Study of Fluid Flows within a Square Cavity by Using FEM

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1. Finance – Mathematical models
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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
In
MATHEMATICS

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<u>**Certificate**</u>

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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2016

To my Parents, Brother & Sister

and

My beloved, dedicated friends
Irfan Mustafa, Shakaib Arslan Gursal
& Muhammad Usman Rahim

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Preface

Most of the problems encountered in the field of science and engineering are developed in terms of non-linear differential equations [1, 2]. It is a well-known fact that these differential equations cannot be integrated analytically in most cases. It is necessary to apply some method of approximation numerically for most reliable solution. A large number of different approximation methods for solving differential equations exist, the most important and famous method is the finite element method [3, 4].

The effects of thermal boundary conditions on natural convection flow of fluid within an enclosure were investigated by different authors so far by using different numerical schemes. The natural convection flow in a square cavity under the influence of uniformly and non-uniformly heated bottom wall and keeping top wall as well insulated while two vertical walls are cooled by means of two constant temperature baths is examined by Basak et al. [5] and yields consistent performance over a wide range of parameters Rayleigh number (*Ra*) and Prandtl number (*Pr*) with respect to Dirichlet boundary conditions. Various aspects of the subject problem have been investigated by Basak and Ayappa [6], Ostrach [7-9], Gebhart [10], Hoogendoorn [11] and Imberger [12]. A comprehensive numerical study of natural convection flows and heat transfer characteristics in an enclosure with different sidewalls temperatures (i.e. one vertical wall of enclosure is heated and another one is cooled wall, whereas top and bottom of the cavity are insulated) has been made previously by Nicolette et al. [13], Hall et al. [14], Hyun and Lee [15], Fusegi et al. [16], Lage and Bejan [17, 18] and Xia and Murthy [19]. November and Nansteel [20] and Valencia and Frederick [21] have examined the natural convection within square cavity, heated from below and/or the top was cooled. Steady natural convection in fluid-filled rectangular enclosure heated from below and symmetrically cooled from the two vertical side walls is studied numerically by Ganzarolli and Milanez [22]. Subsequently, Aydin et al. [23] has inspected the same flow of fluid to acquire the effect of aspect ratio and Rayleigh number on flow pattern and heat transfer in air-filled rectangular enclosure. Experimentally investigation of high Rayleigh number natural convection in a water-filled cubical enclosure heated simultaneously from below and from the side has been made by Kirkpatrick and Bohn [24] and obtained the experimental measurements and observations of the heat transfer, the flow patterns and the mean and fluctuating temperature distribution. Steady laminar natural convection in air-filled rectangular enclosure heated from below and cooled from above is studied numerically by Corcione [25], for a wide variety of thermal boundary conditions at the side walls, and such numerically study was conducted for different values of both width-to-height aspect ratio of the enclosure and Rayleigh number. The numerical and theoretical study of natural convection in square cavity with heated bottom wall, insulated top wall and cooled vertical walls has been examined, which results the discontinuities in temperature distribution occur at bottom wall in response of uniformly heated bottom wall. The discontinuities may be removed by heating the bottom wall non-uniformly, as investigated by Minkowycz et al. [26] for mixed convection flow on a vertical plate (either heated or cooled). In order to assess the accuracy of the numerical procedure, the algorithm based on the grid size (41×41) for a square enclosure with a side wall heated were investigated and are in agreement with the work of Mallinson and Vahl Davis [27] for $Ra = 10^3$–10^6.

The finite element method (FEM) is one of the major numerical solution technique which has major advantage that a general purpose computer program can be developed easily to analyze various kinds of problems. In particular, any complex shape of problem domain with prescribed conditions can be handled with ease. This thesis is useful as a reference tool for researchers using FEM. Also the thesis is intended to serve as a text for students of mathematics, science and engineering who have acquired some knowledge of elementary numerical analysis. The chapter-wise details of the thesis is as follows:

Chapter 1 provides the basic definitions and law regarding fluid mechanics and phenomenon of heat transfer. Chapter 2 has brief explanation and procedure to implement finite element method in partial differential equations for different geometries. Two examples with the application of FEM using triangular elements are solved with brief steps of numerical computations. Laplace equation is also given as additional example in Appendix-B for better and practical understanding of FEM by presenting FEM solution with 4-node rectangular elements. In Chapter 3, the effects of thermal boundary conditions on natural convection flows within a square cavity [5] are reinvestigated. The modelling of the problem is made subject to the boundary conditions due to different temperature situations at different walls of the enclosure. The solution of the developed problem is computed by using Galerkin finite element method by developing code in MATLAB. The results are presented in term of temperature profiles and Nusselt numbers, and discussed in detail.

Chapter - 1

Preliminaries

In this chapter, some basic definitions and fundamental laws related to next chapters are introduced for better understanding of the readers [1, 2].

1.1 Fluids Mechanics

Fluid mechanics is the subject in which we study the applications of the laws of force and motion to fluids including liquids and gases. In other words, it concerned with the statics and dynamics of fluids (both liquids and gases).

1.2 Fluids

A fluid is a substance which deforms continuously, or flows under the action of shearing forces which act tangentially to a surface of fluid. In other words, there is no action of shearing force when fluid is at rest.

Liquid

It is the state of matter in which the molecules are relatively free to change their positions with respect to each other, but restricted by cohesive forces so as to maintain a relatively fixed volume.

Gas

It is the state of matter in which the molecules are practically unrestricted by cohesive forces. Therefore, gases has neither definite shape nor volume.

1.3 Stress

A stress is defined as a force acting per unit area of an infinitesimal surface element.

1.4 Types of Stress

There are two types of stress. These are normal stresses and tangential stresses which are defined as follows:

1.4.1 Normal Stress

The stress which acts perpendicularly to the plane to which a force has been applied.

1.4.2 Tangential Stress

A stress which acts along the surface or parallel to the surface.

1.5 Types of Fluids

Fluids can be classified into four basic types, which are as under:-

1. Ideal Fluid
2. Real Fluid
3. Newtonian Fluid
4. Non-Newtonian Fluid

Details of above each has been given as under:

1.5.1 Ideal Fluids

The fluids which has no resistance in between their molecules are known as ideal fluids. In other words, fluids having zero viscosity are known as ideal fluids. Practically, no ideal fluid exists.

1.5.2 Real Fluids

The Fluids which have some resistance in between their molecules, particles or layers are known as real fluids. They are compressible in nature, and have some viscosity. Kerosene, Petrol and Castor oil are common examples of real fluids.

1.5.3 Newtonian Fluids

Newtonian fluid is a fluid in which the viscous stresses arising from its flow at every point are linearly proportional to the local strain rate (the rate of change of its deformation over time). In other words, fluids which obey the Newton's law of viscosity are called as Newtonian fluids. Newton's law of viscosity is given by

$$\tau = \mu \frac{dv}{dy}.$$

where τ is shear stress, μ is viscosity of the fluid and $\frac{dv}{dy}$ is commonly known by shear rate, rate of strain or velocity gradient. The water, benzene and ethyl alcohol are commonly known as Newtonian fluids.

1.5.4 Non-Newtonian Fluids

A non-Newtonian fluid is a fluid whose viscosity is variable based on applied stress, and such fluids do not obey the Newton's law of viscosity. Common examples of non-Newtonian fluids are ketchup, starch suspensions, paint, blood and shampoo etc.

1.6 Properties of Fluids

Any fluid is characterized by the following properties:

1. Density
2. Viscosity
3. Coefficient of Dynamic Viscosity
4. Kinematic Viscosity

1.6.1 Density

The density (ρ) of a substance is the quantity of matter (mass) contained in a unit volume of the substance. Mathematically, it can be expressed by

$$\rho = \frac{m}{V},$$

where ρ is the density, m is the amount of mass in unit volume V. The unit of density is kgm^{-3} and dimension is M/L^3.

1.6.2 Viscosity

Viscosity (μ) is the property of a fluid, due to cohesion and interaction between the molecules which offers resistance to sheer deformation. Fluid with a high viscosity such as honey or syrup deforms more slowly than that of fluid with a low viscosity such as water.

1.6.3 Coefficient of Dynamic Viscosity

The coefficient of dynamic viscosity (μ) is defined as the shear force per unit area, (or shear stress τ) required to drag one layer of fluid with unit velocity past another layer a unit distance away. Mathematically

$$\mu = \tau \Big/ \frac{dv}{dy} = \frac{\text{Force}}{\text{Area}} \Big/ \frac{\text{Velocity}}{\text{Distance}} = \frac{\text{Force} \times \text{Time}}{\text{Area}} = \frac{\text{Mass}}{\text{Length} \times \text{Area}}.$$

Units of μ are Newton seconds per square meter (Nsm^{-2}) or Kilograms per meter per second $(kgm^{-1}s^{-1})$.

1.6.4 Kinematic Viscosity

Kinematic viscosity (υ) is defined as the ratio of dynamic viscosity to mass density. Mathematically, it can be expressed by

$$\upsilon = \frac{\mu}{\rho}.$$

The unit of υ is square meters per second ($m^2 s^{-1}$) and dimension is L^2/T.

1.7 Types of Fluid Flow

There are many types to classify flow of fluid and describe the state of fluid flow under different circumstances. Some types of fluid flow are as under

1. Uniform Flow
2. Non-Uniform Flow
3. Steady Flow
4. Unsteady Flow
5. Laminar Flow
6. Turbulent Flow

Explanation of each is given below.

1.7.1 Uniform Flow

If the fluid velocity remains same at every point in the flow, then it is said to be uniform flow.

1.7.2 Non-Uniform Flow

If at a given instant, the velocity is not the same at every point, the flow is known as non-uniform.

1.7.3 Steady Flow

A steady flow is one in which the fluid characteristics (velocity, pressure and cross-section) do not change with time.

1.7.4 Unsteady Flow

If at any point in the fluid, the fluid behaviors change with time, the flow is described as unsteady.

1.7.5 Laminar Flow

The flow of a fluid in which particles of the fluid move in parallel layers, each of which has a constant velocity is known as laminar flow.

1.7.6 Turbulent Flow

A fluid flow in which the velocity at a given point varies erratically in magnitude and direction is known as turbulent flow.

1.8 Compressible Fluids

If the density of a fluid varies significantly due to moderate changes in pressure or temperature, such fluid is referred as compressible fluid. Generally, gases and vapours under normal conditions can be classified as compressible fluids.

1.9 Incompressible Fluids

If the variation in temperature or pressure causes a small change in density of a fluid, then the fluid is known as incompressible fluid.

1.10 Streamlines

In analyzing fluid flow, it is useful to visualize the flow pattern by drawing lines joining points of equal velocity *i.e.* velocity contours. These lines are known as streamlines. Here is a simple example of the streamlines around a cross-section of an aircraft wing shaped body:

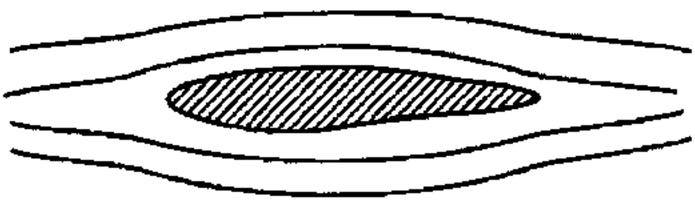


Figure 1.1: Streamlines around a wing shaped body

1.11 Buoyancy Force

The upward force that a fluid exerts on an object which is completely or partly submerged in it is called buoyancy force. This force causes the objects to float. Moreover, buoyancy allows boat to float on water and provides lift for balloons.

1.12 Convection

Convection is the process in which heat moves through a gas or a liquid. In other words, the mode of heat transfer in liquids and gases is known as convection. Transfer of heat through convection is categorized in three different types, each one is explained below.

1.12.1 Natural Convection

Natural convection or free convection is a mechanism of heat transfer in which the fluid motion is generated due to density difference in the fluid occurring due to temperature gradients.

1.12.2 Forced Convection

Forced convection is a mechanism in which the fluid motion results from external surface forces such as fan or pumps. Forced convection may happen by natural means. For example, fluid radiator, heating and cooling of parts of the body by blood circulation are familiar examples of forced convection.

1.12.3 Mixed Convection

Mixed convection occurs when natural convection and forced convection mechanisms act together to transfer heat. This is also defined as situations where both pressure forces and buoyant forces interact.

1.13 Non-dimensional Quantities

The following numbers are the common non-dimensional numbers used in fluid mechanics.

1.13.1 Nusselt Number (Nu)

A dimensionless parameter defined as the ratio of convection heat transfer to fluid conduction heat transfer under the same conditions. Mathematically

$$Nu_L = \frac{\text{Convective heat transfer}}{\text{Conductive heat transfer}} = \frac{hL}{k},$$

where h is the convective heat transfer coefficient of the flow, L is the characteristic length and k is the thermal conductivity of the fluid.

In contrast to the definition given above, average Nusselt number and local Nusselt number are defined by taking the length to be the distance from the surface boundary to the local point of interest. *i.e.*

$$Nu_x = \frac{h_x x}{k}.$$

The mean or average Nusselt number is obtained by integrating the expression over the range of interest, such as

$$\overline{Nu} = \frac{1}{H}\int_0^H Nu(y)\,dy.$$

1.13.2 Rayleigh Number (Ra)

The Rayleigh number for a fluid is a dimensionless number associated with buoyancy driven flow. When the Rayleigh number is below the critical value for that fluid, heat transfer is primarily in the form of conduction; when it exceeds the critical value, heat transfer is primarily in the form of convection. The magnitude of the Rayleigh number is a good indicator as to whether the natural convection boundary layer is laminar or turbulent. Mathematically, it is the product of Grashof number Gr and the Prandtl number Pr, *i.e.* $Ra = GrPr$.

1.13.3 Prandtl Number (Pr)

The Prandtl number is another dimensionless number defined as the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity. Mathematically, it can be defined as:

$$\Pr = \frac{\upsilon}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{c_p \mu}{k},$$

where υ be the kinematic viscosity, α be the thermal diffusivity, μ be the dynamic viscosity, k be the thermal conductivity and c_p be the specific heat.

1.14 Momentum Equation

Linear momentum equation for fluids are developed due to Newton's second law which states that sum of all forces must equal the time rate of change of the momentum, $\sum F = d(mv)/dt$. This is easy to apply in particle mechanics but for fluids, it gets more complex due to the

Arbitrary Volume

control volume (and not individual particles). The change of momentum will have two parts, momentum inside the control volume, and momentum passing through the surface. This concept can be written as

$$\sum F = \frac{\partial}{\partial t}\int_{CV} \rho V \, dV + \int_{CS} V \rho V . n dA ,$$

where V is the velocity vector, n is the outward unit normal vector, and $\sum F$ represents the sum of all forces (body and surface forces) applied to the control volume.

1.15 Thermal Conductivity

Thermal conductivity is a material property which describes the ability to conduct heat. More appropriately, it is defined as the quantity of heat transmitted through a unit thickness of a material in a direction normal to a surface of unit area due to a unit temperature gradient under steady state conditions. Its unit is $W/(mK)$ in the SI system.

1.16 Thermal Diffusivity

In heat transfer analysis, thermal diffusivity is the thermal conductivity divided by the product of density and specific heat capacity at constant pressure. Mathematically, it is denoted by α and defined as

$$\alpha = \frac{k}{\rho c_p},$$

where k be thermal conductivity, ρ be density and c_p be the specific heat capacity.

1.17 No Slip Condition

The fluid has zero velocity at the boundary of solid with witch it is in contact. It occurs due to the strong force of attraction between the fluid particles and solid particles (Adhesive Forces), such condition of viscous fluids is known as no slip condition.

Chapter - 2

Fundamental of the Finite Element Method

The aim of this chapter is to discuss the procedure involved in using the finite element method to solve any partial differential equation subject to the boundary conditions in two dimensional space [3, 4]. The two and three dimensional finite elements used in discretization process of the geometry, shape functions w.r.t different number of nodes, calculation of element stiffness and global stiffness matrices, implementation of boundary conditions and post-processing are discussed in reasonable detail. Moreover, two examples solved with detail calculations are provided in this chapter for better understanding the implementation of finite element method [4].

2.1 Introduction

Many physical phenomena occurring in engineering and daily life can be modeled in terms of partial differential equations subject to some boundary conditions. It is observed that solution of these equations for arbitrary domain is impossible by using classical analytical methods. In this situation, the finite element method (FEM) is an extremely reliable computation technique used to obtain approximate solution of these partial differential equations. For this purpose, converting the given domain into a number of non-overlapping small pieces connected by nodes is required. These small pieces are called finite elements. Then the governing equations for every element are solved to get numerical solution within each element. At last, combining the solutions at all such elements gives the approximate solution for entire domain of given problem. The accuracy of the computed solution may be achieved by increasing the number of elements as well as number of nodes.

In order to obtain the solution over the domain, methods of residuals are used, which are explained as follows.

2.2 Methods of Weighted Residual

The method of weighted residual can be described in its generality by assuming the partial differential equation in operator form as,

$$D(u) = g \quad \text{in } \Omega, \tag{2.1}$$

where D is linear/non-linear differential operator acting on dependent variable u, g is a given function and Ω is a two dimensional domain.

In this method, the solution of equation (2.1) can be approximated by linear combination of basis/shape functions taken from linearly independent set as follows

$$\tilde{u} = \sum_{i=1}^{n} b_i \phi_i . \tag{2.2}$$

In which, b_i's are unknown constants required to determine and ϕ_i are linearly independent basis functions. Substitution of the approximate solution $\tilde{u}$ into the left hand side of Eq. (2.1), the result $D(\tilde{u})$, in general, is not equal to specified function g due to the fact that solution (2.2) is not the exact solution of the problem (2.1). The difference $D(\tilde{u}) - g \neq O$, is known as the residual of the approximation, and is

$$R = D(\tilde{u}) - g = D\left(\sum_{i=1}^{n} b_i \phi_i\right) - g \neq 0 \tag{2.3}$$

To evaluate unknown constants b_i, choose weight functions W_i, set the weighted average of the residual over the problem domain to zero, i.e.

$$\int_{\Omega} W_i(x) R(x, b_i)\, dx dy = 0, \qquad (i = 1,2,3,\ldots,n). \tag{2.4}$$

In general, the choice of the weight functions W_i are not the same as the basis function ϕ_i, but they (W_i) are exactly equal the number of unknown constants b_i. Following methods of weighted residual are classified in terms of the choice of weight function W_i:

1. Collocation Method,
2. Least Squares Method,
3. Galerkin Method and
4. Method of moments

Each of these has been explained below.

2.2.1 Collocation Method

In this method, the Dirac Delta function $\delta(x - x_i)$ is used as weight function, defined as follow

$$\delta(x - x_i) = \begin{cases} 1 \ , & x = x_i \\ 0 \ , & \text{otherwise} \end{cases} \tag{2.5}$$

where the point x_i must be within domain Ω, the unknown constants b_i can be evaluated by setting the weighted average of the residual in Eq. (2.4) equal to zero at specific points in the domain. That is

$$\int_{\Omega} \delta(x - x_i) R(x, b_i)\, dx = 0 \qquad \text{or} \qquad R(x_i, b_i) = 0 . \tag{2.6}$$

2.2.2 Least Squares Method

In this method, the integral of square of residual is minimized by setting its derivative with respect to parameters b_i equal to zero. That is

$$\frac{\partial}{\partial b_i} \int_{\Omega} R^2(x, b_i)\, dx = 0 \quad \Rightarrow \quad \int_{\Omega} R \frac{\partial R}{\partial b_i} dx = 0 . \tag{2.7}$$

Comparison of above integral with Eq. (2.4) imply

$$W_i = \frac{\partial R}{\partial b_i} . \tag{2.8}$$

Therefore, the weight functions are just the derivatives of the residual with respect to the unknown constants b_i.

2.2.3 Galerkin Method

In Galerkin Method, the weight functions W_i are chosen equal to basis functions ϕ_i, *i.e.*

$$W_i = \frac{\partial \tilde{u}}{\partial b_i} = \phi_i . \tag{2.9}$$

2.2.4 Method of Moments

In the method of moments, weight functions are selected from the family of polynomials,

$$W_i = x^i , \qquad i = 0, 1, 2, \ldots, n-1 . \tag{2.10}$$

In order to find the unknown constants b_i through weighted average residual equations (2.4) by choosing suitable weighted functions given in above method, it is required to integrate Eq. (2.4) once analytically is called weak formulation, which is explained as below.

2.3 Weak Formulation

The differential equation along with boundary conditions of the given problem is referred as strong form. These differential equations are difficult to solve due to presence of higher order derivatives, and basis functions in this situation are required to be high order differentiable and smooth. To avoid this exertion, there is a need to remove or decrease the order of differential equations by one through integration is known as weak formulations. It is further noted that the manipulation of the problem in weak form is comparatively easy as that of strong form. In Finite Element Method, the weighted average residual are required to integrate over the finite elements obtained through discretization process in the domain.

2.4 Discretization and Element Mesh of the Domain

The process to divide the geometry or physical domain of the problem into finite number of non-overlapping elements of any shape is known as discretization. The collection of finite elements in a domain is called the finite element mesh of the domain.

The simple meshing of square geometry divided into triangular elements is shown in Figure 2.1. It is important that the solution of the desired problem by using finite element method tends to highly accurate by increasing the number of finite elements.

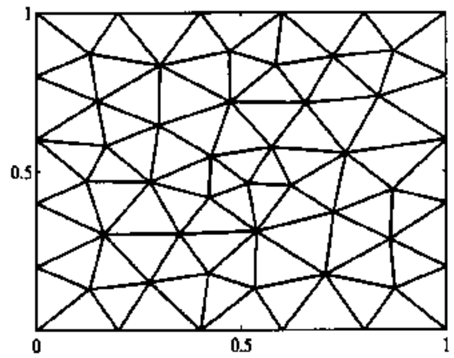


Figure 2.1: Meshing of square geometry in triangular elements

2.5 Types of Elements

Generally, straight-line segments are used as elements in one dimensional case, triangles, rectangles or elements with algebraic curves are used in two dimensional case, and tetrahedron or hexahedron shape of elements are used in three dimensions space. They are explained in detail as follows

2.5.1 Line Segment Element

We divide the interval $[a,b]$ in one dimensional space into non-overlapping subintervals $R_i=[x_i,x_{i+1}]$, $0\le i\le N$, with $x_0=a$ and $x_{N+1}=b$. Each interval $[x_i,x_{i+1}]$ is an element and we represent it by (e) as shown in Figure 2.2,

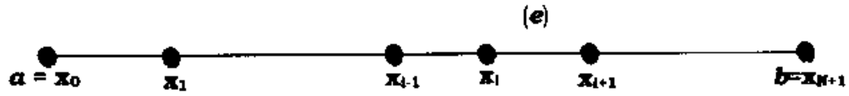
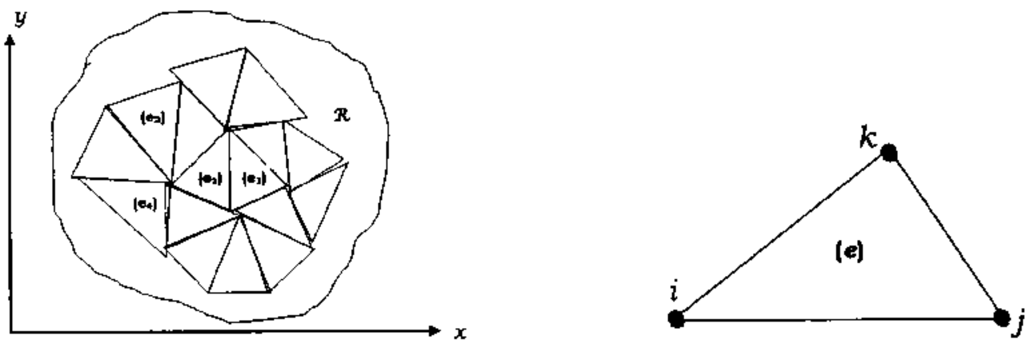


Figure 2.2: Division of an interval into line segment elements

2.5.2 Triangular Element

The region $\mathcal{R}$ in two dimensional space can be divided into triangular elements as shown in Figure 2.3(a). Here each element is a triangle with nodes *i, j, k* numbered anticlockwise as signified in Figure 2.3(b). We assume that the nodes of the region $\mathcal{R}$ are consecutively numbered from 1 to *N*. Further, we denote the value of the function $u(x,y)$ at the node *i* by u_i.



(a) (b)

Figure 2.3: (a) Division of two-dimensional region into triangular elements and (b) Triangular element with allocated nodes at its vertices

Moreover, each triangular element may be categorize by four noded, six noded and ten noded triangular element.

2.5.3 Rectangular Element

The simplest rectangular element has 4 nodes at the vertices with 1 degree of freedom per node, as shown in Figure 2.4. The figure also shows the local node numbering system $(1, 2, 3, 4)$, the nodal coordinates (x_i^e, y_i^e) and the nodal degrees of freedom (dof) T_i^e of local node *i*. The local numbering system usually starts from bottom left corner and is counterclockwise. This is called as local notation. A physical problem is solved by using 4-node rectangular element and is explained in Appendix B for better and practical understanding of finite element method.

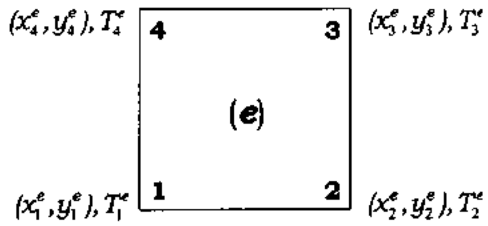


Figure 2.4: Four Noded Rectangular Element

Rectangular element may further be categorized by eight noded, nine noded, twelve noded and sixteen noded rectangular elements.

2.5.4 Quadrilateral Element

A general quadrilateral element (e) with four nodes, one at each corner is shown in Figure 2.5. The coordinates of the vertices at the node i of element e is represented by (x_i^e, y_i^e).

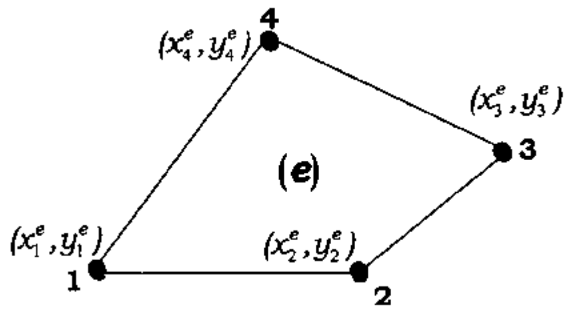


Figure 2.5: Quadrilateral Element

Likewise triangular and rectangular elements, quadrilateral element may also be occurred in eight and twelve noded quadrilateral elements.

2.5.5 Curved Boundary Element

Any physical domain $\mathcal{R}$ in two dimensional space with curved boundaries as shown in Figure 2.6(a) is discretized by triangular elements as shown in Figure 2.6(b). If some of the boundary ∂R of the domain $\mathcal{R}$ is curved, then we may either approximate it by a polygon and use the triangle and quadrilaterals for discretization as shown in Figure 2.6(b) and 2.6(c) or we use triangular elements with at least one curved side as shown in Figure 2.6(d).

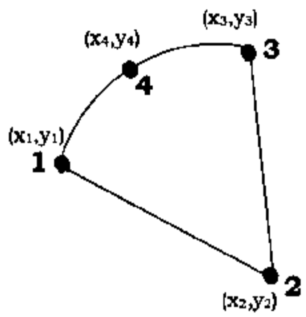


Figure 2.6 (d): Triangular Element with one curved side

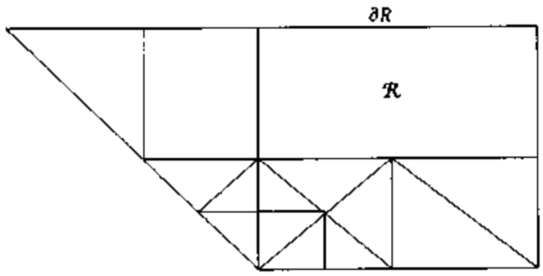


Figure 2.6 (c): Division of domain R with polygon boundary ∂R

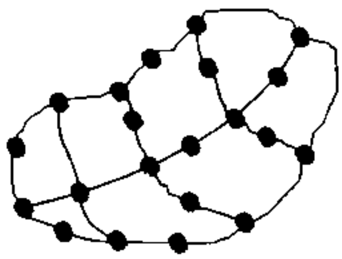


Figure 2.6 (a): Division of domain R with curved-sided elements

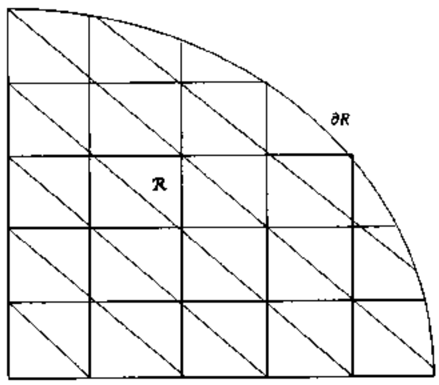


Figure 2.6 (b): Division of domain R with one curved side boundary ∂R

2.6 Shape Functions

In finite element analysis, the model of continuous body is divided in finite elements containing a many number of nodes, the shape of the body between these nodes is estimated by functions, these functions are called shape functions. Moreover, shape function interpolates the solution between the discrete values obtained at mesh nodes.

The characteristics of shape functions are

1. The shape function at any node has a value of 1 at that node and a value of zero at all other nodes, *i.e.*

$$N_i(x_j, y_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

2. The sum of all the shape functions, evaluated at any point must be unity, *i.e.*

$$\sum_i N_i(x, y) = 1$$

2.7 Local and Global Nodes of Element

Consider the following geometry containing four triangular elements represented by the number **1**, **2**, **3** and **4**, each one is defined by three nodes. Red colored numeric numbers (1, 2, 3, 4, 5 and 6) located on the boundary, represent the global nodes. The alphabets p, q and r represented counter-clockwise at inner corner of each triangular element are local node labels for respective element. as shown in Figure 2.7.

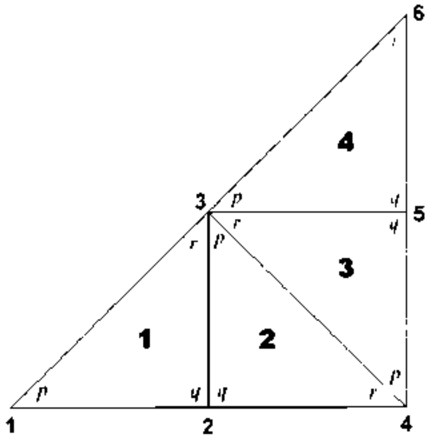


Figure 2.7: Local & Global Nodes in triangular geometry

2.8 Local and Global Stiffness Matrices

The coefficient matrix obtained from the weak form of given differential equation corresponding to an element is referred as local stiffness matrix/element matrix. The assembly of all local stiffness matrices using the equivalence between local and global nodes, is termed as Global stiffness matrix. They both are square symmetric matrices.

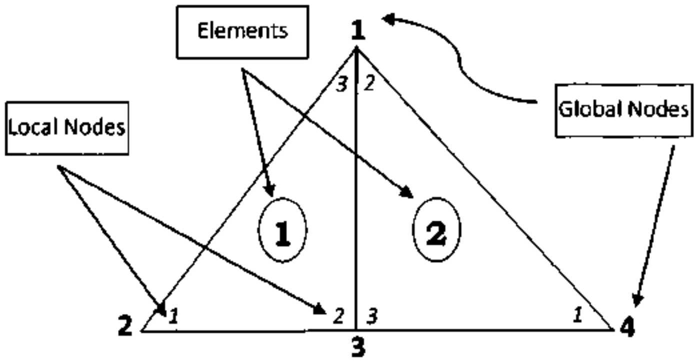


Figure 2.8: Triangular geometry allocated into two elements, symbolize with local & global nodes

Consider the geometry consisting of two triangular elements as shown in Figure 2.8. To get philosophy of subject topic, consider assumed values of local stiffness matrices corresponding to each element as under:

Element 1

$$\begin{array}{c} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{1} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{2} & \mathbf{3} & \mathbf{1} \end{array} \\ \begin{pmatrix} 2 & 5 & 1 \\ 0 & 4 & 5 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

Element 2

$$\begin{array}{c} \\ \mathbf{4} \\ \mathbf{1} \\ \mathbf{3} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{4} & \mathbf{1} & \mathbf{3} \end{array} \\ \begin{pmatrix} 2 & 5 & 1 \\ 0 & 4 & 5 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

Assembly of above element matrices generates a global stiffness matrix as follow:

$$\begin{array}{c|cccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & \mathit{0}+\mathbf{4} & \mathit{1} & \mathit{1}+\mathbf{5} & \mathbf{0} \\ \mathbf{2} & \mathit{1} & \mathit{2} & \mathit{5} & \\ \mathbf{3} & \mathit{5}+\mathbf{1} & \mathit{0} & \mathit{4}+\mathbf{0} & \mathbf{1} \\ \mathbf{4} & \mathbf{5} & & \mathbf{1} & \mathbf{2} \end{array}$$

Italic numbers represent the entries of first element matrix and **bold** entries inside the above matrix belong to second element matrix. Blank locations corresponds to where no entry is allocated from both (elements) matrices, zero entry will be allotted there. Finally, the required global matrix is obtained, which is

$$\begin{pmatrix} 4 & 1 & 6 & 0 \\ 1 & 2 & 5 & 0 \\ 6 & 0 & 4 & 1 \\ 5 & 0 & 1 & 2 \end{pmatrix}.$$

2.9 Solution procedure using FEM

To compute the solution of the problem by using finite element method, we undergo the following steps:

- Discretization of the domain into a set of finite elements.
- Define an approximate solution of given differential equation over an element, such defined solution must satisfy the given boundary conditions.
- Define shape functions as per type of element per number of nodes.
- Choose weight function through using methods of weighted residual.
- Set up a weak formulation of given differential equation.
- Evaluate the weak form of given differential equation for each element using given boundary conditions, and obtain value of local stiffness matrix corresponding to each element.

- Assemble all elementwise local stiffness matrices to generate global stiffness matrix.
- Solve the algebraic system of equations to get desired solution by using any direct/indirect/iterative method.
- and post-processing (This final operation displays the solution to system equations in tabular, graphical or pictorial form. Other meaningful quantities may be derived from the solution and also displayed).

2.10 Examples

Examples with the implementation of FEM are given below for explanation of the method.

2.10.1 **<u>Problem</u>**. Solve the boundary value problem using Finite Element Method

$$\begin{aligned} \nabla^2 u &= -1, \quad |x| \le 1, \ |y| \le 1 \\ u &= 0, \qquad |x| = 1, \ |y| = 1 \end{aligned} \tag{2.11}$$

with $h = \frac{1}{2}$ by using three nodes 32 triangular elements as shown in Figure 2.9.

<u>Solution</u>. The solution of the boundary value problem satisfies the symmetry conditions,

$u(-x,y) = u(x,y)$, $\quad u(x,-y) = u(x,y)$, $\quad u(y,x) = u(x,y)$

Therefore, we shall consider only one eighth of the square as shown in bold black patch of the Figure 2.9.

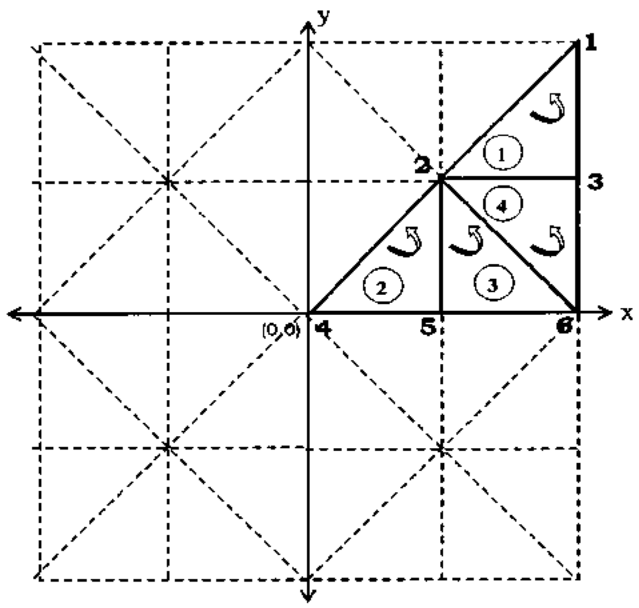


Figure 2.9: Representation of elements with nodal points

After discretizing, the length of each triangular element is $\frac{1}{2}$ (*i.e.* $h=\frac{1}{2}$). There are four elements which are numbered 1, 2, 3 and 4. For given problem, the element functional is

$$J^e=\frac{1}{2}\iint_e\left[\left(\frac{\partial u^e}{\partial x}\right)^2+\left(\frac{\partial u^e}{\partial y}\right)^2-2u\right]dxdy, \tag{2.12}$$

where superscript e denotes an element with nodes *i, j* and *k* marked in counterclockwise as represented in Figure 2.10.

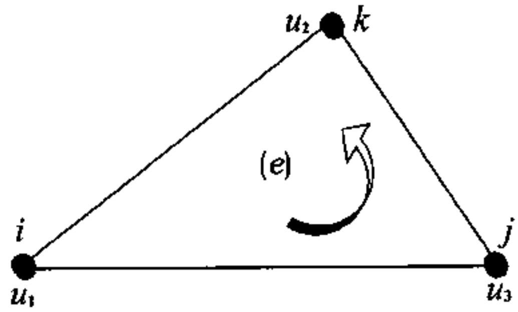


Figure 2.10: Three Noded Triangular Element

The element equation is $\frac{\partial J^e}{\partial u^i}=0$, *i.e.*

$$\frac{\partial J^e}{\partial u^i}=\frac{1}{2}\iint_e\left[\left\{\left(\frac{\partial H_i}{\partial x}\right)^T\frac{\partial H_i}{\partial x}+\left(\frac{\partial H_i}{\partial y}\right)^T\frac{\partial H_i}{\partial y}\right\}u_i-2H_i\right]dxdy=0. \tag{2.13}$$

We have value of u in terms of nodal variables (c_1, c_2, c_3 and c_4),

$$u=c_1H_1(x,y)+c_2H_2(x,y)+c_3H_3(x,y), \tag{2.14}$$

where $H_i(x,y)$; $i=1,2,3$ are shape functions for linear triangular element, which are given below

$$H_1=\frac{1}{2A}\left[(x_2y_3-x_3y_2)+(y_2-y_3)x+(x_3-x_2)y\right], \tag{2.15}$$

$$H_2=\frac{1}{2A}\left[(x_3y_1-x_1y_3)+(y_3-y_1)x+(x_1-x_3)y\right], \tag{2.16}$$

$$H_3=\frac{1}{2A}\left[(x_1y_2-x_2y_1)+(y_1-y_2)x+(x_2-x_1)y\right]. \tag{2.17}$$

The shape functions satisfy the following conditions

$$H_i(x_j,y_j)=\delta_{ij}=\begin{cases}1 & if\ i=j\\ 0 & if\ i\neq j\end{cases} \quad \text{and} \quad \sum_{i=1}^{3}H_i=1, \tag{2.18}$$

where,

$$A=\frac{1}{2}\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}.$$

Magnitude of A is equal to the area of the linear triangular element. However, its value is positive if the element node numbering is in counter-clockwise direction and negative otherwise. For the finite element computation, the element nodal sequence must be in the same direction for every element in the domain.

First two terms on right side of integral (2.13) implies

$$\frac{1}{2}\iint_e\left[\left\{\left(\frac{\partial H_i}{\partial x}\right)^T\frac{\partial H_i}{\partial x}+\left(\frac{\partial H_i}{\partial y}\right)^T\frac{\partial H_i}{\partial y}\right\}u_i\right]dxdy=\frac{1}{2}\iint_e\left[\left\{\begin{pmatrix}\frac{\partial H_1}{\partial x}\\ \frac{\partial H_2}{\partial x}\\ \frac{\partial H_3}{\partial x}\end{pmatrix}\left(\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial x}\frac{\partial H_3}{\partial x}\right)+\begin{pmatrix}\frac{\partial H_1}{\partial y}\\ \frac{\partial H_2}{\partial y}\\ \frac{\partial H_3}{\partial y}\end{pmatrix}\left(\frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial y}\frac{\partial H_3}{\partial y}\right)\right\}u_i\right]dxdy. \tag{2.19}$$

Performing integration after substituting the shape functions, we obtained the following matrix

$$\left[K^e\right]=\begin{bmatrix}k_{11} & k_{12} & k_{13}\\ k_{21} & k_{22} & k_{23}\\ k_{31} & k_{32} & k_{33}\end{bmatrix}. \tag{2.20}$$

In which,

$$k_{11}=\frac{1}{4A}\left[(x_3-x_2)^2+(y_3-y_2)^2\right],$$

$$k_{12}=\frac{1}{4A}\left[(x_3-x_2)(x_1-x_3)+(y_2-y_3)(y_3-y_1)\right]=k_{21},$$

$$k_{13}=\frac{1}{4A}\left[(x_3-x_2)(x_2-x_1)+(y_2-y_3)(y_1-y_2)\right]=k_{31},$$

$$k_{22}=\frac{1}{4A}\left[(x_1-x_3)^2+(y_3-y_1)^2\right],$$

$$k_{23}=\frac{1}{4A}\left[(x_1-x_3)(x_2-x_1)+(y_3-y_1)(y_1-y_2)\right]=k_{32},$$

$$k_{33}=\frac{1}{4A}\left[(x_2-x_1)^2+(y_1-y_2)^2\right].$$

The element-wise local stiffness matrices are computed as follows:

For 1st Element

$$A=\frac{1}{2}\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}=\frac{1}{8},$$

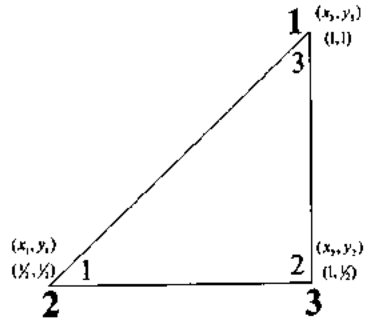


Figure 2.11: 1st element with allocated Local & Global Nodes

$$k_{11}=2\left[(1-1)^2+\left(\frac{1}{2}-1\right)^2\right]=\frac{1}{2},$$

$$k_{12}=2\left[(1-1)\left(\frac{1}{2}-1\right)+\left(\frac{1}{2}-1\right)\left(1-\frac{1}{2}\right)\right]=-\frac{1}{2}=k_{21},$$

$$k_{13}=2\left[(1-1)\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-\frac{1}{2}\right)\right]=0=k_{31},$$

$$k_{22}=2\left[\left(\frac{1}{2}-1\right)^2+\left(1-\frac{1}{2}\right)^2\right]=1,$$

$$k_{23}=2\left[\left(\frac{1}{2}-1\right)\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}\right)\right]=-\frac{1}{2}=k_{32},$$

$$k_{33}=2\left[\left(1-\frac{1}{2}\right)^2+\left(\frac{1}{2}-\frac{1}{2}\right)^2\right]=\frac{1}{2}.$$

Using matrix Eq. (2.20), we obtained local stiffness matrix for 1st element is

$$K^{(1)}=\begin{array}{c} \\ \mathbf{2}\\ \mathbf{3}\\ \mathbf{1}\end{array}\begin{array}{c}\begin{array}{ccc}\mathbf{2} & \mathbf{3} & \mathbf{1}\end{array}\\ \begin{pmatrix}0.5 & -0.5 & 0\\ -0.5 & 1 & -0.5\\ 0 & -0.5 & 0.5\end{pmatrix}\end{array}. \qquad (2.21)$$

For 2nd Element

$$k_{11}=2\left[\left(\frac{1}{2}-\frac{1}{2}\right)^2+\left(0-\frac{1}{2}\right)^2\right]=\frac{1}{2},$$

$$k_{12}=2\left[0+\left(0-\frac{1}{2}\right)\left(\frac{1}{2}-0\right)\right]=-\frac{1}{2}=k_{21},$$

$$k_{13}=2\left[0+0\right]=0=k_{31},$$

$$k_{22}=2\left[\left(0-\frac{1}{2}\right)^2+\left(\frac{1}{2}-0\right)^2\right]=1$$

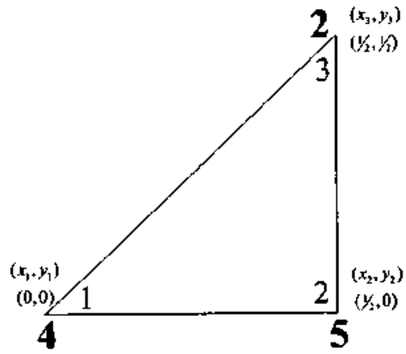


Figure 2.12: 2nd element with allocated Local & Global Nodes

$$k_{23}=2\left[\left(0-\frac{1}{2}\right)\left(\frac{1}{2}-0\right)+\left(\frac{1}{2}-0\right)(0-0)\right]=-\frac{1}{2}=k_{32},$$

$$k_{33}=2\left[\left(\frac{1}{2}-0\right)^2+0\right]=\frac{1}{2}.$$

We obtained element matrix for 2nd element is

$$K^{(2)} = \begin{matrix} & \mathbf{4} & \mathbf{5} & \mathbf{2} \\ \mathbf{4} & 0.5 & -0.5 & 0 \\ \mathbf{5} & -0.5 & 1 & -0.5 \\ \mathbf{2} & 0 & -0.5 & 0.5 \end{matrix} \quad (2.22)$$

For 3rd Element

$$k_{11} = 2\left[\left(\frac{1}{2}-1\right)^2 + \left(0-\frac{1}{2}\right)^2\right] = 1,$$

$$k_{12} = 2\left[\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-\frac{1}{2}\right) + \left(0-\frac{1}{2}\right)\left(\frac{1}{2}-0\right)\right] = -\frac{1}{2} = k_{21},$$

$$k_{13} = 2\left[\left(\frac{1}{2}-1\right)\left(1-\frac{1}{2}\right) + \left(0-\frac{1}{2}\right)(0-0)\right] = -\frac{1}{2} = k_{31},$$

$$k_{22} = 2\left[0 + \left(\frac{1}{2}-0\right)^2\right] = \frac{1}{2},$$

$$k_{23} = 2[0+0] = 0 = k_{32},$$

$$k_{33} = 2\left[\left(1-\frac{1}{2}\right)^2 + 0\right] = \frac{1}{2}.$$

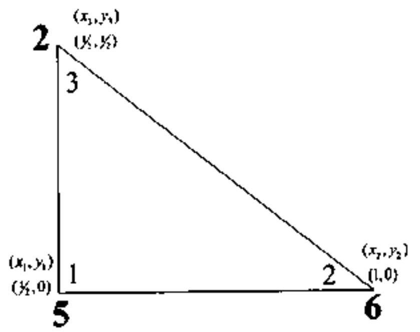


Figure 2.13: 3rd element with allocated Local & Global Nodes

We obtained local stiffness matrix for 3rd element is

$$K^{(3)} = \begin{matrix} & \mathbf{5} & \mathbf{6} & \mathbf{2} \\ \mathbf{5} & 1 & -0.5 & -0.5 \\ \mathbf{6} & -0.5 & 0.5 & 0 \\ \mathbf{2} & -0.5 & 0 & 0.5 \end{matrix} \quad (2.23)$$

For 4th Element

$$k_{11} = 2\left[\left(\frac{1}{2}-1\right)^2 + 0^2\right] = \frac{1}{2},$$

$$k_{12} = 2\left[\left(\frac{1}{2}-1\right)\left(1-\frac{1}{2}\right) + 0\right] = -\frac{1}{2} = k_{21},$$

$$k_{13} = 2\left[\left(\frac{1}{2}-1\right)(1-1) + 0\right] = 0 = k_{31},$$

$$k_{22} = 2\left[\left(1-\frac{1}{2}\right)^2 + \left(\frac{1}{2}-0\right)^2\right] = 1,$$

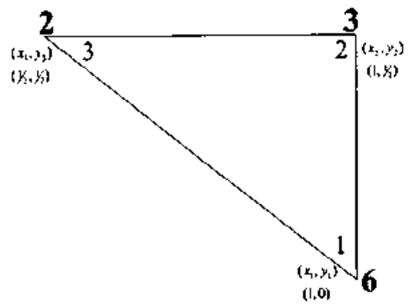


Figure 2.14: 4th element with allocated Local & Global Nodes

$$k_{23}=2\left[\left(1-\frac{1}{2}\right)(1-1)+\left(\frac{1}{2}-0\right)\left(0-\frac{1}{2}\right)\right]=-\frac{1}{2}=k_{32}\ ,$$

$$k_{33}=2\left[0+\left(0-\frac{1}{2}\right)^{2}\right]=\frac{1}{2}.$$

Thus, we obtained the following local stiffness matrix for 4[th] element is

$$K^{(4)}=\begin{array}{c} \\ \mathbf{6} \\ \mathbf{3} \\ \mathbf{2} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{6} & \mathbf{3} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{pmatrix} \end{array}. \tag{2.24}$$

Now, the assembly of all local stiffness matrices using Eqs. (2.21 to 2.24) by connecting the elements corresponding to global nodes generate a global stiffness matrix as under:

$$\begin{array}{c|cccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \hline \mathbf{1} & 0.5 & 0 & -0.5 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 0.5+0.5+0.5+0.5 & -0.5-0.5 & 0 & -0.5-0.5 & 0+0 \\ \mathbf{3} & -0.5 & -0.5-0.5 & 1+1 & 0 & 0 & -0.5 \\ \mathbf{4} & 0 & 0 & 0 & 0.5 & -0.5 & 0 \\ \mathbf{5} & - & -0.5-0.5 & 0 & -0.5 & 1+1 & -0.5 \\ \mathbf{6} & 0 & 0+0 & -0.5 & 0 & -0.5 & 0.5+0.5 \end{array}$$

After simplifying, we get the value of Eq. (2.19) as

$$\frac{1}{2}\iint_{e}\left[\left\{\left(\frac{\partial H_i}{\partial x}\right)^{T}\frac{\partial H_i}{\partial x}+\left(\frac{\partial H_i}{\partial y}\right)^{T}\frac{\partial H_i}{\partial y}\right\}u_i\right]dxdy=\begin{pmatrix} 0.5 & 0 & -0.5 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 \\ -0.5 & -1 & 2 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0.5 & -0.5 & 0 \\ 0 & -1 & 0 & -0.5 & 2 & -0.5 \\ 0 & 0 & -0.5 & 0 & -0.5 & 1 \end{pmatrix}. \tag{2.25}$$

Now, consider the third term on right side of Eq. (2.13)

$$\frac{1}{2}\iint_{e}2H_i\,dxdy=\iint_{e}H_i\,dxdy=b^{(e)}=\frac{A}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}, \tag{2.26}$$

where, $A=\text{Area of Element}=\frac{1}{8}$.

Steps of simplification for result obtained in above Eq. (2.26) are explained in Appendix A.

Using Eq. (2.26), the values corresponding to all four elements are

$$b_1=\frac{1}{24}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{matrix}\mathbf{2}\\\mathbf{3}\\\mathbf{1}\end{matrix},\quad b_2=\frac{1}{24}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{matrix}\mathbf{4}\\\mathbf{5}\\\mathbf{2}\end{matrix},\quad b_3=\frac{1}{24}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{matrix}\mathbf{5}\\\mathbf{6}\\\mathbf{2}\end{matrix},\quad b_4=\frac{1}{24}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{matrix}\mathbf{6}\\\mathbf{3}\\\mathbf{2}\end{matrix}. \tag{2.27}$$

Using Eq. (2.27), the assembly of element matrices generates the global assembled matrix, as given below

$$\frac{1}{24}\begin{pmatrix}1+0+0+0\\1+1+1+1\\1+1\\1\\1+1\\1+1\end{pmatrix}\begin{matrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\\\mathbf{4}\\\mathbf{5}\\\mathbf{6}\end{matrix}=\frac{1}{24}\begin{pmatrix}1\\4\\2\\1\\2\\2\end{pmatrix}. \tag{2.28}$$

Upon using Eqs. (2.25, 2.28) into Eq. (2.13), we may write in compact form as

$$\begin{pmatrix}0.5 & 0 & -0.5 & 0 & 0 & 0\\0 & 2 & -1 & 0 & -1 & 0\\-0.5 & -1 & 2 & 0 & 0 & -0.5\\0 & 0 & 0 & 0.5 & -0.5 & 0\\0 & -1 & 0 & -0.5 & 2 & -0.5\\0 & 0 & -0.5 & 0 & -0.5 & 1\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}=\frac{1}{24}\begin{pmatrix}1\\4\\2\\1\\2\\2\end{pmatrix}. \tag{2.29}$$

The given boundary conditions give $u_1=0$, $u_3=0$, $u_6=0$.

We incorporate these nodal values in matrix Eq. (2.29) by deleting the rows & columns corresponding to u_1, u_3 and u_6, and obtain the following system of equations,

$$\begin{pmatrix}2 & 0 & 1\\0 & 0.5 & 0.5\\-1 & -0.5 & 2\end{pmatrix}\begin{pmatrix}u_2\\u_4\\u_5\end{pmatrix}=\frac{1}{24}\begin{pmatrix}4\\1\\2\end{pmatrix}. \tag{2.30}$$

The solution of above system (2.30) yields $u_2=0.1875$, $u_4=0.29167$ and $u_5=0.20833$, which are the required values of u at global nodes *2, 4* and *5*.

2.10.2 <u>Problem</u>. Compute the element equation for six noded triangular element for Boundary Value Problem represent by Partial Differential Equation given in (2.31) using Finite Element Method.

Solution: Consider the partial differential equation

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right)+r=0 \ \ in\ \mathcal{R} \tag{2.31}$$

with Dirichlet condition

$$u=g(x,y) \quad on\ \partial\mathcal{R}, \tag{2.32}$$

where p and r may be constants or functions of x and y only. The variational formulation of differential equation (2.31) in term of the functional is reduced to simple minimizing problem by assuming an approximate function (or approximate solution),

$$J=\frac{1}{2}\iint_{\mathcal{R}}\left[p\left(\frac{\partial u}{\partial x}\right)^2+p\left(\frac{\partial u}{\partial y}\right)^2-2ur\right]dxdy=\text{minimum}, \tag{2.33}$$

where the boundary condition (2.32) is to be satisfied. We divide the domain $\mathcal{R}$ in six noded triangular elements. The approximate solution $u(x,y)$ for the whole domain $\mathcal{R}$ is

$$u(x,y)=\sum_{e=1}^{M}N^{(e)}\phi^{(e)}=\sum_{i=1}^{K}N_i\phi_i=N\phi, \tag{2.34}$$

where M represents the number of the elements with K nodes in $\mathcal{R}$, and N & ϕ are

$N=[N_1\ N_2\ \ldots\ N_K]$, $\qquad \phi=[\phi_1\ \phi_2\ \ldots\ \phi_K]^T$.

The shape functions N_i satisfy the following conditions

$$N_i(x,y)=\begin{cases}N_i^{(e)}(x,y), & \text{if}\,(x,y)\in(e),\\ 0, & \text{otherwise}\end{cases} \tag{2.35}$$

and $\phi^{(e)}$ are the nodal values associated with the element (e). Substituting the approximate solution from Eq. (2.34) into Eq. (2.33), we get

$$J=\frac{1}{2}\iint_{\mathcal{R}}\left\{p\left(\sum_{e=1}^{M}\frac{\partial N^{(e)}}{\partial x}\phi^{(e)}\right)^2+p\left(\sum_{e=1}^{M}\frac{\partial N^{(e)}}{\partial y}\phi^{(e)}\right)^2-2r\sum_{e=1}^{M}N^{(e)}\phi^{(e)}\right\}dxdy. \tag{2.36}$$

Using Eq. (2.35), assume that Eq. (2.36) can be written in the form

$$J=\sum_{e=1}^{M}J^{(e)}, \tag{2.37}$$

where

$$J^{(e)}=\frac{1}{2}\iint_{(e)}\left\{p\left(\frac{\partial N^{(e)}}{\partial x}\phi^{(e)}\right)^2+p\left(\frac{\partial N^{(e)}}{\partial y}\phi^{(e)}\right)^2-2rN^{(e)}\phi^{(e)}\right\}dxdy \tag{2.38}$$

is the contribution of the element (e) to the functional J. The conditions for minimization of the functional J in Eq. (2.37) with respect to nodal values $\phi_i, i=1,2,3\ldots K$ give the following system of equations

$$\frac{\partial J}{\partial \phi_i}=\sum_{e=1}^{M}\frac{\partial J^{(e)}}{\partial \phi_i}=0, \qquad i=1,2,3\ldots K$$

or

$$\frac{\partial J}{\partial \phi}=\sum_{e=1}^{M}\frac{\partial J^{(e)}}{\partial \phi^{(e)}}=0\,. \tag{2.39}$$

Since $J^{(e)}$ depends on the nodal values associated with the element (e) only. The equation $\frac{\partial J^{(e)}}{\partial \phi^{(e)}}=0$ is called the element equation. Usually, it turns out that one term of the summation gives the form for the other terms. Therefore, it is sufficient to explicitly consider the contribution of a typical finite element (e) only. After differentiating Eq. (2.38) with respect to $\phi^{(e)}$, we get the following element equation

$$\frac{\partial J^{(e)}}{\partial \phi^{(e)}}=\iint_{e}\left[p\left\{\left(\frac{\partial N^{(e)}}{\partial x}\right)^{T}\left(\frac{\partial N^{(e)}}{\partial x}\right)+\left(\frac{\partial N^{(e)}}{\partial y}\right)^{T}\left(\frac{\partial N^{(e)}}{\partial y}\right)\right\}\phi^{(e)}-r\left(N^{(e)}\right)^{T}\right]dxdy=0\,. \tag{2.40}$$

Thus, the element equation becomes

$$A^{(e)}\phi^{(e)}-b^{(e)}=0, \tag{2.41}$$

where

$$A^{(e)}=\iint_{(e)}\left[p\left\{\left(\frac{\partial N^{(e)}}{\partial x}\right)^{T}\left(\frac{\partial N^{(e)}}{\partial x}\right)+\left(\frac{\partial N^{(e)}}{\partial y}\right)^{T}\left(\frac{\partial N^{(e)}}{\partial y}\right)\right\}\right]dxdy, \tag{2.42}$$

$$b^{(e)}=\iint_{e} r\left(N^{(e)}\right)^{T}dxdy, \tag{2.43}$$

and $\left(N^{(e)}\right)^{T}=\left[N_i\ N_l\ N_j\ N_m\ N_k\ N_n\right]^{T}$.

Assume that the functions p and r are constants over each element and are represented by $p^{(e)}$ and $r^{(e)}$ respectively.

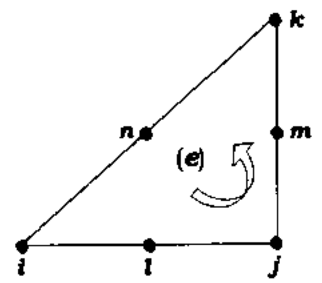


Figure 2.15: Six Noded Triangular Element

The piecewise approximate solution over the element (e) may be assumed as

$$u^{(e)} = N_i u_i + N_l u_l + N_j u_j + N_m u_m + N_k u_k + N_n u_n = N^{(e)} \phi^{(e)}, \tag{2.44}$$

where

$$N^{(e)} = \left[N_i \; N_l \; N_j \; N_m \; N_k \; N_n \right]^T \text{ and } \phi^{(e)} = \left[u_i \; u_l \; u_j \; u_m \; u_k \; u_n \right]^T.$$

The shape functions N_i, N_l, N_j, N_m, N_k and N_n are defined as

$$\begin{aligned} N_i &= 2L_i^2 - L_i, & N_l &= 4L_i L_j, \\ N_j &= 2L_j^2 - L_j, & N_m &= 4L_j L_k, \\ N_k &= 2L_k^2 - L_k, & N_n &= 4L_i L_i, \end{aligned} \tag{2.45}$$

where L_i, L_j and L_k are called Area Coordinates satisfy the following two properties

(i) $L_i + L_j + L_k = 1$ and (2.46)

(ii) $L_i = \frac{1}{2\Delta^{(e)}}(a_i + b_i x + c_i y)$, $L_j = \frac{1}{2\Delta^{(e)}}(a_j + b_j x + c_j y)$, $L_k = \frac{1}{2\Delta^{(e)}}(a_k + b_k x + c_k y)$

in which

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2, & b_1 &= y_2 - y_3, & c_1 &= x_3 - x_2, \\ a_2 &= x_3 y_1 - x_1 y_3, & b_2 &= y_3 - y_1, & c_2 &= x_1 - x_3, \\ a_3 &= x_1 y_2 - x_2 y_1, & b_3 &= y_1 - y_2, & c_3 &= x_2 - x_1, \end{aligned} \tag{2.47}$$

and $$\Delta^{(e)} = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{Area of each triangular element}. \tag{2.48}$$

The differentiation of $N^{(e)}(L_i, L_j, L_k)$ w.r.t x and y may be written as

$$\begin{aligned} \frac{\partial N^{(e)}}{\partial x} &= \frac{\partial N^{(e)}}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial N^{(e)}}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial N^{(e)}}{\partial L_3}\frac{\partial L_3}{\partial x}, \\ \frac{\partial N^{(e)}}{\partial y} &= \frac{\partial N^{(e)}}{\partial L_1}\frac{\partial L_1}{\partial y} + \frac{\partial N^{(e)}}{\partial L_2}\frac{\partial L_2}{\partial y} + \frac{\partial N^{(e)}}{\partial L_3}\frac{\partial L_3}{\partial y}, \end{aligned} \tag{2.49}$$

where

$$\frac{\partial L_i}{\partial x} = \frac{b_i}{2\Delta^{(e)}} \quad \text{and} \quad \frac{\partial L_i}{\partial y} = \frac{c_i}{2\Delta^{(e)}}, \quad i = 1,2,3. \tag{2.50}$$

For integration of polynomial terms in natural coordinates over the element (e), we use the following relation

$$\iint_{\Delta^{(e)}} \left(L_1^r L_2^s L_3^t \right) dx dy = \frac{r!\, s!\, t!\, 2\Delta^{(e)}}{(r+s+t+2)!}. \tag{2.51}$$

Now, consider further evaluation of Eq. (2.42),

$$A^{(e)} = \iint_{(e)} p\left\{\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_l}{\partial x} \\ \frac{\partial N_j}{\partial x} \\ \frac{\partial N_m}{\partial x} \\ \frac{\partial N_k}{\partial x} \\ \frac{\partial N_n}{\partial x} \end{pmatrix}\begin{pmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_l}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_m}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_n}{\partial x} \end{pmatrix} + \begin{pmatrix} \frac{\partial N_i}{\partial y} \\ \frac{\partial N_l}{\partial y} \\ \frac{\partial N_j}{\partial y} \\ \frac{\partial N_m}{\partial y} \\ \frac{\partial N_k}{\partial y} \\ \frac{\partial N_n}{\partial y} \end{pmatrix}\begin{pmatrix} \frac{\partial N_i}{\partial y} & \frac{\partial N_l}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_m}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_n}{\partial y} \end{pmatrix}\right\} dxdy \; . \tag{2.52}$$

Performing integration on first part of integral (2.52) by substituting the shape functions, we get

$$A_x = \iint_{(e)} \left\{\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_l}{\partial x} \\ \frac{\partial N_j}{\partial x} \\ \frac{\partial N_m}{\partial x} \\ \frac{\partial N_k}{\partial x} \\ \frac{\partial N_n}{\partial x} \end{pmatrix}\begin{pmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_l}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_m}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_n}{\partial x} \end{pmatrix}\right\} dxdy \tag{2.53}$$

or

$$A_x = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{pmatrix}, \tag{2.54}$$

In which,

$$A_{11} = \iint_{(e)} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_i}{\partial x}\right) dxdy . \tag{2.55}$$

Consider

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial L_i}\frac{\partial L_i}{\partial x} + \frac{\partial N_i}{\partial L_j}\frac{\partial L_j}{\partial x} + \frac{\partial N_i}{\partial L_k}\frac{\partial L_k}{\partial x} = (4L_i - 1)\left(\frac{b_i}{2\Delta}\right) + 0 + 0 \quad \text{(using Eqs. 2.45 and 2.46)}$$

By substituting above result, the integral (2.55) implies

$$A_{11} = \iint\limits_{(e)} \left[\frac{b_i^2}{4\Delta^2}\left(16L_i^2 + 1 - 8L_i\right) \right] dxdy$$

$$= \frac{b_i^2}{4\Delta^2}\left\{ 16\iint\limits_{(e)} L_i^2 dxdy + \iint\limits_{(e)} (1)\, dxdy - 8\iint\limits_{(e)} L_i dxdy \right\}$$

$$= \frac{b_i^2}{4\Delta^2}\left\{ 16\left(\frac{2!\,2\Delta}{(2+2)!}\right) + \Delta - 8\left(\frac{1!\,2\Delta}{(1+2)!}\right) \right\} \qquad (\text{using Eq.(2.51)})$$

$$= \frac{b_i^2}{4\Delta^2}\left(\frac{8\Delta}{3} + \Delta - \frac{8\Delta}{3}\right) = \frac{b_i^2}{4\Delta},$$

with the same contrast, the evaluation for other terms of matrix A_x are as follow,

$$A_{12} = \iint\limits_{(e)} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_l}{\partial x} \right) dxdy$$

$$= \iint\limits_{(e)} \left[\left\{ (4L_i - 1)\frac{b_i}{2\Delta} \right\} \times \left\{ \frac{2}{\Delta}\left(L_j b_i + L_i b_j\right) \right\} \right] dxdy$$

$$= \frac{b_i}{\Delta^2}\iint\limits_{(e)} \left(4L_iL_jb_i + 4L_i^2 b_j - L_jb_i - L_ib_j\right) dxdy$$

$$= \frac{b_i}{\Delta^2}\left[4b_i\left(\frac{\Delta}{12}\right) + 4b_j\left(\frac{\Delta}{6}\right) - b_i\left(\frac{\Delta}{3}\right) - b_j\left(\frac{\Delta}{3}\right) \right]$$

$$= \frac{b_i}{\Delta^2}\left[\frac{b_i}{3} + \frac{2b_j}{3} - \frac{b_i}{3} - \frac{b_j}{3} \right] = \frac{b_ib_j}{3\Delta},$$

$$A_{13} = \iint\limits_{(e)} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} \right) dxdy$$

$$= \iint\limits_{(e)} \left[\left\{ (4L_i - 1)\frac{b_i}{2\Delta} \right\} \times \left\{ (4L_j - 1)\frac{b_j}{2\Delta} \right\} \right] dxdy$$

$$= \frac{b_ib_j}{4\Delta^2}\iint\limits_{(e)} \left(16L_iL_j - 4L_i - 4L_j + 1\right) dxdy$$

$$= \frac{b_ib_j}{4\Delta^2}\left[16\left(\frac{\Delta}{12}\right) - 4\left(\frac{\Delta}{3}\right) - 4\left(\frac{\Delta}{3}\right) + \Delta \right] = -\frac{b_ib_j}{12\Delta},$$

$$A_{14} = \iint\limits_{(e)} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_m}{\partial x} \right) dxdy$$

$$= \iint\limits_{(e)} \left[\left\{ (4L_i - 1)\frac{b_i}{2\Delta} \right\} \times \left\{ \frac{2}{\Delta}\left(L_k b_j + L_j b_k\right) \right\} \right] dxdy$$

$$= \frac{b_i}{\Delta^2}\iint\limits_{(e)} \left(4L_iL_kb_j + 4L_iL_jb_k - L_kb_j - L_jb_k\right) dxdy$$

$$A_{14}=\frac{b_i}{\Delta^2}\left[4b_j\left(\frac{\Delta}{12}\right)+4b_k\left(\frac{\Delta}{12}\right)-b_j\left(\frac{\Delta}{3}\right)-b_k\left(\frac{\Delta}{3}\right)\right]$$
$$=\frac{b_i}{\Delta^2}\left[\frac{b_j}{3}+\frac{b_k}{3}-\frac{b_j}{3}-\frac{b_k}{3}\right]=0,$$

$$A_{15}=\iint_{(e)}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x}\right)dxdy$$
$$=\iint_{(e)}\left[\left\{(4L_i-1)\frac{b_i}{2\Delta}\right\}\times\left\{(4L_k-1)\frac{b_k}{2\Delta}\right\}\right]dxdy$$
$$=\frac{b_ib_k}{4\Delta^2}\iint_{(e)}(16L_iL_k-4L_i-4L_k+1)dxdy$$
$$=\frac{b_ib_j}{4\Delta^2}\left[16\left(\frac{\Delta}{12}\right)-4\left(\frac{\Delta}{3}\right)-4\left(\frac{\Delta}{3}\right)+\Delta\right]=-\frac{b_ib_k}{12\Delta},$$

$$A_{16}=\iint_{(e)}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_n}{\partial x}\right)dxdy$$
$$=\iint_{(e)}\left[\left\{(4L_i-1)\frac{b_i}{2\Delta}\right\}\times\left\{\frac{2}{\Delta}(L_kb_i+L_ib_k)\right\}\right]dxdy$$
$$=\frac{b_i}{\Delta^2}\iint_{(e)}\left(4L_iL_kb_i+4L_i^2b_k-L_kb_i-L_ib_k\right)dxdy$$
$$=\frac{b_i}{\Delta^2}\left[4b_i\left(\frac{\Delta}{12}\right)+4b_k\left(\frac{\Delta}{6}\right)-b_i\left(\frac{\Delta}{3}\right)-b_k\left(\frac{\Delta}{3}\right)\right]$$
$$=\frac{b_i}{\Delta}\left[\frac{b_i}{3}+\frac{2b_k}{3}-\frac{b_i}{3}-\frac{b_k}{3}\right]=\frac{b_ib_k}{3\Delta},$$

$$A_{21}=\iint_{(e)}\left(\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\right)dxdy=A_{12}=\frac{b_ib_j}{3\Delta},$$

$$A_{22}=\iint_{(e)}\left(\frac{\partial N_l}{\partial x}\frac{\partial N_l}{\partial x}\right)dxdy$$
$$=\iint_{(e)}\left[\left\{\frac{2}{\Delta}(L_jb_i+L_ib_j)\right\}\times\left\{\frac{2}{\Delta}(L_jb_i+L_ib_j)\right\}\right]dxdy$$

$$A_{22}=\frac{4}{\Delta^2}\iint_{(e)}\left(b_i^2L_j^2+b_j^2L_i^2+2b_ib_jL_iL_j\right)dxdy$$
$$=\frac{4}{\Delta^2}\left[b_i^2\left(\frac{\Delta}{6}\right)+b_j^2\left(\frac{\Delta}{6}\right)+2b_ib_j\left(\frac{\Delta}{12}\right)\right]=\frac{2}{3\Delta}\left[b_i^2+b_j^2+b_ib_j\right],$$

$$A_{23} = \iint_{(e)} \left(\frac{\partial N_l}{\partial x} \frac{\partial N_j}{\partial x} \right) dxdy$$

$$= \iint_{(e)} \left[\left\{ \frac{2}{\Delta} \left(L_j b_i + L_i b_j \right) \right\} \times \left\{ \frac{b_j}{2\Delta} \left(4L_j - 1 \right) \right\} \right] dxdy$$

$$= \frac{b_j}{\Delta^2} \iint_{(e)} \left(4b_i L_j^2 - b_i L_j + 4b_j L_i L_j - b_j L_i \right) dxdy$$

$$= \frac{b_j}{\Delta^2} \left[4b_i \left(\frac{\Delta}{6} \right) - b_i \left(\frac{\Delta}{3} \right) + 4b_j \left(\frac{\Delta}{12} \right) - b_j \left(\frac{\Delta}{3} \right) \right]$$

$$= \frac{b_j}{\Delta} \left[\frac{2b_i}{3} - \frac{b_i}{3} + \frac{b_j}{3} - \frac{b_j}{3} \right] = \frac{b_i b_j}{3\Delta},$$

$$A_{24} = \iint_{(e)} \left(\frac{\partial N_l}{\partial x} \frac{\partial N_m}{\partial x} \right) dxdy$$

$$= \iint_{(e)} \left[\left\{ \frac{2}{\Delta} \left(L_j b_i + L_i b_j \right) \right\} \times \left\{ \frac{2}{\Delta} \left(L_k b_j + L_j b_k \right) \right\} \right] dxdy$$

$$= \frac{4}{\Delta^2} \iint_{(e)} \left(b_i b_j L_j L_k + b_i b_k L_j^2 + b_j^2 L_i L_k - b_j b_k L_i L_j \right) dxdy$$

$$= \frac{4}{\Delta^2} \left[b_i b_j \left(\frac{\Delta}{12} \right) + b_i b_k \left(\frac{\Delta}{6} \right) + b_j^2 \left(\frac{\Delta}{12} \right) + b_j b_k \left(\frac{\Delta}{12} \right) \right]$$

$$= \frac{4}{\Delta} \left[\frac{b_i b_j}{12} + \frac{b_i b_k}{6} + \frac{b_j^2}{12} + \frac{b_j b_k}{12} \right] = \frac{1}{3\Delta} \left[b_j b_k + b_j^2 + 2b_i b_k + b_i b_j \right],$$

$$A_{25} = \iint_{(e)} \left(\frac{\partial N_l}{\partial x} \frac{\partial N_k}{\partial x} \right) dxdy$$

$$= \iint_{(e)} \left[\left\{ \frac{2}{\Delta} \left(L_j b_i + L_i b_j \right) \right\} \times \left\{ \frac{b_k}{2\Delta} \left(4L_k - 1 \right) \right\} \right] dxdy$$

$$= \frac{b_k}{\Delta^2} \iint_{(e)} \left(4b_i L_j L_k - b_i L_j + 4b_j L_i L_k - b_j L_i \right) dxdy$$

$$= \frac{b_k}{\Delta^2} \left[4b_i \left(\frac{\Delta}{12} \right) - b_i \left(\frac{\Delta}{3} \right) + 4b_j \left(\frac{\Delta}{12} \right) - b_j \left(\frac{\Delta}{3} \right) \right]$$

$$= \frac{b_k}{\Delta} \left[\frac{b_i}{3} - \frac{b_i}{3} + \frac{b_j}{3} - \frac{b_j}{3} \right] = 0,$$

$$A_{26} = \iint_{(e)} \left(\frac{\partial N_l}{\partial x} \frac{\partial N_n}{\partial x} \right) dxdy$$

$$= \iint_{(e)} \left[\left\{ \frac{2}{\Delta} \left(L_j b_i + L_i b_j \right) \right\} \times \left\{ \frac{2}{\Delta} \left(L_k b_i + L_i b_k \right) \right\} \right] dxdy$$

$$A_{26} = \frac{4}{\Delta^2}\iint_{(e)}\left(b_i^2 L_j L_k + b_i b_k L_i L_j + b_i b_j L_i L_k + b_j b_k L_i^2\right)dxdy$$

$$= \frac{4}{\Delta^2}\left[b_i^2\left(\frac{\Delta}{12}\right) + b_i b_k\left(\frac{\Delta}{12}\right) + b_i b_j\left(\frac{\Delta}{12}\right) + b_j b_k\left(\frac{\Delta}{6}\right)\right]$$

$$= \frac{4}{\Delta}\left[\frac{b_i^2}{12} + \frac{b_i b_k}{12} + \frac{b_i b_j}{12} + \frac{b_j b_k}{6}\right] = \frac{1}{3\Delta}\left[b_i b_k + b_i^2 + 2b_j b_k + b_i b_j\right],$$

$$A_{31} = \iint_{(e)}\left(\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\right)dxdy = A_{13} = -\frac{b_i b_j}{12\Delta},$$

$$A_{32} = \iint_{(e)}\left(\frac{\partial N_j}{\partial x}\frac{\partial N_l}{\partial x}\right)dxdy = A_{23} = \frac{b_i b_j}{3\Delta},$$

$$A_{33} = \iint_{(e)}\left(\frac{\partial N_j}{\partial x}\frac{\partial N_j}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{b_j}{2\Delta}(4L_j - 1)\right\}\times\left\{\frac{b_j}{2\Delta}(4L_j - 1)\right\}\right]dxdy$$

$$= \frac{b_j^2}{4\Delta^2}\iint_{(e)}\left(16L_j^2 + 1 - 8L_j\right)dxdy$$

$$= \frac{b_j^2}{4\Delta^2}\left[16\left(\frac{\Delta}{6}\right) + \Delta - 8\left(\frac{\Delta}{3}\right)\right] = \frac{b_j^2}{4\Delta},$$

$$A_{34} = \iint_{(e)}\left(\frac{\partial N_j}{\partial x}\frac{\partial N_m}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{b_j}{2\Delta}(4L_j - 1)\right\}\times\left\{\frac{2}{\Delta}(L_k b_j + L_j b_k)\right\}\right]dxdy$$

$$= \frac{b_j}{\Delta^2}\iint_{(e)}\left(4b_j L_j L_k + 4b_k L_j^2 - b_j L_k - b_k L_j\right)dxdy$$

$$= \frac{b_j}{\Delta^2}\left[4b_j\left(\frac{\Delta}{12}\right) + 4b_k\left(\frac{\Delta}{6}\right) - b_j\left(\frac{\Delta}{3}\right) - b_k\left(\frac{\Delta}{3}\right)\right]$$

$$= \frac{b_j}{\Delta}\left[\frac{b_j}{3} + \frac{2b_k}{3} - \frac{b_j}{3} - \frac{b_k}{3}\right] = \frac{b_j b_k}{3\Delta},$$

$$A_{35} = \iint_{(e)}\left(\frac{\partial N_j}{\partial x}\frac{\partial N_k}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{b_j}{2\Delta}(4L_j - 1)\right\}\times\left\{\frac{b_k}{2\Delta}(4L_k - 1)\right\}\right]dxdy$$

$$A_{35} = \frac{b_j b_k}{4\Delta^2}\iint_{(e)}\left(16L_jL_k - 4L_j - 4L_k + 1\right)dxdy$$

$$= \frac{b_j^2}{4\Delta^2}\left[16\left(\frac{\Delta}{12}\right) - 4\left(\frac{\Delta}{3}\right) - 4\left(\frac{\Delta}{3}\right) + \Delta\right] = -\frac{b_j b_k}{12\Delta},$$

$$A_{36} = \iint_{(e)}\left(\frac{\partial N_j}{\partial x}\frac{\partial N_n}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{b_j}{2\Delta}\left(4L_j - 1\right)\right\} \times \left\{\frac{2}{\Delta}\left(L_k b_i + L_i b_k\right)\right\}\right]dxdy$$

$$= \frac{b_j}{\Delta^2}\iint_{(e)}\left(4b_i L_j L_k + 4b_k L_i L_j - b_i L_k - b_k L_i\right)dxdy$$

$$= \frac{b_j}{\Delta^2}\left[4b_i\left(\frac{\Delta}{12}\right) + 4b_k\left(\frac{\Delta}{12}\right) - b_i\left(\frac{\Delta}{3}\right) - b_k\left(\frac{\Delta}{3}\right)\right]$$

$$= \frac{b_j}{\Delta}\left[\frac{b_i}{3} + \frac{b_k}{3} - \frac{b_i}{3} - \frac{b_k}{3}\right] = 0,$$

$$A_{41} = \iint_{(e)}\left(\frac{\partial N_m}{\partial x}\frac{\partial N_i}{\partial x}\right)dxdy = A_{14} = 0,$$

$$A_{42} = \iint_{(e)}\left(\frac{\partial N_m}{\partial x}\frac{\partial N_i}{\partial x}\right)dxdy = A_{24} = \frac{1}{3\Delta}\left[b_j b_k + b_j^2 + 2b_i b_k + b_i b_j\right],$$

$$A_{43} = \iint_{(e)}\left(\frac{\partial N_m}{\partial x}\frac{\partial N_i}{\partial x}\right)dxdy = A_{34} = \frac{b_j b_k}{3\Delta},$$

$$A_{44} = \iint_{(e)}\left(\frac{\partial N_m}{\partial x}\frac{\partial N_m}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{2}{\Delta}\left(L_k b_j + L_j b_k\right)\right\} \times \left\{\frac{2}{\Delta}\left(L_k b_j + L_j b_k\right)\right\}\right]dxdy$$

$$= \frac{4}{\Delta^2}\iint_{(e)}\left(b_j^2 L_k^2 + b_k^2 L_j^2 + 2b_j b_k L_j L_k\right)dxdy$$

$$= \frac{4}{\Delta^2}\left[b_j^2\left(\frac{\Delta}{6}\right) + b_k^2\left(\frac{\Delta}{6}\right) + 2b_j b_k\left(\frac{\Delta}{12}\right)\right] = \frac{2}{3\Delta}\left[b_j^2 + b_k^2 + b_j b_k\right],$$

$$A_{45} = \iint_{(e)}\left(\frac{\partial N_m}{\partial x}\frac{\partial N_k}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{2}{\Delta}\left(L_k b_j + L_j b_k\right)\right\} \times \left\{\frac{b_k}{2\Delta}\left(4L_k - 1\right)\right\}\right]dxdy$$

$$= \frac{b_k}{\Delta^2}\iint_{(e)}\left(4b_j L_k^2 - b_j L_k + 4b_k L_j L_k - b_k L_j\right)dxdy$$

$$A_{45} = \frac{b_k}{\Delta^2}\left[4b_j\left(\frac{\Delta}{6}\right) - b_j\left(\frac{\Delta}{3}\right) + 4b_k\left(\frac{\Delta}{12}\right) - b_k\left(\frac{\Delta}{3}\right)\right]$$

$$= \frac{b_k}{\Delta}\left[\frac{2b_j}{3} - \frac{b_j}{3} + \frac{b_k}{3} - \frac{b_k}{3}\right] = \frac{b_j b_k}{3\Delta},$$

$$A_{46} = \iint_{(e)}\left(\frac{\partial N_m}{\partial x}\frac{\partial N_n}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{2}{\Delta}\left(L_k b_j + L_j b_k\right)\right\} \times \left\{\frac{2}{\Delta}\left(L_k b_i + L_i b_k\right)\right\}\right]dxdy$$

$$= \frac{4}{\Delta^2}\iint_{(e)}\left(b_i b_j L_k^2 + b_j b_k L_i L_k + b_i b_k L_j L_k + b_k^2 L_i L_j\right)dxdy$$

$$= \frac{4}{\Delta^2}\left[b_i b_j\left(\frac{\Delta}{6}\right) + b_j b_k\left(\frac{\Delta}{12}\right) + b_i b_k\left(\frac{\Delta}{12}\right) + b_k^2\left(\frac{\Delta}{12}\right)\right]$$

$$= \frac{1}{3\Delta}\left[2b_i b_j + b_j b_k + b_i b_k + b_k^2\right],$$

$$A_{51} = \iint_{(e)}\left(\frac{\partial N_k}{\partial x}\frac{\partial N_i}{\partial x}\right)dxdy = A_{15} = -\frac{b_i b_k}{12\Delta},$$

$$A_{52} = \iint_{(e)}\left(\frac{\partial N_k}{\partial x}\frac{\partial N_l}{\partial x}\right)dxdy = A_{25} = 0,$$

$$A_{53} = \iint_{(e)}\left(\frac{\partial N_k}{\partial x}\frac{\partial N_j}{\partial x}\right)dxdy = A_{35} = -\frac{b_j b_k}{12\Delta},$$

$$A_{54} = \iint_{(e)}\left(\frac{\partial N_k}{\partial x}\frac{\partial N_m}{\partial x}\right)dxdy = A_{45} = \frac{b_j b_k}{3\Delta},$$

$$A_{55} = \iint_{(e)}\left(\frac{\partial N_k}{\partial x}\frac{\partial N_k}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{b_k}{2\Delta}\left(4L_k - 1\right)\right\} \times \left\{\frac{b_k}{2\Delta}\left(4L_k - 1\right)\right\}\right]dxdy$$

$$= \frac{b_k^2}{4\Delta^2}\iint_{(e)}\left(16L_k^2 + 1 - 8L_k\right)dxdy$$

$$= \frac{b_k^2}{4\Delta^2}\left[16\left(\frac{\Delta}{6}\right) + \Delta - 8\left(\frac{\Delta}{3}\right)\right] = \frac{b_k^2}{4\Delta},$$

$$A_{56} = \iint_{(e)}\left(\frac{\partial N_k}{\partial x}\frac{\partial N_n}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left\{\frac{b_k}{2\Delta}\left(4L_k - 1\right)\right\} \times \left[\left\{\frac{2}{\Delta}\left(L_k b_i + L_i b_k\right)\right\}\right]dxdy$$

$$A_{56} = \frac{b_k}{\Delta^2}\iint_{(e)}\left(4b_iL_k^2 + 4b_kL_kL_i - b_iL_k - b_kL_i\right)dxdy$$

$$= \frac{b_k}{\Delta^2}\left[4b_i\left(\frac{\Delta}{6}\right) + 4b_k\left(\frac{\Delta}{12}\right) - b_i\left(\frac{\Delta}{3}\right) - b_k\left(\frac{\Delta}{3}\right)\right]$$

$$= \frac{b_k}{\Delta}\left[\frac{2b_i}{3} + \frac{b_k}{3} - \frac{b_i}{3} - \frac{b_k}{3}\right] = \frac{b_ib_k}{3\Delta},$$

$$A_{61} = \iint_{(e)}\left(\frac{\partial N_n}{\partial x}\frac{\partial N_i}{\partial x}\right)dxdy = A_{16} = \frac{b_ib_k}{3\Delta},$$

$$A_{62} = \iint_{(e)}\left(\frac{\partial N_n}{\partial x}\frac{\partial N_l}{\partial x}\right)dxdy = A_{26} = \frac{1}{3\Delta}\left[b_ib_k + b_i^2 + 2b_jb_k + b_ib_j\right],$$

$$A_{63} = \iint_{(e)}\left(\frac{\partial N_n}{\partial x}\frac{\partial N_j}{\partial x}\right)dxdy = A_{36} = 0,$$

$$A_{64} = \iint_{(e)}\left(\frac{\partial N_n}{\partial x}\frac{\partial N_m}{\partial x}\right)dxdy = A_{46} = \frac{1}{3\Delta}\left[2b_ib_j + b_jb_k + b_ib_k + b_k^2\right],$$

$$A_{65} = \iint_{(e)}\left(\frac{\partial N_n}{\partial x}\frac{\partial N_k}{\partial x}\right)dxdy = A_{56} = \frac{b_ib_k}{3\Delta},$$

$$A_{66} = \iint_{(e)}\left(\frac{\partial N_n}{\partial x}\frac{\partial N_n}{\partial x}\right)dxdy$$

$$= \iint_{(e)}\left[\left\{\frac{2}{\Delta}\left(L_kb_i + L_ib_k\right)\right\}\times\left\{\frac{2}{\Delta}\left(L_kb_i + L_ib_k\right)\right\}\right]dxdy$$

$$= \frac{4}{\Delta^2}\iint_{(e)}\left(b_i^2L_k^2 + b_k^2L_i^2 + 2b_ib_kL_iL_k\right)dxdy$$

$$= \frac{4}{\Delta^2}\left[b_i^2\left(\frac{\Delta}{6}\right) + b_k^2\left(\frac{\Delta}{6}\right) + 2b_ib_k\left(\frac{\Delta}{12}\right)\right]$$

$$= \frac{4}{\Delta}\left[\frac{b_i^2}{6} + \frac{b_k^2}{6} + \frac{b_ib_k}{6}\right] = \frac{2}{3\Delta}\left(b_i^2 + b_k^2 + b_ib_k\right).$$

Substituting all values in Eq. (2.54), we get

$$A_x = \begin{pmatrix} \frac{b_i^2}{4\Delta} & \frac{b_ib_j}{3\Delta} & -\frac{b_ib_j}{12\Delta} & 0 & -\frac{b_ib_k}{12\Delta} & \frac{b_ib_k}{3\Delta} \\ \frac{b_ib_j}{3\Delta} & \frac{2}{3\Delta}\left[b_i^2+b_j^2+b_ib_j\right] & \frac{b_ib_j}{3\Delta} & \frac{1}{3\Delta}\left[b_jb_k+b_j^2+2b_ib_k+b_ib_j\right] & 0 & \frac{1}{3\Delta}\left[b_ib_k+b_i^2+2b_jb_k+b_ib_j\right] \\ -\frac{b_ib_j}{12\Delta} & \frac{b_ib_j}{3\Delta} & \frac{b_j^2}{4\Delta} & \frac{b_jb_k}{3\Delta} & -\frac{b_jb_k}{12\Delta} & 0 \\ 0 & \frac{1}{3\Delta}\left[b_jb_k+b_j^2+2b_ib_k+b_ib_j\right] & \frac{b_jb_k}{3\Delta} & \frac{2}{3\Delta}\left[b_j^2+b_k^2+b_jb_k\right] & \frac{b_jb_k}{3\Delta} & \frac{1}{3\Delta}\left[2b_ib_j+b_jb_k+b_ib_k+b_k^2\right] \\ -\frac{b_ib_k}{12\Delta} & 0 & -\frac{b_jb_k}{12\Delta} & \frac{b_jb_k}{3\Delta} & \frac{b_k^2}{4\Delta} & \frac{b_ib_k}{3\Delta} \\ \frac{b_ib_k}{3\Delta} & \frac{1}{3\Delta}\left[b_ib_k+b_i^2+2b_jb_k+b_ib_j\right] & 0 & \frac{1}{3\Delta}\left[2b_ib_j+b_jb_k+b_ib_k+b_k^2\right] & \frac{b_ib_k}{3\Delta} & \frac{2}{3\Delta}\left(b_i^2+b_k^2+b_ib_k\right) \end{pmatrix} \tag{2.56}$$

Using the same procedure as described above for the evaluation of A_x, value of second part in integral (2.52) may be computed as shown in succeeding steps

$$A_y = \iint_{(e)} \left\{ \begin{pmatrix} \frac{\partial N_i}{\partial y} \\ \frac{\partial N_l}{\partial y} \\ \frac{\partial N_j}{\partial y} \\ \frac{\partial N_m}{\partial y} \\ \frac{\partial N_k}{\partial y} \\ \frac{\partial N_n}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial y} & \frac{\partial N_l}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_m}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_n}{\partial y} \end{pmatrix} \right\} dxdy, \tag{2.57}$$

i.e.

$$A_y = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{pmatrix} \tag{2.58}$$

$$A_y=\begin{pmatrix} \frac{c_i^2}{4\Delta} & \frac{c_ic_j}{3\Delta} & -\frac{c_ic_j}{12\Delta} & 0 & -\frac{c_ic_k}{12\Delta} & \frac{c_ic_k}{3\Delta} \\ \frac{c_ic_j}{3\Delta} & \frac{2}{3\Delta}\left[c_i^2+c_j^2+c_ic_j\right] & \frac{c_ic_j}{3\Delta} & \frac{1}{3\Delta}\left[c_jc_k+c_j^2+2c_ic_k+c_ic_j\right] & 0 & \frac{1}{3\Delta}\left[c_ic_k+c_i^2+2c_jc_k+c_ic_j\right] \\ -\frac{c_ic_j}{12\Delta} & \frac{c_ic_j}{3\Delta} & \frac{c_j^2}{4\Delta} & \frac{c_jc_k}{3\Delta} & -\frac{c_jc_k}{12\Delta} & 0 \\ 0 & \frac{1}{3\Delta}\left[c_jc_k+c_j^2+2c_ic_k+c_ic_j\right] & \frac{c_jc_k}{3\Delta} & \frac{2}{3\Delta}\left[c_j^2+c_k^2+c_jc_k\right] & \frac{c_jc_k}{3\Delta} & \frac{1}{3\Delta}\left[2c_ic_j+c_jc_k+c_ic_k+c_k^2\right] \\ -\frac{c_ic_k}{12\Delta} & 0 & -\frac{c_jc_k}{12\Delta} & \frac{c_jc_k}{3\Delta} & \frac{c_k^2}{4\Delta} & \frac{c_ic_k}{3\Delta} \\ \frac{c_ic_k}{3\Delta} & \frac{1}{3\Delta}\left[c_ic_k+c_i^2+2c_jc_k+c_ic_j\right] & 0 & \frac{1}{3\Delta}\left[2c_ic_j+c_jc_k+c_ic_k+c_k^2\right] & \frac{c_ic_k}{3\Delta} & \frac{2}{3\Delta}\left(c_i^2+c_k^2+c_ic_k\right) \end{pmatrix}. \tag{2.59}$$

Now, consider the evaluation of Eq. (2.43),

$$b^{(e)}=\iint_{(e)} r\left(N^{(e)}\right)^T dxdy=\iint_{(e)} r\left[N_i\ \ N_l\ \ N_j\ \ N_m\ \ N_k\ \ N_n\right]^T dxdy. \tag{2.60}$$

Consider the first term of above integral (2.60), we get

$$\iint_{(e)} rN_i\,dxdy=r\iint_{(e)}\left(2L_i^2-L_i\right)dxdy \qquad (\text{using}(2.45))$$

$$=r\left[2\left(\frac{\Delta}{6}\right)-\frac{\Delta}{3}\right]=0. \qquad (\text{using}(2.51))$$

Similarly, evaluation for other terms of integral (2.60) are as follow,

$$\iint_{(e)} rN_l\,dxdy=r\iint_{(e)}\left(4L_iL_j\right)dxdy=4r\left(\frac{\Delta}{12}\right)=r\frac{\Delta}{3},$$

$$\iint_{(e)} rN_j\,dxdy=r\iint_{(e)}\left(2L_j^2-L_j\right)dxdy=r\left[2\left(\frac{\Delta}{6}\right)-\frac{\Delta}{3}\right]=0,$$

$$\iint_{(e)} rN_m\,dxdy=r\iint_{(e)}\left(4L_jL_k\right)dxdy=4r\left[\frac{\Delta}{12}\right]=r\frac{\Delta}{3},$$

$$\iint_{(e)} rN_k\,dxdy=r\iint_{(e)}\left(2L_k^2-L_k\right)dxdy=r\left[2\left(\frac{\Delta}{6}\right)-\frac{\Delta}{3}\right]=0\,,$$

and

$$\iint_{(e)} rN_n\,dxdy=r\iint_{(e)}\left(4L_kL_i\right)dxdy=4r\left[\frac{\Delta}{12}\right]=r\frac{\Delta}{3}.$$

Using above all values in Eq. (2.60), we get

$$b^{(e)} = \begin{pmatrix} 0 \\ r\dfrac{\Delta}{3} \\ 0 \\ r\dfrac{\Delta}{3} \\ 0 \\ r\dfrac{\Delta}{3} \end{pmatrix} = r\frac{\Delta}{3}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{2.61}$$

After substituting the values from Eqs. (2.56, 2.59) in Eq. (2.52), we get the value of $A^{(e)}$, use this value along with the value of $b^{(e)}$ from Eq. (2.61) in Eq. (2.41), we may be able to obtain the solution of element equations.

Chapter - 3

Effects of Thermal Boundary Conditions on Natural Convection Flow within a Square Cavity

In this chapter, we revised the study of the effects of thermal boundary conditions on natural convection inside a square cavity [5]. The modelling of the governing equation and boundary conditions is presented. The governing equations are reduced to the non-dimensional form by using dimensionless variables. The Galerkin finite element method is used to obtain the solution of the governing equations. For this purpose, the domain is discretized by using triangular element and the shape functions are computed using quadratic triangular elements. The complete procedure for non-linear partial differential equations is described in detail. Results are computed for the wide range of parameters $Ra = 10^3 - 10^5$ and $Pr = 0.7 - 10$, and presented graphically in terms of stream functions, temperature profile, local and average nusselt numbers. The analysis is also made to discuss the effects of thermal boundary conditions on natural convection flow.

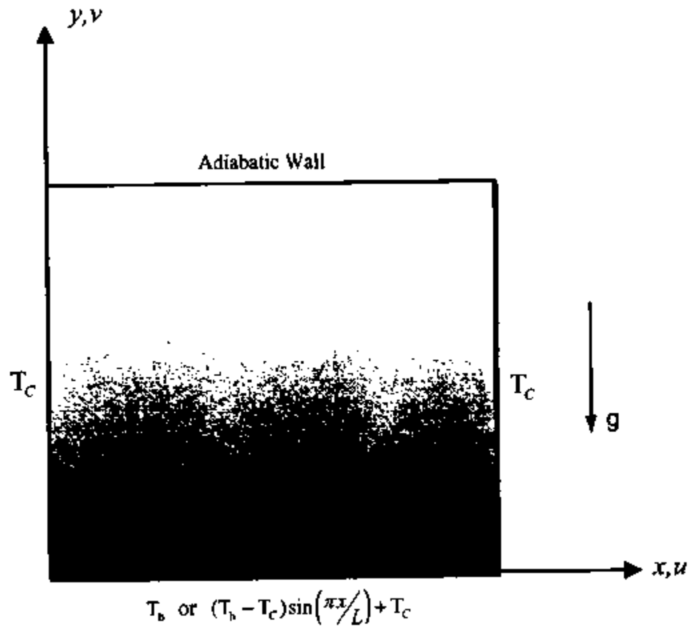


Figure 3.1: Schematic diagram of the physical system

3.1 Problem Description

Let us consider the laminar, steady flow of viscous fluid caused by the heated lower wall and adiabatic upper wall inside the square cavity. It is assumed that the lower/bottom wall of cavity is heated either uniformly or non-uniformly, while two vertical right and left walls are maintained at cool constant temperature. The physical domain in which the fluid is flowing is shown in Figure 3.1. It is further assumed that all the fluid properties are constant except the density of the fluid.

3.2 Mathematical Formulation

The flow model is based on the assumptions that the fluid is Newtonian and the body force term in the momentum equation is temperature dependent. The Boussinesq approximation is invoked for the fluid properties to relate density changes to temperature changes, and to couple in this way, the temperature field to the flow field. The governing equations for natural convection flow using conservation of mass, momentum and energy can be written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{3.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{3.2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+g\beta\left(T-T_c\right), \tag{3.3}$$

and

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right). \tag{3.4}$$

The assumed boundary conditions are

$$u(x,0)=u(x,L)=u(0,y)=u(L,y)=0\,,$$

$$v(x,0)=v(x,L)=v(0,y)=v(L,y)=0\,,$$

$$T(x,0)=T_h \quad \text{or} \quad T(x,0)=(T_h-T_c)\sin\left(\frac{\pi x}{L}\right)+T_c\,,$$

$$\frac{\partial T}{\partial y}(x,L)=0, \qquad T(0,y)=T(L,y)=T_c\,, \tag{3.5}$$

where x and y are the distances measured along the horizontal and vertical directions respectively, u and v are the velocity components in the x- and y-directions respectively, T denotes the temperature, υ and α are the kinematic viscosity and the thermal diffusivity of the fluid respectively, p is the pressure and ρ is the density, T_h and T_c are the temperatures at hot bottom wall and cold vertical walls respectively and L is the side of the square cavity.

Upon using the following change of variables:

$$X=\frac{x}{L},\quad Y=\frac{y}{L},\quad U=\frac{uL}{\alpha},\quad V=\frac{vL}{\alpha},\quad \theta=\frac{T-T_c}{T_h-T_c},$$

$$P=\frac{pL^2}{\rho\alpha^2},\ Pr=\frac{\upsilon}{\alpha},\quad Ra=\frac{g\beta(T_h-T_c)L^3\,\mathrm{Pr}}{\upsilon^2},\tag{3.6}$$

the governing equations (3.1 – 3.4) reduce to non-dimensional form as follow:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0,\tag{3.7}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+Pr\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right),\tag{3.8}$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+Pr\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)+RaPr\theta,\tag{3.9}$$

$$U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2},\tag{3.10}$$

with boundary conditions

$$U(X,0)=U(X,1)=U(0,Y)=U(1,Y)=0,$$
$$V(X,0)=V(X,1)=V(0,Y)=V(1,Y)=0,$$
$$\theta(X,0)=1\quad\text{or}\quad\theta(X,0)=\sin(\pi X),$$
$$\frac{\partial\theta}{\partial Y}(X,1)=0,\qquad \theta(0,Y)=\theta(1,Y)=0.\tag{3.11}$$

Here X and Y are dimensionless coordinates along horizontal and vertical directions respectively, U and V are dimensionless velocity components in the X- and Y-directions respectively, θ is the dimensionless temperature, P is the dimensionless pressure, Ra and Pr are Rayleigh and Prandtl numbers respectively.

3.3 Numerical Computations

To investigate the problem, it is required to solve the partial differential equations (3.7 - 3.10) subject to the boundary conditions (3.11). For this, the physical domain is discretize into finite number of triangular elements by using computation software MATLAB with built-in commands. Shape functions over triangular elements are generated by using 6-nodes, and they are quadratic for every elements. The meshing of given geometry yields 1312 six noded triangular elements with 2705 number of nodes.

The momentum and energy equations (3.8 – 3.10) are solved using the Galerkin finite element method. The continuity equation (3.7) is used as a constraint due to mass conservation and we obtained the pressure distribution as given by Basak and Ayappa [6]. In other words, to solve equations (3.8 – 3.10), we use the penalty finite element method, where the pressure P is eliminated by a penalty parameter γ and the incompressibility criteria given by Eq. (3.7) (see Reddy [3]) which results in:

$$P = -\gamma\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right). \tag{3.12}$$

The continuity equations (3.7) is automatically satisfied for large values of γ. Typical values of γ yield consistent solutions are 10^7.

After using Eq. (3.12), the momentum equations (3.8) and (3.9) reduce to:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \gamma\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + Pr\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right), \tag{3.13}$$

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = \gamma\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + Pr\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) + RaPr\theta. \tag{3.14}$$

We solve Eqs. (3.10), (3.13) and (3.14) to get solution of given physical problem. These three equations consist of highest order derivative terms, and the Eqs. (3.13) and (3.14) include the non-linear terms. The non-linearity in these equations make the problem difficult to solve. To deal with this difficulty, the iterative method such as Newton Raphson has been used. Whereas to remove the highest order derivative terms, the weak form of Eqs. (3.10), (3.13) and (3.14) is developed by using weak formulation.

We assume the approximated solution of velocity components U & V and temperature θ, as given below

$$U = \sum_{k=1}^{N} U_k\phi_k(X,Y),\ V = \sum_{k=1}^{N} V_k\phi_k(X,Y) \text{ and } \theta = \sum_{k=1}^{N} \theta_k\phi_k(X,Y) \quad \text{for } 0 \le X, Y \le 1, \tag{3.15}$$

where N is the total number of nodes, ϕ_k is the vector of shape functions, U_k and V_k are the vectors of nodal values of velocity components, and θ_k is the vector of nodal values of temperature. Using Galerkin's Method, the weight functions for all approximated functions (3.15) is same as follow

$$W_\theta = \frac{d\theta}{d\theta_k} = \phi_k, \qquad W_U = \frac{dU}{dU_k} = \phi_k \quad \text{and} \quad W_V = \frac{dV}{dV_k} = \phi_k. \tag{3.16}$$

For the evaluation of unknown nodal variables U_k, V_k and θ_k, the integral of weighted residual over the problem domain is set to zero. Thus, in the light of approximated functions (3.15) and weight function (3.16), the weak form of Eqs. (3.10), (3.13) and (3.14) are expressed as

$$\int_\Omega \phi_k\left(U\frac{\partial\theta}{\partial X} + V\frac{\partial\theta}{\partial Y}\right)dXdY - \int_\Omega \phi_k\left(\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2}\right)dXdY = 0,$$

$$\int_{\Omega}\left(\phi_k\left(\phi_k{}^{I}U_k\right)\frac{\partial\phi_k{}^{I}}{\partial X}+\phi_k\left(\phi_k{}^{I}V_k\right)\frac{\partial\phi_k{}^{I}}{\partial Y}\right)\theta_k dXdY+\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k{}^{I}}{\partial X}+\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k{}^{I}}{\partial X}\right)\theta_k dXdY-$$

$$\oint_{\Gamma}\left(n_X\frac{\partial\theta}{\partial X}+n_Y\frac{\partial\theta}{\partial Y}\right)\phi_k ds=0, \tag{3.17}$$

$$\int_{\Omega}\phi_k\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right)dXdY-\gamma\int_{\Omega}\phi_k\left[\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY-\Pr\int_{\Omega}\phi_k\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)dXdY=0,$$

$$\int_{\Omega}\left[\phi_k\left(\phi_k{}^{I}U_k\right)\frac{\partial\phi_k{}^{I}}{\partial X}+\phi_k\left(\phi_k{}^{I}V_k\right)\frac{\partial\phi_k{}^{I}}{\partial Y}\right]U_k dXdY+\gamma\left[\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k{}^{I}}{\partial X}\right)U_k dXdY+\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k{}^{I}}{\partial Y}\right)V_k dXdY\right]$$

$$+\Pr\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k{}^{I}}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k{}^{I}}{\partial Y}\right)U_k dXdY-\Pr\oint_{\Gamma}\left(n_X\frac{\partial U}{\partial X}+n_Y\frac{\partial U}{\partial Y}\right)\phi_k ds=0, \tag{3.18}$$

$$\int_{\Omega}\phi_k\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right)dXdY-\gamma\int_{\Omega}\phi_k\left[\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY-\Pr\int_{\Omega}\phi_k\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)dXdY-$$

$$Ra\Pr\int_{\Omega}\phi_k\theta\, dXdY=0,$$

$$\int_{\Omega}\left[\phi_k\left(\phi_k{}^{I}U_k\right)\frac{\partial\phi_k{}^{I}}{\partial X}+\phi_k\left(\phi_k{}^{I}V_k\right)\frac{\partial\phi_k{}^{I}}{\partial Y}\right]U_k dXdY+\gamma\left[\int_{\Omega}\left(\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k{}^{I}}{\partial X}\right)U_k dXdY+\int_{\Omega}\left(\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k{}^{I}}{\partial Y}\right)V_k dXdY\right]$$

$$+\Pr\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k{}^{I}}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k{}^{I}}{\partial Y}\right)V_k dXdY-Ra\Pr\int_{\Omega}\phi_k\left(\phi_k{}^{I}\theta_k\right)dXdY-\Pr\oint_{\Gamma}\left(n_X\frac{\partial V}{\partial X}+n_Y\frac{\partial V}{\partial Y}\right)\phi_k ds=0. \tag{3.19}$$

Reduced form of integrals obtained in Eqs. (3.17 – 3.19) are appended in Appendix A with brief steps of simplification. To avoid the complication of solving non-linear terms in Eqs. (3.18) and (3.19), Newton-Raphson method is used to compute non-linear coefficient matrices, which are function of unknown velocity components (Reddy [3]). Using Galerkin finite element method, the following nonlinear residual equations for Eqs. (3.17), (3.18) and (3.19) respectively, are being made over domain Ω,

$$R_i^{(1)}=\sum_{k=1}^{N}U_k\int_{\Omega}\left[\left(\sum_{k=1}^{N}U_k\phi_k\right)\frac{\partial\phi_k}{\partial X}+\left(\sum_{k=1}^{N}V_k\phi_k\right)\frac{\partial\phi_k}{\partial Y}\right]\phi_i dXdY+\gamma\left[\sum_{k=1}^{N}U_k\int_{\Omega}\frac{\partial\phi_i}{\partial X}\frac{\partial\phi_k}{\partial X}dXdY+\right.$$

$$\left.\sum_{k=1}^{N}V_k\int_{\Omega}\frac{\partial\phi_i}{\partial X}\frac{\partial\phi_k}{\partial Y}dXdY\right]+\Pr\sum_{k=1}^{N}U_k\int_{\Omega}\left[\frac{\partial\phi_i}{\partial X}\frac{\partial\phi_k}{\partial X}+\frac{\partial\phi_i}{\partial Y}\frac{\partial\phi_k}{\partial Y}\right]dXdY, \tag{3.20}$$

$$R_i^{(2)} = \sum_{k=1}^{N} V_k \int_\Omega \left[\left(\sum_{k=1}^{N} U_k \phi_k \right) \frac{\partial \phi_k}{\partial X} + \left(\sum_{k=1}^{N} V_k \phi_k \right) \frac{\partial \phi_k}{\partial Y} \right] \phi_i dXdY + \gamma \left[\sum_{k=1}^{N} U_k \int_\Omega \frac{\partial \phi_i}{\partial Y} \frac{\partial \phi_k}{\partial X} dXdY + \right.$$
$$\left. \sum_{k=1}^{N} V_k \int_\Omega \frac{\partial \phi_i}{\partial Y} \frac{\partial \phi_k}{\partial Y} dXdY \right] + \Pr \sum_{k=1}^{N} V_k \int_\Omega \left[\frac{\partial \phi_i}{\partial X} \frac{\partial \phi_k}{\partial X} + \frac{\partial \phi_i}{\partial Y} \frac{\partial \phi_k}{\partial Y} \right] dXdY -$$
$$Ra \Pr \int_\Omega \left(\sum_{k=1}^{N} \theta_k \phi_k \right) \phi_i dXdY, \tag{3.21}$$

$$R_i^{(3)} = \sum_{k=1}^{N} \theta_k \int_\Omega \left[\left(\sum_{k=1}^{N} U_k \phi_k \right) \frac{\partial \phi_k}{\partial X} + \left(\sum_{k=1}^{N} V_k \phi_k \right) \frac{\partial \phi_k}{\partial Y} \right] \phi_i dXdY + \sum_{k=1}^{N} \theta_k \int_\Omega \left[\frac{\partial \phi_i}{\partial X} \frac{\partial \phi_k}{\partial X} + \frac{\partial \phi_i}{\partial Y} \frac{\partial \phi_k}{\partial Y} \right] dXdY, \tag{3.22}$$

In order to solve above residual equations, quadratic triangular elements with six nodes are used as interpolation functions. Thus, the approximate functions for velocity components U and V, and temperature θ corresponding to six noded triangular element (e) may be expressed as

$$U = \sum_{k=1}^{6} U_k \phi_k(X,Y) = U^{(e)} \phi^{(e)}, \quad V = \sum_{k=1}^{6} V_k \phi_k(X,Y) = V^{(e)} \phi^{(e)} \text{ and } \theta = \sum_{k=1}^{6} \theta_k \phi_k(X,Y) = \theta^{(e)} \phi^{(e)}. \tag{3.23}$$

The interpolations or shape functions (ϕ_k) as already defined in Chapter-2 are

$$\phi_k = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix} = \begin{bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{bmatrix}, \qquad \text{for } k = 1,2,\ldots,6, \tag{3.24}$$

Consider,

$$\begin{aligned} L_1(2L_1 - 1) &= 2L_1^2 - L_1 \\ &= L_1^2 + L_1^2 - L_1 \\ &= L_1^2 + L_1(L_1 - 1) \\ &= L_1^2 + L_1(-L_2 - L_3) \qquad \text{using Eq. (2.46)} \end{aligned}$$

$$L_1(2L_1 - 1) = L_1^2 - L_1L_2 - L_1L_3 \tag{3.25}$$

Similarly, we may write

$$L_2(2L_2 - 1) = L_2^2 - L_2L_3 - L_1L_2, \tag{3.26}$$

$$L_3(2L_3 - 1) = L_3^2 - L_1L_3 - L_2L_3, \tag{3.27}$$

Using above results (3.25) - (3.27), the matrix (3.24) implies,

$$\phi_k = \begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\end{bmatrix} = \begin{bmatrix} L_1^2 - L_1L_2 - L_1L_3 \\ L_2^2 - L_2L_3 - L_1L_2 \\ L_3^2 - L_1L_3 - L_2L_3 \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix} = [A][R], \tag{3.28}$$

where,

$$[A] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad [R] = \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix}.$$

Differentiate Eq. (3.28) w.r.t x, we get

$$\frac{\partial \phi_k}{\partial x} = [A]\frac{\partial}{\partial x}\begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix} = [A]\begin{bmatrix} 2L_1\left(\frac{b_1}{2\Delta^{(e)}}\right) \\ 2L_2\left(\frac{b_2}{2\Delta^{(e)}}\right) \\ 2L_3\left(\frac{b_3}{2\Delta^{(e)}}\right) \\ L_1\left(\frac{b_2}{2\Delta^{(e)}}\right) + L_2\left(\frac{b_1}{2\Delta^{(e)}}\right) \\ L_2\left(\frac{b_3}{2\Delta^{(e)}}\right) + L_3\left(\frac{b_2}{2\Delta^{(e)}}\right) \\ L_3\left(\frac{b_1}{2\Delta^{(e)}}\right) + L_1\left(\frac{b_3}{2\Delta^{(e)}}\right) \end{bmatrix} = [A]\frac{1}{2\Delta^{(e)}}\begin{bmatrix} 2L_1b_1 \\ 2L_2b_2 \\ 2L_3b_3 \\ L_1b_2 + L_2b_1 \\ L_2b_3 + L_3b_2 \\ L_3b_1 + L_1b_3 \end{bmatrix},$$

i.e.

$$\frac{\partial \phi_k}{\partial x} = [A]\frac{1}{2\Delta^{(e)}}\begin{bmatrix} 2b_1 & 0 & 0 \\ 0 & 2b_2 & 0 \\ 0 & 0 & 2b_3 \\ b_2 & b_1 & 0 \\ 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = [A][B][L], \tag{3.29}$$

where,

$$[B] = \frac{1}{2\Delta^{(e)}}\begin{bmatrix} 2b_1 & 0 & 0 \\ 0 & 2b_2 & 0 \\ 0 & 0 & 2b_3 \\ b_2 & b_1 & 0 \\ 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \end{bmatrix} \quad \text{and} \quad [L] = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}.$$

Similarly, differentiating Eq. (3.28) w.r.t y, we get

$$\frac{\partial \phi_k}{\partial y} = [A]\frac{1}{2\Delta^{(e)}}\begin{bmatrix} 2c_1 & 0 & 0 \\ 0 & 2c_2 & 0 \\ 0 & 0 & 2c_3 \\ c_2 & c_1 & 0 \\ 0 & c_3 & c_2 \\ c_3 & 0 & c_1 \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = [A][C][L], \tag{3.30}$$

where,

$$[C] = \frac{1}{2\Delta^{(e)}}\begin{bmatrix} 2c_1 & 0 & 0 \\ 0 & 2c_2 & 0 \\ 0 & 0 & 2c_3 \\ c_2 & c_1 & 0 \\ 0 & c_3 & c_2 \\ c_3 & 0 & c_1 \end{bmatrix}.$$

Consider the product of $[L]$ with $[L]^T$ as

$$[L][L]^T = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}\begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} = \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ L_2L_1 & L_2^2 & L_2L_3 \\ L_3L_1 & L_3L_2 & L_3^2 \end{bmatrix}. \tag{3.31}$$

Using formula (2.51), the integration of Eq. (3.31) over the element (e) generates result as follow

$$H = \int_{\Delta^{(e)}} [L][L]^T d\Delta = \int_{\Delta^{(e)}} \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ L_2L_1 & L_2^2 & L_2L_3 \\ L_3L_1 & L_3L_2 & L_3^2 \end{bmatrix} d\Delta = \begin{bmatrix} \frac{\Delta}{6} & \frac{\Delta}{12} & \frac{\Delta}{12} \\ \frac{\Delta}{12} & \frac{\Delta}{6} & \frac{\Delta}{12} \\ \frac{\Delta}{12} & \frac{\Delta}{12} & \frac{\Delta}{6} \end{bmatrix},$$

i.e.

$$H = \int_{\Delta^{(e)}} [L][L]^T d\Delta = \frac{\Delta}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \tag{3.32}$$

The product of $[R]$ and $[R]^T$ yields result

$$[R][R]^T = \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix}\begin{bmatrix} L_1^2 & L_2^2 & L_3^2 & L_1L_2 & L_2L_3 & L_3L_1 \end{bmatrix},$$

i.e.

$$[R][R]^T = \begin{bmatrix} L_1^4 & L_1^2L_2^2 & L_1^2L_3^2 & L_1^3L_2 & L_1^2L_2L_3 & L_1^3L_3 \\ L_1^2L_2^2 & L_2^4 & L_2^2L_3^2 & L_1L_2^3 & L_2^3L_3 & L_1L_2^2L_3 \\ L_1^2L_3^2 & L_2^2L_3^2 & L_3^4 & L_1L_2L_3^2 & L_2L_3^3 & L_1L_3^3 \\ L_1^3L_2 & L_1L_2^3 & L_1L_2L_3^2 & L_1^2L_2^2 & L_1L_2^2L_3 & L_1^2L_2L_3 \\ L_1^2L_2L_3 & L_2^3L_3 & L_2L_3^3 & L_1L_2^2L_3 & L_2^2L_3^2 & L_1L_2L_3^2 \\ L_1^3L_3 & L_1L_2^2L_3 & L_1L_3^3 & L_1^2L_2L_3 & L_1L_2L_3^2 & L_1^2L_3^2 \end{bmatrix}. \tag{3.33}$$

With the help of formula (2.51), following result may be obtained in response of integrating matrix (3.33) over the element (e) as follows

$$Q = \int_{\Delta^{(e)}} [R][R]^T d\Delta = \int_{\Delta^{(e)}} \begin{bmatrix} L_1^4 & L_1^2L_2^2 & L_1^2L_3^2 & L_1^3L_2 & L_1^2L_2L_3 & L_1^3L_3 \\ L_1^2L_2^2 & L_2^4 & L_2^2L_3^2 & L_1L_2^3 & L_2^3L_3 & L_1L_2^2L_3 \\ L_1^2L_3^2 & L_2^2L_3^2 & L_3^4 & L_1L_2L_3^2 & L_2L_3^3 & L_1L_3^3 \\ L_1^3L_2 & L_1L_2^3 & L_1L_2L_3^2 & L_1^2L_2^2 & L_1L_2^2L_3 & L_1^2L_2L_3 \\ L_1^2L_2L_3 & L_2^3L_3 & L_2L_3^3 & L_1L_2^2L_3 & L_2^2L_3^2 & L_1L_2L_3^2 \\ L_1^3L_3 & L_1L_2^2L_3 & L_1L_3^3 & L_1^2L_2L_3 & L_1L_2L_3^2 & L_1^2L_3^2 \end{bmatrix} d\Delta,$$

$$Q = \int_{\Delta^{(e)}} [R][R]^T d\Delta = \begin{bmatrix} \frac{2\Delta}{30} & \frac{2\Delta}{180} & \frac{2\Delta}{180} & \frac{2\Delta}{120} & \frac{2\Delta}{360} & \frac{2\Delta}{120} \\ \frac{2\Delta}{180} & \frac{2\Delta}{30} & \frac{2\Delta}{180} & \frac{2\Delta}{120} & \frac{2\Delta}{120} & \frac{2\Delta}{360} \\ \frac{2\Delta}{180} & \frac{2\Delta}{180} & \frac{2\Delta}{30} & \frac{2\Delta}{360} & \frac{2\Delta}{120} & \frac{2\Delta}{120} \\ \frac{2\Delta}{120} & \frac{2\Delta}{120} & \frac{2\Delta}{360} & \frac{2\Delta}{180} & \frac{2\Delta}{360} & \frac{2\Delta}{360} \\ \frac{2\Delta}{360} & \frac{2\Delta}{120} & \frac{2\Delta}{120} & \frac{2\Delta}{360} & \frac{2\Delta}{180} & \frac{2\Delta}{360} \\ \frac{2\Delta}{120} & \frac{2\Delta}{360} & \frac{2\Delta}{120} & \frac{2\Delta}{360} & \frac{2\Delta}{360} & \frac{2\Delta}{180} \end{bmatrix} = \frac{2\times\Delta}{360} \begin{bmatrix} 12 & 2 & 2 & 3 & 1 & 3 \\ 2 & 12 & 2 & 3 & 3 & 1 \\ 2 & 2 & 12 & 1 & 3 & 3 \\ 3 & 3 & 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & 1 & 2 & 1 \\ 3 & 1 & 3 & 1 & 1 & 2 \end{bmatrix}. \tag{3.34}$$

Let us consider the integral F_x (used in succeeding phases of computation) comprising combination of identified matrices over the element (e), which results a matrix of order 6×3. Since its calculations are too lengthy and complicated, such steps are given in Appendix A with brief detail. Simplified form is written in the following form

$$F_x = \int_{\Delta^{(e)}} \left(ARR^T A^T UL^T\right) d\Delta$$

$$= \frac{2\times\Delta^{(e)}}{5040} \begin{bmatrix} 60u_1-8u_2-8u_3+24u_4-8u_5+24u_6 & 12u_1-8u_2+2u_3-16u_4-24u_5-8u_6 & 12u_1+2u_2-8u_3-8u_4-24u_5-16u_6 \\ -8u_1+12u_2+2u_3-16u_4-8u_5-24u_6 & -8u_1+60u_2-8u_3+24u_4+24u_5-8u_6 & 2u_1+12u_2-8u_3-8u_4-16u_5-24u_6 \\ -8u_1+2u_2+12u_3-24u_4-8u_5-16u_6 & 2u_1-8u_2+12u_3-24u_4-16u_5-8u_6 & -8u_1-8u_2+60u_3-8u_4+24u_5+24u_6 \\ 24u_1-16u_2-24u_3+192u_4+64u_5+96u_6 & -16u_1+24u_2-24u_3+192u_4+96u_5+64u_6 & -8u_1-8u_2-8u_3+64u_4+64u_5+64u_6 \\ -8u_1-8u_2-8u_3+64u_4+64u_5+64u_6 & -24u_1+24u_2-16u_3+96u_4+192u_5+64u_6 & -24u_1-16u_2+24u_3+64u_4+192u_5+96u_6 \\ 24u_1-24u_2-16u_3+96u_4+64u_5+192u_6 & -8u_1-8u_2-8u_3+64u_4+64u_5+64u_6 & -16u_1-24u_2+24u_3+64u_4+96u_5+192u_6 \end{bmatrix} \tag{3.35}$$

Similarly, consider the integral $F_y = \int_{\Delta^{(e)}} \left(ARR^T A^T V L^T \right) d\Delta$, generates the matrix of order 6×3. Its steps of simplification are described in Appendix A. Final step after integrating and simplifying F_y, is given below

$$F_y = \int_{\Delta^{(e)}} \left(ARR^T A^T V L^T \right) d\Delta$$

$$= \frac{2 \times \Delta^{(e)}}{5040} \begin{bmatrix} 60v_1-8v_2-8v_3+24v_4-8v_5+24v_6 & 12v_1-8v_2+2v_3-16v_4-24v_5-8v_6 & 12v_1+2v_2-8v_3-8v_4-24v_5-16v_6 \\ -8v_1+12v_2+2v_3-16v_4-8v_5-24v_6 & -8v_1+60v_2-8v_3+24v_4+24v_5-8v_6 & 2v_1+12v_2-8v_3-8v_4-16v_5-24v_6 \\ -8v_1+2v_2+12v_3-24v_4-8v_5-16v_6 & 2v_1-8v_2+12v_3-24v_4-16v_5-8v_6 & -8v_1-8v_2+60v_3-8v_4+24v_5+24v_6 \\ 24v_1-16v_2-24v_3+192v_4+64v_5+96v_6 & -16v_1+24v_2-24v_3+192v_4+96v_5+64v_6 & -8v_1-8v_2-8v_3+64v_4+64v_5+64v_6 \\ -8v_1-8v_2-8v_3+64v_4+64v_5+64v_6 & -24v_1+24v_2-16v_3+96v_4+192v_5+64v_6 & -24v_1-16v_2+24v_3+64v_4+192v_5+96v_6 \\ 24v_1-24v_2-16v_3+96v_4+64v_5+192v_6 & -8v_1-8v_2-8v_3+64v_4+64v_5+64v_6 & -16v_1-24v_2+24v_3+64v_4+96v_5+192v_6 \end{bmatrix} \tag{3.36}$$

where $U = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{bmatrix}^T$, $V = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}^T$ are the vectors of nodal values of velocity components associated with six noded triangular element (e). Using approximate functions (3.23), the integrals (3.20 - 3.22) over the element domain $\Omega^{(e)}$ may be written as

$$R_1^{(e)} = \int_{\Omega^e} \left(\phi^{(e)} \left(\left(\phi^{(e)}\right)^T U^{(e)} \right) \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T U^{(e)} \right) dXdY + \int_{\Omega^e} \left(\phi^{(e)} \left(\left(\phi^{(e)}\right)^T V^{(e)} \right) \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T U^{(e)} \right) dXdY +$$
$$\gamma \left[\int_{\Omega^e} \left(\frac{\partial \phi^{(e)}}{\partial X} \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T \right) U^{(e)} dXdY + \int_{\Omega^e} \left(\frac{\partial \phi^{(e)}}{\partial X} \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T \right) V^{(e)} dXdY \right] +$$
$$Pr \int_{\Omega^e} \left(\frac{\partial \phi^{(e)}}{\partial X} \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T + \frac{\partial \phi^{(e)}}{\partial Y} \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T \right) U^{(e)} dXdY, \tag{3.37}$$

$$R_2^{(e)} = \int_{\Omega^e} \left(\phi^{(e)} \left(\left(\phi^{(e)}\right)^T U^{(e)} \right) \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T V^{(e)} \right) dXdY + \int_{\Omega^e} \left(\phi^{(e)} \left(\left(\phi^{(e)}\right)^T V^{(e)} \right) \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T V^{(e)} \right) dXdY +$$
$$\gamma \left[\int_{\Omega^e} \left(\frac{\partial \phi^{(e)}}{\partial Y} \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T \right) U^{(e)} dXdY + \int_{\Omega^e} \left(\frac{\partial \phi^{(e)}}{\partial Y} \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T \right) V^{(e)} dXdY \right] +$$
$$Pr \int_{\Omega^e} \left(\frac{\partial \phi^{(e)}}{\partial X} \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T + \frac{\partial \phi^{(e)}}{\partial Y} \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T \right) V^{(e)} dXdY - RaPr \int_{\Omega^e} \left(\phi^{(e)} \left(\left(\phi^{(e)}\right)^T \theta^{(e)} \right) \right) dXdY. \tag{3.38}$$

and

$$R_3^{(e)} = \int_{\Omega^e} \left(\phi^{(e)} \left(\left(\phi^{(e)}\right)^T U^{(e)} \right) \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T \theta^{(e)} \right) dXdY + \int_{\Omega^e} \left(\phi^{(e)} \left(\left(\phi^{(e)}\right)^T V^{(e)} \right) \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T \theta^{(e)} \right) dXdY +$$
$$\int_{\Omega^e} \left(\frac{\partial \phi^{(e)}}{\partial X} \left(\frac{\partial \phi^{(e)}}{\partial X} \right)^T + \frac{\partial \phi^{(e)}}{\partial Y} \left(\frac{\partial \phi^{(e)}}{\partial Y} \right)^T \right) \theta^{(e)} dXdY. \tag{3.39}$$

The matrix form of above integrals (3.37 – 3.39) may be obtained by using Eqs. (3.28 – 3.36). First consider Eq. (3.37) as follows

$$R_1^{(e)} = \left\{ \int_{\Omega^e} \left(ARR^T A^T U^{(e)} L^T \right) dXdY \right\} B^T A^T U^{(e)} + \left\{ \int_{\Omega^e} \left(ARR^T A^T V^{(e)} L^T \right) dXdY \right\} C^T A^T U^{(e)} +$$
$$\gamma \left[AB \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} B^T A^T U^{(e)} + AB \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} C^T A^T V^{(e)} \right] +$$
$$Pr \left(AB \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} B^T A^T + AC \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} C^T A^T \right) U^{(e)}$$

or

$$R_1^{(e)} = F_x B^T A^T U^{(e)} + F_y C^T A^T U^{(e)} + \gamma \left[ABHB^T A^T U^{(e)} + ABHC^T A^T V^{(e)} \right] +$$
$$Pr \left(ABHB^T A^T + ACHC^T A^T \right) U^{(e)} \tag{3.40}$$

Now Eq. (3.38) implies

$$R_2^{(e)} = \left\{ \int_{\Omega^e} \left(ARR^T A^T U^{(e)} L^T \right) dXdY \right\} B^T A^T U^{(e)} + \left\{ \int_{\Omega^e} \left(ARR^T A^T V^{(e)} L^T \right) dXdY \right\} C^T A^T U^{(e)} +$$
$$\gamma \left[AC \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} B^T A^T U^{(e)} + AC \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} C^T A^T V^{(e)} \right] +$$
$$Pr \left(AB \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} B^T A^T + AC \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} C^T A^T \right) V^{(e)} - RaPr \left[A \left\{ \int_{\Omega^e} \left(RR^T \right) dXdY \right\} A^T \theta^{(e)} \right]$$

or

$$R_2^{(e)} = F_x B^T A^T U^{(e)} + F_y C^T A^T U^{(e)} + \gamma \left[ACHB^T A^T U^{(e)} + ACHC^T A^T V^{(e)} \right] +$$
$$Pr \left(ABHB^T A^T + ACHC^T A^T \right) V^{(e)} - RaPr \left(AQA^T \theta^{(e)} \right) \tag{3.41}$$

Similarly, Eq. (3.39) in matrix form may be written as

$$R_3^{(e)} = \left\{ \int_{\Omega^e} \left(ARR^T A^T U^{(e)} L^T \right) dXdY \right\} B^T A^T \theta^{(e)} + \left\{ \int_{\Omega^e} \left(ARR^T A^T V^{(e)} L^T \right) dXdY \right\} C^T A^T \theta^{(e)} +$$
$$\left(AB \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} B^T A^T + AC \left\{ \int_{\Omega^e} \left(LL^T \right) dXdY \right\} C^T A^T \right) \theta^{(e)}$$

or

$$R_3^{(e)} = F_x B^T A^T \theta^{(e)} + F_y C^T A^T \theta^{(e)} + \left(ABHB^T A^T + ACHC^T A^T \right) \theta^{(e)}. \tag{3.42}$$

Let's start with the differentiation of Eq. (3.40) w.r.t u_i ; $i = 1, 2, \ldots, 6$, we get a square matrix of order 6.

$$J_{11}=\frac{\partial R_1^{(e)}}{\partial u_i}=\begin{bmatrix}\frac{\partial R_1^{(e)}(1,1)}{\partial u_1} & \frac{\partial R_1^{(e)}(1,1)}{\partial u_2} & \frac{\partial R_1^{(e)}(1,1)}{\partial u_3} & \frac{\partial R_1^{(e)}(1,1)}{\partial u_4} & \frac{\partial R_1^{(e)}(1,1)}{\partial u_5} & \frac{\partial R_1^{(e)}(1,1)}{\partial u_6}\\ \frac{\partial R_1^{(e)}(2,1)}{\partial u_1} & \frac{\partial R_1^{(e)}(2,1)}{\partial u_2} & \frac{\partial R_1^{(e)}(2,1)}{\partial u_3} & \frac{\partial R_1^{(e)}(2,1)}{\partial u_4} & \frac{\partial R_1^{(e)}(2,1)}{\partial u_5} & \frac{\partial R_1^{(e)}(2,1)}{\partial u_6}\\ \frac{\partial R_1^{(e)}(3,1)}{\partial u_1} & \frac{\partial R_1^{(e)}(3,1)}{\partial u_2} & \frac{\partial R_1^{(e)}(3,1)}{\partial u_3} & \frac{\partial R_1^{(e)}(3,1)}{\partial u_4} & \frac{\partial R_1^{(e)}(3,1)}{\partial u_5} & \frac{\partial R_1^{(e)}(3,1)}{\partial u_6}\\ \frac{\partial R_1^{(e)}(4,1)}{\partial u_1} & \frac{\partial R_1^{(e)}(4,1)}{\partial u_2} & \frac{\partial R_1^{(e)}(4,1)}{\partial u_3} & \frac{\partial R_1^{(e)}(4,1)}{\partial u_4} & \frac{\partial R_1^{(e)}(4,1)}{\partial u_5} & \frac{\partial R_1^{(e)}(4,1)}{\partial u_6}\\ \frac{\partial R_1^{(e)}(5,1)}{\partial u_1} & \frac{\partial R_1^{(e)}(5,1)}{\partial u_2} & \frac{\partial R_1^{(e)}(5,1)}{\partial u_3} & \frac{\partial R_1^{(e)}(5,1)}{\partial u_4} & \frac{\partial R_1^{(e)}(5,1)}{\partial u_5} & \frac{\partial R_1^{(e)}(5,1)}{\partial u_6}\\ \frac{\partial R_1^{(e)}(6,1)}{\partial u_1} & \frac{\partial R_1^{(e)}(6,1)}{\partial u_2} & \frac{\partial R_1^{(e)}(6,1)}{\partial u_3} & \frac{\partial R_1^{(e)}(6,1)}{\partial u_4} & \frac{\partial R_1^{(e)}(6,1)}{\partial u_5} & \frac{\partial R_1^{(e)}(6,1)}{\partial u_6}\end{bmatrix}. \tag{3.43}$$

Similarly, derivative of $R_1^{(e)}$ (using Eq. (3.40)) w.r.t v_i and θ_i ; $i=1,2,\ldots,6$ is as under,

$$J_{12}=\frac{\partial R_1^{(e)}}{\partial v_i}=\begin{bmatrix}\frac{\partial R_1^{(e)}(1,1)}{\partial v_1} & \frac{\partial R_1^{(e)}(1,1)}{\partial v_2} & \frac{\partial R_1^{(e)}(1,1)}{\partial v_3} & \frac{\partial R_1^{(e)}(1,1)}{\partial v_4} & \frac{\partial R_1^{(e)}(1,1)}{\partial v_5} & \frac{\partial R_1^{(e)}(1,1)}{\partial v_6}\\ \frac{\partial R_1^{(e)}(2,1)}{\partial v_1} & \frac{\partial R_1^{(e)}(2,1)}{\partial v_2} & \frac{\partial R_1^{(e)}(2,1)}{\partial v_3} & \frac{\partial R_1^{(e)}(2,1)}{\partial v_4} & \frac{\partial R_1^{(e)}(2,1)}{\partial v_5} & \frac{\partial R_1^{(e)}(2,1)}{\partial v_6}\\ \frac{\partial R_1^{(e)}(3,1)}{\partial v_1} & \frac{\partial R_1^{(e)}(3,1)}{\partial v_2} & \frac{\partial R_1^{(e)}(3,1)}{\partial v_3} & \frac{\partial R_1^{(e)}(3,1)}{\partial v_4} & \frac{\partial R_1^{(e)}(3,1)}{\partial v_5} & \frac{\partial R_1^{(e)}(3,1)}{\partial v_6}\\ \frac{\partial R_1^{(e)}(4,1)}{\partial v_1} & \frac{\partial R_1^{(e)}(4,1)}{\partial v_2} & \frac{\partial R_1^{(e)}(4,1)}{\partial v_3} & \frac{\partial R_1^{(e)}(4,1)}{\partial v_4} & \frac{\partial R_1^{(e)}(4,1)}{\partial v_5} & \frac{\partial R_1^{(e)}(4,1)}{\partial v_6}\\ \frac{\partial R_1^{(e)}(5,1)}{\partial v_1} & \frac{\partial R_1^{(e)}(5,1)}{\partial v_2} & \frac{\partial R_1^{(e)}(5,1)}{\partial v_3} & \frac{\partial R_1^{(e)}(5,1)}{\partial v_4} & \frac{\partial R_1^{(e)}(5,1)}{\partial v_5} & \frac{\partial R_1^{(e)}(5,1)}{\partial v_6}\\ \frac{\partial R_1^{(e)}(6,1)}{\partial v_1} & \frac{\partial R_1^{(e)}(6,1)}{\partial v_2} & \frac{\partial R_1^{(e)}(6,1)}{\partial v_3} & \frac{\partial R_1^{(e)}(6,1)}{\partial v_4} & \frac{\partial R_1^{(e)}(6,1)}{\partial v_5} & \frac{\partial R_1^{(e)}(6,1)}{\partial v_6}\end{bmatrix}, \tag{3.44}$$

$$J_{13}=\frac{\partial R_1^{(e)}}{\partial \theta_i}=\begin{bmatrix}\frac{\partial R_1^{(e)}(1,1)}{\partial \theta_1} & \frac{\partial R_1^{(e)}(1,1)}{\partial \theta_2} & \frac{\partial R_1^{(e)}(1,1)}{\partial \theta_3} & \frac{\partial R_1^{(e)}(1,1)}{\partial \theta_4} & \frac{\partial R_1^{(e)}(1,1)}{\partial \theta_5} & \frac{\partial R_1^{(e)}(1,1)}{\partial \theta_6}\\ \frac{\partial R_1^{(e)}(2,1)}{\partial \theta_1} & \frac{\partial R_1^{(e)}(2,1)}{\partial \theta_2} & \frac{\partial R_1^{(e)}(2,1)}{\partial \theta_3} & \frac{\partial R_1^{(e)}(2,1)}{\partial \theta_4} & \frac{\partial R_1^{(e)}(2,1)}{\partial \theta_5} & \frac{\partial R_1^{(e)}(2,1)}{\partial \theta_6}\\ \frac{\partial R_1^{(e)}(3,1)}{\partial \theta_1} & \frac{\partial R_1^{(e)}(3,1)}{\partial \theta_2} & \frac{\partial R_1^{(e)}(3,1)}{\partial \theta_3} & \frac{\partial R_1^{(e)}(3,1)}{\partial \theta_4} & \frac{\partial R_1^{(e)}(3,1)}{\partial \theta_5} & \frac{\partial R_1^{(e)}(3,1)}{\partial \theta_6}\\ \frac{\partial R_1^{(e)}(4,1)}{\partial \theta_1} & \frac{\partial R_1^{(e)}(4,1)}{\partial \theta_2} & \frac{\partial R_1^{(e)}(4,1)}{\partial \theta_3} & \frac{\partial R_1^{(e)}(4,1)}{\partial \theta_4} & \frac{\partial R_1^{(e)}(4,1)}{\partial \theta_5} & \frac{\partial R_1^{(e)}(4,1)}{\partial \theta_6}\\ \frac{\partial R_1^{(e)}(5,1)}{\partial \theta_1} & \frac{\partial R_1^{(e)}(5,1)}{\partial \theta_2} & \frac{\partial R_1^{(e)}(5,1)}{\partial \theta_3} & \frac{\partial R_1^{(e)}(5,1)}{\partial \theta_4} & \frac{\partial R_1^{(e)}(5,1)}{\partial \theta_5} & \frac{\partial R_1^{(e)}(5,1)}{\partial \theta_6}\\ \frac{\partial R_1^{(e)}(6,1)}{\partial \theta_1} & \frac{\partial R_1^{(e)}(6,1)}{\partial \theta_2} & \frac{\partial R_1^{(e)}(6,1)}{\partial \theta_3} & \frac{\partial R_1^{(e)}(6,1)}{\partial \theta_4} & \frac{\partial R_1^{(e)}(6,1)}{\partial \theta_5} & \frac{\partial R_1^{(e)}(6,1)}{\partial \theta_6}\end{bmatrix}. \tag{3.45}$$

In a similar manner, derivative of $R_2^{(e)}$ and $R_3^{(e)}$ w.r.t nodal variables (u_i, v_i, θ_i for $i=1,2,\ldots,6$) may be obtained, thus we have

$$\begin{aligned}J_{21}&=\frac{\partial R_2^{(e)}}{\partial u_i}, \quad J_{22}=\frac{\partial R_2^{(e)}}{\partial v_i}, \quad J_{23}=\frac{\partial R_2^{(e)}}{\partial \theta_i},\\ J_{31}&=\frac{\partial R_3^{(e)}}{\partial u_i}, \quad J_{32}=\frac{\partial R_3^{(e)}}{\partial v_i}, \quad J_{33}=\frac{\partial R_3^{(e)}}{\partial \theta_i}.\end{aligned} \tag{3.46}$$

Thus by using Eqs. (3.43 – 3.46), the jacobian J of the element (e) may be obtained as formulated below, which is a square matrix of order 18

$$J^{(e)} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}. \tag{3.47}$$

Moreover, the residue R associated with element (e) is evaluated using Eqs. (3.40 – 3.42), which is a column vector of residuals $\left(R_1^{(e)}, R_2^{(e)}, R_3^{(e)}\right)$ generates a matrix of order 18×1, *i.e.*

$$R^{(e)} = \begin{bmatrix} R_1^{(e)} \\ R_2^{(e)} \\ R_3^{(e)} \end{bmatrix}. \tag{3.48}$$

Since there are total 1312 elements, similar procedure will be followed for each element to get their respective jacobian and residue. Also there are 18 nodal variables $\left(u_i, v_i, \theta_i \ ; i = 1,2,\ldots,6\right)$ in all, corresponding to each element. In other words, three nodal variables u, v and θ are associated with each node. Thus, the assembly of jacobian matrices correspond to each element generate a global stiffness matrix $\boldsymbol{W}$ (say) of order 8115×8115. Whereas, combining residue matrices related to each element, an assembled matrix $\boldsymbol{N}$ (say) of order 8115×1 is obtained. Incorporate the given boundary conditions in above mentioned global stiffness matrices $\boldsymbol{W}$ and $\boldsymbol{N}$, we get a matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ (say) respectively, using these two matrices, approximated solution may be obtained by Newton Raphson method. In this context, Newton Raphson method defines as,

$$p_{n+1} = p_n - J^{-1}(p_n)R(p_n), \tag{3.49}$$

where J is the jacobian matrix, R shows the residual matrix, p denotes the unknown vectors, n is the iterative index at previous step and $n+1$ is index for unknown variable.

In present case, we have $J(p_n) = Y(p_n)$ & $R(p_n) = Z(p_n)$. Thus Eq. (3.49) may be written as

$$p_{n+1} = p_n - Y^{-1}(p_n)Z(p_n). \tag{3.50}$$

To proceed further, first assume the initial guess

$$p_0 = \begin{bmatrix} u_i^0 \\ v_i^0 \\ \theta_i^0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.01 \end{bmatrix} \quad ; \quad i = 1, 2, \ldots, 2705.$$

Substituting initial guess in Eq. (3.50), the iteration generates the p_1, then use this value in Eq. (3.50), p_2 will be obtained after the execution of second iteration. The process of successive iteration is continued until the maximum difference of the variables u, v, θ between two consecutive terms became less than 10^{-6} is achieved.

3.3.1 Evaluation of Stream Function

The stream function ψ can be defined in term of the velocity vectors U and V as follows

$$U=\frac{\partial\psi}{\partial Y} \quad \text{and} \quad V=-\frac{\partial\psi}{\partial X}, \tag{3.51}$$

Using Eq. (3.51), following result may be obtained

$$\frac{\partial^2\psi}{\partial X^2}+\frac{\partial^2\psi}{\partial Y^2}=\frac{\partial U}{\partial Y}-\frac{\partial V}{\partial X}. \tag{3.52}$$

Define approximate function for stream function ψ, we have

$$\psi=\sum_{k=1}^{6}\psi_k\phi_k\left(X,Y\right)=\psi^{(e)}\phi^{(e)}. \tag{3.53}$$

Using approximate functions for U, V and ψ from Eqs. (3.23 and 3.53), weak form of differential Eq. (3.52) is

$$\int_{\Omega^e}\left(\frac{\partial\phi^{(e)}}{\partial X}\left(\frac{\partial\phi^{(e)}}{\partial X}\right)^T+\frac{\partial\phi^{(e)}}{\partial Y}\left(\frac{\partial\phi^{(e)}}{\partial Y}\right)^T\right)\psi^{(e)}dXdY=\int_{\Omega^e}\left[-\phi^{(e)}\left(\frac{\partial\phi^{(e)}}{\partial Y}\right)^T U^{(e)}+\phi^{(e)}\left(\frac{\partial\phi^{(e)}}{\partial X}\right)^T V^{(e)}\right]dXdY. \tag{3.54}$$

Before transforming the above integral in matrix form, the product of R and L^T shall be evaluated as follows

$$[R][L]^T=\begin{bmatrix}L_1^2\\L_2^2\\L_3^2\\L_1L_2\\L_2L_3\\L_3L_1\end{bmatrix}\begin{bmatrix}L_1 & L_2 & L_3\end{bmatrix}=\begin{bmatrix}L_1^3 & L_1^2L_2 & L_1^2L_3\\L_1L_2^2 & L_2^3 & L_2^2L_3\\L_1L_3^2 & L_2L_3^2 & L_3^3\\L_1^2L_2 & L_1L_2^2 & L_1L_2L_3\\L_1L_2L_3 & L_2^2L_3 & L_2L_3^2\\L_1^2L_3 & L_1L_2L_3 & L_1L_3^2\end{bmatrix}. \tag{3.55}$$

Using formula (2.51), the integral of matrix (3.55) over the element (e) gives result as under

$$G=\int_{\Delta^{(e)}}[R][L]^T d\Delta=\int_{\Delta^{(e)}}\begin{bmatrix}L_1^3 & L_1^2L_2 & L_1^2L_3\\L_1L_2^2 & L_2^3 & L_2^2L_3\\L_1L_3^2 & L_2L_3^2 & L_3^3\\L_1^2L_2 & L_1L_2^2 & L_1L_2L_3\\L_1L_2L_3 & L_2^2L_3 & L_2L_3^2\\L_1^2L_3 & L_1L_2L_3 & L_1L_3^2\end{bmatrix}d\Delta=\frac{2\times\Delta}{120}\begin{bmatrix}6&2&2\\2&6&2\\2&2&6\\2&2&1\\1&2&2\\2&1&2\end{bmatrix}. \tag{3.56}$$

Now, transform the integral (3.54) in matrix form, we get

$$\int_{\Omega^e}\left(ABLL^TB^TA^T+ACLL^TC^TA^T\right)\psi^{(e)}dXdY=\int_{\Omega^e}\left(-ARL^TC^TA^TU^{(e)}+ARL^TB^TA^TV^{(e)}\right)dXdY,$$

or

$$\left[AB\left\{\int_{\alpha}\left(LL^T\right)dXdY\right\}B^TA^T + AC\left\{\int_{\alpha}\left(LL^T\right)dXdY\right\}C^TA^T\right]\psi^{(e)} = -A\left\{\int_{\alpha}\left(RL^T\right)dXdY\right\}C^TA^TU^{(e)} + A\left\{\int_{\alpha}\left(RL^T\right)dXdY\right\}B^TA^TV^{(e)},$$

or

$$\left[ABHB^TA^T + ACHC^TA^T\right]\psi^{(e)} = -AGC^TA^TU^{(e)} + AGB^TA^TV^{(e)},$$

$$[M]\psi^{(e)} = [N], \tag{3.57}$$

where $\psi^{(e)} = [\psi_1^{(e)} \quad \psi_2^{(e)} \quad \psi_3^{(e)} \quad \psi_4^{(e)} \quad \psi_5^{(e)} \quad \psi_6^{(e)}]^T$ is column vector of order 6×1 of nodal values of stream function associated with six noded triangular element (e).

$[M] = ABHB^TA^T + ACHC^TA^T$ and $[N] = -AGC^TA^TU^{(e)} + AGB^TA^TV^{(e)}$ are matrices of order 6×6 and 6×1 respectively.

The values of velocity components U and V obtained from Eq. (3.50) are used in Eq. (3.57) for estimation of stream function. Stream functions ($\psi_i^{(e)}$; $i = 1,2,...,6$) corresponding to element (e) is evaluated by solving system of Eqs. (3.57). Same procedure for the evaluation of stream functions is adopted for 1312 elements. At the end, assembly of all those elements is made to get system of equations, and their solution give the stream functions at each node. Graphs of stream function are represented in sections (3.5 and 3.6) for different cases of uniform and non-uniform heating.

3.3.2 Evaluation of Nusselt Number

Nusselt number is a dimensionless parameter used in calculations of heat transfer between a moving fluid and a solid body. Here, local Nusselt number is evaluated at the bottom wall denoted by Nu_b, and Nu_s is a local Nusselt number estimated at the side wall. Formulation for both are presented below

$$Nu_b = -\sum_{k=1}^{N}\theta_k\frac{\partial\phi_k}{\partial Y} \quad \text{and} \quad Nu_s = -\sum_{k=1}^{N}\theta_k\frac{\partial\phi_k}{\partial X}. \tag{3.58}$$

Consider the evaluation of Nusselt number Nu_b at each node of a six noded triangular element (e) by using Eq. (3.58) and transforming such expression in matrix form, we have

$$(Nu_b)^{(e)} = -\sum_{k=1}^{6}\theta_k\frac{\partial\phi_k}{\partial Y}. \tag{3.59}$$

Matrix form of above expression becomes

$$(Nu_b)^{(e)} = -\left(\theta^{(e)}\right)^T\frac{\partial\phi_k}{\partial Y} = -\left(\theta^{(e)}\right)^T ACL_{(i)}, \qquad i = 1, 2, \ldots, 6. \tag{3.60}$$

where $L_{(i)}$ ($i=1,2,...,6$) is a vector of area coordinates (L_1, L_2, L_3) as defined earlier, and gives distinct value for each node (i) at point (x_i, y_i) of an element (e), as follows

$$L_{(1)}=\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}_{(1)}=\frac{1}{2\Delta^{(e)}}\begin{bmatrix} a_1+b_1x_1+c_1y_1 \\ a_2+b_2x_1+c_2y_1 \\ a_3+b_3x_1+c_3y_1 \end{bmatrix},$$

Note. (x_1, y_1) is a point corresponding to node '1'.

$$L_{(2)}=\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}_{(2)}=\frac{1}{2\Delta^{(e)}}\begin{bmatrix} a_1+b_1x_2+c_1y_2 \\ a_2+b_2x_2+c_2y_2 \\ a_3+b_3x_2+c_3y_2 \end{bmatrix}.$$

Note. (x_2, y_2) is a point corresponding to node '2'.

Thus Eq. (3.60) may be written as

$$(Nu_b)^{(e)}=-\begin{bmatrix} (\theta^{(e)})^T ACL_{(1)} \\ (\theta^{(e)})^T ACL_{(2)} \\ (\theta^{(e)})^T ACL_{(3)} \\ (\theta^{(e)})^T ACL_{(4)} \\ (\theta^{(e)})^T ACL_{(5)} \\ (\theta^{(e)})^T ACL_{(6)} \end{bmatrix}.$$

Same procedure is followed for other elements for calculation of Nusselt number at their nodes. Our interest is to acquire values of Nusselt number only at those nodes which lie at the bottom wall. In current investigation, there are 41 nodes which lie at the bottom wall. Thus, an assembled matrix of local Nusselt number at bottom wall is obtained of order 41×1.

On the similar line, the computation of local Nusselt number Nu_s at the side wall may be made. Using Eq. (3.58), value of Nu_s corresponding to six noded triangular element (e) may be written as

$$(Nu_s)^{(e)}=-\sum_{k=1}^{6}\theta_k\frac{\partial\phi_k}{\partial X}=-(\theta^{(e)})^T\frac{\partial\phi_k}{\partial X}=-(\theta^{(e)})^T ABL_{(i)}, \qquad i=1, 2, \dots, 6. \tag{3.61}$$

or
$$(Nu_s)^{(e)}=-\begin{bmatrix} (\theta^{(e)})^T ABL_{(1)} \\ (\theta^{(e)})^T ABL_{(2)} \\ (\theta^{(e)})^T ABL_{(3)} \\ (\theta^{(e)})^T ABL_{(4)} \\ (\theta^{(e)})^T ABL_{(5)} \\ (\theta^{(e)})^T ABL_{(6)} \end{bmatrix}.$$

In present analysis, there are 41 number of nodes at side wall of square cavity on which local Nusselt number is being evaluated. Therefore, an assemble matrix of local Nusselt number at side wall is obtained of order 21×1.

3.4 Results and Discussion

The geometry of given problem consists of 1312 quadratic triangular elements with 2705 number of nodes. Numerical investigation for wide range of parameters $Ra = 10^3 - 10^5$ and $Pr = 0.7 - 10$ have been made with uniform and non-uniform heated bottom wall, keeping cool the vertical walls and insulated top wall. In such numerical computation, a problem occurs to evaluate the temperature at the corner nodes of the domain due to two different temperatures on the adjacent walls. The suitable technique to overcome such difficulty is that average value of temperatures on two adjacent walls is assumed at corner nodes whereas the values of other nodes lie on the walls are taken equal to respective wall temperature.

The special benefit on calculation of local Nusselt number at vertical and bottom walls is made by using finite element method due to the basis function used to calculate the heat flux.

3.5 Effects of Rayleigh number when the bottom wall is under the influence of uniform heating

The stream function and isotherm contours have been illustrated for different values of $Ra = 10^3 - 10^5$ and $Pr = 0.7 - 10$ when the bottom wall is uniformly heated. Since the vertical walls are cooled, which results the fluids rise up from middle portion of bottom wall and flow down along the two vertical walls making two symmetric rolls with clockwise and anti-clockwise circulation in the cavity. The stream function has very low magnitude at $Ra = 10^3$ and initially the heat transfer is due to conduction. During conduction dominant heat transfer, the temperature contours for $\theta = 0.1$ occur symmetrically near the side walls of the square cavity. The other temperature contours with $\theta \geq 0.2$ being as a smooth curves span the whole enclosure and are generally observed symmetric to the vertical center line. The temperature contours shown in Figure 3.2 remains invariant up to $Ra < 5 \times 10^3$.

For Rayleigh number $Ra = 5 \times 10^3$, the durable circulation occurs near the central regimes and subsequently, the temperature contour with $\theta = 0.2$ starts getting shift towards the side wall and break into two symmetric contour lines as shown in Figure 3.3. Existence of significant

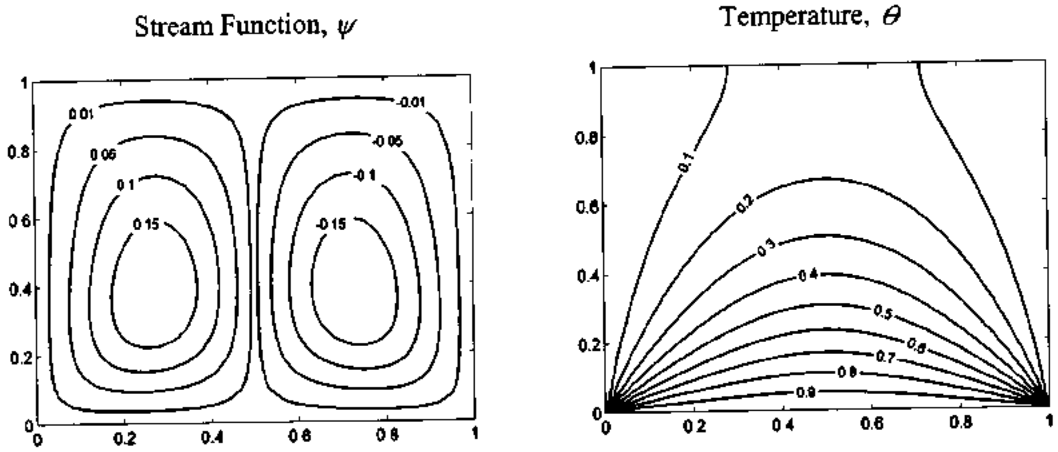


Figure 3.2: Contour plots for uniform bottom heating, $\theta(X,0)=1$, with Pr = 0.7 and Ra = 10^3. Clockwise and anti-clockwise flows are shown via negative and positive signs of stream functions, respectively.

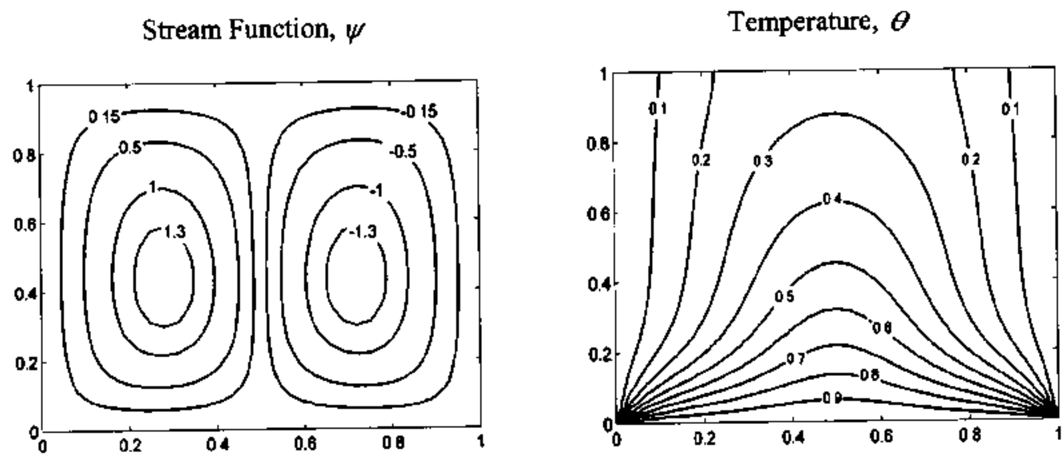


Figure 3.3: Contour plots for uniform bottom heating, $\theta(X,0)=1$, with Pr = 0.7 and $Ra=5\times10^3$. Clockwise and anti-clockwise flows are shown via negative and positive signs of stream functions, respectively.

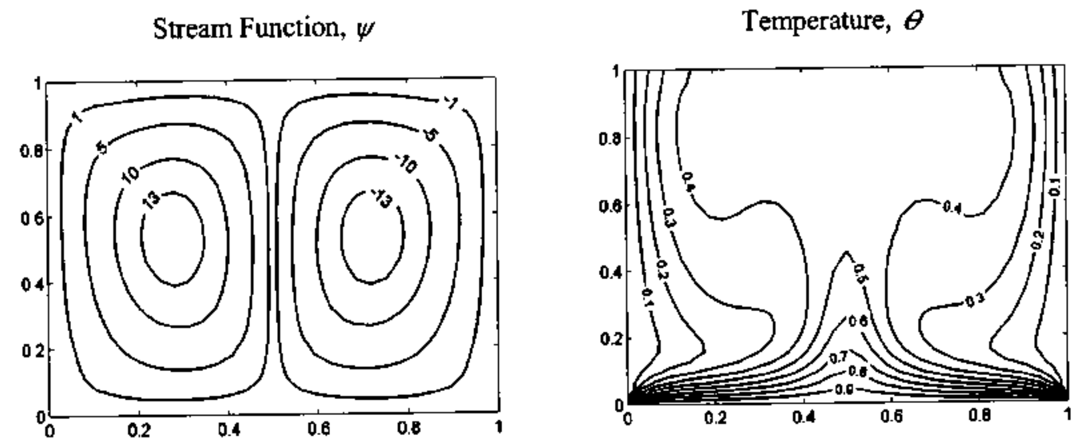


Figure 3.4: Contour plots for uniform bottom heating, $\theta(X,0)=1$, with Pr = 0.7 and $Ra=10^5$. Clockwise and anti-clockwise flows are shown via negative and positive signs of stream functions, respectively.

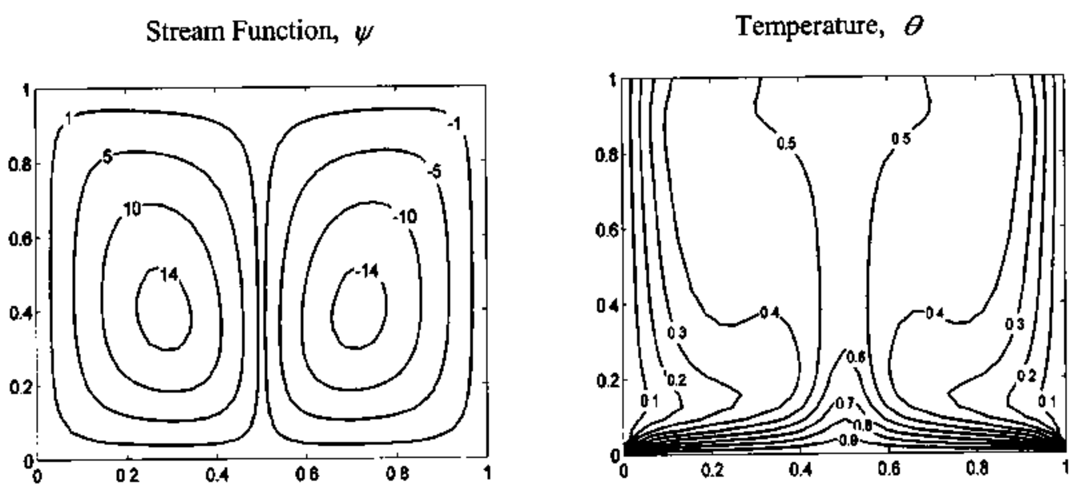


Figure 3.5: Contour plots for uniform bottom heating, $\theta(X,0)=1$, with Pr = 10 and $Ra=10^5$. Clockwise and anti-clockwise flows are shown via negative and positive signs of stream functions, respectively.

convection is also presented in other temperature contour lines which start getting deformed and pushed towards the top plate.

As Rayleigh number increases to 10^5, the buoyancy driven circulation inside the cavity also increases as seen from the greater magnitudes of the stream functions as shown in Figure 3.4. The circulations are greater near the center and least at the wall due to no slip boundary conditions. Consequently, at $Ra=10^5$, the temperature gradients near both the bottom and side walls tend to be significant leading to the development of a thermal boundary layer. Figure 3.2 shows that the thermal boundary layer develops in approximately 80% of the cavity for $Ra=10^3$ whereas for $Ra=10^5$, the isotherms presented in Figure 3.4 indicate that, the thermal boundary layer develops almost throughout the entire cavity.

The values of stream function and isotherms in the cavity increases with the increasing of Pr from 0.7 to 10, comparison is illustrated in Figures 3.4 and 3.5. The greater circulation near the central regime of each half distributes greater heat, resulting in greater temperature near the central symmetric vertical plane as shown in Figure 3.5. It may be noted that the temperature varies within 0.4–0.5 for Pr = 0.7 (Figure 3.4) near the central core regime at the top half of the enclosure whereas the temperature varies within 0.5–0.6 for Pr = 10 as seen in Figure 3.5. Due to greater circulation at $Pr=10$, the zone of stratification of temperature at the central symmetric line is reduced.

3.6 Effects of Rayleigh number when the bottom wall is under the influence of non-uniform heating

Stream function contours and isotherms are shown in Figures 3.6 – 3.8 for $Ra = 10^3 - 10^5$ and $Pr = 0.7 - 10$ when the bottom wall is non-uniformly heated. As seen earlier, uniform heating of the bottom wall causes a finite discontinuity in Dirichlet type boundary conditions for the temperature distribution at the edges of the bottom wall. In contrast, the non-uniform heating removes the singularities at the edges of the bottom wall and provides a smooth temperature distribution in the entire cavity. Due to the non-uniform heating of the bottom wall for $Ra = 10^3$ and $Pr = 0.7$, thermal boundary layer develops only over 60% of the cavity as shown in Figure 3.6, which is small in magnitude as compared to that of the uniform heating case. The conduction dominant heat transfer mode is observed up to

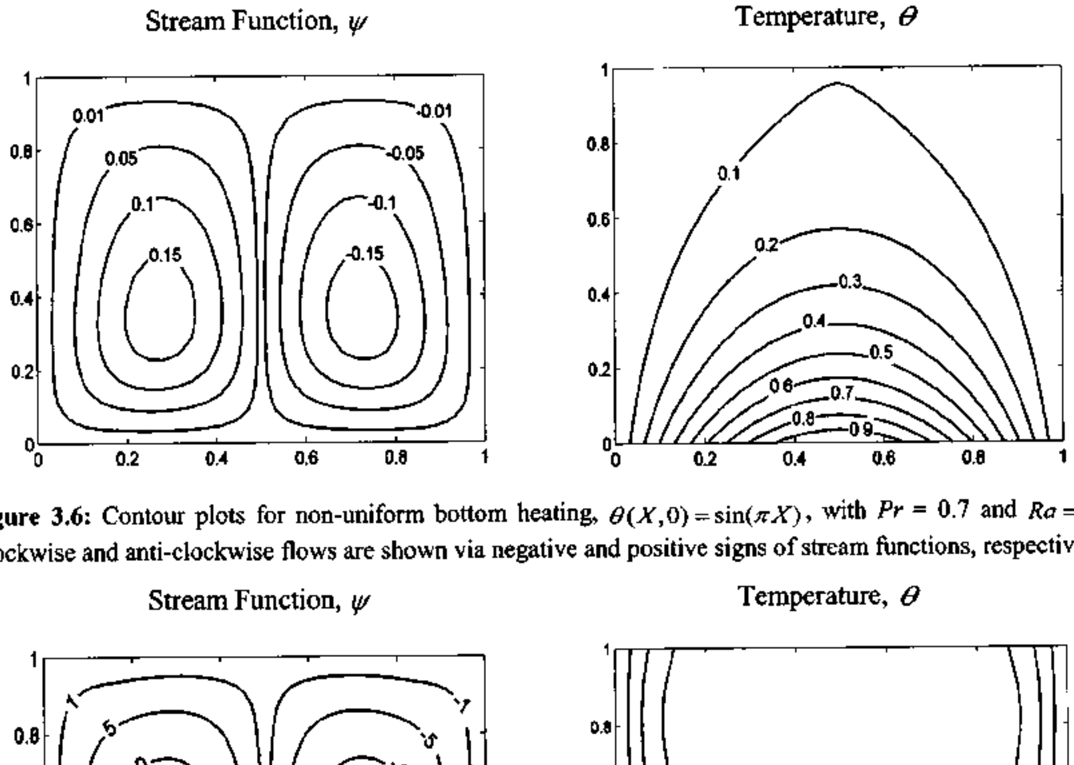


Figure 3.6: Contour plots for non-uniform bottom heating, $\theta(X,0) = \sin(\pi X)$, with Pr = 0.7 and $Ra = 10^3$. Clockwise and anti-clockwise flows are shown via negative and positive signs of stream functions, respectively.

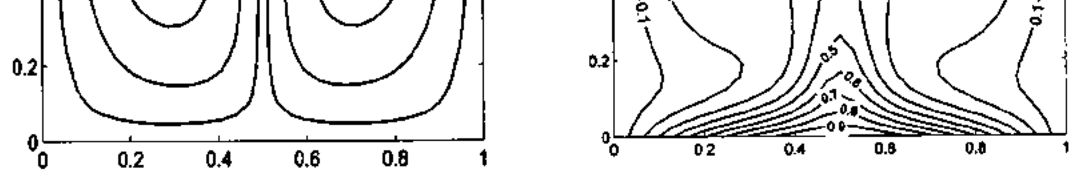


Figure 3.7: Contour plots for non-uniform bottom heating, $\theta(X,0) = \sin(\pi X)$, with Pr = 0.7 and $Ra = 10^5$. Clockwise and anti-clockwise flows are shown via negative and positive signs of stream functions, respectively.

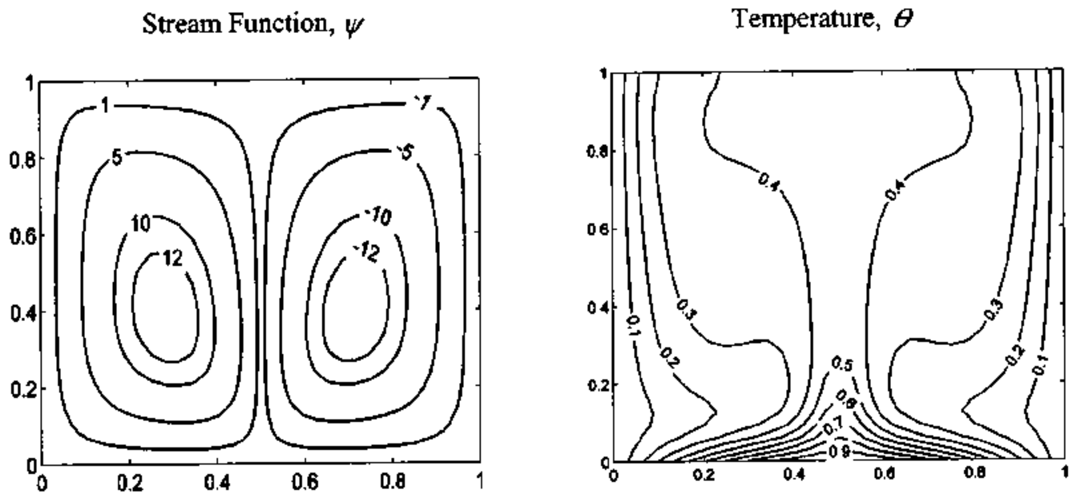


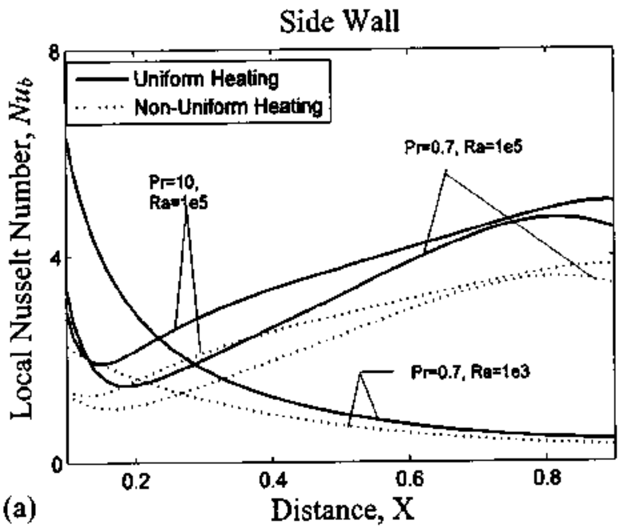
Figure 3.8: Contour plots for non-uniform bottom heating, $\theta(X,0)=\sin(\pi X)$, with Pr = 10 and $Ra = 10^5$. Clockwise and anti-clockwise flows are shown via negative and positive signs of stream functions, respectively.

$Ra = 2\times10^4$ which is consistent with that of uniform heating case, where the critical Rayleigh number is around 5000. It may be noted that the temperature at the bottom wall is non-uniform and a maximum temperature difference occurs at the center.

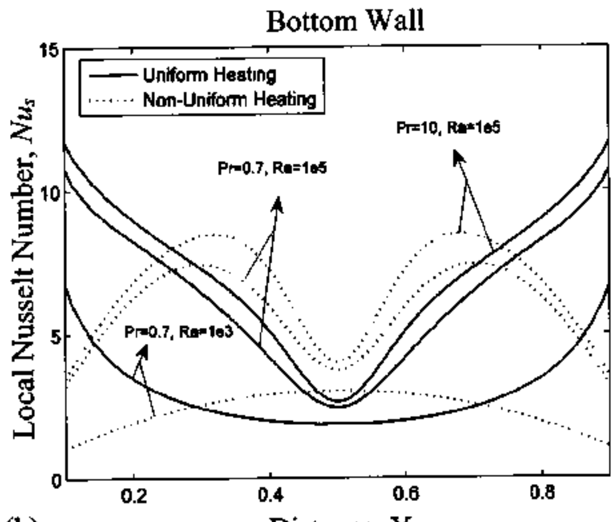
At $Ra = 10^5$, the circulation pattern is qualitatively similar to that of the uniform heating case as shown in Figure 3.7. Due to non-uniform heated bottom wall, the heating rate near the wall is generally minimum which induces less buoyancy resulting in lower thermal gradient throughout the domain. The uniformity in temperature distribution and least temperature gradient are still observed at the central core regime within the top half of the domain. The lower buoyancy effect also leads to a large zone of stratification of temperature at the vertical line of symmetry as shown in Figure 3.7. The effect of Prandtl number is also pronounced for $Ra = 10^5$ as seen in Figure 3.8, where the greater circulation causes more heat to be distributed in the central regime. However, as compared to that of uniform heating cases, the values of temperature contours are lower near the central and top portion of the enclosure for non-uniform heating. The temperature contours are highly dense near the bottom wall which may indicate a lower heating rate at the top as well as central regions of the enclosure.

3.7 Heat Transfer Rates – Local and Average Nusselt Numbers

Figure 3.9 shows the effects of Ra and Pr on the local Nusselt numbers at the bottom Nu_b and side wall Nu_s. For uniform heating of the bottom wall, the heat transfer rate Nu_b is very high at the edges of the bottom wall due to the discontinuities present in the temperature boundary conditions at the edges. It reduces towards the center of the bottom wall with the minimum value at the center as shown in Figure 3.9(a). On the contrary, for $Ra = 10^3$ with non-uniformly heated bottom wall, Nu_b increases from zero at both the edges of the bottom wall towards the center with its maximum value their. Further at $Ra = 10^5$, non-uniform heating produces a sinusoidal type of local heat transfer rate with minimum values at the edges as well as at the center of the bottom wall. The physical reason for this type of behavior is due to the higher values of the stream function (*i.e.* high flow rate) for $Ra = 10^5$ in the middle of



(a)



(b)

Figure 3.9: Variation of local Nusselt number with distance (a) at the bottom wall (b) at the side wall for uniform heating (—) and non-uniform heating (- - -).

the first and second half of the cavity. As Pr increases from 0.7 to 10, the local Nusselt number at the bottom wall Nu_b increases slightly as seen in Figure 3.9(a). It may be noted that for all values of Prandtl *Pr* and Rayleigh number *Ra*, non-uniform heating enhances the heat transfer at the central regime only. The temperature contours diverge from the corner points toward the central vertical line for uniform heating, and therefore local Nusselt number is a monotonically decreasing function with distance. In contrast, for non-uniform heating, the temperature contours are compressed around the intermediate zones between corners and the vertical line of symmetry, and local Nusselt number is maximum at around X = 0.3 and 0.7. Figure 3.9(b) illustrates the heat transfer rate at the side wall. The local Nusselt number at side wall Nu_s decreases with distance at the cold side wall for $Ra = 10^3$, $Pr = 0.7$ for both uniform and non-uniform heating. It may be noted that the heat transfer rate initially decreases and later increases with distance for $Ra = 10^5$ with $Pr = 0.7$ and 10. At higher Rayleigh numbers, the significant circulation has been observed which can be seen in Figures 3.4, 3.5, 3.7 and 3.8

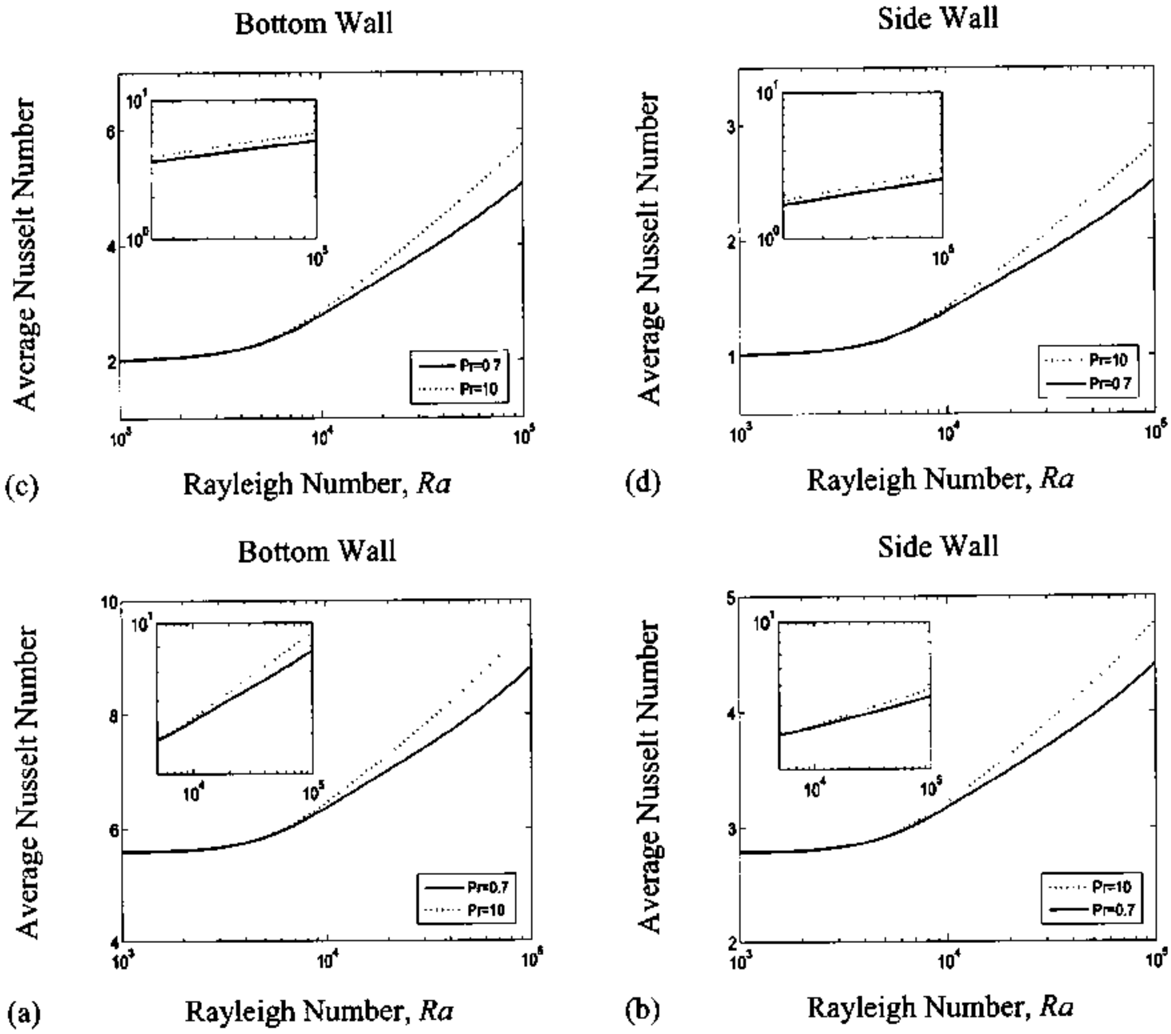


Figure 3.10: Variation of average Nusselt number with Rayleigh number for uniform heating [(a) and (b)] and non-uniform heating [(c) and (d)] with Pr = 0.7; (—) and Pr = 10; (- - -). The insets show the log–log plot of average Nusselt number versus Rayleigh number for convection dominant regimes.

results in highly dense contours at the top portion of the side walls and these dense temperature contours are in contrast with the conduction dominant cases as seen in Figures 3.2 and 3.6. Further, it is observed that the temperature contours are compressed towards the side walls away from the corner points at the bottom. Therefore, the heat fluxes are enhanced at the regions away from the bottom corner points. The heat transfer rates are qualitatively similar, but reduced for non-uniform heating of bottom wall as compared to uniform heating.

The overall effect on the heat transfer rates are shown in Figure 3.10(a)–(d), where the distributions of the average Nusselt number of bottom and side walls respectively, are plotted versus the logarithmic Rayleigh number. Figures 10(a) and (b) (cases a and b) illustrate uniform heating and Figures 10(c) and (d) (cases c and d) illustrate non-uniform heating. For all these cases, it is observed that average Nusselt numbers for both the bottom and side walls remain constant up to $Ra = 5000$ for uniform heating and up to $Ra = 2\times10^4$ for non-uniform heating. Hence, dominant heat conduction mode corresponding to larger range of Rayleigh numbers produces overall lower heat transfer rates against non-uniform heating. The insets show the log–log plot for average Nusselt number versus Rayleigh number for convection dominant regimes. The log–log linear plot is obtained with more than 20 data set. A least square curve is fitted and the overall error is within 1%. The following correlations are obtained for cases a, b, c and d as follows:

Cases a and b: Uniform heating ($Ra \geq 5000$)

$$\begin{aligned}\overline{Nu_b} &= 2\overline{Nu_s}\\ &= 1.6219Ra^{0.145}, \quad Pr = 0.7\\ &= 1.2238Ra^{0.177}, \quad Pr = 10\end{aligned}$$

Cases c and d: Non-Uniform heating ($Ra \geq 2\times10^4$)

$$\begin{aligned}\overline{Nu_b} &= 2\overline{Nu_s}\\ &= 0.2939Ra^{0.249}, \quad Pr = 0.7\\ &= 1.2238Ra^{0.289}, \quad Pr = 10\end{aligned}$$

3.8 Conclusions

The prime objective of this chapter is to reinvestigate the effect of Dirichlet boundary conditions on the flow and heat transfer characteristics due to natural convection within a square enclosure studied by T. Basak et al [5]. The penalty finite element method helps to obtain smooth solutions in terms of stream functions and isotherm contours for wide ranges of parameters *Pr* and *Ra* with uniform and non-uniform heating of the bottom wall. It has been demonstrated that the formation of boundary layers for both the heating cases occurs. It is also observed that thermal boundary layer develops over approximately 80% of the cavity for uniform heating, whereas the boundary layer is approximately 60% for non-uniform heating when $Ra = 10^3$. The heat transfer rate is very high at the edges of the bottom wall and decreases to a minimum value at the center due to uniform heating which is consistent with the lower heat transfer rate at the edges due to non-uniform heating for $Ra = 10^3$. The conduction dominant heat transfer modes occurs at $Ra \leq 5\times10^3$ during uniform heating of bottom wall whereas it occurs at $Ra \leq 2\times10^4$ for non-uniform heating.

At the onset of convection dominant mode, the temperature contour lines get compressed toward the side walls and they tend to get deformed towards the upward direction. During $Ra = 10^5$, the thermal boundary layer develops near the bottom and side walls, and the central regime near the top surface has least temperature gradient for both uniform and non-uniform heating. The local Nusselt numbers at the bottom and side walls represent various interesting heating features. The local Nusselt number at the bottom wall is least at the center for uniform heating and there are two minimum heat transfer zones at the center and corner points for non-uniform heating. The non-uniform heating exhibits greater heat transfer rates at the center of the bottom wall than that with uniform heating for all Rayleigh numbers. The local Nusselt number at the side wall is found to decrease with distance for conduction dominant heat transfer whereas due to highly dense contour lines near the top portion of the side wall, the local Nusselt number is found to increase for both uniform and non-uniform heating cases. The average Nusselt number indicates overall lower heat transfer rates for non-uniform heating. The average Nusselt number is found to follow a power law variation with Rayleigh number for convection dominant regimes.

Appendix A

A.1 Calculation of integral $b^{(e)} = \iint\limits_{(e)} H_i dxdy\,; \qquad i = 1,2,3$

According to problem (2.10.1), the shape functions for linear triangular element (e) are defined as

$$H_1 = \frac{1}{2A}\left[\left(x_2y_3 - x_3y_2\right) + \left(y_2 - y_3\right)x + \left(x_3 - x_2\right)y\right],$$

$$H_2 = \frac{1}{2A}\left[\left(x_3y_1 - x_1y_3\right) + \left(y_3 - y_1\right)x + \left(x_1 - x_3\right)y\right],$$

$$H_3 = \frac{1}{2A}\left[\left(x_1y_2 - x_2y_1\right) + \left(y_1 - y_2\right)x + \left(x_2 - x_1\right)y\right]$$

Without generality, above shape functions may simply be written as (as illustrated in Eq. 2.46)

$H_1 = L_1\,, \qquad H_2 = L_2$ and $H_3 = L_3$

Therefore, given integral becomes

$$b^{(e)} = \iint\limits_{(e)} \left[H_1 \;\; H_2 \;\; H_3\right]^T dxdy \qquad \text{(A.1)}$$

Consider the first term of above integral

$$\iint\limits_{(e)} H_1 dxdy = \iint\limits_{(e)} L_1 dxdy = \frac{\Delta}{3} \qquad \text{(using Eq. 2.51)}$$

Similarly, values for other two terms of integral (A.1) are

$$\iint\limits_{(e)} H_2 dxdy = \iint\limits_{(e)} L_2 dxdy = \frac{\Delta}{3} \qquad \text{(using Eq. 2.51)}$$

$$\iint\limits_{(e)} H_3 dxdy = \iint\limits_{(e)} L_3 dxdy = \frac{\Delta}{3} \qquad \text{(using Eq. 2.51)}$$

Using above values in integral (A.1), we get

$$b^{(e)} = \begin{bmatrix} \Delta/3 \\ \Delta/3 \\ \Delta/3 \end{bmatrix} = \frac{\Delta}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

In problem (2.10.1), Δ is represented by A.

A.2 Evaluation of integral $F_x = \int_{\Delta^{(e)}} \left(ARR^T A^T U L^T\right) d\Delta$

$$F_x = \int_{\Delta^{(e)}} \left(ARR^T A^T U L^T\right) d\Delta$$

$$= \int_{\Delta^{(e)}} \left(\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix} \begin{bmatrix} L_1^2 & L_2^2 & L_3^2 & L_1L_2 & L_2L_3 & L_3L_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 & 4 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \right) d\Delta$$

With the help of computational software program ***Mathematica 8***, the product of matrices written above gives a single matrix of order 6x3, that is

$$F_x = \int_{\Delta^{(e)}} \begin{pmatrix} F_{1,1} & F_{1,2} & F_{1,3} \\ F_{2,1} & F_{2,2} & F_{2,3} \\ F_{3,1} & F_{3,2} & F_{3,3} \\ F_{4,1} & F_{4,2} & F_{4,3} \\ F_{5,1} & F_{5,2} & F_{5,3} \\ F_{6,1} & F_{6,2} & F_{6,3} \end{pmatrix} d\Delta\,, \qquad \text{(A.2)}$$

where,

$$\begin{aligned} F_{1,1} = L_1((&L_1^2(L_1^2 - L_1L_2 - L_1L_3) - L_1L_2(L_1^2 - L_1L_2 - L_1L_3) - L_1L_3(L_1^2 - L_1L_2 - L_1L_3))u_1 \\ &+ (-L_1L_2(L_1^2 - L_1L_2 - L_1L_3) + L_2^2(L_1^2 - L_1L_2 - L_1L_3) - L_2L_3(L_1^2 - L_1L_2 - L_1L_3))u_2 \\ &+ (-L_1L_3(L_1^2 - L_1L_2 - L_1L_3) - L_2L_3(L_1^2 - L_1L_2 - L_1L_3) + L_3^2(L_1^2 - L_1L_2 - L_1L_3))u_3 \\ &+ 4L_1L_2(L_1^2 - L_1L_2 - L_1L_3)u_4 + 4L_2L_3(L_1^2 - L_1L_2 - L_1L_3)u_5 + 4L_1L_3(L_1^2 - L_1L_2 - L_1L_3)u_6), \end{aligned}$$

$$\begin{aligned} F_{1,2} = L_2((&L_1^2(L_1^2 - L_1L_2 - L_1L_3) - L_1L_2(L_1^2 - L_1L_2 - L_1L_3) - L_1L_3(L_1^2 - L_1L_2 - L_1L_3))u_1 \\ &+ (-L_1L_2(L_1^2 - L_1L_2 - L_1L_3) + L_2^2(L_1^2 - L_1L_2 - L_1L_3) - L_2L_3(L_1^2 - L_1L_2 - L_1L_3))u_2 \\ &+ (-L_1L_3(L_1^2 - L_1L_2 - L_1L_3) - L_2L_3(L_1^2 - L_1L_2 - L_1L_3) + L_3^2(L_1^2 - L_1L_2 - L_1L_3))u_3 \\ &+ 4L_1L_2(L_1^2 - L_1L_2 - L_1L_3)u_4 + 4L_2L_3(L_1^2 - L_1L_2 - L_1L_3)u_5 + 4L_1L_3(L_1^2 - L_1L_2 - L_1L_3)u_6), \end{aligned}$$

$$\begin{aligned} F_{1,3} = L_3((&L_1^2(L_1^2 - L_1L_2 - L_1L_3) - L_1L_2(L_1^2 - L_1L_2 - L_1L_3) - L_1L_3(L_1^2 - L_1L_2 - L_1L_3))u_1 \\ &+ (-L_1L_2(L_1^2 - L_1L_2 - L_1L_3) + L_2^2(L_1^2 - L_1L_2 - L_1L_3) - L_2L_3(L_1^2 - L_1L_2 - L_1L_3))u_2 \\ &+ (-L_1L_3(L_1^2 - L_1L_2 - L_1L_3) - L_2L_3(L_1^2 - L_1L_2 - L_1L_3) + L_3^2(L_1^2 - L_1L_2 - L_1L_3))u_3 \\ &+ 4L_1L_2(L_1^2 - L_1L_2 - L_1L_3)u_4 + 4L_2L_3(L_1^2 - L_1L_2 - L_1L_3)u_5 + 4L_1L_3(L_1^2 - L_1L_2 - L_1L_3)u_6), \end{aligned}$$

$$\begin{aligned} F_{2,1} = L_1((&L_1^2(-L_1L_2 + L_2^2 - L_2L_3) - L_1L_2(-L_1L_2 + L_2^2 - L_2L_3) - L_1L_3(-L_1L_2 + L_2^2 - L_2L_3))u_1 \\ &+ (-L_1L_2(-L_1L_2 + L_2^2 - L_2L_3) + L_2^2(-L_1L_2 + L_2^2 - L_2L_3) - L_2L_3(-L_1L_2 + L_2^2 - L_2L_3))u_2 \\ &+ (-L_1L_3(-L_1L_2 + L_2^2 - L_2L_3) - L_2L_3(-L_1L_2 + L_2^2 - L_2L_3) + L_3^2(-L_1L_2 + L_2^2 - L_2L_3))u_3 \\ &+ 4L_1L_2(-L_1L_2 + L_2^2 - L_2L_3)u_4 + 4L_2L_3(-L_1L_2 + L_2^2 - L_2L_3)u_5 + 4L_1L_3(-L_1L_2 + L_2^2 - L_2L_3)u_6), \end{aligned}$$

$$F_{2,2} = L_2((L_1^2(-L_1L_2 + L_2^2 - L_2L_3) - L_1L_2(-L_1L_2 + L_2^2 - L_2L_3) - L_1L_3(-L_1L_2 + L_2^2 - L_2L_3))u_1$$
$$+ (-L_1L_2(-L_1L_2 + L_2^2 - L_2L_3) + L_2^2(-L_1L_2 + L_2^2 - L_2L_3) - L_2L_3(-L_1L_2 + L_2^2 - L_2L_3))u_2$$
$$+ (-L_1L_3(-L_1L_2 + L_2^2 - L_2L_3) - L_2L_3(-L_1L_2 + L_2^2 - L_2L_3) + L_3^2(-L_1L_2 + L_2^2 - L_2L_3))u_3$$
$$+ 4L_1L_2(-L_1L_2 + L_2^2 - L_2L_3)u_4 + 4L_2L_3(-L_1L_2 + L_2^2 - L_2L_3)u_5 + 4L_1L_3(-L_1L_2 + L_2^2 - L_2L_3)u_6),$$

$$F_{2,3} = L_3((L_1^2(-L_1L_2 + L_2^2 - L_2L_3) - L_1L_2(-L_1L_2 + L_2^2 - L_2L_3) - L_1L_3(-L_1L_2 + L_2^2 - L_2L_3))u_1$$
$$+ (-L_1L_2(-L_1L_2 + L_2^2 - L_2L_3) + L_2^2(-L_1L_2 + L_2^2 - L_2L_3) - L_2L_3(-L_1L_2 + L_2^2 - L_2L_3))u_2$$
$$+ (-L_1L_3(-L_1L_2 + L_2^2 - L_2L_3) - L_2L_3(-L_1L_2 + L_2^2 - L_2L_3) + L_3^2(-L_1L_2 + L_2^2 - L_2L_3))u_3$$
$$+ 4L_1L_2(-L_1L_2 + L_2^2 - L_2L_3)u_4 + 4L_2L_3(-L_1L_2 + L_2^2 - L_2L_3)u_5 + 4L_1L_3(-L_1L_2 + L_2^2 - L_2L_3)u_6),$$

$$F_{3,1} = L_1((L_1^2(-L_1L_3 - L_2L_3 + L_3^2) - L_1L_2(-L_1L_3 - L_2L_3 + L_3^2) - L_1L_3(-L_1L_3 - L_2L_3 + L_3^2))u_1$$
$$+ (-L_1L_2(-L_1L_3 - L_2L_3 + L_3^2) + L_2^2(-L_1L_3 - L_2L_3 + L_3^2) - L_2L_3(-L_1L_3 - L_2L_3 + L_3^2))u_2$$
$$+ (-L_1L_3(-L_1L_3 - L_2L_3 + L_3^2) - L_2L_3(-L_1L_3 - L_2L_3 + L_3^2) + L_3^2(-L_1L_3 - L_2L_3 + L_3^2))u_3$$
$$+ 4L_1L_2(-L_1L_3 - L_2L_3 + L_3^2)u_4 + 4L_2L_3(-L_1L_3 - L_2L_3 + L_3^2)u_5 + 4L_1L_3(-L_1L_3 - L_2L_3 + L_3^2)u_6),$$

$$F_{3,2} = L_2((L_1^2(-L_1L_3 - L_2L_3 + L_3^2) - L_1L_2(-L_1L_3 - L_2L_3 + L_3^2) - L_1L_3(-L_1L_3 - L_2L_3 + L_3^2))u_1$$
$$+ (-L_1L_2(-L_1L_3 - L_2L_3 + L_3^2) + L_2^2(-L_1L_3 - L_2L_3 + L_3^2) - L_2L_3(-L_1L_3 - L_2L_3 + L_3^2))u_2$$
$$+ (-L_1L_3(-L_1L_3 - L_2L_3 + L_3^2) - L_2L_3(-L_1L_3 - L_2L_3 + L_3^2) + L_3^2(-L_1L_3 - L_2L_3 + L_3^2))u_3$$
$$+ 4L_1L_2(-L_1L_3 - L_2L_3 + L_3^2)u_4 + 4L_2L_3(-L_1L_3 - L_2L_3 + L_3^2)u_5 + 4L_1L_3(-L_1L_3 - L_2L_3 + L_3^2)u_6),$$

$$F_{3,3} = L_3((L_1^2(-L_1L_3 - L_2L_3 + L_3^2) - L_1L_2(-L_1L_3 - L_2L_3 + L_3^2) - L_1L_3(-L_1L_3 - L_2L_3 + L_3^2))u_1$$
$$+ (-L_1L_2(-L_1L_3 - L_2L_3 + L_3^2) + L_2^2(-L_1L_3 - L_2L_3 + L_3^2) - L_2L_3(-L_1L_3 - L_2L_3 + L_3^2))u_2$$
$$+ (-L_1L_3(-L_1L_3 - L_2L_3 + L_3^2) - L_2L_3(-L_1L_3 - L_2L_3 + L_3^2) + L_3^2(-L_1L_3 - L_2L_3 + L_3^2))u_3$$
$$+ 4L_1L_2(-L_1L_3 - L_2L_3 + L_3^2)u_4 + 4L_2L_3(-L_1L_3 - L_2L_3 + L_3^2)u_5 + 4L_1L_3(-L_1L_3 - L_2L_3 + L_3^2)u_6),$$

$$F_{4,1} = L_1((4L_1^3L_2 - 4L_1^2L_2^2 - 4L_1^2L_2L_3)u_1 + (-4L_1^2L_2^2 + 4L_1L_2^3 - 4L_1L_2^2L_3)u_2$$
$$+ (-4L_1^2L_2L_3 - 4L_1L_2^2L_3 + 4L_1L_2L_3^2)u_3 + 16L_1^2L_2^2u_4 + 16L_1L_2^2L_3u_5 + 16L_1^2L_2L_3u_6),$$

$$F_{4,2} = L_2((4L_1^3L_2 - 4L_1^2L_2^2 - 4L_1^2L_2L_3)u_1 + (-4L_1^2L_2^2 + 4L_1L_2^3 - 4L_1L_2^2L_3)u_2$$
$$+ (-4L_1^2L_2L_3 - 4L_1L_2^2L_3 + 4L_1L_2L_3^2)u_3 + 16L_1^2L_2^2u_4 + 16L_1L_2^2L_3u_5 + 16L_1^2L_2L_3u_6),$$

$$F_{4,3} = L_3((4L_1^3L_2 - 4L_1^2L_2^2 - 4L_1^2L_2L_3)u_1 + (-4L_1^2L_2^2 + 4L_1L_2^3 - 4L_1L_2^2L_3)u_2$$
$$+ (-4L_1^2L_2L_3 - 4L_1L_2^2L_3 + 4L_1L_2L_3^2)u_3 + 16L_1^2L_2^2u_4 + 16L_1L_2^2L_3u_5 + 16L_1^2L_2L_3u_6),$$

$$F_{5,1} = L_1((4L_1^2L_2L_3 - 4L_1L_2^2L_3 - 4L_1L_2L_3^2)u_1 + (-4L_1L_2^2L_3 + 4L_2^3L_3 - 4L_2^2L_3^2)u_2$$
$$+ (-4L_1L_2L_3^2 - 4L_2^2L_3^2 + 4L_2L_3^3)u_3 + 16L_1L_2^2L_3u_4 + 16L_2^2L_3^2u_5 + 16L_1L_2L_3^2u_6),$$

$$F_{5,2} = L_2((4L_1^2L_2L_3 - 4L_1L_2^2L_3 - 4L_1L_2L_3^2)u_1 + (-4L_1L_2^2L_3 + 4L_2^3L_3 - 4L_2^2L_3^2)u_2$$
$$+ (-4L_1L_2L_3^2 - 4L_2^2L_3^2 + 4L_2L_3^3)u_3 + 16L_1L_2^2L_3u_4 + 16L_2^2L_3^2u_5 + 16L_1L_2L_3^2u_6),$$

$$F_{5,3} = L_3((4L_1^2L_2L_3 - 4L_1L_2^2L_3 - 4L_1L_2L_3^2)u_1 + (-4L_1L_2^2L_3 + 4L_2^3L_3 - 4L_2^2L_3^2)u_2$$
$$+ (-4L_1L_2L_3^2 - 4L_2^2L_3^2 + 4L_2L_3^3)u_3 + 16L_1L_2^2L_3u_4 + 16L_2^2L_3^2u_5 + 16L_1L_2L_3^2u_6),$$

$$F_{6,1} = L_1((4L_1^3L_3 - 4L_1^2L_2L_3 - 4L_1^2L_3^2)u_1 + (-4L_1^2L_2L_3 + 4L_1L_2^2L_3 - 4L_1L_2L_3^2)u_2 + (-4L_1^2L_3^2 - 4L_1L_2L_3^2 + 4L_1L_3^3)u_3 + 16L_1^2L_2L_3u_4 + 16L_1L_2L_3^2u_5 + 16L_1^2L_3^2u_6),$$

$$F_{6,2} = L_2((4L_1^3L_3 - 4L_1^2L_2L_3 - 4L_1^2L_3^2)u_1 + (-4L_1^2L_2L_3 + 4L_1L_2^2L_3 - 4L_1L_2L_3^2)u_2 + (-4L_1^2L_3^2 - 4L_1L_2L_3^2 + 4L_1L_3^3)u_3 + 16L_1^2L_2L_3u_4 + 16L_1L_2L_3^2u_5 + 16L_1^2L_3^2u_6),$$

and

$$F_{6,3} = L_3((4L_1^3L_3 - 4L_1^2L_2L_3 - 4L_1^2L_3^2)u_1 + (-4L_1^2L_2L_3 + 4L_1L_2^2L_3 - 4L_1L_2L_3^2)u_2 + (-4L_1^2L_3^2 - 4L_1L_2L_3^2 + 4L_1L_3^3)u_3 + 16L_1^2L_2L_3u_4 + 16L_1L_2L_3^2u_5 + 16L_1^2L_3^2u_6)$$

Substitute above values in Eq. (A.2) and then integrate using formula (2.51), resulting the Eq. (3.35).

Following mathematical code of ***Mathematica 8*** has been used for execution of above results,

```
a1=(x2 y3-x3 y2); a2=(x3 y1-x1 y3); a3=(x1 y2-x2 y1);
b1=(y2-y3); b2=(y3-y1); b3=(y1-y2);
c1=(x3-x2); c2=(x1-x3); c3=(x2-x1);
a={{1,0,0,-1,0,-1},{0,1,0,-1,-1,0},{0,0,1,0,-1,-1},{0,0,0,4,0,0},
{0,0,0,0,4,0},{0,0,0,0,0,4}};
at=Transpose[a];
r={{L1^2},{L2^2},{L3^2},{L1 L2},{L3 L2},{L1 L3}};        rt=Transpose[r];
u={{u1},{u2},{u3},{u4},{u5},{u6}};   h={{L1},{L2},{L3}}; ht=Transpose[h];
F1=(a.r.rt.at.u.ht);
F1[[2,1]]; (*this command is used to get result of an element located at
second row and first column of matrix F1*)
Dimensions[F1]; (*gives dimension of matrix F1 i.e 6x3*)
MatrixForm[F1]
```

A.3 Evaluation of integral $F_y = \int_{\Delta^{(e)}} \left(ARR^T A^T V L^T\right) d\Delta$

$$F_y = \int_{\Delta^{(e)}} \left(ARR^T A^T V L^T\right) d\Delta$$

$$= \int_{\Delta^{(e)}} \left(\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix} \begin{bmatrix} L_1^2 & L_2^2 & L_3^2 & L_1L_2 & L_2L_3 & L_3L_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 & 4 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \right) d\Delta$$

For evaluation and simplification of above integral, similar procedure as described in A.2 may be followed, just replace velocity component u by v whereas remaining terms will be unchanged.

A.4 Simplification of integral $\int_{\Omega}\phi_k\left(U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}\right)dXdY-\int_{\Omega}\phi_k\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)dXdY=0$

Solution.

$$\int_{\Omega}\phi_k\left(U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}\right)dXdY-\int_{\Omega}\phi_k\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)dXdY=0 \tag{A.3}$$

Consider $\int_{\Omega}\phi_k\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)dXdY$

Since, $\frac{\partial}{\partial X}\left(\phi_k\frac{\partial\theta}{\partial X}\right)=\frac{\partial\phi_k}{\partial X}\frac{\partial\theta}{\partial X}+\phi_k\frac{\partial^2\theta}{\partial X^2}$

$$\Rightarrow\quad \phi_k\frac{\partial^2\theta}{\partial X^2}=\frac{\partial}{\partial X}\left(\phi_k\frac{\partial\theta}{\partial X}\right)-\frac{\partial\phi_k}{\partial X}\frac{\partial\theta}{\partial X}$$

Taking area integral on both sides, we get

$$\int_{\Omega}\phi_k\frac{\partial^2\theta}{\partial X^2}dXdY=\int_{\Omega}\frac{\partial}{\partial X}\left(\phi_k\frac{\partial\theta}{\partial X}\right)dXdY-\int_{\Omega}\frac{\partial\phi_k}{\partial X}\frac{\partial\theta}{\partial X}dXdY \tag{A.4}$$

Converting area integral $\int_{\Omega}\frac{\partial}{\partial X}\left(\phi_k\frac{\partial\theta}{\partial X}\right)dXdY$ into line integral, implies that

$$\int_{\Omega}\frac{\partial}{\partial X}\left(\phi_k\frac{\partial\theta}{\partial X}\right)dXdY=\oint_{\Gamma}\left(\phi_k\frac{\partial\theta}{\partial X}n_X\right)ds$$

Using above relation, Eq. (A.4) implies

$$\int_{\Omega}\phi_k\frac{\partial^2\theta}{\partial X^2}dXdY=\oint_{\Gamma}\left(\phi_k\frac{\partial\theta}{\partial X}n_X\right)ds-\int_{\Omega}\frac{\partial\phi_k}{\partial X}\frac{\partial\theta}{\partial X}dXdY \tag{A.5}$$

Similarly,

$$\int_{\Omega}\phi_k\frac{\partial^2\theta}{\partial Y^2}dXdY=\oint_{\Gamma}\left(\phi_k\frac{\partial\theta}{\partial Y}n_Y\right)ds-\int_{\Omega}\frac{\partial\phi_k}{\partial Y}\frac{\partial\theta}{\partial Y}dXdY \tag{A.6}$$

Addition of Eqs. (A.5) and (A.6) generates the following result

$$\int_{\Omega}\phi_k\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)dXdY=-\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\theta}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial\theta}{\partial Y}\right)dXdY+\oint_{\Gamma}\left(n_X\frac{\partial\theta}{\partial X}+n_Y\frac{\partial\theta}{\partial Y}\right)\phi_k ds \tag{A.7}$$

Using integral (A.7), Eq. (A.3) reduces to following form

$$\int_{\Omega}\phi_k\left(U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}\right)dXdY+\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\theta}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial\theta}{\partial Y}\right)dXdY-\oint_{\Gamma}\left(n_X\frac{\partial\theta}{\partial X}+n_Y\frac{\partial\theta}{\partial Y}\right)\phi_k ds=0$$

Using approximated functions (3.15), above integral becomes

$$\int_{\Omega}\left(\phi_k\left(\phi_k^T U_k\right)\frac{\partial \phi_k^T}{\partial X}+\phi_k\left(\phi_k^T V_k\right)\frac{\partial \phi_k^T}{\partial Y}\right)\theta_k dXdY+\int_{\Omega}\left(\frac{\partial \phi_k}{\partial X}\frac{\partial \phi_k^T}{\partial X}+\frac{\partial \phi_k}{\partial X}\frac{\partial \phi_k^T}{\partial X}\right)\theta_k dXdY$$

$$-\oint_{\Gamma}\left(n_X\frac{\partial \theta}{\partial X}+n_Y\frac{\partial \theta}{\partial Y}\right)\phi_k ds=0$$

A.5 Evaluate the following integral

$$\int_{\Omega}\phi_k\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right)dXdY-\gamma\int_{\Omega}\phi_k\left[\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY-\Pr\int_{\Omega}\phi_k\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)dXdY=0 \quad \text{(A.8)}$$

Solution.

Consider $\frac{\partial}{\partial X}\left[\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]=\frac{\partial \phi_k}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)+\phi_k\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)$

$$\Rightarrow \quad \phi_k\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)=\frac{\partial}{\partial X}\left[\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]-\frac{\partial \phi_k}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)$$

Taking area integral on both sides, we get

$$\int_{\Omega}\phi_k\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)dXdY=\int_{\Omega}\frac{\partial}{\partial X}\left[\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY-\int_{\Omega}\frac{\partial \phi_k}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)dXdY$$

$$\int_{\Omega}\phi_k\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)dXdY=\int_{\Omega}\frac{\partial}{\partial X}\left[\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY-\int_{\Omega}\frac{\partial \phi_k}{\partial X}\frac{\partial U}{\partial X}dXdY-\int_{\Omega}\frac{\partial \phi_k}{\partial X}\frac{\partial V}{\partial Y}dXdY \quad \text{(A.9)}$$

Converting area integral $\int_{\Omega}\frac{\partial}{\partial X}\left[\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY$ into line integral, we get

$$\int_{\Omega}\frac{\partial}{\partial X}\left[\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY=\oint_{\Gamma}\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)n_X ds \quad \text{(A.10)}$$

Using integral (A.10) in Eq. (A.9), implies that

$$\int_{\Omega}\phi_k\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)dXdY=\oint_{\Gamma}\phi_k\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)n_X ds-\int_{\Omega}\frac{\partial \phi_k}{\partial X}\frac{\partial U}{\partial X}dXdY-\int_{\Omega}\frac{\partial \phi_k}{\partial X}\frac{\partial V}{\partial Y}dXdY$$

By Eq. (3.7), $\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0$, therefore above integral becomes

$$\int_{\Omega}\phi_k\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)dXdY=-\int_{\Omega}\frac{\partial \phi_k}{\partial X}\frac{\partial U}{\partial X}dXdY-\int_{\Omega}\frac{\partial \phi_k}{\partial X}\frac{\partial V}{\partial Y}dXdY \quad \text{(A.11)}$$

Using Eq. (A.8), we may write

$$\int_{\Omega}\phi_k\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)dXdY=-\int_{\Omega}\left(\frac{\partial \phi_k}{\partial X}\frac{\partial U}{\partial X}+\frac{\partial \phi_k}{\partial Y}\frac{\partial U}{\partial Y}\right)dXdY+\oint_{\Gamma}\left(n_X\frac{\partial U}{\partial X}+n_Y\frac{\partial U}{\partial Y}\right)\phi_k ds \quad \text{(A.12)}$$

Using Eqs. (A.11) & (A.12), Eq. (A.8) becomes

$$\int_{\Omega}\phi_k\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right)dXdY+\gamma\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial U}{\partial X}+\frac{\partial\phi_k}{\partial X}\frac{\partial V}{\partial Y}\right)dXdY+\Pr\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial U}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial U}{\partial Y}\right)dXdY$$
$$-\Pr\oint_{\Gamma}\left(n_X\frac{\partial U}{\partial X}+n_Y\frac{\partial U}{\partial Y}\right)\phi_k ds=0$$

Approximated functions (3.15) reduces to above integral in following form

$$\int_{\Omega}\left[\phi_k\left(\phi_k^TU_k\right)\frac{\partial\phi_k^T}{\partial X}+\phi_k\left(\phi_k^TV_k\right)\frac{\partial\phi_k^T}{\partial Y}\right]U_k dXdY+\gamma\left[\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k^T}{\partial X}\right)U_k dXdY+\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k^T}{\partial Y}\right)V_k dXdY\right]$$
$$+\Pr\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k^T}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k^T}{\partial Y}\right)U_k dXdY-\Pr\oint_{\Gamma}\left(n_X\frac{\partial U}{\partial X}+n_Y\frac{\partial U}{\partial Y}\right)\phi_k ds=0$$

A.6 Compute the integral

$$\int_{\Omega}\phi_k\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right)dXdY-\gamma\int_{\Omega}\phi_k\left[\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right]dXdY-\Pr\int_{\Omega}\phi_k\left(\frac{\partial^2V}{\partial X^2}+\frac{\partial^2V}{\partial Y^2}\right)dXdY-Ra\Pr\int_{\Omega}\phi_k\theta\, dXdY=0 \tag{A.13}$$

Solution.

In the light of Eqs. (A.11) & (A.12), we have

$$\int_{\Omega}\phi_k\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)dXdY=-\int_{\Omega}\frac{\partial\phi_k}{\partial Y}\frac{\partial U}{\partial X}dXdY-\int_{\Omega}\frac{\partial\phi_k}{\partial Y}\frac{\partial V}{\partial Y}dXdY \tag{A.14}$$

$$\int_{\Omega}\phi_k\left(\frac{\partial^2V}{\partial X^2}+\frac{\partial^2V}{\partial Y^2}\right)dXdY=-\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial V}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial V}{\partial Y}\right)dXdY+\oint_{\Gamma}\left(n_X\frac{\partial V}{\partial X}+n_Y\frac{\partial V}{\partial Y}\right)\phi_k ds \tag{A.15}$$

Using Eqs. (A.14) & (A.15), Eq. (A.13) reduces to following form

$$\int_{\Omega}\phi_k\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right)dXdY+\gamma\left[\int_{\Omega}\frac{\partial\phi_k}{\partial Y}\frac{\partial U}{\partial X}dXdY+\int_{\Omega}\frac{\partial\phi_k}{\partial Y}\frac{\partial V}{\partial Y}dXdY\right]+\Pr\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial V}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial V}{\partial Y}\right)dXdY$$
$$-\Pr\oint_{\Gamma}\left(n_X\frac{\partial V}{\partial X}+n_Y\frac{\partial V}{\partial Y}\right)\phi_k ds-Ra\Pr\int_{\Omega}\phi_k\theta\, dXdY=0$$

Using of approximated functions (3.15), above integral gives

$$\int_{\Omega}\left[\phi_k\left(\phi_k^TU_k\right)\frac{\partial\phi_k^T}{\partial X}+\phi_k\left(\phi_k^TV_k\right)\frac{\partial\phi_k^T}{\partial Y}\right]U_k dXdY+\gamma\left[\int_{\Omega}\left(\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k^T}{\partial X}\right)U_k dXdY+\int_{\Omega}\left(\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k^T}{\partial Y}\right)V_k dXdY\right]$$
$$+\Pr\int_{\Omega}\left(\frac{\partial\phi_k}{\partial X}\frac{\partial\phi_k^T}{\partial X}+\frac{\partial\phi_k}{\partial Y}\frac{\partial\phi_k^T}{\partial Y}\right)V_k dXdY-Ra\Pr\int_{\Omega}\phi_k\left(\phi_k^T\theta_k\right)dXdY-\Pr\oint_{\Gamma}\left(n_X\frac{\partial V}{\partial X}+n_Y\frac{\partial V}{\partial Y}\right)\phi_k ds=0$$

Appendix B

B.1 Finite Element Solution of the Laplace Equation with 4-Node Rectangular Element

Consider a simple form of the steady state heat conduction problem in a rectangular domain (shown in Figure B.1) with Dirichlet boundary conditions defined by Laplace Equation (all material properties are set to unity).

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(B.1)}$$

for $x = [0, a], y = [0, b]$, with $a = 4,\ b = 2$

where, $u(x, y)$ is the steady state temperature distribution in the domain.

The boundary conditions are

$$\left.\begin{array}{l} u(0, y) = 100 \\ u(4, y) = 250 \end{array}\right\} \text{Imposed temperatures on the left \& right boundaries}$$

$$\left.\begin{array}{l} u(x, 0) = 50 \\ u(x, 2) = 200 \end{array}\right\} \text{Imposed temperatures on the top \& bottom boundaries}$$

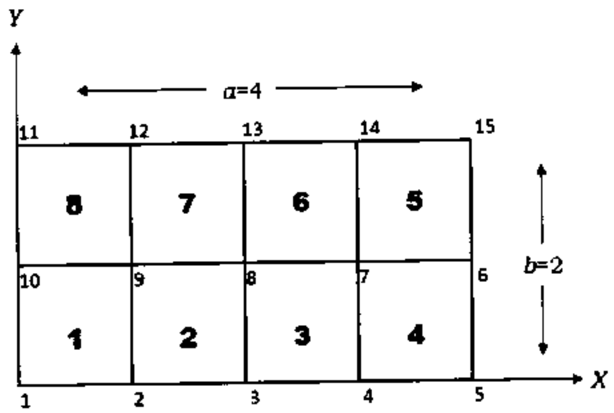


Figure B.1: Discretization of given geometry into 8 elements (each one is 4-node rectangular element) by signifying global nodes at vertex of each element

For weak formulation of governing Eq. (B.1), multiply Eq. (B.1) by an arbitrary weight function $w(x, y)$, and integrate over an arbitrary domain Ω^e, whose boundary is Γ^e. The arbitrary domain could represent an n-node element within the solution domain Ω with boundary Γ, as shown in Figure B.2.

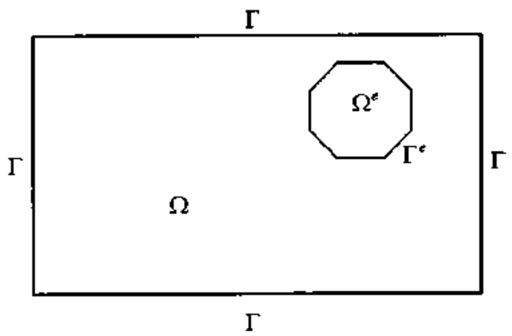


Figure B.2

The equation obtained is

$$\int_{\Omega^e} w(x,y)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)dxdy = 0$$

Using eq. (A.7), above integral may be written as

$$-\int_{\Omega^e}\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}\right)dxdy+\oint_{\Gamma^e}\left(n_x\frac{\partial u}{\partial x}+n_y\frac{\partial u}{\partial y}\right)wds = 0$$

$$\Rightarrow \int_{\Omega^e}\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}\right)dxdy=\oint_{\Gamma^e}\left(n_x\frac{\partial u}{\partial x}+n_y\frac{\partial u}{\partial y}\right)wds \qquad \text{(B.2)}$$

Define the flux term (q_n) as

$$q_n \equiv n_x\frac{\partial u}{\partial x}+n_y\frac{\partial u}{\partial y}$$

In view of above result, Eq. (B.2) becomes

$$\int_{\Omega^e}\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}\right)dxdy=\oint_{\Gamma^e} q_n w\, ds \qquad \text{(B.3)}$$

The approximate solution of equation (B.3) for an arbitrary, *n-node* element is defined by

$$u^e(x,y)=\sum_{j=1}^{n} u_j^e \psi_j^e(x,y)$$

where, u_j^e is nodal value for $u(x,y)$ at node *j* corresponding to element *e*

$\psi_j^e(x,y)$ is interpolation function for $u(x,y)$ at node *j* within the element *e*

Moreover, the weight function $w(x,y)$ represents a variation of primary variable $u(x,y)$, and thus takes on the nodal values $w_i=\psi_i^e$, $i=1,2,...,n$. Thus, Eq. (B.3) yields the following form

$$\sum_{j=1}^{n} u_j^e \int_{\Omega^e}\left(\frac{\partial w}{\partial x}\frac{\partial \psi_j^e}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial \psi_j^e}{\partial y}\right)dxdy=\oint_{\Gamma^e}\psi_i^e q_n\, ds$$

In matrix form, it becomes

$$\sum_{j=1}^{n} K_{ij}^{e} u_j^{e} = Q_i^{e} \quad ; \; i = 1,2,\ldots\ldots n$$

Where, $K_{ij}^{e} = \int\limits_{\Omega^e} \left(\frac{\partial w}{\partial x}\frac{\partial \psi_j^e}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \psi_j^e}{\partial y} \right) dxdy$

$$Q_i^e = \oint\limits_{\Gamma^e} \psi_j^e q_n \, ds$$

$$u_j^e = \left[u_1^e \;\; u_2^e \; \ldots\ldots \; u_n^e \right]^T$$

Since there is no flux (q_n) given at all nodes of the problem domain, therefore value of column vector Q_i^e will be assumed as zero vector. Thus, the weak form for an n-node element in condensed form may be written as

$$\sum_{j=1}^{n} K_{ij}^{e} u_j^{e} = F_i^{e} \quad ; \; i = 1,2,\ldots\ldots n \tag{B.4}$$

where,

$$K_{ij}^{e} = \int\limits_{\Omega^e} \left(\frac{\partial \psi_i^e}{\partial x}\frac{\partial \psi_j^e}{\partial x} + \frac{\partial \psi_i^e}{\partial y}\frac{\partial \psi_j^e}{\partial y} \right) dxdy \tag{B.5}$$

which is element coefficient matrix, also called the element stiffness matrix.

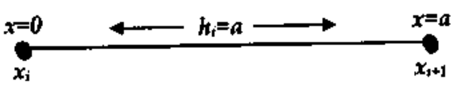
$$u_j^e = [u_1 \, u_2 \, u_3 \ldots\ldots\ldots u_n]^T$$

$$F_i^e = [0 \; 0 \; 0 \ldots\ldots\ldots 0]^T$$

Now consider a 4-node rectangular element and interpolation functions for this element are

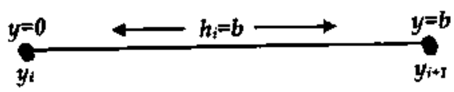
$$\phi_1(x) = \frac{x_{i+1} - x}{h_i} = \frac{a - x}{a} \qquad \phi_2(x) = \frac{x - x_i}{h_i} = \frac{x - 0}{a}$$

Linear shape functions in x - direction



$$\zeta_1(y) = \frac{y_{i+1} - y}{h_i} = \frac{b - y}{b} \qquad \zeta_2(y) = \frac{y - y_i}{h_i} = \frac{y - 0}{b}$$

Linear shape functions in y - direction



The product of two sets of above mentioned shape functions results an interpolation functions in terms of local coordinates (x, y), which are as under

$$\psi_1^e(x,y) = \phi_1(x)\zeta_1(y) = \left(\frac{a-x}{a}\right)\left(\frac{b-y}{b}\right) = \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$$

$$\psi_2^e(x,y)=\phi_2(x)\zeta_2(y)=\frac{x}{a}\left(\frac{b-y}{b}\right)=\frac{x}{a}\left(1-\frac{y}{b}\right)$$

$$\psi_3^e(x,y)=\phi_3(x)\zeta_3(y)=\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)=\frac{xy}{ab}$$

$$\psi_4^e(x,y)=\phi_4(x)\zeta_4(y)=\left(\frac{a-x}{a}\right)\left(\frac{y}{b}\right)=\left(1-\frac{x}{a}\right)\frac{y}{b}$$

Using eq. (B.5), local stiffness matrix corresponding to 4-node rectangular element may be evaluated as

$$K_{ij}^e=\int_{\Omega^e}\left\{\begin{bmatrix}\frac{\partial\psi_1}{\partial x}\\ \frac{\partial\psi_2}{\partial x}\\ \frac{\partial\psi_3}{\partial x}\\ \frac{\partial\psi_4}{\partial x}\end{bmatrix}\begin{bmatrix}\frac{\partial\psi_1}{\partial x} & \frac{\partial\psi_2}{\partial x} & \frac{\partial\psi_3}{\partial x} & \frac{\partial\psi_4}{\partial x}\end{bmatrix}+\begin{bmatrix}\frac{\partial\psi_1}{\partial y}\\ \frac{\partial\psi_2}{\partial y}\\ \frac{\partial\psi_3}{\partial y}\\ \frac{\partial\psi_4}{\partial y}\end{bmatrix}\begin{bmatrix}\frac{\partial\psi_1}{\partial y} & \frac{\partial\psi_2}{\partial y} & \frac{\partial\psi_3}{\partial y} & \frac{\partial\psi_4}{\partial y}\end{bmatrix}\right\}dxdy$$

Performing integral after substituting the above interpolation functions, the computed element stiffness matrix is

$$K_{ij}^e=\begin{bmatrix}A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ A_{31} & A_{32} & A_{33} & A_{34}\\ A_{41} & A_{42} & A_{43} & A_{44}\end{bmatrix} \qquad \text{(B.6)}$$

Where,

$$A_{11}=\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_1}{\partial y}\frac{\partial\psi_1}{\partial y}\right)dxdy$$

$$=\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left(-\frac{b-y}{ab}\right)\left(-\frac{b-y}{ab}\right)+\left(-\frac{a-x}{ab}\right)\left(-\frac{a-x}{ab}\right)\right\}dxdy$$

$$=\frac{1}{a^2b^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{(b-y)^2+(a-x)^2\right\}dxdy$$

$$=\frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{(b-y)^2|x|_0^a+\left|-\frac{(a-x)^3}{3}\right|_0^a\right\}dy$$

$$=\frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{a(b-y)^2+\frac{a^3}{3}\right\}dy$$

$$=\frac{1}{a^2b^2}\left\{a\left|-\frac{(b-y)^3}{3}\right|_0^b+\frac{a^3}{3}|y|_0^b\right\}=\frac{a^2+b^2}{3ab},$$

$$A_{12} = \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial x} + \frac{\partial\psi_1}{\partial y}\frac{\partial\psi_2}{\partial y}\right)dxdy$$

$$= \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left(-\frac{b-y}{ab}\right)\left(\frac{b-y}{ab}\right) + \left(-\frac{a-x}{ab}\right)\left(-\frac{x}{ab}\right)\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{-(b-y)^2 + \left(ax - x^2\right)\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{-(b-y)^2(a) + a\left(\frac{a^2}{2}\right) - \frac{a^3}{3}\right\}dy$$

$$= \frac{1}{ab^2}\int_{y=0}^{y=b}\left\{-(b-y)^2 + \frac{a^2}{6}\right\}dy$$

$$= \frac{1}{ab^2}\left\{-\left|-\frac{(b-y)^3}{3}\right|_0^b + \frac{a^2}{6}|y|_0^b\right\} = \frac{a^2 - 2b^2}{6ab},$$

$$A_{13} = \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_1}{\partial y}\frac{\partial\psi_3}{\partial y}\right)dxdy$$

$$= \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left(-\frac{b-y}{ab}\right)\left(\frac{y}{ab}\right) + \left(-\frac{a-x}{ab}\right)\left(\frac{x}{ab}\right)\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left(y^2 - by\right) + \left(x^2 - ax\right)\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{\left(y^2 - by\right)(a) + \frac{a^3}{3} - \frac{a^3}{2}\right\}dy$$

$$= \frac{1}{ab^2}\int_{y=0}^{y=b}\left\{y^2 - by - \frac{a^2}{6}\right\}dy$$

$$= \frac{1}{ab^2}\left\{\frac{b^3}{3} - \frac{b^3}{2} - \frac{a^2b}{6}\right\} = -\frac{a^2 + b^2}{6ab},$$

$$A_{14} = \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_1}{\partial y}\frac{\partial\psi_4}{\partial y}\right)dxdy$$

$$= \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left(-\frac{b-y}{ab}\right)\left(-\frac{y}{ab}\right) + \left(-\frac{a-x}{ab}\right)\left(\frac{a-x}{ab}\right)\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left(by - y^2\right) - (a-x)^2\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{\left(by - y^2\right)(a) - \frac{a^3}{3}\right\}dy$$

$$A_{14} = \frac{1}{ab^2} \int_{y=0}^{y=b} \left\{ by - y^2 - \frac{a^2}{3} \right\} dy$$

$$= \frac{1}{ab^2} \left\{ \frac{b^3}{2} - \frac{b^3}{3} - \frac{a^2 b}{3} \right\} = \frac{b^2 - 2a^2}{6ab},$$

Since element coefficient matrix is symmetric, then we have

$$A_{21} = A_{12} = \frac{a^2 - 2b^2}{6ab},$$

$$A_{22} = \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left(\frac{\partial \psi_2}{\partial x} \frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_2}{\partial y} \frac{\partial \psi_2}{\partial y} \right) dxdy$$

$$= \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \frac{(b-y)^2}{a^2 b^2} + \frac{x^2}{a^2 b^2} \right\} dxdy$$

$$= \frac{1}{a^2 b^2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ (b-y)^2 + x^2 \right\} dxdy$$

$$= \frac{1}{a^2 b^2} \int_{y=0}^{y=b} \left\{ (b-y)^2 (a) + \frac{a^3}{3} \right\} dy$$

$$= \frac{1}{ab^2} \left\{ \frac{b^3}{3} + \frac{a^2 b}{3} \right\} = \frac{a^2 + b^2}{3ab},$$

$$A_{23} = \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left(\frac{\partial \psi_2}{\partial x} \frac{\partial \psi_3}{\partial x} + \frac{\partial \psi_2}{\partial y} \frac{\partial \psi_3}{\partial y} \right) dxdy$$

$$= \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \left(\frac{b-y}{ab} \right) \left(\frac{y}{ab} \right) + \left(-\frac{x}{ab} \right) \left(\frac{x}{ab} \right) \right\} dxdy$$

$$= \frac{1}{a^2 b^2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ by - y^2 - x^2 \right\} dxdy$$

$$= \frac{1}{a^2 b^2} \int_{y=0}^{y=b} \left\{ aby - ay^2 - \frac{a^3}{3} \right\} dy$$

$$= \frac{1}{ab^2} \left\{ \frac{b^3}{2} - \frac{b^3}{3} - \frac{a^2 b}{3} \right\} = \frac{b^2 - 2a^2}{6ab},$$

$$A_{24} = \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left(\frac{\partial \psi_2}{\partial x} \frac{\partial \psi_4}{\partial x} + \frac{\partial \psi_2}{\partial y} \frac{\partial \psi_4}{\partial y} \right) dxdy$$

$$= \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \left(\frac{b-y}{ab} \right) \left(-\frac{y}{ab} \right) + \left(-\frac{x}{ab} \right) \left(\frac{a-x}{ab} \right) \right\} dxdy$$

$$= \frac{1}{a^2 b^2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ y^2 - by - ax + x^2 \right\} dxdy$$

$$A_{24} = \frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{ay^2 - aby - \frac{a^3}{2} + \frac{a^3}{3}\right\}dy$$

$$= \frac{1}{ab^2}\int_{y=0}^{y=b}\left\{y^2 - by - \frac{a^2}{6}\right\}dy$$

$$= \frac{1}{ab^2}\left\{\frac{b^3}{3} - \frac{b^3}{2} - \frac{a^2b}{6}\right\} = -\frac{a^2+b^2}{6ab},$$

$$A_{31} = A_{13} = -\frac{a^2+b^2}{6ab},$$

$$A_{32} = A_{23} = \frac{b^2-2a^2}{6ab},$$

$$A_{33} = \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left(\frac{\partial\psi_3}{\partial x}\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y}\frac{\partial\psi_3}{\partial y}\right)dxdy$$

$$= \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\frac{y^2}{a^2b^2} + \frac{x^2}{a^2b^2}\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{y^2 + x^2\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{ay^2 + \frac{a^3}{3}\right\}dy$$

$$= \frac{1}{ab^2}\int_{y=0}^{y=b}\left\{y^2 + \frac{a^2}{3}\right\}dy$$

$$= \frac{1}{ab^2}\left\{\frac{b^3}{3} + \frac{a^2b}{3}\right\} = \frac{a^2+b^2}{3ab},$$

$$A_{34} = \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left(\frac{\partial\psi_3}{\partial x}\frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_3}{\partial y}\frac{\partial\psi_4}{\partial y}\right)dxdy$$

$$= \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{-\frac{y^2}{a^2b^2} + \frac{x(a-x)}{a^2b^2}\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{-y^2 + ax - x^2\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{-ay^2 + \frac{a^3}{2} - \frac{a^3}{3}\right\}dy$$

$$= \frac{1}{ab^2}\int_{y=0}^{y=b}\left\{-y^2 + \frac{a^2}{6}\right\}dy$$

$$= \frac{1}{ab^2}\left\{-\frac{b^3}{3} + \frac{a^2b}{6}\right\} = \frac{a^2-2b^2}{6ab},$$

$$A_{41} = A_{14} = \frac{b^2 - 2a^2}{6ab},$$

$$A_{42} = A_{24} = -\frac{a^2 + b^2}{6ab},$$

$$A_{43} = A_{34} = \frac{a^2 - 2b^2}{6ab},$$

$$A_{44} = \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left(\frac{\partial\psi_4}{\partial x}\frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_4}{\partial y}\frac{\partial\psi_4}{\partial y}\right)dxdy$$

$$= \int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\frac{y^2}{a^2b^2} + \frac{(a-x)^2}{a^2b^2}\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{y^2 + (a-x)^2\right\}dxdy$$

$$= \frac{1}{a^2b^2}\int_{y=0}^{y=b}\left\{ay^2 + \frac{a^3}{3}\right\}dy$$

$$= \frac{1}{ab^2}\int_{y=0}^{y=b}\left\{y^2 + \frac{a^2}{3}\right\}dy$$

$$= \frac{1}{ab^2}\left\{\frac{b^3}{3} + \frac{a^2b}{3}\right\} = \frac{a^2 + b^2}{3ab}$$

Substitution of above all values in Eq. (B.6) yields

$$K_{ij}^e = \begin{bmatrix} \frac{a^2+b^2}{3ab} & \frac{a^2-2b^2}{6ab} & -\frac{a^2+b^2}{6ab} & \frac{b^2-2a^2}{6ab} \\ \frac{a^2-2b^2}{6ab} & \frac{a^2+b^2}{3ab} & \frac{b^2-2a^2}{6ab} & -\frac{a^2+b^2}{6ab} \\ -\frac{a^2+b^2}{6ab} & \frac{b^2-2a^2}{6ab} & \frac{a^2+b^2}{3ab} & \frac{a^2-2b^2}{6ab} \\ \frac{b^2-2a^2}{6ab} & -\frac{a^2+b^2}{6ab} & \frac{a^2-2b^2}{6ab} & \frac{a^2+b^2}{3ab} \end{bmatrix}$$

$$K_{ij}^e = \frac{1}{6ab}\begin{bmatrix} 2(a^2+b^2) & a^2-2b^2 & -(a^2+b^2) & b^2-2a^2 \\ a^2-2b^2 & 2(a^2+b^2) & b^2-2a^2 & -(a^2+b^2) \\ -(a^2+b^2) & b^2-2a^2 & 2(a^2+b^2) & a^2-2b^2 \\ b^2-2a^2 & -(a^2+b^2) & a^2-2b^2 & 2(a^2+b^2) \end{bmatrix} \tag{B.7}$$

Now consider a value of above matrix for a given domain which is discretized into eight equal segments (as demonstrated in Figure B.1), each one is four noded rectangular element. Each element has length $a = 1$ and width $b = 1$. 6^{th} element is illustrated in Figure B.3 with 1-4 local nodes whereas 8, 7, 14 and 13 represent the global nodes.

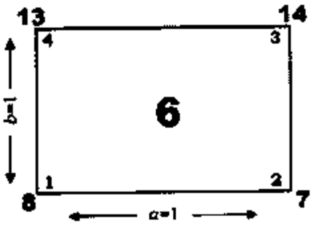


Figure B.3: Dimensions of 6[th] Element with Symbolization of Local and Global Nodes

By putting $a=1$ and $b=1$ in local stiffness matrix (B.7), the value of 6[th] rectangular element is obtained as under

$$K_6 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

Since all the elements are equal in length and width. Therefore, value of local stiffness matrix corresponding to each element will be same as mentioned above (for 6[th] element).

Local stiffness matrices of each rectangular element by allocating global nodes (corresponding to their local nodes) in anti-clockwise direction are given below

$$K_1 \quad \begin{array}{c|cccc} & \mathbf{1} & \mathbf{2} & \mathbf{9} & \mathbf{10} \\ \hline \mathbf{1} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \mathbf{2} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ \mathbf{9} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ \mathbf{10} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}$$

$$K_2 \quad \begin{array}{c|cccc} & \mathbf{2} & \mathbf{3} & \mathbf{8} & \mathbf{9} \\ \hline \mathbf{2} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \mathbf{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ \mathbf{8} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ \mathbf{9} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}$$

$$K_3 \quad \begin{array}{c|cccc} & \mathbf{3} & \mathbf{4} & \mathbf{7} & \mathbf{8} \\ \hline \mathbf{3} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \mathbf{4} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ \mathbf{7} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ \mathbf{8} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}$$

$$K_4 \quad \begin{array}{c|cccc} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \hline \mathbf{4} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \mathbf{5} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ \mathbf{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ \mathbf{7} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}$$

$$K_5 \quad \begin{array}{c|cccc} & \mathbf{7} & \mathbf{6} & \mathbf{15} & \mathbf{14} \\ \hline \mathbf{7} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \mathbf{6} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ \mathbf{15} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ \mathbf{14} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}$$

$$K_6 \quad \begin{array}{c|cccc} & \mathbf{8} & \mathbf{7} & \mathbf{14} & \mathbf{13} \\ \hline \mathbf{8} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \mathbf{7} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ \mathbf{14} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ \mathbf{13} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}$$

$$
K_7 = \begin{array}{c} \\ 9 \\ 8 \\ 13 \\ 12 \end{array}
\begin{array}{c}
\begin{array}{cccc} 9 & 8 & 13 & 12 \end{array} \\
\begin{bmatrix}
\frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\
-\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\
-\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3}
\end{bmatrix}
\end{array}
\qquad
K_8 = \begin{array}{c} \\ 10 \\ 9 \\ 12 \\ 11 \end{array}
\begin{array}{c}
\begin{array}{cccc} 10 & 9 & 12 & 11 \end{array} \\
\begin{bmatrix}
\frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\
-\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\
-\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3}
\end{bmatrix}
\end{array}
$$

Assembling all of above element matrices generates an assembled global stiffness matrix $[K]$ of order 15×15 as given below

$$
K = \begin{array}{c|ccccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
2 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} \\
7 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
8 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
9 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\
10 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} \\
15 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{2}{3}
\end{array}
$$

Also the value of column matrix $\left[F_i^e\right]$ on the right side of Eq. (B.4) for the whole domain gives matrix F of order 15×1 as under

$$F = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

The assembled equation is of the form

$$[K][U]=[F]$$

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{4}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{B.8}$$

where, $[U]$ is a vector of nodal values of temperature.

At the singular points (*i.e* global nodes 1, 5, 15 and 11), the specified nodal values are handled either by average of the two specified values or the higher of the two specified values of u. (Note that the points occur at corners of problem domain are referred as singular points). Therefore, values of boundary conditions at nodes on the boundary of the domain are

$$u_1 = \frac{100+50}{2} = 75 \qquad u_2 = u_3 = u_4 = 50$$

$$u_5 = \frac{250+50}{2} = 150 \qquad u_6 = 250$$

$$u_{10} = 100 \qquad u_{11} = \frac{100+200}{2} = 150$$

$$u_{12} = u_{13} = u_{14} = 200 \qquad u_{15} = \frac{200+250}{2} = 225$$

The nodal solution vector $[U]$ becomes

$$[U] = [\,75 \quad 50 \quad 50 \quad 50 \quad 150 \quad 250 \quad u_7 \quad u_8 \quad u_9 \quad 100 \quad 150 \quad 200 \quad 200 \quad 200 \quad 225\,]^T$$

Above vector shows that the unknown values of $[U]$ occur at global nodes 7, 8 and 9.

For evaluation of unknown parameters (u_7, u_8 and u_9), eliminate rows 1-6 and 10-15 of global stiffness matrix K. All known quantities are moved from left side of the matrix equation to the right side to obtain the condensed equations. Thus, Eq. (B.8) implies

$$\begin{bmatrix} 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 75 \\ 50 \\ 50 \\ 50 \\ 150 \\ 250 \\ u_7 \\ u_8 \\ u_9 \\ 100 \\ 150 \\ 200 \\ 200 \\ 200 \\ 225 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Necessary simplification generates result as follow

$$\begin{bmatrix} 2.6667 & -0.3333 & 0 \\ -0.3333 & 2.6667 & -0.3333 \\ 0 & -0.3333 & 2.6667 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 375 \\ 250 \\ 275 \end{bmatrix}$$

Above system of equations gives the following solution at unknown nodes,

$$\begin{bmatrix} u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 156.6532 \\ 128.2258 \\ 119.1532 \end{bmatrix}$$

Graphical illustration of nodal solution has been shown in contour plot as under

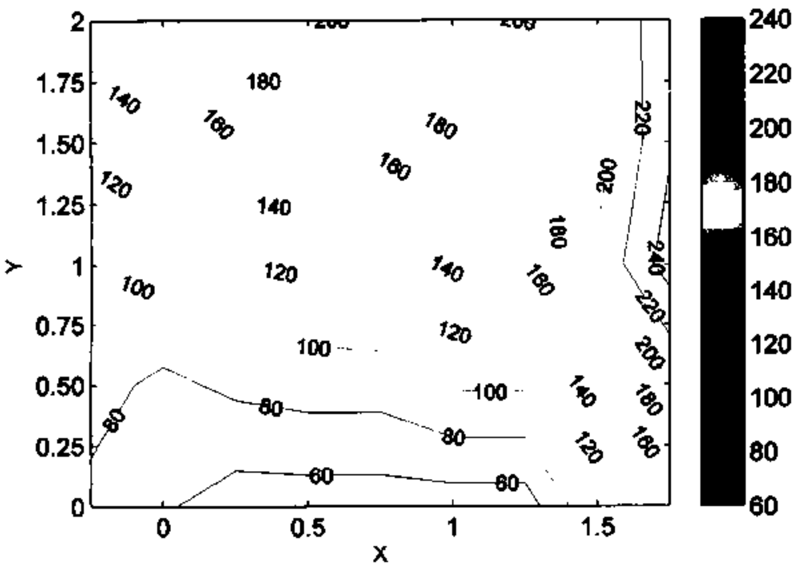


Figure B.4: Contour Plot executed using Finite Element Solution of Laplace Equation

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