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On the Boundary Layer Flow of Ferrofluids

By

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1. Boundary layer

2. Fluid dynamics

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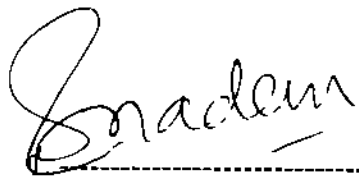
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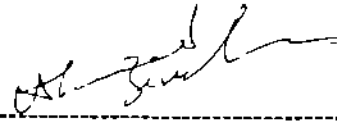
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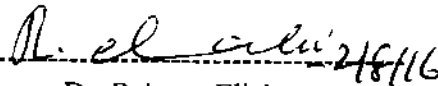We accept this as conforming to the required standard

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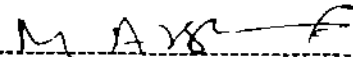
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Muzammal Hameed Tariq

M Phil Scholar (Mathematics)

210-FBAS/MSMA/F-14

Dedication

This thesis is dedicated to Holy prophet

Hazrat Muhammad (P.B.U.H)

&

My family

Acknowledgements

Preface

On the boundary layer flow of ferrofluids are important in many processes For instance, researchers have prepared ferrofluids, which have the fluid properties of a liquid and the magnetic properties of a solid The ferrofluids actually contain tiny particles (~10 nm diameter) of a magnetic solid suspended in a liquid medium Ferrofluids were originally discovered in the 1960s at the NASA research center, where scientists were investigating different possible methods of controlling liquids in space The benefits of a magnetic fluid were immediately obvious The location of the fluid could be precisely controlled through the application of a magnetic field and by varying the strength of the field, the fluids could be forced to flow Researchers have prepared ferrofluids containing small particles of ferromagnetic metals, such as cobalt and iron, as well as magnetic compounds, like manganese zinc ferrite, $Zn_xMn_{1-x}Fe_2O_4$ ($0 < x < 1$; the family of solid solutions) But by far, the most work has been conducted on ferrofluids containing small particles of magnetite, Fe_3O_4 One of many fascinating features of ferrofluid is the prospect of influencing flow by a magnetic field and vice- versa [1,2]

Morever, ferrofluids have found a wide variety of applications, including use in rotating shaft seals These rotating shaft seals are found in rotating anode X-ray generators and in vacuum chambers used in the semiconductor industry Ferrofluid seals are used in high-speed computer disk drives to eliminate harmful dust particles or other impurities that can cause the data-reading heads to crash into the disks under engineering application

The major application of ferrofluids in electrical field is improving the performance of loudspeakers Finally, there is much hope for future biomedical applications of ferrofluids For example, researchers are attempting to design ferrofluids that can carry medications to specific locations in the body through the use of applied magnetic fields Ferrofluid have been investigated by a number of workers Some recent investigations in this direction are presented in the refs [3-10]

In view of the above discussion the present dissertation is arranged as follows Chapter one includes some relevant definations and equations of the subsequent chapters

Chapter two is a review work of [11]. In this paper the theoretical investigation of the effect of rotation on ferrofluid flow due to rotating disk by solving boundary layer equation is investigated The couple of nonlinear differential equation have been

solved by power series approximations Expressions for the components of velocity and pressure profile are obtained by cylindrical co-ordinate system by considering the z-axis as the axis of rotation The results for all above variables are obtained numerically and discussed graphically

Chapter three is the extension of chapter two in which the unsteady ferrofluid flow revolving a stretchable rotating disk on a boundary layer under the influence of low oscillating magnetic field is investigated The governing partial differential equations are first converted into non-linear ordinary differential equations and then a convergent series solution is obtained using homotopy analysis method (HAM) [12-22] Heat transfer analysis, velocity profile, pressure profile are discussed in the presence of particle concentration and magnetic isolatry In addition, expression of Skin fraction, Nusselt number, angle between the wall and revolving ferrofluid, boundary layer thickness and volume flowing through the axial direction are discussed Graphs are also presented to analyze the behavior of effective parameters on the velocity, temperature and pressure profiles

Contents

Chapter 1

Basic of fluid dynamic

This chapter comprises of some elementary definitions which may be useful for the better understanding of the next two chapters

1.1 Definition

1.1.1 Fluid

Any substance which shows resistance to its internal molecular structure when external force applies Liquids and gases are identified as fluid since they deform continuously is response to shear stress

1.2 Types of fluids

1.2.1 Newtonian fluid

Such fluid which obeys Newton's law of viscosity is called Newtonian fluid Newton's law of viscosity is given by

$$\tau = \mu \frac{du}{dy} \tag{1 1}$$

Where, τ is the shear stress, μ is the viscosity of fluid, du dy is the shear rate or velocity of gradient Gases and most common liquids are tends to Newtonian fluids Most common examples are Water, thin motor oil, air, sugar solutions, silicone, etc

1.2.2 Non-Newtonian fluids

Fluid which do not obey the Newton's law of viscosity's is called non-Newtonian fluid Common examples are paints, flour dough, coal tar, ketchup, shampoo, chewing gum and fruit juice

1.2.3 Ferrofluid

A ferrofluid is a liquid that becomes strongly magnetized in the presence of a magnetic field

3

Fig. 1.1: Ferrofluid on glass with a magnet underneath

1.3 Magnetohydrodynamics (MHD)

MHD is a discipline which studies the dynamics of electrically conducting fluids The word magnetohydrodynamics (MHD) is derived from magneto-meaning magnetic field, hydro-meaning liquid, and dynamics meaning movements The field of MHD was initiated by Hannes Alfve'n for which he received the Noble prize in physics in 1970 Examples of such fluids include plasmas, liquid metal and salwater

1.3.1 Magnetism

The study of nature, and cause of magnetic force fields and how different substances are affected by them is called magnetism

1.3.2 Magnetic field

The region or space around a magnet in which it exerts a force on other magnet, is called its magnetic field It is mostly strong near poles The path of small magnet in a magnetic field is called magnetic field lines The magnetic field is vector quantity and has SI unit of Tesla

1.3.3 Magnetic permability

This is the property of material that is equal to magnetic flux density B established within the material by magnetizing field divided by magnetic field strength H of the magnetizing field Mathematically one can write it as

$$\mu_m = \frac{B}{H} \tag{1 2}$$

Where μ_m is the magnetic permability

4

1.3.4 Maxweel's equations

The Maxwell's equations are the set of four foundamental equations governing electro-magnetism, i e , the behavior of electro and magnetic fields They were first written down in complete form by James Clerk Maxwell For the varying fields the differential form of those equations in MKS are given as

$$
\left.\begin{array}{l}
\nabla E = \dfrac{\rho}{\varepsilon_0}, \\[2mm]
\nabla \times E = -\dfrac{\partial B}{\partial t}, \\[2mm]
\nabla B = 0, \\[2mm]
\nabla \times B = \mu_e J + \varepsilon_0 \mu_e \dfrac{\partial E}{\partial t}
\end{array}\right\}
\tag{1 3}
$$

Where E is the electric field, B is the magnetic field. J is the charge density. $\varepsilon_0 = 8\ 854187817 \times 10^{-12} Fm^{-1}$ is the permittivity of free space and $\mu_e = 1\ 2566370614 \times 10^{-6} NA^{-2}$ is the permeability of free space

1.4 Force associate with fluid in motion

1.4.1 Force

Force is defined as an action that alters or tends to alter a body's state of rest or uniform motion in a straight line Its magnitude and direction is given by vector F

1.4.2 Inertial force

If several forces act on a particle of mass m and produced an acceleration a, the magnitude F of the vector sum of applied forces is equal to ma Thus

$$
F - ma = 0 \tag{1 4}
$$

Where, $-ma$ is called inertial force Examples are the Coriolis force and the centrifugal force that appears in rotating coordinate system

1.4.3 Coriolis force

The Coriolis force is an inertial force that acts on objects which are in motion relative to a rotating reference frame In a reference frame with clockwise rotation, the force acts to the left of the motion of the object In one with anticlockwise rotation, the force acts to the right The Coriolis forces are always perpendicular to the direction of flow so the pressure gradient is also perpendicular to the flow

The vector formula for the magnitude and direction of the Coriolis acceleration is

$$a_c = -2\Omega \times v \qquad\qquad (1\ 5)$$

In above. a_t is the acceleration of the particle in the rotating system, v is the velocity of the particle with respect to the rotating system, Ω is the angular velocity vector of magnitude equal to the rotation rate ω directed along the axis of rotation in rotating reference frame of reference and the $\times$ symbol represents the cross product operator

1.4.4 Centrifugal force

The term centrifugal force is refer to an inertial force directed away from the axis of rotation that appears to act on all objects when viewed in a rotating reference frame The concept of centrifugal force can be applied in rotating devices such as centrifuges, centrifugal pumps and centrifugal governors etc

1.4.5 External force

A force applied at an element of area dA tends to distort some other element of area dB and is transmitted throughout the body For example, body forces and surface forces These forces are also called long-range forces

1.4.6 Internal force

A force exerted by matter on one side to that on the other side is termed on internal force One of its examples is stress It can be resolved in to components normal and tangential to the area that are called tensile stress component and the shear stress component respectively These forces are also called short-range forces

1.5 Heat transfer

In physics, heat is the energy transferred from one body or system to another due to difference in temperature The discipline of heat transfer is concerned with only two things one of them is temperature and the other is flow heat Temperature represents the amount of thermal energy available, whereas, heat flow represent the movement of thermal energy from place to place

1.5.1 Modes of heat flow

Modes of heat transfer are mainly classified into three categories as given below

i   Conduction

ii Convection

iii Radiation

6

1.5.2 Conduction

Conduction is a heat flow through solids without any visible movement of the particles It is because of temperature difference in which heat transportations takes place from the more energetic particles to energetic fluid particles only by collision or because of molecular vibrations and the energy movement by free electrons Heat transfer occurs in Silver, Copper, Mercury and Iron through conduction Heat conduction rate depends upon the medium's geometry, relative thickness and the type of material It can be easily measured in one dimension as flows

$$Heat\ conduction\ rate \propto \frac{(area)(temprature\ difference)}{thickness}.$$

(1 6)

$$Q_{cond}^{\cdot} = k_f A \frac{T_1 - T_2}{\Delta x}$$

The above equation is also called heat conduction equation in which proportionality constant k_f stand for material's thermal conductivity which measures the material's heat conducting capability When $\Delta x \rightarrow 0$ then the above equation becomes

$$Q_{cond}^{\cdot} = -k_f A \frac{dt}{dx}$$

(1 7)

The above equality recognized as Fourier's law of heat conduction

1.5.3 Convection

Heat convection is the transportation and exchange of heat, due to mixing motion of different parts of fluid It is governed by the laws of fluid dynamics in combination with the laws of heat conduction Examples of convection are hot water, the cooling system of an automobile engine, the flow of the blood in human body etc

1.5.4 Radiation

Radiation is heat transfer mechanism in which heat transfer take place from any material through emission or absorption of electromagnetic waves Especially radiation is significant during combustion processes where temperature is very high, but can also be favorable at room's temperature The transfer of heat due to radiation can occur through gasses, fluids or a vacuum Distinct from conductive and convective processes, heat transfer by radiative processes does not require to propagate any material

7

1.6 Physical description of dimensionless numbers

1.6.1 Reynolds number (Re)

This dimensionless quantity estimates the correspondence of inertial forces by viscous forces Reynolds number is used to characterize fluid behavior in the boundary layer flow Mathematically, it is given by

$$Re = \frac{\rho u^2 \; l}{\mu u \; l^2} = \frac{lu}{\upsilon}$$ (1 8)

Where u is the fluid velocity, μ is the dynamic viscosity, l is the characteristic length and υ is the kinematic viscosity Laminar flow is characterized by low Reynolds numbers (Dominance of viscous forces) whereas turbulent flow is characterized by high Reynolds numbers (Dominance of inertial forces)

1.6.2 Prandtl number (Pr)

The Prandtl number is a dimensionless number approximating the ratio of momentum diffusivity and thermal diffusivity Mathematically, it is expressed by the following relation

$$Pr = \frac{Viscousd \; diffusion \; rate}{Thermal \; diffusion \; rate},$$

$$Pr = \frac{\mu C_p}{k}$$ (1 9)

In heat transfer problem, the Prandtl number controls the relative thickness of the momentum and thermal boundary layer When Pr is small it means that heat diffuses very quickly compared to the velocity (momentum) In above equation C_p is the specific heat

1.6.3 Nusselt number

The convective to conductive heat transfer ratio across a boundary layer is referred to Nusselt number In mathematical notation, it can be written as

$$Nu = \frac{h\Delta T}{K\Delta T \; l} = \frac{hl}{k}$$ (1 10)

Here, Nu, h, l and k are the Nusselt number, heat transfer coefficient, characteristic length and thermal conductivity respectively

8

1.6.4 Skin friction

When fluid and surface are in relative motion, the drag force exerted by fluid on surface in contact is called skin friction It can be expressed as

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho u^2} \tag{1 11}$$

In above, C_f, τ_w, ρ and u stand for the skin friction, wall shear stress, density and velocity of moving object respectively

Chapter 2

On the revolving ferrofluids flow due to rotating disk

This chapter deals with review investigation of the effect of rotation on ferrofluid flow due to rotating disk by solving boundary layer equation [11] Here, the couple of nonlinear differential equation are solved by power series approximations Expressions for the components of velocity and pressure profile are obtained by cylindrical co-ordinate system by considering the z -axis as the axis of rotation It is observed that there is significant increment in the thickness of the boundary layer over rotating due to comparison to the ordinary case of viscous fluid flow without rotation

2.1 Mathematical formulation of the problem

2.1.1 Flow modelling

The flow is isotropic medium is considered to be study $\left(\partial \ \partial t = 0\right)$ and axi-symmetric $\left(\partial_/\partial \theta = 0\right)$ excluding the thermal effects. The fluid and ferrous particles have the same velocity The fluid and disk are assumed to be electrically non-conducting The whole system is rotating with angular velocity $\Omega = (0,0,\Omega)$ along the vertical axis, which is taken z-axis as shown in **Fig. 2.1.**
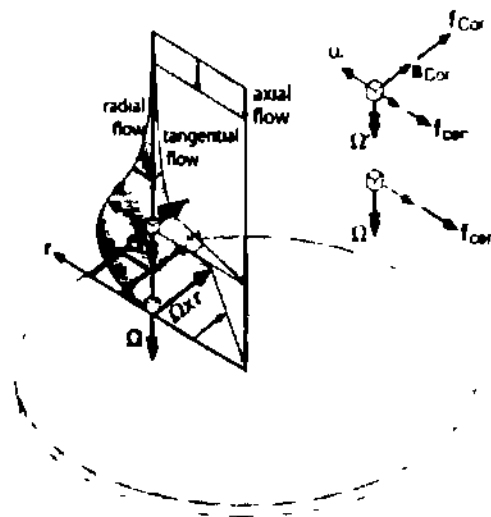

Fig. 2.1: Geometry of the problem

One additional simplification is assumed in momentum equation that the viscosity is independent of magnetic-field intensity The continuity and momentum equations for incompressible ferromagnetic fluid with constant viscosity in a rotating frame of reference

$$\nabla \cdot V = 0,$$ (2 1)

$$\rho\left[\frac{\partial v}{\partial t}+(V \cdot \nabla)V\right]=-\nabla p +\mu_0(M \cdot \nabla)H+\mu_r\nabla^2 V$$
$$+ 2\rho(\Omega \times V)-\frac{\rho}{2}\nabla|\Omega \times r|^2$$ (2 2)

In Eq (2 2) the effect of rotation includes Centrifugal force $-\frac{\rho}{2}\nabla|\Omega \times r|^2$ and

Coriolis acceleration $2\rho(\Omega \times V)$ Here $P -\frac{\rho}{2}\nabla|\Omega \times r|^2 = p$ is the reduce pressure.

in which P stands for fluid pressure The Maxweel's equations, simplified for a non-conducting fluid with no displacement currents are as follows

$$\nabla \times H = 0, \qquad \nabla \cdot (H + 4\pi M) = 0,$$ (2 3)

with assumptions

$$M = \chi H, \quad \chi = \frac{\mu_f}{\mu_0} - 1, \quad M \times H = 0$$ (2 4)

Attentions have been drawn to the fact that the external magnetic field interacts with magnetic particles, not only through forces but also through couples The velocity component v_z is less as compared to v_r and v_θ The velocity components v_r, v_θ and v_z respectively along r, θ and z direction can be written as

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}$$
$$=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\upsilon\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right],$$ (2 5)

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}$$
$$=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\upsilon\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right],$$ (2 6)

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}$$

$$= F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] \qquad (2\ 7)$$

Here $\partial_z \partial t = 0$, $\partial_z \partial \theta = 0$ and neglecting body force, the Eqs (2 5)-(2 7) are obtained as

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu_0}{\rho}|M|\frac{\partial}{\partial r}|H| + \upsilon\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_r}{r}\right) + \frac{\partial^2 v_r}{\partial z^2}\right] + 2\Omega v_\theta, \qquad (2\ 8)$$

$$v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = \upsilon\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right] - 2\Omega v_\theta, \qquad (2\ 9)$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu_0}{\rho}|M|\frac{\partial}{\partial r}|H| + \upsilon\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right], \qquad (2\ 10)$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \qquad (2\ 11)$$

The associated initial and boundary condition for the flow due to rotation of an infinitely long disk ($z = 0$) with constant angular velocity ω are given by

$$\begin{aligned} at \quad z &= 0, \quad v_r = 0, \quad v_\theta = r\omega, \quad v_z = 0 \\ at \quad z &\to \infty, \quad v_r = 0, \quad v_\theta = 0 \end{aligned} \Bigg\} \qquad (2\ 12)$$

Using the boundary layer approximation $-\frac{1}{\rho}\frac{\partial}{\partial r} + \frac{\mu_0}{\rho}|M|\frac{\partial}{\partial r}H| = -r\omega^2$ [23] with

very less variation of magnetic-field in z-direction along with the following transformations

$$v_r = r\omega E(\alpha), \ v_\theta = r\omega F(\alpha), \ v_z = r\omega G(\alpha), \ p = \rho\omega P(\alpha), \ \alpha = z\sqrt{\frac{\omega}{\upsilon}}, \quad (2\ 13)$$

the Eqs (2 8)-(2 11) in term of E, F, G, and P take the following form

$$E^{''} - GE^{'} - E^2 + F^2 + 2F - 1 = 0, \qquad (2\ 14)$$

$$F^{''} - GF^{'} - 2EF - 2E = 0, \qquad (2\ 15)$$

$$P^{'} - G^{'} + GG^{'} = 0, \qquad (2\ 16)$$

$$G^{'} + 2E = 0 \qquad (2\ 17)$$

Corresponding boundary condition of Eq (2 12) become

$$E(0) = G(0) = 0, \; F(0) = 1, \; P(0) - P_0 = 0, \\ E(\infty) = F(\infty) = 0 \Bigg\}$$

(2 18)

Note that G must tend to finite limit, say -c, as $\alpha \to \infty$, i e, $G(\infty) = -c, (c > 0)$

2.1.2 Parameters of physical interest

Boundary layer displacement thickness

Thickness of boundary layer can be found by calculated the distance over which inertia and viscous forces are comparable The velocity in the boundary layer attains a value which is very close to the external velocity at a small distance from the wall The boundary layer displacement thickness is defined as

$$d = \frac{1}{r\omega} \int_{z=0}^{\infty} v_\theta dz = \int_{\alpha=0}^{\infty} F(\alpha)d\alpha = 1\ 3456145$$

(2 19)

Further, total volume flowing outword the z-axis

$$Q = 2\pi R \int_0^{\infty} v_r dz = 2\pi R \int_0^{\infty} r\omega E(\alpha)\sqrt{\upsilon\ \omega} d\alpha$$

$$= -\pi R^2 \sqrt{\omega\upsilon} G(\infty) = 2\ 786094 R^2 \sqrt{\omega\upsilon} = 2\ 786094 R^2 \upsilon \frac{\alpha}{z}$$

(2 20)

Hence the total volume flowing outward the z-axis is proportional to the dimensionless parameter α

The fluid is taken to rotate at a large distance from the wall, in this case the angle becomes

$$\tan \varphi_0 = -\left(\frac{\frac{\partial v_r}{\partial z}}{\frac{\partial v_\theta}{\partial \theta}} \right) = -\frac{E\ (0)}{F\ (0)} = \frac{0\ 54}{0\ 62} = 0\ 870967$$

(2 21)

$$\Rightarrow \varphi_0 = 41^0$$

2.2 Solution of the problem

The value of E, F, G and P are compared graphically with their corresponding value in classical case Cochran indicated that formal asymptotic expansions (for large α) of the system of Eqs (2 14)-(2 17) are the power series in $\exp(-ic\alpha)$, i e,

$$E(\alpha) \approx \sum_{i=1}^{\infty} A_i \exp(-ic\alpha),$$

(2 22)

13

$$F(\alpha) \approx \sum_{i=1}^{\infty} B_i \exp(-ic\alpha), \qquad (2\ 23)$$

$$G(\alpha) \approx G(\infty) + \sum_{i=1}^{\infty} C_i \exp(-ic\alpha), \qquad (2\ 24)$$

$$P(\alpha) - P_0 \approx \sum_{i=1}^{\infty} D_i \exp(-ic\alpha) \qquad (2\ 24)$$

Integrating the Eqs (2 14) and (2 15) between 0 to ∞, and using the relations

$$\left. \begin{array}{l} \displaystyle\int_0^{\infty} GE\,d\alpha = [GE]_0^{\infty} - \int_0^{\infty} G'E\,d\alpha = 2\int_0^{\infty} E^2\,d\alpha, \\[4mm] \displaystyle\int_0^{\infty} GF\,d\alpha = [GF]_0^{\infty} - \int_0^{\infty} G'F\,d\alpha = 2\int_0^{\infty} EF\,d\alpha \end{array} \right\} \qquad (2\ 25)$$

together with $E(\infty) = 0$ and $F(\infty) = 0$, we get

$$\left. \begin{array}{l} \displaystyle -E'(0) = \int_0^{\infty} (3E^2 - F^2 - 2F + 1)\,d\alpha, \\[4mm] \displaystyle -F'(0) = \int_0^{\infty} (4EF + 2E)\,d\alpha \end{array} \right\} \qquad (2\ 26)$$

Using the supposition $E(0) = a$ and $F(0) = b$ in Eqs (2 14)-(2 18), we get the following additional boundary conditions for the approximate solution

$$\left. \begin{array}{ll} E'(0) = -2, & E''(0) = -4b, \\ F(0) = 0, & F'(0) = 4a, \\ G(0) = 0, & G'(0) = -2a, \quad G''(0) = 4, \\ P(0) = -2a, & P'(0), \quad P''(0) = 8b \end{array} \right\} \qquad (2\ 27)$$

First four coefficients in each Eqs (2 22)-(2 24) with the help of Eqs (2 18) and (2 27) are as follow

$$\left. \begin{aligned} A_1 + A_2 + A_3 + A_4 &= 0 \\ A_1 + 2A_2 + 3A_3 + 4A_4 &= \frac{-a}{c} \\ A_1 + 4A_2 + 9A_3 + 16A_4 &= -2 \\ A_1 + 8A_2 + 27A_3 + 64A_4 &= \frac{4b}{c^3} \end{aligned} \right\} , \qquad \left. \begin{aligned} B_1 + B_2 + B_3 + B_4 &= 1 \\ B_1 + 2B_2 + 3B_3 + 4B_4 &= \frac{-b}{c} \\ B_1 + 4B_2 + 9B_3 + 16B_4 &= 0 \\ B_1 + 8B_2 + 27B_3 + 64B_4 &= \frac{-4a}{c^3} \end{aligned} \right\} ,$$

$$\left. \begin{aligned} -c + C_1 + C_2 + C_3 + C_4 &= 0 \\ C_1 + 2C_2 + 3C_3 + 4C_4 &= 0 \\ C_1 + 4C_2 + 9C_3 + 16C_4 &= \frac{-20}{C^2} \\ C_1 + 8C_2 + 27C_3 + 64C_4 &= \frac{-4}{c^3} \end{aligned} \right\} , \qquad \left. \begin{aligned} D_1 + D_2 + D_3 + D_4 &= 0 \\ D_1 + 2D_2 + 3D_3 + 4D_4 &= \frac{2a}{c} \\ D_1 + 4D_2 + 9D_3 + 16D_4 &= \frac{4}{c^2} \\ D_1 + 8D_2 + 27D_3 + 64D_4 &= \frac{-8b}{c^3} \end{aligned} \right\} \right\} \tag{2 28}$$

Solving Eq (2 28), we have

$$\left. \begin{aligned} A_1 &= \left(-\frac{2b}{3c^3} - \frac{3}{c^2} + \frac{13a}{3c} \right) & A_2 &= \left(-\frac{2b}{c^3} + \frac{8}{c^2} - \frac{19a}{2c} \right) \\ A_3 &= \left(-\frac{2b}{c^3} - \frac{7}{c^3} + \frac{7a}{c} \right) & A_4 &= \left(\frac{2b}{3c^3} + \frac{2}{c^2} - \frac{11a}{6c} \right) \\ B_1 &= \left(\frac{2a}{3c^3} + \frac{13b}{3c} + 4 \right) & B_2 &= \left(-\frac{2a}{c^3} - \frac{19b}{2c} - 6 \right) \\ B_3 &= \left(\frac{2a}{c^3} + \frac{7b}{c} + 4 \right) & B_4 &= \left(-\frac{2a}{3c^3} - \frac{11b}{6c} - 1 \right) \end{aligned} \right\} , \tag{2 29}$$

$$\left. \begin{aligned} C_1 &= \left(\frac{2}{3c^3} - \frac{3a}{c^2} + 4c \right) & C_2 &= \left(-\frac{2}{c^3} + \frac{8a}{c^2} - 6c \right) \\ C_3 &= \left(\frac{2}{c^3} - \frac{7a}{c^2} + 4c \right) & C_4 &= \left(-\frac{2}{3c^3} + \frac{2a}{c^2} - c \right) \\ D_1 &= \left(\frac{4b}{3c^3} + \frac{6}{c^2} - \frac{26a}{3c} \right) & D_2 &= \left(-\frac{4b}{c^3} - \frac{16}{c^2} + \frac{19a}{c} \right) \\ D_3 &= \left(\frac{4b}{c^3} + \frac{14}{c^2} - \frac{14a}{c} \right) & D_4 &= \left(-\frac{4b}{3c^3} - \frac{4}{c^2} + \frac{11a}{3c} \right) \end{aligned} \right\} , \tag{2 30}$$

Using the values $a = 0\ 54$, $b = -0\ 62$, and $c = 0\ 866$ [24], one can calculate the numerical values of the coefficient A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 C_1, C_2, C_3, C_4, D_1, D_2, D_3 and D_4, as a result the velocity components and asymptotic pressure with the dimension less parameter α can be obtained easily

15

2.3 Results and discussion

The problem considered here involves a number of parameters, on the basis of which, a wide range of numerical results have been derived Of these results, a small section is presented here for brevity The numerical results for the velocity profiles, commonly known as radial, tangential and axial are shown in Figs (2 2)-(2 4) respectively

Also, we have calculated the angle between the wall and revolving ferrofluid which is 41^0 In Fig (2 2), the curve E_2 represent the effect of rotation on radial velocity in case of ferrofluid flow due to rotating disk, whereas E_1 indicates the radial velocity profile of the Cochran's study of ordinary viscous fluid flow Due to rotation the radial velocity reaches its maximum value near the surface of the disk with magnitude 0 08894 at $\alpha = 0 4$, whereas, in Cochran case, the maximum value 0 181 of radial velocity is attained comparatively at distant point $\alpha = 0 9$ Here, it is noticed that the radial velocity E_2 has very less peak value in comparison to E_1 because of thickening of the fluid layer due to the rotation of the whole system Thus, the effect of rotation is more pronounced then the force of magnetization in the sense of fluid thickening Also it is quite interesting to see that in case of rotation before converging to zero, the radial velocity once become negative

Fig (2 3) is the graphical comparison of the tangential velocity of ferrofluid flow with rotation F_2 to that of Cochran's ordinary viscous fluid F_1 In our case, we increase the value of α the tangential velocity F_2 decreases continuously and tends to zero for large value of α It is observed from the table (2 1), the value of tangential velocity is 0 49762 at $\alpha = 1$, whereas for the ordinary viscous fluid, the tangential velocity is 0 46800 for the same value of α Therefore at $\alpha = 1$, there is an approximate increment of 6 7% in the value of tangential velocity in comparison to that of Cochran value From the figures, It is clear that F_1 converges to zero little faster than F_2 however, both the curves have similar trends

Fig (2 4) show that axial velocity profile, which is zero in the beginning and tends to finite value -0 886 for $\alpha = 14 8$ onwards When we increase the value of α, it decreases continuously in the negative region Our axial velocity value is -0 23858 at $\alpha = 1$, whereas for Cochran, the axial velocity is -0 266 for the same value of α

Meaning thereby, for the same value of α the axial velocity component acquires large value then the value in ordinary case

The pressure profile $P(\alpha) - P_0$ with the initial pressure P_0 at $\alpha = 0$ is shown in Fig (2 5) Here pressure goes to negative region near the surface of the disk, and at $\alpha = 0\,4$ it goes to maximum negative value -0 17909 Onwards to $\alpha = 0\,4$, pressure starts increasing with increasing the value of α and $\alpha = 1\,3$, it enters in the positive region and attains maximum value 0 07667 at $\alpha = 2\,2$ Finally $P(\alpha) - P_0$ converges to zero i e, $P(\alpha)$ converges to P_0

Comparing Fig (2 3) and Fig (2 5), we conclude that when radial velocity increases, the pressure of the ferrofluid decreases and when radial velocity decreases, ferrofluid pressure increases i e, they are converse in convergence behavior Also, tangential velocity diminishes slower than axial velocity component The change in the curve of radial velocity is faster due to effect of external magnetic field resulting in reducing the time required for velocity profile to reach there convergence level In Fig (2 6) and Fig (2 7), derivatives of radial and tangential components of velocities are shown

In nut shell, the rotation of the disk along with revolution of the ferrofluid results in an increased displacement thickness 1 34562 more than that of [25]
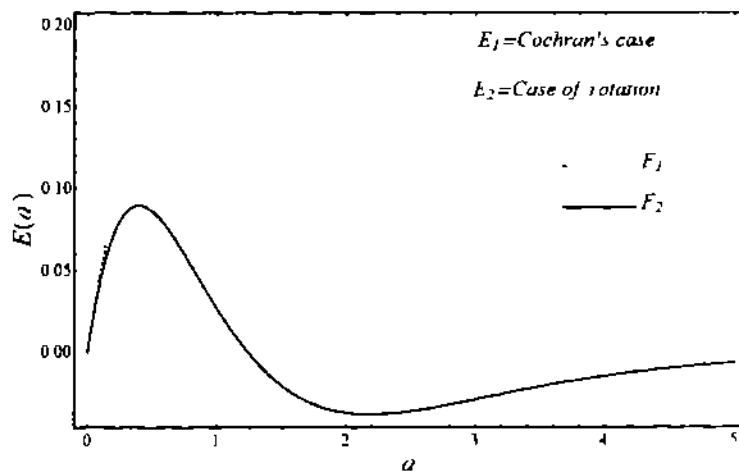

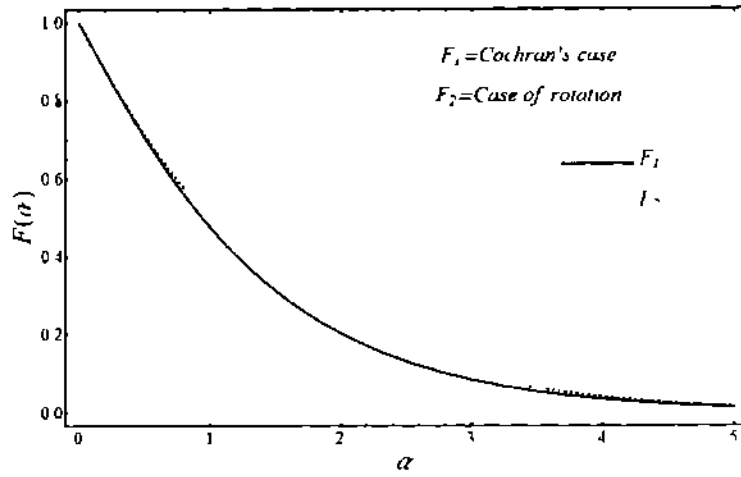Fig. 2.2: Effect of rotation on radial velocity profile

Fig. 2.3: Effect of rotation on tangential velocity profile
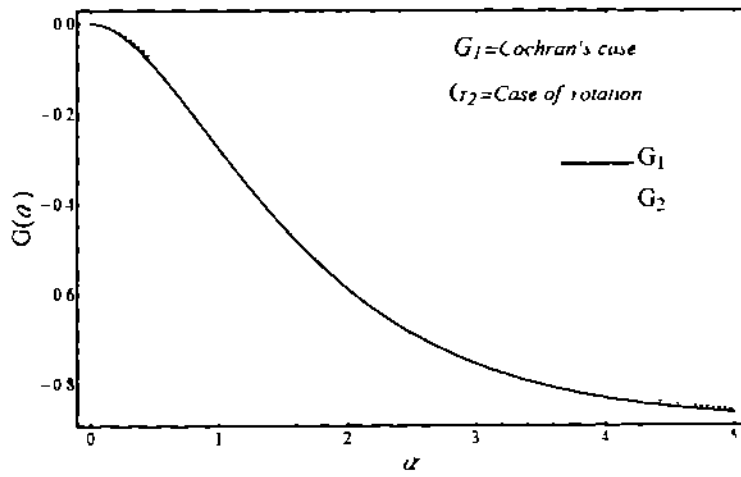

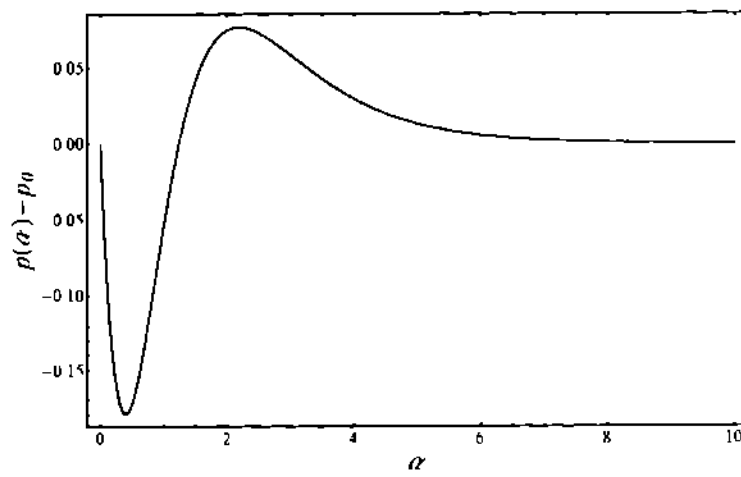**Fig. 2.4:** Effect of rotation on axial velocity profile

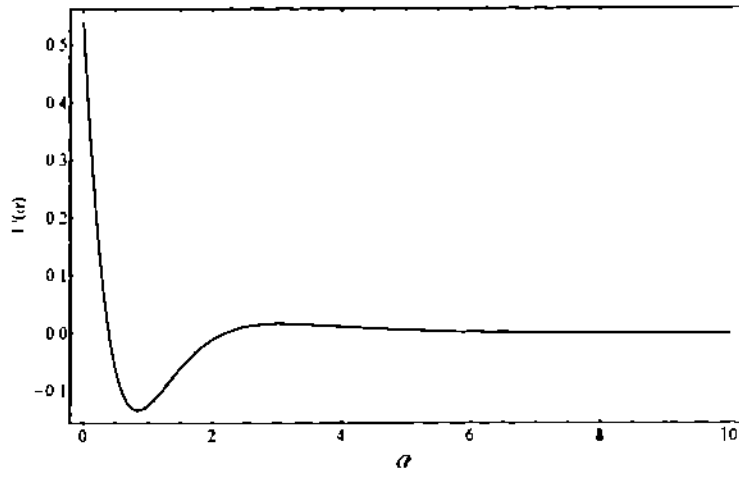
Fig. 2.5: Effect of rotation on pressure profile

18

Fig. 2.6: Derivative of radial velocity
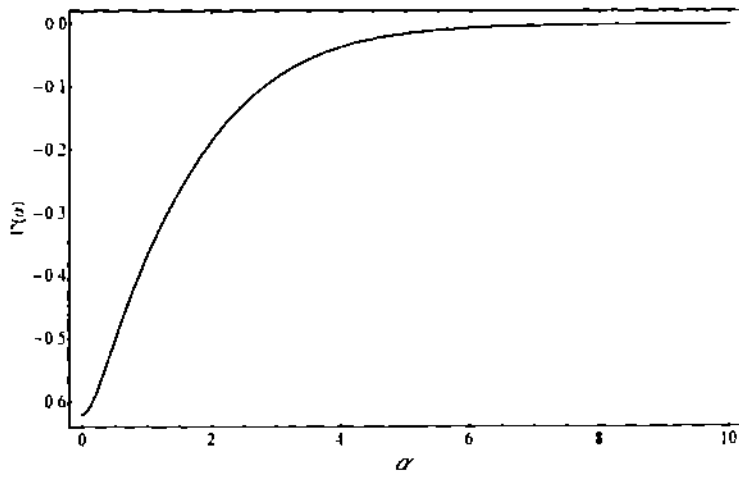
Fig. 2.7: Derivative of tangential velocity

19

The present numerical results are presented in table (2 1) that give very good approximate solution of the above system of non-linear differential equations

Table 2.1: The study state velocity field and pressure as a function of α

α	E	F	G	$P(\alpha)-P_0$	E	F
0	0	1	0	0	0 54	-0 6200
0 4	0 08894	0 765722	-0 05778	-0 17909	-0 0046	-0 53308
0 8	0 0528076	0 576094	-0 17309	-0 10562	-0 1332	-0 41783
1 2	0 0035476	0 4286140	-0 30498	-0 00710	-0 10210	-0 32301
1 6	-0 0262658	0 3152410	-0 43171	0 05253	-0 04825	-0 24658
2 0	-0 0373981	0 229349	-0 54162	0 074796	-0 01078	-0 18521
4 0	-0 0148249	0 04218	-0 81773	0 029650	0 01132	-0 03673
6 0	-0 0028179	0 00727434	-0 87408	0 005635	0 00244	-0 00643
8 0	-0 0006941	0 00124	-0 88397	0 000976	0 00043	-0 00110
12 0	-0 00001415	0 0000358	-0 88594	0 000028	1 3E-05	-3 2E-05
∞	0	0	-0 886	0	0	0

2.4 Concluding remarks

From these result we conclude that magnetization force $i\,e$, $\mu_0(M\,\nabla)H$ reduce the pressure Also, it has been observed that magnetic field intensity increases the radial velocity, whereas, the fluid rotation has reverse effect The effect of rotation is more pronounced then the force of magnetization due to which the radial velocity takes very less peak value in comparison to ordinary viscous fluid flow case Due to the rotation, the retardation of radial velocity increase the thickness of the magnetic field layer

Chapter 3

On the boundary layer flow of ferrofluids

The purpose of the present problem is to theoretically examine the magneto-viscous effects on axisymmetric unsteady flow of an incompressible non-conducting nano-Ferrofluid on a rotating disk under the influence of low oscillating magnetic field The problem has been formulated by employing the basic idea of the Shliomis theory to the equation of motion and magnetization equation for nsano-Ferrofluid flow due to a rotating disk The system of nonlinear partial differential equations governing the unsteady flow is expressed in cylindrical coordinates taking z-axis as axis of rotation are further solved with homotpy analysis method (HAM) based Bvph 2 0 package by taking the initial guess and linear operator The nature of velocity profile and temperature distribution in the presence of low oscillating magnetic field with different values of particle concentration and effective magnetization parameters are discussed graphically The physical parameters likes shear stress at wall, heat transfer rate through wall, boundary layer thickness and volume flow rate in axial direction are presented by tabular form

3.1 Formulation of the problem

3.1.1 Flow modelling

Consider the axially symmetric laminar and non-conducting flow of an incompressible nano-Ferrofluid past a stretchable rotating disk The disk has stretching speed $\Omega_v r / 1 - \beta t$ which is proportional to the radius r and has an angular velocity $\alpha \Omega_r r / 1 - \beta t$ varying with time The coordinate system and geometry of the problem are shown in **Fig. 3.1.**

21
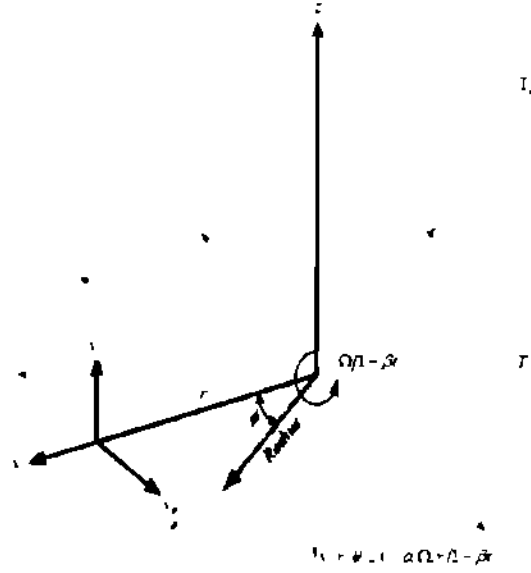
Fig. 3.1: Geometry of the problem

The system of Ferro-hydrodynamic governing equations consisting of equation of continuity, equation of motion, temperature equation, magnetization equation and equation of rotational motion in vector form are

$$\nabla \cdot \mathbf{V} = 0, \tag{3 1}$$

$$\rho_{nf} \frac{d\mathbf{V}}{dt} = -\nabla p + \mu_{nf} \nabla^2 \mathbf{V} + \mu_0 (\mathbf{M} \cdot \nabla) H + \frac{1}{2\tau_s} \nabla \times (\omega_p - \Omega), \tag{3 2}$$

$$(\rho C_p)_{nf} \frac{dT}{dt} = k_{nf} \nabla^2 T, \tag{3 3}$$

$$\frac{d\mathbf{M}}{dt} = \omega_p \times \mathbf{M} - \frac{1}{\tau_s} (\mathbf{M} - \mathbf{M}_0), \tag{3 4}$$

$$I \frac{d\omega_p}{dt} = \mathbf{M} \times H - \frac{1}{\tau_s} (\omega_p - \Omega) \tag{3 5}$$

In above, $\mathbf{V} = (v_r, v_\phi, v_z)$ is velocity, T is temperature, M is magnetization of the fluid, H is strength magnetic field, τ_s is Relaxation time parameter, τ_B is Brownian relaxation time, μ_0 is permeability of free space, I is sum of moments of inertia of the particles per unit volume, ω_p is internal angular momentum due to the self-rotation of particles and Ω is the vorticity of the flow. The complete set of equations also includes the Maxwell's equations

$$\nabla \times H = 0, \nabla \cdot (H + 4\pi M) = 0 \tag{3 6}$$

22

The magnetization of the fluid M with the relation of strength magnetic field H is given as

$$M = M_s \left(\coth \xi - \frac{1}{\xi} \right),$$ (3 7)

where $M_S = \Phi M_0$ is the saturation magnetization of the liquid, being determined by the volume concentration of the magnetic component Φ and its instantaneous magnetization M_0 [26-28] At $\tau_B = 0$, the instantaneous equilibrium magnetization M_0 is given as

$$M_0 = nmL(\xi)\frac{H}{H}, \quad \xi = \frac{mH}{kT}, \quad L(\xi) = \coth \xi - \xi^{-1},$$ (3 8)

where n is the number of particle, m is the magnetic moment of the particle, $L(\xi)$ is the Langevin function, ξ is the Langevin parameter (ratio of magnetization energy to the thermal energy) and k is the Boltzmann constant Since, τ_s is small, the inertial term is negligible in comparison with relaxation term i e , $I\frac{d\omega_p}{dt} \ll I\frac{\omega_p}{\tau_s}$, therefore, Eq (3 5) can be written as

$$\omega_p = \Omega + \frac{\tau_s}{I}(M \times H)$$ (3 9)

Now, Eqs (3 2) and (3 4) in view of Eq (3 9) can written as

$$\rho_{nf}\frac{dV}{dt} = -\nabla p + (M \nabla)H + \mu_{nf}\nabla^2 V + \frac{1}{2}\nabla \times (M \times H),$$ (3 10)

$$\frac{dM}{dt} = \Omega \times M - \frac{1}{\tau_B}(M - M_0) + \frac{\tau_s}{I}M \times (M \times H)$$ (3 11)

Magnetic torque and viscous torque are acting on the particles denoted by $M \times H$ and $(\omega_p - \Omega)$ respectively The equilibrium of both torques, leads to the interference of the particle rotation, can be written as

$$M \times H = -6\mu_{nf}\phi(\Omega - \omega_p),$$ (3 12)

and at equilibrium

$$M = \frac{\phi M_s L(\xi_e)\xi_e}{\xi}$$ (3 13)

23

Where ϕ is the volume fraction and ξ_e denotes the effective magnetic field parameter (the ratio of effective magnetic force to the thermal force i e , $\xi_i = \dfrac{mH_e}{kT}$)

In the presence of slow oscillating magnetic field Eq (3 11) can be written as [29-33]

$$\frac{d}{dt}\left(L_e \frac{\xi_e}{\xi} \right) = \Omega \times \left(L_e \frac{\xi_e}{\xi} \right) - \frac{1}{\tau_B} \frac{L_e}{\xi_e} (\xi_e - \xi) - \frac{L_e^2}{2\tau_B \xi_e} \xi_e \times (\xi_i - \xi) \qquad (3\ 14)$$

The effective field can be expressed by the equation of zero approximation from the Eq (3 14) as

$$\tau_B \frac{d\xi_e}{dt} = -\left(\frac{d \log L_e}{d\xi_e} \right)^{-1} \left(1 - \frac{\xi_0}{\xi_e} \cos \omega_0 t \right) \qquad (3\ 15)$$

Here, the parameter ξ_0 is the amplitude of the real magnetic field, L_i is the effective Langevin function, ω_0 is the frequency of the field From [34], using linear approximations in $\Omega \tau_B$ and the expression

$$M^{(1)} = M^{(0)} R(\xi_e) \tau_B \Omega \times h, \qquad (3\ 16)$$

where $M^{(0)} = \phi M_s L(\xi)$ and h is unit vector along the applied field, Eq (3 14) reduces to

$$\tau_B \frac{dR(\xi_e)}{dt} = \left(1 - \frac{1}{2}\left(\frac{1}{L_e} - \frac{1}{\xi_e} \right) \xi_0 R(\xi_e) \cos \omega_0 t \right) \qquad (3\ 17)$$

Since magnetic torque is not equal to zero, so from Eqs (3 12) and (3 17), the expression for mean magnetic torque becomes

$$\overline{M \times H} = -6\mu\phi\Omega g, \quad g = \frac{1}{2}\xi_0 \overline{\cos \omega_0 t L(\xi_e) R(\xi_e)} \qquad (3\ 18)$$

Here, $g(\xi_0, \omega_0 \tau_B)$ is the effective magnetization parameter, if $\omega_0 \tau_B < 1$ (in the low oscillating field) the Magnetic fluid viscosity under and alternating magnetic field become larger than the Magnetic fluid viscosity in zero field

$$\frac{1}{2}\nabla \times \overline{M \times H} = \frac{1}{2}\nabla \times -6\mu_{nf}\phi g\Omega = -\frac{3}{2}\mu_{nf}\phi g\nabla(\nabla V) + \frac{3}{2}\mu_{nf}\phi g\nabla^2 V$$
$$= \frac{3}{2}\mu_{nf}\phi g\nabla^2 V \qquad (3\ 19)$$

Now the Eq (3 10) with the help of Eq (3 19) can be written as

$$\rho_{nf} \frac{dV}{dt} = -\nabla p + (M \nabla)H + \mu_{nf}\left(1 + \frac{3}{2}\phi g \right)\nabla^2 V, \qquad (3\ 20)$$

24

$$\rho \frac{dV}{dt} = -\nabla \tilde{p} + \left(1 + \frac{3}{2}\phi g\right)\mu_{nf}\nabla^2 V \tag{3 21}$$

As $-\nabla\tilde{p} = -\nabla p + (M\ \nabla)H$ i e , is reduced pressure In the investigation, unsteady, axially- symmetric, incompressible flow of an electrically non-conducting fluid has been considered, assuming that the viscosity is dependent on the magnetic field and thermal effects are not taken under consideration The disk rotates about z -axis with constant angular velocity ω, where z is the vertical -axis in the cylindrical coordinate system with r and θ are the radial and tangential axis respectively, and the flow is considered axis-symmetric and incompressible, so the equation of motion and continuity can be written in the cylindrical form as [35,36]

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0, \tag{3 22}$$

$$-\frac{\partial \tilde{p}}{\partial r} + \mu_{nf}\left(1 + \frac{3}{2}\phi g\right)\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_r}{r}\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$
$$= \rho_{nf}\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right], \tag{3 23}$$

$$\mu_{nf}\left(1 + \frac{3}{2}\phi g\right)\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$
$$= \rho_{nf}\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right]. \tag{3 24}$$

$$-\frac{\partial \tilde{p}}{\partial z} + \mu_{nf}\left(1 + \frac{3}{2}\phi g\right)\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right]$$
$$= \rho_{nf}\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right]. \tag{3 25}$$

$$\left(\rho C_p\right)_{nf}\left[\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + v_z\frac{\partial T}{\partial z}\right] = k_{nf}\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right], \tag{3 26}$$

along the following boundaries conditions

$$at\ z = 0, \quad v_r = \frac{\alpha \Omega_r r}{1 - \beta t}, \quad v_\theta = \frac{\Omega_r r}{1 - \beta t}, \quad v_z = 0, \quad T(r,\theta,z) = T_u$$
$$at\ z = \infty, \quad v_r = 0, \quad v_\theta = 0, \quad T(r,\theta,z) = T_\infty \tag{3 27}$$

Now, introduce the following similarity transformation

$$\eta = \sqrt{\frac{\Omega}{v_f}} \frac{z}{\sqrt{1-\beta t}}, \quad v_r(r,\theta,z) = \frac{\Omega r}{1-\beta t} F(\eta), \quad v_\theta(r,\theta,z) = \frac{\Omega r}{1-\beta t} G(\eta),$$

$$v_z(r,\theta,z) = \sqrt{\frac{\Omega v_f}{1-\beta t}} E(\eta), \quad \theta(\eta) = \frac{T-T_\infty}{T_w - T_\infty}, \quad \frac{p}{\rho_f} = -\frac{\Omega v_f}{1-\beta t} P(\eta) \qquad (3\ 28)$$

By using Eq (3 28) into Eq (3 22) to (3 26) and get the system of nonlinear ordinary differential equations in dimensionless form as

$$2F(\eta) + E'(\eta) = 0, \qquad (3\ 29)$$

$$\frac{\rho_{nf}}{\rho_f}\left[F^2 - G^2 + EF' + S\left(F + \frac{\eta}{2}F' \right) \right] = \frac{\mu_{nf}}{\mu_f}\left(1 + \frac{3}{2}\phi g \right) F'', \qquad (3\ 30)$$

$$\frac{\rho_{nf}}{\rho_f}\left[EG' + 2FG + S\left(G + \frac{\eta}{2}G' \right) \right] = \frac{\mu_{nf}}{\mu_f}\left(1 + \frac{3}{2}\phi g \right) G''. \qquad (3\ 31)$$

$$\frac{\rho_{nf}}{\rho_f}\left[EE' + \frac{S}{2}(E + \eta E') \right] = -\frac{\partial P}{\partial \eta} + \frac{\mu_{nf}}{\mu_f}\left(1 + \frac{3}{2}\phi g \right) E''. \qquad (3\ 32)$$

$$\frac{(\rho C_p)_{nf}}{(\rho C_p)_f} \Pr\left[E\theta' + S\frac{\eta}{2}\theta' \right] = \frac{k_{nf}}{k_f} \theta''. \qquad (3\ 33)$$

and corresponding boundary condition are

$$\left. \begin{array}{l} F(0) = \alpha, \quad G(0) = 1, \quad E(0) = 0, \quad \theta(0) = 1, \\ F(\infty) = 0, \quad G(\infty) = 0, \quad \theta(\infty) = 0 \end{array} \right\} \qquad (3\ 34)$$

Where, $\Pr = \dfrac{\mu_f C_{pf}}{k_f}$ is modified Prandtl number and $S = \dfrac{\beta}{\Omega}$ is the unsteadiness

parameter

3.1.2 Thermodynamics properties

In the Eqs (3 22) to (3 26), the effective density is ρ_{nf}, heat capacitance is $(C_p)_{nf}$,

viscosity is μ_{nf}, and thermal conductive k_{nf} of the nano-ferrofluid defined as

$$\rho_{nf} = (1-\phi_a)\rho_f + \phi\rho_a, \qquad (3\ 35)$$

$$(\rho C_p)_{nf} = (1-\phi_a)(\rho C_p)_f + \phi_a (\rho C_p)_a, \qquad (3\ 36)$$

$$k_{nf} = \frac{k_a + 2k_f + 2\phi_a(k_a - k_f)}{k_a + 2k_f - \phi_a(k_a - k_f)} k_f, \qquad (3\ 37)$$

$$\mu_{nf} = \mu_f\left(1 - \frac{\phi_a}{\phi_{max}} \right)^{-2.5\phi_{max}} \qquad (3\ 38)$$

26

Here, ϕ_a is the solid volume fraction of a particles aggregate, ρ_f, ρ_a, $(C_p)_f$ and $(C_p)_a$ are the densities and heat capacitances of base fluid and aggregate respectively The μ_f is viscosity of the base fluid, k_f and k_a are thermal conductivities of base fluid and particle aggregation respectively The thermal properties of particle aggregation are defined as

$$\rho_a = (1-\phi_{int})\rho_f + \phi_{int}\rho_s,$$ (3 39)

$$(C_p)_a = (1-\phi_{int})(C_p)_f + \phi_{int}(C_p)_s$$ (3 40)

Above $(C_p)_s$ is heat capacitance of the particle, and ϕ_{int} is the volume fraction of the nanoparticles in the aggregate or the cluster The thermal conductivities of aggregates are estimated by separating them into two components, the percolation contributing backbone, and non-percolation contributing dead-end particles Considering the interfacial thermal resistance (kaptiza resistance), the effective thermal conductivity of dead-end particles based suspension (including the basefluid and dead-end particles) is given by using Bruggeman modle

$$(1-\phi_{nk})(k_f - k_{nk})/(k_f + 2k_m) + \phi_{nc}(k_s - k_m)\ (k_s + 2k_{nc}) = 0$$ (3 41)

The effective thermal conductivity of aggregate, k_a is determined by using composite theory for misoriented ellipsoidal particles for the backbone, in a matrix of non-percolation contributing portion, the following equations are used

$$k_a = k_{nc}\frac{3 + \phi_c\left[2\beta_{11}(1 - L_{11}) + \beta_{33}(1 - L_{33})\right]}{3 - \phi_c\left[2\beta_{11}L_{11} + \beta_{33}L_{31}\right]},$$ (3 42)

where

$$\left.\begin{array}{l}L_{11} = 0\ 5p^2{}_f\left(p^2 - 1\right) - 0\ 5p\cosh^{-1}p\ \left(p^2 - 1\right)^{15}, \\ L_{33} = 1 - 2L_{11}, \\ \beta_n = \left(k_{11}^c - k_{nc}\right)_f\left[k_m - L_{11}\left(k_{11}^c - k_{nc}\right)\right]\end{array}\right\}$$ (3 43)

Interfacial resistance is accounted for in the term

$$k_{11}^c = k_s{}_f\left(1 + \gamma L_{11}k_s\ k_f\right)$$ (3 44)

Here $\gamma = (2 + 1/p)\alpha$, $\alpha = A_k\ a_1$ and A_k is the kapitza radius In Eq (3 44) p is aspect ratio, for which the cluster spanning chain is given by $p = R_g\ a_1$ In particular, our treatment allows the effect of cluster morphology to the evaluated in terms of the

27

average radius of gyration R_g, the fractal dimensions d_f and chemical dimension d_l of the aggregates, respectively Following the definition of the fractal dimension d_f, the number of particle in aggregation is given by

$$N_{int} = \left(R_g \, a_1 \right)^{d_f},$$ (3 45)

Where a_1 is the radius of the primary particle Due to number conservation of the nanoparticles, $\phi = \phi_{int}\phi_a$, where ϕ is the volume fraction of the nanoparticles It can be shown that $\phi_{int} = \left(R_g \, a_1 \right)^{1\,d_f,-3}$ and $\left(R_g / a_1 \right)_{max} = \phi^{1\,d_f-3}$ for which $\phi_a = 1$ The number of particles in belonging to backbone N_c is the chemical dimension is given by

$$N_c = \left(R_g \, a_1 \right)^{d_l}.$$ (3 46)

Where d_l ranges between one and d_f When $d_l = d_f$ all of the particles belong to the backbone, and there are no dead ends Therefore, the volume fraction of backbone particles ϕ_c in the aggregate is given by $\phi_c = \left(R_g / a_1 \right)^{d_l-3}$ The volume fraction of the particles belonging to dead ends ϕ_{nc} is given by

$$\phi_{nc} = \phi_{int} - \phi_c$$ (3 47)

3.1.3 Parameters of physical interest

Skin friction coefficient and Nusselt number

The skin fraction coefficient C_f and the Nusselt number Nu are physical quantities which are given by

$$C_f = \frac{\sqrt{\tau_{wr}^2 + \tau_{w\phi}^2}}{\rho_f \left(\dfrac{a\Omega r}{1 - \beta t} \right)^2}, \quad Nu = \frac{r q_w}{k_f (T_w - T_\infty)}$$ (3 48)

Where τ_{wr} and $\tau_{w\phi}$ are the radial and the transversal skin fraction or shear stress at the surface of disk, respectively, and q_w is the surface heat flux, introduced as

$$\tau_{wr} = \mu_{nf} \left(1 + \frac{3}{2}\phi g \right)\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial \theta} \right)_{z=0}, \quad \tau_{w\phi} = \mu_{nf}\left(1 + \frac{3}{2}\phi g \right)\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta} \right)_{z=0},$$ (3 49)

$$q_w = -k_{nf}(T_z)_{z=0}$$

28

Substituting Eqs (3 28) and (3 38) in Eq (3 49) and using Eq (3 48), we have

$$Re^{\frac{1}{2}} C_f = \frac{\sqrt{F'(0)^2 + G'(0)^2}}{(1-\phi)^{2.5}}, \qquad Re^{-\frac{1}{2}} Nu = -\frac{k_{nf}}{k_f} \theta'(0) \qquad (3\ 50)$$

Here $Re = \Omega_s r^2\ v_f (1 - \beta t)$ is the rotational Reynolds number

Boundary layer displacement thickness

Thickness of boundary layer can be found by finding the separation over inertia and viscous forces are practically identical The velocity in the boundary layer accomplishes a value which is near the external velocity as of now at a little separation from the wall The boundary layer displacement thickness is ascertained as

$$d = \frac{1 - \beta t}{\Omega r} \int_{z=0}^{\infty} v_\theta dz = r \, Re^{-1/2} \int_{\eta=0}^{\infty} G(\eta) d\eta \qquad (3\ 51)$$

Total volume flowing outward the z-axis,

$$Q = 2\pi r \int_0^{\infty} v_r dz = 2\pi r v_f \, Re^{1/2} \int_0^{\infty} F(\eta) d\eta \qquad (3\ 52)$$

The fluid is taken to rotate at a large distance from the wall, the angle becomes

$$\tan(\psi_0) = -\left(\frac{\frac{\partial v_r}{\partial v_z}}{\frac{\partial v_\theta}{\partial v_z}} \right) = -\frac{F'(\eta)}{G'(\eta)} = -10\ 67 = -84\ 6'' \qquad (3\ 53)$$

3.2 Solution of the problem

Due to nonlinear nature of Eqs (3 29)-(3 33), an exact solution is not possible Now, we opted to go for analytic solution To this end, the Mathematica package BVPh 2 0 which is based on the homotopy analysis method employed for solving nonlinear ordinary differential equation using computational software Mathematica 9 For simplicity, the BVPh 2 0 needs to input the governing equations along with corresponding boundary conditions and choose proper initial guess of solutions and auxiliary linear operators for under consideration linear sub-problems In this package, one has great freedom to choose the auxiliary linear operator and initial guess, thus we choose the auxiliary linear operators and initial guess for the desire solutions as follow

29

$$\mathcal{L}_E(H) = \frac{dE}{d\eta}, \quad \mathcal{L}_F(F) = \frac{dF}{d\eta} + \frac{d^2F}{d\eta^2},$$

$$\mathcal{L}_G(G) = \frac{dG}{d\eta} + \frac{d^2G}{d\eta^2}, \quad \mathcal{L}_\theta(\theta) = 2\frac{d\theta}{d\eta} + \frac{d^2\theta}{d\eta^2}, \qquad (354)$$

$$\left.\begin{array}{l} E_o = 0, \quad F_o(\eta) = \alpha e^{-\eta}, \\ G_o(\eta) = e^{-\eta}, \quad \theta_o(\eta) = e^{-2\eta} \end{array}\right\}. \qquad (3\,55)$$

so far, we have defined all the input of this problem properly, except the convergence-control parameters Usually, the optimal values of the convergence-control parameters are obtained by minimizing the squared residual error as follow

$$E_H = \int_0^\infty \left(2F(\eta) + H'(\eta)\right)^2 d\eta, \qquad (3\,56)$$

$$E_F = \int_0^\infty \left(\frac{\rho_{nf}}{\rho_f}\left[F^2 - G^2 + HF' + S\left(F + \frac{\eta}{2}F'\right)\right] - \frac{\mu_{nf}}{\mu_f}(1 + \frac{3}{2}\phi g)F''\right)^2 d\eta \qquad (3\,57)$$

$$E_G = \int_0^\infty \left(\frac{\rho_{nf}}{\rho_f}\left[HG' + 2FG + S\left(G + \frac{\eta}{2}G'\right)\right] - \frac{\mu_{nf}}{\mu_f}(1 + \frac{3}{2}\phi g)G''\right)^2 d\eta. \qquad (3\,58)$$

$$E_\theta = \int_0^\infty \left(\frac{(\rho C_p)_{nf}}{(\rho C_p)_f} \Pr\left[H\theta' + S\frac{\eta}{2}\theta'\right] - \frac{k_{nf}}{k_f}\theta''\right)^2 d\eta \qquad (3\,59)$$

In addition to this, the convergence of homotopy series can be accelerated by using either the iteration approach or by applying the homotopy-pade approximations Both of these techniques can also be employed to obtain the problem solution by directly using some simple commands Further, the results for velocity components and temperature up to first iteration are as follow

$$F = \left(\begin{array}{l} \frac{1}{2} + \frac{83}{400}\left(\frac{\mu_{nf}}{\mu_f}\right) + \frac{83}{800}\left(\frac{\rho_{nf}}{\rho_f}\right) - \\ \frac{83S}{1600}\left(\frac{\rho_{nf}}{\rho_f}\right) - \frac{249}{800}\left(\frac{\mu_{nf}}{\mu_f}\right)g\phi \end{array}\right)e^{-z}$$

$$+ \left(\begin{array}{l} \frac{-83}{400}\left(\frac{\mu_{nf}}{\mu_f}\right) + \frac{83S}{1600}\left(\frac{\rho_{nf}}{\rho_f}\right) - \\ \frac{83S}{800}\left(\frac{\rho_{nf}}{\rho_f}\right)z - \frac{249}{800}\left(\frac{\mu_{nf}}{\mu_f}\right)g\phi \end{array}\right)e^{-2z} - \left(\frac{83}{800}\left(\frac{\rho_{nf}}{\rho_f}\right)\right)e^{-3z}, \qquad (3\,60)$$

30

$$G = \left(\begin{array}{c} \left(1 + \dfrac{43}{200}\left(\dfrac{\mu_{nf}}{\mu_f}\right) - \dfrac{43}{600}\left(\dfrac{\rho_{nf}}{\rho_f}\right) - \right. \\ \left. \dfrac{43S}{800}\left(\dfrac{\rho_{nf}}{\rho_f}\right) + \dfrac{192}{400}\left(\dfrac{\mu_{nf}}{\mu_f}\right)g\phi \right) \end{array} \right) e^{-z}$$

$$+ \left(\begin{array}{c} \left(-\dfrac{43}{200}\left(\dfrac{\mu_{nf}}{\mu_f}\right) + \dfrac{43S}{800}\left(\dfrac{\rho_{nf}}{\rho_f}\right) - \right. \\ \left. \dfrac{43S}{400}\left(\dfrac{\rho_{nf}}{\rho_f}\right)z - \dfrac{192}{400}\left(\dfrac{\mu_{nf}}{\mu_f}\right)g\phi \right) \end{array} \right) e^{-2z} - \left(\dfrac{43}{600}\left(\dfrac{\rho_{nf}}{\rho_f}\right)\right) e^{-3z}, \tag{3 61}$$

$$E = e^{-2z}\left(-\frac{73}{100} + \frac{73}{100}\right), \tag{3 62}$$

$$T = \left(-\frac{53}{75}\left(\frac{k_{nf}}{k_f}\right) - \frac{53}{225}\Pr S - \frac{53}{300}\Pr S\left(\frac{(\rho C_p)_{nf}}{(\rho C_p)_f}\right)z\right)e^{-3z}$$

$$+ \left(1 + \frac{53}{75}\left(\frac{k_{nf}}{k_f}\right) + \frac{53}{225}\Pr S\left(\frac{(\rho C_p)_{nf}}{(\rho C_p)_f}\right)\right)e^{-2z} \tag{3 63}$$

3.3 Results and discussion

In this chapter, the behavior of emerging parameters involved in the expression of velocity and temperature distributions are examined through Figs (3 2)-(3 9) with water based nano-ferrofluid contained Iron nanoparticles in water In this study, low isolating magnetic field effect is taken to account For preparing these figures, consider the radius of gyration is 200 nm there as each sphere shape particles has 10nm radius In single aggregation, the total particles are 200 in which 50 particles belong to backbone Figs (3 2)-(3 4) represent radial, tangential and axial velocitiy profiles for different value of particle concentration against a dimensionless parameter η In these Figs, it is noticed that several velocity lines have been publicized corresponding to different concentration of nanoparticles By diverse concentration, unalike collisions between neighboring particles in a fluid are produced different velocity lines It is also noticed that when the nanoparticle concentration is enhanced the all velocity components are declined This is accordance with the physical expectation because heighten of resistance between adjacent layers of moving fluid, the base fluid has much velocity as compared to nanofluid Fig (3 5) shows the effect of particle concentrations on temperature profile It is seen that the temperature of

31

nanofluid is enhanced by cumulative of nanoparticles volume fraction When the nanoparticle volume fraction increases, thermal conduction of the fluid increases that make a cause of enhancement in temperature

Figs (3 6)-(3 8) characterize the radial, tangential and axial velocities profiles for various value of effective magnetization parameters It is observed that when the effective magnetization parameter is boosted, the all velocity components are increased When magnetization parameter is decreased, the angular velocity of the particle increased In addition, the free rotation of the particle in the flow is damped and increases the fluid flow resistance, consequently result in increased velocity Fig (3 9) shows the effect of magnetization parameter on temperature profile In this Fig it is seen that the temperature is amplified with in boosted influence of magnetization parameter

The numerical sets of values show the results for parameters of physical interest The impact of particle volume fraction on skin-friction coefficient, local Nusselt, boundary layer thickness and volume flowing outward the z -axis are shown in table (3 1) Skin friction arises when a fluid flows over a solid surface The fluid is in contact with the surface of the body, resulting in a friction force exerted on the surface It is seen that when nanoparticle volume fraction enhances shear stress at wall is increased The shear stress depends on the dynamic viscosity and the gradient of the velocity it is found that when particle concentration is increased, the gradient of the velocity and dynamic viscosity increases that are caused in enhancement in shear stress at wall In table (3 1), it is observed that when nanoparticle volume fraction enhances, heat transfer rate at wall increases Thermal conductivity of fluid plays an important role in the heat transfer rate For example, enhancement in thermal conductivity of fluid is found 25 13% and 54 85% at 5% and 10% concentration of nanoparticles Then with same trend, enhancement in heat transfer rate at wall is found 22 39% and 46 93% at 5% and 10% concentrations With the increasing effect of concentration, total volume flowing outward the z -axis is decreased. Boundary layer thickness is frequently used boundary layer property describes the difference between the case with hypothetical flow over a surface without a boundary layer and the actual flow with a boundary layer It is noticed that boundary layer thickness decreases corresponding to concentration of particle Furthermore, the total volume flowing outward the z -axis is decreased In table (3 2), the result of effective magnetization parameter on skin-friction coefficient, local Nusselt, boundary layer thickness and volume flowing

32

outward the z-axis is publicized It is perceived that with augmentation of magnetization parameter declines, there the shear stress and heat transfer rate at the wall In the other sundry parameters, displacement thickness is increased because of enrichment in boundary layer thickness due to consequences of magnetization parameter Furthermore, the total volume flowing outward the z-axis is also increased
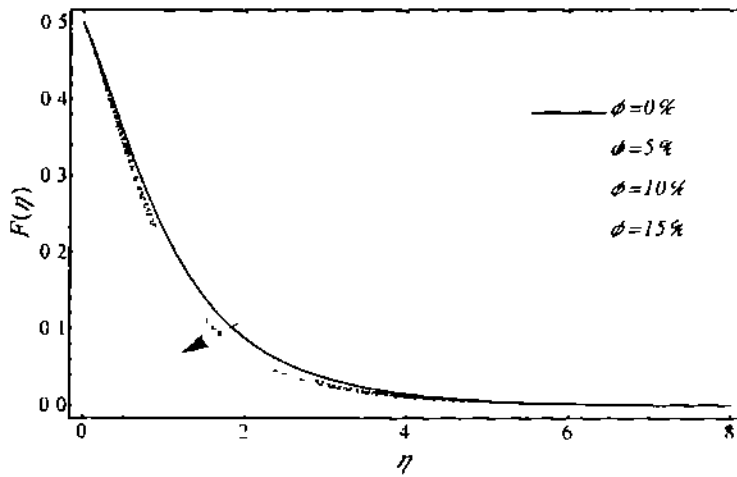

Fig. 3.2: Radial velocity profile for various values of nanoparticle volume fraction
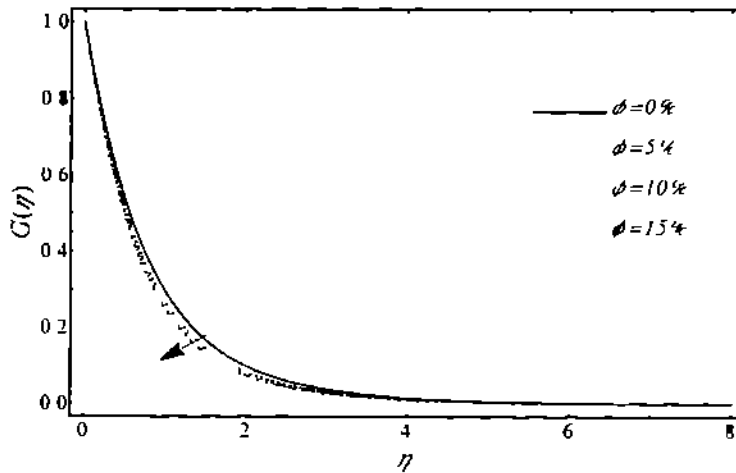

Fig. 3.3: Tangential velocity profile for various values of nanoparticle volume fraction
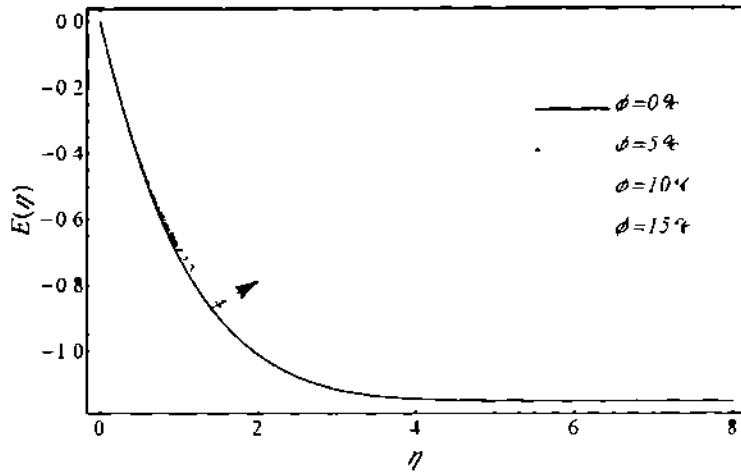
Fig. 3.4: Axial velocity profile for various values of nanoparticle volume fraction
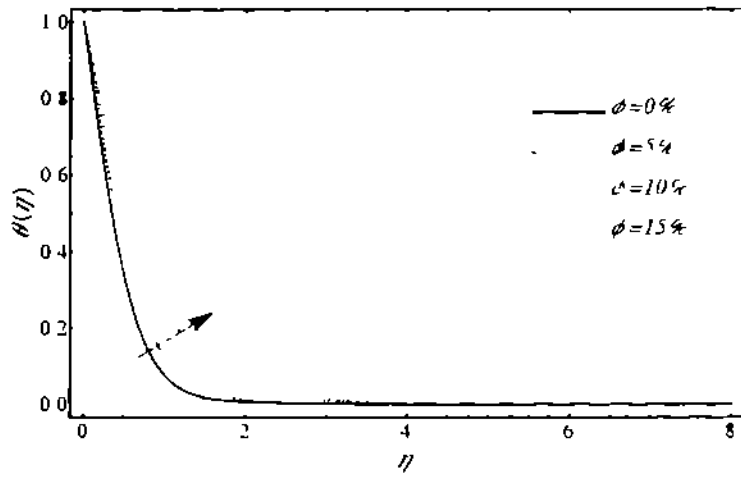

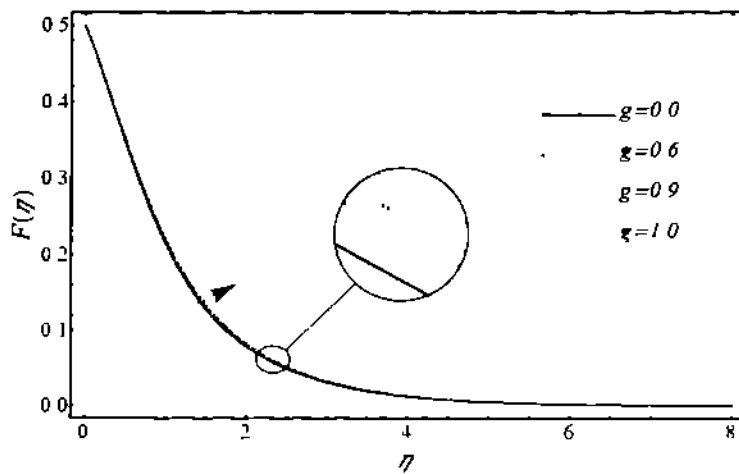**Fig. 3.5:** Temperature profile for various values of nanoparticle volume fraction

Fig. 3.6: Radial velocity profile for various values of effective magnetization parameter
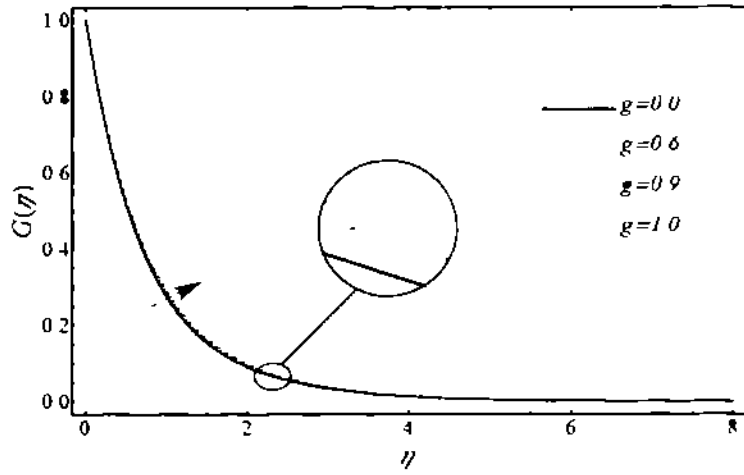
34

Fig. 3.7: Tangential velocity profile for various values of effective magnetization parameter
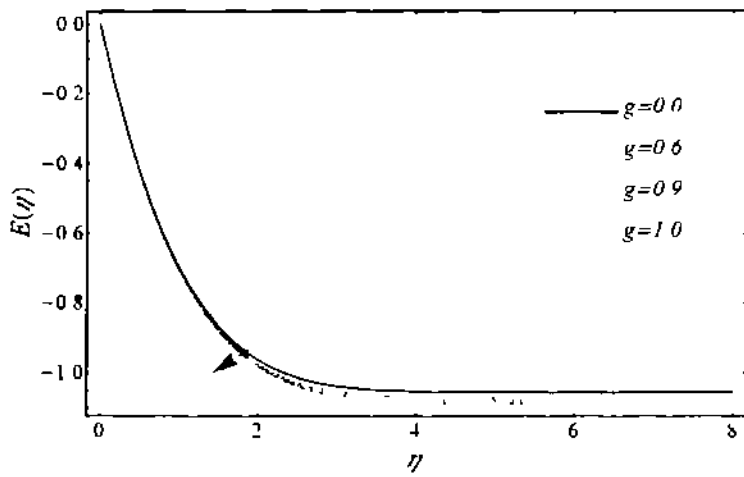

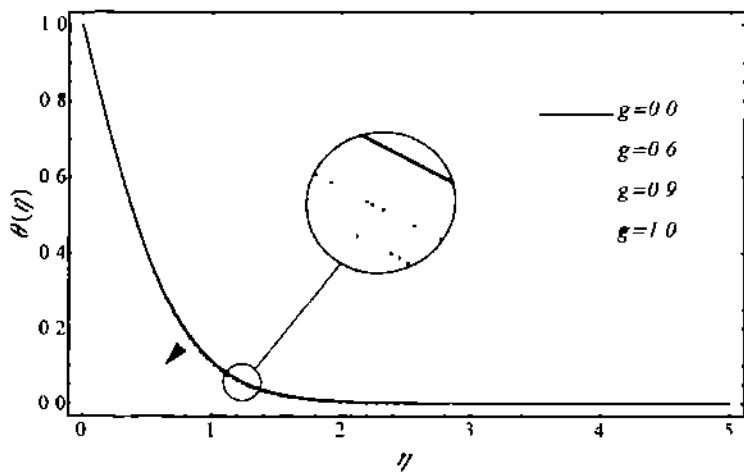**Fig. 3.8:** Axial velocity profile for various values of effective magnetization parameter

Fig. 3.9: Temperature profile for various values of effective magnetization parameter

35

Table. 3.1: Values of skin-friction coefficient, local Nusselt, boundary layer thickness, volume flowing outward the z-axis and angle for various values of nanoparticle volume fraction

ϕ	$Re^{1/2}C_f$	$Re^{1/2}Nu$	d	Q
0%	2 5251	1 5891	0 8580	0 6026
5%	3 1222	1 7102	0 7941	0 5591
10%	3 7483	1 8085	0 7628	0 5372
15%	4 4134	1 9174	0 7488	0 5274

Table. 3.2: Values of skin-friction coefficient, local Nusselt, boundary layer thickness, volume flowing outward the z-axis and angle for various values of effective magnetization parameter

g	$Re^{1/2}C_f$	$Re^{1/2}Nu$	d	Q
0	3 1258	1.7097	0 7934	0 5585
0 6	3 0578	1 7194	0 8101	0 5699
0 9	3 0254	1 7239	0 8183	0 5756
01	3 0148	1 7253	0 8210	0 5775

3.4 Concluding remarks

In this chapter, unsteady ferrofluid flow revolving a streachable rotating disk on a boundary layer is presented Analytical expressions of velocity and temperature profiles are obtained analytically The numerical values are found to see the impact of physical parameter g and ϕ on Skin fraction, Nusselt number, boundary layer thickness and Volume flowing through out word the axial-direction through Mathematica wolfram It is observed that enhancement of ferro-particles concentration changes the physical properties of fluid which distresses the fluid velocity and temperature distribution as well as physical parameters Both shear stress and heat transfer rate at wall are increased by enrichment of particles concentrations, but boundary layer displacement is decreased Additionally, it is perceived that with augmentation of magnetization parameter reduces shear stress and heat transfer rate at wall Moreover, displacement thickness is also increased

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