Heat transfer analysis in an exponentially stretching flow

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We accept this dissertation as conforming to the required standard.

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2010.

dedicated to

my mother,father

& my family members.

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To begin with the name of Almighty ALLAH the creator of the universe, who bestowed his blessing on me to complete this dissertation. I offer my humblest, sincerest and millions of Drood to the Holy Prophet Hazrat MUHAMMAD (peace be upon him), who exhorts his followers to seek for knowledge from cradle to grave.

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Hussain Ahmad
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Preface

Boundary layer flows induced by stretching sheet has gained significant importance for the last three decades. All of it is due to prominent applications of such flows in manufacturing processes, in industry such as the aerodynamic extrusion of the plastic plate, cooling of an infinite metallic plate in a cooling bath, boundary layer along a liquid thin film and condensation process and many more. Large number of mathematicians, physicists, modeler, and engineers are attracted by theses applications and are investigating such flows in many different ways for the last so many years. The primeval researches on boundary layer flow by a continuously moving solid surface with constant velocity have been done by Sakiadis [1, 2]. After this Crane [3] investigated the continuously moving solid surface with linear velocity. Later, C. Y. Wang discussed the steady three dimensional flows due to stretching of the sheet [4] and then liquid film on an unsteady stretching sheet [5]. Till know, extensive literature is available on the linear stretching flow disused by different authors [6-14]. But unfortunately, a very little attention has been given to the nonlinear stretching flows. Vajravelu [15] discussed the fluid flow over a non-linear stretching sheet first time and then Vajravelu and Cannon [16] studied the existence and behavior of solutions of different equations arising in viscous flow over a nonlinear stretching sheet. In addition to the stretching surfaces, the heat and mass flow has driven many industrial applications due to thermal diffusion, concentration difference and due to chemical reaction. Thereafter, the viscous flow and heat transfer characteristics over a nonlinear stretching sheet have been discussed by Cortell [17]. In recent, analytic solution has been obtained for flow and diffusion of chemically reactive species over a nonlinear stretching sheet immersed in a porous medium by Ziabakhsh [18]. Motivated by the above facts, the aim of this dissertation is to investigate the effects of porous medium on the fluid flow over a nonlinear rotating stretching sheet. The dissertation is arranged as follows:

Chapter 1 includes some basic definitions, concept of boundary layer and its equations [19], equation of motion of fluid in rotating frame [20] for the convenience and better understanding of the reader. The contents of chapter 2 are based on the work of Ziabakhsh et al [18]. All the results are reproduced by shooting method [21] and by famous implicit finite difference scheme, Keller-Box Scheme [22]. In chapter 3 the work of reference [18] is generalized by taking the whole in rigid body rotation immersed in a porous medium. The similarity transformations for the rotating frame are introduced to the partial differential equations. As a result governing nonlinear ordinary differential equations are then solved by two well known methods namely shooting method [21] and by Keller-Box method [22]. Results obtained by both the methods are compared for different values of emerging parameters and found in excellent agreement. The influence of rotation and porous medium parameter are analyzed through graphs.

Contents

Chapter 1

Preliminaries

1.0.1 Introduction

In this chapter we present some definitions and basic concepts relating the problem. Further the shooting method is explained with a simple example.

1.1 Definitions

1.1.1 Flow

A material goes under deformation when different forces act on it. If this deformation increases continuously without limit then the phenomenon is known as flow.

1.1.2 Fluid

It is a substance that deforms continuously under the influence of shear stress.

1.2 Types of fluid

1.2.1 Ideal fluid

A fluid that has no viscosity, it can support no shearing stress, and flows without energy dissipation is called ideal fluid or nonviscous fluid or perfect fluid.

1.2.2 Real fluid

The flow in which the effects of tangential or shearing forces are taken into account ; these forces give rise to fluid friction because they oppose the sliding of one particle upon another is called a real flow and a fluid exhibiting this flow pattern is called real fluid.

1.2.3 Newtonian fluids

The Newtonian or viscous fluid is a fluid for which the shear stress is directly and linearly proportional to the rate of deformation i.e.

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.1}$$

where τ_{yx} is the shear stress acting on the plane normal to y-axis and μ is the constant of proportionality, called absolute or dynamic viscosity. Water and gasoline are examples of Newtonian fluids under normal conditions.

1.2.4 Non-Newtonian fluids

The fluids in which the relation between shear stress and the strain rate is non-linear and can even be time dependent. Therefore a constant coefficient of viscosity cannot be defined i.e.

$$\tau_{yx} = \mu_1 \left(\frac{du}{dy}\right)^n , (n \neq 1), \tag{1.2}$$

where n and μ_1 denote the flow behavior index and consistency index, respectively. Paints blood, shampoo etc. are the common examples of non-Newtonian fluids.

1.2.5 Compressible fluids

If the density of a fluid varies with space coordinates or time or both then it is called compressible fluid. All gases are examples of compressible fluid.

1.2.6 Incompressible fluids

If density of a fluid changes so that we can ignore the change then it known as incompressible fluid. All liquids are assumed as incompressible fluids.

1.3 Types of flow

1.3.1 Uniform flow

A flow in which fluid particles possess same velocities at each section of a channel or a pipe is called uniform flow

1.3.2 Non-uniform flow

A flow in which velocities of fluid particles are different at each section of a channel or a pipe is called non-uniform flow.

1.3.3 Laminar flow

A flow in which each fluid particle has a definite path and the paths of individual particles do not cross each other is called laminar flow.

1.3.4 Turbulent flow

A flow in which each fluid particle does not have a definite path and the paths of individual particles also cross each other is called a turbulent flow.

1.3.5 Steady flow

A flow in which properties associated with the motion of fluid are independent of time or flow pattern remains unchanged with the time is called a steady flow. Mathematically it can be written as

$$\frac{\partial \eta}{\partial t} = 0, \tag{1.3}$$

where η represents any fluid property that may be velocity, density, pressure etc. Flow through a gently flowing stream is an example of steady flow.

1.3.6 Unsteady flow

All those flows in which properties associated with the motion of fluid depend on time so that flow pattern varies with the time are called unsteady flows. Mathematically it can be expressed as,

$$\frac{\partial \eta}{\partial t} \neq 0, \tag{1.4}$$

Flow in ocean tides is an example of unsteady flow.

1.4 Heat

The total molecular kinetic energy in a system is called the heat of system. In thermodynamics, heat is the process of energy transfer from one body or system to another due to the thermal contact.

1.4.1 Temperature

The average kinetic energy of the particles in a substance is called the temperature of the substance.

1.4.2 Specific heat

It is amount of heat required to raise the temperature of one gram of a substance to 1^0C. The relationship between amount of heat transferred, specific heat and change in temperature is defined as

$$Specific\ heat = \frac{\text{heat transferred}}{\text{mass} \times \text{change in temperature}} = \frac{Q}{m \times \Delta T}, \tag{1.5}$$

1.4.3 Heat transfer mechanism

Conduction

The way in which the heat is transferred from one part of the system to another part or from one system to an adjacent system by inter collisions of interconnected molecules is called conduction. This phenomenon occurs usually in solids.

Convection

Convection is the way in which the heat is transferred by the motion of heated molecules of the system. Convection usually occurs in liquids and gases.

Radiation

The way in which the heat is transferred by the electromagnetic waves and it does not require any medium to access the target is known as radiation.

1.4.4 Types of convection

Free or natural convection

Natural convection is a type of heat transport in which fluid motion does not require any external agent or source, it occurs only due to the difference in temperature from place to place.

Forced convection

If the heat transfer occurs only due to an external agent or source then this type of heat transport is called force convection.

Mixed convection

If heat transfer is due to both forced and natural convection then it is called mixed convection.

1.5 Dimensionless numbers

A dimensionless number is the number without any unit associated with it. It is the ratio of the quantities having same unit. It is usually used to simplify our procedure and various quantities are replaced by a single number saving a lot of time and work. There is a lot of dimensionless numbers but here we mention only those being used in this work.

1.5.1 Reynold number

It is the ratio of force of inertia to the viscous force. Mathematically it can be written as,

$$\mathrm{Re} = \frac{LU}{\nu}, \tag{1.6}$$

where L is the characteristic length and U is the typical velocity.

1.5.2 Prandtl number

The Prandtl number is the ratio of kinematic viscosity and thermal diffusivity. It is denoted by Pr .

1.6 Miscellaneous

1.6.1 Density

Density of a fluid is defined as mass per unit volume. Mathematically the density ρ at a point P may be defined as

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}, \tag{1.7}$$

where δV is total volume element around the point P and δm is the mass of the fluid within δV. The dimension of density is $[M/L^3]$

1.6.2 Viscosity

It is the measure of internal resistance of a fluid to flow or it may be thought of a measure of fluid friction and it is denoted by μ.In other wards it is defined as the ratio of shear stress to the rate of shear strain, i.e.

$$\mu = \frac{shear\ stress}{rate\ of\ shear\ strain}, \tag{1.8}$$

where μ is the coefficient of viscosity or dynamic viscosity or simply viscosity. The dimension of viscosity is $[M/LT]$.

Kinematic viscosity

It is defined as the ratio of dynamic viscosity to density of a fluid. Mathematically it is expressed as

$$\nu = \frac{\mu}{\rho}, \tag{1.9}$$

where μ is the dynamic viscosity and ρ is the fluid density. The dimension of kinematic viscosity is $[L^2/T]$

1.6.3 Skin friction

When a fluid moves across a surface, a certain amount of friction called skin friction occurs between the fluid and the surface which tends to slow down the motion of fluid.

1.6.4 Thermal conductivity

It is measure of the ability of any substance to conduct heat and it is denoted by k.

1.6.5 Thermal diffusivity

Thermal diffusibility α is the rate at which temperature at one point in a body travels to another point. It is expressed as

$$\alpha = \frac{k}{\rho C_p}. \tag{1.10}$$

1.6.6 Thermal radiation

Electromagnetic radiation emitted from a heated material is known as thermal radiation. The radiation emitted by an electric heater or household radiator are common examples of thermal radiation.

1.7 The continuity equation

Let us consider a differential control volume $\Delta x \Delta y \Delta z$ in a cube. We take the density and the velocity as function of time and space. The mass flow rates through each face of the cube in three directions, that is along x, y and $z-axes$ respectively are

$$-\left(\frac{\partial(\rho u)}{\partial x}\Delta x\right)\Delta y\Delta z, -\left(\frac{\partial(\rho v)}{\partial y}\Delta y\right)\Delta x\Delta z, -\left(\frac{\partial(\rho w)}{\partial z}\Delta z\right)\Delta x\Delta y, \tag{1.11}$$

then net mass flow rate through the control volume is given by sum of flow rates along the three directions

$$-\left(\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right)\Delta x\Delta y\Delta z. \tag{1.12}$$

Since the instantaneous rate of change of mass within the control volume is equal to net mass flow rate through the control volume i.e.

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z) = -\left(\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right)\Delta x\Delta y\Delta z, \tag{1.13}$$

As the fixed control volume $\Delta x\Delta y\Delta z$ is independent of time, therefore, Eq. (1.13) takes the following form

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0, \tag{1.14}$$

which is called equation of continuity.For an incompressible fluid, we have $\rho=$ Constant, therefore Eq. (1.14) gets the shape as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1.15}$$

or

$$\nabla.\mathbf{V}=0 \text{ or } \operatorname{div}\mathbf{V}=0, \tag{1.16}$$

where

$$\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right). \tag{1.17}$$

1.8 The momentum equation

The equation of motion in vector form is

$$\rho\frac{d\mathbf{V}}{dt}=\operatorname{div}\mathbf{T}+\rho\mathbf{b}, \tag{1.18}$$

where **b** is the body force per unit mass and **T** is the Cauchy stress tensor given by

$$\mathbf{T} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}, \tag{1.19}$$

where τ_{xx}, τ_{yy} and τ_{zz} are the normal stresses and $\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{zx}$ and τ_{zy} are shear stresses.

1.9 Runge-Kutta method

There are many different schemes for solving initial value problems relating to ordinary differential equations numerically, but due to the highest order of accuracy i.e. of O(4) we prefer to use the Runge-Kutta method.

The general equation of second order initial value problem can be written as

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \tag{1.20}$$

subject to initial conditions

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = a. \tag{1.21}$$

In order to solve above problem, we need to convert second order initial value problem to the system of first order initial value problems by defining

$$\frac{dy}{dx} = z = g(x, y, z), \tag{1.22}$$

so we will have

$$\frac{dz}{dx} = f(x, y, z), \tag{1.23}$$

with initial conditions

$$y(x_0) = y_0, \quad z(x_0) = a. \tag{1.24}$$

Now the Runge-Kutta method of order 4 for above system of first order differential Eqs.

(1.22) and (1.23) is defined as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{1.25}$$

and

$$z_{n+1} = z_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \tag{1.26}$$

where

$$\begin{aligned}
k_1 &= hg(x_n, y_n, z_n), \quad l_1 = hf(x_n, y_n, z_n),\\
k_2 &= hg\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right), \quad l_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right),\\
k_3 &= hg\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right), \quad l_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right),\\
k_4 &= hg(x_n + h, y_n + k_3, z_n + l_3), \quad l_4 = hf(x_n + h, y_n + k_3, z_n + l_3).
\end{aligned} \tag{1.27}$$

where h is uniform step size defined as

$$h = \frac{x_n - x_0}{n},$$

n is number of steps.

1.10 Shooting method

Shooting method is an iterative technique which is very popular for the two points boundary value problems[21]. In this technique boundary value problem of higher order is first reduced to the system of first order initial value problems by letting the missing initial condition. Then our goal is to find the solution of initial value problem instead of the given boundary value problem directly. For this purpose, any scheme for the solution of initial value problem can be used. Runge-Kutta method of order 4 is used for this purpose in this thesis. For illustration, let us consider a second order boundary value problem

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \tag{1.28}$$

with boundary conditions

$$y(0) = 0, \quad y(L) = A, \tag{1.29}$$

where f is an arbitrary function and data is prescribed at $x = 0$ and $x = L$. The same differential equation describes an initial value problem if data is prescribed as

$$y(0) = 0, \quad y'(0) = s. \tag{1.30}$$

To solve the boundary value problem we reduce it into a system of two first order differential equations as

$$\frac{dy}{dx} = u, \quad \frac{du}{dx} = f(x, y, u), \tag{1.31}$$

with initial conditions

$$y(0) = 0, \quad y'(0) = u(0) = s, \tag{1.32}$$

where s denotes the missing initial conditions which will be assigned an initial value. Next we will find the actual value of s such that the solution of equations (1.31) subject to initial conditions (1.32) satisfies the boundary conditions (1.29). In other words, if the solutions of the initial value problems are denoted by $y(x, s)$ and $u(x, s)$, one searches for the value of s such that

$$y(L, s) - A = 0 = \phi(s) \quad (let). \tag{1.33}$$

Here Newton's formula [21] can be used to find the value of s as we are to choose a root of linear algebraic Eq. (1.33) as

$$s^{(n+1)} = s^{(n)} - \frac{\phi\left(s^{(n)}\right)}{\frac{d\phi}{ds}\left(s^{(n)}\right)}, \tag{1.34}$$

which implies that

$$s^{(n+1)} = s^{(n)} - \frac{y(L, s^{(n)}) - A}{\frac{\partial y}{\partial s}(L, s^{(n)})}. \tag{1.35}$$

To find the derivative of y with respect to s Eqs. (1.31) and (1.32) are differentiated with respect to s and we get

$$\frac{dY}{dx} = U, \frac{dU}{dx} = \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial u}U \tag{1.36}$$

where

$$Y = \frac{\partial y}{\partial s}, \quad U = \frac{\partial u}{\partial s},$$

and initial conditions take the following form

$$Y(0) = 0, U(0) = 1. \tag{1.37}$$

The solution of equation (1.28) subject to the boundary conditions (1.29) can therefore be obtained by the following steps:

(i) We choose the initial guess for the missing initial condition (1.32) and denote it by $s^{(1)}$.

(ii) Solve the system of equations (1.31) subject to initial conditions (1.32) from $x = 0$ to $x = L$.

(iii) Integrate the system of equations (1.36) subject to initial conditions (1.37) from $x = 0$ to $x = L$.

(iv) Substituting the value of $y(L, s^{(1)})$ obtained by step (ii) and $Y(L, s^{(1)})$ obtained by step (iii) into equation (1.35) as

$$s^{(2)} = s^{(1)} - \frac{y(L, s^{(1)}) - A}{Y(L, s^{(1)})}. \tag{1.38}$$

So the next approximation of missing initial condition $s^{(2)}$ is obtained.

(v) We repeat the steps (i) to (iv) until the value of s is within the specified degree of accuracy or the solution $y(L, s^{(n)})$ satisfies the prescribed boundary condition (1.29).

Chapter 2

Heat and mass transfer in the boundary layers on an exponentially stretching continuous surface

2.1 Introduction

In this chapter, we have discussed the similarity solution describing flow and heat transfer in the boundary layers on an exponentially stretching continuous surface. The governing boundary value problem has been reduced to a system of five simultaneous equations of first order. Then this system is solved by employing a well-known shooting technique (for two unknown initial conditions) with fourth order Runge - Kutta integration scheme. The dimensionless velocity and temperature fields are computed for different values of the Prandtl number Pr and wall temperature-distribution parameter a. The computed values of stream function and temperature field are discussed through graphs. This chapter is a review of the paper by Magyary and Keller [**17**]. Some missing details are also incorporated.

2.2 Mathematical formulation

We consider a laminar thermal boundary layer flow on an impermeable plane wall. The wall is stretching continuously with velocity $U_w = U(x)$ and at a given temperature distribution

$q_w = q_w(x)$ and in moving through quiescent incompressible fluid of constant temperature q_∞.

The governing equations for such flow and heat transfer are

$$\nabla . \mathbf{V} = 0, \tag{2.1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T}, \tag{2.2}$$

$$\rho \frac{de}{dt} = \mathbf{T} \cdot \mathbf{L} - \nabla \cdot \boldsymbol{\sigma}, \tag{2.3}$$

where $\mathbf{V}$ is the fluid velocity, $\mathbf{T}$ the Cauchy stress tensor, $e = C_p q$ is the energy in which q is temperature and $\nabla \cdot \boldsymbol{\sigma} = -k\partial^2 q/\partial y^2$.

For viscous fluids $\mathbf{T}$ is given by

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1, \tag{2.4}$$

in which p is pressure, $\mathbf{I}$ is a unit tensor, μ is viscosity and $\mathbf{A}_1$ is first Rivlin Ericksen tensor which is defined as

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{2.5}$$

and

$$\mathbf{L} = \operatorname{grad} \mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}. \tag{2.6}$$

For steady two dimensional flow we define velocity and temperature fields as

$$\mathbf{V} = [u(x,y), v(x,y), 0], \tag{2.7}$$

$$q = q(x,y). \tag{2.8}$$

Using Eqs. (2.7) and (2.8), Eqs. (2.1) - (2.3) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.9}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{2.10}$$

$$u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}=\alpha\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right), \tag{2.11}$$

where ν and α are kinematic viscosity and thermal diffusivity of the fluid respectively. With the usual boundary layer approximations, the governing Eqs. (2.9) - (2.11) for the velocity and temperature fields reduce to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2.12}$$

$$u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}=\alpha\frac{\partial^2 q}{\partial y^2}. \tag{2.13}$$

Here we assume that pressure gradient is absent because the flow is driven by the stretching sheet. The boundary conditions are

$$u(x,0)=U_w(x),\ v(x,0)=0,\ q(x,0)=q_w(x), \tag{2.14}$$

$$u(x,\infty)=0,\ q(x,\infty)=q_\infty. \tag{2.15}$$

The direction of motion is along $x-$axis due to the reason that the surface is stretching continuously along this direction and u and v are x and y components of the velocity respectively. The flow of fluid is independent of temperature so the Equations (2.12), (2.14) and (2.15) constitute an independent flow boundary value problem and the Equations (2.13), (2.14) and (2.15) a forced thermal convection problem.

2.3 Similarity solution for the exponential stretching

Introducing the similarity transformation corresponding to an exponential stretching and exponential temperature distribution of the continuous surface as suggested by Magyary and Keller

[17]

$$u(x,y) = U_0 e^{x/L} f'(\eta), \tag{2.16}$$

$$v(x,y) = -\frac{\nu}{L}\left(\frac{\mathrm{Re}}{2}\right)^{1/2} e^{x/(2L)}[f(\eta) + \eta\ f'(\eta)], \tag{2.17}$$

$$q(x,y) = q_\infty + q_0 e^{ax/(2L)}\theta(\eta), \tag{2.18}$$

$$\eta = \left(\frac{\mathrm{Re}}{2}\right)^{1/2} \frac{y}{L} e^{ax/(2L)}. \tag{2.19}$$

This solution corresponds to stretching and heating prescriptions

$$U_w(x) = U_0 e^{x/L}, \tag{2.20}$$

$$q_w(x) = q_\infty + q_0 e^{ax/(2L)}, \tag{2.21}$$

of the continuous surface respectively. The Eqs. (2.12) and (2.13) takes the following form in terms of dimensionless functions $f(\eta)$ and $\theta(\eta)$

$$f''' + ff'' - 2f'^2 = 0, \tag{2.22}$$

$$\theta'' + \mathrm{Pr}(f\ \theta' - af'\theta) = 0, \tag{2.23}$$

subject to the following boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \tag{2.24}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0, \tag{2.25}$$

where primes denote the differentiation with respect to η, $\mathrm{Pr} = \nu/\alpha$ denotes the Prandtl number and $\mathrm{Re} = LU_0/\nu$ the Reynold number with $L > 0$ as reference length, $U_0 > 0$ as velocity parameter of stretching. Also q_0 and a are parameters of the temperature distribution in the stretching surface. It is noted that, in terms of the dimensionless function $f(\eta)$, the boundary value problems Eqs. (2.22) and (2.25) is independent of all those parameters. However dimensionless temperature field θ is dependent on a and Pr i.e.

$$\theta = \theta(\eta; a, \mathrm{Pr}).$$

The following flow and heat transfer characteristic will be of interest. The skin friction acting on the stretching surface in contact with the ambient incompressible fluid is

$$\tau_w = \rho\nu \frac{\partial u(x,0)}{\partial y} = \frac{\rho\nu U_0}{L}\left(\frac{\mathrm{Re}}{2}\right)^{1/2} e^{3x/(2L)} f''(0) \tag{2.26}$$

The local surface - heat flux through the wall is

$$T_w(x) = -k\frac{\partial q(x,0)}{\partial y} = \frac{-kq_0}{L}\left(\frac{\mathrm{Re}}{2}\right)^{1/2} \exp((a+1)x/(2L))\theta'(0; a, \mathrm{Pr}). \tag{2.27}$$

The local Reynolds and Nusselt numbers are

$$\mathrm{Re}_x = \frac{U_w(x)x}{\nu} \tag{2.28}$$

$$Nu_x = \frac{T_w(x)x}{k(q_w(x) - q_\infty)} \tag{2.29}$$

respectively. In the present case

$$\frac{Nu_x}{\sqrt{\mathrm{Re}_x}} = -\left(\frac{x}{2L}\right)^{1/2} \theta'(0; a, \mathrm{Pr}) \tag{2.30}$$

holds.

2.4 Numerical solution of the boundary value problem

Since the boundary value problem Eqs. (2.22- 2.25) is non-linear, so it is impossible to find its analytical exact solution. In this situation, usual practice is to convert boundary value problem to the system of first order initial value problem as

$$f' = g, \tag{2.31}$$

$$g' = h, \tag{2.32}$$

$$h' = 2g^2 - fh, \tag{2.33}$$

$$\theta' = w, \tag{2.34}$$

$$w' = (ag\theta - fw), \tag{2.35}$$

and solve them subject to the initial condition

$$f(0) = 0, \quad g(0) = 1, \quad h(0) = s_1 = \left(f''(0)\right), \tag{2.36}$$

$$\theta(0) = 1, \quad w(0) = s_2 = \left(\theta'(0)\right), \tag{2.37}$$

where s_1 and s_2 are missing condition, which will be measured in such a way that solution satisfy boundary conditions (2.24) and (2.25). Furthermore, these two missing conditions are associated with the skin friction (2.26) and to the heat flux (2.27). Differentiating Eqs. (2.31) – (2.33) and (2.36) with respect to s_1 and Eqs. (2.34), (2.35) and (2.37) with respect to s_2 respectively, we get

$$F' = G, \tag{2.38}$$

$$G\,' = H, \tag{2.39}$$

$$H\,' = 2gG - Fh - fH, \tag{2.40}$$

$$F(0) = 0, \quad G(0) = 0, \quad H(0) = 1, \tag{2.41}$$

$$\Psi' = W, \tag{2.42}$$

$$W\,' = \Pr\left(ag\Psi - fW\right), \tag{2.43}$$

$$\Psi(0) = 0, \quad W(0) = 1, \tag{2.44}$$

where

$$F = \frac{\partial f}{\partial s_1}, G = \frac{\partial g}{\partial s_1}, H = \frac{\partial h}{\partial s_1},$$

$$\Psi = \frac{\partial \theta}{\partial s_2}, \text{and } W = \frac{\partial w}{\partial s_2}.$$

As the problem Eq. (2.22) does not involve any parameter so the value of $s_1 = f''(0)$ is same for each value of the parameter Pr and a i.e. $s_1 = -1.2818085$. But on the other hand problem Eq. (2.23) involves parameters Pr and a, so the wall temperature field θ and s_2 depends on the values of Pr and a. So different values of s_2 and temperature field θ are obtained for different values of Pr and a .

2.5 Analytical solution

To find analytical solution of BVP's Eqs. (2.22) and (2.23) subject to the boundary conditions Eqs. (2.24) and (2.25) we first find approximate analytical solution of the problem Eq. (2.22) then by using this approximate solution in problem Eq. (2.23) we solve it exactly. Integrating Eq. (2.22), we get

$$f'' + ff' = 3\int_0^{\eta} f'^2(\eta_1)d\eta_1 + C_1, \tag{2.45}$$

where C_1 is a constant of integration. By using boundary conditions Eq. (2.24a) we can find value of C_1 as

$$C_1 = f''(0) = s_1, \tag{2.46}$$

and then Eq. (2.45) can be written as

$$f'' + ff' = 3\int_0^{\eta} f'^2(\eta_1)d\eta_1 + s_1. \tag{2.47}$$

After using the boundary condition Eq. (2.24c), we get

$$s_1 = -3\int_0^{\infty} f'^2(\eta_1)d\eta. \tag{2.48}$$

Again integrating Eq. (2.47), we get

$$f' + \frac{1}{2}f^2 = 3\int_0^{\eta}\left(\int_0^{\eta_2} f'^2(\eta_1)d\eta_1\right)d\eta_2 + s_1\eta + C_2, \tag{2.49}$$

where C_2 is constant of integration which is obtained as

$$C_2 = 1, \tag{2.50}$$

by using the boundary condition (2.24b).

Hence Eq. (2.49) becomes

$$f' + \frac{1}{2}f^2 = 1 + s_1\eta + 3\int_0^{\eta}\int_0^{\eta_2} f'^2(\eta_1)d\eta_1 d\eta_2. \tag{2.51}$$

The analytical approximate solution can be obtained by setting iteration on the integral equation (2.51) as

$$f_n' + \frac{1}{2}{f_n}^2 = 1 + s_1\eta + 3\int_0^{\eta}\int_0^{\eta_2} {f_{n-1}'}^2 d\eta_1 d\eta_2. \tag{2.52}$$

On right hand side of Eq. (2.52) we take the suitable zeroth order approximation $f_0'(\eta)$ for $f'(\eta)$. The boundary conditions Eq. (2.24) and the shape of graph of $f'(\eta)$ in Fig. (2.1) suggest the initial guess as

$$f_0'(\eta) = e^{s_1\eta}, \tag{2.53}$$

after integration and using boundary condition (2.25a), we get

$$f_0(\eta) = \frac{e^{s_1\eta} - 1}{s_1}. \tag{2.54}$$

By substituting this into the right hand side of Eq. (2.52) and bounding the first iterate f_1 on left hand side to satisfy the boundary conditions Eq. (2.24), we obtain the zeroth order approximation

$$s_1 \equiv s_{10} = -\sqrt{\frac{3}{2}} \simeq -1.22, \tag{2.55}$$

$$f_0''(0) = s_{10}, \tag{2.56}$$

and Eq. (2.54) gives the value

$$f_0(\infty) = -\frac{1}{s_{10}} \simeq 0.82. \tag{2.57}$$

The equation of first order iterate f_1 becomes

$$f_1' + \frac{1}{2}{f_1}^2 = \frac{1}{2}\left(1 + e^{-\sqrt{6}\eta}\right). \tag{2.58}$$

By putting here $\eta = \infty$ in Eq. (2.58) and after using the boundary condition (2.24c), one immediately obtains as

$$f_1(\infty) = 1. \tag{2.59}$$

Now the arithmetic mean of $f_0(\infty)$ and $f_1(\infty)$ is given by

$$A.M = 0.908248. \tag{2.60}$$

Thus arithmetic mean coincides with the exact value $f(\infty) = 0.905639$ to within an inaccuracy of 0.003.

Now by using for f and f' the zeroth order approximations given by Eqs. (2.54) and (2.53) respectively with $s_1 \equiv s_{10} = -1.22$, we now turn to solve the Eq. (2.23). After integrating with respect to η,

$$\theta' + \Pr f\theta = (a+1)\Pr \int_0^\eta f'\theta d\eta_1 + C_3, \tag{2.61}$$

where C_3 is constant of integration which is obtained by using the boundary condition (2.25a)

$$C_3 = \theta'(0) = s_2. \tag{2.62}$$

The Eq. (2.61) will take the form as

$$\theta' + \Pr f\theta = s_2 + (a+1)\Pr \int_0^\eta f'\theta d\eta_1. \tag{2.63}$$

At $\eta \to \infty$ the boundary condition (2.25) further augmented as

$$\theta'(\infty) = 0. \tag{2.64}$$

By using Eqs. (2.25b) and (2.64) in Eq. (2.63) we get

$$s_2 = -(a+1)\,\mathrm{Pr}\int_0^{\infty} f'\theta d\eta_1. \tag{2.65}$$

For $a = -1$, the Eq. (2.65) shows that

$$s_2\left(=\theta'(0)\right) = 0, \tag{2.66}$$

that is there is no flow of heat between stretching surface and the ambient for all values of Pr. So in this adiabatic flow, the temperature field is given by the Eq. (2.63) as

$$\theta' + \mathrm{Pr}\, f\theta = 0, \tag{2.67}$$

The solution of Eq. (2.67) is of the simple form

$$\theta(\eta; -1, \mathrm{Pr}) = \exp\left(-\,\mathrm{Pr}\int_0^{\eta} f(\eta_1)d\eta_1\right), \tag{2.68}$$

which reduces to

$$\theta(\eta; -1, \mathrm{Pr}) = \exp\left(\frac{2\,\mathrm{Pr}}{3} + \frac{\mathrm{Pr}}{s_{10}^2}\eta - \frac{2\,\mathrm{Pr}}{3}e^{s_{10}\eta}\right). \tag{2.69}$$

Now introducing new parameter and variable z here

$$b = \frac{\mathrm{Pr}}{s_{10}^2} = \frac{2\,\mathrm{Pr}}{3}, \tag{2.70}$$

$$z = be^{s_{10}\eta}. \tag{2.71}$$

By applying zeroth order approximation for f given by the Eq. (2.54), the temperature field in the adiabatic case reduces to

$$\theta(\eta; -1, \mathrm{Pr}) = \exp(b + bs_{10}\eta - z). \tag{2.72}$$

Again solving Eq. (2.23) for $a \neq 1$, explicitly by using for f and f' the zeroth order approximation with

$$s_1 \equiv s_{10} = -\sqrt{\frac{3}{2}}. \tag{2.73}$$

After some standard mathematical manipulation the solution can be expressed in terms of Kummer's function $M(\alpha, \beta, z)$ and $\theta(\eta; -1, \mathrm{Pr})$ as

$$\theta(\eta; a, \mathrm{Pr}) = \frac{M(a+1, b+1, z)}{M(a+1, b+1, b)}\theta(\eta; -1, \mathrm{Pr}). \tag{2.74}$$

The approximation formula for wall-temperature gradient $\theta'(0) = s_2(a, \mathrm{Pr})$ is

$$s_2(a, \mathrm{Pr}) = -(a+1)\,|s_{10}|\,\frac{b}{b+1}\frac{M(a+2, b+2, b)}{M(a+1, b+1, b)}. \tag{2.75}$$

2.6 Results and discussion

The features of heat and mass transfer in boundary layers on an exponentially stretching continuous surface have been observed and the effects of Prandtl Number Pr and temperature parameter a have been examined. Fig. 2.1-2.7 are drawn to analyze the behavior of prescribed parameters on the obtained solution. Fig. 2.1 is displayed to show the graphical results of dimensionless functions f, f', θ, $f + \eta f'$ against η. All the solutions are drawn corresponding to those missing values of s_1 and s_2 for which an accuracy upto 10^{-6} is achieved. In Fig. 2.2, the curvature of θ at $\eta = 0$ for $a = 0$, $a < 0$ and $a > 0$. In other words, it can be described as the value of $\theta'(0)$ or the heat flux at the wall for $a = 0$ changes its sign from negative to positive opposite to sign of a. At the same time thermal boundary layer thickness is decreasing with the increase in the values of a. Fig. 2.3 is drawn to observe the effect of Prandtl number Pr on dimensionless temperature field. It is observed that with the increase in Pr, boundary layer thickness decreases. However the temperature hills increases with the increase in Pr. It is further observed that the solution of temperature field obtained by the shooting method for large value of Pr i.e. $\mathrm{Pr} = 3, 8$ coincides with the analytical solution given in Eq. (2.74) . However for the small Pr number, $\mathrm{Pr} = 0.5$, a small but alternating deviation between the two solutions is observed. It can be stated that for larger Pr number, the accuracy of the formula Eq. (2.74) and Eq. (2.75) is better. The effect of temperature field θ is shown against η for $a = -3$ and $b = 9$. It is observed that when $a = -3$, for $b < 2$, both θ and s_2 are positive, however for $b > 2$, both θ and s_2 changes from positive to negative. Thus the solution is physically consistent only for $0 < b < 2$. In this range $s_2(-3, \mathrm{Pr}) > 0$, thus the surface heat flow is reversed as shown in Fig. 2.5. Moreover it seen that when $a = -3$, on passing over the border $b = 2$ of the physical range, all profiles in Fig. 2.5 started at $\theta(0) = 1$ is in agreement with the boundary condition Eq. (2.25 a). The physical ones ($b < 2$) remain positive for any $\eta > 0$, but the nonphysical ones ($b > 2$) changes it signs from positive to negative immediately after $\eta = 0$ and thus they are not physical. Figure 2.6 illustrates the comparison of the approximation formula Eq. (2.75) by comparing its plot against Pr number for various values of a with the corresponding missing condition $\theta'(0)$ of the shooting method. For $a = -1$, $s_2(a, \mathrm{Pr}) = 0$ for any Pr and the plot coincides with the Pr-axis. The formula (2.75) becomes exact in this case for any Pr, but for $a \neq -1$, the formula (2.75) agreed very well with the numerical results. Table 1 is constructed to compare the values of the formula (2.75) with the numerical results for various values of Pr and a. It is concluded that the accuracy of the formula for larger values of Pr number and a is comparatively better then for smaller values.

a/Pr	0.5	1.0	3.0	5.0	8.0	10.0
-1.5	0.204050	0.377427	0.923886	1.359274	1.888544	2.200127
	0.189479	0.356496	0.898542	1.330560	1.870950	2.185672
-0.5	-0.175817	-0.299871	-0.634096	-0.870400	-1.150290	-1.308575
	-0.167073	-0.290743	-0.628598	-0.867346	-1.149380	-1.308612
0	-0.330493	-0.549641	-1.122065	-1.521216	-1.991817	-2257402
	-0.316573	-0.536774	-1.116180	-1.518833	-1.992238	-2.258994
1.0	-0.594345	-0.954784	-1.869071	-2.500126	-3.242110	-3.660355
	-0.575950	-0.940890	-1.865517	-2.500451	-3.245271	-3.664652
3.0	-1.008417	-1.560310	-2.938540	-3.886555	-5.000459	-5.628188
	-0.990315	-1.550413	-2.939387	-3.890628	-5.006768	-5.635369

Table 1. The wall-temperature gradient $s_2(a, \mathrm{Pr})$ calculated by shooting method (upper numbers) and by the approximation formula (2.75) [lower numbers].

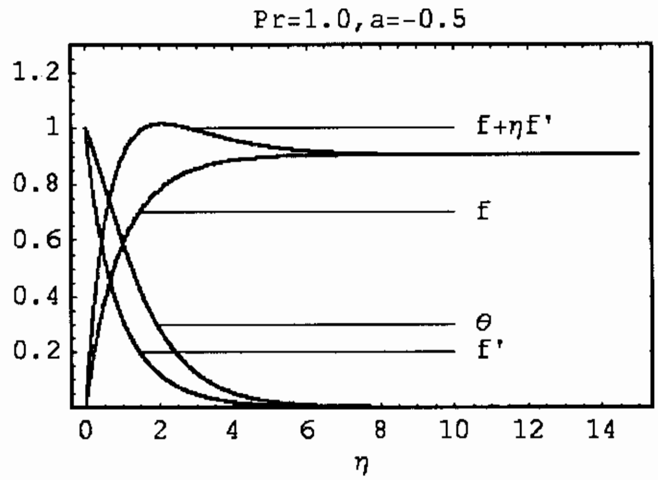

Fig. 2.1: The functions f, $f\,'$, θ and $f+\eta f\,'$are plotted against η obtained by numerical solution of the boundary value problems (2.22) and (2.24).

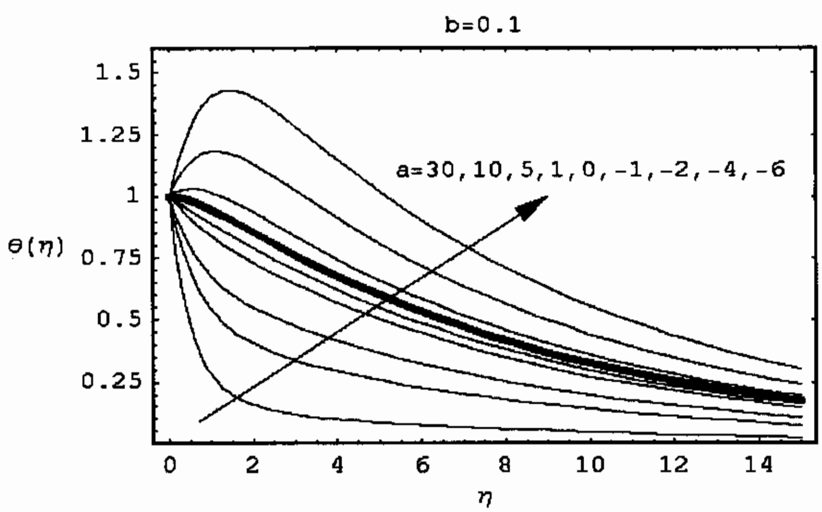

Fig. 2.2: The temperature profiles $\theta(\eta; a, \mathrm{Pr})$ given by (2.74) for $b = 0.1$ and $a = -6, -4, -2, -1, 0, 1, 5, 10$ and 30.

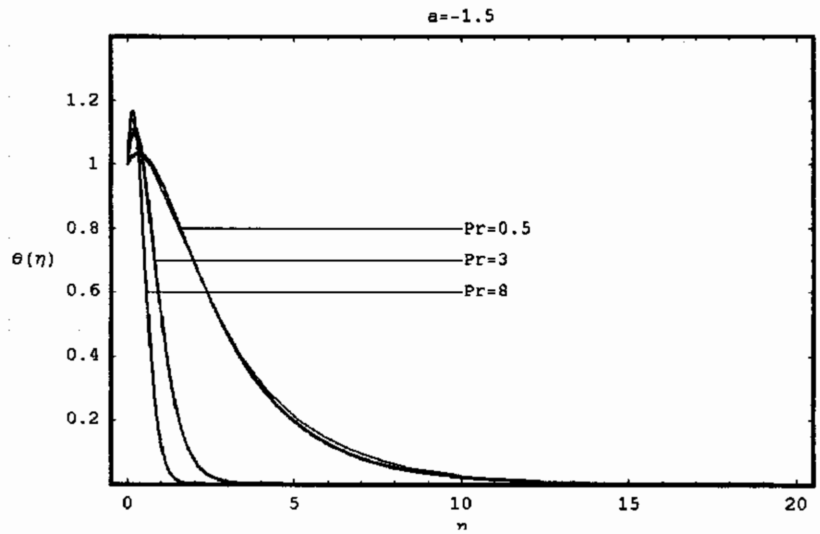

Fig. 2.3: The dimensionless wall temperature θ plotted against η for $a = -1.5$ and Pr $=$ 0.5, 3,and 8. The curves have been obtained bot by numerical solution of boundary value problem (2.23) and (2.25) and according to approximation formula (2.74).

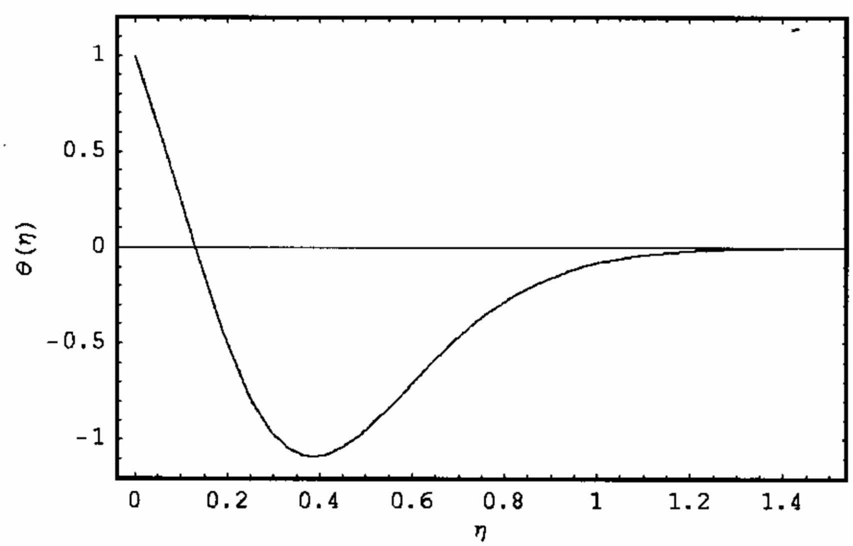

Fig. 2.4: The unphysical solution $\theta(\eta, -3, \mathrm{Pr})$ of the boundary value problem (2.23) and (2.25) plotted against η for $b = 9$ according to Eq.(2.74).

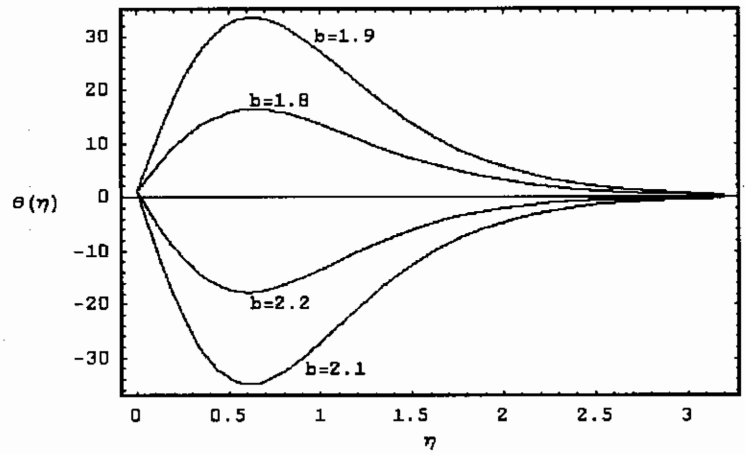

Fig. 2.5: Temperature profiles $\theta(\eta; -3, \mathrm{Pr})$ given by (2.77) for $b = 1.9, 1.8, 2.2$ and 2.1.

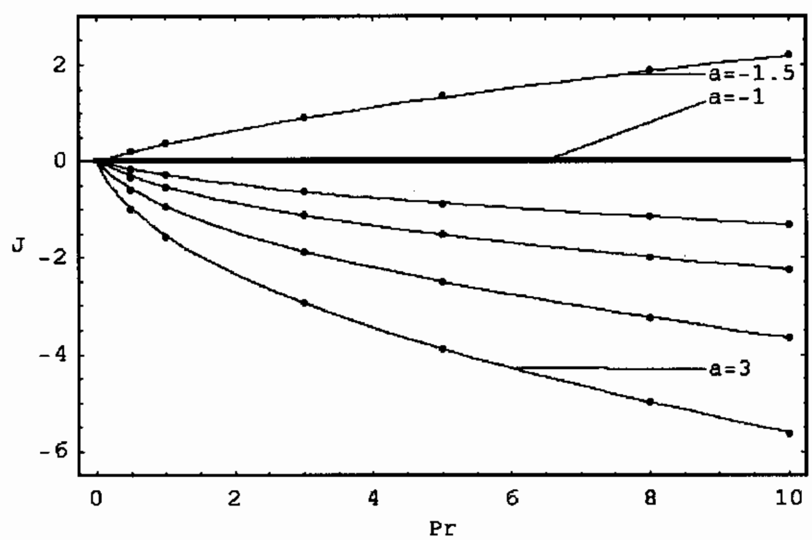

Fig. 2.6: The dimensionless wall temperature gradient $s_2(a, \mathrm{Pr})$ plotted according to (2.75) against the Prandtl number Pr for $a = -1.5, -1, -0.5, 0, 1$ and 3 (listed from top down) and compared with results of the numerical calculations(dots corresponding to $\mathrm{Pr} = 0.5, 1, 3, 5, 8$ and 10 respectively), $s_2(-1; \mathrm{Pr})$ coincides with $\mathrm{Pr}-axis$.

2.7 Conclusions

The heat and mass transfer characteristics have been examined and the main results of this chapter are arranged as follows

- Since the dimensionless stream function $f(\eta)$ does not contain any parameter, so mass transfer remains unaffected by Prandtl number Pr and temperature parameter a.

- The variation of Prandtl number Pr and parameter of temperature distribution a effect the dimensionless temperature field $\theta(\eta)$ and dimensionless surface temperature gradient $s_2 = \theta'(0)$ i.e.

$$\theta = \theta(\eta, a, \mathrm{Pr}) \text{ and } s_2 = \theta'(0) = s_2(a, \mathrm{Pr}). \tag{2.76}$$

- The physical variation ranges of Pr is as a result of the requirement that dimensionless temperature field $\theta(\eta, a, \mathrm{Pr})$ remains positive in the region $0 < \eta < \infty$.

- If $a > -1$ then $\theta'(0, a, \mathrm{Pr}) = s_2 < 0$ for all values of Pr and the heat flows from stretching surface to the surroundings.

- If $a = -1$ then $\theta'(0, a, \mathrm{Pr}) = s_2 = 0$ for all values of $\mathrm{Pr} > 0$ i.e. the stretching surface is "adiabatic".

- If $-2 \leq a < -1$, then $\theta'(0, a, \mathrm{Pr}) = s_2 > 0$ for all values of $\mathrm{Pr} > 0$ and the surface heat flow becomes reversed and Sparrow Gregg type hills appear in the $\theta(\eta)$ profile as is observed from Fig. 2.2.

- For $a < -2$, the physical situation changes dramatically. In this case for certain values of b, in the approximate solution $\theta(\eta; a, \mathrm{Pr})$, zeros of Kummer's function occurs that it becomes unphysical. In addition at zeros of denominator, both $\theta(\eta; a, \mathrm{Pr})$ and $s_2(a, \mathrm{Pr})$ become singular which means that infinite sources of energy have been generated spontaneously in the fluid. To illustrate this we have

$$\theta(\eta, -3, \mathrm{Pr}) = \frac{b^2 e^{2s_{10}\eta} - 2b(b+2)e^{s_{10}\eta} + (b+1)(b+2)}{2-b}\theta(\eta, -1, \mathrm{Pr}), \tag{2.77}$$

and

$$s_2(-3,\mathrm{Pr}) = \frac{4b\,|s_{10}|}{2-b}, \tag{2.78}$$

which are obtained by taking $a = -3$ in approximation formulae (2.74) and (2.75). The singularity mentioned above, in this case, is located at $b = 2$. For $b < 2$ both $\theta(\eta; -3.\,\mathrm{Pr})$ and $s_2(-3, \mathrm{Pr})$ are positive, whereas for $b > 2$ the solution $\theta(\eta; -3.\,\mathrm{Pr})$ changes sign from positive to negative at

$$\eta = \left(\frac{2}{3}\right)^{\frac{1}{2}} \ln\left(\frac{b[b+2+(b+2)^{\frac{1}{2}}}{(b+1)(b+2)}\right), \tag{2.79}$$

as shown in Fig.2.4. Therefore, the solution $\theta(\eta; -3.\,\mathrm{Pr})$ is physically consistent only for $0 < b < 2$. In [Fig.2.5] we see that what happens for $a = -3$ on passing over the border $b = 2$ of the physical range. All profiles in [Fig.2.5] start at $\theta(\eta; -3.\,\mathrm{Pr}) = 1$.The physical ones ($b < 0$) are positive for any $\eta > 0$. For $b > 2$ the profiles change signs at the value of η given by Eq. (2.76) and thus they are unphysical. As b approaches the limiting value $b = 2$ the denominator in Eqs.(2.77) and (2.78) for $a = -3$ vanishes and maximum and minimum value of $\theta(\eta; -3.\,\mathrm{Pr})$ approaches $\pm\infty$ respectively. Similar results are obtained for the case $a < -3$.

- In both the regions i.e. $a > -1$ and $a < -1$ the magnitude of $\theta'(0) = s_2$ increases with the increase in $|a|$ and with increasing in Pr .
- The boundary layer thickness decreases with the increase in value of a for all Pr. It also decreases with the increasing Pr for all a.

Chapter 3

Heat and mass transfer analysis in the boundary layers on an exponentially stretching continuous surface with thermal radiations

3.1 Introduction

In this chapter we have examined the heat and mass transfer in the boundary layers on an exponentially stretching continuous surface with thermal radiations. The resulting boundary value problem has been converted into system of five simultaneous first order initial value problems for five unknowns. The system is then solved numerically by well known shooting method for two unknown initial conditions with Runge Kutta of order 4. The dimensionless stream function $f\ (\eta)$ and temperature field $\theta(\eta)$ are computed for different values of Prandtl Number Pr and Radiation Number K. At the end this numerical solution is discussed through graphs for the variation of Pr and K.

3.2 Mathematical formulation

The basic problem is same as we have discussed in chapter two, however, in this case we have added thermal radiations. Under usual boundary layer approximations, the flow and heat transfer in the presence of radiation effects are governed by the Equations (2.9), (2.12) and

$$\rho C_p\left(u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}\right)=k\frac{\partial^2 q}{\partial y^2}-\frac{\partial q_r}{\partial y}. \tag{3.1}$$

Here viscous dissipation term $\mu\left(\frac{\partial u}{\partial y}\right)^2$ is neglected and q_r is radioactive heat flux. The boundary conditions are given by Eqs. (2.14), (2.15), (2.20) and (2.21). Employing Rossiland approximation of radiation for an optically thick layer, one has

$$q_r=-\frac{4\sigma^*}{3k'}\frac{\partial q^{\,4}}{\partial y}, \tag{3.2}$$

where k' is the mean absorption coefficient and σ^* is the Stefan Boltzmann constant. $q^{\,4}$can be expressed as

$$q^{\,4}=4q_\infty{}^3q-3q_\infty{}^4. \tag{3.3}$$

Using Equations (3.2) and (3.3) in Equation (3.1) then we have

$$\rho C_p\left(u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}\right)=\left(k+\frac{16\sigma^* q_\infty{}^3}{3k'}\right)\frac{\partial^2 q}{\partial y^2}. \tag{3.4}$$

Now using the following smilarity transformations given in Eqs. (2.16), (2.17),and

$$q(x,y)=q_\infty+q_0e^{x/(2L)}\theta(\eta), \tag{3.5}$$

$$\eta=\left(\frac{\mathrm{Re}}{2}\right)^{1/2}\frac{y}{L}e^{x/(2L)}, \tag{3.6}$$

the Eq. (2.9) is identically satisfied and Eqs. (2.10) and (3.4) are reduced to Eqs. (2.22) and

$$\left(1+\frac{4}{3}K\right)\theta''+\Pr(f\theta'-f'\theta)=0, \tag{3.7}$$

where $\Pr=\frac{\mu C_p}{k}$ is the Prandtl Number, $K=\frac{4\sigma^* q_\infty^3}{k'k}$ is the Radiation Number and prime denotes differentiation with respect to η. The boundary conditions are given by Eqs. (2.24) and (2.25).

3.3 Numerical solution of the problem:

We have solved the above mentioned problem by the same technique as in the Chapter Two. So we transform the given problem to a system of first order initial value problem.

Let

$$f' = p, \tag{3.8}$$

$$p' = z, \tag{3.9}$$

$$z' = 2p^2 - fz, \tag{3.10}$$

$$\theta' = r, \tag{3.11}$$

$$r' = \frac{3\,\mathrm{Pr}}{3+4K}[\,p\theta - fr], \tag{3.12}$$

subject to the initial conditions:

$$f(0) = 0, \quad p(0) = 1, \quad z(0) = m, \tag{3.13}$$

$$\theta(0) = 1, \quad r(0) = n. \tag{3.14}$$

After differentiating Eq. (3.8) to Eq. (3.10) and Eq. (3.13) with respect to m and Equations (3.11), (3.12).and (3.14) with respect to n respectively we get the following system of Equations.

$$F' = P, \tag{3.15}$$

$$P' = Z, \tag{3.16}$$

$$Z' = 2pP - Fz - fZ, \tag{3.17}$$

subject to

$$F(0) = 0, \quad P(0) = 0, \quad Z(0) = 1, \tag{3.18}$$

and

$$W\,' = R, \tag{3.19}$$

$$R\,' = \frac{3\,\mathrm{Pr}}{3+4K}[pW - fR], \tag{3.20}$$

subject to initial conditions:

$$W(0) = 0, \quad R(0) = 1. \tag{3.21}$$

where F ,P and Z denote derivative of f, p and z with respect to m and W and R denote derivative of θ and r with respect to n.

3.4 Solution by the homotopy analysis method

In order to provide a HAM solution we assume the following initial guess and auxiliary initial guess and auxiliary linear operator

$$f_0(\eta) = \left(1 - e^{-2\eta}\right)/2 \tag{3.22}$$

$$\theta_0(\eta) = e^{-\eta} \tag{3.23}$$

$$\mathcal{L}_1(f) = f''' - 4f', \tag{3.24}$$

$$\mathcal{L}_2(\theta) = \theta'' - \theta, \tag{3.25}$$

with the property

$$\mathcal{L}_1\left(C_1 e^{2\eta} + C_2 e^{-2\eta} + C_3\right) = 0,\ \mathcal{L}_2\left(D_1 e^{\eta} + D_2 e^{-\eta}\right) = 0, \tag{3.26}$$

where C_i $(i = 1-3)$ and D_j $(j = 1, 2)$ are arbitrary constants be determined. If $p_1 \in [0,1]$ the embedding parameter and $\hbar_i$ $(i = 1, 2)$ the nonzero auxiliary parameters then the zeroth order problem is

$$(1 - p_1)\,\mathcal{L}_1\left[F(\eta, p_1) - f_0(\eta)\right] = p_1 \hbar_1 \mathcal{N}_1\left[F(\eta, p_1)\right] \tag{3.27}$$

$$(1 - p_1)\,\mathcal{L}_2\left[\Theta(\eta, p_1) - \theta_0(\eta)\right] = p_1 \hbar_2 \mathcal{N}_2\left[F(\eta, p_1), \Theta(\eta, p_1)\right] \tag{3.28}$$

$$F(0,p_1)=0, F'(0,p_1)=1, F'(\infty,p_1)=0, \tag{3.29}$$

$$\Theta(0,p_1)=1,\ \Theta(\infty,p_1)=0, \tag{3.30}$$

$$\mathcal{N}_1[F(\eta,p_1)]=F'''(\eta,p_1)+F(\eta,p_1)F''(\eta,p_1)-2F'^2(\eta,p_1), \tag{3.31}$$

$$\mathcal{N}_2[F(\eta,p_1),\theta(\eta,p_1)]=(1+4K/3)\Theta''(\eta,p_1)+\Pr\left(F(\eta,p_1)\Theta'(\eta,p_1)-F'(\eta,p_1)\Theta(\eta,p_1)\right), \tag{3.32}$$

For $p_1=0$ and 1, the above expressions becomes

$$F(\eta,0)=f_0(\eta), \Theta(\eta,0)=\theta_0(\eta) \text{ and } F(\eta,1)=f(\eta), \Theta(\eta,1)=\theta(\eta) \tag{3.33}$$

and as p_1 increases from 0 to 1, $F(\eta,p_1), \Theta(\eta,p_1)$ change an initial guesses $f_0(\eta), \theta_0(\eta)$,to the solution $f(\eta)$ and $\theta(\eta)$ respectively. Expanding F and Θ with the help of Maclaurin's series, we obtain

$$F(\eta,p_1)=f_0(\eta)+\sum_{m=1}^{\infty} f_m(\eta)\ p_1^m \text{ and } \Theta(\eta,p_1)=\theta_0(\eta)+\sum_{m=1}^{\infty} u_m(\eta)\ p_1^m, \tag{3.34}$$

where

$$f_m(\eta)=\frac{1}{m!}\frac{\partial^m F(\eta,p_1)}{\partial p_1^m}\bigg|_{p_1=0},\ \theta_m(\eta)=\frac{1}{m!}\frac{\partial^m \Theta(\eta,p_1)}{\partial p_1^m}\bigg|_{p_1=0}.$$

Assuming that the above series is convergent for $p_1=1$, we can write

$$f(\eta)=f_0(\eta)+\sum_{m=1}^{\infty} f_m(\eta) \text{ and } \theta(\eta)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta). \tag{3.35}$$

The mth order deformation problems are

$$\mathcal{L}_1[f_m(\eta)-\chi_m f_{m-1}(\eta)]=\hbar_1\mathcal{R}_m^f(\eta), \tag{3.36}$$

$$\mathcal{L}_2[\theta_m(\eta)-\chi_m\theta_{m-1}(\eta)]=\hbar_2\mathcal{R}_m^\theta(\eta), \tag{3.37}$$

$$f_m(0)=0,\quad f_m'(0)=0,\ f_m'(\infty)\rightarrow 0,$$
$$\theta_m(0)=0 \text{ and } \theta_m(\infty)\rightarrow 0, \tag{3.38}$$

$$\mathcal{R}^f_m(\eta) = f'''_{m-1} + \sum_{k=0}^{m-1} \left(f_{m-1-k} f''_k - 2 f'_{m-1-k} f'_k \right), \tag{3.39}$$

$$\mathcal{R}^\theta_m(\eta) = (1 + 4K/3)\,\theta''_{m-1} + \Pr \sum_{k=0}^{m-1} \left(f_{m-1-k} \theta'_k - f'_{m-1-k} \theta_k \right) \tag{3.40}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.41}$$

Now the nonlinear boundary value problem Eqs. (2.22) and (3.7) subject to the boundary conditions Eqs. (2.24) and (2.25) is reduced to the system of linear nonhomogeneous problem Eqs. (3.36) and (3.37) with the boundary conditions Eq. (3.38). The reduced system is solved with the symbolic computation software MATHEMATICA.

3.5 Results and discussion

The Equations (2.22) and (3.7) with the given boundary conditions (2.24) and (2.25) are solved numerically by using shooting method and analytically by using Homotopy analysis method [22]. The convergence of the series solution obtained by homotopy analysis method is ensured through $\hbar-$curve drawn in Fig. 3.1. As $\hbar_1$ and $\hbar_2$ are the auxiliary parameters corresponding to f and θ respectively. As pointed by Liao [22], if the series solution obtained by the homotopy analysis method is convergent then it must be one of the solution to the given boundary value problem. The valid region of $\hbar_1$ and $\hbar_2$ is that for which the curve in Fig. 3.1 become parallel to the $\hbar$-axis, as pointed out by Liao. It is observed from the Fig. 3.1, that the valid region for $\hbar_1$ is $-0.75 < \hbar_1 < -0.25$. Since there is no parameter involve in the boundary value problem Eq. (2.22), so one can choose any value of $\hbar_1$ belongs to the interval. However, the valid region of $\hbar_2$ is $-1.0 < \hbar_2 < -0.1$ for $\Pr = 1, K = 1/2$.

$K/\Pr$	1	1.5	2	3	5
0	0.9548	1.2348	1.4715	1.8691	2.5001
	0.9548		1.4714	1.8691	
0.1	0.8795	1.1421	1.3642	1.7376	2.3302
0.2	0.8166	1.0645	1.2745	1.6275	2.1879
0.3	0.7631	0.9984	1.1979	1.5335	2.0665
0.5	0.6765	0.8912	1.0735	1.3807	1.8691
	0.6765		1.0735	1.3807	
0.7	0.6092	0.8071	0.9760	1.2609	1.7140
1.0	0.5316	0.7099	0.8628	1.1214	1.5335
	0.5315		0.8627	1.1214	

Table 2 : The values of heat transfer coefficient, $-\theta'(0)$ calculated by shooting method (upper numbers) and by Keller box method (lower numbers)

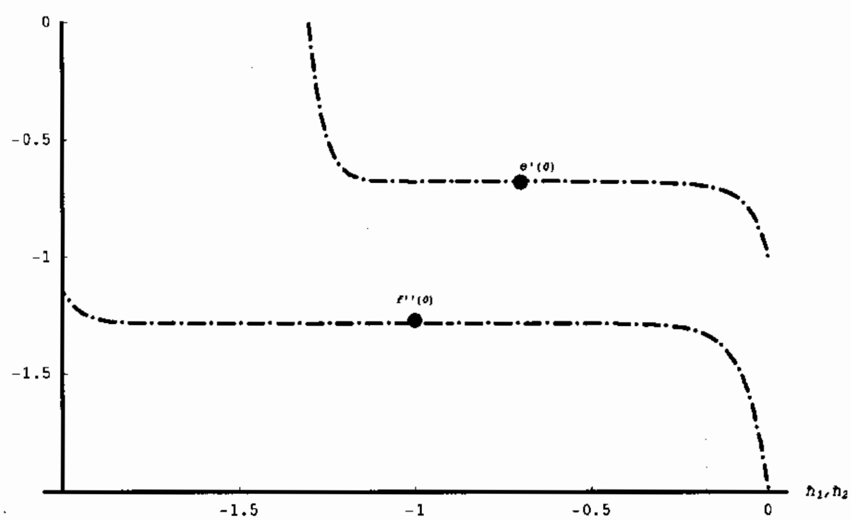

Fig. **3.1**: The 20th order approximate solution $f''(0)$ and $\theta'(0)$ against the auxiliary parameters $\hbar_1$, $\hbar_2$ for $\Pr = 1.0$ and $K = 0.5$. Dots denote the corresponding values obtained by shooting method.

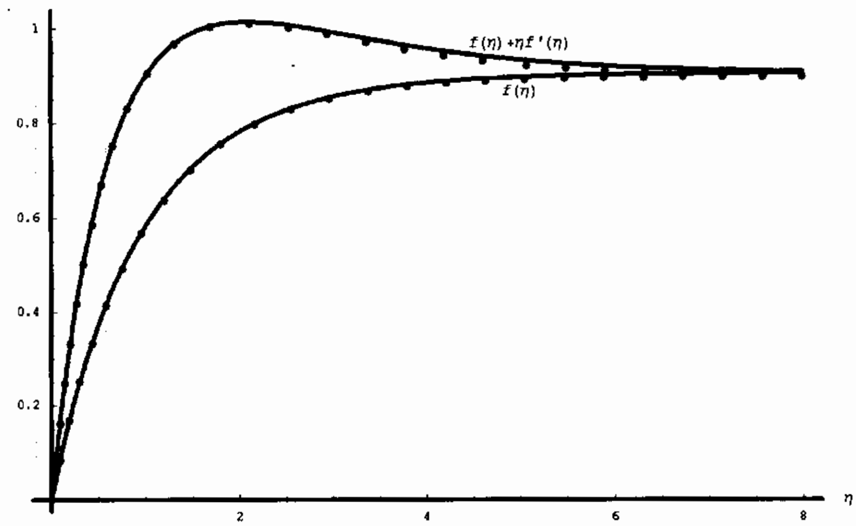

Fig. **3.2**: The plot of f and $f+\eta f'$ against the dimensionless variable η, dotted curves represent 30th order HAM solution and solid line is obtained by shooting method.

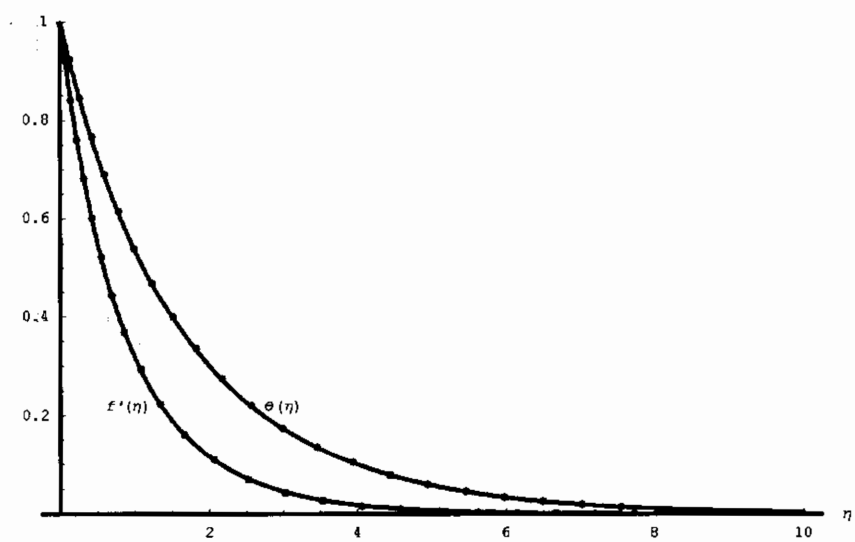

Fig. **3.3**: The plot of f' and θ against the dimensionless variable η,dotted curves represent 30th order HAM solution and solid line is obtained by shooting method for $\mathrm{Pr} = 1.0$ and $K = 0.5$.

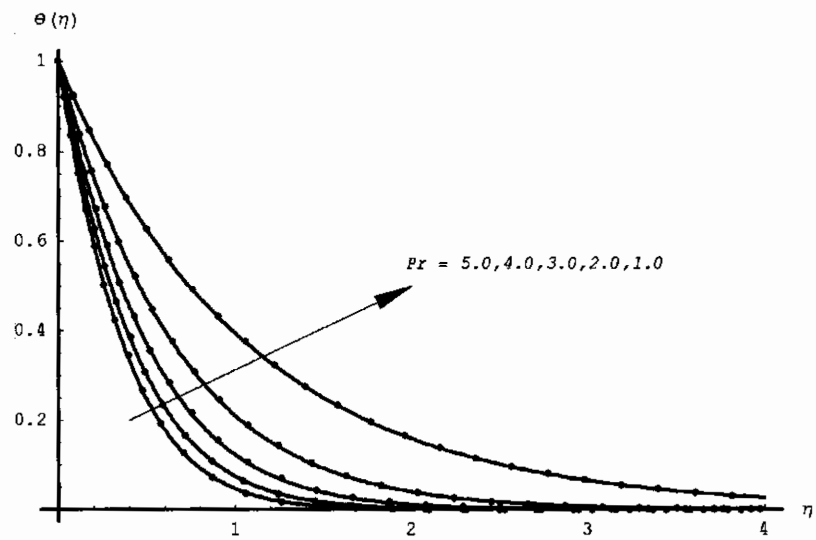

Fig. **3.4**: The variations of Prandtl number on temperature profile $\theta(\eta)$ plotted against η for $K = 0$. Dotted curves represent 30th order HAM solution and solid line is obtained by shooting method.

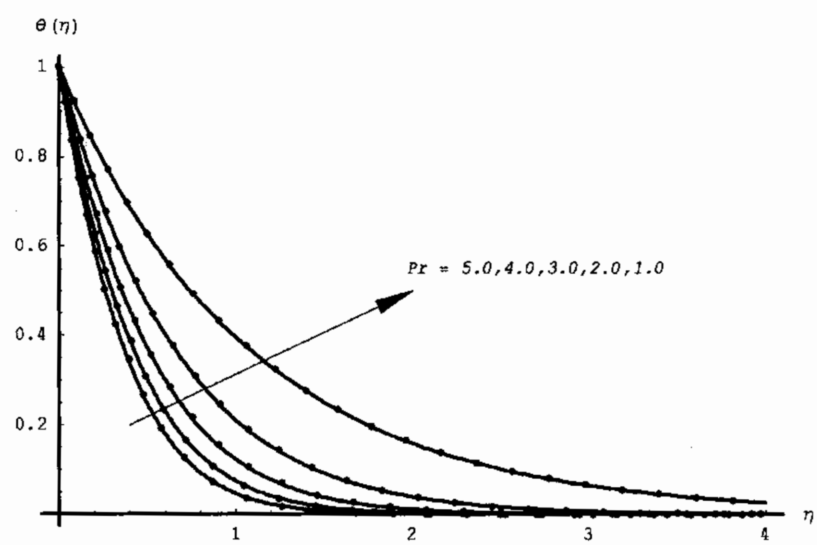

Fig. **3.5**: The effect of Prandtl number Pr on temperature profile $\theta(\eta)$ plotted against η for $K = 0.5$. Dotted curves represent 30th order HAM solution and solid line is obtained by shooting method.

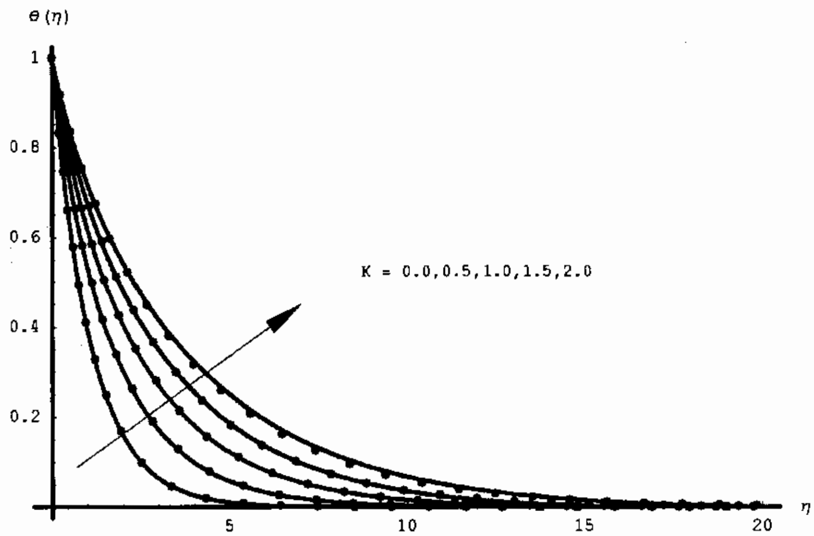

Fig. **3.6**: The effect of Radiation number K on temperature profile $\theta(\eta)$ plotted against η for $\mathrm{Pr} = 1$. Dotted curves represent 30th order HAM solution and solid line is obtained by shooting method.

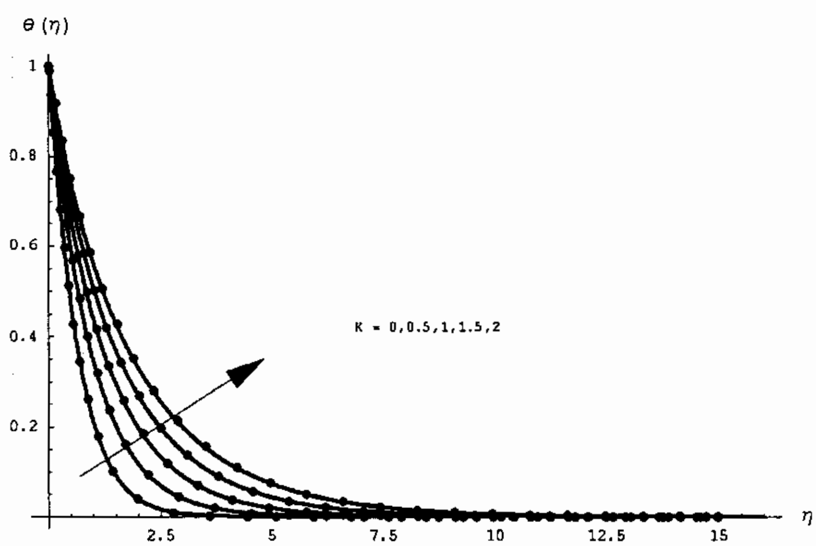

Fig. **3.7**: The effect of Radiation number K on temperature profile $\theta(\eta)$ plotted against η for $\mathrm{Pr} = 2$. Dotted curves represent 30th order HAM solution and solid line is obtained by shooting method.

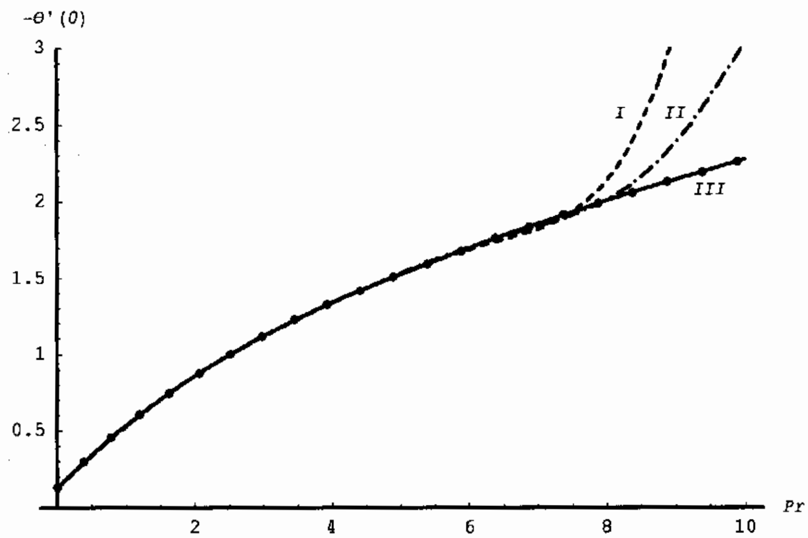

Fig. **3.8**: The comparison of the 15th order approximate solution $-\theta'(0)$ against Pr number for $K = 1$. Case I is for $\hbar_2 = -3/4$, Case II is for $\hbar_2 = -1/2$ and Case III is for $\hbar_2 = -1/(2 + \Pr/2)$. Dotted curve represents the solution $-\theta'(0)$ obtained by shooting method.

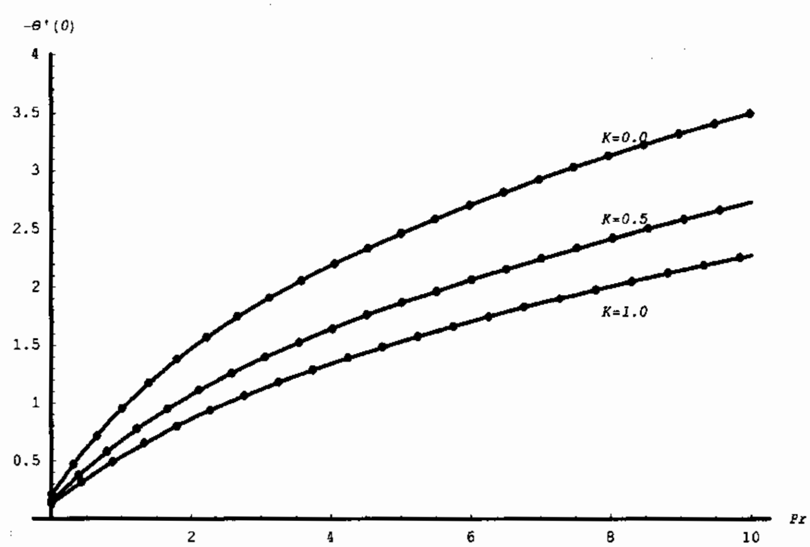

Fig. **3.9**: The variation of $-\theta'(0)$ against Prandtl number Pr for different values of radiation number K.

Since the dimensionless temperature field comprising the parameters Pr and K, so the profile of θ and θ' will be changed by the variation in the values of these parameters. In the same way, proper value of $\hbar_2$ may be chosen according to the values of all Pr and K. In figure 3.1, dots represent the corresponding values obtained by the shooting method. It is worthmentioning here that the missing conditions in shooting method is iterated until the accuracy of 10^{-6} is achieved. The profile of $f + \eta f'$ and f are shown in the Fig. 3.2. Solid curves represent the solution obtained by shooting method and dotted curve is for 30th order approximate solution obtained by homotopy analysis method through the Fig. 3.2 to 3.9. Fig. 3.3 depicts the effects of the temperature profile θ and stream function f' against η, when $K = 0$. To analyze the effect of the Pr number on the temperature field θ, Fig. 3.4 is prepared. It is observed that by increasing the prandtl number Pr, temperature profile and thermal boundary layer thickness are decreasing. Same effects are observed qualitatively by taking radiation parameter K to be equal to 0.5 as shown in Fig. 3.5 but with larger magnitude. To show the effects of the radiation parameter on temperature filed θ, Fig. 3.6 and 3.7 are drawn. It is quite interesting to see that the temperature field and the thermal boundary layer thickness is increasing with the increase in radiation parameter K as shown in Fig. 3.6 for $\mathrm{Pr} = 1$. Same profile of θ is observed with $\mathrm{Pr} = 2$ qualitatively, but with larger in quantitatively. Table 2 is constructed to show the effect of the temperature gradient measured by shooting method with the variation of radiation parameter K and Prandtl number Pr . Furthermore, few values are compared with the solution obtained by implicit finite difference scheme. It is seen from table 2 that values obtained by both the techniques are within good agreement.

3.6 Conclusions

In this chapter we study the heat and mass transfer analysis in the boundary layers on an exponentially stretching continuous surface with thermal radiations. The governing ordinary differential equation are solved using both shooting method and approximately by homotopy analysis method. It is observed that the solution obtained by both the techniques are in excellent agreement with each other. Since there is no parameter in the dimensionless stream function f, so the behavior of mass transfer is universal. Dimensionless temperature field and its gradient are both changed with the change in the radiation parameter K and Prandtl number Pr. Increase in the radiation parameter tends to increase the thermal boundary layer thickness, however, Pr helps us to decrease the thermal boundary layer thickness.

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