

Fixed Point Approximations using Iterative Procedures in Generalized Metric Spaces



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**Department of Mathematics & Statistics
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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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DEDICATION

To my parents,
for their unconditional love, endless patience, and constant support throughout every step of my life.

To my brothers,
Mati Ur Rahman and Saeed Ur Rahman,
for always standing beside me with steadfast encouragement and brotherly love.

To my best student and friend,
Qasim Jan,
whose encouragement inspired me to seek admission at IIUI—a journey that ultimately culminated in this achievement. I am deeply grateful for his belief in me, his support, and his timely motivation during crucial moments. Thank you, Qasim, for being a part of this journey.

And to my sweet niece,
Noora Saeed,
whose innocent smile brings endless joy to my heart.

To all of them,
I lovingly dedicate this work.

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“With humble hearts, we bow before **Almighty Allah** for His countless blessings.”

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Latif Ur Rahman

June 8, 2025

Preface

Fixed point theory is a magnificent blending of analysis, geometry and topology. It reckons with fixed point existence of a mapping Ψ on a set $\mathcal{X}$, i.e., the functional equation that depends on the operator may admits a certain fixed point solution in the form $x = \Psi x$. The basis of fixed point theory was laid down in the 20th century by celebrated Polish mathematician Stefan Banach. The maps satisfying various fixed point contractive conditions on the entire domain play an important role and have a wide range of applications in functional equations, differential equations and integral equations, which are used in parametrize estimation problems, recurrent network convergence, adaptive control systems, nonlinear and fractal image decoding, and computing magneto-metric fields in a nonlinear medium. Numerous mathematicians have generalized fixed point theory in different ways. We direct readers to the books for more information to [87] and [79].

Nevertheless, suppose the presence of a fixed point is assured through the use of complete or compact space, contraction, or non-expansive mappings. In that case, then determining the value of that fixed point is a difficult problem, which is why we compute them using iterative procedures.

It would take a lot of work to discuss all of the numerous iterative procedures that have been developed over time. The most useful fixed point result which is known in the literature as a Banach contraction theorem point out the Picard iteration technique to approximate the fixed point(Throughout this thesis, we will refer to the "iteration process" by I.P.). Other well-known iterative processes include: Ishikawa [63], PicardMann [90], Mann [18], Agarwal [57], Noor [19], SP [101], S^* [85], CR [81], Normal-S [94], Abbas [20], Picard-S [25], Thakur New [69], Vatan, Step two [86] and so forth. We recommend reading the book [80] to thoroughly describe iteration procedures. An I.P must possess attributes like

”fastness” and ”stability” to be chosen over another I.P Rhoades said that for decreasing function, the Mann I.P converges more quickly than the Ishikawa I.P, but Ishikawa claimed that I.P is superior to the Mann I.P in [69]. Remember that Mann I.P. is independent of the initial estimate (see [102] for details). The Agarwal I.P. converges quicker than the Mann I.P. for contraction maps, according to Agarwal et al.’s claim in [57], and at the same pace as the Picard I.P.

According to Abbas et al., Abbas I.P converges faster than Agarwal I.P in [20]. For quasi-contractive operators in Banach spaces, Chugh et al. shown in [81] that C.R. I.P. is faster and equivalent to Picard which is basic iteration, Mann which contains one set of parameter, Ishikawa which contains two set of parameters, Agarwal which has similar structure as the Ishikawa iteration, Noor which is known as first three-step fixed point scheme, and S.P. iterative processes. Additionally, Karakaya et al. demonstrated in [92] that C.R. I.P. converges more quickly than S^* I.P. for the class of contraction maps. In [25], Gursoy and Karakaya reported that for contraction maps, Picard-S I.P converges more quickly than all of Picard, Mann, Ishikawa, Noor, SP, CR, Agarwal, S^* , Abbas, and Normal-S. In [69], Thakur and colleagues demonstrated using

Numerical proof shows that Thakur New I.P. converges more quickly than Picard iteration, the iteration due to Mann, a basic two-step Ishikawa fixed point scheme, Agarwal two-step iteration, three-step basic Noor iteration, and Abbas iteration three-step iteration for the class of Suzuki nonlinear nonexpansive maps. Similarly, Karakaya et al. demonstrated in [86] that the Vatan Two-step I.P. is quicker for weak contraction maps than the Picard-S, CR, S.P., and Picard-Mann iteration processes.

In this dissertation, we present several new iteration processes in the context of $CAT(0)$ and Banach spaces, and we then demonstrate how much faster our processes are. Our new method is based on the class of mappings with the (KSC) condition and the M^* -iterative scheme. First, we use a M^* -iterative approach to obtain various Δ and strong convergence theorems. In the broad scenario of

CAT(0) spaces, we show convergence findings on the F iteration technique for generalized α -nonexpansive mappings.

For the sequence created by some of the provided modified iteration schemes, we aimed to establish a different types of results associated with weak, Δ , and strong convergence in nonlinear settings with supportive and basic examples by considering large class of nonlinear and nonexpansive maps. We employ Banach space and CAT(0) as our underlying spaces. Furthermore, we present a brand-new idea in Banach spaces: extended φ -Suzuki nonexpansive mappings. This concept provides an effective way to approximate common fixed points.

This dissertation consists of five chapters.

Chapter 1 presents basic definitions, a thorough study of CAT(0) spaces and a concise overview of current iteration procedures, along with several key ideas and findings employed throughout this thesis.

Chapter 2 Within the context of CAT(0) spaces, we provide a few iterative constructions of fixed points that demonstrate and discuss an application of these outcomes to solve a wide range equation relation to fractional derivative.

Our new method is the modification of M^* scheme and is connected to nonlinear maps that has a property called in the literature as a (KSC) condition. First, we use a M^* -iterative approach to obtain various Δ and strong convergence theorems. The concept in fixed point and our outcomes of this chapter are combined and a FDE is solved as an application. Finally, it has been shown by supportive and basic example which clearly shows the wideness of our results. Eventually, M^* iteration is seen as a highly accurate numerical method relating and comparing with some known and faster methods of the current literature in approximation methods of fixed points. Our findings are novel and extend the scope of several related findings in fixed point theory and related fields.

Chapter 3 This chapter describes the successful application of an efficient iterative approximation approach to discover fixed points in the nonlinear setting of CAT(0) spaces for a general class of operators. Finding solutions through the

suggestion of fixed-point processes for issues of various kinds (such as differential and integral) and functional problems is one of the more challenging tasks. This is particularly true when examining approximation techniques when the domain of definition lacks linearity.

Several examples are used to demonstrate the results. There are also some numerical calculations offered. Finally, we demonstrate that our novel findings can be used to resolve split feasibility issues.

Chapter 4 We provide convergence findings on the F iteration method for generalized α -nonexpansive mappings in the general case of CAT(0) spaces. To bolster our findings, we provide examples and numerical data. One implementation of our primary findings is also given. The results complement the comparable fixed point iteration findings in the present literature and are new in the literature.

Chapter 5 As an improvement over earlier research, we provide a novel idea called the A^* -iterative scheme, which broadens our understanding of convergence rate analysis.

Furthermore, we present a brand-new idea in the category of Banach spaces extended φ -Suzuki nonexpansive mappings. This concept provides an effective way to approximate common fixed points.

Our theoretical analysis's dependability is demonstrated by numerical examples and its application to delay nonlinear fractional differential equations and basins of attraction. Our study builds on the foundation established by earlier research in this area by offering a complete and comprehensive analysis of this expanded iterative strategy.

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Chapter 1

Basic Concepts

Several fundamental ideas and terms that are utilized throughout the dissertation are covered in this chapter. To keep the chapter within a manageable length, several known findings that are currently in existence are offered without evidence.

A few definitions, theorems, assertions, and lemmas that are being reviewed in Section 1.1 will be utilized in the upcoming chapters. We present CAT(0) space and a few of its fundamental characteristics that are necessary for fixed points in mappings to exist, in Section 1.2. A brief introduction and history of various iteration systems are provided in Section 1.3.

1.1 Preliminaries

Definition 1.1.1. [82] *A uniform convex Banach space namely $\mathcal{X}$ is a complete norm space having a property that $\|\frac{u+v}{2}\| \leq 1 - \zeta$ where $\zeta > 0$ whenever $\|u\|, \|v\| \leq 1$ and $\|v - u\| > \xi$ for some $0 < \xi \leq 2$.*

Definition 1.1.2. [3] *A Banach space $\mathcal{X}$ equipped with Opial condition will have the property that a sequence $\{v_i\}$ which converge weakly to a weak limit v_0 in $\mathcal{X}$ in such way*

$$\lim_{i \rightarrow \infty} \sup \|v_i - v_0\| \leq \lim_{i \rightarrow \infty} \sup \|v_i - a\|$$

holds, for every $a \in \mathcal{X}$ with $v_0 \neq a$.

The details of Banach's contraction principle (BCP) which guarantees the existence

and as well as the uniqueness of contraction mappings in complete metric space, are as follows.

Theorem 1.1.1. [13] Suppose a selfmapping defined as $\Psi : \mathcal{X} \rightarrow \mathcal{X}$ whose domain is a certain space $(\mathcal{X}, d)$ which is complete. The Banach's contraction satisfied by the selfmapping which is defined on a complete metric space $(\mathcal{X}, d)$ as $\Psi : \mathcal{X} \rightarrow \mathcal{X}$ and

$$d(\Psi u, \Psi v) \leq \theta d(u, v), \forall u, v \in \mathcal{X}$$

where $\theta \in (0, 1)$, then the equation $u = \Psi u$ must be satisfied by sequence generated by the Picard iteration will surely converge to a point u .

The set of fixed points of the selfmapping Ψ define over the Banach space $\mathcal{X}$ satisfying the equation $u = \Psi u$ will be denoted through out the dissertation by F_Ψ .

Definition 1.1.3. The mapping $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ will be called nonexpansive, if the value of θ is transformed to unity in Theorem 1.1.1.

It is evident that nonexpansive maps are not covered by Banach's Contraction Principle, and this issue has existed for about 40 years. Eventually, this dilemma was solved by Kirk [96], Browder [93], and Göhde [15] separately in 1965, which followed by the spate of papers in the literature that gave a new form to the fixed point theory.

Theorem 1.1.2. A closed, bounded convex subset $\mathcal{K}$ of uniform convex Banach space $\mathcal{X}$ on which nonexpansive a selfmap Ψ is defined, has a fixed point.

Sometimes as we know, extension of fixed point results needed a larger setting of nonlinear maps that are not a straightforward research in analysis. As we know, the condition that a certain map will be nonexpansive is simple and can be easily managed in existence and in iterative construction results. In 2008, one of the attempts to extend this idea was provided by Suzuki as follows.

Definition 1.1.4. [68] The map Ψ is called Suzuki mapping (SM) if satisfies Suzuki's (C) condition

$$\frac{1}{2}d(u, \Psi u) \leq d(u, v) \Rightarrow d(\Psi u, \Psi v) \leq d(u, v),$$

for all points u, v in the subset $\mathcal{K}$ of $\mathcal{X}$.

Compared to the nonexpansiveness property of any operator Ψ , the (C) condition is effectively weaker. [68] provides an example.

Author Karapinar [64] proposed an additional requirement for mappings, which was greatly inspired by Suzuki [68].

Definition 1.1.5. [64] *The Kannan-Suzuki condition (KSC) is being satisfied by the selfmap Ψ which is defined on set $\mathcal{K}$.*

$$\frac{1}{2}d(u, \Psi v) \leq d(u, v) \Rightarrow d(\Psi u, \Psi v) \leq \frac{1}{2}d(u, \Psi v) + d(v, \Psi u), \text{ for all } u, v \in \mathcal{K}.$$

The following are several characteristics of SGN maps.

Proposition 1.1.1. [111] *Let $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ be any mapping and let $\emptyset \neq \mathcal{K} \subset \mathcal{X}$ (a Banach space).*

- (i) *It can be shown that Ψ is an SGN map if it is a nonexpansive map.*
- (ii) *An SGN map Ψ having a fixed point defined over $\mathcal{K}$ is regarded as quasi-nonexpansive.*
- (iii) *The subsequent inequality satisfies only if Ψ is an SGN map for every $u, v \in \mathcal{K}$.*

$$\|u - \Psi v\| \leq 3 \|\Psi u - u\| + \|u - v\|.$$

Lemma 1.1.1. [111] *An SGN Ψ is defined over the subset $\mathcal{K}$ of $\mathcal{X}$ which is a Banach space $\mathcal{X}$ having the Opial property and with $\lim_{i \rightarrow \infty} \|\Psi u_i - u_i\| = 0$. if the sequence $\{u_i\}$ converges weakly to u after that $\Psi u = u$.*

Lemma 1.1.2. [111] *Let an SGN map Ψ defined on a domain that is compact in weak sense denoted by $\mathcal{K}$ and is essentially a subset of a uniformly convex Banach space. As consequent, there exists a point that is fixed for the mapping Ψ .*

A new concept regarding mappings of Reich-Suzuki type nonexpansive came into view in the recent past. The following is the information about such mappings.

Definition 1.1.6. [36] *Suppose we have a distance function d over a metric space $\mathcal{X}$ and take $\mathcal{K}$ as a subset of a metric space $\mathcal{X}$. On the subset $\mathcal{K}$ define a selfmap Ψ , provided that for all $u, v \in \mathcal{K}$ there would be a value c in the interval $[0, 1)$, then Ψ*

is said to be as a nonexpansive Reich-Suzuki type mapping if the consequent inequality $\frac{d(u, \Psi u)}{2} \leq d(u, v)$ implies $d(\Psi u, \Psi v) \leq cd(u, \Psi v) + cd(v, \Psi u) + (1 - 2c)d(u, v)$ holds.

It is critical to acknowledge that nonexpansive operators of the Reich-Suzuki type possess the subsequent characteristics.

Proposition 1.1.2. [37] Assume that $\mathcal{K}$ is the subset of a metric space $\mathcal{X}$. Let us take Ψ be a selfmap defined on subset $\mathcal{K}$. The collection of all fixed points of Ψ is represented by $F_\Psi = \{y_1 \in \mathcal{K} : y_1 = \Psi y_1\} \neq \emptyset$. Subsequently, the subsequent qualities are considered valid.

- (a) If Ψ is a Reich-Suzuki type nonexpansive operator, then the condition $d(\Psi y, \Psi y_1) \leq d(y, y_1)$ holds for every $y \in \mathcal{K}$ and any $y_1 \in F_\Psi$.
- (b) If the function Ψ is a Suzuki map, then it can be classified as a nonexpansive function of Reich-Suzuki type.

Definition 1.1.7. The Ψ mapping is called generalized α -nonexpansive (GAN) provided that

$$\frac{1}{2}d(u, \Psi u) \leq d(u, y) \Rightarrow d(\Psi u, \Psi y) \leq \alpha d(u, \Psi y) + \alpha d(y, \Psi u) + (1 - 2\alpha)d(u, y),$$

for some $\alpha \in [0, 1)$ and for all point u, y in the subset $\mathcal{K}$ of $\mathcal{X}$.

Definition 1.1.8. [80] Suppose two convergent sequences are given $\{u_i\}_{i=0}^\infty$ and $\{v_i\}_{i=0}^\infty$ with the limits u and v respectively then the comparison of their rate of convergence can be made in such way that if

$$\lim_{i \rightarrow \infty} \frac{\|u_i - u\|}{\|v_i - v\|} = 0$$

then we say that the convergence of $\{u_i\}_{i=0}^\infty$ is faster than that of $\{v_i\}_{i=0}^\infty$.

Definition 1.1.9. [80] Suppose we have fixed point iteration scheme for which two sequences $\{u_i\}_{i=0}^\infty$ and $\{v_i\}_{i=0}^\infty$ are generated, which are converging to the fixed point p also $\{a_i\}_{i=0}^\infty$ and $\{b_i\}_{i=0}^\infty$ are two positive real numbers null sequences. This occurs when $\|u_i - p\| \leq a_i$ and $\|v_i - p\| \leq b_i$ for all $i \geq 0$. If the convergence of $\{a_i\}_{i=0}^\infty$ is greater than that of $\{b_i\}_{i=0}^\infty$, then the convergence of $\{u_i\}_{i=0}^\infty$ is also greater than $\{v_i\}_{i=0}^\infty$, with respect to p .

Definition 1.1.10. [3] Any Banach space $\mathcal{X}$ equipped with the Opial condition will have the property that a weak convergent sequence $\{v_i\}$ in $\mathcal{X}$ that possess a weak limit v , with the result that.

$$\limsup_{i \rightarrow \infty} \|v_i - v\| \leq \limsup_{i \rightarrow \infty} \|v_i - a\|$$

holds, for every $a \in \mathcal{X}$ with $v \neq a$.

Lemma 1.1.3. [68] Suppose Ψ which denotes a map having domain a nonempty set $\mathcal{K}$ in a Banach space. If the underlying set admits the Opial property and Ψ is Suzuki map, then the weak convergence of any $\{v_i\}$ to some point v^* with $\lim_{i \rightarrow \infty} \|\Psi v_i - v_i\| = 0$, implies that Ψ has a single fixed point i.e $I - \Psi$ is essentially demiclosed on the point zero.

Lemma 1.1.4. [68] Suppose a selfmap Ψ defined on weakly compact subset $\mathcal{K}$ of $\mathcal{X}$, where $\mathcal{X}$ is Banach space which is uniformly convex. Then, let's assume that Ψ meets the condition (C). Only one fixed point exists for Ψ when this happens.

Lemma 1.1.5. [5] Let's consider that a Banach space $\mathcal{X}$ that exhibits uniform convexity and for each $i \in \mathbb{N}$ there is sequence a_i such that $0 < p \leq a_i \leq q < 1$. Let us consider $\{u_i\}$ and $\{v_i\}$ are two sequences of $\mathcal{X}$ that $\lim_{i \rightarrow \infty} \sup \|u_i\| \leq \gamma$, $\lim_{i \rightarrow \infty} \sup \|v_i\| \leq \gamma$ and $\lim_{i \rightarrow \infty} \sup \|a_i u_i + (1 - a_i) v_i\| = \gamma$ hold for some $\gamma \geq 0$. Then, $\lim_{i \rightarrow \infty} \|u_i - v_i\| = 0$.

Proposition 1.1.3. [68] The following hypothesis is true if $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ and $\mathcal{K} \neq \emptyset \subset \mathcal{X}$

(a) : Ψ satisfies (C), if Ψ is non-expansive.

(b) : For a mapping Ψ to be considered quasi-non-expansive, it must have a fixed point and satisfy (C).

(c) : For every $u, a \in \mathcal{K}$, $\|u - \Psi a\| \leq 3 \|\Psi u - u\| + \|u - a\|$ if Ψ satisfies (C).

Definition 1.1.11. [6] If h is a non-decreasing function and $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ is a mapping being referred to as 1.1 inequality, then $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $h(0) = 0$ and $h(t) > 0$ for any $t > 0$ in a way that

$$h(d(u, F(\Psi))) \leq d(u, \Psi u) \quad \forall u \in \mathcal{K}, \quad (1.1)$$

whereas $d(u, F(\Psi)) = \inf_{v \in F(\Psi)} d(u, v)$ and $F(\Psi)$ stands for the collection of all fixed points of Ψ .

1.2 CAT(0) space and some of its important properties

A word geodesic is used common in many area of research that has meaning a shortest possible between two points. But when a certain metric space, in which we can define a map that can connect two points will be called a geodesic and the space will be referred to as a geodesic metric space or simple a geodesic space (see [74]). The concept of such spaces has a clear impact of science and many ideas related to famous general relativity heavily rely on such concepts. Notice that in such type of spaces, a geodesic three sided figure is must be thinner or same in corresponding shapes of complex plane. We also know that Riemannian manifold is one of the most studied example of such spaces. Other examples of such spaces can be widely found in the work in [75] and included Hilbert spaces and R-trees as a special examples.

The fixed point study was only known in Hilbert, Banach and complete metric space initially. But in such type nonlinear CAT(0) or geodesic spaces, it was challenge problem. This study was researched by many authors but no effective outcomes was achieved. In the paper due Kirk [77] eventually got a breakthrough in this setting by establishing a bridge between fixed points and geodesic spaces. This breakthrough motivated many new researchers and new fixed point outcomes along with new applications were initiated.

Verifying the similarity between the metric fixed discoveries in Hilbert and CAT(0) spaces is a relatively simple. The author of the proposed work is motivated by the above revelation and became interested in investigating the possibility of constructing CAT(0) spaces, essentially nonlinear equivalents of spaces like l_p . This study assumes that the comparison triangle lies outside the Euclidean plane in a more general Banach space, extending Gromov's notion. To the author's knowledge, no such generalized statement

has been presented in literature so far. We have obtained some novel results for p more than two and Banach space being a classical sequence space l_p .

1.2.1 Basic Definitions and Properties

Consider a geodesic metric space represented by $(\mathcal{X}, d)$. In geodesic metric space, a path is a continuous map that maps the closed interval $[0, 1]$ to the set of points in $\mathcal{X}$ in a continuous way. Let the map which connects two sets is define as $q : [0, 1] \rightarrow \mathcal{X}$ if we choose any two points s and r from the interval $[0, 1]$ then their corresponding distance be measured by the formula $d(q(s), q(r)) = |(s - r)|d(q(0), q(1))$ in space $\mathcal{X}$. The set of image points forms a path, which is called a geodesic path. If for any two points in space, $\mathcal{X}$ exists a geodesic path that connects them, it is called geodesic metric space. If a and b are two points corresponding to points 0 and 1 of the map q in the space $\mathcal{X}$, then the path which connects a to b is denoted by $[a, b]$ is called a geodesic. if c is any point from the interval $[a, b]$ then the notation $c = ((1 - r)a \oplus rb)$ will be used to determine the value of c where r is a values from $(0, 1)$. If a unique geodesic exists between any two points of space $\mathcal{X}$, then such a space is known as uniquely geodesic. If $\mathcal{K}$ is a subset of $\mathcal{X}$, the $\mathcal{K}$ is said to be convex if the geodesic $[a, b]$ is contained in set $\mathcal{K}$ if a and b are any two points of $\mathcal{K}$.

A triangle $\Delta(t_1, t_2, t_3)$ in geodesic space $\mathcal{X}$ is called a geodesic triangle if the three vertices of it are represented by points t_1, t_2, t_3 and the sides are formed by geodesics $[t_1, t_2]$, $[t_2, t_3]$ and $[t_3, t_1]$ respectively. A comparison triangle for $\Delta(t_1, t_2, t_3)$ is defined as a triangle $\bar{\Delta}(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ in the plane $\mathbb{R}^2$ if and only if

$$d_{\mathbb{R}^2}(\bar{t}_1, \bar{t}_2) = d(t_1, t_2), d_{\mathbb{R}^2}(\bar{t}_2, \bar{t}_3) = d(t_2, t_3) \text{ and } d_{\mathbb{R}^2}(\bar{t}_3, \bar{t}_1) = d(t_3, t_1).$$

A comparison between the points of the geodesic segment and an interval in a plane is made in such a way that a point $\bar{e} \in [\bar{t}_1, \bar{t}_2]$ is regarded as comparison point of $e \in [t_1, t_2]$ if the subsequent condition is satisfied $d(t_1, e) = d_{\mathbb{R}^2}(\bar{t}_1, \bar{e})$. Similarly, we can also compare the points in $[\bar{t}_1, \bar{t}_3]$ and $[\bar{t}_3, \bar{t}_2]$.

Definition 1.2.1. [43] *The geodesic triangle $\Delta(t_1, t_2, t_3)$ is said to possess the CAT(0) property in a metric space $(\mathcal{X}, d)$ if the comparison is made between the corresponding*

points in the two triangles $\Delta(t_1, t_2, t_3)$ and $\overline{\Delta}(\overline{t}_1, \overline{t}_2, \overline{t}_3)$ by choosing two points from each in such way $t, t' \in \Delta$ and $\overline{t}, \overline{t}' \in \overline{\Delta}$ then the following condition must be obeyed.

$$d(t, t') \leq d_{\mathbb{R}^2}(\overline{t}, \overline{t}').$$

The most basic types of geodesic spaces are polyhedral complexes with piecewise constant curvature, complete Riemannian manifolds, and normed vector spaces. In the previous two instances where the presence of it is not always clear whether such spaces are uniquely geodesic, finding such paths is also a difficult task. It is considerably simpler in the case of normed vector spaces. [74].

In the purview of geodesic metric spaces, if a geodesic triangle lies within metric space $\mathcal{X}$, then the metric space deemed $\mathcal{X}$ to be a $CAT(0)$ space if the said triangle adheres to the $CAT(0)$ property. To obtain more information regarding the exact definition of $CAT(0)$ attributes, please consult reference [108]. It is crucial to emphasize that every $CAT(0)$ space is fundamentally geodesically unique. Pre-Hilbert spaces and metric trees are commonly cited as instances of $CAT(0)$ spaces. We suggest referring to sources [43, 91, 107, 108] for a more comprehensive examination of this subject. In addition, $CAT(0)$ spaces exhibit several fascinating characteristics that merit further exploration.

Suzuki [111] infers that the requirement of non-expansiveness is stronger than the condition (C). Consequently, the class of the maps that satisfy condition (C) is a superclass of the non-expansive class. Suzuki generalized non-expansive maps are frequently referred to as SGN. For mappings of the exact nature, the existence and convergence of the fixed points are established in the paper [111]. For both uniformly convex Banach and $CAT(0)$ spaces, Phuengrattana proved the convergence theorems for SGN mappings using the Ishikawa iterations method [77]. This study focuses on the fixed point theorems for SGN mappings. For in-depth scrutiny, the readers are referred to the work of Thakur [69] and the work cited therein. Additional information on $CAT(0)$ spaces is available in the books [61]. Now, we present some information from [45].

Lemma 1.2.1. [11] *Let us consider a $CAT(0)$ space $\mathcal{X}$, which is complete, and take a nonempty subset $\mathcal{K}$ of $\mathcal{X}$. Then, the following statements are followed due to the*

CAT(0) space properties.

- (a) *Then according to the geodesic property, there exists a unique representation of point q in the geodesic $[u, v]$ where $u, v \in \mathcal{K}$ and θ represents a fixed element in the interval $[0, 1]$ on the real line. Then*

$$d(u, q) = \theta d(u, v) \text{ and } d(v, q) = (1 - \theta)d(u, v). \quad (1.2)$$

Any point q in the geodesic $[u, v]$ has a unique representation $q = (1 - \theta)u \oplus \theta v$ which satisfies (1.2).

- (b) *If $\theta \in [0, 1]$ is fixed and $u, v, w \in \mathcal{K}$, then one has*

$$d(w, \theta u \oplus (1 - \theta)v) \leq \theta d(w, u) + (1 - \theta)d(w, v).$$

To figure out the primary result, we additionally require certain concepts and information. In the forthcoming chapters we may denote $\mathcal{X}$ as a CAT(0) space and $\mathcal{K}$ is a nonempty convex, closed subset of $\mathcal{X}$

Let us suppose that a bounded sequence $\{v_i\} \in \mathcal{X}$ and that a subset of a Banach space $\mathcal{X}$ that is $\mathcal{K} \neq \emptyset$ is convex and closed. We define $\gamma(u, \{v_i\}) = \lim_{i \rightarrow \infty} \sup \|u - v_i\|$. Associative to $\mathcal{K}$ the asymptotic radius of $\{v_i\}$ is and may be determined by using

$$\gamma(\mathcal{K}, \{v_i\}) = \inf \{\gamma(u, \{v_i\}) : u \in \mathcal{K}\}$$

and an asymptotic centre of $\{v_i\}$ associated with $\mathcal{K}$ is defined as

$$\Omega(\mathcal{K}, \{v_i\}) = \{u \in \mathcal{K} : \gamma(u, \{v_i\}) = \gamma(\mathcal{K}, \{v_i\})\}.$$

The $\Omega(\mathcal{K}, \{v_i\})$ has exactly one point, as observed within the framework of CAT(0) space. Additionally, this set is convex and nonempty in cases of weak compactness and convexity of $\mathcal{K}$ (see [88, 89]).

Definition 1.2.2. [46] *Consider $\{v_i\} \subseteq \mathcal{X}$, and $\mathcal{X}$ to be a CAT(0) space. A Δ limit of $\{v_i\}$ is defined as the point y_0 only if it serves as the sole asymptotic center for $\{s_i\}$,*

where $\{s_i\}$ represents any subsequence of $\{v_i\}$.

Strong convergence theorems for iterative sequences are typically proven for specific types of mappings using condition (I), which is stated as follows and was presented by Senter and Dotson in.

Definition 1.2.3. [41] Let us have $\mathcal{X}$ as $CAT(0)$ space and $\mathcal{K}$ as a subset of $\mathcal{X}$ on which the selfmap Ψ is defined then Ψ is said to satisfy condition (I) if the inequality $d(v, \Psi v) \geq \mu(\text{dist}(v, F_\Psi))$ is true for every point $v \in \mathcal{K}$ where μ is a function such that $\mu(0) = 0$ and $\mu(u) > 0$ for every $u > 0$ and $\text{dist}(v, F_\Psi)$ represents the distance between the set F_Ψ and the point v .

The Opial's property [3] can be expressed in the term of $CAT(0)$ space as follows.

Definition 1.2.4. A space $\mathcal{X}$ that represents $CAT(0)$ space Possesses the Opial's property if any sequence $\{v_i\} \subseteq \mathcal{X}$ which is Δ -convergent to $y_0 \in \mathcal{X}$, the following condition holds:

$$\limsup_{i \rightarrow \infty} d(v_i, y_0) < \limsup_{i \rightarrow \infty} d(v_i, x_0),$$

$$\forall x_0 \in \mathcal{X} - \{y_0\}.$$

It has long been known that every $CAT(0)$ space possesses this attribute.

Lemma 1.2.2. [3] Assume that a space $\mathcal{X}$, which is a complete $CAT(0)$ space, and let $\{v_i\} \subseteq \mathcal{X}$ is a bounded sequence. In this scenario, the sequence $\{v_i\}$ possesses a subsequence that is Δ -convergent.

The deduction of the following Lemma has been made from the definition Reich-Suzuki type nonexpansive mapping.

Lemma 1.2.3. [37] Suppose we have $CAT(0)$ space $\mathcal{X}$ and $\mathcal{K}$ is a subset of $\mathcal{X}$, now define a mapping as $\Psi : \mathcal{K} \rightarrow \mathcal{K}$. If Ψ is a nonexpansive function of Reich-Suzuki type and y and z are elements of $\mathcal{K}$.

$$\text{Subsequently } d(y, \Psi z) \leq \frac{(\alpha+3)}{(1-\alpha)} d(y, \Psi y) + d(y, z).$$

The idea of a Δ convergence in $CAT(0)$ space exhibits similarity with the weak convergence in a Banach space.

Definition 1.2.5. [107] In a $CAT(0)$ space $\mathcal{X}$ which is complete a bounded sequence $\{v_i\}$ is said to be Δ -convergent if it has a unique asymptotic center z for every subsequence $\{s_i\}$ of $\{v_i\}$ and it is denoted by $\Delta - \lim_i v_i = z$ where $z \in \mathcal{X}$ to which $\{v_i\}$ converges is the only asymptotic centre of it.

Lemma 1.2.4. [108] Assume that $\mathcal{X}$, the $CAT(0)$ space, is complete. Once this is established, any bounded sequence $\{v_i\} \subseteq \mathcal{K}$ admits a subsequence that is Δ -convergent.

Lemma 1.2.5. [110] Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$ which is a convex nonempty closed subset of $\mathcal{X}$ then for every bounded sequence $\{v_i\}$ in $\mathcal{K}$, the asymptotic center is contained in the subset $\mathcal{K}$.

Lemma 1.2.6. [64] Assume that $\mathcal{X}$ and $\emptyset \neq \mathcal{K} \subseteq \mathcal{X}$ are any $CAT(0)$ spaces. Assume that $F_\Psi \neq \emptyset$ and that Ψ is a selfmap of $\mathcal{K}$ that satisfies the (KSC) requirement. For any $u \in \mathcal{K}$ and $v \in F_\Psi$, one then possesses the following property:

$$d(\Psi u, v) \leq d(u, v).$$

Lemma 1.2.7. [64] Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$, which is a nonempty subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ which satisfies the (KSC) criterion. Then one has the following property holds for each $u, v \in \mathcal{K}$.

$$d(u, \Psi v) \leq 5d(u, \Psi u) + d(u, v).$$

Lemma 1.2.8. [64] Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$, which is a nonempty subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ which satisfies the (KSC) criterion. The subsequent property is valid.

$$\{v_i\} \subseteq \mathcal{K}, \quad \Delta - \lim_i v_i = v, \quad d(v_i, \Psi v_i) \rightarrow 0 \Rightarrow \quad \Psi v = v.$$

We require some evidence from [17] on this basis, and the $CAT(0)$ space interpretation follows.

Proposition 1.2.1. Assume that $\mathcal{X}$ is a metric space and any subset $\mathcal{K}$ of $\mathcal{X}$. Let Ψ a map from $\mathcal{K}$ to itself. Next, we present the following pieces of information.

(i) When Ψ is SM then Ψ GAN.

(ii) When Ψ is a GAN map having a fixed point, it follows that $d(\Psi s, l^*) \leq d(s, l^*)$, where $s \in \mathcal{K}$ and l^* is any chosen point which acts as fixed point of Ψ .

(iii) When Ψ is a mapping of GAN and the set of its fixed points F_G is nonempty. Then F_Ψ is the subset of Ψ that must be closed set.

(iv) When Ψ is a GAN map, then it follows that for all $s, r \in \mathcal{K}$,

$$d(s, \Psi r) \leq \left(\frac{\alpha + 3}{1 - \alpha} \right) d(s, \Psi s) + d(s, r).$$

(v) Since the $CAT(0)$ space satisfies the $CAT(0)$ style of Opial condition. So, if Ψ is a GAN map with $\{v_i\}$ is Δ -convergent sequence to l^* and $\lim_{t \rightarrow \infty} d(\Psi v_i, v_i) = 0$, then one has $l^* \in F_\Psi$.

1.3 Iteration processes

In the forthcoming chapters we shall use four real sequences $\{\kappa_i\}, \{\lambda_i\}, \{\gamma_i\}$ and $\{\mu_i\}$ for ($i \geq 0$) all of which are entirely lying in the interval $[0, 1]$. Furthermore a nonlinear selfmap Ψ defined on a subset $\mathcal{K}$ of $\mathcal{X}$ where $\mathcal{K}$ is nonempty closed convex subset of the space $\mathcal{X}$ under the consideration.

As for as iterative sequence is concern we shall use $\{v_i\}$ throughout the dissertation unless and otherwise. In order to demonstrate Banach's Contraction Principle, the most fundamental I.P utilized is Picard I.P, which is as follows:

$$\begin{cases} v_0 \in \mathcal{K} \\ v_{i+1} = \Psi v_i. \end{cases} \quad (1.3)$$

This technique is also known as the successive substitution method or the Richardson iteration.

Hassan et al. have presented the following iterative technique. [38]:

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K}, \\ w_i = \Psi((1 - \kappa_i)v_i + \kappa_i\Psi v_i), \\ x_i = \Psi((1 - \mu_i)w_i + \lambda_i\Psi w_i), \\ y_i = \Psi((1 - \lambda_i)x_i + \lambda_i\Psi x_i), \\ v_{i+1} = \Psi((1 - \gamma_i)y_i + \gamma_i\Psi y_i), i \geq 1, \end{array} \right. \quad (1.4)$$

where $\gamma_i, \lambda_i, \mu_i, \kappa_i \in (0, 1)$.

The initial generalization of Picard's iterative process, which is defined for a fixed $(\lambda > 0)$, is known as Krasnoselskij's iterative process.

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ v_{i+1} = (1 - \lambda)v_i + \lambda\Psi v_i. \end{array} \right. \quad (1.5)$$

M^* -iteration is a newly designed iteration by Ullah and Arshad [70]. The following sequence $\{v_i\}$ is produced by this iteration:

$$\left\{ \begin{array}{l} v_1 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i\Psi v_i \\ x_i = \Psi[(1 - \kappa_i)v_i + \kappa_i\Psi w_i] \\ v_{i+1} = \Psi x_i. \end{array} \right. \quad (1.6)$$

Halpern in [83] proposed an iteration method that uses a fixed value of $u \in \mathcal{K}$ and a single set of parameters that is the first one-step I.P.

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ v_{i+1} = (1 - \kappa_i)u + \kappa_i\Psi v_i. \end{array} \right. \quad (1.7)$$

The following is the one-step Mann iterative process:

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ v_{i+1} = (1 - \kappa_i)v_i + \kappa_i\Psi v_i. \end{array} \right. \quad (1.8)$$

If we consider $\kappa_i \in [0, 1]$ For all i , the one-step Mann I.P. reduces to (1.8). The

one-step mann I.P. also becomes (1.3) for all i when $\kappa_i = 1$, in a similar fashion.

Ishikawa I.P, which is the initial two-step I.P is defined in [63] and provided by:

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i\Psi v_i \\ v_{i+1} = (1 - \kappa_i)v_i + \kappa_i\Psi v_i. \end{cases} \quad (1.9)$$

If we assume that $\lambda_i = 0$, the Ishikawa I.P falls to (1.8). for each of i .

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i\Psi v_i \\ v_{i+1} = (1 - \kappa_i)w_i + \kappa_i\Psi w_i. \end{cases} \quad (1.10)$$

Presented in [57], Agarwal I.P (sometimes called S I.P) is described as follows:

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i\Psi v_i \\ v_{i+1} = (1 - \kappa_i)\Psi w_i + \kappa_i\Psi w_i. \end{cases} \quad (1.11)$$

They demonstrated that, for contraction maps, the convergence rate of their novel I.P. is quicker than (1.8) and equal to that of (1.3).

The initial three-step I.P. was presented in 2000 in [19] and was called Noor I.P. It was described as follows:

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)v_i + \lambda_i\Psi w_i \\ v_{i+1} = (1 - \kappa_i)v_i + \kappa_i\Psi x_i. \end{cases} \quad (1.12)$$

The value of Noor I.P falls to (1.8) when we set $\lambda_i = \mu_i = 0$ for all i , and to (1.9) when we assume $\mu_i = 0$ for all i . Subsequently, other authors developed three-step iteration procedures.

SP I.P [101] is given by;

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)w_i + \lambda_i\Psi w_i \\ v_{i+1} = (1 - \kappa_i)x_i + \kappa_i\Psi x_i. \end{array} \right. \quad (1.13)$$

The rate of convergence of (1.13) for non-decreasing and continuous functions faster than that of (1.8), (1.9) , and (1.12) as shown by Phnengrattana and Suantai demonstrated in [101].

It is evident that (1.13) and (1.9) are instances of (1.12).

The Abbas I.P. was presented by Abbas et al. [20] and is defined as follows:

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)\Psi v_i + \lambda_i\Psi w_i \\ v_{i+1} = (1 - \kappa_i)\Psi x_i + \kappa_i\Psi w_i. \end{array} \right. \quad (1.14)$$

The process converges more quickly than all of mentioned above, as they demonstrated.

The source of CR I.P, as reported in [81], is:

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)\Psi v_i + \lambda_i\Psi w_i \\ v_{i+1} = (1 - \kappa_i)x_i + \kappa_i\Psi x_i. \end{array} \right. \quad (1.15)$$

A subclass of contraction maps known as quasi-contractive maps, the convergence of (1.13) is faster than that of (1.3), (1.6), (1.7), (1.9), (1.10), and (1.11) as demonstrated by Chugh et al. [81].

A novel three-step I.P is presented by Thakur et al. [69] and is described as:

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)w_i + \lambda_i\Psi w_i \\ v_{i+1} = (1 - \kappa_i)\Psi x_i + \kappa_i\Psi x_i. \end{array} \right. \quad (1.16)$$

Additionally, the authors demonstrated both analytically and numerically that their novel I.P converges more quickly than all of the contraction map (1.3), (1.6), (1.7), (1.9), (1.10), and (1.12).

The three-step I.P utilized by Karahan and Ozdemir [85] is called S^* I.P.

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)\Psi v_i + \lambda_i\Psi w_i \\ v_{i+1} = (1 - \kappa_i)\Psi v_i + \kappa_i\Psi x_i. \end{array} \right. \quad (1.17)$$

Karakaya et al. In [92] shown that (1.13) is converging to the fixed point of contraction map faster than S^* I.P.

A three step I.P known as P I.P [102] is presented in the following way.

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)w_i + \lambda_i\Psi w_i \\ v_{i+1} = (1 - \kappa_i)\Psi w_i + \kappa_i\Psi x_i. \end{array} \right. \quad (1.18)$$

For continuous and non-decreasing functions, if S-iterations 1.9 converge, then the P-iteration also converges but at a faster rate than the S-iterations showed by Sainuan [102].

Introduced by Sintunavavat and Pitea [109], S_i I.P is provided by:

$$\left\{ \begin{array}{l} v_0 \in \mathcal{K} \\ w_i = (1 - \mu_i)v_i + \mu_i\Psi v_i \\ x_i = (1 - \lambda_i)v_i + \lambda_i w_i \\ u_{i+1} = (1 - \kappa_i)\Psi x_i + \kappa_i\Psi w_i. \end{array} \right. \quad (1.19)$$

Through a rigorous comparison of the convergence rates of S_i I.P with (1.6), (1.7), and (1.9) using numerical examples, we were able to precisely demonstrate the excellent convergence rate of S_i I.P, solidifying its superiority over other iteration processes.

The definition of a two-step Picard Mann hybrid I.P is:

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \kappa_i)\Psi v_i + \kappa_i\Psi x_i \\ u_{i+1} = \Psi w_i. \end{cases} \quad (1.20)$$

A new I.P named Picard-S I.P was introduced in 2014 by Gursoy and Karakaya [25].

It works as follows:

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i\Psi v_i \\ x_i = (1 - \kappa_i)\Psi v_i + \kappa_i\Psi w_i \\ v_{i+1} = \Psi x_i. \end{cases} \quad (1.21)$$

The fixed point estimation for contraction maps employing the Picard-S iterative procedure is rapidly converging as compared to the iteration in the literature that includes the two-step Mann, Normal-S, S, S^* , (1.12), (1.6), (1.7), (1.10), (1.11), (1.13), and (1.1) processes. This is proved by providing an example.

Taking into consideration the previous I.P., Yildirim, and Kadioglu [84] introduced a new I.P.

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i\Psi v_i \\ x_i = (1 - \kappa_i)w_i + \kappa_i\Psi w_i \\ u_{i+1} = \Psi x_i. \end{cases} \quad (1.22)$$

It has been shown that their newly introduced I.P is more quicker than all of iteration process (1.3), (1.6), (1.7) and (1.9).

We shall refer to the two-step I.P. that Karakaya et al. [86] provided as Vatan's

two-step I.P, which is defined as:

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = \Psi((1 - \lambda_i)v_i + \lambda_i \Psi v_i) \\ v_{i+1} = \Psi((1 - \kappa_i)w_i + \kappa_i \Psi w_i). \end{cases} \quad (1.23)$$

In 2015, Thakur et al. [69] employed the subsequent new I.P after this.

$$\begin{cases} v_0 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i \Psi v_i \\ x_i = \Psi((1 - \kappa_i)v_i + \kappa_i w_i) \\ v_{i+1} = \Psi x_i. \end{cases} \quad (1.24)$$

We'll refer to it as Thakur New I. P. They demonstrated that (1.22) for Suzuki generalized nonexpansive maps is quicker than (1.3), (1.6), (1.7), (1.9), (1.10), and (1.12) using numerical examples. The leading I.P., (1.19), is not compared to this I.P. We will discover their convergence speeds in the upcoming chapters.

The idea of practically finding solutions either in original form directly or in the transform (for example in fixed point theory) by employing approximation methodologies in order to effectively estimate their values is highly searched area in analysis. This area of research mainly focus on the problem when the analytical solutions become impossible to locate. Notice that, many methods to estimate the accurate and semi closed form solutions often depends on complicated initial conditions that are not easy to fulfill in real cases. One of such methods are homology analysis method. The aspect of this analysis includes conditions under which one can guarantee the existence of solutions which is also not available widely in other fields of research. However, in fixed point theory, both the existence and solution approximation is easy to get once the corresponding operator is constructed. This research will focus on the concept of fixed point existence and fixed point estimation under weak assertion and conditions with real-world problems in various linear and CAT(0) nonlinear settings.

Chapter 2

Approximation of Fixed Points for Mappings with (KSC) Conditions in CAT(0) Spaces

Fixed point theory has recently provided numerous effective techniques for addressing nonlinear problems. For more detailed information, refer to the survey study authored by Karapinar [?, 56]. Studying functional equations and iterative solutions for fractional differential equations (FDEs) is now a highly active research field. To obtain the solution of Functional equations or fractional differential equations (FDEs), it is often expressed in the form of an operator, which may be linear or nonlinear, operating on subsets of suitable metric spaces under the observation, then the fixed point of operator leads to the solution of the given problems. For further information, refer to sources such as [55] and others. It is always preferable for the fixed point of this operator to exist and be produced iteratively.

As we are aware, finding the fixed point of an operator is not a difficult task, but evolving an appropriate algorithm to accurately calculate the value of the fixed point of an operator is challenging (see to, for example, [?, ?] and other relevant sources). An approach to estimate the values of this distinct fixed point [67] for contraction is to employ the Picard iteration [60], represented by $\nu_i = \Psi\nu_i$ as endorsed by the Banach Contraction Principle (BCP) [13]. The Browder-Gohde-Kirk have shown

that a nonexpansive mapping has a fixed point if defined on a convex, closed and bounded subset of a uniformly convex Banach space (UCBS). These inferences have been elaborated in their work in Gohde [15], Browder [14] and Kirk [65]. A selfmap Ψ defined over a subset $\mathcal{K}$ of a metric space then the mapping Ψ is said to be contraction if

$$d(\Psi u, \Psi v) \leq \alpha d(u, v), \quad \text{for all } u, v \in \mathcal{K} \quad (2.1)$$

wherever $\alpha \in [0, 1)$.

The fixed point of mapping Ψ is any point $u \in \mathcal{K}$ which satisfies $u = \Psi u$, and the set of all fixed points of mapping Ψ will be represented as F_Ψ throughout this chapter. The mapping Ψ will become nonexpansive if (2.1) holds for $\alpha = 1$.

We present an example of nonexpansive mapping that does not converge for the sequence of Picard iteration.

Example 2.0.1. *Let us define a nonexpansive map like $\Psi u = 1 - u$ for each u in $\mathcal{K} = [0, 1]$ and the set of all fixed point i.e $F_\Psi = \{0.5\}$ The iteration produced does not converge for initial values different from 0.5.*

Example 2.0.1 presents alternative iterative methods that guarantee convergence for nonexpansive mappings (or even generalized nonexpansive mappings) as an alternative to Picard iteration [60]. The (C) criterion 1.1.4 was initially introduced for mappings by Suzuki [68] in 2008.

The literature contains a large number of iterative schemes that are widely used to approximate fixed points in various mapping settings (see, for example, Mann [18], Ishikawa [63], Agarwal et al. [57], S -iteration, Noor [19], abbas [20], Thakur et al. [69] and others).

In [70], Ullah and Arshad introduced an iteration which is a new version of M -iteration called M^* -iteration. As compared to the prior iterations in the literature, the present one is stable and generates more accurate results. The sequence of iteration $\{v_i\}$ which has been generated is presented as.

$$\begin{cases} v_1 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i + \lambda_i\Psi v_i \\ x_i = \Psi[(1 - \kappa_i)v_i + \kappa_i\Psi w_i] \\ v_{i+1} = \Psi x_i. \end{cases} \quad (2.2)$$

In the above scheme (2.2) Ψ is defined as a selfmap on the set $\mathcal{K}$ and the sequences $\{\kappa_i\}$ and $\{\lambda_i\}$ are taken from the interval $(0, 1)$. Nevertheless, Ullah and Arshad (AKIT) have shown that the scheme introduced in (2.2) converges in the case of contraction mappings. In this context, we extend their primary finding to encompass mappings that satisfy the (KSC)-condition on a wider scale. The convergence of the iteration indicated above can be proven using the same evidence and procedures. We establish that the iteration scheme M^* produces precise results that align with the other iterations in this novel mapping configuration, as demonstrated by a non-trivial example.

2.1 Introduction

Initially, the $CAT(0)$ space iteration of the M^* scheme is defined as follows (2.2):

$$\begin{cases} v_1 \in \mathcal{K} \\ w_i = (1 - \lambda_i)v_i \oplus \lambda_i\Psi v_i \\ x_i = \Psi[(1 - \kappa_i)v_i \oplus \kappa_i\Psi w_i] \\ v_{i+1} = \Psi x_i. \end{cases} \quad (2.3)$$

2.2 Main Results

We establish our primary result by utilizing (2.3). An initial lemma is presented, which will subsequently assume an extremely important position.

Lemma 2.2.1. *Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$ which is convex nonempty closed subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ with the set*

of fixed points i.e $F_\Psi \neq \emptyset$ which satisfies (KSC) criterion. Then the sequence $\{v_i\}$ generated from M^* -iteration (2.3) satisfies that $\lim_{i \rightarrow \infty} d(v_i, z)$ exists for each $z \in F_\Psi$.

Proof. Lemma 1.2.6 can be applied to consider any point $z \in F_\Psi$. From there, one has

$$\begin{aligned}
d(v_{i+1}, z) &= d(\Psi x_i, z) \\
&\leq d(x_i, z) \\
&= d(\Psi[(1 - \kappa_i)v_i \oplus \kappa_i \Psi w_i], z) \\
&\leq d((1 - \kappa_i)v_i \oplus \kappa_i \Psi w_i, z) \\
&\leq (1 - \kappa_i)d(v_i, z) + \kappa_i d(\Psi w_i, z) \\
&\leq (1 - \kappa_i)d(v_i, z) + \kappa_i d(w_i, z) \\
&= (1 - \kappa_i)d(v_i, z) + \kappa_i (d(1 - \lambda_i)v_i + \lambda_i d(\Psi v_i, z)) \\
&\leq (1 - \kappa_i)d(v_i, z) + \kappa_i ((1 - \lambda_i)d(v_i, z) + \lambda_i d(\Psi v_i, z)) \\
&\leq (1 - \kappa_i)d(v_i, z) + \kappa_i ((1 - \lambda_i)d(v_i, z) + \lambda_i d(v_i, z)) \\
&= (1 - \kappa_i)d(v_i, z) + \kappa_i d(v_i, z) \\
&= d(v_i, z).
\end{aligned}$$

As a consequence of the above calculation we acquired the inequality $d(v_{i+1}, z) \leq d(v_i, z)$ for every $z \in F_\Psi$ which suggest that the set $\{d(v_i, z)\}$ is both bounded and non-increasing. So we concluded that $\lim_{i \rightarrow \infty} d(v_i, z)$ exists for all $z \in F_\Psi$.

No numerical method work, until existence of solution is not available. Existence of solutions and their computation is not an easy task unless several conditions are not imposed. This chapter includes one existence of fixed point related result, that is based on the some conditions in nonlinear CAT(0) spaces. We essentially give the proof of the result by assuming the following criteria.

Theorem 2.2.1. *Let us take a complete CAT(0) space $\mathcal{X}$ and select a set $\mathcal{K}$ which is convex nonempty closed subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ which satisfies (KSC) criterion. The sequence $\{v_i\}$ generated from M^* -iteration (2.3). The condition*

that the set of all fixed points i.e $F_\Psi \neq \emptyset$ holds true if and only if the sequence $\{v_i\}$ generated is bounded and satisfies equation $\lim_{i \rightarrow \infty} d(\Psi v_i, v_i) = 0$.

Proof. Initially, we will assume that the set $F_\Psi \neq \emptyset$. We will then demonstrate that $\{v_i\}$ is bounded and showing that $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0$. Lemma 2.2.1 indicates that $\{v_i\}$ is bounded and that $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i)$ exists in this regard.

Put

$$\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = r, \quad (2.4)$$

assuming that $r \in \mathbb{R}^+$. for $r > 0$, which is the nontivial situation, as assumed. In light of Lemma 2.2.1's proof, $d(w_i, z) \leq d(v_i, z)$. Consequently,

$$\limsup_{i \rightarrow \infty} d(w_i, z) \leq \limsup_{i \rightarrow \infty} d(v_i, z) = r. \quad (2.5)$$

Now $d(\Psi v_i, z) \leq d(v_i, z)$ from the Lemma 1.2.6. So,

$$\limsup_{i \rightarrow \infty} d(\Psi v_i, z) \leq \limsup_{i \rightarrow \infty} d(v_i, z) = r. \quad (2.6)$$

From the proof of Lemma 2.2.1, we may observe once more that $d(v_{i+1}, z) \leq (1 - \kappa_i)d(v_i, z) + \kappa_i d(w_i, z)$. Consequently, $d(v_{i+1}, z) \leq d(w_i, z)$. Thus,

$$r = \liminf_{i \rightarrow \infty} d(v_{i+1}, z) \leq \liminf_{i \rightarrow \infty} d(w_i, z). \quad (2.7)$$

Thus from (2.5) and (2.7), we have

$$\lim_{i \rightarrow \infty} d(w_i, z) = r. \quad (2.8)$$

From (2.8), we have

$$r = \lim_{i \rightarrow \infty} d((1 - \lambda_i)v_i \oplus \lambda_i \Psi v_i, z). \quad (2.9)$$

Now applying Lemma 1.1.5 on (2.4), (2.6) and (2.9), we get

$$\lim_{i \rightarrow \infty} d(\Psi v_i, v_i) = 0.$$

Finally, let us assume that the set $\{v_i\}$ is bounded and satisfies the property $\lim_{i \rightarrow \infty} d(\Psi v_i, v_i) = 0$. We will now demonstrate that the set F_Ψ is not empty. To address this, we can consider any point, denoted as z , that belongs to the set $\mathcal{A}(\mathcal{K}, \{v_i\})$. According to Lemma 1.2.7, we obtain

$$\begin{aligned}
R(\Psi z, \{v_i\}) &= \limsup_{i \rightarrow \infty} d(v_i, \Psi z) \\
&\leq \limsup_{i \rightarrow \infty} (5d(\Psi v_i, v_i) + d(v_i, z)) \\
&\leq 0 + \limsup_{i \rightarrow \infty} d(v_i, z) \\
&= R(z, \{v_i\}).
\end{aligned}$$

This indicates that $\Psi z \in \mathcal{A}(\mathcal{K}, \{v_i\})$. Since the collection $\mathcal{A}(\mathcal{K}, \{v_i\})$ has just one element, it follows that $\Psi z = z$ and so F_Ψ is not empty. $\square$

First, we propose a result on Δ convergence.

Theorem 2.2.2. *Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$ which is convex nonempty closed subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ with the set of fixed points i.e $F_\Psi \neq \emptyset$ which satisfies (KSC) criterion. Then the sequence $\{v_i\}$ generated from M^* -iteration (2.3) will Δ -converges to a fixed point of ψ if the space under consideration satisfies the Opial's property.*

Proof. The sequence of iterates $\{v_i\}$ is bounded in the set $\mathcal{K}$ and meets the condition $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0$, according to Theorem 2.2.1. With $\{s_i\}$ denoting any subsequence $\{v_i\}$, set $\omega_\Delta(\{v_i\}) = \bigcup A(\{s_i\})$. $\omega_\Delta(\{v_i\}) \subseteq F_\Psi$ is demonstrated. Let $s \in \omega_\Delta(\{v_i\})$ in order to accomplish the goal. As a result, a subsequence $\{s_i\}$ of $\{v_i\}$ may be found such that $A(\{s_i\}) = \{s\}$. Now applying Lemmas 1.2.4 and 1.2.5 we get a Δ -convergent subsequence $\{r_i\}$ which converges to to a point r in $\mathcal{X}$ of $\{s_i\}$. We now get $\lim_{i \rightarrow \infty} d(r_i, \Psi r_i) = 0$ using Theorem 2.2.1. Additionally, Ψ has the (KSC)-condition, as a result

$$d(r_i, \Psi r) \leq 5d(r_i, \Psi r_i) + d(r_i, r). \quad (2.10)$$

It follows that $r \in F_\Psi$ when the limit is applied to (3.16). Therefore, it is proven that $\lim_{i \rightarrow \infty} d(r_i, r)$ exists by use of Lemma 2.2.1. As a second objective, we must ensure

that $s = r$. To demonstrate this, we will suppose that $s \neq r$, which will lead us to a contradiction. One must bear in mind that asymptotic centers are unique and

$$\begin{aligned}
\limsup_{i \rightarrow \infty} d(r_i, r) &< \limsup_{i \rightarrow \infty} d(r_i, s) \leq \limsup_{i \rightarrow \infty} d(s_i, s) \\
&< \limsup_{i \rightarrow \infty} d(s_i, r) = \limsup_{i \rightarrow \infty} d(v_i, r) \\
&= \limsup_{i \rightarrow \infty} d(r_i, r).
\end{aligned}$$

Thus we acquire $\limsup_{i \rightarrow \infty} d(r_i, r) < \limsup_{i \rightarrow \infty} d(r_i, r)$ which is a straightforward contradiction so we come to the conclusion that $s = r \in F_\Psi$ and that $\omega_\Delta(\{v_i\}) \subseteq F_\Psi$. Eventually we will demonstrate that the sequence $\{v_i\}$ Δ -converges to a point which is fixed point of Ψ . We will proceed to our goal by showing that $\omega_\Delta(\{v_i\})$ is a singleton set. Let us take a subsequence $\{s_i\}$ of $\{v_i\}$ now applying Lemmas 1.2.4 and 1.2.5 we get a Δ -convergent subsequence $\{r_i\}$ which converges to to a point r in $\mathcal{X}$ of $\{s_i\}$. It is assumed that $\mathcal{A}(\{s_i\}) = \{s\}$ and $\mathcal{A}(\{v_i\}) = \{q\}$. Since $s = r$ and $r \in F_\Psi$ have already been shown, we may assert that $q = r$. One obtain $\lim_{i \rightarrow \infty} d(v_i, r)$ if $q \neq r$ with the help of uniqueness of asymptotic center.

$$\begin{aligned}
\limsup_{i \rightarrow \infty} d(r_i, r) &< \limsup_{i \rightarrow \infty} d(r_i, q) \leq \limsup_{i \rightarrow \infty} d(s_i, q) \\
&< \limsup_{i \rightarrow \infty} d(v_i, r) = \limsup_{i \rightarrow \infty} d(r_i, r).
\end{aligned}$$

that blatantly contradicts itself. Hence we deduced that $q = r \in F_\Psi$ so we conclude that $\omega_\Delta(\{v_i\}) = \{q\}$ as result $\{v_i\}$ Δ -converges to a fixed point of Ψ .

□

The upcoming theorem rests on compactness principles.

Theorem 2.2.3. *Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$ which is convex nonempty compact subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ with the set of fixed points i.e $F_\Psi \neq \emptyset$ which satisfies (KSC) criterion. Consequently M^* -iteration sequence generated from (2.3) strongly converge to fixed point of Ψ .*

Proof. As given that the set $\mathcal{K}$ is compact and convex so the sequence of iterates $\{v_i\}$ lies inside the set $\mathcal{K}$, due to the compactness of set $\mathcal{K}$ the sequence $\{v_i\}$ has subsequence. Strong convergence to $v \in \mathcal{K}$ is achieved by $\{v_{i_k}\}$ of $\{v_i\}$. Additionally, we have $\lim_{i \rightarrow \infty} d(\Psi v_{i_k}, v_{i_k}) = 0$ in the context of Theorem 2.2.1. Thus, by combining these facts with Lemma 1.2.7, we have

$$d(v_{i_k}, \Psi v) \leq 5d(\Psi v_{i_k}, v_{i_k}) + d(v_{i_k}, v) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Then $\Psi v = v$ so according to Lemma 2.2.1 the existence of $\lim_{i \rightarrow \infty} d(v_i, v)$ follows hence $\{v_i\}$ is strongly convergent to v . $\square$

In the absence of compactness of the domain the strong convergence of the mappings is the following.

Theorem 2.2.4. *Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$ which is convex nonempty closed subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ with the set of fixed points i.e $F_\Psi \neq \emptyset$ which satisfies (KSC) criterion. The M^* -iteration sequence generated from (2.3) strongly converge some fixed point of Ψ as long as $\liminf_{i \rightarrow \infty} dist(v_i, F_\Psi) = 0$.*

Proof. We will skip the proof since it is straightforward to demonstrate this result. $\square$

Theorem 2.2.5. *Let us take a complete $CAT(0)$ space $\mathcal{X}$ and select a set $\mathcal{K}$ which is convex nonempty closed subset of $\mathcal{X}$. Now choose a selfmap Ψ on the $\mathcal{K}$ with the set of fixed points $F_\Psi \neq \emptyset$ i.e which satisfies (KSC) criterion. The M^* -iteration sequence generated from (2.3) strongly converge to fixed point of Ψ if condition (I) is being satisfied by Ψ .*

Proof. According to theorem 2.2.1 that $\liminf_{i \rightarrow \infty} d(x_i, \Psi x_i) = 0$. $\liminf_{i \rightarrow \infty} dist(v_i, F_\Psi) = 0$ is the result of Ψ 's condition (I). Hence, $\{v_i\}$ is strongly convergent in F_Ψ according to Theorem 2.2.4. $\square$

2.3 Numerical example

The current section begins with the numerical demonstration of the mapping that fulfills the requirements of (KSC) -condition but fails to obey the (C) -condition. Subsequently, we establish that the M^* -iteration process produces a sequence $\{v_i\}$ that achieves convergence at a faster pace compared to many other widely recognized iteration methods.

Example 2.3.1. *Let Ψ be a mapping defined on the interval $[-1, 1]$ in the following manner:*

$$\Psi v = \begin{cases} -\frac{v}{2}, & \text{if } v \in [-1, 0) / \{-\frac{1}{2}\}, \\ 0, & \text{if } v = \{-\frac{1}{2}\}, \\ -\frac{v}{4}, & \text{if } v \in [0, 1]. \end{cases}$$

Now, it is evident that the above selfmap Ψ is not enriched with the condition C . For instance, if $v = -\frac{1}{2}$ and $w = -\frac{4}{5}$ are selected then the condition C is not satisfied by Ψ . We ultimately prove that this map has (KSC) -condition enrichment. The following non-trivial circumstances are taken into consideration in order to accomplish the goal, while certain elementary cases have been left out. **C1:** When $v, w \in [-1, 0) / \{-\frac{1}{2}\}$, we have

$$\begin{aligned} d(\Psi v, \Psi w) &= d\left(\frac{v}{2}, \frac{w}{2}\right) \leq \frac{3}{4}[d(v, w)] \leq \frac{1}{2}\left[\left|\frac{3v}{2}\right| + \left|\frac{3w}{2}\right|\right] \\ &= \frac{1}{2}\left[\left|\left(\frac{-v}{2}\right) - v\right| + \left|w - \left(\frac{-w}{2}\right)\right|\right] = \frac{1}{2}[|v - \Psi v| + |w - \Psi w|] \\ &= \frac{1}{2}[d(v, \Psi v) + d(w, \Psi w)]. \end{aligned}$$

C2: When $v, w \in [0, 1]$, we have

$$\begin{aligned}
d(\Psi v, \Psi w) &= d\left(\frac{v}{4}, \frac{w}{4}\right) \leq \frac{1}{4}[|v| + |w|] \leq \frac{3}{8}[|v| + |w|] \\
&= \frac{1}{2}\left[\left|\frac{3v}{4}\right| + \left|\frac{3w}{4}\right|\right] = \frac{1}{2}\left[\left|\frac{v}{4} - v\right| + \left|w - \frac{w}{4}\right|\right] \\
&= \frac{1}{2}[|v - \Psi v| + |w - \Psi w|] \\
&= \frac{1}{2}[d(v, \Psi v) + d(w, \Psi w)].
\end{aligned}$$

C3: When $v \in [-1, 0)/\{-\frac{1}{2}\}$ and $w \in [0, 1]$ we have

$$\begin{aligned}
d(\Psi v, \Psi w) &= d\left(\frac{v}{2}, \frac{w}{4}\right) \leq \frac{1}{2}|v| + \frac{1}{4}|w| \leq \frac{3}{4}|v| + \frac{3}{8}|w| \\
&= \frac{1}{2}\left[\left|\frac{3v}{2}\right| + \left|\frac{3w}{4}\right|\right] = \frac{1}{2}\left[\left|\left(\frac{-v}{2}\right) - v\right| + \left|w - \frac{w}{4}\right|\right] \\
&= \frac{1}{2}[|v - \Psi v| + |w - \Psi w|] \\
&= \frac{1}{2}[d(v, \Psi v) + d(w, \Psi w)].
\end{aligned}$$

C4: When $v \in [-1, 0)/\{-\frac{1}{2}\}$ and $w \in \{-\frac{1}{2}\}$, we have

$$\begin{aligned}
d(\Psi v, \Psi w) &= d\left(\frac{v}{2}, 0\right) = \left|\frac{v}{2}\right| \leq \left|\frac{3v}{4}\right| \leq \left|\frac{3v}{4}\right| + \left|\frac{w}{2}\right| \\
&= \frac{1}{2}\left[\left|\frac{3v}{2}\right| + |w|\right] = \frac{1}{2}\left[\left|\left(\frac{-v}{2}\right) - v\right| + |w - 0|\right] \\
&= \frac{1}{2}[|v - \Psi v| + |w - \Psi w|] \\
&= \frac{1}{2}[d(v, \Psi v) + d(w, \Psi w)].
\end{aligned}$$

C5: When $v \in [0, 1]$ and $w \in \{-\frac{1}{2}\}$, we have

$$\begin{aligned}
d(\Psi v, \Psi w) &= d\left(\frac{v}{4}, 0\right) = \left|\frac{v}{4}\right| \leq \left|\frac{3v}{8}\right| \leq \left|\frac{3v}{8}\right| + \left|\frac{w}{2}\right| \\
&= \frac{1}{2}\left[\left|\frac{3v}{4}\right| + |w|\right] = \frac{1}{2}\left[\left|\frac{v}{4} - v\right| + |w - 0|\right] \\
&= \frac{1}{2}[|v - \Psi v| + |w - \Psi w|] \\
&= \frac{1}{2}[d(v, \Psi v) + d(w, \Psi w)].
\end{aligned}$$

To design a table and graph of the sequence $\{v_i\}$ of M^* -iteration scheme which demonstrates the more quick convergence to fixed point 0 of Ψ as compared to the Ishikawa, Abbas, Thukar and Noor and iterative schemes. Let us take the values of

a_i , b_i , and c_i are 0.70, 0.65, and 0.90, respectively. The results concluded from the iteration are furnished in Table 2.1 whereas the way of behaving of iterates is drawn in Figure 2.1. The M^* -iterative scheme efficiency is observable from the table and graph simultaneously.

We will put an end to this chapter by giving an example that is non-trivial.

Example 2.3.2. Let $B_1 = \{(v, 0) : v \in \mathbb{R}\}$ and $B_2 = \{(0, w) : w \in \mathbb{R}\}$. Put $B = B_1 \cup B_2$. Clearly, $B \subset \mathbb{R}^2$. Define d on B as follows:

$$d((v_1, v_2), (w_1, w_2)) = \begin{cases} |v_1 - w_1| & \text{if } v_2 = 0 = w_2 \\ |v_2 - w_2| & \text{if } v_1 = 0 = w_1 \\ |v_1| + |w_2| & \text{if } v_2 = 0 = w_1. \end{cases}$$

The space (B, d) is categorised as a $CAT(0)$ space in the present context, but it does not qualify the criteria to be thought about as a Banach space [59]. Furthermore, it should be noted that B possesses the properties of being both closed and convex. Now let Ψ be the metric projection on B , then by a well-known result (see, p178 in [10]) that Ψ is nonexpansive and hence it satisfies (KSC)-condition. So according to our principle results, the sequence generated by (2.3) converges to a point which is the fixed point of Ψ .

2.4 Application to differential equations

Here, we look at the solution of an FDE using our freshly made mapping configuration. Several authors have considered this matter concerning nonexpansive mappings [95] and other spaces [71, 72]. Always remember that our strategy is unique and is based on the category of mappings with (KSC). The conventional approaches and our approaches to the problem are fundamentally different, and for our approach, KSC mappings are not necessarily continuous across domains. In addition, our iterative method uses fewer iterations to provide highly accurate numerical results, making it more efficient overall. For the purpose of achieving our goal, we will employ the approach suggested by [73].

The following broad category of fractional calculus boundary value issues is

Table 2.1: Iterative numerical fixed points of the selfmap in Example 4.3.1.

Value of i	M^* iteration	Thakur iteration	Abbas	Noor iteration	Ishikawa
1	0.99200000	0.99200000	0.99200000	0.99200000	0.99200000
2	0.07208438	0.07858126	0.13037101	0.13631063	0.28833801
3	0.00524863	0.00623737	0.01716821	0.01876827	0.08397831
4	0.00038217	0.00049508	0.00226084	0.002584157	0.02445801
5	0.00002783	0.00003928	0.00029773	0.00035581	0.00712358
6	0.00000203	0.00000312	0.00003921	0.00004898	0.00207476
7	0.00000013	0.00000023	0.00000517	0.00000673	0.00060428
8	0	0.00000002	0.00000068	0.00000093	0.00017598
9	0	0	0.00000009	0.00000012	0.00005127
10	0	0	0.00000001	0.00000002	0.00001493
11	0	0	0	0	0.00000433
12	0	0	0	0	0.00000127
13	0	0	0	0	0.00000037
14	0	0	0	0	0.00000011
15	0	0	0	0	0.00000003
16	0	0	0	0	0

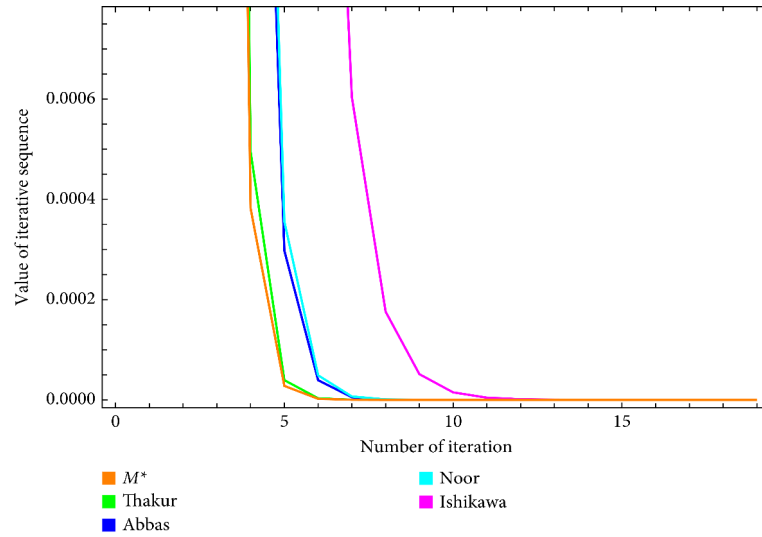


Figure 2.1: Graphical illustration of the iterates.

examined:

$$\left. \begin{aligned} D^\gamma h(t) + \Omega(u, h(t)) &= 0, \\ h(0) &= h(1) = 0, \end{aligned} \right\} \quad (2.11)$$

where D^γ , $(1 < \gamma < 2)$, and $(0 \leq t \leq 1)$ represent the Caputo fractional derivative with order γ and $\Omega : [0, 1] \times \mathbb{R}$, respectively.

where D^γ denote the Caputo fractional derivative of order γ where as γ lies in $(1 < \gamma < 2), (0 \leq t \leq 1)$ and $\Omega : [0, 1] \times \mathbb{R}$, respectively.

A certain function connected to 2.11 is now aim to establish wiith the spaces $B = C[0, 1]$. Because in the case of BVPs, we need such functions for the purpose to construct a certain operator equation in way that the fixed point can become same as a solution of the constructed operator. This function is called Green's function and now taken the following form:

$$G(t, s) = \begin{cases} \frac{1}{\Gamma(\xi)} (t(1-s)^{(\xi-1)} - (t-s)^{(\xi-1)}), & \text{if } s \leq t \leq 1, \\ \frac{t(1-s)^{(\xi-1)}}{\Gamma(\xi)}, & \text{if } t \leq s \leq 1. \end{cases}$$

All setting is now set for the main outcomes of this section. We consider the following result that completed the research of this section.

Theorem 2.4.1. *Construct a selfmap $\Psi : B \rightarrow B$ given as:*

$$\Psi(v(t)) = \int_0^1 G(t, s) \Omega(s, v(s)) ds, \quad \text{for each } v(t) \in B.$$

When

$$|\Omega(v, h(v)) - \Omega(v, g(v))| \leq \frac{1}{2} (|h(v) - \Psi(h(v))| + |g(v) - \Psi(g(v))|),$$

if $\liminf_{i \rightarrow \infty} \text{dist}(v_i, S) = 0$ such that S is a a set that contains some solutions of (2.11) and M iteration sequence (2.3). This sequence has a strong limit which is a solution of this problem.

Proof. The desired answer can be put in the integral form by representing G as Green's function in our problem according to [58] follows.

$$h(u) = \int_0^1 G(u, v) \Omega(v, h(v)) dv.$$

As a result it lead to that each selection of $h, g \in B$ and $0 \leq u \leq 1$,

$$\begin{aligned}
d(\Psi(h(u)), \Psi(g(u))) &\leq d\left(\int_0^1 G(u, v)\Omega(v, h(v))dv, \int_0^1 G(u, v)\Upsilon(v, g(v))dv\right) \\
&= \left|\int_0^1 G(u, v)[\Omega(v, h(v)) - \Omega(v, g(v))]dv\right| \\
&\leq \int_0^1 G(u, v)|\Omega(v, h(v)) - \Omega(v, g(v))|dv \\
&\leq \int_0^1 G(u, v)\left(\frac{1}{2}|h(v) - \Psi(h(v))| + \frac{1}{2}|g(v) - \Psi(g(v))|\right)dv \\
&\leq \left(\frac{1}{2}\|h(v) - \Psi(h(v))\| + \frac{1}{2}\|g(v) - \Psi(g(v))\|\right) \\
&\quad \left(\sup_{t \in [0,1]} \int_0^1 G(u, v)dv\right) \\
&\leq \frac{1}{2}d(h(v), \Psi(h(v))) + \frac{1}{2}d(g(v), \Psi(g(v))) \\
&= \frac{1}{2}(d(h(v)), \Psi(h(v))) + d(g(v), \Psi(g(v))).
\end{aligned}$$

As a consequence, we obtain

$$d(\Psi(h), \Psi(g)) \leq \frac{1}{2}(d(h, \Psi(h)) + d(g, \Psi(g))).$$

Therefore, according to the Theorem 2.2.4, Ψ satisfies (KSC) condition, and The sequence of the M iterates converges to a fixed point of Ψ which leads to the solution of the given equation. $\square$

2.5 Conclusions

Under the iterative method M^* in a CAT(0) space setting, existence as well as iterative constructional for the class of mappings fulfilling the (KSC)-condition are established. For these mappings, we established Δ and strong convergence findings under certain mild assumptions. The class of mappings meeting the (KSC)-condition differs from the class of mappings satisfying the (C)-condition, as an example has demonstrated. In the end, we carried out a comparative numerical experiment and demonstrated the superiority of the M^* iterative scheme over the several alternative iterative schemes in the class of (KSC) mappings. Additionally, one application is completed. Our findings extend and enhance some of Ullah and Arshad [70]'s primary findings from the (C)-

condition example to the broader (KSC)-condition situation. Similar to how Abbas [20], Agarwal [57], Noor [19] Thakur [69], and others' findings are expanded upon by ours.

Chapter 3

Approximation of Fixed Points for Reich-Suzuki type Nonexpansive Mappings in CAT(0) Spaces

3.1 Introduction

Let us take $\mathcal{X}$ as metric space and consider a nonempty subset $\mathcal{K}$ of $\mathcal{X}$. Assume that Ψ represents a selfmap of $\mathcal{K}$.

$$\text{Determine } y_0 \in \mathcal{K} \text{ such that } \Psi y_0 = y_0. \quad (3.1)$$

F_Ψ will designate the set of all fixed points of Ψ in $\mathcal{K}$.

A highly fruitful area of research in mathematics has been fixed point problems, which involve the mapping of an element within a set back to itself via a function. This chapter explores the Problem (3.1). The prime objective of it is to find a fixed point of Ψ , which is a selfmap on a nonempty subset $\mathcal{K}$ of a metric space $\mathcal{X}$. In a recent study by Sahu et al., [20], the Problem of quasi-expansive mappings was examined, and its applicability in convex programming and feasibility problems was demonstrated.

Usurelu et al. [20] examined Problem (3.1) in the context of hybrid mappings, illustrating its relationship to split equilibrium problems. Similarly, Yao et al. [33] examined the issue using monotone mappings and established a connection with split equilibrium problems. Building upon these developments, this chapter further explores the Problem (3.1) by examining a broader range of nonlinear mappings. We aim to expand the scope of fixed-point theory by investigating this issue within a wider framework and showcasing its practicality in the domain of split feasibility problems.

Once a result of a fixed point is established in a linear space (e.g. Hilbert or Banach space), then its applications are vehemently desirous. In a nonlinear space (e.g. geodesic metric space) Nonetheless, the principle of convexity is needed for this expansion. A significant contribution was made by Takahashi [34] that established the concept of convexity in nonlinear domains, which paved the way for finding fixed points of mappings in the fixed-point theory and the related subjects in these broader contexts. (For further information, see [103–106]).

Remark 1. *As we know from the literature of fixed point theory, nonexpansive mappings and Suzuki nonexpansive mappings are two vastly researched categories. As demonstrated by Suzuki [68], all nonexpansive mappings are Suzuki nonexpansive, but the converse is not true.*

There is an abundance of findings in the scientific literature regarding two categories, Suzuki nonexpansive and nonexpansive mappings. In a uniform convex Banach space, the nonexpansive mappings invariably attain fixed points depending upon some underlying assumptions shown by Gohde [15] and Browder [35]. Suzuki demonstrated that this result holds true for all Suzuki maps in existence. Kirk demonstrated that the Browder and Gohde result holds true in nonlinear space.

Hassan et al. [38] introduced the technique described below:

$$\left\{ \begin{array}{l} v_1 \in \mathcal{K}, \\ w_i = \Psi((1 - \xi_i)v_i + \xi_i\Psi v_i), \\ x_i = \Psi((1 - \mu_i)w_i + \mu_i\Psi w_i), \\ y_i = \Psi((1 - \lambda_i)x_i + \lambda_i\Psi x_i), \\ v_{i+1} = \Psi((1 - \kappa_i)y_i + \kappa_i\Psi y_i), i \geq 1, \end{array} \right. \quad (3.2)$$

where $\kappa_i, \lambda_i, \mu_i, \xi_i \in (0, 1)$.

This chapter explores fixed point iteration strategies in the context of CAT(0) spaces, which are a broader category of nonlinear spaces. Fixed point techniques have demonstrated their worth in the field of applied analysis, namely in the domains of image processing and computer science [42]. Our research enhances the outcomes achieved in prior iterative systems by expanding their application to the wider context of CAT(0) spaces. In a previous study conducted by Hassan et al. [38], it was demonstrated that a particular approach was more effective when compared to several other options like Mann [18], Picard [60], Noor [19], Agarwal [57], Ishikawa [63], Abbas [20], Thakur et al. [69] Ullah et al. [39] have recently expanded their study to include nonexpansive mappings of the Reich-Suzuki type. For more information, see to [40, 41].

3.2 Convergence results

We are going ahead to deduce many important convergence results in the term of CAT(0) spaces. Firstly, we proceed to change the scheme (3.2) into CAT(0) spaces using the following iteration.

$$\begin{cases} v_1 \in \mathcal{K}, \\ w_i = \Psi((1 - \xi_i)v_i \oplus \xi_i \Psi v_i), \\ x_i = \Psi((1 - \mu_i)w_i \oplus \mu_i \Psi w_i), \\ y_i = \Psi((1 - \lambda_i)x_i \oplus \lambda_i \Psi x_i), \\ v_{i+1} = \Psi((1 - \kappa_i)y_i \oplus \kappa_i \Psi y_i), i \geq 1, \end{cases} \quad (3.3)$$

where $\kappa_i, \lambda_i, \mu_i, \xi_i \in (0, 1)$. Here, it is essential to highlight that $\mathcal{X}$ acts as complete CAT(0) space in the present context.

With the subsequent Lemma, we commence our principal result.

Lemma 3.2.1. *Let us suppose that a selfmap Ψ on a nonempty convex, closed subset $\mathcal{K}$ of $\mathcal{X}$. If the sequence $\{v_i\}$ generated by the use of equation (3.3) and a nonexpansive mapping Ψ of Reich-Suzuki type with the set of all fixed points, i.e. $F_\Psi \neq \emptyset$. Then for any choice of $y_0 \in F_\Psi$ the existence of $\lim_{i \rightarrow \infty} d(v_i, y_0)$ will follow.*

Proof. Suppose that y_0 be an element of the set F_Ψ . By employing (3.3) in conjunction

with Proposition 1.1.2(i), we obtain

$$\begin{aligned}
d(w_i, y_0) &= d(\Psi((1 - \xi_i)v_i \oplus d_i \Psi v_i), y_0) \\
&\leq d((1 - \xi_i)v_i \oplus \xi_i \Psi v_i, y_0) \\
&\leq (1 - \xi_i)d(v_i, y_0) + \xi_i d(\Psi v_i, y_0) \\
&\leq (1 - \xi_i)d(v_i, y_0) + \xi_i d(v_i, y_0) \\
&\leq d(v_i, y_0).
\end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}
d(x_i, y_0) &= d(\Psi((1 - \mu_i)w_i \oplus \mu_i \Psi w_i), y_0) \\
&\leq d((1 - \mu_i)w_i \oplus \mu_i \Psi w_i, y_0) \\
&\leq (1 - \mu_i)d(w_i, y_0) + \mu_i d(\Psi w_i, y_0) \\
&\leq (1 - \mu_i)d(w_i, y_0) + \mu_i d(w_i, y_0) \\
&\leq d(w_i, y_0).
\end{aligned} \tag{3.5}$$

Also

$$\begin{aligned}
d(y_i, y_0) &= d(\Psi((1 - \lambda_i)x_i \oplus \lambda_i \Psi p_i), y_0) \\
&\leq d((1 - \lambda_i)x_i \oplus \lambda_i \Psi x_i, y_0) \\
&\leq (1 - \lambda_i)d(x_i, y_0) + \lambda_i d(\Psi x_i, y_0) \\
&\leq (1 - \lambda_i)d(x_i, y_0) + \lambda_i d(x_i, y_0) \\
&\leq d(x_i, y_0).
\end{aligned} \tag{3.6}$$

Now (3.4), (3.5) and (3.6) suggest that

$$\begin{aligned}
d(v_{i+1}, y_0) &= d(\Psi((1 - \kappa_i)y_i \oplus \kappa_i \Psi y_i), y_0) \\
&\leq d((1 - \kappa_i)y_i \oplus \kappa_i \Psi y_i, y_0) \\
&\leq (1 - \kappa_i)d(y_i, y_0) + \kappa_i d(\Psi y_i, y_0) \\
&\leq (1 - \kappa_i)d(y_i, y_0) + \kappa_i d(y_i, y_0) \\
&\leq d(y_i, y_0) \leq d(x_i, y_0) \leq d(w_i, y_0) \\
&\leq d(v_i, y_0).
\end{aligned} \tag{3.7}$$

Therefore, we have demonstrated that for every y_0 belonging to F_Ψ , the inequality $d(v_{i+1}, y_0) \leq d(v_i, y_0)$ holds true. The set $\{d(v_i, y_0)\}$ is both bounded and nonincreasing. Hence, the limit of $d(v_i, y_0)$ as i approaches infinity occurs for every y_0 belonging to the set F_Ψ . $\square$

Now, we are able to present a fundamental theorem in this chapter that will assist us in proving the convergence theorem in the following part.

Theorem 3.2.1. *Let us suppose that a selfmap Ψ on a nonempty convex, closed subset $\mathcal{K}$ of $\mathcal{X}$. If the sequence $\{v_i\}$ is generated by the use of equation (3.3) and a nonexpansive mapping Ψ of Reich-Suzuki type. The condition $F_\Psi \neq \emptyset$ holds if and only if the set $\{v_i\}$ is bounded and the limit $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i)$ approaches zero.*

Proof. Assume that F_Ψ is not empty and the objective is to prove that the set $\{v_i\}$ is bounded, with $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0$. Let $y_0 \in F_\Psi$ be any point. According to Lemma 3.2.1, it is known that $\{v_i\}$ is bounded and that $\lim_{i \rightarrow \infty} d(v_i, y_0)$ exists.

The only thing left to do to show that $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0$. Suppose

$$\lim_{i \rightarrow \infty} d(v_i, x_0) = \eta, \tag{3.8}$$

Where η is a constant that can take any value from the interval $[0, \infty)$, we only investigate the scenario where $\eta > 0$. Now, in accordance with equation (3.4),

$$d(w_i, y_0) \leq d(v_i, y_0),$$

$$\Rightarrow \limsup_{i \rightarrow \infty} d(w_i, y_0) \leq \limsup_{i \rightarrow \infty} d(v_i, y_0) = \eta. \quad (3.9)$$

Also by Proposition 1.1.2(a), we have

$$\begin{aligned} d(\Psi v_i, y_0) &\leq d(v_i, y_0), \\ \Rightarrow \limsup_{i \rightarrow \infty} d(\Psi v_i, y_0) &\leq \limsup_{i \rightarrow \infty} d(v_i, y_0) = \eta. \end{aligned} \quad (3.10)$$

Now from (3.7), we have

$$d(v_{i+1}, y_0) \leq d(w_i, y_0).$$

By utilising this combined with equation (4.3), we deduce

$$\eta \leq \liminf_{i \rightarrow \infty} d(w_i, y_0). \quad (3.11)$$

By utilising equations (4.4) and (4.6), we derive

$$\lim_{i \rightarrow \infty} d(w_i, y_0) = \eta. \quad (3.12)$$

Using (4.7), we get

$$\begin{aligned} \eta &= \lim_{i \rightarrow \infty} d(w_i, y_0) = \lim_{i \rightarrow \infty} d(\Psi((1 - \xi_i)v_i \oplus \xi_i \Psi v_i), y_0) \\ &\leq \lim_{i \rightarrow \infty} d((1 - \xi_i)v_i \oplus \xi_i \Psi v_i, y_0) \\ &\leq \lim_{i \rightarrow \infty} (1 - \xi_i)d(v_i, y_0) + \lim_{i \rightarrow \infty} \xi_i d(\Psi v_i, y_0) \\ &\leq \lim_{i \rightarrow \infty} (1 - \xi_i)d(v_i, y_0) + \lim_{i \rightarrow \infty} \xi_i d(v_i, y_0) \\ &= \lim_{i \rightarrow \infty} d(v_i, y_0) \\ &= \eta. \end{aligned}$$

Consequently, we have

$$\eta = \lim_{i \rightarrow \infty} \xi_i((1 - \xi_i)v_i \oplus \xi_i \Psi v_i, y_0). \quad (3.13)$$

By utilising equations (4.3), (4.5), and (4.8) and applying Lemma 1.1.5, we derive the following facts

$$\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0.$$

In the converse scenario, we assume that $\{v_i\}$ is a bounded sequence with the $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0$. Demonstrate that F_Ψ is not an empty set. In order to accomplish this, we make the assumption that any $y_0 \in A(\mathcal{X}, \{v_i\})$, and then we proceed to use Lemma 1.2.3 in the following manner.

$$\begin{aligned} A(\Psi y_0, \{v_i\}) &= \limsup_{i \rightarrow \infty} d(v_i, \Psi y_0) \\ &\leq \frac{(\alpha + 3)}{(1 - \alpha)} \limsup_{i \rightarrow \infty} d(v_i, \Psi v_i) + \limsup_{i \rightarrow \infty} d(v_i, y_0) \\ &= \limsup_{i \rightarrow \infty} d(v_i, y_0) \\ &= A(y_0, \{v_i\}). \end{aligned}$$

Consequently, we obtain $\Psi y_0 \in \mathcal{A}(\mathcal{X}, \{v_i\})$ and we know that the set $\mathcal{A}(\mathcal{X}, \{v_i\})$ contains only one element, thus we come to the conclusion that the point y_0 is an element of F_Ψ , as stated by the equation $y_0 = \Psi y_0$. Thus, the set of fixed points F_Ψ is not empty. □

In the coming section, we will discuss the introductory convergence theorem.

Theorem 3.2.2. *Let us suppose that a selfmap Ψ on a nonempty convex, closed subset $\mathcal{K}$ of $\mathcal{X}$. If the conditions provided in Theorem 3.2.1 are satisfied by Ψ and the sequence $\{v_i\}$ and the set of all fixed points i.e. $F_\Psi \neq \emptyset$. Then, the strong convergence to the fixed point of F_Ψ of the sequence $\{v_i\}$ will occur.*

Proof. Given that the sequence $\{v_i\}$ generated by Ψ lies within the compact set $\mathcal{K}$, so by the compactness property there exists a subsequence $\{v_{i_k}\}$ of $\{v_i\}$ and a point $q_0 \in \mathcal{K}$ such that the $d(v_{i_k}, q_0) \rightarrow 0$. It is sufficient to establish that the strong limit of the sequence $\{v_i\}$ is q_0 .

In order to accomplish this, we employ Lemma to acquire the desired outcome.

$$d(v_{i_k}, \Psi q_0) \leq \frac{(\alpha + 3)}{(1 - \alpha)} d(v_{i_k}, \Psi v_{i_k}) + d(v_{i_k}, q_0) \quad (3.14)$$

Since Theorem 3.2.1 suggests that $d(v_{i_k}, \Psi v_{i_k}) \rightarrow 0$. Hence, (3.14) gives us $\lim_{k \rightarrow \infty} d(v_{i_k}, \Psi q_0) = 0$. It follows that $\Psi q_0 = q_0$, that is, $q_0 \in F_\Psi$. Lemma 3.2.1 guarantees the existence of $\lim_{i \rightarrow \infty} d(v_i, q_0)$. Therefore, we may conclude that q_0 is the strong limit of $\{v_i\}$. $\square$

Theorem 3.2.3. *Let us suppose that a selfmap Ψ on a nonempty convex, closed subset $\mathcal{K}$ of $\mathcal{X}$. If the conditions provided in Theorem 3.2.1 are satisfied by Ψ and the sequence $\{v_i\}$ and the set of all fixed points i.e. $F_\Psi \neq \emptyset$. Then the strong convergence to the fixed point of F_Ψ of the sequence $\{v_i\}$ will be occurred, provided that $\liminf_{i \rightarrow \infty} d(v_i, F_\Psi) = 0$.*

Proof. The specific details of this result are omitted because proving it does not require much effort. $\square$

A convergence theorem is now desired that does not require the domain to be compact. It is necessary to have the following criterion.

Definition 1. [6] *A mapping Ψ is said to satisfy the condition I if Ψ is a selfmap on a nonempty convex closed subset $\mathcal{K}$ of $\mathcal{X}$ if there exists μ with $\mu(k) > 0$ for all $k > 0$, $\mu(0) = 0$ and $d(u, \Psi u) \geq \mu(d(u, F_\Psi))$ for all elements $u \in \mathcal{K}$.*

Theorem 3.2.4. *Let us suppose that a selfmap Ψ on a nonempty convex, closed subset $\mathcal{K}$ of $\mathcal{X}$. If the conditions provided in Theorem 3.2.1 are satisfied by Ψ and the sequence $\{v_i\}$ and the set of all fixed points i.e. $F_\Psi \neq \emptyset$. If Ψ satisfies condition (I) also, then the strong convergence to the fixed point of F_Ψ of the sequence $\{v_i\}$ will occur.*

Proof. The results of the theorem 3.2.1 give us

$$\liminf_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0. \quad (3.15)$$

The condition (I) of Ψ suggests

$$d(v_i, \Psi v_i) \geq \mu(d(v_i, F_\Psi)).$$

By applying condition I and (3.15), one has

$$\liminf_{i \rightarrow \infty} \mu(d(v_i, F_\Psi)) = 0.$$

However, for each option of $k > 0$, the non-decreasing map $\mu : [0, \infty) \rightarrow [0, \infty)$, with $\mu(k) > 0$ and $\mu(0) = 0$, hence

$$\liminf_{i \rightarrow \infty} d(v_i, F_\Psi) = 0.$$

All the necessary conditions of Theorem 4.2.4 have been fulfilled. Therefore, based on its results, the sequence $\{v_i\}$ exhibits strong convergence within the set F_Ψ .

□

The following represents the ultimate outcome, which proves the Δ -convergence of $\{v_i\}$ as produced by (3.3).

Theorem 3.2.5. *Let us suppose that a selfmap Ψ on a nonempty convex closed subset $\mathcal{K}$ of $\mathcal{X}$. If the conditions provided in Theorem 3.2.1 are satisfied by Ψ and the sequence $\{v_i\}$ and the set of all fixed points i.e. $F_\Psi \neq \emptyset$. Δ -converges to the point of F_Ψ of the sequence $\{v_i\}$ will occurred.*

Proof. As a convex set $\mathcal{K}$, it follows that $\{v_i\}$ is contained in $\mathcal{X}$. By Theorem 3.2.1, $\{v_i\}$ is both bounded and satisfies $\lim_{i \rightarrow \infty} d(v_i, \Psi v_i) = 0$. Let $\omega_\Delta(\{v_i\}) = \bigcup A(\{s_i\})$, where $\{s_i\}$ is any subsequence of $\{v_i\}$. The objective is to demonstrate that $\omega_\Delta(\{v_i\}) \subseteq F_\Psi$. Given $s \in \omega_\Delta(\{v_i\})$, we can select a sub-sequence, denoted as $\{s_i\}$, from the sequence $\{v_i\}$ such that $A(\{s_i\}) = \{s\}$. Lemma 2.4 and Lemma 2.5 state that it is possible to find a sub-sequence $\{e_i\}$ of $\{s_i\}$ that has the Δ -limit e in B . However, Theorem 3.2.1 implies that $\lim_{i \rightarrow \infty} d(e_i, \Psi e_i) = 0$.

By Lemma 1.2.3,

$$d(e_i, \Psi e) \leq \frac{(3 + \alpha)}{(1 - \alpha)} d(e_i, \Psi e_i) + d(e_i, e). \quad (3.16)$$

By applying the limit supremum to both sides of the equation (3.16), we may conclude that e belongs to the set F_Ψ . According to Lemma 3.2.1, $\lim_{i \rightarrow \infty} d(e_i, e)$ exists. We must demonstrate that the variable s is equal to the variable e . Let us assume, on the contrary, that s is not equal to e . Therefore, due to the unique nature of asymptotic

centres, one can conclude

$$\begin{aligned}
\limsup_{i \rightarrow \infty} d(e_i, e) &< \limsup_{i \rightarrow \infty} d(e_i, s) \leq \limsup_{i \rightarrow \infty} d(s_i, s) \\
&< \limsup_{i \rightarrow \infty} d(s_i, e) = \limsup_{i \rightarrow \infty} d(v_i, e) \\
&= \limsup_{i \rightarrow \infty} d(e_i, e).
\end{aligned}$$

We have demonstrated that $\limsup_{i \rightarrow \infty} d(e_i, e) < \limsup_{i \rightarrow \infty} d(e_i, e)$. Consequently, we may deduce that $s = e$ in the set F_Ψ and that $\omega_\Delta(\{r_i\}) \subseteq F_\Psi$.

We aim to demonstrate that the sequence $\{v_i\}$ fundamentally Δ -converges within the space F_Ψ . In order to accomplish the objective, we will demonstrate that $\omega_\Delta(\{v_i\})$ is comprised of a single point. Suppose that a given sub-sequence $\{s_i\}$ of $\{v_i\}$, according to Lemmas 1.2.4 and 1.2.5, we may get the conclusion that there exists a sub-sequence of $\{s_i\}$ which is denoted by $\{e_i\}$ that is Δ -convergent sub-sequence with the Δ -lim $e \in \mathcal{K}$. Let's presume that $\mathcal{A}(\{s_i\}) = \{s\}$ and $\mathcal{A}(\{v_i\}) = \{g\}$. It is essential to remember that we previously demonstrated that s and e are equal and that e is an element of the set F_Ψ . If that is not the case, then the $\lim_{i \rightarrow \infty} d(v_i, e)$ exists, and the asymptotic centres are likewise singletons. In this scenario, we have:

$$\begin{aligned}
\limsup_{i \rightarrow \infty} d(e_i, e) &< \limsup_{i \rightarrow \infty} d(e_i, g) \leq \limsup_{i \rightarrow \infty} d(v_i, g) \\
&< \limsup_{i \rightarrow \infty} d(v_i, e) = \limsup_{i \rightarrow \infty} d(e_i, g).
\end{aligned}$$

Therefore, since it is a contradiction, we can conclude that $g = e \in F_\Psi$. Therefore, the set $\omega_\Delta(\{s\}) = \{g\}$. This provides conclusive evidence. $\square$

An example is provided to conclude this section.

Example 3.2.1. Let $\mathcal{X}$ be the set defined as $\mathcal{X} = \mathbb{K}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : \langle y, y \rangle = -1, y_4 > 0\}$. $\mathbb{K}^3$ is a three-dimensional space and $\mathbb{R}^4$, is augmented with the Lorentz distance, $\langle \cdot, \cdot \rangle$.

$$y \cdot z = y_1 z_1 + y_2 z_2 + y_3 z_3 - y_4 z_4, \quad y = (y_1, y_2, y_3, y_4), \quad z = (z_1, z_2, z_3, z_4) \in \mathbb{K}^3.$$

The geodesic map q , which is deduced from the equation $q(r) = \cosh(r)y + \sinh(y)v$ and v represents a unit vector in this case. Define a mapping Ψ on $\mathbb{K}$ using the formula:

$$\Psi(y_1, y_2, y_3, y_4) = (-y_1, -y_2, -y_3, y_4).$$

we have noticed that Ψ has a fixed point $y_0 = (0, 0, 0, 1)$ which is nonexpansive Reich-Suzuki type mapping. Based on our primary findings, the sequence $\{v_i\}$ exhibits convergence towards the value y_0 .

3.3 Examples to illustrate

The following cases provide evidence for the theoretical findings:

Example 3.3.1. Let us define a mapping Ψ that correlates the values of $\mathcal{K} = [7, 9]$ in the following way. $\Psi 9 = 6$ and $\Psi y = \frac{y+42}{7}$ for $7 \leq y < 9$. Let c be equal to $\frac{1}{2}$. Now, let's examine the several scenarios listed below.

(i): Given that y and z are both in the interval $[7, 9)$, we may define $\Psi y = \frac{y+42}{7}$ and $\Psi z = \frac{z+42}{7}$. By applying the triangle inequality, we obtain

$$\begin{aligned} c|y - \Psi y| + c|z - \Psi z| + (1 - 2c)|y - z| &= \frac{1}{2}|y - (\frac{y+42}{7})| + \frac{1}{2}|z - (\frac{z+42}{7})| \\ &= \frac{1}{2}|\frac{6y-42}{7}| + \frac{1}{2}|\frac{7z-42}{7}| \\ &\geq \frac{1}{2}|(\frac{6y-42}{7}) - (\frac{6z-42}{7})| \\ &= \frac{1}{2}|\frac{6y-6z}{7}| \\ &= \frac{3}{7}|y - z| \\ &\geq \frac{1}{7}|y - z| = |\Psi y - \Psi z|. \end{aligned}$$

(ii) When $y \in [7, 9)$ and $z \in \{9\}$. Then $\Psi y = \frac{y+42}{7}$ and $\Psi z = 6$. Accordingly, we

have

$$\begin{aligned}
c|y - \Psi y| + c|z - \Psi z| + (1 - 2c)|y - z| &= \frac{1}{2}|y - (\frac{y+42}{7})| + \frac{1}{2}|9 - 6| \\
&= \frac{1}{2}|\frac{6y - 42}{7}| + \frac{1}{2}|3| \\
&\geq \frac{1}{2}|3| \\
&> \frac{9}{7} \\
&\geq |\frac{y}{7}| = |\Psi y - \Psi z|.
\end{aligned}$$

(iii) When $z \in [7, 9)$ and $y \in \{9\}$. Then $\Psi z = \frac{z+42}{7}$ and $\Psi y = 6$. Accordingly, we have

$$\begin{aligned}
c|y - \Psi y| + c|z - \Psi z| + (1 - 2c)|y - z| &= \frac{1}{2}|9 - 6| + \frac{1}{2}|z - (\frac{z+42}{7})| \\
&= \frac{1}{2}|3| + \frac{1}{2}|\frac{6z - 42}{5}| \\
&\geq \frac{1}{2}|3| \\
&> \frac{7}{5} \\
&\geq |\frac{z}{5}| = |\Psi y - \Psi z|.
\end{aligned}$$

(iv) When $y, z \in \{9\}$. Then $\Psi y = \Psi z = 6$. Accordingly, we have

$$\begin{aligned}
c|y - \Psi y| + c|z - \Psi z| + (1 - 2c)|y - z| &\geq 0 \\
&= |\Psi y - \Psi z|.
\end{aligned}$$

Considering the aforementioned situations, it can be inferred that F is a map of Reich-Suzuki type with $c = \frac{1}{2}$. However, Ψ does not belong to the Suzuki type. By choosing the values $y = 8$ and $z = 9$, it can be demonstrated that $\frac{1}{2}|y - \Psi y| < 1 = |y - z|$ and $|\Psi y - \Psi z| > 1 = |y - z|$. For each positive integer i , assign the values $a_i = 0.95$, $b_i = 0.65$, and $c_i = 0.85$, with an initial value of $r_1 = 8.91$. Upon examining Tables 3.1–3.2 and Figure 3.1, it becomes evident that the current iteration under study exhibits superior performance when compared to previous iterative systems.

Here, we present a CAT(0) space example, which is not a linear Banach space.

Table 3.1: The schemes of Ullah, Hassan, and Thakur converge to the fixed point $y_0 = 7$ of the selfmap Ψ as shown in Example 5.2.1.

i	Ullah	Hassan	Thakur
1	8.91	8.91	8.91
2	7.00553935860057	7.00003282209952	7.038775510204087
3	7.00003691367663	7.00000000106116	7.000621764740881
4	7.00000023061437	7.000000000000003	7.000010272109422
5	7.00000000132067	7	7.000000175943571
6	7.000000000000703	7	7.000000003098253
7	7.000000000000004	7	7.000000000055701
8	7	7	7.000000000001021
9	7	7	7.0000000000000021
10	7	7	7

Table 3.2: The Mann, Agarwal, and Ishikawa schemes converge to the selfmap Ψ fixed point $y_0 = 7$ as shown in Example 5.2.1.

i	Mann	Agarwal	Ishikawa
1	8.91	8.91	8.91
2	7.27142857142858	7.27142857142858	7.27142857142858
3	7.15510204081634	7.03046647230322	7.14679300291546
4	7.11078717201167	7.00352333353167	7.10085777751333
5	7.08704706372346	7.00042244049996	7.07692978437881
6	7.07212470994228	7.00005207225756	7.06223462556686
7	7.06182117995054	7.00000655331131	7.05228555277216
8	7.05425123954841	7.00000083792861	7.04509928354481
9	7.04843860673968	7.00000010848184	7.03966320918896
10	7.04382540609778	7.00000001418546	7.0354060846956

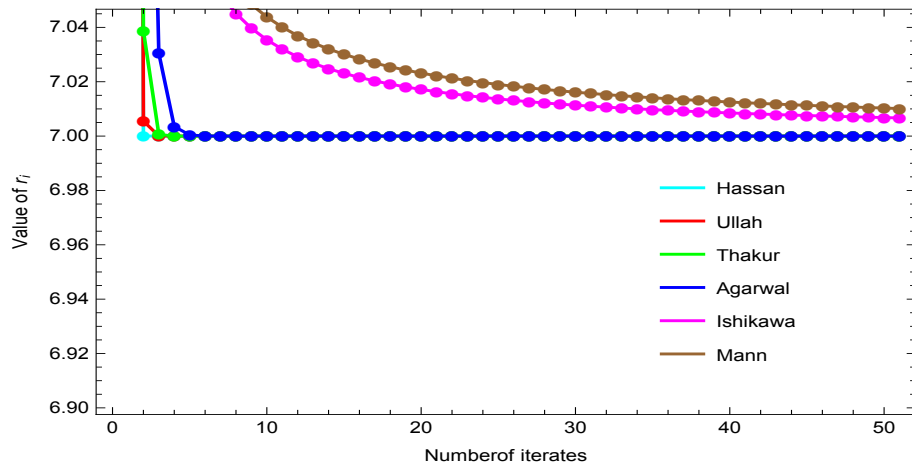


Figure 3.1: The behaviours of iterates for the schemes proposed by various writers for Example 5.2.1 are examined for the values $r_1 = 8.91$ and $i = 50$.

Subsequently, we demonstrate a nonexpansive mapping of the Reich-Suzuki type, which illustrates our primary findings.

Example 3.3.2. *Let d be the metric induced by the normal norm on $\mathbb{R}^2$, where $\mathbb{R}^2$ be a space. The pair $(\mathbb{R}^2, d)$ constitutes a metric space that is complete. We will now substitute d with a different sort of metric, denoted as ρ , which is defined by the subsequent formula:*

$$\rho((y_1, y_2), (z_1, z_2)) = \begin{cases} |y_1 - z_1| & \text{if } y_2 = 0 = z_2 \\ |y_2 - z_2| & \text{if } y_1 = 0 = z_1 \\ |y_1| + |z_2| & \text{if } y_2 = 0 = z_1. \end{cases}$$

The mathematical structure of $(\mathbb{R}^2, \rho)$ is categorized as geodesic $CAT(0)$ space in this example, but it does not fulfil the requirements to be taken into account as Banach space. For that reason, we introduce $\mathcal{K}_1 = \{(y, 0) : y \in \mathbb{R}\}$, and $\mathcal{K}_2 = \{(0, z) : z \in \mathbb{R}\}$. Let take $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. As a consequence, it can be inferred that $\mathcal{K}$ is a closed convex nonempty subset of $(\mathbb{R}^2, \rho)$ which is a complete $CAT(0)$ space. Let a metric projection on $\mathcal{K}$ be denoted by Ψ and is known to be nonexpansive, so according to a well-established result (see page 178 in [108]), it can be categorized as a Reich-Suzuki type nonexpansive projection. The iterated sequence in equation (3.3) converges to a fixed point of Ψ , according to our findings.

3.4 Application

We demonstrate the applicability of our novel result in resolving an SFP. As a way to accomplish our goal, assume that two Hilbert spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ and take two closed convex subsets such that $C \subseteq \mathcal{X}_1$ and $Q \subseteq \mathcal{X}_2$. We make the additional assumption that there exists a map $\mathcal{T}$ which is both bounded and linear, and it maps elements from $\mathcal{X}_1$ to $\mathcal{X}_2$. In this instance, we establish the SFP [48] as a problem characterized by the following:

$$\text{Compute } y_0 \in C : \mathcal{T}y_0 \in Q. \tag{3.17}$$

In this scenario, we make the assumption that the SFP (3.17) has at least one

solution. We refer to the collection of all solutions as the set S . If θ is a positive real integer and P^C and P^Q are the corresponding nearest point projections (NPP) onto sets C and Q , respectively, then $\mathcal{T}^*$ effectively serves as the adjoint operator for $\mathcal{T}$.

The point $y_0 \in C$ can be authenticated that it is a solution of SFP (3.17) if and only if it is the solution of the equation that follows (refer to [49] and other sources):

$$y = P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})y.$$

Due to the Picard iteration limitations for nonexpansive mappings, the author of the paper [?] proposed a connection between the SFP (3.17) and the class of nonexpansive mappings. They then developed an iterative approach to obtain the desired solution.

$$v_{i+1} = P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})v_i.$$

However, he only achieves weak convergence with the given procedure. Nevertheless, it is well-established that there is a strong desire to achieve a strong convergence [49] whenever weak convergence results of an iterative scheme for a peculiar problem is attained. Contrary to Byrne's [?] approach, which depends on nonexpansive mappings, this topic elaborates an analysis of strong convergence employing Reich-Suzuki-type nonexpansive mappings. Additionally, we employ a more comprehensive iterative technique to achieve strong convergence. The most important results of this section are presented below.

Theorem 3.4.1. *The solution of the SFP (3.17) with $S \neq \emptyset$, $0 < \theta < \frac{2}{\delta}$, and $P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})$ is any nonexpansive selfmap of the Reich-Suzuki type that fulfils the condition (I), then the sequence $\{v_i\}$ converges strongly to a solution of the SFP (3.17) induced by (3.3).*

Proof. All Hilbert spaces are CAT(0) spaces, as is widely stated in the literature. The map Ψ is just a nonexpansive selfmap of the Reich-Suzuki type. Thus we can write it as $\Psi = P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})$. This means that in the set F_Ψ , the strong convergence of $\{v_i\}$ can be achieved by using the Theorem 3.2.4. As $F_\Psi = S$ so we can deduce that $\{v_i\}$ converges to the solution of the SFP (3.17). $\square$

3.5 Conclusion

It is also advantageous for any operator whose domain is a subset of a nonlinear domain, such as $\text{CAT}(0)$ space, to have an approximation of a fixed point. Takahashi [34] was the first to fundamentally propose the concept of convexity in the nonlinear context of metric space for the study of fixed point theory of nonexpansive operators, as was previously indicated in the work. In essence, this work presented the $\text{CAT}(0)$ space variant of a recently proposed iterative technique credited to Hassan et al. [38]. In the $\text{CAT}(0)$ context, we looked at a number of convergence theorems for the broader class of nonlinear operators. New cases finally corroborate the primary result. Ultimately, a solution for solving SFP within the broader context of mappings is acquired.

Chapter 4

Fixed point approximations in the CAT(0) spaces for generalized nonexpansive mappings with an application

4.1 Introduction

This chapter's goal is to examine the class of GAN mapping in CAT(0) spaces, a nonlinear environment [10,11]. Even if it is established that an operator equation has a solution, conventional analytical techniques frequently fail to produce that solution. Approximating the solution becomes required in these kinds of situations. To achieve this, the operator equation must be rearranged to take the shape of a fixed point equation. We employ an iterative approach suitable for fixed point equations to solve the operator equation. With the assistance of this method, we will attain our objective fixed point. With contraction maps, the basic Picard iteration $v_{i+1} = \Psi v_i$ is given by the Banach fixed point theorem (BFPT). Picard iterations, while effective in many cases, may not always converge to a fixed point for nonexpansive mappings. As a result, other iterative techniques with different parameters and phases are used. The Mann fixed point method, presented in [18], the Ishikawa fixed point method proposed in [63], the

three-step Noor iteration introduced in [19], the two-step Normal-S presented in [23], the Picard-Mann hybrid due to Khan [24], the Krasnoselskii-Mann introduced in [29], the three-step Abbas and Nazir method, the Thakur fixed point method [69], and Picard-S, a hybrid of Picard and S iteration [25] are notable approaches. For SMs, Ullah and Arshad introduced the M iteration procedure [27] in 2018, showing faster convergence than the previously discussed techniques.

An innovative iterative procedure for generalized contraction has just been demonstrated by Ali and Ali [112]. We describe the strategy as follows:

$$\begin{cases} v_1 \in \mathcal{K}, \\ w_i = \Psi((1 - \kappa_i)v_i + \kappa_i\Psi v_i), \\ x_i = \Psi w_i, \\ v_{i+1} = \Psi x_i, t \geq 1, \end{cases} \quad (4.1)$$

where $\kappa_i \in (0, 1)$. In the context of generalized contractions, they proved that the F iteration (4.2) has better stability and a faster rate of convergence than other iterations. This chapter examines the CAT(0) variant of the aforementioned technique for GAN maps, demonstrating its convergence in both weak and strong convergence senses, illustrating its convergence in the sense of both strong and weak. We also bring forth a numerical example along with one of its applications. Fixed points for nonexpansive maps were shown to exist in uniformly convex Banach spaces (UBS) in 1965 by Browder [14] and Gphde [15], while Kirk [65] established a similar result in reflexive Banach spaces (RBS). Suzuki [68] thus developed a weaker definition of a map than that of a nonexpansive map and gave analogous fixed-point theorems in Banach spaces. They established the existence of the SM class as well. Suzuki's study prompted Pant and Shukla [17] to demonstrate in 2017 that the idea behind GAN maps is weaker than that of SMs. They also examined fixed point results for these maps using the S iteration [57]. The M iterative technique for GAN in Banach spaces was recently used by Ullah et al. [28]. This work aims to provide a single appropriate implementation of the F iteration for GAN maps in CAT(0) space.

4.2 Main Results

Now, we establish the subsequent CAT(0) version of the F-iteration.

$$(4.1) \quad \begin{cases} v_1 \in \mathcal{K}, \\ w_i = \Psi((1 - \kappa_i)v_i \oplus \kappa_i \Psi v_i), \\ x_i = \Psi w_i, \\ v_{i+1} = \Psi x_i, i \geq 1, \end{cases} \quad (4.2)$$

where $\kappa_i \in (0, 1)$. Consider first the subsequent result.

Lemma 4.2.1. *Let a CAT(0) space $\mathcal{X}$ and $\mathcal{K}$ nonempty convex closed in $\mathcal{X}$ with $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ is a GAN map. If F_Ψ is nonempty and $\{v_i\}$ a sequence produced by (4.2). Then $\lim_{i \rightarrow \infty} d(v_i, l^*)$ exists for any $l^* \in F_\Psi$.*

Proof. Suppose $l^* \in F_\Psi$. According to Proposition 1.2.1(ii), one has

$$\begin{aligned} d(w_i, l^*) &= d(\Psi((1 - \kappa_i)v_i + \kappa_i \Psi v_i), l^*) \\ &\leq d((1 - \kappa_i)v_i + \kappa_i \Psi v_i, l^*) \\ &\leq (1 - \kappa_i)d(v_i, l^*) + \kappa_i d(\Psi v_i, l^*) \\ &\leq (1 - \kappa_i)d(v_i, l^*) + \kappa_i d(v_i, l^*) \\ &\leq d(v_i, l^*). \end{aligned}$$

One has

$$\begin{aligned} d(v_{i+1}, l^*) &= d(\Psi x_i, l^*) \\ &\leq d(x_i, l^*) = d(\Psi w_i, l^*) \\ &\leq d(w_i, l^*) \leq d(v_i, l^*). \end{aligned}$$

Hence, we obtain that the sequence $\{d(v_{i+1}, l^*)\}$ is a real numbers sequence such that it is bounded and nonincreasing in Ψ . From this, we conclude that $\lim_{i \rightarrow \infty} d(v_i, l^*)$ exists for all choices of $l^* \in F_\Psi$.

We also have the helpful results shown below.

Theorem 4.2.1. *Given a $CAT(0)$ space $\mathcal{X}$ and $\mathcal{K}$ nonempty convex closed in $\mathcal{X}$ with $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ is a GAN map. If a sequence $\{v_i\}$ is generated by (4.2). Then, $F_\Psi \neq \emptyset$ if and only if $\lim_{i \rightarrow \infty} d(\Psi v_i, v_i) = 0$ and $\{v_i\}$ is bounded in Ψ as well.*

Proof. Assume that F_Ψ is a nonempty set. Then, we can take any point, namely, l^* , in it. By previous lemma, $\lim_{i \rightarrow \infty} d(v_i, l^*)$ exists. Therefore, we may conclude that this limit is equal to some real number. Hence, we put

$$\lim_{i \rightarrow \infty} d(v_i, l^*) = k. \quad (4.3)$$

In Lemma 4.2.1's proof, we observe

$$d(w_i, l^*) \leq d(v_i, l^*).$$

Combining this with (4.3), one has

$$\limsup_{i \rightarrow \infty} d(w_i, l^*) \leq \limsup_{i \rightarrow \infty} d(v_i, l^*) = k. \quad (4.4)$$

Now the point $l^* \in F_\Psi$, hence by Proposition 1.2.1(ii), one has

$$\begin{aligned} d(\Psi v_i, l^*) &\leq d(v_i, l^*), \\ \Rightarrow \limsup_{i \rightarrow \infty} d(\Psi v_i, l^*) &\leq \limsup_{i \rightarrow \infty} d(v_i, l^*) = k. \end{aligned} \quad (4.5)$$

In proof of Lemma 4.2.1, one has

$$\begin{aligned} d(v_{i+1}, l^*) &\leq d(w_i, l^*). \\ \Rightarrow k &= \liminf_{i \rightarrow \infty} d(v_{i+1}, l^*) \leq \liminf_{i \rightarrow \infty} d(w_i, l^*). \end{aligned} \quad (4.6)$$

By (4.4) and (4.6), it follows that

$$k = \lim_{i \rightarrow \infty} d(w_i, l^*). \quad (4.7)$$

By (4.7) and (4.2), Consequently, it is as follows:

$$\begin{aligned}
k = \lim_{i \rightarrow \infty} d(w_i, l^*) &= \lim_{i \rightarrow \infty} d(\Psi((1 - \kappa_i)v_i + \kappa_i \Psi v_i), l^*) \\
&\leq \lim_{i \rightarrow \infty} d((1 - \kappa_i)v_i + \kappa_i \Psi v_i, l^*) \\
&\leq \lim_{i \rightarrow \infty} [(1 - \kappa_i)d(v_i, l^*) + \kappa_i d(\Psi v_i, l^*)] \\
&\leq \lim_{i \rightarrow \infty} (1 - \kappa_i)d(v_i, l^*) + \lim_{i \rightarrow \infty} \kappa_i d(\Psi v_i, l^*) \\
&\leq \lim_{i \rightarrow \infty} (1 - \kappa_i)d(v_i, l^*) + \lim_{i \rightarrow \infty} \kappa_i d(v_i, l^*) \\
&= \lim_{i \rightarrow \infty} d(v_i, l^*) \\
&\leq k.
\end{aligned}$$

if and only if

$$k = \lim_{i \rightarrow \infty} d((1 - \kappa_i)v_i \oplus \kappa_i \Psi v_i, l^*). \quad (4.8)$$

Hence, we obtain by applying Lemma 1.1.5, the following

$$\lim_{t \rightarrow \infty} d(\Psi v_i, v_i) = 0.$$

Now we assume in converse that $\lim_{i \rightarrow \infty} d(\Psi v_i, v_i) = 0$. and want to establish that F_Ψ has one fixed point at least. For this aim, we choose any $l^* \in A(\mathcal{K}, \{v_i\})$. Then from Proposition 1.2.1(iv), we have

$$\begin{aligned}
r(\Psi l^*, \{v_i\}) &= \limsup_{i \rightarrow \infty} d(v_i, \Psi l^*) \\
&\leq \left(\frac{\alpha + 3}{1 - \alpha} \right) \limsup_{i \rightarrow \infty} d(\Psi v_i, v_i) + \limsup_{t \rightarrow \infty} d(v_i, l^*) \\
&= \limsup_{i \rightarrow \infty} d(v_i, l^*) \\
&= r(l^*, \{v_i\}).
\end{aligned}$$

From the above, we see that $\Psi l^* \in A(\mathcal{K}, \{v_i\})$. However, this set in the CAT(0) space setting has only one element. Hence we may write $\Psi l^* = l^*$, this shows that $F_\Psi \neq \emptyset$, which was the required target. This concludes the proof.

Now is presented the Δ -convergence theorem.

Theorem 4.2.2. *Let $\mathcal{X}$ be a CAT(0) space and $\mathcal{K}$ be a convex closed subset of $\mathcal{X}$.*

So let's presume that $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ that represents the GAN map. Assuming $\{v_i\}$ is a sequence produced by (4.2) and F_Ψ is not-empty. Then, there is a particular fixed point of the mapping Ψ to which the sequence $\{v_i\}$ Δ converges.

Proof. As the proof is rudimentary in nature, it is omitted.

A convergence theorem on a compact set is established as follows.

Theorem 4.2.3. Assume $\mathcal{K}$ is a nonempty convex compact in $\mathcal{X}$ with a GAN map $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ and $\mathcal{X}$ is a $CAT(0)$ space. If F_Ψ is not empty, then (4.2) generates a sequence $\{v_i\}$. This implies that $\{v_i\}$ converges strongly to a fixed point of Ψ .

Proof. Given that Ψ is compact within $\mathcal{K}$ and the sequence $\{v_i\}$ is a subset of $\mathcal{K}$, the distance between $\{v_{i_k}\}$ and some l^{**} in $\mathcal{K}$ approaches zero as k approaches infinity which is a sub-sequence of $\{v_i\}$ due to the compactness. Considering Theorem 4.2.1, it follows that the limit as k approaches infinity so $d(\Psi v_{i_k}, v_{i_k})=0$. By applying Proposition 1.2.1(iv), we can conclude that

$$d(v_{i_k}, \Psi l^{**}) \leq \left(\frac{\alpha + 3}{1 - \alpha} \right) d(v_{i_k}, \Psi v_{i_k}) + d(v_{i_k}, l^{**}).$$

Now in the above estimate, if we consider $k \rightarrow \infty$, one get $\Psi l^{**} = l^{**}$. The required aim that l^{**} is a point such that $\{v_i\}$ converges strongly to it now follows from $\lim_{t \rightarrow \infty} d(v_i, l^{**})$.

Theorem 4.2.4. A GAN map is $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ considering that $\mathcal{K}$ is a nonempty convex closed set in $\mathcal{X}$ and that $\mathcal{X}$ is a $CAT(0)$ space. A sequence $\{v_i\}$ is generated by (4.2) if F_Ψ is nonempty. Strong convergence of $\{v_i\}$ to a fixed point of Ψ occurs if $\liminf_{i \rightarrow \infty} d(v_i, F_\Psi) = 0$.

Proof. It is proven in Lemma 4.2.1 that $\lim_{t \rightarrow \infty} d(v_i, l^*)$ exists. We conclude that $\lim_{i \rightarrow \infty} d(v_i, F_\Psi)$ exists. Hence

$$\lim_{i \rightarrow \infty} d(v_i, F_\Psi) = 0.$$

From this, one has subsequences namely $\{v_{i_k}\}$ and $\{l_k\}$ of $\{v_i\}$ and F_Ψ with

$$d(v_{i_k}, l_k) \leq \frac{1}{2^k} \text{ for any choice of } k \geq 1.$$

In the proof Lemma 4.2.1, one has $\{v_i\}$ is nonincreasing, hence

$$d(v_{i_{k+1}}, l_k) \leq d(v_{i_k}, l_k) \leq \frac{1}{2^k}.$$

One has

$$\begin{aligned} d(l_{k+1}, l_k) &\leq d(l_{k+1}, v_{i_{k+1}}) + d(v_{i_{k+1}}, l_k) \\ &\leq \frac{1}{2^{k+1}} \\ &\leq \frac{1}{2^{k-1}} \rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned}$$

Hence, we have $\lim_{k \rightarrow \infty} d(l_{k+1}, l_k) = 0$ so that, therefore, we may say that $\{l_r\}$ is a Cauchy sequence in F_Ψ and that it converges to l^{**} . From Proposition 1.2.1(iii), F_Ψ is closed, it follows that $l^{**} \in F_\Psi$. Now using Lemma 4.2.1, $\lim_{i \rightarrow \infty} d(l_i, p^{**})$ exists which proves that l^{**} is also the strong limit of for our $\{v_i\}$ sequence.

Theorem 4.2.5. *It is assumed that $\mathcal{X}$ is a $CAT(0)$ space and $\mathcal{K}$ is a closed convex subset of $\mathcal{X}$. Given a GAN map $\Psi : \mathcal{K} \rightarrow \mathcal{K}$, we may assume the following. F_Ψ is not empty, and $\{v_i\}$ is a sequence generated by (4.2). In order for a $\{v_i\}$, sequence is said to strongly converge to a fixed point of Ψ if and only if Ψ satisfies condition (I).*

Proof. From our main results of Theorem 4.2.1, it follows

$$\liminf_{i \rightarrow \infty} d(\Psi v_i, v_i) = 0. \quad (4.9)$$

But our selfmap $\mathcal{K}$ has a condition (I),

$$d(v_i, \Psi v_i) \geq U(d(v_i, F_\Psi)). \quad (4.10)$$

Combine (4.9) with (4.10), one has

$$\liminf_{i \rightarrow \infty} U(d(v_i, F_\Psi)) = 0.$$

Therefore,

$$\liminf_{i \rightarrow \infty} d(w_i, F_\Psi) = 0.$$

It is shown that the sequence $\{v_i\}$ generated converges strongly to a fixed point of Ψ by using Theorem 4.2.4.

4.3 Example

To substantiate our main findings, we present an instance of GAN selfmaps that diverges from an SM. Through this instance, we conduct a numerical comparison of F with alternative iterations within the framework of GAN mappings. This numerical analysis serves to bolster our main conclusions.

Example 4.3.1. *We use the subsequent rule on the set $\mathcal{K} = [7, 13]$ to build a selfmap on Ψ .*

$$\Psi s = \begin{cases} \frac{s+8}{2} & \text{if } s < 13 \\ 8 & \text{if } s = 13. \end{cases}$$

We now select $\kappa = \frac{1}{2}$ and consider the cases as follows.

Case No: A: *Assume $s = 13 = r$, one has*

$$\frac{1}{2}d(s, \Psi r) + \frac{1}{2}d(r, \Psi s) + (1 - 2(\frac{1}{2}))d(s, r) \geq 0 = d(\Psi s, \Psi r).$$

Case No:B: *Assume $s, r < 13$, we have*

$$\begin{aligned} \frac{1}{2}d(s, \Psi r) + \frac{1}{2}d(r, \Psi s) + (1 - 2(\frac{1}{2}))d(s, r) &= \frac{1}{2} \left| s - \left(\frac{r+8}{2} \right) \right| + \frac{1}{2} \left| r - \left(\frac{s+8}{2} \right) \right| \\ &\geq \frac{1}{2} \left| \left(s - \left(\frac{r+8}{2} \right) \right) - \left(r - \left(\frac{s+8}{2} \right) \right) \right| \\ &\geq \frac{1}{2} |s - r| = d(\Psi s, \Psi r). \end{aligned}$$

Case No:C: *Assume $s = 13$ and $r < 13$, we have*

$$\begin{aligned} \frac{1}{2}d(s, \Psi r) + \frac{1}{2}d(r, \Psi s) + (1 - 2(\frac{1}{2}))d(s, r) &= \frac{1}{2} |s - 8| + \frac{1}{2} \left| r - \left(\frac{s+7}{2} \right) \right| \\ &\geq \frac{1}{2} |s - 8| \\ &= \left| \frac{s - 8}{2} \right| \\ &= d(\Psi s, \Psi r) \end{aligned}$$

Hence, our map is GAN, but it is easy to see that Ψ is not an SM. We now connect our fixed point iterations with this example to support of claims and results.

We currently evaluate the effectiveness of the iterative algorithm F [112] in comparison to the well-known M [27] and Picard [25] methods, as well as the fundamental S [57], Ishikawa [63], and Mann [18] approximation techniques. Given that $\kappa_i = 0.85$ and $\beta_i = 0.65$, and with an initial value of $a_1 = 8.8$, we may see the specific values in Table 4.1. Furthermore, Figure 4.1 visually represents the performance of these prominent schemes. The superiority of the F iterative scheme over other methods is clearly evident when considering the broader context of generalized α -nonexpansive maps.

4.4 Applications of Variational Inequalities Problems

Variational Inequality Problems (VIPs) represent a broad category of mathematical optimization challenges applicable across various real-world domains like economics, engineering, and physics. The central aim within a variational inequality problem involves identifying a solution vector from a predefined set, ensuring that a specific inequality condition holds true under all conceivable perturbations. This is essentially about locating a point where a designated functional is minimized while adhering to a set of inequality constraints. VIPs provide a foundational framework for depicting and addressing equilibrium problems, crucial in scenarios where different

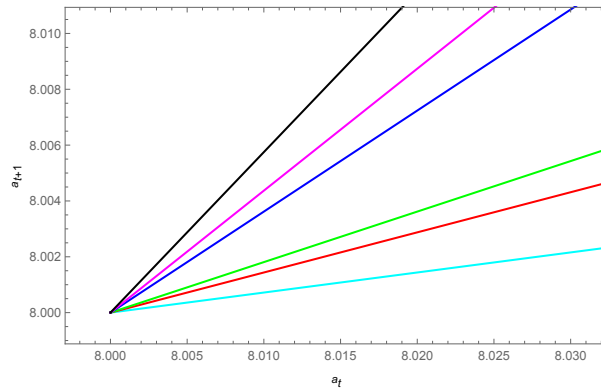


Figure 4.1: Graphical convergence for F, M, Picard-S,S (Agarwal), Ishikawa and Mann schemes.

Table 4.1: Using Example 4.3.1 and the convergence of schemes.

	F	M	Picard-S	S	Ishikawa	Mann
1	8.8	8.8	8.8	8.8	8.8	8.8
2	8.06468750	8.12937500	8.16284375	8.3256875	8.3931875	8.51750000
3	8.00464941	8.01859766	8.02946454	8.11785816	8.17177379	8.29756250
4	8.00033418	8.00267341	8.00533124	8.04264992	8.07504367	8.17109844
5	8.00002402	8.00038430	8.00096462	8.01543394	8.03278471	8.09838160
6	8.00000173	8.00005524	8.00017454	8.00558516	8.01432282	8.05656942
7	8.00000012	8.00000794	8.00003158	8.00202113	8.00625728	8.03252742
8	8.00000001	8.00000114	8.00000571	8.0007314	8.00273365	8.01870326
9	8	8.00000016	8.00000103	8.00026467	8.00119426	8.01075438
10	8	8.00000002	8.00000019	8.00009578	8.00052174	8.00618377
11	8	8	8.00000003	8.00003466	8.00022794	8.00355567
12	8	8	8.00000001	8.00001254	8.00009959	8.00204451
13	8	8	8	8.00000454	8.00004350	8.00117559
14	8	8	8	8.00000164	8.00001901	8.00067597
15	8	8	8	8.00000059	8.00000830	8.00038868
16	8	8	8	8.00000022	8.00000363	8.00022349
17	8	8	8	8.00000008	8.00000158	8.00012851
18	8	8	8	8.00000003	8.00000069	8.00007389
19	8	8	8	8.00000001	8.00000030	8.00004249
20	8	8	8	8	8.00000013	8.00002443
21	8	8	8	8	8.00000006	8.00001405
22	8	8	8	8	8.00000003	8.00000808
23	8	8	8	8	8.00000001	8.00000464
24	8	8	8	8	8	8.00000260

entities or influences interact, necessitating the attainment of a state of equilibrium. The exploration of variational inequality problems has spurred the development of robust mathematical methodologies and computational algorithms, underscoring their indispensable role in optimization theory and their wide-ranging practical applications.

This section aims to explore the category of VIPs within a Hilbert space framework and achieve a robust convergence outcome by utilizing our principal result. To accomplish this, we initially posit two Hilbert spaces, denoted as J_1 and J_2 , such that $C \subseteq J_1$ and $Q \subseteq J_2$ are both convex and compact. Under these conditions, assuming $\mathcal{B}$ is monotonic from J_1 to J_2 , a VIP can be formulated as follows:

$$\text{Compute } \theta_0 \in C : \langle \mathcal{B}\theta_0, \theta - \theta_0 \rangle \geq 0. \quad (4.11)$$

To achieve our goal, we examine a variational inequality problem (VIP) referenced as (4.11), with its solution set denoted as $\mathcal{S}$. Let P^C represent the closest point projection onto C and P^Q represent the closest point projection onto Q , respectively, and considering a constant $c > 0$, prior studies, outlined in [31], establish that any point θ_0 within set C solves (4.11) if and only if the corresponding θ_0 is the solution to subsequent equation:

$$a = P^C(I - c\mathcal{B})a.$$

The following approximation method is introduced in [30] that converges weakly to the solutions of VIPs.

$$v_{i+1} = P^C(I - c\mathcal{B})v_i.$$

For variational inequality problems (VIPs) in Hilbert spaces, the ultimate objective of this section is to attain strong convergence results. However, transitioning from weak to strong convergence typically necessitates additional conditions, as observed in [31] concerning VIPs. This chapter establishes strong convergence based on our primary outcome for VIPs. Our distinctive approach relies on a more efficient iterative technique known as F iteration, analogous to the one utilized in [30]. It is noteworthy that mappings are recognized for their continuity across their entire domains. Furthermore, our method introduces the notion of GAN mappings, which proves to be broader in scope than the concept of nonexpansive mappings, as demonstrated in this investigation.

Theorem 4.4.1. *Suppose VIP (4.11) satisfies $\mathcal{S} \neq \emptyset$, $c > 0$ and $P^C(I - c\mathcal{B})$ is GAN mapping. Then $\{v_i\}$ produced by (4.2) is strongly convergent to some sought solution of (4.11).*

Proof. Assume that $\Psi = P^C(I - c\mathcal{B})$. As assumed, the mapping Ψ is GAN. But $F_\Psi = \mathcal{S}$, by Theorem 4.2.3, the sequence $\{v_i\}$ approaches the given problem's solution θ_0 in a strong sense. (4.11). □

4.5 Conclusions

We suggested the CAT(0) space version of F iteration and proved strong and Δ -convergence results for a larger class of mappings. We also proved that numerically our new approach is faster convergent via new examples of GAN maps. We provide one application of our main outcome. Our results are new and extend the corresponding results from the linear setting to the nonlinear setting of CAT(0) spaces.

Chapter 5

Approximation of Fixed Points for a Class of Nonlinear Nonexpansive Mappings in Banach Spaces

5.1 Introduction

Recent times have seen the iterative codification of fixed points significantly impact the solutions to many problems related to stability, data dependency, convergence rate analysis, and related topics.

Control sequences that are seen in different fields of study. Look at a class of Banach spaces $\mathcal{X}$ in which any of these spaces has a nonempty closed convex subset K .

Make sure that for any $u, y \in \mathcal{K}$, the given mapping $\Psi : \mathcal{K} \rightarrow \mathcal{X}$ is non-expansive by defining $\|\Psi u - \Psi y\| \leq \|u - y\|$. If the set $F(\Psi)$ is not empty and $\|\Psi u - v\| \leq \|u - v\|$ for every $u \in \mathcal{K}$ and $v \in F(\Psi)$, then Ψ is considered quasi-nonexpansive [1]. Based on the previous understanding, any operator that is nonexpansive with regard to a fixed point is equivalent to a quasi-nonexpansive operator. Simply put, if $\mathcal{X}$ is uniformly convex and $\mathcal{K}$ is a convex closed bounded subset of $\mathcal{X}$, then $F(\Psi) \neq \emptyset$ holds for non-expensive operators.

An algorithmic approach to determining the possible value of a fixed point is appealing, provided that beforehand, it is shown that such a mapping exists and is unique. The Picard iterative codification is the starting point for the examination of fixed point optimisation algorithms [8,9], which has a considerable amount of literature on the subject of its structure. The Banach contraction prescript declares that the Picard iterative system can obtain the contraction mapping fixed point. Here, the sequence v_i is produced from an arbitrarily chosen point $v_1 \in \mathcal{K}$. Using the following formula: $v_{i+1} = \xi_i v_i + (1 - \xi_i) \Psi v_i, i \in \mathbb{N}$.

Thereafter, a non-expansive mapping's consecutive iterative nature is incompatible with the contraction mapping, and a fixed point will inevitably be reached by the mapping. In this context, a previous finding for the non-expansive operator's fixed point approximation was put forth by Krasnosel'skii [2]. If the self operator Ψ of $\mathcal{X}$ is non-expansive and $\mathcal{X}$ is a uniformly convex Banach space, then the function $(\frac{I+\Psi}{2})$ will reach to a fixed point of Ψ which is the point of convergence of it, according to his proposal.

Many iteration strategies have been developed to approximate non-expansive mappings in the years following the failure of this hypothesis. Nearly all non-expansive mappings have utilised Mann's iteration approach [18] to get close to the fixed point. The iterative system is like this: the sequence $\{v_i\}$ is

generated using the following method, starting from any point $v_1 \in \mathcal{K}$:

$$\left\{ \begin{array}{l} v_1 \in \mathcal{K}, \\ v_{i+1} = \xi_i v_i + (1 - \xi_i) \Psi v_i, \\ i \in \mathbb{N} \text{ and } \xi_i \in (0, 1). \end{array} \right. \quad (5.1)$$

Ishikawa conducted further research on the unique iterative approach that has been extensively employed to estimate the fixed point of non-expansive mappings. The sequence $\{v_i\}$ is generated iteratively from

$$\left\{ \begin{array}{l} v_1 \in \mathcal{K}, \\ w_i = (1 - \lambda_i) v_i + \lambda_i \Psi v_i, \\ v_{i+1} = (1 - \xi_i) v_i + \xi_i \Psi w_i, \\ i \in \mathbb{N} \text{ and } \xi_i, \lambda_i \in (0, 1). \end{array} \right. \quad (5.2)$$

Further, Noor [19] established an iterative system where for each. fixed value $v_1 \in \mathcal{K}$, $i \in \mathbb{N}$ and $\xi_i, \lambda_i, \kappa_i \in (0, 1)$ and the sequence $\{v_i\}$ is initiated by

$$\left\{ \begin{array}{l} v_1 \in \mathcal{K}, \\ w_i = (1 - \kappa_i) v_i + \kappa_i \Psi v_i, \\ x_i = (1 - \lambda_i) v_i + \lambda_i \Psi w_i, \\ v_{i+1} = (1 - \xi_i) v_i + \xi_i \Psi x_i. \end{array} \right. \quad (5.3)$$

In the following section, the iteration system was presented by Agrawal et al. [57]. The authors stated that the rate of analysis convergence is the same as that of The Picard iterative procedure, which is more efficient than the Mann iterative scheme when dealing with contractions. In this system, $\{v_i\}$ is the sequence that is produced by

$$\left\{ \begin{array}{l} v_1 \in \mathcal{K}, \\ w_i = (1 - \lambda_i) v_i + \lambda_i \Psi v_i, \\ v_{i+1} = (1 - \xi_i) \Psi v_i + \xi_i \Psi w_i, \\ i \in \mathbb{N} \text{ and } \xi_i, \lambda_i \in (0, 1). \end{array} \right. \quad (5.4)$$

Abbas and Nazir [20] examine the following iterative fashion and declare that this procedure faster converges than Agrawal et al. [57] iteration process, where for each fixed value $v_1 \in \mathcal{K}$, $i \in \mathbb{N}$ and $\kappa_i, \xi_i, \lambda_i \in (0, 1)$ and the sequence $\{v_i\}$ is construct by

$$\left\{ \begin{array}{l} v_1 \in \mathcal{K}, \\ w_i = (1 - \kappa_i) v_i + \kappa_i \Psi v_i, \\ x_i = (1 - \lambda_i) \Psi v_i + \lambda_i \Psi w_i, \\ v_{i+1} = (1 - \xi_i) \Psi x_i + \xi_i \Psi w_i, \end{array} \right. \quad (5.5)$$

B. S. Thakur et al. [69] suggested a new iterative method for estimating the fixed point of non-expansive mappings. The operation is applicable to all fixed values $v_1 \in \mathcal{K}$, $i \in \mathbb{N}$ and $\xi_i, \lambda_i \in (0, 1)$ and the sequence $\{v_i\}$ is construct by

$$\begin{cases} v_1 \in \mathcal{K}, \\ w_i = (1 - \lambda_i) v_i + \lambda_i \Psi v_i, \\ x_i = \Psi((1 - \xi_i) v_i + \xi_i w_i), \\ v_{i+1} = \Psi x_i, \end{cases} \quad (5.6)$$

However, Suzuki also proposed the idea of extended non-expansive mappings, which are defined by a mapping condition (C) . Let $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$ and $\mathcal{X}$ be a Banach space. Condition (C) is met by a mapping $\Psi : \mathcal{K} \rightarrow \mathcal{K}$ if

$$\frac{1}{2} \|u - \Psi u\| \leq \|u - a\| \Rightarrow \|\Psi u - \Psi a\| \leq \|u - a\| \text{ for each } u, a \in \mathcal{K}.$$

The mapping that fulfils condition (C) is more effective than quasi-non-expansive and less robust than non-expansive, as demonstrated by Suzuki [68]. In 2011, Phuengrattana [4] proved convergence rate theorems for mappings that satisfy condition (C) by utilising the iterative system of Ishikawa in the category of uniformly convex Banach spaces and CAT(0) spaces.

Certain researchers have recently looked into common fixed point results and fixed point results for mappings that meet criterion (C) . For further information, please refer to the following references: [52, 54, 68, 69, 114].

In this chapter, we derive inspiration from the concept by B. S. Thakura et al. [69] and present a unique iterative methodology. In addition, we offer a thorough examination of the rates at which the mapping, which fulfils condition (C) , converges weakly and strongly in the context of uniformly convex Banach spaces. In addition, we offer specific instances of a mapping that fulfils condition (C) and analyse the numerical characteristics of the convergence of our suggested iterative system compared to established models.

5.2 Main Results

An innovative extended iterative approach (A^*) for estimating the common fixed point of extended nonexpansive Suzuki mappings meeting condition (C) in the class of uniformly convex Banach spaces is the goal of the current study. where for each fixed value $v_1 \in \mathcal{K}$, $i \in \mathbb{N}$ and $\xi_i, \lambda_i \in (0, 1)$.

$$\begin{cases} v_1 \in \mathcal{K}, \\ w_i = \Psi_1(1 - \lambda_i)v_i + \lambda_i\Psi_1v_i, \\ x_i = \Psi_2(1 - \xi_i)v_i + \xi_i\Psi_2w_i, \\ v_{i+1} = \Psi_3x_i. \end{cases} \quad (5.7)$$

This exposition's first significant basic Lemma is then presented.

Lemma 5.2.1. *Assume that a convex closed subset called $\mathcal{K}$ of a Banach space $\mathcal{X}$. Consider a family of mappings of Suzuki-generalized nonexpansive type. Ψ_1, Ψ_2 , and $\Psi_3 : \mathcal{K} \rightarrow \mathcal{K}$, let $F_{i=1,2,3} = F(\Psi_1 \cap \Psi_2 \cap \Psi_3)$ is nonempty. Let $\{v_i\}$ be the sequence obtained by the extended iterative technique (A^*) with an arbitrary configuration $v_1 \in \mathcal{K}$. Then, implies the existence of $\lim_{i \rightarrow \infty} \|v_i - v\|$ for every $v \in F_i$*

Proof. Take $v \in F_i$ and $a \in \mathcal{K}$ according to the the premise of this relation. Since Ψ_i satisfies condition (C) , precisely we have

$$\frac{1}{2} \|v - \Psi_i v\| = 0 \leq \|x - a\| \Rightarrow \|\Psi_i v - \Psi_i a\| \leq \|x - a\|. \quad (5.8)$$

In the light of Proposition 1.1.3 (i), we have

$$\begin{aligned} \|w_i - v\| &= \|\Psi_1[(1 - \lambda_i)v_i + \lambda_i\Psi_1v_i - v]\| \\ &\leq \|(1 - \lambda_i)v_i + \lambda_i\Psi_1x_i - v\| \\ &\leq (1 - \lambda_i)\|v_i - v\| + \lambda_i\|\Psi_1x_i - v\| \\ &\leq (1 - \lambda_i)\|v_i - v\| + \lambda_i\|v_i - v\| \\ &\leq \|v_i - v\| \end{aligned} \quad (5.9)$$

and having these precise, one-write

$$\begin{aligned}
\|x_i - v\| &= \|\Psi_2[(1 - \xi_i)v_i + \xi_i\Psi_2w_i - v]\| \\
&\leq \|(1 - \xi_i)v_i + \xi_i\Psi_2w_i - v\| \\
&\leq (1 - \xi_i)\|v_i - v\| + \xi_i\|\Psi_2w_i - v\| \\
&\leq (1 - \xi_i)\|v_i - v\| + \xi_i\|w_i - v\| \\
&\leq (1 - \xi_i)\|v_i - v\| + \xi_i\|v_i - v\| \\
&\leq \|v_i - v\|
\end{aligned} \tag{5.10}$$

Using the inequalities 5.9 and 5.10, one gets

$$\begin{aligned}
\|v_{i+1} - v\| &= \|\Psi_3x_i - v\| \\
&\leq \|x_i - v\| \\
&\leq \|w_i - v\| \\
&\leq \|v_i - v\|.
\end{aligned} \tag{5.11}$$

Ultimately, we deduce from the previous observation that, for each $v \in F_i$, $\{\|v_i - v\|\}$ is bounded and non-increasing. It follows that the limit $\lim_{i \rightarrow \infty} \|v_i - v\|$ exists. $\square$

Theorem 5.2.1. *Let a closed convex subset $\mathcal{K}$ of a uniformly Banach space $\mathcal{X}$. Consider a family of mappings $\Psi_{i=1,2,3} : \mathcal{K} \rightarrow \mathcal{K}$ which are Suzuki-generalized nonexpansive and each F_i is a nonempty set.*

For arbitrary setting $v_1 \in \mathcal{K}$, let the sequence $\{v_i\}$ is generated by the extended iterative procedure (A^) for every $i \in \mathbb{N}$. Additionally, where $\{\xi_i\}$ and $\{\lambda_i\}$ are the real numbers sequence in $[r_1, r_2]$ to some r_1, r_2 with respect to $0 < r_1 \leq r_2 < 1$. Then $F_i \neq \emptyset$ if and only if $\lim_{i \rightarrow \infty} \|\Psi_i v_i - v_i\| = 0$ and $\{v_i\}$ is bounded sequence.*

Proof. Consider the cases when $a \in \mathcal{K}$ and $v \in F_i \neq \emptyset$. Given that Ψ_i fulfills condition (C), we have

$$\frac{1}{2} \|v - \Psi_i v\| = 0 \leq \|x - a\| \Rightarrow \|\Psi_i v - \Psi_i a\| \leq \|x - a\|. \tag{5.12}$$

Then, owing to Lemma 5.2.1, the fact $\lim_{i \rightarrow \infty} \|v_i - v\|$ exists and $\{v_i\}$ is bounded. For given some $\varepsilon > 0$, setting

$$\lim_{i \rightarrow \infty} \|v_i - v\| = \varepsilon. \quad (5.13)$$

Based on 5.9 and 5.13, we have

$$\limsup_{i \rightarrow \infty} \|w_i - v\| \leq \limsup_{i \rightarrow \infty} \|v_i - v\| = \varepsilon. \quad (5.14)$$

In view of Proposition 1.1.3 (i), we have

$$\limsup_{i \rightarrow \infty} \|\Psi_i v_i - v\| \leq \limsup_{i \rightarrow \infty} \|v_i - v\| = \varepsilon. \quad (5.15)$$

On the other hand,

$$\begin{aligned} \|v_{i+1} - v\| &\leq \|w_i - v\| \\ &= \|\Psi_1[(1 - \lambda_i)v_i + \lambda_i \Psi_1 v_i] - v\| \\ &\leq \|(1 - \lambda_i)v_i + \lambda_i \Psi_1 v_i - v\| \\ &\leq (1 - \lambda_i)\|v_i - v\| + \lambda_i\|\Psi_1 v_i - v\| \\ &\leq (1 - \lambda_i)\|v_i - v\| + \lambda_i\|v_i - v\| \\ &\leq \|v_i - v\| - \lambda_i\|v_i - v\| + \lambda_i\|v_i - v\| \\ &\leq \|v_i - v\| \end{aligned} \quad (5.16)$$

By 5.13, we have

$$\varepsilon = \liminf_{i \rightarrow \infty} \|v_{i+1} - v\| \leq \liminf_{i \rightarrow \infty} \|w_i - v\|. \quad (5.17)$$

Further, applying (5.13) and (5.17), we obtain

$$\lim_{i \rightarrow \infty} \|w_i - v\| = \varepsilon. \quad (5.18)$$

Next, owing to (5.16) and (5.18), we write

$$\begin{aligned}
\varepsilon &= \lim_{i \rightarrow \infty} \|w_i - v\| = \lim_{i \rightarrow \infty} \|\Psi_1[(1 - \lambda_i)x_i + \lambda_i\Psi_1v_i] - v\| \\
&\leq \lim_{i \rightarrow \infty} \|(1 - \lambda_i)v_i + \lambda_i\Psi_1v_i - v\| \\
&\leq \lim_{i \rightarrow \infty} \|(1 - \lambda_i)\|v_i - v\| + \lim_{i \rightarrow \infty} \lambda_i\|\Psi_1v_i - v\| \\
&\leq \lim_{i \rightarrow \infty} \|(1 - \lambda_i)\|v_i - v\| + \lim_{i \rightarrow \infty} \lambda_i\|v_i - v\| \\
&= \lim_{k \rightarrow \infty} \|v_i - v\| \\
&= \varepsilon,
\end{aligned} \tag{5.19}$$

if and only if

$$\varepsilon = \lim_{i \rightarrow \infty} \|(1 - \lambda_i)v_i + \lambda_i\Psi_1v_i - v\|. \tag{5.20}$$

In the light of Lemma 1.1.5, we get

$$\lim_{i \rightarrow \infty} \|\Psi_1v_i - v_i\| = 0. \tag{5.21}$$

So, we can easily conclude that $\lim_{i \rightarrow \infty} \|\Psi_2v_i - v_i\| = 0$ and $\lim_{i \rightarrow \infty} \|\Psi_3v_i - v_i\| = 0$. Further, in the sense of converse, taking bounded sequence $\{v_i\} \in \mathcal{K}$ and $\lim_{i \rightarrow \infty} \|\Psi_1v_i - x_i\| = \lim_{i \rightarrow \infty} \|\Psi_2v_i - x_i\| = \lim_{i \rightarrow \infty} \|\Psi_3v_i - x_i\| = 0$. To prove that $F_i \neq \emptyset$. For this, we take $v \in \Omega(\mathcal{K}, \{v_i\})$. Applying Proposition 1.1.3 (c), subsequently the following inequalities are evident.

$$\begin{aligned}
r(\Psi_1v, \{v_i\}) &= \limsup_{i \rightarrow \infty} \|v_i - \Psi_1v\| \\
&\leq \limsup_{i \rightarrow \infty} 3\|\Psi_1v_i - x_i\| + \limsup_{i \rightarrow \infty} \|v_i - v\| \\
&= \limsup_{i \rightarrow \infty} \|v_i - v\| \\
&= r(v, \{v_i\}).
\end{aligned} \tag{5.22}$$

Similarly, by 5.22 one can prove that

$$r(\Psi_2v, \{v_i\}) = r(v, \{v_i\}) \tag{5.23}$$

and

$$r(\Psi_3 v, \{v_i\}) = r(v, \{v_i\}) \quad (5.24)$$

From equations (5.22), (5.23) and (5.24), we obtain $\Psi_i v \in \Omega(\mathcal{K}, \{v_i\})$. Since $\mathcal{K}$ is uniformly convex, so $\Omega(\mathcal{K}, \{v_i\})$ is singleton, we get $\Psi_i v = v$. Hence $F_i \neq \emptyset$. $\square$

Weak Convergence Rate Analysis with respect to iterative procedure (A^*) .

Theorem 5.2.2. *Assume that a uniform Banach space $\mathcal{X}$ that is convex and equipped with Opial property. Let a closed convex subset $\mathcal{K}$ of $\mathcal{X}$. Consider a family of mappings $\Psi_{i=1,2,3} : \mathcal{K} \rightarrow \mathcal{K}$ which are Suzuki-generalized nonexpansive and each F_i is a nonempty set. For arbitrary setting $v_1 \in \mathcal{K}$, let the sequence $\{v_i\}$ is generated by the extended iterative procedure (A^*) for every $i \in \mathbb{N}$. Additionally, where $\{\xi_i\}$ and $\{\lambda_i\}$ are the real numbers sequence in $[r_1, r_2]$ to some r_1, r_2 with respect to $0 < r_1 \leq r_2 < 1$ in such a way that $F_i \neq \emptyset$. Following that, the sequence $\{v_i\}$ weakly converges at the common fixed point of Ψ_i .*

Proof. Since $F_i \neq \emptyset$, taking into considration of the Theorem 5.2.1, one can prove that $\{v_i\}$ is bounded and $\lim_{i \rightarrow \infty} \|\Psi_i v_i - v_i\| = 0$. Based on the given hypothesis, a subset $\mathcal{K}$ of uniformly convex Banach space is closed and convex. Eberlin's theorem specifies that there is a subsequence $\{v_{i_w}\}$ of $\{v_i\}$ that weakly converges to $\lim_{w \rightarrow \infty} \|v_{i_w} - \Psi_i v_{i_w}\| = 0$ because of the reflexive nature of $\mathcal{K}$.

Under Mazur's theorem, $v_1 \in \Lambda$, given that $\mathcal{K}$ is convex and closed, we have $v_1 \in F_i$ by Lemma (1.1.3).

Moreover, we demonstrate the weak convergence of $\{v_i\}$ to v_1 . Specifically, if this isn't the case, then there has to be another subsequence of $\{v_{i_\nu}\}$ of $\{v_i\}$ that converges weakly to $v_2 \in \Lambda$, such that $v_1 \neq v_2$.

In the sense of Lemma (1.1.3), we have $v_2 \in F_i$. Since $\lim_{i \rightarrow \infty} \|v_i - v\| = 0$ exists

for each $v \in F_i$. Further, by Theorem 5.2.1 and by Opial's fashion, we have

$$\begin{aligned}
\lim_{i \rightarrow \infty} \|v_i - v_1\| &= \lim_{w \rightarrow \infty} \|v_{i_w} - v_1\| \\
&< \lim_{w \rightarrow \infty} \|v_{i_w} - v_2\| \\
&= \lim_{i \rightarrow \infty} \|v_i - v_2\| \\
&= \lim_{\nu \rightarrow \infty} \|v_{i_\nu} - v_2\| \\
&< \lim_{\nu \rightarrow \infty} \|v_{i_\nu} - v_1\| \\
&= \lim_{i \rightarrow \infty} \|v_i - v_1\|.
\end{aligned}$$

Which is contradiction, so we have $v_1 = v_2$. As a result, at the common fixed point of Ψ_i , the sequence $\{v_i\}$ converges weakly. $\square$

Furthermore, Strong Convergence Rate Analysis with respect to iterative procedure (A^*) .

Theorem 5.2.3. *A Banach space $\mathcal{X}$ which is a uniformly convex, consider a subset $\mathcal{K}$ of this space which is nonempty compact and convex. Let $\Psi_{i=1,2,3} : \mathcal{K} \rightarrow \mathcal{K}$ be a family of nonexpansive Suzuki-generalized mappings, where F_i is nonempty. For arbitrary setting $v_1 \in \mathcal{K}$, let the sequence $\{v_i\}$ is generated by the extended iterative procedure (A^*) for every $i \in \mathbb{N}$. Additionally, where $\{\xi_i\}$ and $\{\lambda_i\}$ are the real numbers sequence in $[r_1, r_2]$ Considering a set of r_1, r_2 such that $0 < r_1 \leq r_2 < 1$. The strong convergence of the sequence $\{v_i\}$ occurs at the fixed point of Ψ_i .*

Proof. According to Theorem (5.2.3), one may write $\lim_{i \rightarrow \infty} \|\Psi_i v_i - v_i\| = 0$. Lemma 1.1.4 gives us $F_i \neq \emptyset$. As $\mathcal{K}$ is compact, $\{v_{i_w}\}$ of $\{v_i\}$ is a sub-sequence such that $\{v_{i_w}\}$ strongly converges to v for some $v \in \mathcal{K}$. Using Proposition 1.1.3 (c), we arrive to

$$\|v_{i_w} - \Psi_i v\| \leq 3\|\Psi_i v_{i_w} - v_{i_w}\| + \|v_{i_w} - v\| \quad \forall i \geq 1. \quad (5.25)$$

Taking the limit as $w \rightarrow \infty$, then $\Psi_i v = v$, that is $v \in F_i$. In the light of Lemma 5.2.1, for every $v \in F_i$, there exists $\lim_{i \rightarrow \infty} \|v_i - v\|$. As a result, v_i strongly converges to v . $\square$

Additionally, Strong Convergence Rate Analysis based on non-compact iterative procedure (A^*) .

Theorem 5.2.4. *Consider a convex, nonempty and closed subset $\mathcal{K}$ of uniform Banach space of $\mathcal{X}$. Let $\Psi_{i=1,2,3} : \mathcal{K} \rightarrow \mathcal{K}$ be a family of mappings which are non-expansive Suzuki-generalized type and F_i be a nonempty set. For arbitrary setting $v_1 \in \mathcal{K}$, let the sequence $\{v_i\}$ is generated by the extended iterative procedure (A^*) for every $i \in \mathbb{N}$. Additionally, where $\{\xi_i\}$ and $\{\lambda_i\}$ are real number sequences in $[r_1, r_2]$ for some r_1, r_2 such that $F_i \neq \emptyset$ and $0 < r_1 \leq r_2 < 1$. If Ψ_i fulfil the subsequent style.*

$$h(d(v, F(\Psi_i))) \leq \|v - \Psi_i v\| \quad \forall v \in \mathcal{K}, \quad (5.26)$$

at the common fixed point of Ψ_i , the sequence $\{v_i\}$ converges strongly.

Proof. Lemma (5.2.1) allows us to conclude that for any $v \in F(\Psi_i)$, $\lim_{i \rightarrow \infty} \|v_i - v\|$ exists and it follows that $\lim_{i \rightarrow \infty} d(v_i, F(\Psi_i))$ exists. Let us assume that $\lim_{i \rightarrow \infty} \|v_i - v\| = l$ for some $l \geq 0$, then our proof of Theorem (5.2.4) proceeds as follows. Suppose that there is $l > 0$, by appealing the given hypothesis and (5.26) we can write

$$h(d(v_i, F(\Psi_i))) \leq \|\Psi_i v_i - v_i\|. \quad (5.27)$$

Since $F_i \neq \emptyset$, in view of Theorem (5.2.2) we find that

$$\lim_{i \rightarrow \infty} \|\Psi_i v_i - v_i\| = 0. \quad (5.28)$$

By appealing to (5.27), we can write

$$\lim_{i \rightarrow \infty} h(d(v_i, F(\Psi_i))) = 0. \quad (5.29)$$

Since h is a non-decreasing function, we obtain

$$\lim_{i \rightarrow \infty} d(v_i, F(\Psi_i)) = 0. \quad (5.30)$$

Thus, we find a subsequence $\{v_{i_j}\}$ of $\{v_i\}$. From this, a sequence $\{s_i\} \subset F(\Psi_i)$ and

together with the equation (5.11) such that

$$\|v_{i_{j+1}} - s_j\| \leq \|v_{i_j} - s_j\| \leq \frac{1}{2^j} \text{ for all } j \geq 1. \quad (5.31)$$

Thus, we can clearly write,

$$\begin{aligned} \|s_{j+1} - s_j\| &\leq \|s_{j+1} - v_{i_{j+1}}\| + \|v_{i_{j+1}} - s_j\| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}} \rightarrow 0, \text{ as } j \rightarrow \infty. \end{aligned} \quad (5.32)$$

Hence, the Cauchy sequence $\{s_j\}$ that converges to point v in $F(\Psi_i)$.

$\{v_{i_j}\}$ is a subsequence that strongly converges to point v since $F(\Psi_i)$ is closed. Consequently, $v \in F(\Psi_i)$. Since $\lim_{i \rightarrow \infty} \|v_i - v\|$ exists, we can write $v_i \rightarrow v \in F(\Psi_i)$. $\square$

From now on, we will provide an intriguing example that showcases the numerical and graphical behaviour that serves as motivation for the investigations described in this work.

Example 5.2.1. Define a selfmappings $\Psi_{i=1,2,3}$ on $\mathcal{K} = [1, 2]$ by

$$\Psi_1 x = \begin{cases} \frac{x+16}{9} \text{ for } 1 \leq x < \frac{10}{9}, \\ 3 - x \text{ for } \frac{10}{9} \leq x \leq 2, \end{cases}$$

$$\Psi_2 e = \begin{cases} \frac{e+32}{10} \text{ for } 1 \leq e < \frac{10}{9}, \\ 4 - e \text{ for } \frac{10}{9} \leq e \leq 2, \end{cases}$$

$$\Psi_3 e = \begin{cases} \frac{x+64}{11} \text{ for } 1 \leq x < \frac{10}{9}, \\ 5 - e \text{ for } \frac{10}{9} \leq x \leq 2. \end{cases}$$

In this case, Ψ_i are extended versions of mappings of type Suzuki's non-expansive as defined above; yet, Ψ_i are not non-expansive mappings because

$$\|\Psi_1 e_1 - \Psi_1 e_2\| = 0.24 > 0.001 = \|e_1 - e_2\|,$$

$$\|\Psi_2 e_1 - \Psi_2 e_2\| = 0.80 > 0.001 = \|e_1 - e_2\|,$$

and

$$\|\Psi_3 e_1 - \Psi_3 e_2\| = 1.80 > 0.001 = \|e_1 - e_2\|$$

for $e_1 = 1.11$ and $e_2 = 1.111$. Further, we examine that $\Psi_{i=1,2,3}$ are extended Suzuki's non-expansive mapping. Now, let's proceed with the discussion of the following scenarios.

$$(i) \frac{1}{2}\|e_1 - \Psi_1 e_1\| \leq \|x_1 - e_2\| \Rightarrow \|\Psi_1 e_1 - \Psi_1 e_2\| \leq \|e_1 - e_2\|.$$

$$(ii) \frac{1}{2}\|e_1 - \Psi_2 e_1\| \leq \|e_1 - e_2\| \Rightarrow \|\Psi_2 e_1 - \Psi_2 e_2\| \leq \|e_1 - e_2\|.$$

$$(iii) \frac{1}{2}\|e_1 - \Psi_3 e_1\| \leq \|e_1 - e_2\| \Rightarrow \|\Psi_3 e_1 - \Psi_3 e_2\| \leq \|e_1 - e_2\|.$$

Case I: Setting $1 \leq e_1, e_1 < \frac{10}{9}$ or $\frac{10}{9} \leq e_1, e_1 \leq 2$, both cases involve Ψ_i being a non-expansive mapping, thereby establishing Ψ_i as an expansion of mappings of type Suzuki's non-expansive.

Case II: Take $1 \leq x < \frac{10}{9}$, so $\frac{1}{2}\|x - \Psi_1 x\| = \frac{1}{2}\|\frac{x+16}{9} - x\| = \|\frac{16-8x}{18}\| \in [0, \frac{64}{162}]$. As $\frac{1}{2}\|x_1 - \Psi_1 e_1\| \leq \|e_1 - e_2\|$, one writes $\frac{16-8x}{18} \leq e_1 - e_2$. Let us now discuss the following two cases:

(a): Consider $e_1 < e_2$, then $\frac{10x+16}{18} \leq e_2$ this implies $e_2 \in [\frac{244}{162}, 2]$. Which yield

$$\|\Psi_1 e_1 - \Psi_1 e_2\| = \frac{1}{9}\|e_1 - e_2\|.$$

Hence,

$$\frac{1}{2}\|e_1 - \Psi_1 e_1\| \leq \|e_1 - e_2\| \Rightarrow \|\Psi_1 e_1 - \Psi_1 e_2\| \leq \|e_1 - e_2\|.$$

(b): Consider $e_1 > e_2$, then $x_2 \leq \frac{26x-16}{18}$ implies $e_2 \leq \frac{116}{162}$, and $e_2 \leq 2$, so $e_2 \in [1, 2]$. As $e_2 \in [1, 2]$, and $e_2 \leq \frac{26e_2-16}{18}$, which implies $e_1 \geq \frac{18e_2+16}{26}$. Thus, taking $e_1 \in [\frac{34}{26}, 2]$ and $e_2 \in [1, \frac{10}{9})$, so

$$\|\Psi_1 e_1 - \Psi_1 e_2\| < \frac{1}{9},$$

and

$$\|e_1 - e_2\| > \frac{1}{9}.$$

Hence,

$$\frac{1}{2}\|e_1 - \Psi_1 e_1\| \leq \|e_1 - e_2\| \Rightarrow \|\Psi_1 e_1 - \Psi_1 e_2\| \leq \|e_1 - e_2\|.$$

Case III: Consider $\frac{10}{9} \leq x \leq 2$, then $\frac{1}{2}\|x - \Psi_1 x\| = \frac{1}{2}\|x - (3-x)\| = \|2e-3\| \in (\frac{7}{18}, \frac{1}{2}]$. As $\frac{1}{2}\|e_1 - \Psi_1 e_1\| \leq \|e_1 - e_2\|$, one writes $2e-3 \leq e_1 - e_2$. This implies that $e_2 \geq \frac{3}{2}$, so $e_2 \in [\frac{3}{2}, 2]$ and we obtain

$$\|\Psi_1 e_1 - \Psi_1 e_2\| < \frac{1}{9}$$

and

$$\|e_1 - e_2\| > \frac{1}{9}.$$

Hence,

$$\frac{1}{2}\|e_1 - \Psi_1 e_1\| \leq \|e_1 - e_2\| \Rightarrow \|\Psi_1 e_1 - \Psi_1 e_2\| \leq \|e_1 - e_2\|.$$

Similarly, we easily obtain the following condition:

$$(ii) \frac{1}{2}\|e_1 - \Psi_2 e_1\| \leq \|e_1 - e_2\| \Rightarrow \|\Psi_2 e_1 - \Psi_2 e_2\| \leq \|e_1 - e_2\|.$$

(iii) $\frac{1}{2}\|e_1 - \Psi_3 e_1\| \leq \|e_1 - e_2\| \Rightarrow \|\Psi_3 e_1 - \Psi_3 e_2\| \leq \|e_1 - e_2\|$. Thus, Ψ_i as an extended Suzuki's non-expansive mapping

5.3 Analysis Basins of Attraction

The objective of this section is to incorporate Newton's famous iterative scheme [13] into the A^* iterative scheme (5.7), along with various well-known iterative methods from existing literature. This will result in the visualization of multiple basins of attraction. Let's consider P as a complex polynomial. It's important to highlight that, when starting with any initial value $z_0 \in C$, the iterative process of Newton's scheme can be expressed in the following manner:

$$z_{i+1} = z_i - \frac{P(z_i)}{P'(z_i)}, \text{ for } i = 0, 1, 2, \dots$$

The expression $P'(z_i)$ represents the first derivative of the function P . The Newton iterative scheme can be reformulated into a fixed point iterative scheme in the following manner:

$$z_{i+1} = Tz_i$$

Table 5.1: A^* -iterative scheme (5.7) for $\xi_i = 0.1$ and $\zeta_i = 0.2$ in Example 5.2.1

k	Scheme (5.7)	Scheme (5.7)	Scheme (5.7)
1	1	1.5	2
2	7.00	6.50	6.00
3	8.00	8.00	8.00
4	8.00	8.00	8.00
5	8.00	8.00	8.00
6	8.00	8.00	8.00
7	8.00	8.00	8.00
8	8.00	8.00	8.00
9	8.00	8.00	8.00
10	8.00	8.00	8.00
11	8.00	8.00	8.00
12	8.00	8.00	8.00

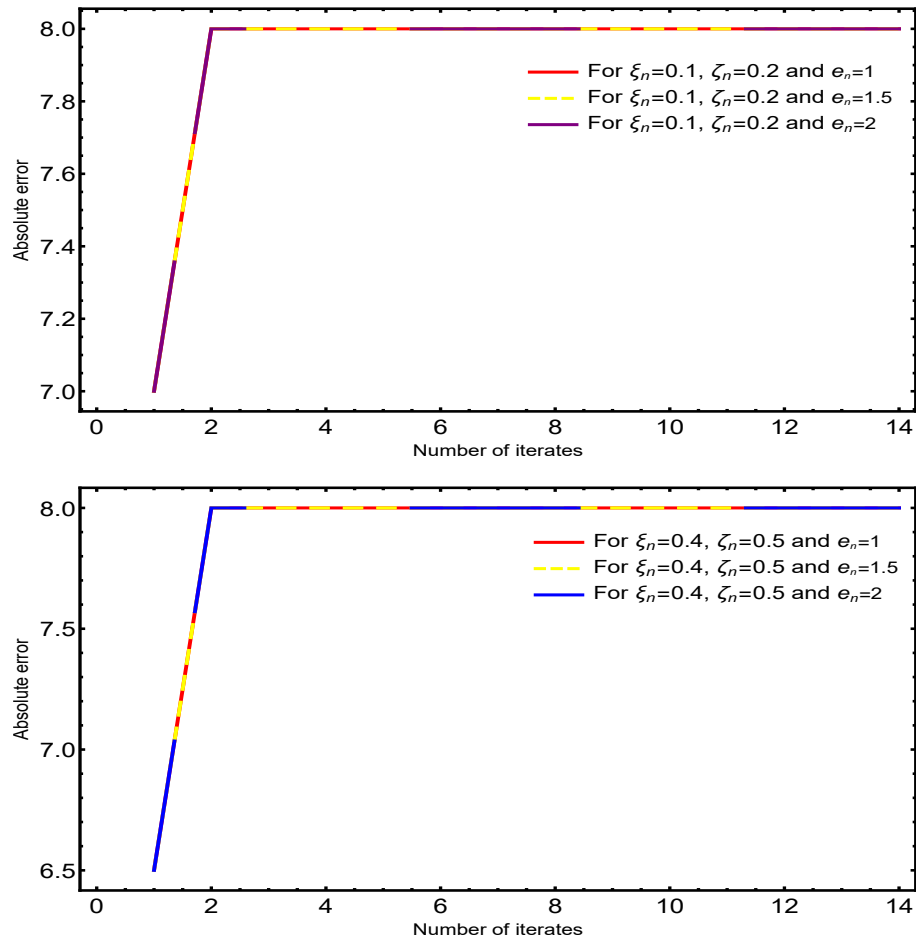


Figure 5.1: The scheme (5.7) is analyzed graphically for various parameter and starting point combinations.

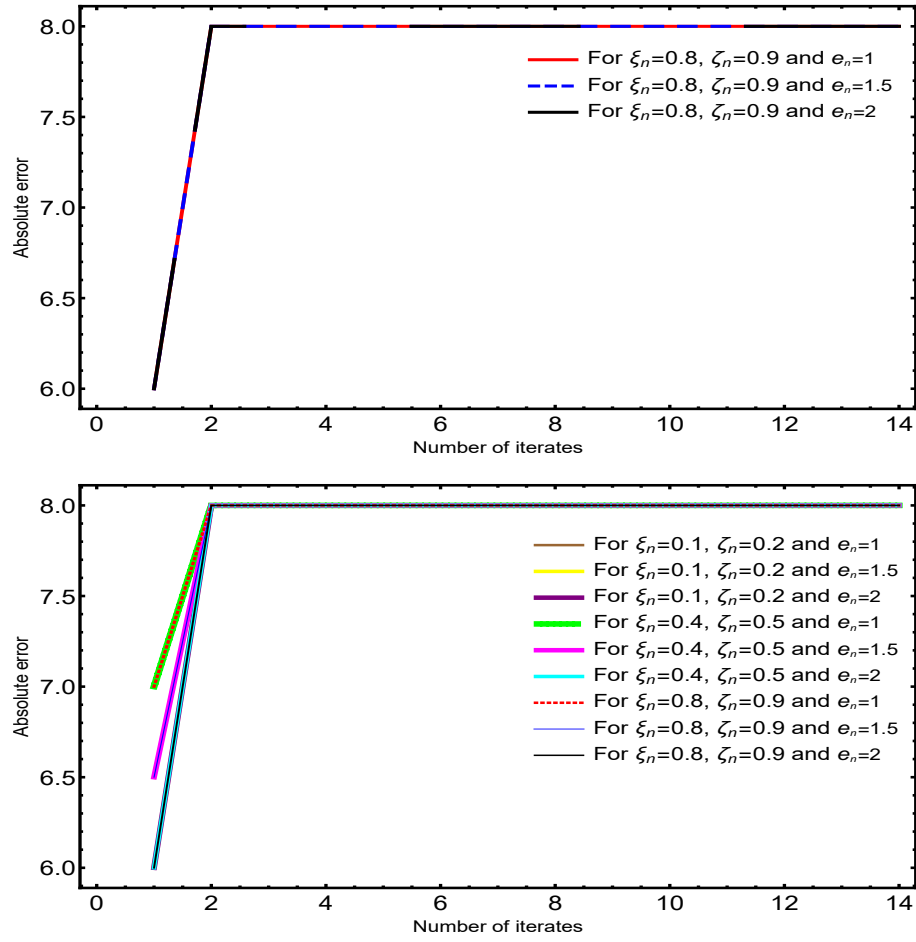


Figure 5.2: An illustration of our system (5.7) using several parameter and beginning points

Table 5.2: A^* -iterative scheme (5.7) for $\xi_i = 0.4$ and $\zeta_i = 0.4$ in Example 5.2.1

k	Scheme (5.7)	Scheme (5.7)	Scheme (5.7)
1	1	1.5	2
2	7.00	6.50	6.00
3	8.00	8.00	8.00
4	8.00	8.00	8.00
5	8.00	8.00	8.00
6	8.00	8.00	8.00
7	8.00	8.00	8.00
8	8.00	8.00	8.00
9	8.00	8.00	8.00
10	8.00	8.00	8.00
11	8.00	8.00	8.00
12	8.00	8.00	8.00

Table 5.3: A^* -iterative scheme (5.7) for $\xi_i = 0.8$ and $\zeta_i = 0.9$ in Example 5.2.1

k	Scheme (5.7)	Scheme (5.7)	Scheme (5.7)
1	1	1.5	2
2	7.00	6.50	6.00
3	8.00	8.00	8.00
4	8.00	8.00	8.00
5	8.00	8.00	8.00
6	8.00	8.00	8.00
7	8.00	8.00	8.00
8	8.00	8.00	8.00
9	8.00	8.00	8.00
10	8.00	8.00	8.00
11	8.00	8.00	8.00
12	8.00	8.00	8.00

In the event that the preceding iterative process approaches a stable point, denoted as z within operation T , the following relationship holds:

$$z = Tz = z - \frac{P(z)}{P'(z)}$$

Consequently, we might conclude the expression $\frac{P(z)}{P'(z)} = 0$, leading us to the conclusion that $P(z) = 0$ as well. This establishes that z is indeed a root of the function P . Now, let's analyze the comparison between the iterative procedure utilizing Newton's operator and the utilization of the aforementioned iteration procedure aimed at determining the roots of a complex polynomial. We can consider the following complex polynomials and their respective iteration processes:

$$\text{Polynomial: } P_1(z) = z^4 - 1;$$

$$\text{Polynomial: } P_2(z) = z^7 - 1;$$

$$\text{Polynomial: } P_3(z) = z^9 + z^8 + z^7 + \dots + z^1;$$

Consider a complex number represented as $z = a + ib$. To create regions of attraction, known as basins of attraction, using three specific complex polynomials, various iterative schemes are employed, including the A^* iterative scheme (5.7). In this process, the parameter ϱ_i is selected with a fixed value of 0.99 for different values of i , specifically

when $(i = 0, 1, 2)$. Now, let's focus on the specified domain depicted below:

$$D_1 = [-2, 2] \times [-2, 1], \text{ and } D_2 = D_3 = [-2, 2] \times [-2, 2]$$

respectively, where their centers are located at the origin. To proceed, we divide these regions into grids of size 250×250 . Using Newton's operator and employing the Mann, Khan, and A^* iterative scheme (5.7), we generate a sequence of complex values denoted as $\{\xi_i\}$. This sequence originates from each grid point, starting with an initial guess ξ_0 . If, during a maximum of iteration $i \leq 12$ iterations, the iterates $\{\xi_i\}$ substantially converge to a root with a precision of 10^{-8} , we attribute a specific color to the corresponding ξ_0 . In instances where $\{\xi_i\}$ does not converge towards any root, we designate the color white for $\{\xi_i\}$. The collection of ξ_0 points that converge to the same root forms what we refer to as a basin of attraction. Through this process, we generate various basins of attraction corresponding to the complex polynomials we have defined. For visual representation, please refer to Figures 5, 6 and 7.

5.4 An Application to Fractional Differential Equation

In this portion, we employ the A^* iterative scheme (5.7) to show that Delay Nonlinear fractional differential equations have solutions(DNFDE). Before proceeding, we review the notion of solutions' existence in the context of DNFDE.

$${}^c\mathcal{D}^y(w(q)) = h(q, w(q), w(q - \sigma)), q \in [e, B] \quad (5.33)$$

with with based on an the initial conditions

$$w(q) = \psi(q), q \in [e - \mu, e]. \quad (5.34)$$

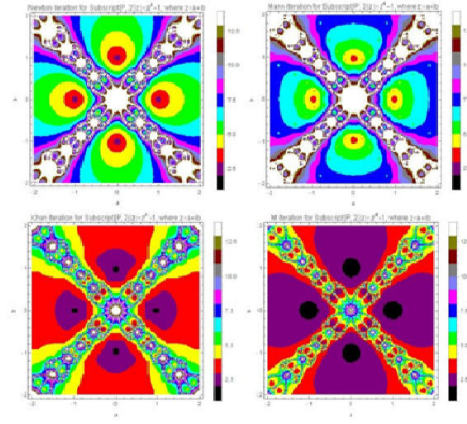


Figure 5.3: Basins of attraction of different iteration using polynomial P1.

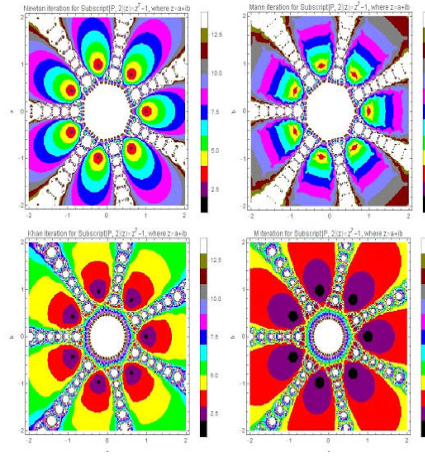


Figure 5.4: Basins of attraction of different iteration using polynomial P2

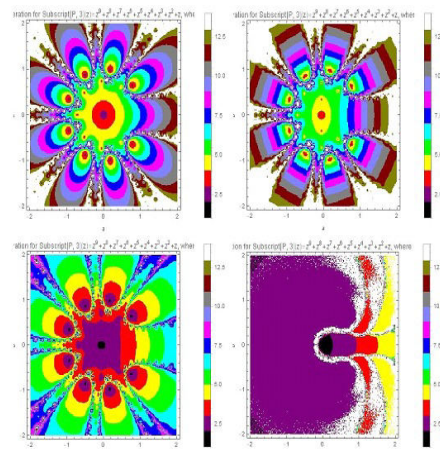


Figure 5.5: Basins of attraction of different iteration using polynomial P3.

Where, $\sigma > 0$, $B > 0$, $\mu > 0$, $w \in \mathbb{R}^k$, $\psi \in C([e - \sigma, e] : \mathbb{R}^k)$, and an continuous mapping $h : [e, B] \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$. Define the integral equation with respect to ${}^c\mathcal{D}^y(w(q))$ by

$$w(q) = \psi(e) + \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} h(\mu, w(q), w(q - \sigma)) d\mu, \quad \forall q \in [e, B]. \quad (5.35)$$

Define the norm $\|\cdot\|_{\delta L}$ on $C([e - \sigma, e] : \mathbb{R}^k)$ by

$$\|\psi\|_{\delta L} = \frac{\sup \|\psi(q)\|}{E_y(\delta_L q_y)}, \quad \text{for all } \psi \in C([e - \sigma, e] : \mathbb{R}^k) \quad (5.36)$$

where E_y consider is Mittag-Leffler function and is defined as follows

$$E_y(q) = \sum_{k=0}^{\infty} \frac{q^k}{\Gamma(y_{k+1})}, \quad \text{for all } q \in \mathbb{R} \quad (5.37)$$

Therefore, the pair $(C([e - \sigma, e] : \mathbb{R}^k), \|\cdot\|_{\delta L})$ forms a Banach space.

Theorem 1. *Assume that:*

(D1): *there exists $L_h > 0$ such that*

$$\|h(q, \check{s}_1, \check{o}_1) - h(q, \check{s}_2, \check{o}_2)\| \leq L_h(\|\check{s}_1 - \check{o}_1\| + \|\check{s}_2 - \check{o}_2\|), \quad \forall \check{s}_1, \check{o}_1, \check{s}_2, \check{o}_2 \in \mathbb{R}^k. \quad (5.38)$$

(D2): *there exists $L_h > 0$ with respect to $\frac{2L}{\delta L} < 1$ such that*

$$m \in (C([e - \sigma, e] : \mathbb{R}^k)) \cap C^1([e - \sigma, e] : \mathbb{R}^k) \quad (5.39)$$

which satisfying (5.33) and (5.34). Then, the sequence of $\check{s}^*$ iterative scheme (5.7) have a unique solution.

Proof. Define an operator Λ as

$$\Lambda w(q) = \begin{cases} \psi(e) + \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} h(\mu, w(\mu), w(\mu - \sigma)) d\mu, & \forall q \in [e, B]; \\ \psi(q), & q \in [e - \mu, e] \end{cases} \quad (5.40)$$

Let us now discuss the following two cases for showing that the sequence $x_n \rightarrow m$ as $n \rightarrow \infty$:

Case 1. If $q \in [e - \mu, e]$, then clearly we have $x_n \rightarrow m$ as $n \rightarrow \infty$.

Case 2. If $q \in [e, B]$, then in the light of (5.7), Lemma (5.2.1), and conditions (D1) and (D2), one writes

$$\begin{aligned} \|x_{n+1} - m\| &= \|(1 - \varrho_n)\Lambda x_n + \varrho_n\Lambda x_n - m\| \\ &\leq (1 - \varrho_n) \|x_n - m\| + \varrho_n \|\Lambda x_n - m\|. \end{aligned} \quad (5.41)$$

Taking into account supremum, we have

$$\begin{aligned} \sup_{q \in [e - \sigma, B]} \|x_{n+1} - m\| &\leq \sup_{q \in [e - \sigma, B]} ((1 - \varrho_n) \|x_n - m\| + \varrho_n \|\Lambda x_n - \Lambda m\|) \\ &\leq (1 - \varrho_n) \sup_{q \in [e - \sigma, B]} \|x_n - m\| + \varrho_n \sup_{q \in [e - \sigma, B]} \|\Lambda x_n - \Lambda m\|. \end{aligned}$$

This implies

$$\begin{aligned} \sup_{q \in [e - \sigma, B]} \|x_{n+1} - m\| &\leq (1 - \varrho_n) \sup_{q \in [e - \sigma, B]} \|x_n - m\| \\ &\quad + \varrho_n \sup_{q \in [e - \sigma, B]} \left\| \begin{aligned} &\psi(e) + \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} h(\mu, x_n(\mu), x_n(\mu - \sigma)) d\mu \\ &-\psi(e) - \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} h(\mu, m(\mu), m(\mu - \sigma)) d\mu \end{aligned} \right\| \\ &\leq (1 - \varrho_n) \sup_{q \in [e - \sigma, B]} \|x_n - m\| \\ &\quad + \varrho_n \sup_{q \in [e - \sigma, B]} \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} \left\| \begin{aligned} &h(\mu, x_n(\mu), x_n(\mu - \sigma)) \\ &-h(\mu, m(\mu), m(\mu - \sigma)) \end{aligned} \right\| d\mu. \end{aligned}$$

Which implies

$$\begin{aligned} \sup_{q \in [e - \sigma, B]} \|x_{n+1} - m\| &\leq (1 - \varrho_n) \sup_{q \in [e - \sigma, B]} \|x_n - m\| + \varrho_n \sup_{q \in [e - \sigma, B]} \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} d\mu \\ &\quad \times L_h(\|x_n - m(\mu)\| + \|x_n(\mu - \sigma) - m(\mu - \sigma)\|) \\ &\leq (1 - \varrho_n) \sup_{q \in [e - \sigma, B]} \|x_n - m\| + \varrho_n \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} d\mu \\ &\quad \times L_h\left(\sup_{q \in [e - \sigma, B]} \|x_n - m(\mu)\| + \sup_{q \in [e - \sigma, B]} \|x_n(\mu - \sigma) - m(\mu - \sigma)\|\right). \end{aligned}$$

Dividing by $E_y(\delta_L q_y)$,

$$\begin{aligned} \frac{\sup_{q \in [e-\sigma, B]} \|x_{n+1} - m\|}{E_y(\delta_L q_y)} &\leq \frac{(1 - \varrho_n) \sup_{q \in [e-\sigma, B]} \|x_n - m\|}{E_y(\delta_L q_y)} + \varrho_n \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} d\mu \\ &\quad \times L_h \left(\frac{\sup_{q \in [e-\sigma, B]} \|x_n - m(\mu)\|}{E_y(\delta_L q_y)} \right. \\ &\quad \left. + \frac{\sup_{q \in [e-\sigma, B]} \|x_n(\mu - \sigma) - m(\mu - \sigma)\|}{E_y(\delta_L q_y)} \right). \end{aligned}$$

By virtue of (5.36), we have

$$\begin{aligned} \|x_{n+1} - m\|_{\delta_L} &\leq (1 - \varrho_n) \|x_n - m\|_{\delta_L} + \varrho_n \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} d\mu \\ &\quad \times L_h(\|x_n - m(\mu)\|_{\delta_L} \\ &\quad + \|x_n(\mu - \sigma) - m(\mu - \sigma)\|_{\delta_L}). \end{aligned}$$

This implies

$$\begin{aligned} \|x_{n+1} - m\|_{\delta_L} &\leq (1 - \varrho_n) \|x_n - m\|_{\delta_L} + \varrho_n \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} d\mu \\ &\quad \times 2L_h(\|x_n - m(\mu)\|_{\delta_L}) \\ &= (1 - \varrho_n) \|x_n - m\|_{\delta_L} + \frac{\varrho_n 2L_h}{E_y(\delta_L q_y)} \|x_n - m\|_{\delta_L} \\ &\quad \varrho_n \frac{1}{\Gamma(y)} \int_e^q (q - e)^{y-1} E_y(\delta_L q_y) d\mu. \\ &= (1 - \varrho_n) \|x_n - m\|_{\delta_L} + \frac{\varrho_n 2L_h}{E_y(\delta_L q_y)} \|x_n - m\|_{\delta_L} \\ &= (1 - \varrho_n) \|x_n - m\|_{\delta_L} + \frac{\varrho_n 2L_h}{E_y(\delta_L q_y)} \|x_n - m\|_{\delta_L} \\ &\quad \check{C} \left(\frac{E_y(\delta_L q_y)}{\delta_L} \right) \\ &= (1 - \varrho_n) \|x_n - m\|_{\delta_L} + \frac{\varrho_n 2L_h}{E_y(\delta_L q_y)} \frac{E_y(\delta_L q_y)}{\delta_L} \|x_n - m\|_{\delta_L} \\ &= (1 - \varrho_n) \|x_n - m\|_{\delta_L} + \frac{\varrho_n 2L_h}{\delta_L} \|x_n - m\|_{\delta_L}. \end{aligned}$$

Owing to (D2), we have

$$\|x_{n+1} - m\|_{\delta_L} \leq \|x_n - m\|_{\delta_L}.$$

Setting $\Delta_n = \|x_n - m\|_{\delta_L}$, then

$$\Delta_{n+1} \leq \Delta_n, \quad n \in N.$$

Therefore, Δ_n is a monotone decreasing sequence with respect to real numbers and is bounded from below. Hence,

$$\lim_{n \rightarrow \infty} \Delta_n = \inf\{\Delta_n\} = 0.$$

This implies that $\|x_n - m\|_{\delta L} \rightarrow 0$ as $n \rightarrow \infty$. □

5.5 Conclusion

This work put forth an innovative and groundbreaking concept called extended φ -Suzuki nonexpansive mappings. Through this novel approach, we have significantly improved the understanding of convergence rate analysis in comparison to existing studies in the field. Additionally, we have introduced a new and efficient technique called A^* -iterative scheme, which enables the accurate approximation of common fixed points in the realm of Banach spaces.

Our study's significance lies in the practical implications and the soundness of our theoretical analysis, both of which are demonstrated through concrete numerical examples and real-world applications. By extending the iterative scheme, we have delved deeper into this research area, building upon and expanding the scope of prior literature.

Our comprehensive exploration and contributions hold promise for future advancements in the field of nonexpansive mappings and iterative methods. The proposed extended φ -Suzuki approach and A^* -iterative scheme opens new avenues for more efficient and reliable techniques in solving problems related to fixed points in Banach spaces. We analyze that our work will serve as a cornerstone for future research, inspiring further investigations and innovations in this domain.

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