# HEAT AND MASS TRANSFER IN THE FALKNER-SKAN WEDGE FLOW WITH RADIATION: AN ANALYTIC SOLUTION



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2010.

1. Heat – trasmission – Mathematics

In the name of almighty **ALLAH**,
the most beneficent, the most merciful

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*A Dissertation*
*Submitted in the Partial Fulfillment of the*
*Requirements for the Degree of*
***MASTER OF SCIENCE***
***IN***
***MATHEMATICS***

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# Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTRER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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## dedicated To

*my mother and father*
*& my family members.*

# Acknowledgements

To begin with the name of Almighty ***ALLAH***, Who inculcated the consecration upon me to fulfill the requirement for this dissertation I offer my humblest Darood to the Holy Prophet ***Muhammad*** (Peace be upon Him) who is forever a torch of guidance for humanity.

I express my gratitude to all my teachers whose teachings have brought me to this stage of academic zenith, in particular, I wish to express my profound gratitude to my kind natured, eminent, affectionate, candid and devoted supervisor, ***Dr. Tariq Javed***, who aided me with many inspirational discussions. His many valuable comments and suggestions were most welcome and instructive and greatly improved the clarity of this document. I would never have been able to do it up to the standard without his help. I am placing my earnest thanks to ***Dr Tariq Javed.***

I would also like to thank all my friends and others who directly or indirectly helped me during my research period.

Finally, it would have been impossible for me to complete this work without a great support and understanding of my family. I am placing regards to them.

**Date: September 6, 2010** Muhammad Kashif

# Preface

The study of boundary layer flow [1-7] has attracted many researchers in past three decades. This theory is due to Ludwig Prandtl [8]. According to which the viscous effects are significant in a thin layer near the boundary called as a boundary layer. This assumption not only simplified the Navier-Stokes equations but also resolved the dilemma between the difference in experimental and theoretical results in fluid flow past solid boundaries. The first successful implementation of boundary layer theory was discussed by Blasius in his doctoral thesis [9] who was a student of Prandtl. Blasius problem is one of the famous problems in boundary layer theory. In the Blasius flow the free stream velocity was assumed to be constant. Later on Falkner and Skan [10] discovered a very famous similarity solution for variable free stream velocity. The numerical solution of the transformed equations was given by Hartree [11}.

The Falkner-Skan flow has been studied in different ways extensively in the literature. The study of temperature field associated with two dimensional steady and incompressible boundary layer flow governed by the Falkner-Skan equation has extensive application in the field of hydrodynamics and aerodynamics. Researchers such as Hsu et al [12], Assaithabi [13], Howarth [14], Cebecci and Keller [15] and Sher and Yakhot [16] have studied the Falkner-Skan wedge flow numerically due to difficulties arising in obtaining exact solutions of the problems in close form. As for as Heat transfer analysis of Falkner-Skan wedge flow is concern, Lin and Lin [17] has introduced a similarity solution method for the forced convection heat transfer of any prandtl number and then solve it by using numerical scheme such as shooting method. Later on Kuo [18] studied the heat transfer analysis of the Falkner-Skan wedge flow by converting it into a pair of initial value problem and then solved it by using differential transform method. Since there were problem in finding the solution with the Falkner-Skan wedge flow with numerical scheme, as one has to find the solution for each discrete value of the parameter separately. It took lot of time to calculate the solution of the problem within reasonable range of the parameter. In order to rectify this situation, Liao introduced homotopy analysis method in his book Beyond Perturbation [19] which is rapid convergent series solution approach. Later, Liao [20] has applied this method to give an analytical solution of the temperature distribution in viscous Blasius flow Problem. Since then, lot of work of this field have been investigated by different authors [21-33] and established the fact of homotpy analysis method as rapid convergent method.

In this thesis, Chapter 1 comprises basic definitions and preliminaries of the homotopy analysis method. In chapter 2, temperature distribution in the falkner-Skan wedge flow by Yao [34] is reproduced and compared its results with numerical results in the book [35]. In chapter 3, we extend the analysis of Yao and investigated the heat and mass transfer effects in the Falkner-Skan wedge flow with radiation by using homotopy analysis method.

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# Chapter 1

# Preliminaries

In this chapter, some basic definitions, concepts of different types of fluids and basic equations which govern the flow are given. The basic idea of homotopy analysis method (HAM) and its advantages are explained in this chapter.

## 1.1 Deformation

It is a relative change in position or length of the fluid particles.

## 1.2 Fluid

A fluid is a substance that deforms continuously under the application of shearing, i.e., tangential stress, no matter how small the shearing stress is but in the absence of it there is no change in position of fluid particles.

## 1.3 Flow

In the presence of different forces, material goes under deformation. In many cases, this deformation of fluid is change of position of particles. If this change in position of fluid particles continuously increases without limit, the process is known as flow.

## 1.4 Fluid Mechanics

It is the branch of engineering and physics that deals with the nature and properties of the fluid both in motion and at rest.In practice, the study of fluid-mechanics can be divided into two categories

### 1.4.1 Internal flow systems

Are those where fluid flows through confined spaces e.g., flow through pipes, pumping of blood through blood arteries and water in the channels.

### 1.4.2 External flow systems

Are those where confining boundaries are at relatively larger or at infinite distance such as atmosphere through which airplanes and space vehicles travel.

## 1.5 Stress Field

A field in which surface forces and body forces are in-countered are called stress field. A stress field is a region where the stress (surface forces and body forces) is defined at every point.

## 1.6 Surface Force.

Surface forces include all the forces acting on the boundary through direct contact. Since these forces act only in the short range, therefore these forces are also called short range forces.

## 1.7 Body Forces

Forces which does not require any physical contact and distributed over the whole volume of the fluid are known as body forces. Gravitational and electromagnetic forces are categorized as body forces. These are infect long range forces.

## 1.8 Pressure

Pressure is a surface force that acts normal to the area under consideration. The force per unit area is called pressure.Let $A$ is the surface area of a fluid and $F$ is the magnitude of force acting normal to the surface, then the pressure $p^*$ due to the force on unit area of this surface is defined as

$$p^* = \frac{F}{A}. \tag{1.1}$$

## 1.9 Density

It is defined as the mass per unit volume. Mathematically it can be can written as

$$\rho = \frac{m}{V}. \tag{1.2}$$

Here $\rho$ is the density, $m$ is the mass and $V$ is the volume.

## 1.10 Viscosity

Viscosity is the measure of the resistance of a fluid which is being deformed by either shear stress or tangential stress. In general, viscosity is "thickness" thus, either is "thin" having a lower viscosity, while honey is thick having a higher viscosity. It is mathematically defined as the ratio of shear stress to the rate of shear strain.i.e.

$$\mu = \frac{\text{shear stress}}{\text{rate of shear strain}}, \tag{1.3}$$

where $\mu$ is the coefficient of viscosity.

## 1.11 Kinematic viscosity

The ratio of viscosity to density is referred as kinematic viscosity $\nu$ defined as follows

$$\nu = \frac{\mu}{\rho}. \tag{1.4}$$

## 1.12 Classification of fluids

### 1.12.1 Ideal fluids

The fluids, for which viscosity is zero are termed as ideal fluids. The fluid with zero viscosity offers no resistance to shearing forces and hence during the flow the deformation of the fluid all shear forces are zero. An ideal fluid is fictitious and does not exist in nature however many fluids under certain engineering applications show negligible viscosity effects and can be treated as ideal fluids. All ideal fluids are incompressible.

### 1.12.2 Real fluids

All the fluids for which dynamic viscosity is non-zero are known as real fluids. Real fluids are also known as viscous fluids.

### 1.12.3 Newton's law of viscosity.

According to this law "shear stress is directly and linearly proportional to the rate of deformation".For one dimensional flow we can write as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.5}$$

where $\tau_{yx}$ is the shear stress and $du/dy$ is a deformation rate. Real fluids are further classified into two classes on the basis of Newton's law of viscosity

### 1.12.4 Compressible fluids

All real fluids are said to be compressible to some extent, that is change in pressure or temperature will result in change in density. All the gases are treated as compressible flows.Mathematically $\nabla.\mathrm{v} \neq 0$ .

### 1.12.5 Incompressible fluids

If the change of pressure or temperature is so small and will produce negligible change in density. In this situation the fluid is termed as incompressible. All the liquids are treated as incompressible flows. Mathematically $\frac{d\rho}{dt} = 0 \Rightarrow \rho$ =constant. Where $d/dt$ is called material derivative, i.e, $\rho$ is constant not only with respect to time but also with respect to space coordinates. In this case $\nabla.\mathbf{v} = 0$.

### 1.12.6 Newtonian fluids

The fluids which obey Newton's law of viscosity are categorized as Newtonian fluids. In Newtonian fluids the viscosity coefficient is independent of the rate of deformation. Water, air, gasoline and mercury are some examples of Newtonian fluids.

### 1.12.7 Non-Newtonian fluids

The fluids for which "shear stress is directly but non-linearly proportional to the rate of deformation". Mathematically it is defined as

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n, \qquad n \neq 1 \tag{1.6}$$

or

$$\tau_{yx} = \eta\left(\frac{du}{dy}\right), \tag{1.7}$$

where

$$\eta = \left(\frac{du}{dy}\right)^{n-1} \tag{1.8}$$

is the apparent viscosity which is a function of the rate of deformation. Examples of non-Newtonian fluids are tooth paste, ketchup, gel, shampoo, blood and soaps etc.

## 1.13 Types of flows

### 1.13.1 Uniform flow

The flow in which the velocities of fluid particles at all sections of a pipe or channel are same.

### 1.13.2 Non-uniform flow

The flow in which the velocities of fluid particles are not the same at all sections of the flow domain.

### 1.13.3 Steady flow

A flow in which the quantity of fluid flowing per second is constant . In other words, flow for which fluid properties are independ of time. For such flow

$$\frac{\partial \gamma}{\partial t} = 0, \tag{1.9}$$

where $\gamma$ is the any fluid property and $t$ is the time.

### 1.13.4 Unsteady flow

A flow in which the quantity of fluid flowing per second is not constant .i.e., velocity depends upon time so

$$\frac{\partial \gamma}{\partial t} \neq 0. \tag{1.10}$$

### 1.13.5 Laminar Flow

The smooth flow of a fluid in which adjoining layers of the fluid flow parallel to one another. During laminar flow, all the fluid particles move in distinct and separate layers; there is no mixing between adjacent layers.

### 1.13.6 Porosity

The measure of void spaces in a material is called porosity and is a fraction of volume of voids over the total volume.

### 1.13.7 Porous medium

A porous medium is a material containing pores (voids). Natural porous media include soil, sand, mineral salts, sponge, wood etc. Synthetic porous media include paper, cloth filters, chemical reaction catalysts and membranes.

### 1.13.8 Prandtl number

The Prandtl number $Pr$ is a dimensionless number which is the reaction of the product of dynamic viscosity and specific heat with thermal conductivity $k$, i.e .,

$$P_r = \frac{\mu c_p}{k} \tag{1.11}$$

The Prandtl number serves as to control the relative thickness of the momentum and thermal boundary layers.

### 1.13.9 Schmidt Number

Schmidt number is a dimensionless number defined as the ratio of momentum diffusivity (viscosity) and mass diffusivity and is used to characterize fluid flows in which there are simultaneous momentum and mass diffusion convection processes. It is denoted by Sc and defined as

$$Sc = \frac{\nu}{D} = \frac{\mu}{\rho D} = \frac{\text{viscous diffusion rate}}{\text{molecular (mass) diffusion rate}} \tag{1.12}$$

## 1.14 Falkner-Skan flow.

A Falkner-Skan flow is an external flow with a pressure gradient in which the free stream velocity $u_\infty(x)$ is proportional to $x^m$

$$u_\infty \; \alpha \; x^m \tag{1.13}$$

## 1.15 Falkner-Skan Transformation.

The transformations for incompressible external flows is the Falkner-Skan transformation

$$\Psi = (u_e \nu x)^{1/2} f(\eta), \tag{1.1}$$

$$\eta = \left(\frac{u_e}{\nu x}\right)^{1/2} y, \tag{1.2}$$

used extensively for external boundary-layer flows. The Falkner-Skan transformation can be used to reduce the boundary-layer equations into ordinary differential equations for similar flows. It can also be used for nonsimilar flows for convenience in numerical work.

## 1.16 Heat-Transfer

Heat is the total kinetic energy of all the molecules of the substance transfers from one system to a second system at lower temperature, by virtue of the temperature difference. Heat transfer to a body increases its thermal energy, just as doing work on a body increases its momentum and kinetic energy.

## 1.17 Mass Transfer

Mass transfer is the transfer of mass from high concentration to low concentration. The phrase is commonly used in engineering for physical processes that involve molecular and convective transport of atoms and molecules within physical systems.

## 1.18 Homotopic functions

In topology two functions are said to be homotopic if one function can be transformed continuously into the other. Formally, a homotopy between two continuous functions $f$ and $g$ from a topological space $X$ to a topological space $Y$ is defined to be a continuous function

$$H : X \times [0,1] \to Y, \tag{1.15}$$

from the product of the space $X$ with the unit interval $[0,1]$ to $Y$ such that for all point $x$ in $X$ and

$$H(x,0) = f(x) , \qquad H(x,1) = g(x). \tag{1.16}$$

If we consider second coordinate as time $t$ then at time $t = 0$ we have the initial function $f$ and at $t = 1$ we have the terminal function $g$.

## 1.19 Homotopy analysis method

In fluid mechanics, mostly differential equations representing the fluid flow phenomena are nonlinear. Since it is very difficult to find out the exact solution of those nonlinear problems, so we have no choice other than numerical schemes. In numerical scheme, we need a large amount of time in developing solution for every single value of the parameter involved in nonlinear differential equation. For this reason we preferred to use series solution method for finding the solution. Amongst them homotopy analysis method introduced by Liao [19] is common in practice due to its advantages over rest of the methods the basic idea of HAM is described as follow:

1. Consider a nonlinear equation governed by

$$A(u) \; + \; f(r) = 0, \tag{1.3}$$

where $A$ is a nonlinear operator, $f(r)$ is a known function and $u$ is an unknown function. By means of homotopy analysis method, one first constructs a family of equations

$$(1-q)\mathcal{L}[\widehat{v}(r,p) - u_o(r)] = q\hbar\{A[\widehat{v}(r,p) - f(r)]\}. \tag{1.4}$$

where $u_o(r)$ is an initial guess chosen by using the "Rule of solution expression" such that it satisfies the boundary conditions, $\mathcal{L}$ is an auxiliary linear operator is to be chosen in such a way that it must generate the set of base functions that are used to define the initial guess, $\hbar$ is an auxiliary parameter, $q \in [0,1]$ is an embedding parameter, $\widehat{v}(r,q)$ is an unknown function of $r$ and $q$. Liao [6] expanded $\widehat{v}(r,q)$ in Taylor series about the embedding parameter

$$\widehat{v}(r,p) = u_o(r) + \sum_{m=1}^{\infty} u_m(r) q^m, \tag{1.5}$$

where

$$u_m(r) = \frac{1}{m!}\frac{\partial^m \widehat{v}(r,q)}{\partial q^m}]_{q=0} \tag{1.6}$$

The convergence of the series (1.19) depends upon the auxiliary parameter $\hbar$. If it is convergent at $q = 1$, one has

$$u(r) = u_o(r) + \sum_{m=1}^{\infty} u_m(r) \tag{1.7}$$

Differentiating the zeroth order deformation equation (1.35) $m$-times with respect to $p$ and then dividing them by $m!$ and finally setting $q = 0$ we obtain the following $mth$- order deformation equation

$$\mathcal{L}[u_m(r) - \chi_m u_{m-1}(r)] = \hbar \mathcal{R}_m(r), \tag{1.8}$$

in which

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{1.9}$$

$$\mathcal{R}_m(r) = \frac{1}{(m-1)!} \times \left\{ \frac{d^{k-1}A}{dq^{k-1}} \left[ u_o(r) + \sum_{m=1}^{\infty} u_m(r) q^m \right] \right\}_{p=0} \tag{1.10}$$

There are many different ways to get the higher order deformation equations. However, according to the fundamental theorem in calculus [7], the term $u_m(r)$ in the series (1.19) is unique. Note that HAM contains an auxiliary parameters $\hbar$, which provides us with a simple way to control and adjust the convergence of the series solution (1.21) .

### 1.19.1 Advantages of HAM

The HAM has following advantages,

1. It is valid even if a given non-linear problem does not contain any small/large parameters at all

2. It provides us with a convenient way to control the convergence of approximation series and adjust convergence regions where necessary

3. It can be employed to efficiently approximate a nonlinear problem by choosing different sets of base function.

# Chapter 2

# Heat and mass transfer in the Falkner-Skan wedge flow

## 2.1 Introduction

Baoheng Yao [34] consider the temperature distribution in the Falkner-Skan wedge flow. Hompotopy analysis method is applied to solve the governing nonlinear ordinary differential equation. Convergence of the solution is discussed the effect of the sundry parameters is also discussed in this chapter.

## 2.2 Mathematical description

Let us consider a two-dimensional steady and incopressible boundary layer flow passing a wedge, for a main stream with velocity $U$ varying as $x^k$ , the transformations

$$\eta = y\sqrt{\frac{U(k+1)}{2\nu x}}, \tag{2.1}$$

$$\psi(x,y) = \sqrt{\frac{2\nu U x}{(k+1)}} f(\eta) \tag{2.2}$$

These are the velocity components along and normal to the boundary layer. Now by using these velocity components in two dimensional steady and incompressible laminar flow, the governing

Navier Stock's equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

reduce to a famous nonlinear Falkner-Skan equation

$$f^{'''}(\eta) + f(\eta)f''(\eta) + \beta[1 - f'^2(\eta)] = 0 \tag{2.1}$$

with the boundary conditions

$$f(0) = 0, \qquad f^{'}(0) = 0, \qquad f^{'}(+\infty) = 1 \tag{2.4}$$

Where $\beta = \frac{2k}{(k+1)}$ is a parameter of the stream wise pressure gradient, $\Psi(x, y)$ is a stream function, $x$ and $y$ are coordinates along and normal to the boundary layer, and $\nu$ is the kinematic viscosity. To consider the related heat transfer problem, a non-dimensional temperature is define as

$$\theta = \frac{T_W - T}{T_W - T_\infty} \tag{2.5}$$

Where $T$ denotes the dimensional temperature, and $T_W$ and $T_\infty$ are constant temperature at the boundary and at infinity, respectively. Thus in the absence of frictional heat, $\theta(\eta)$ satisfies the second order differentials equation

$$\theta^{''}(\eta) + \Pr f(\eta)\theta^{'}(\eta) = 0 \tag{2.6}$$

with boundary conditions

$$\theta(0) = 0, \theta(+\infty) = 1 \tag{2.7}$$

where $Pr$ is the Prandtl number which is the ratio of the momentum diffusivity of the fluid and its thermal diffusivity. In order to solve Eq. (2.3) and (2.6) subject to the boundary conditions (2.4) and (2.7), homotopy analysis method is used.

## 2.3 Series solution by HAM

As Liao [19] has proved that, a series solution obtained by HAM must be one of the solutions the given nonlinear problem. Since HAM provides grid freedom to choose the appropriate set of base functions to approximate the solution of nonlinear problem. Since better choice of base functions helps to accelerate the convergence of the series solution. Similarly, an inappropriate choice of base functions result in divergence of the series solution. The chosen basis function also determine the choice of initial guess for HAM solution. Consider the boundary conditions in (2.4) and (2.7), it is appropriate to choose two sets of basis functions for $f$ and $\theta$ respectively

$$\{\eta^m e^{-2n\eta}, m, n \geq 0\}, \tag{2.8}$$

and

$$\{\eta^m e^{-n\eta}, m, n \geq 0\}, \tag{2.9}$$

to express the solutions of Eqs. (2.3) and (2.6) as

$$f(\eta) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_{m,n} \eta^m e^{-2n\eta}, \tag{2.10}$$

and

$$\theta(\eta) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} b_{m,n} \eta^m e^{-n\eta}, \tag{2.11}$$

where $a_{m,n}$ and $b_{m,n}$ are coefficients to be determined.

## 2.4 Zeroth-order deformation equation of HAM

Consider the boundary conditions in (2.4) and (2.7) and the solution expression in (2.10) and (2.11), the initial guess which satisfies the prescribed boundary conditions are chosen to be

$$f_0(\eta) = \eta - \frac{1 - e^{-2\eta}}{2}, \tag{2.12}$$

and

$$\theta_0(\eta) = 1 - e^{-\eta}, \tag{2.13}$$

for $f$ and $\theta$ respectively and correspondingly linear operators $\pounds_f$ and $\pounds_\theta$ are chosen as

$$\pounds_f = \frac{\partial^3}{\partial\eta^3} - 4\frac{\partial}{\partial\eta}, \tag{2.14}$$

and

$$\pounds_\theta = \frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}, \tag{2.15}$$

which satisfies the following conditions

$$\pounds_f[C_1e^{2\eta} + C_2e^{-2\eta} + C_3] = 0 \tag{2.16}$$

and

$$\pounds_\theta[D_1e^{-\eta} + D_2] = 0 \tag{2.17}$$

where $C_i(i = 1-3)$ and $D_j(j = 1,2)$ are integral constants to be determined by boundary conditions. Obviously, $C_1$ should be chosen to be zero because we seek a finite solution. The zeroth-order deformation problems are :

$$(1-q)\pounds_f[F(\eta;q) - f_0(\eta)] = q\hbar_1H_f(\eta)N_1[F(\eta,q)] \tag{2.18}$$

and

$$(1-q)\pounds_\theta[\theta(\eta;q) - \theta_0(\eta)] = q\hbar_2H_\theta(\eta)N_2[F(\eta,q),\theta(\eta,q)] \tag{2.19}$$

with the boundary conditions

$$F(0;q) = 0, \frac{\partial F(\eta;q)}{\partial\eta}\mid_{\eta=0}= 0, \frac{\partial F(\eta;q)}{\partial\eta}\mid_{\eta=+\infty}= 1, \theta(0;q) = 0, \theta(+\infty;q) = 1, \tag{2.20}$$

where

$$N_1[F(\eta,q)] = \frac{\partial^3F(\eta;q)}{\partial\eta^3} + F(\eta;q)\frac{\partial^2F(\eta;q)}{\partial\eta^2} + \beta[1 - (\frac{\partial F(\eta;q)}{\partial\eta})^2]$$

$$N_2[F(\eta,q),\theta(\eta,q)] = \frac{\partial^2\theta(\eta;q)}{\partial\eta^2} + \mathrm{Pr}\,.F(\eta;q)\frac{\partial\theta(\eta;q)}{\partial\eta}$$

and $q \in [0,1]$ is an embedding parameter, $\hbar_1$ and $\hbar_2$ are non-zero auxiliary parameter, $H_f(\eta)$ and $H_\theta(\eta)$ are non-zero auxiliary functions. Clearly, when $q = 0$, the solutions to Eqs. (2.18) and (2.19) are given by

$$F(\eta;0) = f_0(\eta), \ \theta(\eta;0) = \theta_0(\eta) \tag{2.21}$$

and when $q = 1$, because of $\hbar \neq 0$, $H_f(\eta) \neq 0$ and $H_\theta(\eta) \neq 0$ the solutions are equivalent to those of Eqs. (2.3) and (2.6) provided that the conditions

$$F(\eta;1) = f(\eta) \tag{2.22}$$

and

$$\theta(\eta;1) = \theta(\eta) \tag{2.23}$$

are satisfied. Therefore, as the embedding parameter $q$ increases from 0 to 1, $\phi(\eta;q)$ varies continuously from the initial guess solutions $f_0(\eta)$, $\theta_0(\eta)$ to the exact solutions $f(\eta)$ and $\theta(\eta)$ of Eq. (2.3), and so does $\theta(\eta;q)$ from the initial guess solution $\theta_0(\eta)$ to the exact solution $\theta(\eta)$ of Eq. (2.6). This process is so-called deformation in topology. With the aid of terms used in topology, Eqs. (2.18) and (2.19) are correspondingly called zeroth-order deformation equations of HAM. Therefore, $F(\eta;q)$ and $\theta(\eta;q)$ can be expanded in a Taylor series with respect to q as

$$F(\eta;q) = f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta) q^k \tag{2.24}$$

and

$$\theta(\eta;q) = \theta_0(\eta) + \sum_{k=1}^{+\infty} \theta_k(\eta) q^k \tag{2.25}$$

where

$$f_k(\eta) = \frac{1}{k!} \frac{\partial^k F(\eta;q)}{\partial q^k} \Big|_{q=0} \tag{2.26}$$

and

$$\theta_k(\eta) = \frac{1}{k!} \frac{\partial^k \Theta(\eta;q)}{\partial q^k} \Big|_{q=0} \tag{2.27}$$

Obviously, the convergence region of the series (2.24) and (2.25) depends upon the choice of auxiliary linear operators $\mathcal{L}_f$ and $\mathcal{L}_\theta$, the auxiliary parameter $\hbar_1$ & $\hbar_2$, and the two non-zero auxiliary functions $H_f(\eta)$ and $H_\theta(\eta)$. If all of them are properly chosen so that convergences of the above two series solutions at $q = 1$ are guaranteed, the series solutions Eqs. (3) and (6) can be expressed as follows

$$F(\eta; q) = f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta) \tag{2.28}$$

$$\theta(\eta; q) = \theta_0(\eta) + \sum_{k=1}^{+\infty} \theta_k(\eta) \tag{2.29}$$

## 2.5 $k$th-order deformation equation of HAM

For convenience, define the vectors

$$\vec{f}_k = \{f_0(\eta), f_1(\eta), f_2(\eta), ..., f_k(\eta)\}, \tag{2.30}$$

$$\vec{\theta_k} = \{\theta_0(\eta), \theta_1(\eta), \theta_2(\eta), ..., \theta_k(\eta)\}, \tag{2.31}$$

Where $k \in \mathbb{N}$. Differentiating the zero-order deformation Eqs. (2.18) and (2.19) $k$ times with respect to $q$, setting $q = 0$ and then dividing by $k!$, the $k$-th order deformation equation is obtained as

$$\pounds_f[f_k(\eta) - \Psi_k f_{k-1}(\eta)] = \hbar_1 H_f(\eta) R_k^f(\vec{f}_{k-1}, \eta), \tag{2.2}$$

$$\pounds_\theta[\theta_k(\eta) - \Psi_k \theta_{k-1}(\eta)] = \hbar_2 H_\theta(\eta) R_K^\theta(\vec{f_{k-1}}, \vec{\theta_{k-1}}, \eta), \tag{2.3}$$

with the boundary conditions

$$f_k(0) = 0,\ f_k'(0) = 0,\ f_k'(+\infty) = 0, \tag{2.4}$$

$$\theta_k(0) = 0,\ \theta_k(+\infty) = 0, \tag{2.5}$$

where

$$\Psi_k = \begin{cases} 0, k \leq 1 \\ 1, k > 1 \end{cases}, \tag{2.6}$$

$$R_k^f(f_{k-1}, \eta) = f_{k-1}'''(\eta) + \sum_{j=0}^{k-1} \left[f_j(\eta) f_{k-1-j}''(\eta) - \beta f_j'(\eta) f_{k-1-j}'(\eta)\right] + \beta(1 - \Psi_k), \tag{2.36}$$

$$R_k^{\theta}(f_{k-1}, \theta_{k-1,}\eta) = \theta''_{k-1}(\eta) + \Pr \sum_{j=0}^{k-1} f_j(\eta)\theta'_{k-1-j}(\eta). \tag{2.37}$$

Let $f_k^*(\eta)$ and $\theta_k^*(\eta)$ denotes special solution of equations

$$\pounds_f \left[f_k^*(\eta)\right] = \hbar_1 H_f(\eta) R_k^f(f_{k-1}, \eta), \tag{2.38}$$

$$\pounds_\theta \left[\theta_k^*(\eta)\right] = \hbar_2 H_\theta(\eta) R_K^{\theta}(f_{k-1}, \theta_{k-1,}, \eta) \tag{2.39}$$

and then according to the properties of linear operators in Eqs.(2.16) and (2.17), we seek the solutions to Eqs.(2.32) and (2.33) in the following form

$$f_k(\eta) = \Psi_k f_{k-1}(\eta) + f_k^*(\eta) + C_1^K e^{2\eta} + C_2^K e^{-2\eta} + C_3^K \tag{2.40}$$

$$\theta_k(\eta) = \Psi_k \theta_{k-1}(\eta) + \theta_k^*(\eta) + D_1^K e^{-\eta} + D_2^K \tag{2.41}$$

where $C_1^K$, $C_2^K$, and $C_3^K$ are coefficients to be determined by the boundary conditions in (2.34), i.e.

$$C_1^K = 0,$$

$$C_2^K = \frac{f_k^{*\prime}(0)}{2},$$

$$C_3^K = -f_k^*(0) - \frac{f_k^{*\prime}(0)}{2},$$

$$D_1^K = -\theta_k^{*\prime}(0),$$

$$D_2^K = 0 \tag{2.42}$$

Furthermore, according to the rules of the solution expression and coefficient ergodicity by the HAM and as HAM provides great freedom to choose the auxiliary function the proper choice of auxiliary function the appropriate auxiliary function help to accelerate the convergence of the series solution. It is observed that the auxiliary functions $H_f(\eta)$ and $H_\theta(\eta)$ are uniquely determined by

$$H_f(\eta) = 1, \ H_\theta(\eta) = e^{-\eta} \tag{2.43}$$

Thus, the non-linear Eqs.(2.3) and (2.6) are converted into a series of linear boundary value problems as Eqs.(2.32) and (2.33),which can be easily solved by symbolic computation software such as Mathematica and Maple.

## 2.6 Results and analysis

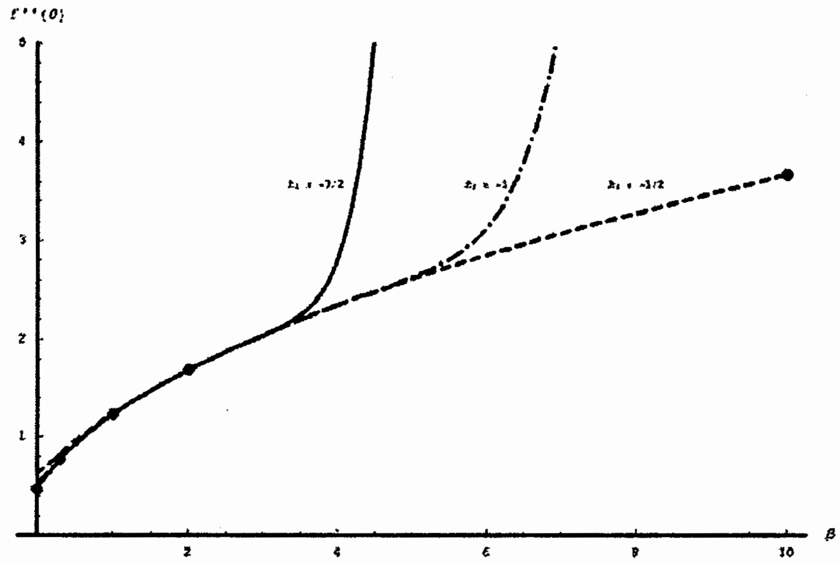


Fig. 2.1: The 10th-order HAM solution of $f''(0)$ with different $\hbar_1$ verses $\beta = 2$ (solid line $\hbar_1 = -3/2$; dotted line $\hbar_1 = -1$; dashed line $\hbar_1 = -1/2$). The dots represent the numerical results of White [35].

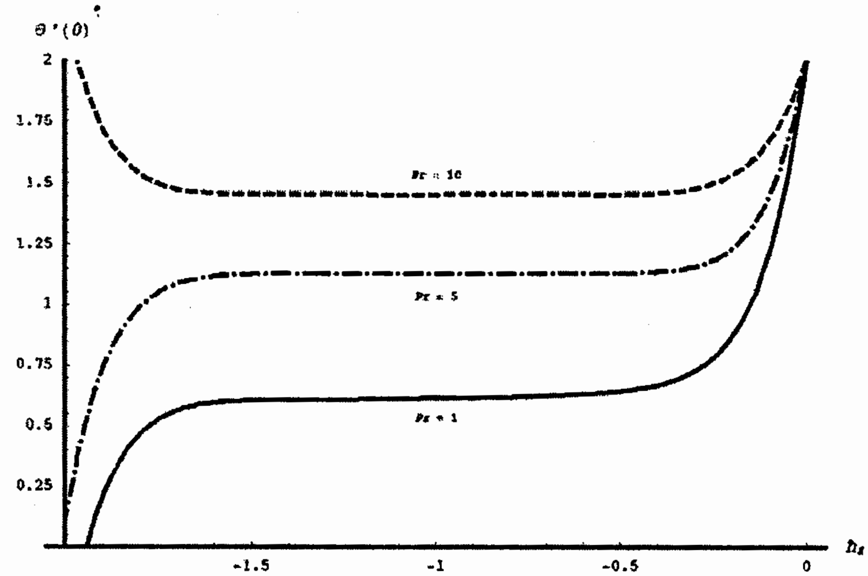


Fig.2.2: The 10th order HAM solution of $\theta'(0)$ as a function of $\hbar_2$ with $\beta = 2$ and $\hbar_1 = -1$ for $Pr = 1, 5,$and 10.

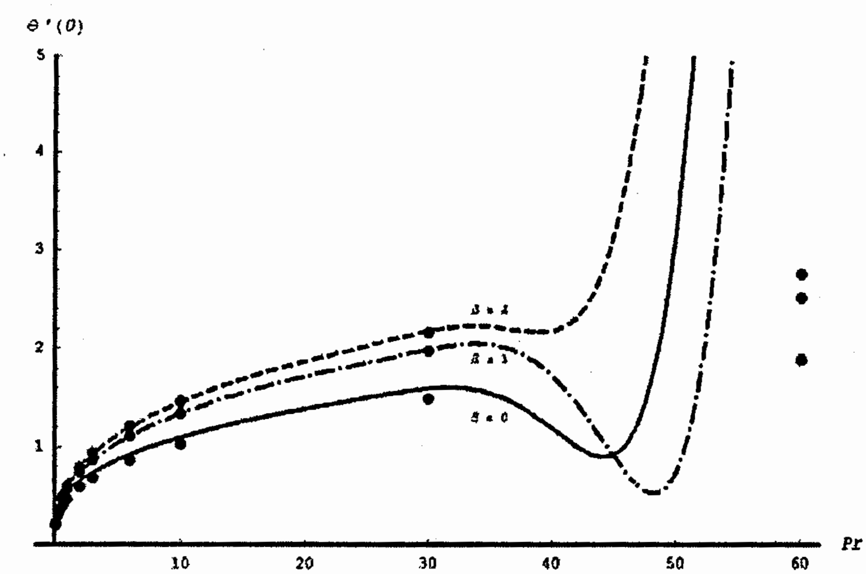


Fig. 2.3: The 10th-order HAM solution of $\theta'(0)$ verses Pr $\beta = 0, 1, 2$ when $\hbar_1 = -1$ and $\hbar_2 = -1/2$. The dots represent the numerical results of White [35].

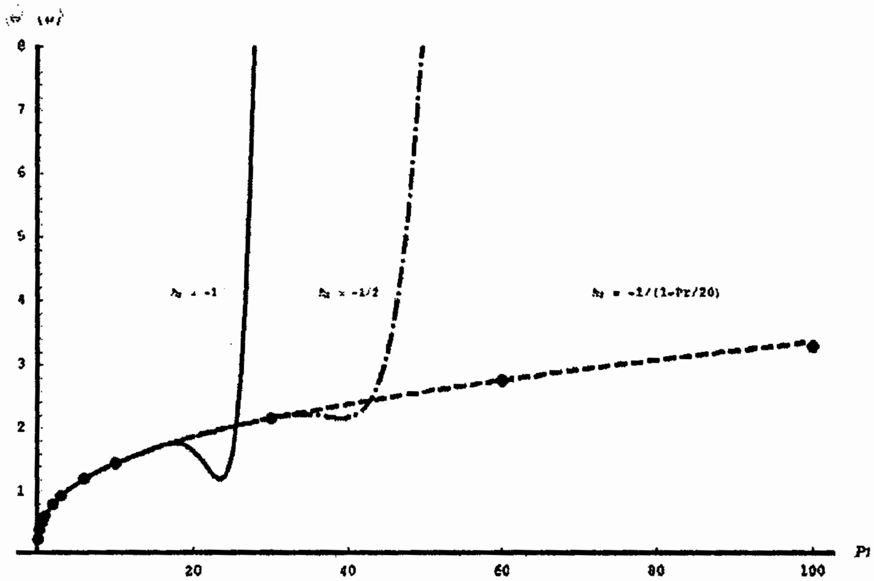


Fig. 2.4: The 10th-order HAM solution $\theta'(0)$ with different $\hbar_2$ verses Pr (solid line $\hbar_2 = -1$; dotted line $\hbar_2 = -1/2$;dashed line $\hbar_2 = -1/(1 + \mathrm{Pr}/20)$), when $\beta = 2$ $\hbar_1 = 1$. The dots represent the numerical results of White [35].

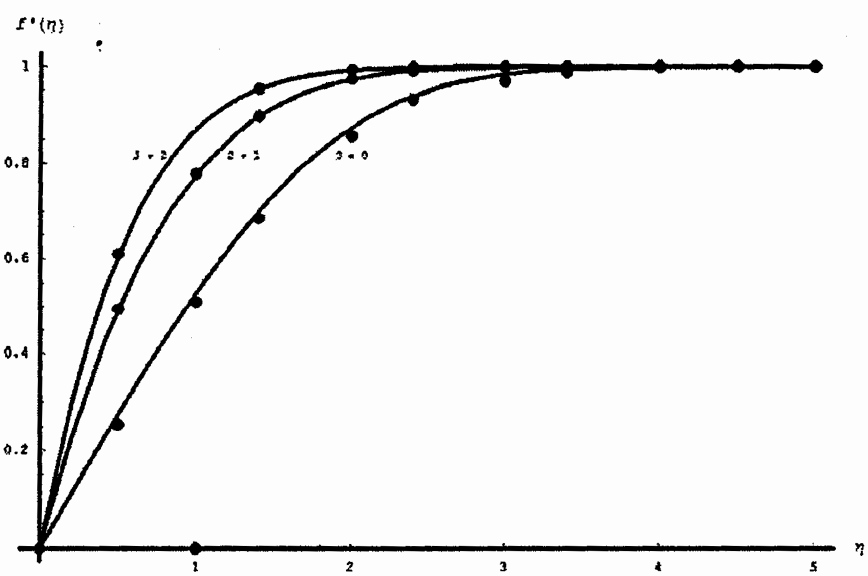


Fig. 2.5: Comparison of the 15th order HAM solution of $f'(\eta)$ verses $\eta$ with the numerical results by White [35] for $\beta = 0, 1$ and 2.The dots represents the numerical results of White [35].

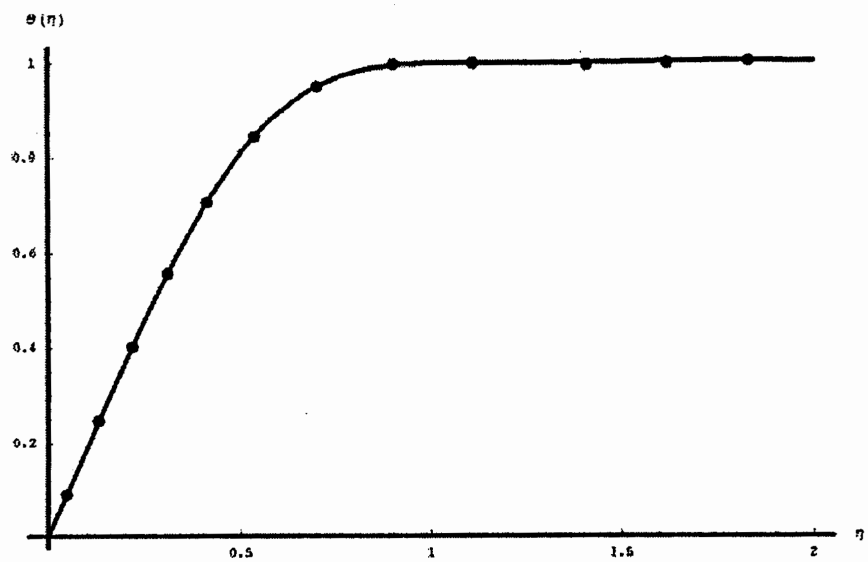


Fig. 2.6: The 15th order analytical solution of $\theta(\eta)$ for the heat analysis of the Falkner-Skan wedge flow with $\beta = 2$, Pr $= 20$ and $\hbar_1 = -1$, $\hbar_2 = -1/2$.The solid line represents the 30th order HAM approximate solution and dots represent 10th order HAM solution.

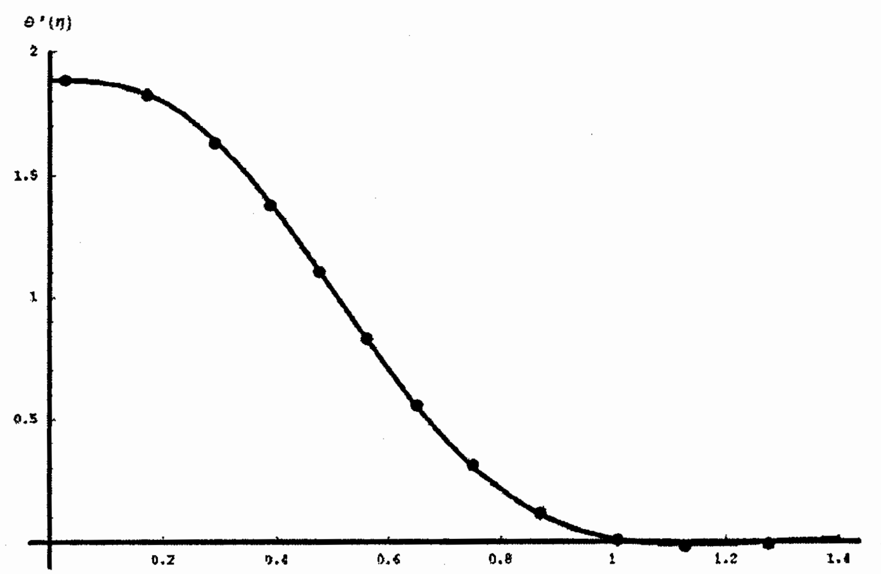


Fig. 2.7: The 15th order analytical solution of $\theta'(\eta)$ as a function of $\eta$ for the heat analysis of the Falkner-Skan wedge flow with $\beta = 2$, Pr $= 20$ *and* $\hbar_1 = -1$, $\hbar_2 = -1/2$. The solid line represents the 15th order HAM solution,and dots represent 10th order HAM solution.

## 2.7 Results and Discussion

In order to assure the convergence of the series solution of the given nonlinear boundary value problem for all required range of the parameters, we need to find the most appropriate value of $\hbar_1$ for the solution $f(\eta)$ particularly. For this purpose Fig. 2.1 is drawn which represents the graph of 10th order HAM solution of $f''(0)$ against $\beta$ for different values of $\hbar_1 = -3/2, -1$ and $-1/2$. It is observed that the values of $f''(0)$ for $\hbar_1 = -3/2$ coincide with the numerical values of White [35] only in the range $\beta < 3$ and for all larger values it deviates from the numerical results. However the values of $f''(0)$ for $\hbar_1 = -1$ coincide with the numerical data upto the range of $\beta < 5$,which clearly indicate that the choice of $\hbar_1 = -1$ is comparatively better than the $\hbar_1 = -3/2$. Most interestingly, It is seen that the values of $f''(0)$ matches with the numerical values upto the range of $\beta = 10$ with $\hbar_1 = -1/2$. In fig. 2.2, $\hbar_2-$curve is drawn for different values of the parameters $Pr = 1, 5$ and 10 of 10th order approximate solution $\theta'(0)$ with $\beta = 2$ with $\hbar_1 = -1$. As pointed out by Liao [19], the region of $\hbar_2$ for which the curve $\theta'(0)$ verses $\hbar_2$ is horizontal is valid region for the auxiliary parameter $\hbar_2$.

It can easily be seen that for all values of Pr $= 1, 5$ and 10, the value of $\hbar_2$ can be used for the interval of $[-1.5, -0.5]$. However, for larger values of Pr, it is observed that the choice of $\hbar_2 = -1/2$ is not appropriate as shown in Fig. 2.3. It is observed that the value of $\theta'(0)$ coincide with the given numerical result for upto Pr $< 30$ for $\beta = 0, 1$ and 2 with the choice of $\hbar_2 = -1/2$. Fig. 2.4 represents the 10th order approximate solution $\theta'(0)$ with different values of $\hbar_2 = -1$, $\hbar_2 = -1/2$ and $\hbar_2 = 1/(1 + \mathrm{Pr}/20)$ against $Pr$. It is observed that with $\hbar_2 = -1$, the value of $\theta'(0)$ coincide with the numerical values only upto Pr $< 20$, the choice of $\hbar_2 = -1/2$ can enhance the convergence of solution upto Pr $< 30$. Furthermore it is observed that the choice of $\hbar_2 = 1/(1 + \mathrm{Pr}/20)$ is most appropriate and can be helpful for the assurance of the convergence for upto very larger value of Pr $= 100$,as with this choice of $\hbar_2$, values of $\theta'(0)$ coincide with the numerical values only upto $0 < \mathrm{Pr} \leq 100$.

A comparison of the 15th order HAM solution $f'(\eta)$ verses $\eta$ with the numerical results by White is shown in Fig. 2.5. It is observed that the 15th order HAM solution shows good agreement with the numerical solution of White for $\beta = 0, 1$ and 2. By increasing the order of approximate solution, accuracy of any order can be achieved. Moreover, comparison of the 15th order HAM solution $\theta(\eta)$ with 10th order HAM solution for $\hbar_2 = -1/2$ is shown in Fig. 2.6

with $\beta = 2$, $Pr = 20$, $\hbar_2 = -1$and $\hbar_2 = -1/2$. Comparison of 15th and 10th order approximate analytical solutions $\theta'(\eta)$ for the heat transfer analysis of the Falkner-Skan wedge flow with $\beta = 2$, Pr $= 20$ and $\hbar = -\frac{1}{2}$ is shown in Fig. 2.7. This further shows the convergence of the series solution to the heat transfer of the Falkner-Skan wedge flow according to the homotopy analysis method.

## 2.8 Conclusions

An efficient homotopy analysis method, has been applied to investigate the temperature field associated with the Falkner-Skan boundary-layer problem and a series solution is presented in this chapter. Interestingly, the choice of $\hbar$ provides an easy way to extend the convergent region of the series solution to heat transfer analysis of the Falkner-Skan wedge flow. The results agree well with those of the numerical method by White. The results of the present work are effective for a very large range of Prandtl numbers ($0 < \mathrm{Pr} \leq 100$), which shows the validity of the series solution.

# Chapter 3

# Heat and mass transfer in the Falkner-Skan wedge flow with radiation

In this chapter heat and mass transfer in the Falkner-Sken wedge flow with radiation is considered. The governing ordinary differential equation is solved by rapid convergent homotopy analysis method. Convergence of the solution is properly analyzed. The effects of the pertinent parameters are shown and discussed through graphs in this chapter.

## 3.1 Mathematical description

The boundary layer equation of the mass concentration is given by

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=D\frac{\partial^2 c}{\partial y^2}, \tag{3.1}$$

where

$$c=c_w\Phi(\eta), \tag{3.2}$$

with the boundary conditions

$$\begin{aligned} c &= c_w, \text{at } y = 0, \\ c &= c_\infty, \text{ as } y \longrightarrow \infty. \end{aligned} \tag{3.3}$$

Now

$$\frac{\partial c}{\partial x} = c_w \Phi'(\eta) \left[ -\frac{y}{2x^{\frac{3}{2}}} \sqrt{\frac{U(k+1)}{2\nu}} \right], \tag{3.4}$$

$$\frac{\partial c}{\partial y} = c_w \Phi'(\eta) \sqrt{\frac{U(k+1)}{2\nu x}}, \tag{3.5}$$

$$\frac{\partial^2 c}{\partial y^2} = c_w \Phi''(\eta) \left[ \frac{U(k+1)}{2\nu x} \right], \tag{3.6}$$

Now replacing the above results and after simplifying

$$\Phi''(\eta) + S_c f(\eta) \Phi'(\eta) = 0 \tag{3.7}$$

where

$$\Phi = \frac{c - c_\infty}{c_w - c_\infty} \text{ and } S_c = \frac{\nu}{D(k+1)} \tag{3.8}$$

where $S_c$ is the Schmidt number. The boundary conditions become

$$\begin{aligned} \Phi(\eta) &= 1 \rightarrow \eta = 0, \\ \Phi(\eta) &= 0 \rightarrow \eta = \infty. \end{aligned} \tag{3.9}$$

To consider the related heat transfer problem, a non-dimensional temperature is define as

$$\theta = \frac{T_W - T}{T_W - T_\infty} \tag{3.10}$$

Where $T$ denotes the dimensional temperature, and $T_W$ and $T_\infty$ are constant temperature at the boundary and at infinity, respectively. Thus in the presence of radiation, $\theta(\eta)$ satisfies the second order differential equation

$$(1+\frac{3}{4}R_d)\theta''(\eta)+\Pr . f(\eta)\theta'(\eta)=0, \tag{3.11}$$

with boundary conditions given in Eq. (2.7). Here $R_d$ is radiation number. To solve the system of differential equation (2.3,3.8) and (3.12) subject to the boundary conditions (2.4), (3.10) and (2.7) respectively, homotopy analysis method is used. The procedure for the solution of the Eq. (2.3) subject to the boundary condition (2.4) is same as given in chapter 2. However, for the solution of Eq. (3.12), homotopy is applied as in the next section.

## 3.2 Series solution by HAM

### 3.2.1 Zeroth-order deformation equation of HAM

Consider the boundary conditions in Eq. (3.10) and the solution expression in (2.11) for the Eq. (3.8), the initial guess which satisfies the prescribed boundary conditions are chosen to be

$$\phi_0(\eta)=e^{-\eta}, \tag{3.12}$$

for $\phi$ corresponding linear operator $\pounds_\phi$ is defined as

$$\pounds_\phi=\frac{\partial^2}{\partial\eta^2}+\frac{\partial}{\partial\eta}, \tag{3.13}$$

which satisfies the following condition

$$\pounds_\phi[E_1e^{-\eta}+E_2]=0, \tag{3.14}$$

where $E_j(j=1,2)$ are constants of integration to be determined by boundary conditions. Obviously, to have finite solution, $E_2$ must be chosen to be zero. Zeroth order deformation equations for HAM is constructed as

$$(1-q)\pounds_\phi[\Phi(\eta;q)-\phi_0(\eta)]=q\hbar_3H_\phi(\eta)N_3[\Phi(\eta;q),F(\eta,q)] \tag{3.15}$$

with the boundary conditions

$$\Phi(0;q) = 1 \text{ and } \Phi(+\infty;q) = 0. \tag{3.16}$$

where

$$N_3[F(\eta,q),\Phi(\eta,q)] = \frac{\partial^2\Phi(\eta;q)}{\partial\eta^2} + ScF(\eta;q)\frac{\partial\Phi(\eta;q)}{\partial\eta}$$

and $q \in [0,1]$ is embedding parameter, $\hbar_3$ is non-zero auxiliary parameter, $H_\phi(\eta)$ is non-zero auxiliary functions. Clearly, when $q$ varies from 0 to 1, $\phi(\eta;q)$ varies continuously from the initial guess solution to the final approximate solution, i.e.

$$\Phi(\eta;0) = \phi_0(\eta), \;\; \Phi(\eta;1) = \phi(\eta). \tag{3.17}$$

Where $\Phi(\eta;q)$ can be written with the help of Taylor series as

$$\Phi(\eta;q) = \phi_0(\eta) + \sum_{k=1}^{+\infty} \phi_k(\eta)q^k \tag{3.18}$$

where

$$\phi_k(\eta) = \frac{1}{k!}\frac{\partial^k\Phi(\eta;q)}{\partial q^k}\,|_{q=0}\,. \tag{3.19}$$

Obviously, the convergence region of the series (3.18) depends upon the choice of auxiliary linear operators $\mathcal{L}_\phi$, the auxiliary parameter $\hbar_1$ & $\hbar_3$, and the auxiliary functions $H_\phi(\eta)$. If all of them are properly chosen in such a way so that convergences of the above two series at $q = 1$ are guaranteed, then due to Eq. (3.17), we get

$$\Phi(\eta;q) = \phi_0(\eta) + \sum_{k=1}^{+\infty} \phi_k(\eta) \tag{3.20}$$

## 3.3 kth-order deformation equation of HAM

For convenience, define the vectors

$$\phi_k = \{\phi_0(\eta), \phi_1(\eta), \phi_2(\eta), ..., \phi_k(\eta)\}\,, \tag{3.21}$$

Where $k \in \mathbb{N}$. Differentiating the zero-order deformation Eqs. (3.15) with respect to the embedding parameter $q$, $k$ times and setting $q = 0$ and then dividing by $k!$, the $k$th order deformation equation is obtained as

$$\mathcal{L}_\phi[\phi_k(\eta) - \Psi_k\phi_{k-1}(\eta)] = \hbar_3 H_\phi(\eta) R_k^\phi(\overrightarrow{\phi_{k-1}, f_{k-1}}, \eta), \tag{3.22}$$

with the boundary conditions

$$\phi_k(0) = 0,\ \phi_k(+\infty) = 0, \tag{3.23}$$

where $\Psi_k$ is defined in Eq. (2.6).

$$R_k^\phi(\phi_{k-1}, f_{k-1}, \eta) = \phi_{k-1}''(\eta) + Sc\sum_{j=0}^{k-1} f_j(\eta)\phi_{k-1-j}'(\eta). \tag{3.24}$$

Let $\phi_k^*(\eta)$ denotes a special solution of the equation defined as

$$\mathcal{L}_\phi\,[\phi_k^*(\eta)] = \hbar_3 H_\phi(\eta) R_K^\phi(f_{k-1}, \phi_{k-1,}, \eta) \tag{3.25}$$

and then according to the properties of linear operators in Eqs.(3.14), we seek the solutions to Eqs.(3.7) in the following form

$$\phi_k(\eta) = \Psi_k\phi_{k-1}(\eta) + \phi_k^*(\eta) + E_1^K e^{-\eta} + E_2^K \tag{3.26}$$

where $E_1^K$ and $E_2^K$ are coefficients to be determined by the boundary conditions in (3.23), i.e.

$$E_1^K = -\theta_k^{*'}(0) \text{ and } E_2^K = 0. \tag{3.27}$$

It is observed that the auxiliary functions $H_\phi(\eta)$ can conveniently be determined by

$$H_\phi(\eta) = e^{-\eta} \tag{3.28}$$

Thus, the nonlinear system of Eqs. (2.1, 3.7) and (3.11) are converted into a series of linear boundary value problems as Eqs.(2.2, 2.3) and (3.25),which are solved very easily by using symbolic computation software Mathematica.

## 3.4 Results and discussions

Since the solution obtained by the Homotopy analysis method is infinite series, it is to ensure first the convergence of the solution. For this purpose we have drawn an $\hbar$−curve in Fig. 3.1. Since homotopy analysis methods provide us great freedom to choose the initial guess, auxiliary linear operator and auxiliary functions. But this should all be chosen in such a way so that the series solution obtained by HAM is convergent. As pointed out by Liao in his book, beyond perturbation [19], if the solution is convergent, then it must be one of the solution of the given boundary value problem. The suitable value of the auxiliary parameter also responsible for the convergence of the solution. Suitable range of the auxiliary parameter $\hbar$ is such that for which $\hbar$−curve become parallel to the $\hbar$−axis. It is observed from Fig. 3.1 that suitable range for $\hbar_3$, which is auxiliary parameter associated with the dimensionless $\phi$, is the interval $-1.75 < \hbar_3 < -0.25$ for $Sc = 1$ and 2. For each value of $\hbar_3$ lie in this interval assured the convergence of the series solution. Since dimensionless concentration field changes with the change of Schmidt number $Sc$ and $\beta$, for each value of the prescribed parameters, suitable value of $\hbar_3$ must be chosen accordingly to ensure the convergence of solution.

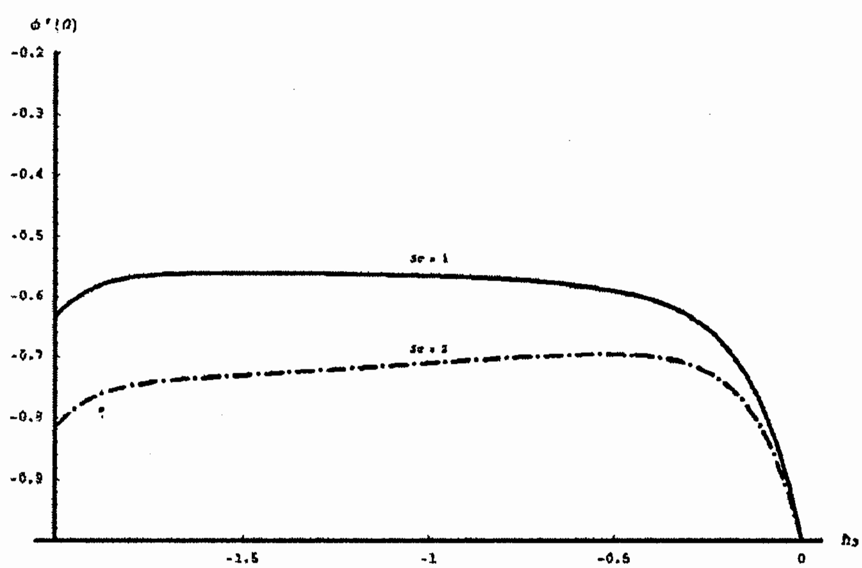


Fig.3.1: Graph showing 15th order HAM solution of $\phi'(0)$ as a function of $\hbar_1$ and $\hbar_3$with $\beta = 2$ and $\hbar_1 = -1$ for $Sc = 1$ and 2.

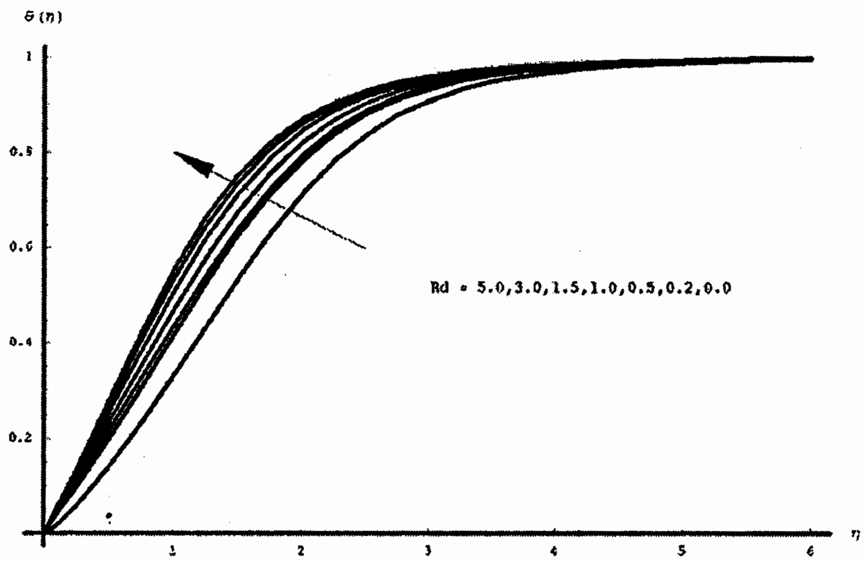


Fig. 3.4: Graph showing 15th order HAM solution $\theta(\eta)$ verses $\eta$ for different values of $Rd$ when $\hbar_3 = -1/2$, $\hbar_2 = -1/(1 + \mathrm{Pr}/20)$, $\beta = 2$ and $\mathrm{Pr} = 0.7$.

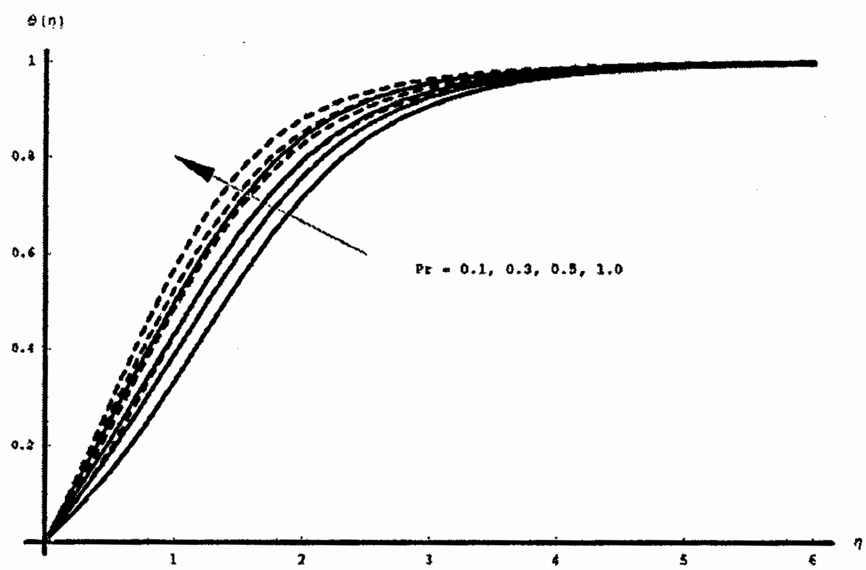


Fig.3.5: Graph showing 15th order HAM solution $\theta(\eta)$ verses $\eta$ for different values of Pr when $\hbar_1 = -1$, $\hbar_2 = -1/(1 + \mathrm{Pr}/20)$, $\beta = 2$,$\mathrm{Pr} = 1/2$, where dashed line is for $Rd = 0$ and solid line is for $Rd = 1$.

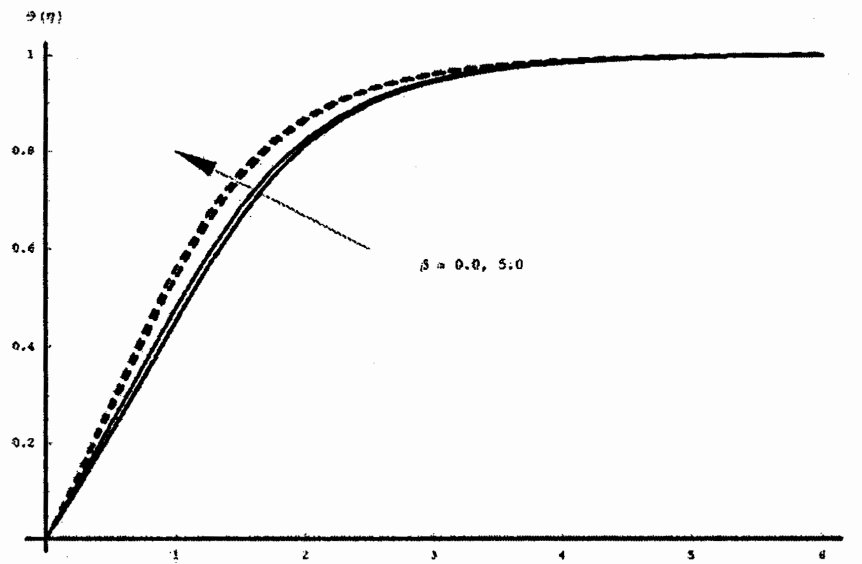


Fig.3.6: Graph showing 15th order HAM solution $\theta(\eta)$ verses $\eta$ for different values of $\beta$ when $\hbar_1 = -1/2$, $\hbar_2 = -1/(1 + \mathrm{Pr}/20)$ ,Pr $= 0.7$, $Sr = 1$.Where dashed line is for $Rd = 0$ and solid line is for $Rd = 1$.

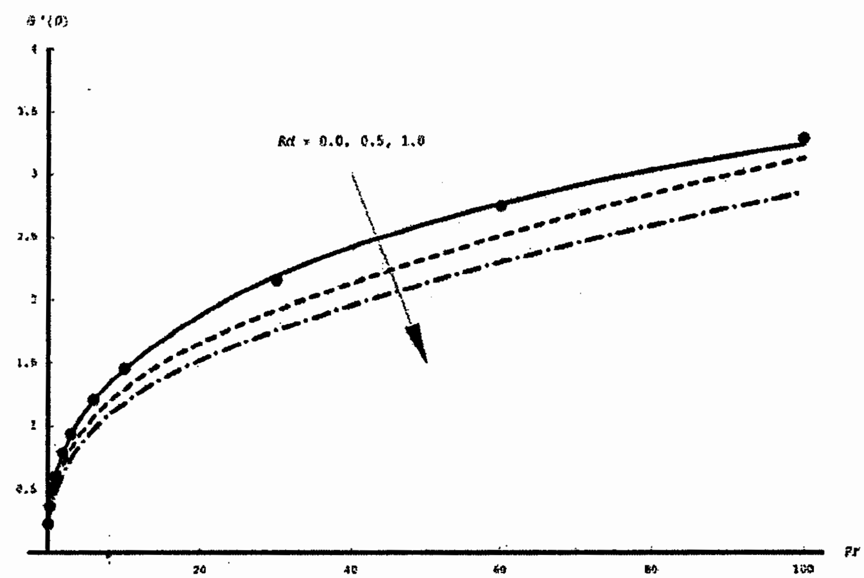


Fig. 3.7: Graph showing 15th order HAM solution $\theta'(0)$ verses $Pr$ for different values of $Rd$ when $\hbar_1 = -1$ and $\beta = 2$.Where dots represent the numerical values of White [35].

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