}}{\partial r} + \frac{1}{r}\frac{\partial u^{(3)}}{\partial \theta} - \frac{v^{(3)}}{r} \right) + \frac{\alpha_1 U}{\mu r^3}\left(-2 f_1' f_2'' + 2 f_1'' f_2' + f_1 f_2' - f_1 f_2''' \right), \tag{2.65}$$

$$S_{\theta\theta}^{(3)} = -2\mu \frac{\partial u^{(3)}}{\partial r} + \frac{\alpha_1 U}{\mu r^3}\left(-4 f_1' f_2' - 2 f_1 f_2'' \right) + \frac{\alpha_2 U}{\mu r^3}\left(2 f_1'' f_2'' + 2 f_1 f_2'' \right). \tag{2.66}$$

To reduce the unknown function, introduce the following stream function $\psi^{(3)}(r, \theta)$

$$u^{(3)} = \frac{1}{r}\frac{\partial \psi^{(3)}}{\partial \theta}, \qquad v^{(3)} = -\frac{\partial \psi^{(3)}}{\partial r}. \tag{2.67}$$

Using stress tensor components and stream function for third order, Eq.(2.62) – (2.63) take the following form

$$\begin{aligned} \frac{\partial p^{(3)}}{\partial r} &= \frac{\mu}{r}\frac{\partial}{\partial \theta}\nabla^2 \psi^{(3)} + \frac{\alpha_1 U}{\mu r^3}\left(-9 f_1 f_2'' - 9 f_1'' f_2'' - 3 f_1' f_2''' + f_1''' f_2' + 5 f_1' f_2' - f_1 f_2^{iv} \right) \\ &\quad + \frac{\alpha_2 U}{\mu r^3}\left(-6 f_1'' f_2'' - 6 f_1 f_2'' \right), \end{aligned} \tag{2.68}$$

$$\begin{aligned} \frac{1}{r}\frac{\partial p^{(3)}}{\partial \theta} &= -\mu \frac{\partial}{\partial r}\nabla^2 \psi^{(3)} + \frac{\alpha_1 U}{\mu r^4}\left(-4 f_1' f_2'' - 5 f_1'' f_2' - f_1 f_2' - f_1 f_2''' \right) \\ &\quad + \frac{\alpha_2 U}{\mu r^4}\left(2 f_1'' f_2''' + 2 f_1''' f_2'' + 2 f_1' f_2'' + 2 f_1 f_2''' \right). \end{aligned} \tag{2.69}$$

After eliminating pressure gradient, one can get following system of BVP

$$\mu\nabla^4\psi^{(3)} = \frac{\alpha_1 U}{\mu r^5}\begin{pmatrix} 16f_1^{'}f_2^{''} + 10f_1^{''}f_2^{'} + 3f_1f_2^{'} + 12f_1f_2^{'''} + 8f_1^{'''}f_2^{''} \\ +12f_1^{''}f_2^{'''} + 4f_1^{'}f_2^{iv} - f_1^{iv}f_2^{'} + f_1f_2^{v} \end{pmatrix}, \tag{2.70}$$

and B.C's in stream function take the following form

$$\psi^{(3)} = 0, \quad \frac{1}{r}\frac{\partial\psi^{(3)}}{\partial\theta} = 0 \quad \text{at} \quad \theta = 0, \tag{2.71}$$

$$\psi^{(3)} = 0, \quad \frac{1}{r}\frac{\partial\psi^{(3)}}{\partial\theta} = 0 \quad \text{at} \quad \theta = \theta_w. \tag{2.72}$$

To find the third order stream function following form of function is assumed

$$\psi^{(3)} = \frac{f_3(\theta)}{\mu^2 r}. \tag{2.73}$$

Using Eq.(2.73), in Eq.(2.70) −Eq.(2.72) ,following BVP can be obtained

$$\begin{aligned} f_3^{iv} + 10f_3^{''} + 9f_3 &= \left(M_1 + M_2\theta + M_3\theta^2 + M_4\theta^3 + M_5\theta^4\right)\cos\theta \\ &+ \left(M_6 + M_7\theta + M_8\theta^2 + M_9\theta^3 + M_{10}\theta^4\right)\cos 3\theta \\ &+ \left(M_{11} + M_{12}\theta + M_{13}\theta^2 + M_{14}\theta^3 + M_{15}\theta^4\right)\cos 2\theta\sin\theta, \end{aligned} \tag{2.74}$$

with B.C's

$$f_3 = 0, \quad f_3^{'} = 0 \quad \text{at} \quad \theta = 0, \theta = \theta_w. \tag{2.75}$$

The solution of above BVP is mentioned as follows:

$$\begin{aligned} f_3(\theta) &= \left(Q_4 + N_1\theta + N_2\theta^2 + N_3\theta^3 + N_4\theta^4 + N_5\theta^5\right)\sin\theta \\ &+ \left(Q_2 + N_6\theta + N_7\theta^2 + N_8\theta^3 + N_9\theta^4 + N_{10}\theta^5\right)\sin 3\theta \\ &+ \left(Q_3 + N_{11}\theta + N_{12}\theta^2 + N_{13}\theta^3 + N_{14}\theta^4 + N_{15}\theta^5\right)\cos\theta \\ &+ \left(Q_1 + N_{16}\theta + N_{17}\theta^2 + N_{18}\theta^3 + N_{19}\theta^4 + N_{20}\theta^5\right)\cos 3\theta. \end{aligned} \tag{2.76}$$

where, $N_k$and Q_j are defined in Appendix. Here, $k = 1...20$, $j = 1...4$.

Now, radial and axial velocity components are :

$$u^{(3)} = \frac{1}{r}\frac{\partial \psi^{(3)}}{\partial \theta} = \frac{1}{\mu^2 r^2} f_3^{'}, \tag{2.77}$$

$$v^{(3)} = -\frac{\partial \psi^{(3)}}{\partial r} = \frac{1}{\mu^2 r^2} f_3. \tag{2.78}$$

To calculate expression of second order pressure, integrating Eq.(2.68) with respect to r and then differentiating the resulting expression with respect to θ, after comparing the expression with Eq.(2.69), one can get the expression of pressure for the second order as follows

$$\begin{aligned} p^{(3)}(r,\theta) \quad = \quad & -\frac{1}{\mu r^3}\left(f_3^{'} + f_3^{''}\right) - \frac{\alpha_1 U}{3\mu r^3}\left(-9 f_1 f_2^{''} - 9 f_1^{''} f_2^{''} - 3 f_1^{'} f_2^{'''} + f_1^{'''} f_2^{'}\right. \\ & \left. +5 f_1^{'} f_2^{'} - f_1 f_2^{iv}\right) - \frac{\alpha_2 U}{3\mu r^3}\left(-6 f_1^{''} f_2^{''} - 6 f_1 f_2^{''}\right). \end{aligned} \tag{2.79}$$

Finally, we obtain tangential and normal stresses as follows

$$T_{r\theta}^{(3)} = \frac{1}{\mu r^3}\left(f_3^{''} - 3 f_3\right) + \frac{\alpha_1 U}{\mu r^3}\left(-2 f_1^{'} f_2^{''} + 2 f_1^{''} f_2^{'} + f_1 f_2^{'} - f_1 f_2^{'''}\right), \tag{2.80}$$

$$T_{rr}^{(3)} = -p^{(3)} - \frac{4}{\mu r^3} f_3^{'} + \frac{2\alpha_1 U}{\mu r^3}\left(3 f_1 f_2^{''} + 2 f_1^{''} f_2^{''}\right) + \frac{2\alpha_2 U}{\mu r^3}\left(f_1 f_2^{''} + f_1^{''} f_2^{''}\right), \tag{2.81}$$

$$T_{\theta\theta}^{(3)} = -p^{(3)} + \frac{4}{\mu r^3} f_3^{'} + \frac{\alpha_1 U}{\mu r^3}\left(-4 f_1^{'} f_2^{'} - 2 f_1 f_2^{''}\right) + \frac{\alpha_2 U}{\mu r^3}\left(2 f_1^{''} f_2^{''} + 2 f_1 f_2^{''}\right). \tag{2.82}$$

By merging first, second and third order solutions, one can get the stream function, velocity profile, pressure distribution and stress components in the following form:

$$\mathbf{V}(r,\theta) = \mathbf{V}^{(1)} + \mathbf{V}^{(2)} + \mathbf{V}^{(3)},$$

$$p(r,\theta) = const + p^{(1)} + p^{(2)} + p^{(3)},$$

$$\psi(r,\theta) = \psi^{(1)} + \psi^{(2)} + \psi^{(3)},$$

$$S(r,\theta) = S^{(1)} + S^{(2)} + S^{(3)}.$$

Following the recursive approach, and using values from previous sections, one can get

$$\psi = Urf_1(\theta) + \frac{f_2(\theta)}{\mu} + \frac{f_3(\theta)}{\mu^2 r}, \tag{2.83}$$

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = Uf_1^{'} + \frac{1}{\mu r}f_2^{'} + \frac{1}{\mu^2 r^2}f_3^{'}, \tag{2.84}$$

$$v = -\frac{\partial\psi}{\partial r} = -Uf_1 + \frac{1}{\mu^2 r^2}f_3, \tag{2.85}$$

$$\begin{aligned} p(r,\theta) \quad = \quad & p_0 - \frac{\mu}{r}U\left(f_1^{'} + f_1^{'''}\right) - \frac{1}{2r^2}f_2^{'''} + \frac{U^2\alpha_2}{2r^2}\left((f_1)^2 + \left(f_1^{''}\right)^2\right) \\ & -\frac{1}{\mu r^3}\left(f_3^{'} + f_3^{''}\right) - \frac{\alpha_1 U}{3\mu r^3}\left(-9f_1 f_2^{''} - 9f_1^{''}f_2^{''} - 3f_1^{'}f_2^{'''}\right. \\ & \left.+f_1^{'''}f_2^{'} + 5f_1^{'}f_2^{'} - f_1 f_2^{iv}\right) - \frac{\alpha_2 U}{3\mu r^3}\left(-6f_1^{''}f_2^{''} - 6f_1 f_2^{''}\right), \end{aligned} \tag{2.86}$$

$$\begin{aligned} T_{r\theta} \quad = \quad & \frac{\mu}{r}U\left(f_1 + f_1^{''}\right) + \frac{1}{r^2}f_2^{''} - \frac{\alpha_1 U^2}{r^2}\left(f_1^{'}f_1^{''} + 2f_1 f_1^{'} + f_1 f_1^{'''}\right) \\ & +\frac{1}{\mu r^3}\left(f_3^{''} - 3f_3\right) + \frac{\alpha_1 U}{\mu r^3}\left(-2f_1^{'}f_2^{''} + 2f_1^{''}f_2^{'} + f_1 f_2^{'} - f_1 f_2^{'''}\right), \end{aligned} \tag{2.87}$$

$$T_{rr} \quad = \quad -p - \frac{2}{r^2}f_2^{'} + \frac{(2\alpha_1 + \alpha_2)U^2}{r^2}\left(f_1 + f_1^{''}\right)^2 - \frac{4}{\mu r^3}f_3^{'} \tag{2.88}$$

$$+\frac{2\alpha_1 U}{\mu r^3}\left(3f_1 f_2^{''} + 2f_1^{''}f_2^{''}\right) + \frac{2\alpha_2 U}{\mu r^3}\left(f_1 f_2^{''} + f_1^{''}f_2^{''}\right), \tag{2.89}$$

$$T_{\theta\theta} \quad = \quad -p + \frac{2}{r^2}f_2^{'} + \frac{\alpha_2 U^2}{r^2}\left((f_1)^2 + \left(f_1^{''}\right)^2\right) + \frac{4}{\mu r^3}f_3^{'} \tag{2.90}$$

$$+\frac{2\alpha_1 U}{\mu r^3}\left(-2f_1^{'}f_2^{'} - f_1 f_2^{''}\right) + \frac{2\alpha_2 U}{\mu r^3}\left(f_1^{''}f_2^{''} + f_1 f_2^{''}\right). \tag{2.91}$$

2.3 Graphs and Discussion

2.3.1 Velocity Field

Figures 2.2$(a-c)$ show the behavior of velocity for the variations of the involved parameters. It can be observed from figures 2.2 that the velocity of the fluid rises when the velocity of the plate and second grade viscoelastic parameter increases. Because the plate velocity reduces the friction and causing to increase the flow in radial direction, also the viscoelasticity help to reduce the resistance in the fluid causing to increase the flow. The fluid flow in reverse direction with the growing values of second grade parameter when the contact angle is $[0, \pi/4]$ and$[\pi/2, 2\pi/3]$. The plate velocity causes to escalate the angular velocity in clock wise direction that can be observed due to negative values of velocity.

2.3.2 Pressure Distribution

Pressure near a corner varies due to variations in different parameters which is shown graphically in figures 2.4. The effect of second grade parameter and plate velocity on pressure is shown in figure 2.4 which explains that viscoelastic nature of the fluid and plate velocity help to build the more pressure in the region $[\pi/4, \pi/2]$ but due to high speed and sharp corner the pressure drops in the region $[0, \pi/4]$ and $[\pi/2, 2\pi/3]$.

2.3.3 Normal and Tangential Stresses

Figures 2.5 explain the graphical behavior of normal and tangential stress to the plate against different parameters. The impact of viscoelastic fluid parameter on normal and tangential stress shows that when viscoelastic fluid parameter increases then viscoelastic forces become dominant near a corner and rate of deformation become high which results to increase the stress near a corner which leads to a rise in normal and tangential stress. Figures 2.5 demonstrate the effect of plate speed U on normal and tangential stress which explains that the velocity gradient enhances as the plate moves with high speed so the shear rate increases and as a result normal and tangential stress also rises.

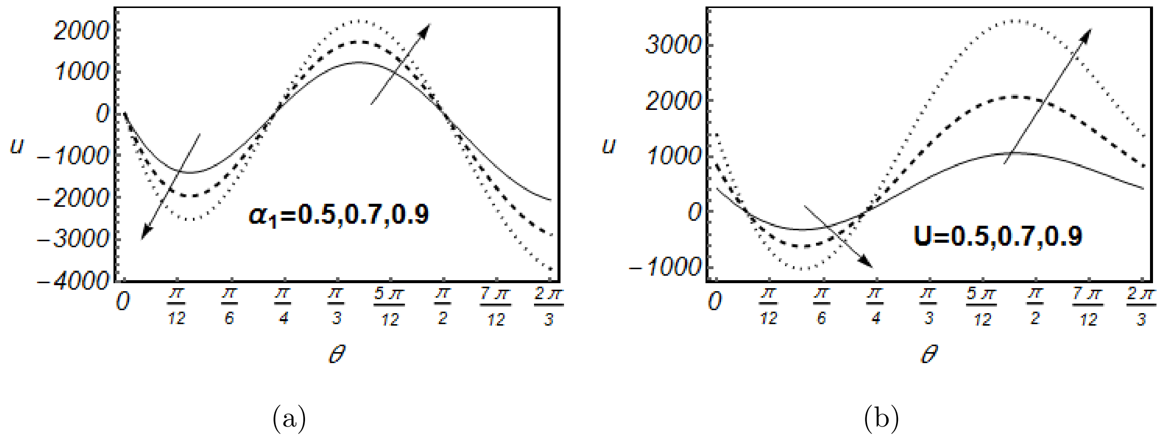


Fig 2.3(a − b) : Fluctuations in velocity u with respect to α_1 and U.

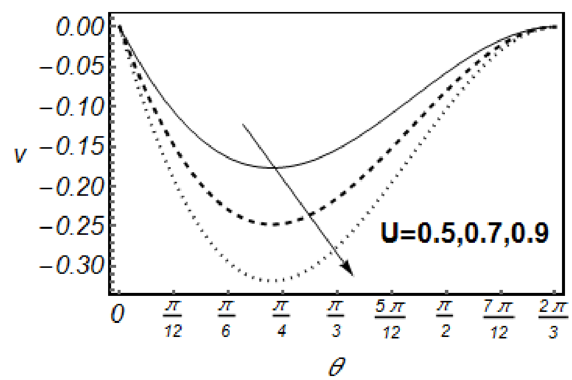


Fig 2.4 : Fluctuations in velocity v with respect to U.

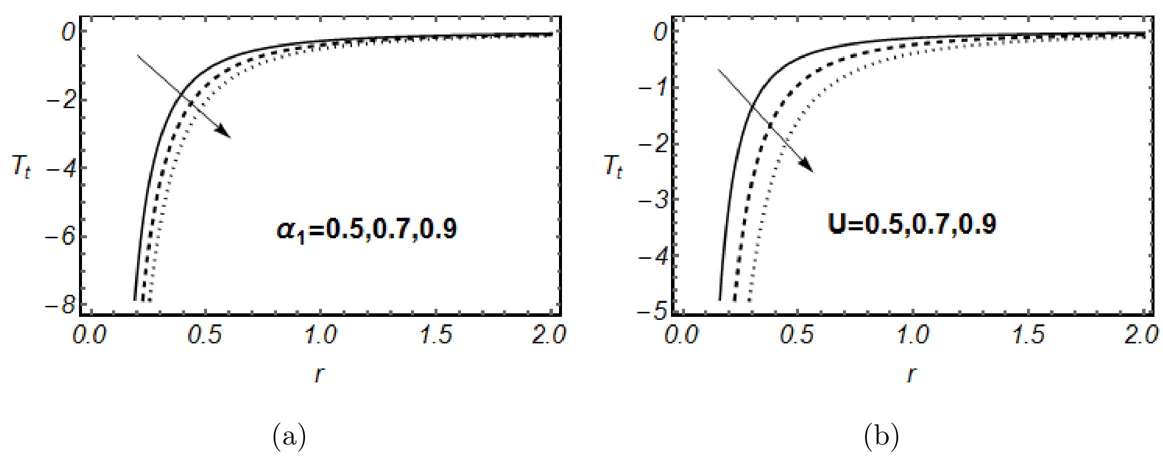


Fig 2.5(a − b) :Variation in tangential stress T_t for $\alpha_1$and U.

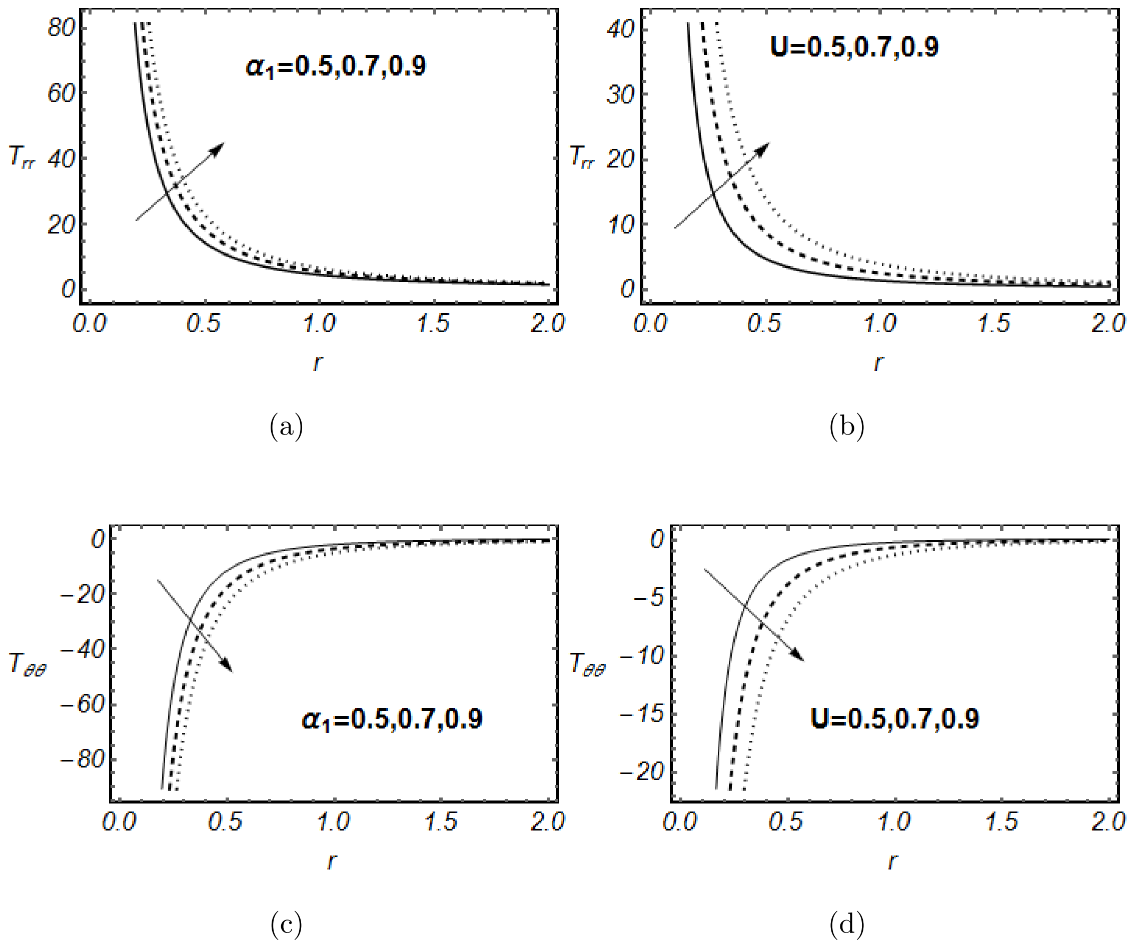


Fig 2.6$(\mathbf{a}-\mathbf{d})$:Variation in normal stresses T_{rr} and $T_{\theta\theta}$ along $\alpha_1$and U.

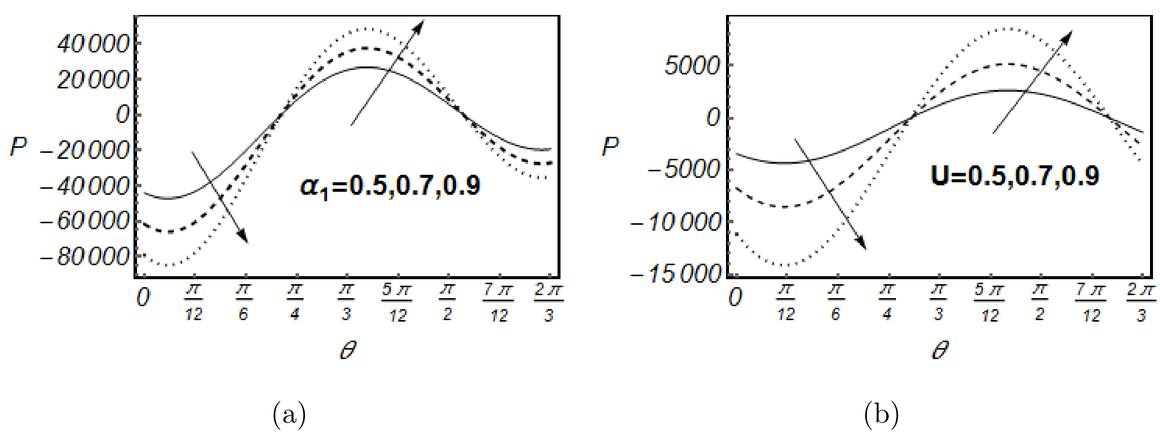


Fig 2.7$(\mathbf{a}-\mathbf{b})$:Variation in pressure P along $\alpha_1$and U.

Chapter 3

Inertial and leakage effects on viscous fluid flow near a corner

This chapter deals with two dimensional , in-compressible flow of second grade fluid between two intersecting walls through a specific angle. Inertial and leakage effects which are already discussed by Riedler and Hancock [21, 22] for Newtonian fluid but in this chapter, we will discuss these effects simultaneously for second grade fluid near the corner. Mathematical model of the problem provides a complex system of non-linear partial differential equations which are solved using Langloi's approach. The resulting boundary value problems are solved by inverse method. Mathematical results for velocity, pressure, stream function, normal and tangential stresses are found in explicit form and displayed the impact of emerging parameters through graphs.

3.1 Governing Equations

Consider two dimensional , incompressible, second grade fluid flow near a corner between two rigid walls that intersect at a constant angle θ_w. Assume that the liquid is moving with wall velocity εU at $\theta = 0$ but fluid and walls are at rest at $\theta = \theta_w$. We further assume a mass source(or sink) of strength εQ at the apex corner due to leakage.

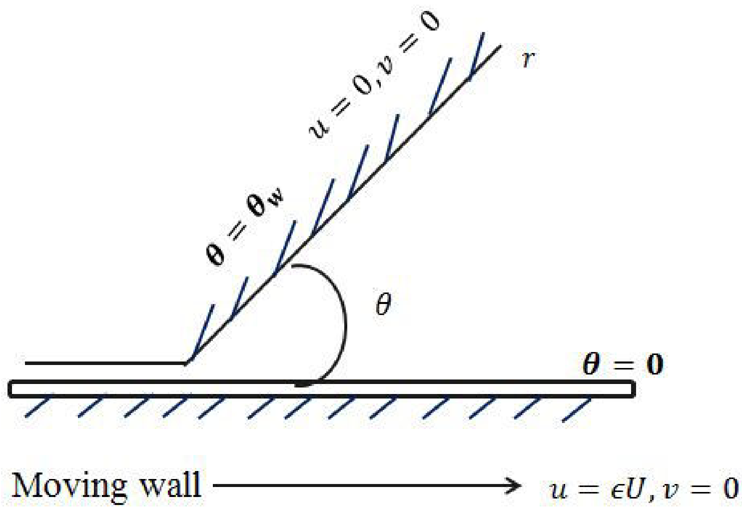


Fig 3.1: Corner flow geometry

The governing equations of motion for an incompressible, steady state, second grade fluid, in the absence of body forces are as follows:

$$\nabla.\mathbf{V} = 0, \tag{3.1}$$

$$\rho\left(\nabla.\mathbf{V}\right)\mathbf{V} = \operatorname{div}\mathbf{T}, \tag{3.2}$$

where $\mathbf{T}$ is briefly explained in Eq.$(2.3)-(2.6)$.

For steady plane flow in polar coordinates, one can choose following velocity and pressure fields.

$$\mathbf{V} = u\left(r,\theta\right)e_r + v\left(r,\theta\right)e_\theta,\ \ p = p\left(r,\theta\right). \tag{3.3}$$

After using tensor and vector properties Eq.(3.2) can be written in the following form:

$$\rho\left(\nabla.\mathbf{V}\right)\mathbf{V} = -\nabla p+\mu\operatorname{div}\mathbf{A}_1+\alpha_1\left(\operatorname{div}\left(\left(\mathbf{V}\cdot\nabla\right)\mathbf{A}_1\right)+\operatorname{div}\left(\mathbf{A}_1\left(\nabla\mathbf{V}\right)\right)\right.$$
$$\left.+\operatorname{div}\left(\left(\nabla\mathbf{V}\right)^T\mathbf{A}_1\right)\right)+\alpha_2\operatorname{div}\left(\mathbf{A}_1\right)^2. \tag{3.4}$$

The following properties can be used for further simplification:

$$\operatorname{div}\left(\left(\mathbf{V}\cdot\nabla\right)\mathbf{A}_1\right)=\left(V\cdot\nabla\right)\operatorname{div}\mathbf{A}_1+\operatorname{div}\left(\mathbf{A}_1\left(\nabla\mathbf{V}\right)^T\right), \tag{3.5}$$

$$\operatorname{div}\left(\mathbf{A}_1\left(\nabla\mathbf{V}\right)\right)=-\operatorname{div}\left(\mathbf{A}_1\left(\nabla\mathbf{V}\right)^T\right)+\operatorname{div}\left(\left(\mathbf{A}_1\right)^2\right), \tag{3.6}$$

$$\operatorname{div}\left(\left(\nabla\mathbf{V}\right)^T\mathbf{A}_1\right)=\left(\nabla\mathbf{V}\right)^T\operatorname{div}\mathbf{A}_1+\mathbf{A}_1\left(\nabla\left(\nabla\mathbf{V}\right)^T\right), \tag{3.7}$$

$$\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}=\frac{\nabla\left|\mathbf{V}\right|^2}{2}-\mathbf{V}\times\left(\nabla\times\mathbf{V}\right). \tag{3.8}$$

After using the above identities, Eq.(3.4) takes the following form:

$$\rho\left(\frac{\nabla\left|\mathbf{V}\right|^2}{2}-\mathbf{V}\times\left(\nabla\times\mathbf{V}\right)\right) = -\nabla p+\mu\operatorname{div}\mathbf{A}_1+\alpha_1\left(\left(V\cdot\nabla\right)\operatorname{div}\mathbf{A}_1\right.$$
$$\left.+\left(\nabla\mathbf{V}\right)^T\operatorname{div}\mathbf{A}_1+\mathbf{A}_1\left(\nabla\left(\nabla\mathbf{V}\right)^T\right)\right)$$
$$+\left(\alpha_1+\alpha_2\right)\operatorname{div}\left(\mathbf{A}_1\right)^2. \tag{3.9}$$

Now, component form of above equation can be written as follows:

$r-$ component of inertial part:

$$\rho\left(\frac{\nabla\left|\mathbf{V}\right|^2}{2}-\mathbf{V}\times\left(\nabla\times\mathbf{V}\right)\right)_r=\rho\left(\frac{1}{2}\frac{\partial}{\partial r}\left(u^2+v^2\right)-v\omega\right), \tag{3.10}$$

$\theta-$ component of inertial part:

$$\rho\left(\frac{\nabla\left|\mathbf{V}\right|^2}{2}-\mathbf{V}\times\left(\nabla\times\mathbf{V}\right)\right)_\theta=\rho\left(\frac{1}{2r}\frac{\partial}{\partial\theta}\left(u^2+v^2\right)+u\omega\right), \tag{3.11}$$

where, vorticity function ω is defined as follows:

$$\omega = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{3.12}$$

$r-$ component of dynamic part:

$$\begin{aligned} (\operatorname{div}\mathbf{T})_r &= -\frac{\partial p}{\partial r} - \frac{\mu}{r}\frac{\partial \omega}{\partial \theta} + \frac{(3\alpha_1 + 2\alpha_2)}{4}\frac{\partial}{\partial r}\left(tr\,(\mathbf{A}_1)^2\right) \\ &\quad + \alpha_1\left(-v\nabla^2\omega\right) + \frac{\partial}{\partial r}\left(v\frac{\partial}{\partial r} - \frac{u}{r}\frac{\partial}{\partial \theta}\right)\omega - \frac{1}{2}\frac{\partial}{\partial r}\left(\omega^2\right), \end{aligned} \tag{3.13}$$

$\theta-$ component of dynamic part:

$$\begin{aligned} (\operatorname{div}\mathbf{T})_\theta &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\frac{\partial \omega}{\partial r} + \frac{(3\alpha_1 + 2\alpha_2)}{4}\frac{1}{r}\frac{\partial}{\partial \theta}\left(tr\,(\mathbf{A}_1)^2\right) \\ &\quad + \alpha_1\left(u\nabla^2\omega\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(v\frac{\partial}{\partial r} - \frac{u}{r}\frac{\partial}{\partial \theta}\right)\omega - \frac{1}{2r}\frac{\partial}{\partial \theta}\left(\omega^2\right). \end{aligned} \tag{3.14}$$

After using Eq.(3.10) – (3.14) , momentum equation (3.9) takes the following form:

$r-$component of momentum equation

$$\begin{aligned} \rho\left(\frac{1}{2}\frac{\partial}{\partial r}\left(u^2 + v^2\right) - v\omega\right) &= -\frac{\partial p}{\partial r} - \frac{\mu}{r}\frac{\partial \omega}{\partial \theta} + \frac{(3\alpha_1 + 2\alpha_2)}{4}\frac{\partial}{\partial r}\left(tr\,(\mathbf{A}_1)^2\right) \\ &\quad + \alpha_1\left(-v\nabla^2\omega\right) + \frac{\partial}{\partial r}\left(v\frac{\partial}{\partial r} - \frac{u}{r}\frac{\partial}{\partial \theta}\right)\omega - \frac{1}{2}\frac{\partial}{\partial r}\left(\omega^2\right), \end{aligned} \tag{3.15}$$

or

$$\begin{aligned} &\frac{\partial}{\partial r}\left(\frac{\rho}{2}\left(u^2 + v^2\right) + p - \frac{(3\alpha_1 + 2\alpha_2)}{4}\left(tr\,(\mathbf{A}_1)^2\right) - \alpha_1\frac{\partial}{\partial r}\left(v\frac{\partial}{\partial r} - \frac{u}{r}\frac{\partial}{\partial \theta}\right)\omega + \alpha_1\frac{1}{2}\left(\omega^2\right)\right) \\ &= \rho v\omega - \frac{\mu}{r}\frac{\partial \omega}{\partial \theta} + \alpha_1\left(-v\nabla^2\omega\right), \end{aligned} \tag{3.16}$$

$\theta-$ component of momentum equation

$$\begin{aligned}\rho\left(\frac{1}{2r}\frac{\partial}{\partial\theta}\left(u^2+v^2\right)+u\omega\right) &= -\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\frac{\partial\omega}{\partial\theta}+\frac{(3\alpha_1+2\alpha_2)}{4}\frac{1}{r}\frac{\partial}{\partial\theta}\left(tr\left(\mathbf{A}_1\right)^2\right)+\alpha_1\left(u\nabla^2\omega\right) \\ &\quad +\frac{1}{r}\frac{\partial}{\partial\theta}\left(v\frac{\partial}{\partial r}-\frac{u}{r}\frac{\partial}{\partial\theta}\right)\omega-\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\omega^2\right), \qquad (3.17)\end{aligned}$$

or

$$\begin{aligned}&\frac{\partial}{\partial r}\left(\frac{\rho}{2}\left(u^2+v^2\right)+p-\frac{(3\alpha_1+2\alpha_2)}{4}\left(tr\left(\mathbf{A}_1\right)^2\right)-\alpha_1\frac{\partial}{\partial r}\left(v\frac{\partial}{\partial r}-\frac{u}{r}\frac{\partial}{\partial\theta}\right)\omega+\alpha_1\frac{1}{2}\left(\omega^2\right)\right) \\ = \;& -\rho u\omega+\mu\frac{\partial\omega}{\partial r}+\alpha_1\left(u\nabla^2\omega\right). \qquad (3.18)\end{aligned}$$

Now, from Eq.(3.16) and Eq.(3.18), we will define following modified pressure $h(r,\theta)$

$$h(r,\theta)=\frac{\rho}{2}\left(u^2+v^2\right)-\frac{(3\alpha_1+2\alpha_2)}{4}\left(trA_1^2\right)-\alpha_1\left(v\frac{\partial}{\partial r}-\frac{u}{r}\frac{\partial}{\partial\theta}\right)\omega+p(r,\theta)+\frac{\alpha_1}{2}\omega^2, \quad (3.19)$$

where,

$$trA_1^2=8\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2. \qquad (3.20)$$

Now using modified pressure $h(r,\theta)$ in Eq.(3.16) and (3.18) one can write the following form

$$\frac{\partial h}{\partial r}-\rho v\omega+\frac{\mu}{r}\frac{\partial\omega}{\partial\theta}+\alpha_1 v\nabla^2\omega=0, \qquad (3.21)$$

$$\frac{1}{r}\frac{\partial h}{\partial\theta}+\rho u\omega-\mu\frac{\partial\omega}{\partial r}-\alpha_1 u\nabla^2\omega=0, \qquad (3.22)$$

where,

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \qquad (3.23)$$

Following non-dimensional quantities are defined for further analysis:

$$\begin{aligned}
r^{'} &= \frac{r}{R},\ u^{'}=\frac{u}{U},\ v^{'}=\frac{v}{U},\ p^{'}=\frac{p}{\rho U^2},\ h^{'}=\frac{h}{\rho U^2}, \\
\alpha_1^{'} &= \frac{\alpha_1 U}{\mu R},\ \alpha_2^{'}=\frac{\alpha_2 U}{\mu R},\ T^{'}=\frac{TR}{\mu U},\ \ \mathrm{Re}=\frac{RU}{v}.
\end{aligned} \tag{3.24}$$

After using the above non-dimensional quantities in Eq (3.21)−(3.22), one can get the following form

$$\frac{\partial h}{\partial r} - v\omega + \frac{1}{\mathrm{Re}}\frac{1}{r}\frac{\partial \omega}{\partial \theta} + \frac{1}{\mathrm{Re}}\alpha_1 v\nabla^2\omega = 0, \tag{3.25}$$

$$\frac{1}{r}\frac{\partial h}{\partial \theta} + u\omega - \frac{1}{\mathrm{Re}}\frac{\partial \omega}{\partial r} - \frac{1}{\mathrm{Re}}\alpha_1 u\nabla^2\omega = 0. \tag{3.26}$$

To reduce the number of unknown functions from above equations, we introduce the following relation of stream function $\psi\left(r,\theta\right)$

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, v = -\frac{\partial \psi}{\partial r}. \tag{3.27}$$

After using stream function, Eq(3.25) − (3.26) take the following form

$$\mathrm{Re}\left(\frac{\partial h}{\partial r} - \frac{\partial \psi}{\partial r}\left(\nabla^2\psi\right)\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\nabla^2\psi\right) + \alpha_1\left(\frac{\partial \psi}{\partial r}\right)\nabla^4\psi = 0, \tag{3.28}$$

$$\mathrm{Re}\left(\frac{1}{r}\frac{\partial h}{\partial \theta} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\left(\nabla^2\psi\right)\right) + \frac{\partial}{\partial r}\left(\nabla^2\psi\right) + \frac{\alpha_1}{r}\left(\frac{\partial \psi}{\partial \theta}\right)\nabla^4\psi = 0, \tag{3.29}$$

after eliminating $h\left(r,\theta\right)$ from Eq.(3.28) and Eq.(3.29), we have

$$-\frac{\mathrm{Re}}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \theta}\left(\nabla^2\psi\right) - \frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\left(\nabla^2\psi\right)\right) - \left(\nabla^4\psi\right) + \frac{\alpha_1}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \theta}\left(\nabla^4\psi\right) - \frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\left(\nabla^4\psi\right)\right) = 0, \tag{3.30}$$

B.C's in stream function are given as follows

$$\psi = 0, \qquad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \varepsilon U \quad \text{at} \quad \theta = 0, \tag{3.31}$$

$$\psi = \varepsilon Q, \qquad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \quad \text{at} \quad \theta = \theta_w. \tag{3.32}$$

3.2 Methodology

To solve the above problem, we will follow the Langiois recursive approach which is briefly explained in chapter. 2

3.2.1 First Order Problem and its Solution

$$\frac{\partial h^{(1)}}{\partial r} - \frac{1}{\mathrm{Re}}\frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2\psi^{(1)} = 0, \tag{3.33}$$

$$\frac{\partial h^{(1)}}{\partial \theta} + \frac{1}{\mathrm{Re}}r\frac{\partial}{\partial r}\nabla^2\psi^{(1)} = 0, \tag{3.34}$$

where,

$$h^{(1)}(r,\theta) = p^{(1)}(r,\theta). \tag{3.35}$$

Eliminating $h^{(1)}(r,\theta)$ from Eq.(3.33) and Eq.(3.34)

$$\nabla^4\psi^{(1)} = 0. \tag{3.36}$$

with B.C's

$$\psi^{(1)} = 0, \quad \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta} = U \quad \text{at} \quad \theta = 0, \tag{3.37}$$

$$\psi^{(1)} = Q, \quad \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta} = 0 \quad at \;\; \theta = \theta_w. \tag{3.38}$$

One can consider $\psi^{(1)}$ as suggested in Ref. [20] in the following form:

$$\psi^{(1)} = Urf_1(\theta) + Qg_1(\theta). \tag{3.39}$$

Using above value of $\psi^{(1)}$, in Eq.(3.36)−(3.38) , one can get the following fourth order boundary value problems in terms of $f_1$and g_1.

$$f_1^{iv} + 2f_1^{''} + f_1 = 0, \tag{3.40}$$

$$f_1 = 0, \quad f_1^{'} = 1 \;\; \text{at} \;\; \theta = 0, \tag{3.41}$$

$$f_1 = 0, \qquad f_1^{'} = 0 \ \text{ at } \ \theta = \theta_w. \tag{3.42}$$

Also,

$$g_1^{iv} + 4g_1^{''} = 0, \tag{3.43}$$

$$g_1 = 0, \qquad g_1^{'} = 0 \ \text{ at } \ \theta = 0, \tag{3.44}$$

$$g_1 = 1, \qquad g_1^{'} = 0 \ \text{ at } \ \theta = \theta_w. \tag{3.45}$$

After a significant amount of effort, solutions of above boundary value problems are obtained and are given as follows:

$$f_1(\theta) = B\sin\theta + C\theta\cos\theta + D\theta\sin\theta, \tag{3.46}$$

and

$$g_1(\theta) = 2K\sin^2\theta + N(\sin 2\theta - 2\theta). \tag{3.47}$$

Where B, C, D, K, N are defined in Appendix.

Now radial and tangential velocity in terms of $f_1(\theta)$, and $g_1(\theta)$ are as follows:

$$u^{(1)} = \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta} = Uf_1^{'} + \frac{Q}{r}g_1^{'}, \tag{3.48}$$

$$v^{(1)} = -\frac{\partial \psi^{(1)}}{\partial r} = -Uf_1. \tag{3.49}$$

Using Eq.(3.33) – (3.35), we obtain pressure field in the following form:

$$p^{(1)}(r,\theta) = p_0 - \frac{1}{\mathrm{Re}}\frac{U}{r}\left(f_1^{'} + f_1^{'''}\right) - \frac{1}{2\,\mathrm{Re}}\frac{Q}{r^2}g_1^{'''}, \tag{3.50}$$

where p_0 is a constant.

Finally, we obtain the normal and tangential stresses in the following form

$$T_{rr}^{(1)} = -\mathrm{Re}\,p^{(1)} - \frac{2}{r^2}Qg_1^{'}, \tag{3.51}$$

$$T_{\theta\theta}^{(1)} = -\mathrm{Re}\,p^{(1)} + \frac{2}{r^2}Qg_1^{'}, \tag{3.52}$$

$$T_{r\theta}^{(1)} = \frac{U}{r}\left(f_1 + f_1^{''}\right) + \frac{Q}{r^2} g_1^{''}, \tag{3.53}$$

3.2.2 Second Order Problem and its Solution

$$\frac{\partial h^{(2)}}{\partial r} - \frac{\partial \psi^{(1)}}{\partial r}\nabla^2\psi^{(1)} - \frac{1}{\mathrm{Re}}\frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2\psi^{(2)} + \frac{1}{\mathrm{Re}}\alpha_1\frac{\partial \psi^{(1)}}{\partial r}\nabla^4\psi^{(1)} = 0, \tag{3.54}$$

$$\frac{1}{r}\frac{\partial h^{(2)}}{\partial \theta} - \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta}\nabla^2\psi^{(1)} + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial r}\nabla^2\psi^{(2)} + \frac{1}{\mathrm{Re}}\frac{\alpha_1}{r}\frac{\partial \psi^{(1)}}{\partial \theta}\nabla^4\psi^{(1)} = 0, \tag{3.55}$$

where,

$$\begin{aligned} h^{(2)}(r,\theta) &= \frac{1}{2}\left(u^{(1)^2} + v^{(1)^2}\right) - \frac{1}{\mathrm{Re}}\left(\frac{3\alpha_1 + 2\alpha_2}{4}\right)\left(trA_1^{(1)2}\right) - \frac{1}{\mathrm{Re}}\alpha_1\left(v^{(1)}\frac{\partial}{\partial r} - \frac{u^{(1)}}{r}\frac{\partial}{\partial \theta}\right)\omega^{(1)} \\ &\quad + p^{(2)}(r,\theta) + \frac{1}{\mathrm{Re}}\frac{\alpha_1}{2}\omega^{(1)2}. \end{aligned} \tag{3.56}$$

Eliminating $h^{(2)}(r,\theta)$ from Eq.(3.54) and Eq.(3.55), we have

$$\nabla^4\psi^{(2)} = \frac{-\mathrm{Re}}{r}\frac{\partial\left(\psi^{(1)}, \nabla^2\psi^{(1)}\right)}{\partial(r,\theta)} + \frac{\alpha_1}{r}\frac{\partial\left(\psi^{(1)}, \nabla^4\psi^{(1)}\right)}{\partial(r,\theta)}, \tag{3.57}$$

with B.C's

$$\psi^{(2)} = 0, \quad \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \tag{3.58}$$

$$\psi^{(2)} = 0, \quad \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta} = 0 \quad \text{at} \quad \theta = \theta_w, \tag{3.59}$$

where,

$$\frac{\partial\left(\psi^{(1)}, \nabla^2\psi^{(1)}\right)}{\partial(r,\theta)} = \frac{\partial \psi^{(1)}}{\partial r}\frac{\partial\left(\nabla^2\psi^{(1)}\right)}{\partial \theta} - \frac{\partial \psi^{(1)}}{\partial \theta}\frac{\partial\left(\nabla^2\psi^{(1)}\right)}{\partial r}. \tag{3.60}$$

The second order system $(3.57)-(3.60)$ contains a fourth order PDE. To change this PDE into a system of ODE, one can consider the stream function $\psi^{(2)}$ in the following form

$$\psi^{(2)}(r,\theta)=\mathrm{Re}\left(r^2U^2f_2(\theta)+rUQf_3(\theta)+Q^2f_4(\theta)\right)+\alpha_1\left(U^2g_2(\theta)+\frac{UQ}{r}g_3(\theta)+\frac{Q^2}{r^2}g_4(\theta)\right). \tag{3.61}$$

Using Eqs.(3.39) and (3.61) in Eq.(3.57) – (3.60) ,we get the following fourth order boundary value problems in terms of f_2, f_3, f_4, g_2, g_3 and g_4 .

$$f_2^{iv}+4f_2^{''}=(\beta_1+\beta_3\theta)\sin 2\theta+(\beta_2+\beta_4\theta)\cos 2\theta, \tag{3.62}$$

$$f_2=0, \qquad f_2^{'}=0 \qquad \text{at} \qquad \theta=0,\theta=\theta_w, \tag{3.63}$$

$$f_3^{iv}+2f_3^{''}+f_3=(T_1+T_5\theta)\cos 3\theta+(T_2+T_6\theta)\sin 3\theta+T_3\cos\theta+T_4\sin\theta, \tag{3.64}$$

$$f_3=0, \qquad f_3^{'}=0 \qquad \text{at} \qquad \theta=0,\theta=\theta_w, \tag{3.65}$$

$$g_2^{iv}+4g_2^{''}=0, \tag{3.66}$$

$$g_2=0, \qquad g_2^{'}=0 \qquad \text{at} \qquad \theta=0,\theta=\theta_w, \tag{3.67}$$

$$g_3^{iv}+10g_3^{''}+9g_3=0, \tag{3.68}$$

$$g_3=0, \qquad g_3^{'}=0 \qquad \text{at} \qquad \theta=0,\theta=\theta_w, \tag{3.69}$$

$$g_4^{iv}+20g_4^{''}+64g_4=0, \tag{3.70}$$

$$g_4=0, \qquad g_4^{'}=0 \qquad \text{at} \qquad \theta=0,\theta=\theta_w. \tag{3.71}$$

After a considerable amount of work, solutions of boundary value problems (3.62) – (3.65) are represented as follows:

$$f_2(\theta)=R_1+R_2\theta+\left(R_3+\beta_5\theta+\beta_7\theta^2\right)\cos 2\theta+\left(R_4+\beta_6\theta+\beta_8\theta^2\right)\sin 2\theta, \tag{3.72}$$

$$\begin{aligned} f_3(\theta) \quad = \quad & \left(A_1+A_3\theta+B_3\theta^2\right)\cos\theta+\left(A_2+A_4\theta+B_4\theta^2\right)\sin\theta+(B_1+B_5\theta)\cos 3\theta \\ & +(B_2+B_6\theta)\sin 3\theta, \end{aligned} \tag{3.73}$$

$$f_4(\theta) = D_1 + D_1\theta + (D_3 + F_1\theta)\cos 2\theta + (D_4 + F_2\theta)\sin 2\theta + F_3\cos 4\theta + F_4\sin 4\theta. \tag{3.74}$$

As in boundary value problem (3.65) – (3.71) , equations and boundary conditions both are homogeneous,so $g_2(\theta) = g_3(\theta) = g_4(\theta) = 0$.

Where $\beta_j, T_l, F_i, R_i, A_i, B_l$, and D_i are constants that are given in Appendix, where $i = 1...4, j = 1...8, l = 1...6$.

The velocity components can be obtained as follows:

$$u^{(2)} = \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta} = \mathrm{Re}\left(rU^2\ f_2' + UQ\ f_3' + \frac{Q^2}{r}\ f_4'\right), \tag{3.75}$$

and

$$v^{(2)} = -\frac{\partial\psi^{(2)}}{\partial r} = \mathrm{Re}\left(-2rU^2 f_2 - UQf_3\right). \tag{3.76}$$

From Eqs.(3.54) and (3.55) and using Eq.(3.56), we obtain pressure field of order two and is given as follows:

$$\begin{aligned} p^{(2)}(r,\theta) &= \left(U^2 C_1(\theta)\log r - \frac{UQ}{r}C_2(\theta) - \frac{Q^2}{2r^2}C_3(\theta)\right) \\ &\quad + \frac{1}{\mathrm{Re}}\frac{U^2}{r^2}C_4(\theta) + \frac{1}{\mathrm{Re}}\frac{UQ}{r^3}C_5(\theta) + \frac{1}{\mathrm{Re}}\frac{Q^2}{r^4}C_6(\theta). \end{aligned} \tag{3.77}$$

Here $C_l(\theta)$ are defined in Appendix , where $l = 1...6$.

Finally, we obtain the normal and tangential stresses as well

$$T_{rr}^{(2)} = -\mathrm{Re}\, p^{(2)} + \mathrm{Re}\, O_1(\theta) + \frac{U^2}{r^2}O_2(\theta) + \frac{UQ}{r^3}O_3(\theta) + \frac{Q^2}{r^4}O_4(\theta), \tag{3.78}$$

$$T_{\theta\theta}^{(2)} = -\mathrm{Re}\, p^{(2)} + \mathrm{Re}\, O_5(\theta) + \frac{U^2}{r^2}O_6(\theta) + \frac{UQ}{r^3}O_7(\theta) + \frac{Q^2}{r^4}O_8(\theta), \tag{3.79}$$

here O_j is defined in Appendix, where $j = 1...8$.

$$T_{r\theta}^{(2)} = \operatorname{Re}\left(U^2 W_1(\theta) + \frac{UQ}{r} W_2(\theta) + \frac{Q^2}{r^2} W_3(\theta)\right) + \tag{3.80}$$

$$\frac{U^2}{r^2} W_4(\theta) + \frac{UQ}{r^3} W_5(\theta) + \frac{Q^2}{r^4} W_6(\theta)\,, \tag{3.81}$$

here W_l is defined in Appendix, where $l = 1...6$.

3.2.3 Third Order Problem

$$\frac{\partial h^{(3)}}{\partial r} - \frac{\partial \psi^{(1)}}{\partial r}\nabla^2\psi^{(2)} - \frac{\partial \psi^{(2)}}{\partial r}\nabla^2\psi^{(1)} - \frac{1}{\operatorname{Re}}\frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2\psi^{(3)}$$
$$+\frac{1}{\operatorname{Re}}\alpha_1\left(\frac{\partial \psi^{(1)}}{\partial r}\nabla^4\psi^{(2)} + \frac{\partial \psi^{(2)}}{\partial r}\nabla^4\psi^{(1)}\right) = 0\,, \tag{3.82}$$

$$\frac{1}{r}\frac{\partial h^{(3)}}{\partial \theta} - \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta}\nabla^2\psi^{(2)} - \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta}\nabla^2\psi^{(1)} + \frac{1}{\operatorname{Re}}\frac{\partial}{\partial r}\nabla^2\psi^{(3)}$$
$$+\frac{1}{\operatorname{Re}}\frac{\alpha_1}{r}\left(\frac{\partial \psi^{(2)}}{\partial \theta}\nabla^4\psi^{(1)} + \frac{\partial \psi^{(1)}}{\partial \theta}\nabla^4\psi^{(2)}\right) = 0\,, \tag{3.83}$$

where,

$$h^{(3)}(r,\theta) = \frac{1}{2}\left(2u^{(1)}u^{(2)} + 2v^{(1)}v^{(2)}\right) - \frac{1}{\operatorname{Re}}\left(\frac{3\alpha_1 + 2\alpha_2}{4}\right)\left(trA_1^{(3)2}\right)$$
$$-\frac{1}{\operatorname{Re}}\alpha_1\left(v^{(2)}\frac{\partial}{\partial r} - \frac{u^{(2)}}{r}\frac{\partial}{\partial \theta}\right)\omega^{(1)} - \frac{1}{\operatorname{Re}}\alpha_1\left(v^{(1)}\frac{\partial}{\partial r} - \frac{u^{(1)}}{r}\frac{\partial}{\partial \theta}\right)\omega^{(2)}$$
$$+p^{(3)}(r,\theta) + \frac{1}{\operatorname{Re}}\frac{\alpha_1}{2}\omega^{(3)2}. \tag{3.84}$$

Eliminating $h^{(3)}(r,\theta)$ from Eq.(3.82) and Eq.(3.83), we have

$$
\begin{aligned}
\nabla^4 \psi^{(3)} &= \frac{-\operatorname{Re}}{r}\left(\frac{\partial \psi^{(1)}}{\partial r}\frac{\partial\left(\nabla^2 \psi^{(2)}\right)}{\partial \theta}+\frac{\partial \psi^{(2)}}{\partial r}\frac{\partial\left(\nabla^2 \psi^{(1)}\right)}{\partial \theta}-\frac{\partial \psi^{(1)}}{\partial \theta}\frac{\partial\left(\nabla^2 \psi^{(2)}\right)}{\partial r}\right. \\
&\left.-\frac{\partial \psi^{(2)}}{\partial \theta}\frac{\partial\left(\nabla^2 \psi^{(1)}\right)}{\partial r}\right)+\frac{\alpha_1}{r}\left(\frac{\partial \psi^{(1)}}{\partial r}\frac{\partial\left(\nabla^4 \psi^{(2)}\right)}{\partial \theta}-\frac{\partial \psi^{(1)}}{\partial \theta}\frac{\partial\left(\nabla^4 \psi^{(2)}\right)}{\partial r}\right), \qquad (3.85)
\end{aligned}
$$

with B.C's

$$
\psi^{(3)}=0, \quad \frac{1}{r}\frac{\partial \psi^{(3)}}{\partial \theta}=0 \quad \text{at} \quad \theta=0, \qquad (3.86)
$$

$$
\psi^{(3)}=0, \quad \frac{1}{r}\frac{\partial \psi^{(3)}}{\partial \theta}=0 \quad \text{at} \quad \theta=\theta_w, \qquad (3.87)
$$

The third order system (3.85) – (3.87) contains a fourth order PDE. To change this PDE into a system of ODE, one can consider the stream function $\psi^{(3)}$ in the following form

$$
\begin{aligned}
\psi^{(3)}(r,\theta) &= -(\operatorname{Re})^2\left(r^3U^3 f_5(\theta)+r^2U^2Q f_6(\theta)+rUQ^2 f_7\left(\theta+Q^3 f_8 \theta\right)\right) \\
&+\alpha_1 \operatorname{Re}\left(rU^3 g_5(\theta)+U^2Q g_6(\theta)+\frac{UQ^2}{r} g_7(\theta)+\frac{UQ^3}{r^2} g_8(\theta)\right). \qquad (3.88)
\end{aligned}
$$

Using expressions for first, second and third order stream functions, pressure fields, velocity components, and stresses computed in previous sections, one can get the following solutions .

$$
\psi(r,\theta)=\psi^{(1)}+\psi^{(2)}+\psi^{(3)}, \qquad (3.89)
$$

$$
u(r,\theta)=u^{(1)}+u^{(2)}+u^{(3)}, \qquad (3.90)
$$

$$
v(r,\theta)=v^{(1)}+v^{(2)}+v^{(3)}, \qquad (3.91)
$$

$$
p(r,\theta)=const+p^{(1)}+p^{(2)}+p^{(3)}, \qquad (3.92)
$$

$$
T_t=T_t^{(1)}+T_t^{(2)}+T_t^{(3)}, \qquad (3.93)
$$

$$
T_{rr}=T_{rr}^{(1)}+T_{rr}^{(2)}+T_{rr}^{(3)}, \qquad (3.94)
$$

$$T_{\theta\theta} = T_{\theta\theta}^{(1)} + T_{\theta\theta}^{(2)} + T_{\theta\theta}^{(3)}. \tag{3.95}$$

3.3 Graphs and Discussion

3.3.1 Velocity Field

Figures 3.2$(a-c)$ show the behavior of velocity for the variations of Reynolds number, leakge at the apex and plate velocity. It can be observed from figures 3.2 that the velocity of the fluid rises when the velocity of the plate, Reynolds number and leakage parameter increases. Because the plate velocity reduces the friction and causing to increase the flow in radial direction, also the dominance of inertial forces reduce the resistance in the fluid causing to increase the flow. The leakage at the apex also reduce the resistive forces and vortices near the corner that makes the flow fast.

Figures3.3 shows that angular velocity also increases with the growing effect of leakage, moving boundary and leakage but the angular flow is in clockwise direction.

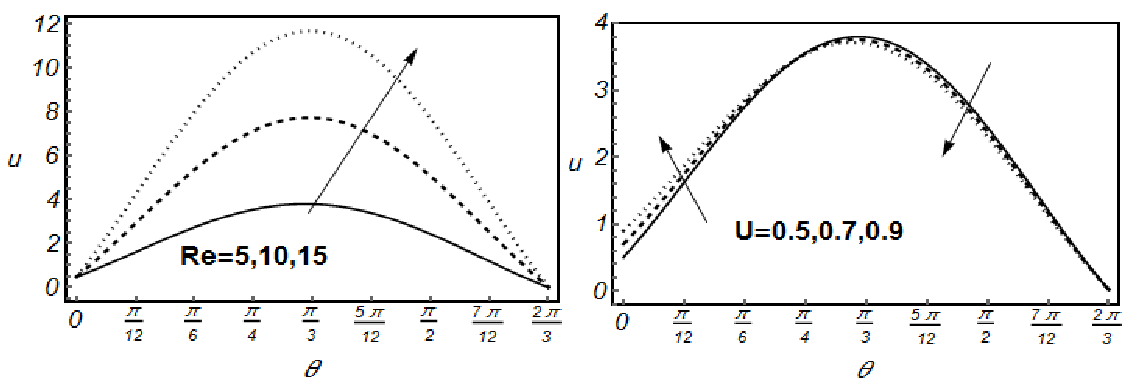
3.3.2 Pressure Distribution

Pressure near a corner varies due to variations in second grade parameter, plate velocity, leakage at the apex and Reynolds number which is shown in figure3.4. Figure 3.4 shows that when inertial effects of fluid become dominant internal pressure reduces due to fast movement of the fluid particles but the leakage, plate speed and viscoelastic property of the fluid causes to enhance the pressure near corner.

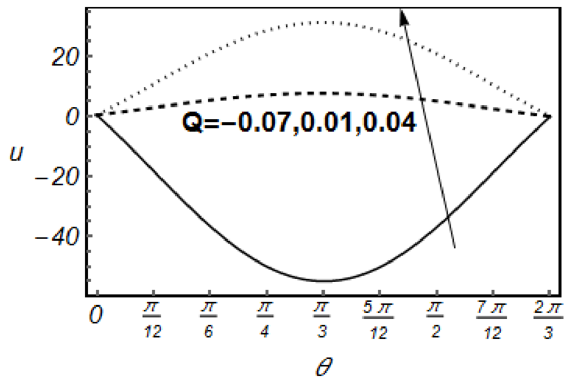
3.3.3 Normal and Tangential Stresses

Figures 3.5 explain the graphical behavior of normal and tangential stress to the plate against second grade parameter, plate velocity, leakage at the apex and Reynolds number. The impact of viscoelastic fluid parameter on normal and tangential stress shows that when viscoelastic fluid parameter increases then viscoelastic forces become dominant near a corner and rate of deformation become high which results to increase the stress near a corner which leads to a rise in normal and tangential stress. Figures 3.5 demonstrate the effect of plate speed U on normal and tangential stress in the presence of inertial forces and leakage which explains that the velocity gradient decay and enhance in different regions as the plate moves with high speed so the shear rate increases and decreases in different regions and as a result normal stress in angular direction rises by the dominance of inertial forces but shows dual behavior due to moving boundary. and tangential stress also rises. The normal stress in radial direction decays by the dominance of inertial and leakage effects but normal stress enhances by the plate velocity and viscoelastic effect.

The effect of the Reynolds number on tangential stress is shown in figure 7 which describes that the inertial effects become dominant over viscous forces due to a rise in Reynolds number then the shear rate enhances near a corner and this leads to a rise in tangential stress but in a reverse direction

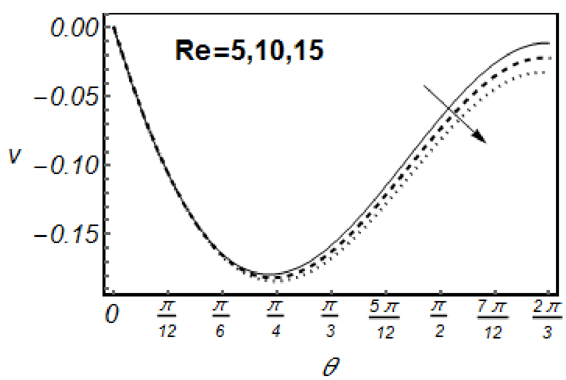


(a) (b)

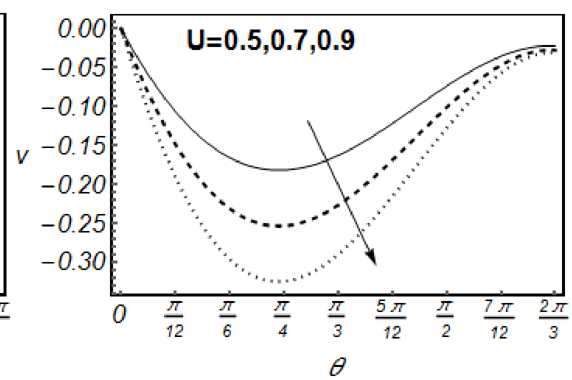


(c)

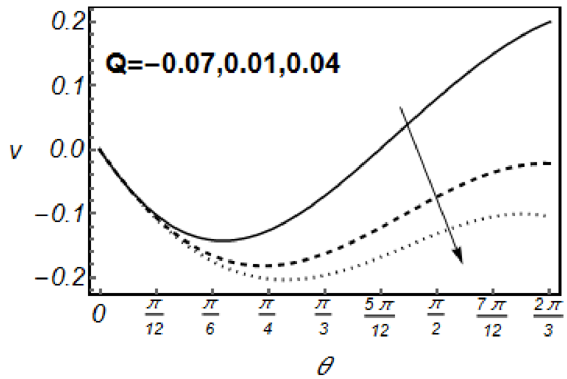
Fig 3.2(a-c): Variation in velocity components u for Re, U, Q.



(a)

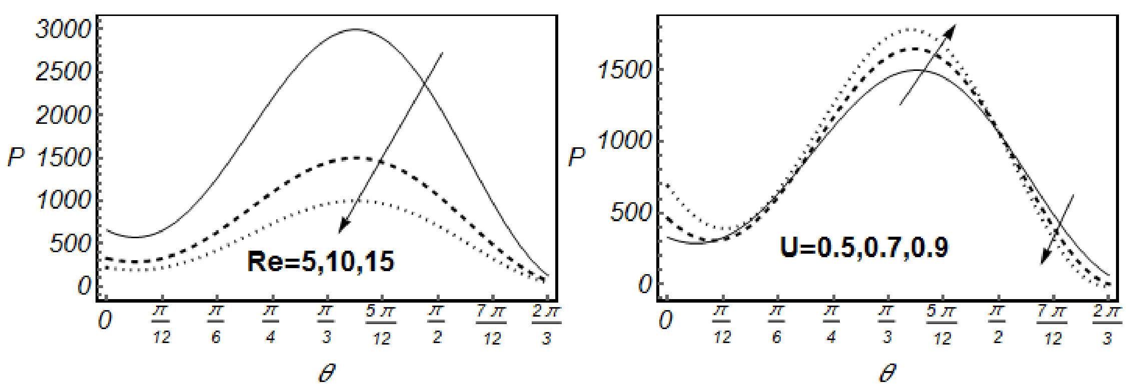


(b)

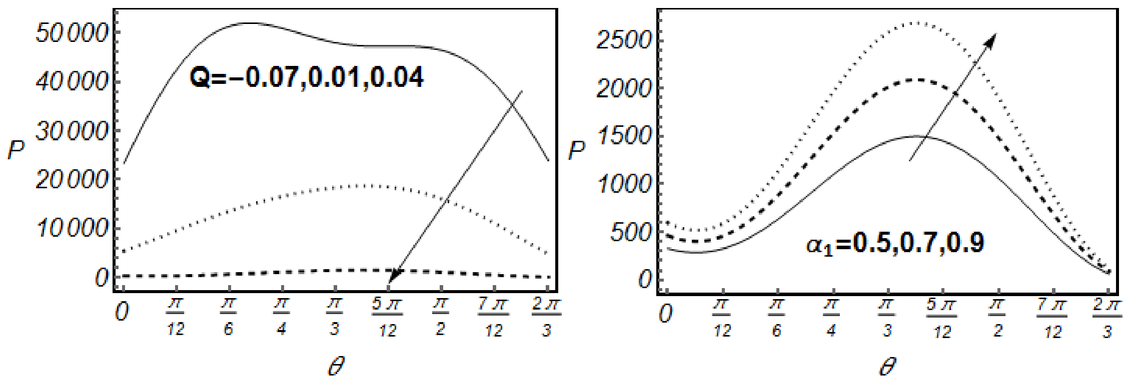


(c)

Fig 3.3(a-c): Variation in velocity components v for Re, U, Q.

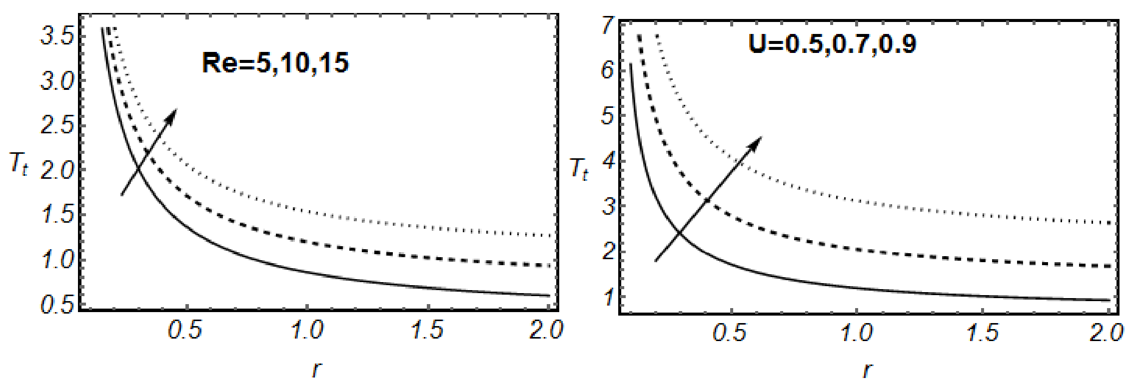


(a) (b)

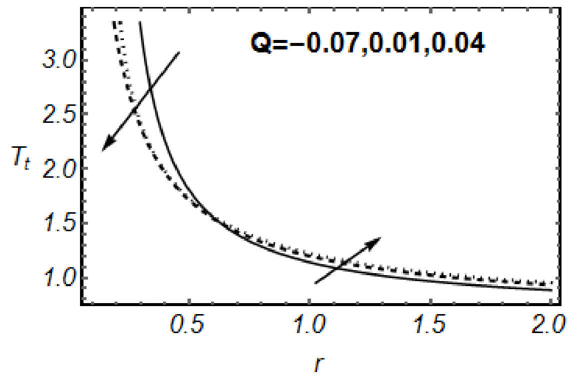


(c) (d)

Fig 3.4(a-d): Variation in pressure p for Re, U, Q and α_1.

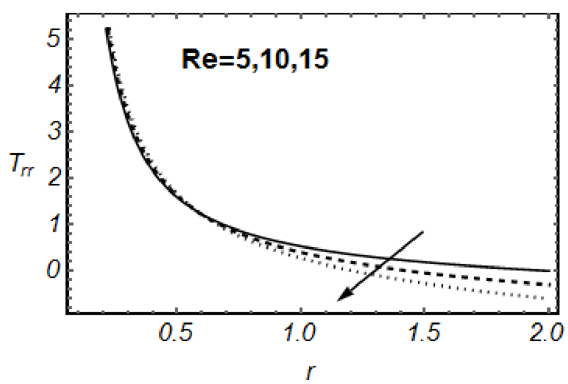


(a) (b)

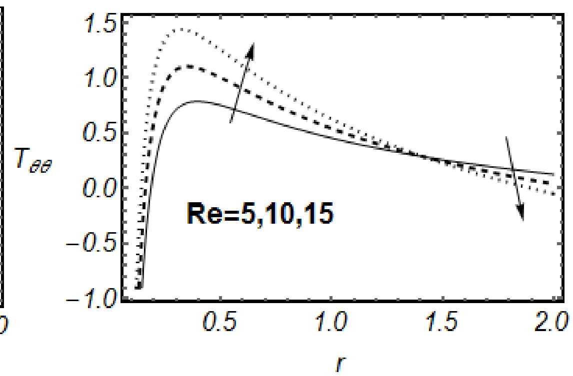


(c)

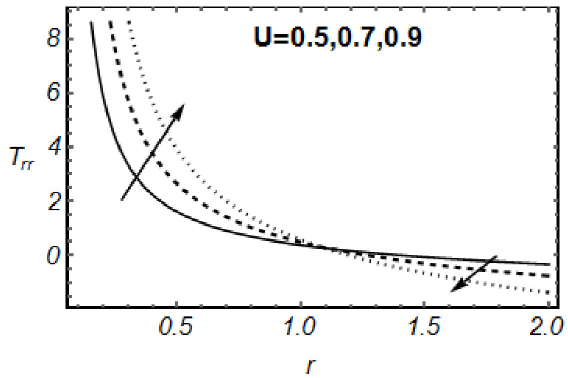
Fig 3.5(a-c): Variation in tangential stress T_t for Re, U, Q.



(a)

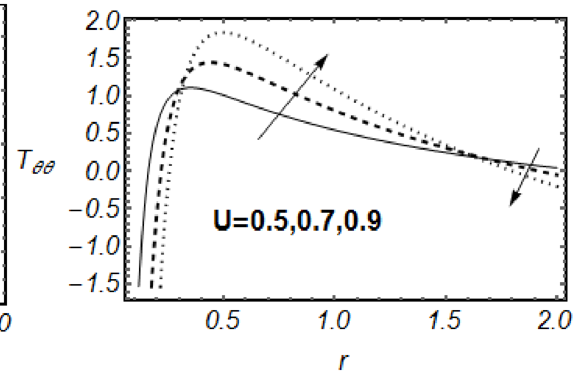


(b)



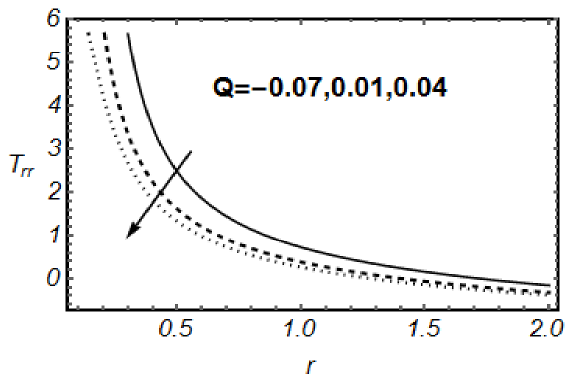
U=0.5,0.7,0.9
T_{rr}
r

(c)



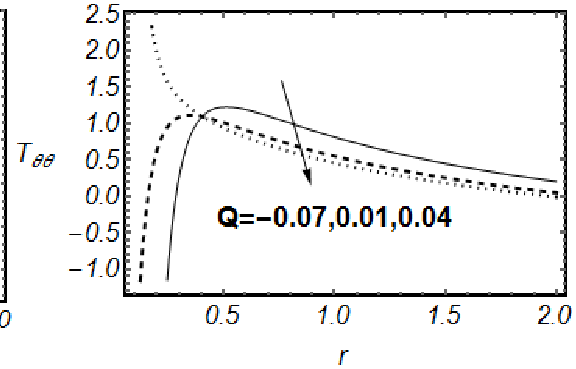
U=0.5,0.7,0.9
$T_{\theta\theta}$
r

(d)



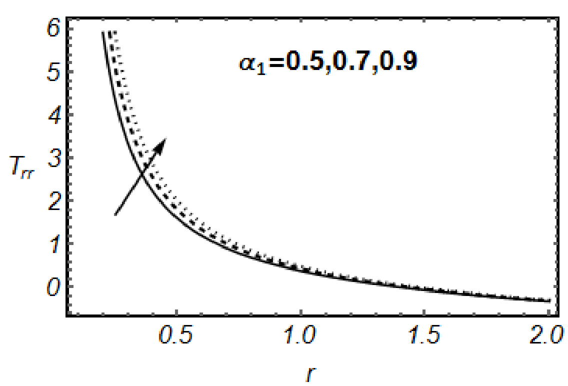
Q=−0.07,0.01,0.04
T_{rr}
r

(e)



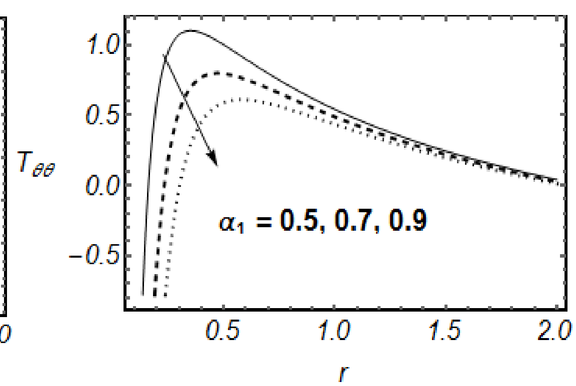
Q=−0.07,0.01,0.04
$T_{\theta\theta}$
r

(f)



α_1=0.5,0.7,0.9
T_{rr}
r

(g)



α_1 = 0.5, 0.7, 0.9
$T_{\theta\theta}$
r

(h)

Fig 3.6(a-h): Variation in normal stresses T_{rr} $T_{\theta\theta}$ for Re, U, Q and α_1.

3.4 Conclusion

In this work, we have emphasized the importance of inertial and non inertial forces of a two dimensional second grade fluid with the effects of leakage at the apex of corner. The mathematical models of second grade fluid are represented by the set of intricate non-linear partial differential equations and their solutions are obtained by Langloi's recursive technique using no slip boundary conditions. The analytical results of various flow characteristics like velocity profile, pressure, and shear stresses are calculated in this study, and graphical results for these flow characteristics are observed for different parameters. It is concluded from graphical results that radial component of velocity rises with the extending values of Reynold's Number (Re), plate velocity (U), and leakage parameter (Q). It is also observed that anngular component of velocity also rises when there is increase in Reynold's Number (Re), plate velocity (U), and leakage parameter (Q) but in clockwise direction. Pressure near a corner varies due to variation in involving parameters. Pressure reduces due to fast movement of the fluid particles but other parameters causes to enhance the pressure near a corner. This research also concludes that wall shear stress rises with the rising values of all emerging parameters.

3.5 Appendix

$$E_1 = U^2 \left[\alpha_1 \left(-20D^2 + 20C^2 + 8BC\right) + \alpha_2 \left(-2B^2 - 6BC - 2C^2 + 2D^2\right)\right],$$

$$E_2 = U^2 \left[\alpha_1 \left(-8BD - 40CD\right) + \alpha_2 \left(6BD + 4CD\right)\right],$$

$$E_3 = U^2 \left[\alpha_1 \left(16CD\right) + \alpha_2 \left(-4BD - 12CD\right)\right],$$

$$E_4 = U^2 \left[\alpha_1 \left(8C^2 - 8D^2\right) + \alpha_2 \left(6D^2 - 6C^2 - 4BC\right)\right],$$

$$E_5 = U^2 \left[\alpha_2 \left(2C^2 - 2D^2\right)\right], E_6 = U^2 \left[\alpha_2 \left(-4CD\right)\right],$$

$$E_7 = U^2 \left[2\alpha_2 \left(C^2 + D^2\right)\right], E_8 = U^2 \left[2\alpha_2 \left(-BD\right)\right],$$

$$\gamma_1 = \frac{E_1}{16} - \frac{5E_4}{64} - \frac{51E_5}{384}, \gamma_5 = \frac{8E_5}{384},$$

$$\gamma_2 = \frac{-E_2}{16} - \frac{5E_3}{64} + \frac{51E_6}{384}, \gamma_6 = \frac{-8E_6}{384},$$

$$\gamma_3 = \frac{2E_3}{64} - \frac{30E_6}{384}, \gamma_7 = \frac{E_8}{24} - \frac{E_7}{96},$$

$$\gamma_4 = \frac{-2E_4}{64} - \frac{30E_5}{384}, \gamma_8 = \frac{E_7}{96},$$

$$V = \gamma_1\theta_w \cos 2\theta_w + \gamma_2\theta_w \sin 2\theta_w + \gamma_3\theta_w^2 \cos 2\theta_w + \gamma_4\theta_w^2 \sin 2\theta_w + \gamma_5\theta_w^3 \cos 2\theta_w + \gamma_6\theta_w^3 \sin 2\theta_w + \gamma_7\theta_w^3 + \gamma_8\theta_w^4,$$

$$\begin{aligned} V_1 \quad = \quad & \gamma_1 \left(\cos 2\theta_w - 2\theta_w \sin 2\theta_w\right) + \gamma_2 \left(\sin 2\theta_w + 2\theta_w \cos 2\theta_w\right) + \gamma_3 \left(2\theta_w \cos 2\theta_w - 2\theta_w^2 \sin 2\theta_w\right) \\ & + \gamma_4 \left(2\theta_w \sin 2\theta_w + 2\theta_w^2 \cos 2\theta_w\right) + \gamma_5 \left(3\theta_w^2 \cos 2\theta_w - 2\theta_w^3 \sin 2\theta_w\right) + \gamma_6 \left(3\theta_w^2 \sin 2\theta_w + 2\theta_w^3 \cos 2\theta_w\right) \\ & + 3\gamma_7\theta_w^2 + 4\gamma_8\theta_w^3, \end{aligned}$$

$$U_1 = \frac{\left[2V - 2V_1\theta_w - \left(\gamma_1 - V_1\right) \sin 2\theta_w + 2\left(V - \gamma_1\theta_w\right) \cos 2\theta_w\right]}{4\left(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1\right)},$$

$$U_2 = \frac{V_1 + \gamma_1 - \left(V_1 + \gamma_1\right) \cos 2\theta_w - 2V_1 \sin 2\theta_w}{2\left(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1\right)},$$

$$U_3 = -U_1 = \frac{-1\left[2V - 2V_1\theta_w - \left(\gamma_1 - V_1\right) \sin 2\theta_w + 2\left(V - \gamma_1\theta_w\right) \cos 2\theta_w\right]}{4\left(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1\right)},$$

$$U_4 = \frac{\gamma_1 - V_1 + \left(V_1 - \gamma_1\right) \cos 2\theta_w + 2\left(V - \gamma_5\right) \sin 2\theta_w}{4\left(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1\right)},$$

$$R_1 = \frac{2X - 2X_1\theta_w - (\beta_5 - X_1)\sin 2\theta}{4(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1)} - \frac{2(X - \beta_5\theta_w)\cos 2\theta_w}{4(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1)},$$

$$R_2 = \frac{X_1 + \beta_5 - (\beta_5 + X_1)\cos 2\theta_w - 2X\sin 2\theta_w}{2(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1)},$$

$$R_3 = -\frac{2X - 2X_1\theta_w - (\beta_5 - X_1)\sin 2\theta}{4(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1)} + \frac{2(X - \beta_5\theta_w)\cos 2\theta_w}{4(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1)},$$

$$R_4 = \frac{\beta_5 - X_1 + (X_1 - \beta_5)\cos 2\theta_w}{4(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1)} + \frac{2(X - \beta_5\theta_w)\sin 2\theta_w}{4(\cos 2\theta_w + \theta_w \sin 2\theta_w - 1)},$$

$$\beta_1 = \frac{\sin^4\theta_w - (\theta_w - \sin\theta_w\cos\theta_w)^2 - 2\theta_w^2\sin^2\theta_w}{\left(\theta_w^2 - \sin^2\theta_w\right)^2},$$

$$\beta_2 = \frac{2(\theta_w - \sin\theta_w\cos\theta_w)}{\left(\theta_w^2 - \sin^2\theta_w\right)^2}, \beta_3 = \frac{4(\theta_w - \sin\theta_w\cos\theta_w)\sin^2\theta_w}{\left(\theta_w^2 - \sin^2\theta_w\right)^2},$$

$$\beta_4 = \frac{2\sin^4\theta_w - 2(\theta_w - \sin\theta_w\cos\theta_w)^2}{\left(\theta_w^2 - \sin^2\theta_w\right)^2}, \beta_5 = \left(\frac{\beta_1}{16} - \frac{5}{64}\beta_4\right),$$

$$\beta_6 = -\left(\frac{\beta_2}{16} + \frac{5}{64}\beta_3\right), \beta_7 = \frac{1}{32}\beta_3, \beta_8 = -\frac{1}{32}\beta_4,$$

$$X = \beta_5\theta_w\cos 2\theta_w + \beta_6\theta_w\sin 2\theta_w + \beta_7\theta_w^2\cos 2\theta_w + \beta_8\theta_w^2\sin 2\theta_w,$$

$$\begin{aligned} X_1 \quad = \quad & \beta_5\cos 2\theta_w - 2\beta_5\theta_w\sin 2\theta_w + \beta_6\sin 2\theta_w + 2\beta_6\theta_w\cos 2\theta_w + 2\beta_7\theta_w\cos 2\theta_w - 2\beta_7\theta_w^2\sin 2\theta_w \\ & + 2\beta_8\theta_w\sin 2\theta_w + 2\beta_8\theta_w^2\cos 2\theta_w, \end{aligned}$$

$$A_1 = -B_1,$$

$$A_2 = -\frac{B_1\theta_w + (3B_2 + B_5)\theta_w^2 - Y(\sin\theta_w + \theta_w\cos\theta_w)}{\left(\theta_w^2 - \sin^2\theta_w\right)} - \frac{(\theta_w Y_1 + B_1\cos\theta_w)\sin\theta_w}{\left(\theta_w^2 - \sin^2\theta_w\right)},$$

$$A_3 = \frac{B_1\theta_w + (3B_2 + B_5)\sin^2\theta_w - Y(\sin\theta_w + \theta_w\cos\theta_w)}{\left(\theta_w^2 - \sin^2\theta_w\right)} + \frac{(\theta_w Y_1 + B_1\cos\theta_w)\sin\theta_w}{\left(\theta_w^2 - \sin^2\theta_w\right)},$$

$$A_4 = \frac{B_1\sin^2\theta_w + (3B_2 + B_5)(\theta_w - \sin\theta_w\cos\theta_w)}{\left(\theta_w^2 - \sin^2\theta_w\right)} - \frac{Y_1(\theta_w\cos\theta_w - \sin\theta_w) + Y\theta_w\sin\theta_w}{\left(\theta_w^2 - \sin^2\theta_w\right)},$$

$$B_1 = \frac{1}{64}\left(T_1 + \frac{3}{2}T_6\right), B_3 = -\frac{1}{8}T_3, B_4 = -\frac{1}{8}T_4,$$

$$B_2 = \frac{1}{64}\left(T_2 - \frac{3}{2}T_5\right), B_5 = \frac{1}{64}T_5, B_6 = \frac{1}{64}T_6,$$

$$T_1 = -8BK - 4CK - 6DN,$$

$$T_2 = 8BN + 2CK + 8CN - 6DK,$$

$$T_3 = -4CK + 6DN, T_4 = CK - 4CN + 2DK,$$

$$T_5 = 8CN - 8DK, T_6 = 8DK + 8DN,$$

$$B = \frac{\theta_w^2}{\theta_w^2 - \sin^2\theta_w},\ C = -\frac{\sin^2\theta_w}{\theta_w^2 - \sin^2\theta_w},\ D = -\frac{(\theta_w - \sin\theta_w\cos\theta_w)}{\theta_w^2 - \sin^2\theta_w},$$

$$K = -M = \frac{2\sin^2\theta_w}{4\sin^4\theta_w + \sin 2\theta_w\left(\sin 2\theta_w - 2\theta_w\right)},$$

$$N = -2L = \frac{\sin 2\theta_w}{4\sin^4\theta_w + \sin 2\theta_w\left(\sin 2\theta_w - 2\theta_w\right)},$$

$$Y = B_1\cos 3\theta_w + B_2\sin 3\theta_w + B_3\theta_w^2\cos\theta_w + B_4\theta_w^2\sin\theta_w + B_5\theta_w\cos 3\theta_w + B_6\theta_w\sin 3\theta_w,$$

$$\begin{aligned} Y_1 \quad = \quad & -3B_1\sin 3\theta_w + 3B_2\cos 3\theta_w - B_3\theta_w^2\sin\theta_w + 2B_3\theta_w\cos\theta_w + B_4\theta_w^2\cos\theta_w + 2B_4\theta_w\sin\theta_w + B_5\cos 3\theta_w \\ & -3B_5\theta_w\sin 3\theta_w + B_6\sin 3\theta_w + 3B_6\theta_w\cos 3\theta_w, \end{aligned}$$

$$S_1 = 16KN,\ S_2 = -16N^2,\ S_3 = -16KN,\ S_4 = 8\left(N^2 - K^2\right),$$

$$F_1 = \frac{1}{16}S_2,\ F_2 = -\frac{1}{16}S_1,\ F_3 = \frac{1}{192}S_3,\ F_4 = \frac{1}{192}S_4,$$

$$D_1 = \frac{4\left(F_3 + Z\right)\sin^2\theta_w - 2\theta_w Z_1}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)} - \frac{\left(F_1 + 4F_4\right)\left(\sin 2\theta_w - 2\theta_w\cos 2\theta_w\right)}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)} - \frac{\left(4F_3\theta_w - Z_1\right)\sin 2\theta_w}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)},$$

$$D_2 = \frac{\left(F_1 + 4F_4 + Z_1\right)\left(1 - \cos 2\theta_w\right)}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)} + \frac{\left(2F_3 - 2Z\right)\sin 2\theta_w}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)},$$

$$D_3 = \frac{4\left(F_3 - Z\right)\sin^2\theta_w + \left(2\theta_w - \sin 2\theta_w\right)Z_1}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)} + \frac{\left(F_1 + 4F_4\right)\left(\sin 2\theta_w - 2\theta_w\cos 2\theta_w\right)}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)},$$

$$D_4\frac{\left(F_1 + 4F_4 + Z_1\right)\left(1 - \cos 2\theta_w\right)}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)} - \frac{\left(2F_3 - 2Z\right)\sin 2\theta}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)} - \frac{2\left(F_1 + 4F_4\right)\left(\theta_w\sin 2\theta_w\right)}{4\left(\cos 2\theta_w + \theta_w\sin 2\theta_w - 1\right)},$$

$$Z = F_1\theta_w\cos 2\theta_w + F_2\theta_w\sin 2\theta_w + F_3\cos 4\theta_w + F_4\sin 4\theta_w,$$

$$Z_1 = F_1 \cos 2\theta_w - 2F_1\theta_w \sin 2\theta_w + F_2 \sin 2\theta_w + 2F_2\theta_w \cos 2\theta_w - 4F_3 \sin 4\theta_w + 4F_4 \cos 4\theta_w,$$

$$C_1(\theta) = \left[(f_1)^2 + f_1 f_1^{''} + f_2^{'''} + 4f_2^{'}\right],$$

$$C_2(\theta) = \left[f_1 g_1^{''} + f_3^{'''} + f_3^{'} + 2f_1^{'} g_1^{'}\right],$$

$$J_3(\theta) = \left[f_4^{'''} + \left(g_1^{'}\right)^2\right],$$

$$C_4(\theta) = \alpha_1 \left[\frac{(f_1)^2}{2} + 2f_1 f_1^{''} + \frac{f_1 f_1^{iv}}{2} + \left(f_1^{'}\right)^2 + f_1^{'} f_1^{'''} + \left(f_1^{''}\right)^2\right] + \alpha_2 \left[(f_1)^2 + \left(f_1^{''}\right)^2 + 2f_1 f_1^{''}\right],$$

$$C_5(\theta) = \alpha_1 \left[\frac{f_1 g_1^{iv}}{3} + \frac{4}{3} f_1 g_1^{''} + f_1^{'} g_1^{'''} + f_1^{'} g_1^{'} + f_1^{'''} g_1^{'} + 2f_1^{''} g_1^{''}\right] + 2\alpha_2 \left[f_1 g_1^{''} + f_1^{''} g_1^{''}\right],$$

$$C_6(\theta) = \alpha_1 \left[g_1^{'} g_1^{'''} + \left(g_1^{''}\right)^2 + 6\left(g_1^{'}\right)^2\right] + \alpha_2 \left[\left(g_1^{''}\right)^2 + 4\left(g_1^{'}\right)^2\right],$$

$$O_1(\theta) = 2\left[U^2 f_2^{'} - \frac{Q^2}{r^4} f_4^{'}\right],$$

$$O_2(\theta) = 2\alpha_1 \left[(f_1)^2 + \left(f_1^{''}\right)^2 + 2f_1 f_1^{''}\right] + \alpha_2 \left[(f_1)^2 + \left(f_1^{''}\right)^2 + 2f_1 f_1^{''}\right],$$

$$O_3(\theta) = 2\alpha_1 \left[2f_1^{'} g_1^{'} + 4f_1 g_1^{''} + 2f_1^{''} g_1^{''}\right] + 2\alpha_2 \left[f_1 g_1^{''} + f_1^{''} g_1^{''}\right],$$

$$O_4(\theta) = 2\alpha_1 \left[\left(g_1^{''}\right)^2 + 4\left(g_1^{'}\right)^2\right] + \alpha_2 \left[\left(g_1^{''}\right)^2 + 4\left(g_1^{'}\right)^2\right],$$

$$O_5(\theta) = -O_1(\theta) = -2\left[U^2 f_2^{'} - \frac{Q^2}{r^4} f_4^{'}\right],$$

$$O_6(\theta) = \alpha_2 \left[(f_1)^2 + \left(f_1^{''}\right)^2\right],$$

$$O_7(\theta) = -2\alpha_1 \left[2f_1^{'} g_1^{'} + f_1 g_1^{''}\right] + 2\alpha_2 \left[f_1^{''} g_1^{''}\right],$$

$$O_8(\theta) = \alpha_2 \left[\left(g_1^{''}\right)^2 + 4\left(g_1^{'}\right)^2\right],$$

$$W_1(\theta) = \left[f_2^{''} - 2f_2\right],\ W_2(\theta) = \left[f_3 + f_3^{''}\right],\ W_2(\theta) = \left[f_3 + f_3^{''}\right],$$

$$W_4(\theta) = \alpha_1 \left[-f_1^{'} f_1^{''} - 2f_1 f_1^{'} - f_1 f_1^{'''}\right],$$

$$W_5(\theta) = \alpha_1 \left[f_1^{''} g_1^{'} + f_1 g_1^{'} - 2f_1^{'} g_1^{''} - f_1 g_1^{'''}\right],$$

$$W_6(\theta) = \alpha_1 \left[-g_1^{'} g_1^{''} \right],$$

$$M_1 = \alpha_1 U \left(\begin{array}{c} 8BR_3 - 12B\beta_2 - 8B\beta_3 - 24B\beta_6 + 96B\beta_8 - 24CR_3 + 52C\beta_2 + 40C\beta_3 \\ +24C\beta_6 + 96C\beta_8 + 24DR_2 - 24DR_4 - 52D\beta_1 + 40D\beta_4 - 24D\beta_5 + 144D\beta_7 \end{array} \right),$$

$$M_2 = \alpha_1 U \left(\begin{array}{c} 8DR_3 - 36D\beta_2 - 112D\beta_3 + 96D\beta_6 + 672D\beta_8 + 8CR_2 + 8CR_4 \\ +36C\beta_1 - 112C\beta_4 - 96C\beta_5 + 48C\beta_7 + 8B\beta_1 - 24B\beta_4 - 24B\beta_5 + 48B\beta_7 \end{array} \right),$$

$$M_3 = \alpha_1 U \left(\begin{array}{c} 8D\beta_1 - 48D\beta_4 - 180D\beta_5 + 120D\beta_7 + 8B\beta_3 - 36B\beta_6 \\ +96B\beta_8 + 8C\beta_2 + 48C\beta_3 - 180C\beta_6 + 96C\beta_8 \end{array} \right),$$

$$M_4 = \alpha_1 U \left(8D\beta_3 - 60D\beta_6 + 192D\beta_8 + 8B\beta_5 - 8C\beta_4 - 60C\beta_5 - 24C\beta_7\right),$$

$$M_5 = \alpha_1 U \left(8D\beta_5 - 8C\beta_6 - 16C\beta_8\right),$$

$$\begin{aligned} M_6 \;\; = \;\; & \alpha_1 U \left(-120DR_4 + 248D\beta_4 + 168D\beta_5 - 212D\beta_1 + 120CR_3 - 248C\beta_3 + 168C\beta_6 \right. \\ & \left. -212C\beta_2 + 24BR_3 - 84B\beta_2 + 72B\beta_6 - 168B\beta_3\right), \end{aligned}$$

$$\begin{aligned} M_7 \;\; = \;\; & \alpha_1 U \left(24DR_3 - 204D\beta_2 - 592D\beta_3 + 960D\beta_6 + 24CR_4 + 204C\beta_1 - 592C\beta_4 \right. \\ & \left. -960C\beta_5 + 24B\beta_1 - 168B\beta_4\right), \end{aligned}$$

$$M_8 = \alpha_1 U \left(24D\beta_1 - 288D\beta_4 - 1140D\beta_5 + 24C\beta_2 + 288C\beta_3 - 1140C\beta_6 - 252B\beta_6 + 24B\beta_3\right),$$

$$M_9 = \alpha_1 U \left(24D\beta_3 - 372D\beta_6 + 24C\beta_4 + 372C\beta_5 + 24B\beta_5\right),$$

$$M_{10} = \alpha_1 U \left(24D\beta_5 + 24C\beta_6\right),$$

$$M_{11} = \alpha_1 U \left(\begin{array}{c} 24\left(6DR_3 - 11D\beta_2 - 12D\beta_3 + 6D\beta_6 + 4D\beta_8\right) - 2\left(4BR_2 + 16BR_4 + 48B\beta_1 - 88B\beta_4 \right. \\ -96B\beta_5 - 12CR_2 + 48CR_4 + 80C\beta_1 - 104C\beta_4 - 96C\beta_5 - 72C\beta_7 + 120CR_4 - 248C\beta_4 \\ \left. -168C\beta_5 + 120DR_3 - 248D\beta_3 + 168D\beta_6 + 24BR_4 + 84B\beta_1 - 168B\beta_4 - 216B\beta_5\right) \end{array} \right),$$

$$M_{12} = \alpha_1 U \left(\begin{array}{c} 8\left(4B\beta_2 + 24B\beta_3 - 66B\beta_6\right) + 8\left(-DR_2 + 4DR_4 + 30D\beta_1 - 88D\beta_4 - 132D\beta_5 \right. \\ -6D\beta_7 - 2CR_3 + 21C\beta_2 + 60C\beta_3 - 108C\beta_6 - 84C\beta_8 - 6CR_3 + 51C\beta_2 + 148C\beta_3 \\ \left. -240C\beta_6 + 6DR_4 + 51D\beta_1 - 148D\beta_4 - 240D\beta_5 + 6B\beta_2 + 42B\beta_3 - 126B\beta_6\right) \end{array} \right),$$

$$M_{13} = 8\alpha_1 U \left(4B\beta_4 + 36B\beta_5 - 3B\beta_7 + 4D\beta_2 + 42D\beta_3\right),$$

$$M_{14} = 8\alpha_1 U \left(4B\beta_6 - 4B\beta_8 + 4D\beta_4 + 54D\beta_5 - 3D\beta_7 - 2C\beta_3 + 39C\beta_6 - 24C\beta_8 \right.$$
$$\left. -6C\beta_3 + 39C\beta_6 + 6D\beta_4 + 93C\beta_5 + 6B\beta_6\right),$$

$$M_{15} = 8\alpha_1 U \left(4D\beta_6 - 4D\beta_8 - 2C\beta_5 - 6C\beta_5 + 6D\beta_6\right),$$

$$N_1 = \left(\frac{M_1}{16} - \frac{10M_3}{256} + \frac{15M_5}{512} - \frac{M_{12}}{128} + \frac{15M_{14}}{1024}\right),$$

$$N_2 = \left(\frac{2M_2}{64} - \frac{30M_4}{512} - \frac{4M_{13}}{512} + \frac{30M_{15}}{1024}\right),$$

$$N_3 = \left(\frac{16}{3}\frac{M_3}{256} + \frac{24M_5}{512} - \frac{8M_{14}}{1024}\right),$$

$$N_4 = \left(\frac{8M_4}{512} + \frac{8M_{15}}{1024}\right), N_5 = \left(\frac{32}{5}\frac{M_5}{512}\right),$$

$$N_6 = \left(\frac{-M_6}{48} + \frac{3368M_8}{9261} - \frac{19094336M_{10}}{1361367} - \frac{44M_{12}}{882} + \frac{126896M_{14}}{129654}\right),$$

$$N_7 = \left(\frac{-21}{2}\frac{M_7}{441} + \frac{35364M_9}{64827} - \frac{924M_{13}}{18522} + \frac{505806M_{15}}{2722734}\right),$$

$$N_8 = \left(\frac{-147M_8}{9261} + \frac{990192M_{10}}{1361367} + \frac{6468M_{14}}{129654}\right),$$

$$N_9 = \left(\frac{-3087}{4}\frac{M_9}{64827} - \frac{135828M_{15}}{2722734}\right),$$

$$N_{10} = \left(\frac{-64827}{5}\frac{M_{10}}{1361367}\right),$$

$$N_{11} = \left(\frac{M_2}{64} - \frac{15M_4}{512} + \frac{M_{11}}{32} - \frac{10M_{13}}{512} + \frac{15M_{15}}{1024}\right),$$

$$N_{12} = \left(\frac{4M_3}{256} - \frac{30M_5}{512} - \frac{30M_{14}}{1024}\right),$$

$$N_{13} = \left(\frac{8M_4}{512} + \frac{16}{3}\frac{M_{13}}{512} + \frac{24M_{15}}{1024}\right),$$

$$N_{14} = \left(\frac{-8M_{15}}{512} + \frac{8M_{14}}{1024}\right), N_{15} = \left(\frac{32}{5}\frac{M_{15}}{1024}\right),$$

$$N_{16} = \left(\frac{-44M_7}{441} + \frac{126898M_9}{64827} + \frac{M_{11}}{96} - \frac{3368M_{13}}{18522} + \frac{19094336M_{15}}{2722734}\right),$$

$$N_{17} = \left(\frac{-924M_8}{9261} + \frac{5058060M_{10}}{1361367} + \frac{M_{12}}{882} - \frac{35364M_{14}}{129654} \right),$$

$$N_{18} = \left(\frac{6468M_9}{64827} + \frac{147M_{13}}{18522} - \frac{990192M_{15}}{2722734} \right),$$

$$N_{19} = \left(\frac{-135828M_{10}}{1361367} + \frac{3087}{4}\frac{M_{14}}{129654} \right),$$

$$N_{20} = \left(\frac{64827}{5}\frac{M_{15}}{2722734} \right),$$

$$\begin{aligned}
P \;=\;& \left(N_1\theta_w + N_2\theta_w^2 + N_3\theta_w^3 + N_4\theta_w^4 + N_5\theta_w^5\right)\sin\theta_w + \left(N_6\theta_w + N_7\theta_w^2 + N_8\theta_w^3 + N_9\theta_w^4\right.\\
&\left. + N_{10}\theta_w^5\right)\sin 3\theta_w + \left(N_{11}\theta_w + N_{12}\theta_w^2 + N_{13}\theta_w^3 + N_{14}\theta_w^4 + N_{15}\theta_w^5\right)\cos\theta_w\\
&+ \left(N_{16}\theta_w + N_{17}\theta_w^2 + N_{18}\theta_w^3 + N_{19}\theta_w^4 + N_{20}\theta_w^5\right)\cos 3\theta_w,
\end{aligned}$$

$$\begin{aligned}
P_1 \;=\;& \left(N_1\theta_w + N_2\theta_w^2 + N_3\theta_w^3 + N_4\theta_w^4 + N_5\theta_w^5\right)\cos\theta_w + \left(N_1 + 2\theta_w N_2 + 3\theta_w^2 N_3 + 4\theta_w^3 N_4\right.\\
&\left. + 5\theta_w^4 N_5\right)\sin\theta_w + \left(N_6 + 2\theta_w N_7 + 3\theta_w^2 N_8 + 4\theta_w^3 N_9 + 5\theta_w^4 N_{10}\right)\sin 3\theta_w + (3N_6\theta_w\\
& + 3\theta_w^2 N_7 + 3\theta_w^3 N_8 + 3\theta_w^4 N_9 + 3\theta_w^5 N_{10}\big)\cos 3\theta_w + \big(N_{11} + 2\theta_w N_{12} + 3\theta_w^2 N_{13}\\
& + 4\theta_w^3 N_{14} + 5\theta_w^4 N_{15}\big)\cos\theta_w - \left(N_{11}\theta_w + N_{12}\theta_w^2 + N_{13}\theta_w^3 + N_{14}\theta_w^4 + N_{15}\theta_w^5\right)\sin\theta_w\\
& + \left(N_{16} + 2\theta_w N_{17} + 3\theta_w^2 N_{18} + 4\theta_w^3 N_{19} + 5\theta_w^4 N_{20}\right)\cos 3\theta_w - \big(3N_{16}\theta_w + 3\theta_w^2 N_{17}\\
& + 3\theta_w^3 N_{18} + 3\theta_w^4 N_{19} + 3\theta_w^5 N_{20}\big)\sin 3\theta_w,
\end{aligned}$$

$$\begin{aligned}
Q_1 \;=\;& \frac{(N_{11} + N_{16})\left(-\cos 2\theta_w - 2(\cos\theta_w)^2\right) - P\left(-3\sin 3\theta_w + \sin\theta_w\right)}{(-6 + 16\cos 4\theta_w - 4\cos 2\theta_w)}\\
&+ \frac{P_1(\cos 3\theta_w - \cos\theta_w)}{(-6 + 16\cos 4\theta_w - 4\cos 2\theta_w)},
\end{aligned}$$

$$Q_2 = \frac{-Q_1(3\cos 3\theta_w - 3\cos\theta_w) + \cos\theta_w(N_{11} + N_{16}) - P_1}{(-3\sin 3\theta_w + \sin\theta_w)},$$

$$Q_3 = -Q_2, Q_4 = -3Q_1 - N_{11} - N_{16}.$$

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