Integral Methods for Boundary Layer Flow over Non-Flat Surface

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A Dissertation
Submitted in the Partial Fulfillment of the
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In
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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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Dedication

To my parents.

The reason of what I become today.

Thanks for your great support and continuous care.

To my loving brothers and sisters.

I am really grateful to both of you.

You have been my inspiration and my soul mates.

This humble work is a sign of my love to you!

Declaration

I hereby declare, that this thesis neither as a whole nor as a part thereof has been copied out from any source It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor No portion of the work presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning

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Preface

Ludwig Prandtl [1] a German fluid dynamist gave the concept of the boundary layer theory in 1904 He proposed that the flow of a real fluid around objects (i e solid bodies) could be divided into two regions a thin region (or layer) near to the object called the boundary layer where the effects of viscous forces are important and an outer region where the effects of viscosity may be neglected The simplest example of the application of the boundary layer is afforded by the flow along a very thin flat plate Historically this was the first example illustrating the application of Prandtl's boundary layer theory Blasius [2] discussed this problem in detail in his doctor's thesis at Goettingen in 1908 The boundary layer problems are modeled through Navier Stokes equations The governing partial differential equations are solved directly or by transforming them into some convenient forms The obtained differential equations are solved by two different approaches, the first one is analytical and other one is numerical approach During the study of the boundary layer, oftenly we deal with such complicated problems whose exact solution is very hard to find Therefore approximate methods such as perturbation methods numerical methods and integral methods are always of our interest Because of less limitations in geometry and boundary conditions, the integral methods are appropriate and suitable to obtain the direct analytical solution

These integral methods are based on the momentum integral equation developed by Th von Karman [3] in 1921 To solve the two dimensional boundary layer equations K Pohlhausen [4] introduced an integral method known as Pohlhausen method after his name This method based on the momentum integral equation and quartic form of velocity profile was found to give good results in non retarted flow but give less satisfactory result in retarted region as first noticed by Schubauer in his experimental study of flow past an elliptical cylinder Soon after the approximate method of Karman and Millikan [5] in which the boundary layer was divided into inner and outer regions with separate solutions was applied to Schubauer's ellipse with reasonable success Bryan Thwaites [6] introduced and integral method in which he used some suitable correlations between boundary layer parameters instead of velocity profile N Curle [9] used quintic velocity profile to modify the Karman-Pohlhausen method Attempts were subsequently made to secure improved accuracy of Karman-Pohlhausen method by assuming a more adequate form of the velocity profiles by Walz [10], Mangler [11] and Timman [12]. Yuan [13] and Lew [14] used exponential function like Timman's [12] velocity profile. Loitsianskii [15] introduced a new method in which he multiplied the momentum integral equation by a small variation of velocity and then integrating the equation across the boundary layer thickness. Two-parameter integral methods were introduced by Sutton, Wieghardt and Head [16-18]. Sakiadis [19] proposed an integral method to investigate the boundary layer behavior on continuous solid surfaces. The von Karman-Pohlhausen method was modified to account for suction and injection by Fox [20]. This method gives acceptable values of the transfer coefficient for heat, mass and momentum transfer for most of the values considered.

In this thesis, a brief introduction to integral methods and their historical background has been discussed. Different forms of integral methods in the perspective of their applications have been presented. Also integral method, particularly method of Thwaites, is utilized to obtain the solution of the non-similar flows past a flat plate and sphere. It is important to note that the solution obtained from aforementioned method agrees well with other methods. Specially in the calculation of separation point in Howarth's retarded flow, the percentage error is almost 2% and in the case of retarded flow past a sphere it is 5% which show the validity of present integral method.

This dissertation is divided into three chapters. Chapter 1 contains the definition and basic concepts related to boundary layer and integral methods. In chapter 2, some prominent and extensively used integral methods have been reviewed. In chapter 3, the well-known integral method, namely Method of Thwaites, has been utilized in the determination of the flow separation of Howarth's retarted flow and flow separation over a sphere.

Chapter 1

Introduction and Basic Definitions

1.1 Introduction

By the end of nineteenth century, there were two major groups of researchers in fluid mechanics First group of researchers was engaged with inviscid fluid flow while the other group was studying hydraulics Hydrodynamics, though mathematically elegant, was unable to predict drag experienced by bodies moving in fluids This is known as D'Alembert's paradox On the other hand, the solution of practical problems offered by hydraulics were based on mainly empirical data Ludwig Prandtl changed this situation by giving the concept of boundary layer In august 1904, Prandtl presented his revolutionary concept of boundary layer through his paper entitled "*Uber Flussigketsbewgung bei sehr Kleiner Reibung*" before a group of mathematicians and scientists, gathered at the third international congress of Mathematics held in Heidelberg In 1905, the paper presented by Ludwig Prandtl, was published in the proceedings of congress

With the help of this theory, Prandtl succeeded to prove that the flow about a solid body can be categorized in to two regions a thin layer in the neighborhood of the body (boundary layer) where the effects of viscous forces are important and friction plays an essential part and the remaining region outside the layer, where friction may be neglected i e the flow can be considered as inviscid outside the boundary layer, while viscous effects stay confined within the boundary layer Thus Prandtl pointed out the important and significant role of viscous forces, no matter how small they are, in the determination of the flow Boundary layer can develop on the objects of different shapes An infinitely long flat plate along which a viscous incompressible fluid flows, is the simplest geometry on which boundary layer is developed Near the leading edge of plate the flow in the boundary layer exhibits all characteristics of laminar flow The boundary layer thickness at the leading edge is zero and it grows continuously towards the rear end of the plate When the pressure increases or surface is highly curved the boundary layer structure would be more complex In such case the boundary layer may detach from the surface a phenomenon known as separation and the simplified form of Navier Stokes equations no longer applies

Mathematically, the boundary layer problems are modeled through Navier Stokes equations The governing partial differential equations are solved directly or by transforming them into differential equations Thus differential equations are tackled by two different approaches, the first one is the analytical and other one is numerical approach During the study of boundary layer, oftenly one deals with such complicated problems that their exact solution is very difficult to find Therefore, the approximate methods such as series solution method, numerical methods and integral methods etc are always of our interest Because of less limitations in terms of geometry and boundary conditions, the integral methods are appropriate and suitable technique to obtain the direct analytical solution It can also be applied to both laminar and turbulence flow situations The integral methods easily provide accurate answer (not exact) for complex problems The boundary layer equations are usually integrated over the boundary layer thickness by assuming a profile for velocity, temperature, and concentration, as needed The better the approximate shape for the velocity and temperature, the better the prediction of drag force and heat transfer (friction coefficient or heat transfer coefficient)

This dissertation consists of three chapters First chapter contains the basic definitions In second chapter we have reviewed some prominent integral methods with their advantages and disadvantages In chapter three we have used the well-known integral method, Method of Thwaites, to investigate the flow separation in Howarth's retarted flow and flow separation over a sphere

1.2 Basics Definitions

1.2.1 Fluid

A fluid is a substance that continuously deforms under an applied shear stress, no matter how small that shear stress may be In simple words fluid is a continuous substance whose molecules move freely past one another and that has the tendency to assume the shape of its container, a liquid or a gas

1.2.2 Fluid Mechanics

Fluid mechanics is the branch of applied mechanics which is concerned with the study of behavior of fluids (liquids or gasses) at rest or in motion Fluid mechanics is further divided into three branches

i. Fluid Statics

ii. Fluid Kinematics

iii. Fluid Dynamics

1.2.2.1 Fluid Statics

Fluid statics is the branch of fluid mechanics which deals with the study of fluid behavior at rest

1.2.2.2 Fluid Kinematics

Fluid kinematics is the branch of fluid mechanics which deals with the study of fluid movements without considering the forces and energies in them

1.2.2.3 Fluid Dynamics

Fluid dynamics is the branch of Fluid mechanics which deals with the study of motion of fluids or that of bodies in contact with fluids

1.2.3 Flow

A fluid continuously deforms when different forces act upon it If the deformation continuously increases without limitation, then this phenomenon is referred as flow

1.2.4 Deformation

The relative change in position or length of the fluid particle is called deformation

1.3 Physical Properties of Fluid

1.3.1 Density

The density of a substance is a measure of mass per unit volume The symbol most often used for density is "ρ" and read as rho Mathematically, the density "ρ" at any point P may be defined as

$$\rho = \lim_{\delta v \to 0} \left(\frac{\delta m}{\delta v}\right);$$

where δv is the volume element around P and δm is the mass of the fluid If the density of the fluid varies then the density at a point is given by

$$\rho = \lim_{\delta v \to \delta v} \left(\frac{\delta m}{\delta v}\right),$$

where δv is the small volume over which the substance can be considered as continuum

1.3.2 Stress

The force per unit area of the surface on which its act is called stress It is designated by ``σ''. Mathematically, the stress at any point of the fluid is given by

$$\sigma = \lim_{\Delta s \to 0} \left(\frac{\Delta F}{\Delta s}\right),$$

where ΔF is the force acting on an element of surface area ΔS enclosing the point P There are two components of stress

i. Normal Stress,
ii. Tangential Stress

1.3.2.1 Normal Stress

The component of stress normal to the surface at point P is said to be the normal stress

1.3.2.2 Tangential Stress

The component of stress tangent to the surface at point P is referred as tangential stress

1.3.3 Pressure

The amount of force acting per unit area is known as pressure It is denoted by P and mathematically it is formulated as

$$P = \frac{\vec{F}}{A},$$

where $\vec{F}$ is the normal force and A is the area of the surface Pascal (Pa) is the unit of pressure in SI

1.3.4 Viscosity

The viscosity of a fluid is a measure of its resistance to deformation i e resisting to a shearing (or tangential) when the fluid is in motion It describes the internal friction of moving fluid It is designated by the symbol μ. Mathematically it is given as

$$\mu = \frac{\tau}{du/dx} = \frac{Shear\ Stress}{Rate\ of\ shear\ strain}$$

1.3.5 Temperature

The measure of intensity of heat is referred as temperature of the body Heat always travel from higher temperature to the lower temperature Several scales and units are exist for measuring temperature The most common are given below

i Celsius (°C),
ii Fahrenheit (°F),
iii Kelvin (K)

1.3.6 Kinematic Viscosity

The ratio of the absolute viscosity μ to the density ρ is known as kinematics viscosity of the fluid It is denoted by ν Mathematically is defined as

$$\nu = \frac{\mu}{\rho},$$

1.4 Types of Fluid

The fluid is classified into following types

i Compressible Fluid
ii Incompressible Fluid
iii Inviscid Fluid
iv Ideal Fluid
v Real Fluid

1.4.1 Compressible Fluid

A fluid in which the density ρ varies significantly with in the flow field when it is subjected to high pressure gradient, is called compressible fluid Gases are considered as compressible fluid

1.4.2 Incompressible Fluid

A fluid in which the density ρ remains constant is known as incompressible fluid It can also be stated as if a fluid requires a large variation in pressure to produce some appreciable variation in density such fluid is called incompressible fluid All liquids are often referred as incompressible fluids

1.4.3 Inviscid Fluid

A fluid which is assumed to have no viscosity μ is called inviscid fluid With zero viscosity the fluid offers no internal resistance to change in shape i e deformation No natural fluid is inviscid all fluid possess a certain degree of viscosity, but in many cases, for example, with water and air, the viscosity is very small and to the reasonable degree of accuracy be treated as inviscid fluid

1.4.4 Ideal Fluid

A fluid which is both inviscid and incompressible is called an ideal (or perfect) fluid

1.4.5 Real Fluid

A real fluid is one which has finite viscosity and thus can exert a tangential (or shearing) stress on a surface with which it is in contact

1.5 Classification of Fluid

Fluid can be classified into following two groups

i Newtonian Fluid

ii Non-Newtonian Fluid

1.5.1 Newtonian Fluid

Newtonian fluid is a fluid which satisfies the Newton's law of viscosity In such type of fluid the shear stress is linearly proportional to the velocity gradient

1.5.2 Non-Newtonian Fluid

A fluid which does not satisfy the Newton's law of viscosity is known as non-Newtonian fluid In this case, the shear stress is non-linearly proportional to the velocity gradient Mathematically it is described as

$$\tau = k(\frac{\partial u}{\partial y})^n , \tag{1 1}$$

where k indicates the consistency index which describes the consistency of fluid and n indicates the flow behavior index which describes a measure of how fluid differ from a Newtonian fluid For $k = \mu$ and $n = 1$ equation (1 1) reduces to the Newton's law of viscosity

1.6 Types of Flow

There are different types of flow which are discussed below

1.6.1 Compressible Flow

The flow of a compressible fluid (i e for which the density does not remain constant) is known as compressible flow

1.6.2 Incompressible Flow

The flow of an incompressible fluid (i e for which the density remains unchanged throughout the fluid) is referred as incompressible flow

1.6.3 Uniform Flow

A flow is said to be uniform when the velocity vector as well as other hydrodynamic parameters remain unchanged from point to point in the fluid For a uniform flow, the velocity is a function of time only, which can be expressed in Eulerian description as

$$V = V(t)$$

1.6.4 Non-Uniform Flow

A flow in which the fluid properties (velocity, density, pressure etc) change from point to point is referred as non-uniform flow

1.6.5 Steady Flow

A flow in which the velocity vector and other fluid properties at every point in a fluid does not change with time so that the flow pattern depend on time change is called steady flow, i e

$$\frac{\partial}{\partial t}(\cdot) = 0$$

1.6.6 Unsteady Flow

A flow in which fluid properties and conditions at any point in fluid change with time is known as unsteady flow, i e

$$\frac{\partial}{\partial t}(\ ) \neq 0$$

1.6.7 Laminar Flow

The flow of a fluid when each particle of the fluid follows a smooth path, paths which never interfere with one another One result of laminar flow is that the velocity of the fluid is constant at any point in the fluid The flow of air on air craft wing is an example of laminar flow

1.6.8 Turbulent Flow

A flow in which the fluid particle moves in irregular manner in all direction is known as turbulent flow In turbulent flow the speed of the fluid at a point is continuously undergoing changes in both direction and magnitude The blood flow in arteries and oil transport in pipelines are common examples of turbulent flow

1.6.9 Ideal Flow

The flow of ideal fluid (i e inviscid and incompressible fluid) is called ideal flow

1.6.10 Internal Flow

A flow for which the fluid is confined by a surface is called internal flow The flow in a pipe is an example of internal flow

1.6.11 External Flow

An external flow is a flow which occurs over the bodies immersed in an unbounded fluid The flow over airfoils, ship hulls and over turbine blades are examples of external flow

1.6.12 Irrotational Flow

Such type of flow in which the fluid particles do not rotate about their own axes is known as irrotational flow The condition for an irrotational flow is

$$\nabla x \vec{V} = 0$$

1.6.13 Rotational Flow

A flow in which the fluid particles rotate about their own axes is called rotational flow The condition for rotational flow is

$$\nabla x \vec{V} \neq 0$$

1.6.14 One-, Two -, and Three-Dimensional Flows

A flow can be classified as one-, two-, and three- dimensional flow depending on the number of space coordinates A flow is said to be one-dimensional flow if the fluid properties are depending only on one space coordinate The flow across a duct is an example of one-dimensional flow A flow in which the flow parameters vary in the direction of flow and in one direction at right angles to this direction is called two dimensional flow The flow between two non-parallel flat plates is considered as two dimensional flow A flow in which the velocity field and other hydrodynamic properties are functions of three space coordinates and for time is referred as three dimensional flow The flow of air on the wing of an airplane is an example of three dimensional flow

1.7 Basic Definitions used in Integral Methods

1.7.1 Boundary Layer Thickness

Boundary layer thickness is the distance from the surface to the point on normal direction where the fluid velocity u equals 99% of the free stream velocity It is designated by δ Mathematically it can be given as

$$\delta = y_{(u=0.99U_s)},$$

Where U_s denote the free stream velocity

1.7.2 Displacement Thickness

It is defined as the distance by which the external potential flow is displaced outwards due to the decrease in the velocity in the boundary layer It is denoted by δ_D Mathematically, it is expressed as

$$\delta_D = \int_0^{\delta}\left(1-\frac{u}{U_s}\right)dy \approx \int_0^{\infty}\left(1-\frac{u}{U_s}\right)dy$$

Since at $y = \delta$, $u \approx U_s$ therefore the integrand vanishes i e zero for $y \geq \delta$

1.7.3 Momentum Thickness

Moment Thickness is the distance from the surface such that momentum flux corresponding to free stream velocity through this distance is equal to the loss in momentum due the formulation of boundary layer It represents the reduction in momentum, it is denoted by δ_m Mathematically, it is represented as

$$\delta_m = \int_0^{\delta} \frac{u}{U_s}\left(1 - \frac{u}{U_s}\right) dy \approx \int_0^{\infty} \frac{u}{U_s}\left(1 - \frac{u}{U_s}\right) dy$$

Since at $y = \delta,\ \ u \approx U_s$, therefore the integrand is zero for $y \geq \delta$

1.7.4 Shape Factor

Shape factor is the ratio of displacement thickness and momentum thickness It is denoted by H and mathematically formulated as

$$H = \frac{\delta_D}{\delta_m}$$

It describes the nature of flow

1.7.5 Skin Friction

The friction between a moving fluid and its enclosing surface is known as skin friction

1.7.6 Skin Friction Coefficient

Skin friction coefficient is defined as

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$$

Where τ_w =Wall shear stress, ρ=Density of the fluid and U_∞= Free stream velocity

1.7.7 Boundary Layer Separation

A phenomenon in which the fluid layer suddenly detaches from the body is known as separation It occurs due to the presence of sufficient adverse pressure gradient within the boundary layer The point at which flow leaves the surface is called separation point At this point the shear stress or the velocity gradient normal to the wall, is zero i e

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$$

1.8 Alternative Forms of Displacement Thickness and Momentum Thickness

Since we know that the displacement thickness and momentum thickness are formulated as

$$\delta_D = \int_0^{\delta} \left(1 - \frac{u}{U_s}\right) dy,$$

$$\delta_m = \int_0^{\delta} \frac{u}{U_s}\left(1 - \frac{u}{U_s}\right) dy,$$

Defining the non-dimensional distance from the wall as

$$\eta = \frac{y}{\delta} \text{ Or } y = \eta\delta,$$

due to which

$$dy = \delta d\eta$$

Also,

$$\text{at } y = 0, \eta = 0 \text{ and at } y = \delta, \eta = 1$$

Therefore, the displacement thickness and the momentum thickness defined above, are modified as below

$$\delta_D = \delta \int_0^{1} \left(1 - \frac{u}{U_s}\right) d\eta,$$

$$\delta_m = \delta \int_0^{1} \frac{u}{U_s}\left(1 - \frac{u}{U_s}\right) d\eta,$$

respectively

Chapter 2

Review of Some Prominent Integral Methods

2.1 Literature Review

After the revolutionary concept of the boundary layer, given by Ludwig Prandtl [1] in 1904, Blasius [2] was the first who succeeded to obtain the expression for the boundary layer thickness $\delta(x)$ and wall sheer stress $\tau_w(x)$ It was found that the velocity profiles were similar when plotted non-dimensionally as $\frac{u}{U}$ versus $\frac{y}{\delta}$ It was not possible to obtain a closed form solution for the velocity profile, a numerical solution was needed

The closed form solution for laminar boundary layers can be obtained by using approximate methods The characteristics and properties of turbulent boundary layer can also be determined by the same approximate methods In this chapter we will discuss some prominent approximate methods (Integral Methods) which will make us able to calculate the boundary layer thickness of laminar and turbulent boundary layer as a function of distance along the wall The equations which are obtained by integrating the momentum equation across the boundary layer thickness (from $y = 0$ to $y = \delta$) are very useful and simple which immediately provide the desired results However, the process involves some further manipulations and construction of some new functions in order to make the resulting mathematics compact The business of making all such efforts is usually named as the integral methods The integral methods are used to calculate the boundary layer parameters approximately These methods, oftenly, are used when the equations of motion are not satisfied everywhere in the fluid but only across the boundary layer thickness These integral methods are based on the momentum integral equation given by Karman [3] in 1921

In 1921, Pohlhausen [4] introduced an integral method for the solution of two dimensional boundary layer equations along with the pressure gradient This method was based on von Karman momentum integral equation This method gives satisfactory results for favorable pressure gradient while for the adverse pressure gradient it gives less satisfactory results To come over the short comings of Pohlhausen method, Karman and Millikan [5] developed a new method (approximate) by using von Mises transformations

Bryan Thwaites [6] introduced an approximate method, in which instead of using polynomial like Pohlhausen, he used the relation between two parameters $\delta_m \left(\frac{\partial u}{\partial y}\right)_{y=0}$ and $\delta_m \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$ This method gives more satisfactory results than the Pohlhausen method To improve the Pohlhausen method which is based on 4th degree polynomial Schlichting and Ulrich [7] developed a method which is based on 6th degree polynomial The main advantage of this method was that the higher derivation of the velocity profiles with respect to distance along the wall are obtained with the more degree of nicety as compared to the fourth order polynomial Weil [8] has applied method [7] to compressible flows with zero heat transfer Curle [9] modified the Karman-Pohlhausen method by using a quintic velocity profile which satisfied an additional boundary condition at the wall

Walz [10] approximated the one-parameter velocity profiles used by Hartree, in fractional exponents To obtain more accurate results different methods, similarly to Pohlhausen method, have been proposed using the velocity profiles which satisfied the more boundary conditions than the Pohlhausen quartic profile Mangler [11] used a one-parameter family of profiles and compare his results with exact results He used the following velocity profile

$$\frac{u}{U} = f(\eta) = 1 - (1 - \eta)^n (1 + a_1\eta + a_2\eta^2) \quad (2\ 1)$$

Timman [12] introduced a method based on momentum integral equation which satisfies more boundary conditions than Pohlhausen's and Mangler's velocity profiles He used the polynomial of the form

$$\frac{u}{U} = 1 - \int_{\eta}^{\delta=\infty} \exp(-\eta^2)(a + c\eta^2 + \quad) d\eta - \exp(-\eta^2)(b + d\eta^2 + \quad), \quad (2\ 2)$$

where $\eta = \frac{y}{\delta}$ Moreover, the exponential functions like Timman's velocity profiles has also been used by Yuan [13] and Lew [14] Loitsianskii [15] modified the Karman-Pohlhausen method by first multiplying the momentum equation by a small variation of velocity and then integrating the equation across the boundary layer thickness Sutton [16] used the method which contains two-parameters He used this method to the case of a flat plate with zero pressure gradient Wieghardt [17] also developed a two-parameter integral method and utilized it for the different cases Head [18] proposed a two-parameter method which satisfied the momentum and energy equation together with the conditions at the wall and at the outer edge of the boundary layer An integral method for the moving continuous flat surface was developed by Sakiadis [19] This method is based on the different velocity profiles and satisfied the suitable boundary conditions Fox [20] developed an integral method for the calculation of the boundary layer over the moving continuous flat surface with injection and suction This method is modified form of the Karman-Pohlhausen method for the injection and suction

In this section we will discuss some prominent and most preferred integral methods to calculate the boundary layer The following integral methods will be discussed

1 Karman-Pohlhausen Method
2 Alternative Polynomial Forms
3 Timman's Method
4 Method of Thwaites

Since all these methods depend upon Von-Karman momentum integral equation Therefore, first we will develop the momentum integral equation for incompressible steady two dimensional flow and axisymmetric flow

2.2 von-Karman Momentum Integral Equation

The boundary layer equations for incompressible, steady and two dimensional flow are given as

Momentum Equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \qquad (2\ 3)$$

Continuity Equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2\ 4)$$

Subject to the following boundary conditions

$$\left.\begin{array}{l} u=v=0 \ at\ y=0, \\ u=U(x)\ \ at\ y\to\infty \end{array}\right\} \qquad (2\ 5)$$

To find the momentum integral equation we integrate momentum equation from $y=0$ to $y=\delta$ with respect to y

$$\int_0^\delta u\frac{\partial u}{\partial x}dy+\int_0^\delta v\frac{\partial u}{\partial y}dy=\int_0^\delta U\frac{\partial U}{\partial x}dy+\int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy \qquad (2\ 6)$$

where

$$U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Evaluating the integrals involved in equation (2 6) Last term on the R H S reads as

$$\int_0^{\delta} \nu \frac{\partial^2 u}{\partial y^2} dy = -\frac{\tau_0}{\rho} \tag{2 7}$$

where $\tau_0 = \mu(\frac{\partial u}{\partial y})_{y=0}$

The second term of the L H S of equation (2 6) can be written as

$$\int_0^{\delta} v\frac{\partial u}{\partial y} dy = \int_0^{\delta} v\frac{\partial}{\partial y}(u-U)\, dy, \tag{2 8}$$

Which upon integration by parts takes the form

$$\int_0^{\delta} v\frac{\partial u}{\partial y} dy = -\int_0^{\delta} (u-U)\frac{\partial v}{\partial y} dy \tag{2 9}$$

Since from continuity equation, we have

$$-\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Therefore, substituting in equation (2 9), we have

$$\int_0^{\delta} v\frac{\partial u}{\partial y} dy = \int_0^{\delta} (u-U)\frac{\partial u}{\partial x} dy \tag{2 10}$$

By invoking equation (2 7) and (2 10) in equation(2 6), we obtain

$$\frac{\partial}{\partial x}\left[U^2\int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] + U\frac{\partial U}{\partial x}\int_0^{\delta}\left(1-\frac{u}{U}\right)dy = \frac{\tau_0}{\rho} \tag{2 11}$$

Equation (2 11) can be written as

$$\frac{\partial}{\partial x}[U^2\delta_m] + U\frac{\partial U}{\partial x}\delta_D = \frac{\tau_0}{\rho}\ , \tag{2 12}$$

where

$$\delta_m = \int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy \tag{2 13}$$

$$\delta_D = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy \tag{2 14}$$

Dividing Equation (2 12) by U^2 on both sides to get

$$\frac{1}{U^2}\frac{d}{dx}[U^2\delta_m] + \frac{1}{U}\frac{\partial U}{\partial x}\delta_D = \frac{\tau_0}{\rho U^2} \quad (2\ 15)$$

Equation (2 15) can be expressed as

$$\frac{d}{dx}(\delta_m) + \left(\frac{2+H}{U}\right)\delta_m\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}, \quad (2\ 16)$$

The above equation simplifies to

$$\frac{d}{dx}(\delta_m) + (2+H)\frac{\delta_m}{U}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2} \quad (2\ 17)$$

where H is called the shape factor and defined as

$$H = \frac{\delta_D}{\delta_m}$$

Thus Equation (2 17) is known as the momentum integral equation for incompressible, steady and two dimensional flow

2.3 von-Karman Momentum Integral Equation for Axisymmetric Flow

The boundary layer equations for incompressible, steady and two dimensional axisymmetric flow are given as

Momentum Equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2}, \quad (2\ 18)$$

Continuity Equation

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0, \quad (2\ 19)$$

Subject to the following boundary conditions

$$\left.\begin{aligned} u = v = 0 \ at\, y = 0, \\ u = U(x) \ \ at\, y \to \infty \end{aligned}\right\} \quad (2\ 20)$$

To obtain the Momentum Integral form integrate the equation (2 18) from $y = 0$ to $y = \delta$ with respect to y Therefore, we have

$$\int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta U\frac{\partial U}{\partial x}dy + \int_0^\delta v\frac{\partial^2 u}{\partial y^2}dy \quad (2\ 21)$$

Evaluating the integrals involved in equation (2 21) Last term on the R H S reads as

$$\int_0^\delta \nu \frac{\partial^2 u}{\partial y^2} dy = -\frac{\tau_0}{\rho} \qquad (2\ 22)$$

where

$\tau_0 = \mu(\frac{\partial u}{\partial y})_{y=0}$

The second term of the L H S can be written as

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = \int_0^\delta v \frac{\partial}{\partial y}(u - U)\, dy, \qquad (2\ 23)$$

Upon integration by parts of equation (2 23) we obtain

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = -\int_0^\delta (u - U) \frac{\partial v}{\partial y} dy \qquad (2\ 24)$$

Since from continuity equation, we have

$$-\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + \frac{u}{r}\frac{dr}{\partial x}, \qquad (2\ 25)$$

Therefore, substituting (2 25) in (2 24), we have

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = \int_0^\delta (u - U) \frac{\partial u}{\partial x} dy + \int_0^\delta (u - U) \frac{u}{r}\frac{dr}{\partial x} dy \qquad (2\ 26)$$

By invoking equation (2 22) and (2 26) in equation(2 21), we obtain

$$\frac{\partial}{\partial x}[U^2 \delta_m] + U\frac{\partial U}{\partial x}\delta_D + \frac{dr}{\partial x}U^2\delta_m = \frac{\tau_0}{\rho}, \qquad (2\ 27)$$

Where δ_m and δ_D are defined in equation (2 13) & (2 14) respectively

Dividing Equation (2 27) by U^2 on both sides to get

$$\frac{1}{U^2}\frac{d}{dx}[U^2\delta_m] + \frac{1}{U}\frac{\partial U}{\partial x}\delta_D + \frac{dr}{\partial x}\delta_m = \frac{\tau_0}{\rho U^2} \qquad (2\ 28)$$

Equation (2 28) can be expressed as

$$\frac{d}{dx}(\delta_m) + \left(\frac{2 + H}{U}\right)\delta_m \frac{dU}{dx} + \frac{1}{r}\frac{dr}{dx}\delta_m = \frac{\tau_0}{\rho U^2}, \qquad (2\ 29)$$

Equation (2 29) can be written

$$\frac{d}{dx}(\delta_m) + (2 + H)\frac{\delta_m}{U}\frac{dU}{dx} + \frac{\delta_m}{r}\frac{dr}{dx} = \frac{\tau_0}{\rho U^2} \qquad (2\ 30)$$

where H is called shape factor and defined as

$$H = \frac{\delta_D}{\delta_m}$$

Thus, Equation (2 30) is the momentum integral equation for the incompressible, steady and two dimensional axisymmetric flow

2.4 Karman-Pohlhausen Method

In 1921 Pohlhausen [4] introduced an integral method for the solution of the boundary layer equations This method based on momentum integral equation developed by von-Karman [3] This method was the first general method and had so many applications in the different branches of fluid dynamics This method doesn't solve the boundary layer equations everywhere in the fluid but at the boundary of boundary layer In his method he assumed the velocity profile in the form

$$\frac{u}{U} = f(\eta)\ , \quad \eta = \frac{y}{\delta} \tag{2 31}$$

For the assumed velocity profile, the skin friction, momentum thickness and displacement thickness are defined as follows

$$\frac{\tau_w}{\rho U^2} = \frac{\nu}{U\delta} f'(0), \tag{2 32}$$

$$\frac{\delta_m}{\delta} = \int_0^1 f(1-f)d\eta, \tag{2 33}$$

$$\frac{\delta_D}{\delta} = \int_0^1 (1-f)d\eta, \tag{2 34}$$

The boundary condition to be satisfied are

$$\left.\begin{aligned} \text{at } y = 0,\quad u = 0,\quad \mu\frac{\partial^2 u}{\partial^2 y} = \frac{dp}{dx} = -\rho UU', \\ \text{Or } \frac{-1}{\rho}\frac{dp}{dx} = UU', \\ \frac{\partial^3 u}{\partial^3 y} = 0, \end{aligned}\right\} \tag{2 35}$$

$$\text{at} \quad y = \delta,\quad u = U,\quad \frac{\partial u}{\partial y}\ ,\ \frac{\partial^2 u}{\partial^2 y},\ \frac{\partial^3 u}{\partial^3 y}\ ,\ \to 0 \tag{2 36}$$

These boundary conditions are transformed to the form

$$\left.\begin{aligned} f(0) = 0,\ \ f''(0) = -A,\ \ f'''(0) = 0, \\ f(1) = f'(1) = f''(1) = f'''(1) = \quad = 0 \end{aligned}\right\} \tag{2 37}$$

where the prime denotes the differentiation w r t 'η' and A $= \frac{\delta^2}{\mu U}\frac{dp}{dx} = \frac{\delta^2}{\nu}U'$ The 'n' number of unknown coefficients involved in assumed velocity profile can be determined by n' of the boundary conditions (2 37) and the other unknown factor δ can be found by the momentum integral equation If first three boundary conditions at $y = 0$ are considered then the skin friction, momentum thickness and displacement thickness are the function of A only Therefore the substitution of (2 31) into momentum integral equation results in the following form

$$\frac{dz}{dx} = U^{-1}M(A) + U''N(A)Z^2, \tag{2 38}$$

where $Z = \frac{\delta^2}{\nu} = \frac{A}{U'}$ The functions $M(A)$ & $N(A)$ are the universal functions and depend upon which of the boundary conditions of (2 35) & (2 36) are satisfied

There are two major shortcomings of Pohlhausen method described above

1 The first criticism is the choice that which boundary conditions shall be satisfied and fact is that at $y = \delta$ the outer boundary conditions are satisfied Pohlhausen assumed a quartic velocity profile which satisfied the first two boundary conditions of (2 35) at the wall and first three boundary conditions of (2 36)are satisfied at the edge of the boundary layer By using these assumptions one leads to a method which, when applies to the region for which the pressure is decreasing, gives accurate results but it gives less satisfactory results when separation is approached To solve this problem Curle [9] in 1958 used a quintic velocity profile which satisfied an additional boundary condition at the wall to obtain the accurate results in the region of the down-stream of the pressure minimum, so that the predicted separation is 6 percent too high than the exact However, the method fails near the stagnation point Earlier, Schlichting and Ulrich [7] assumed a sixth degree polynomial which satisfied one more boundary conditions at the edge of the boundary layer This method gives some better results near to separation, say 15 percent error in the separation distance But this method fails near to a stagnation-point

2 The second shortcoming of this method is the second derivative of free stream velocity U i e U'', which appears in the final equation given by (2 38) Now if we are succeeded to

obtain U from experimental results, but in many cases it will be very hard to obtain the second derivative of the free stream velocity U To overcome this problem Holstein and Bohlen [30] modified the Pohlhausen method so that U'' does not appear in the final equation They used a parameter 'm' defined by

$$m = \frac{\delta_m^2 U'}{\vartheta} = (\frac{\delta_m}{\delta})^2 A \tag{2 39}$$

Consequently, the momentum integral equation given in equation (2 17) is written as follows

$$\frac{\tau_w \delta_m}{\mu U} = \frac{U}{\nu}\delta_m \frac{d\delta_m}{dx} + \frac{\delta_m^2}{\nu}U'\left(2 + \frac{\delta_D}{\delta_m}\right) \tag{2 40}$$

The equation (2 40) further can be re-written as

$$l = \frac{U}{2\nu}\frac{d}{dx}(\delta_m^2) + m(2 + H), \tag{2 41}$$

where

$$l = \frac{\tau_w \delta_m}{\mu U}, \qquad H = \frac{\delta_D}{\delta_m}, \qquad m = \frac{\delta_m^2}{\nu}U' \tag{2 42}$$

The equation (2 41) can also be simplified as

$$U\frac{d}{dx}\left(\frac{\delta_m^2}{\vartheta}\right) = L, \tag{2 43}$$

where

$$L = 2\{l - m(H + 2)\}$$

Since we know that the skin friction τ_w, momentum thickness δ_m, displacement thickness δ_D are the function of 'A' Therefore, from equation (2 42) and (2 43) we can observe that the functions l, H, L are the functions of m Therefore, by using equation (2 39) into equation (2 43) we will obtain the first order equation for m as follows,

$$\frac{d}{dx}\left(\frac{m}{U'}\right) = \frac{L(m)}{U} \tag{2 44}$$

The function $L(m)$ is linearly approximated by Tani in 1941 so that the equation (2 44) can be integrated numerically The linear approximation given by Tani has been used by many other authors, notably the Thwaites [6]

2.5 Alternative Polynomial Forms

In 1944 Mangler [11] used a more complicated velocity profile than the Pohlhausen's velocity profile He proposed a velocity profile of the form

$$\frac{u}{U} = f(\eta) = 1 - (1 - \eta)^n \left(1 + a_1\eta + a_2\eta^2\right) \tag{2 45}$$

The assumed velocity profile satisfies 'n' number of boundary conditions of the form (2 37) at $\eta = 1$ If a_2 is considered to be zero-only the first two boundary conditions $f(0) = 0$, $f''(0) = -A$ are used If a_2 is kept non-zero, then the third boundary condition $f'''(0) = 0$ is used By these boundary conditions the unknown coefficients a_1 & a_2, involved in assumed velocity profile, are determined and hence the functions $l(m), H(m)$ *and* $L(m)$ are obtained

The momentum integral equation can be solved in similar way as in Pohlhausen method The solution obtained by the alternative polynomial forms depends upon the value of 'n' and on either a_2 is kept zero or non-zero Mangler concluded that the polynomial given in equation (2 45) with $n = 12, a_2 \neq 0$ gives better approximation near a leading-edge stagnation point The Pohlhausen quartic profile, similar to velocity profile defined by Mangler with $n = 3, a_2 = 0$ has a very little less accuracy Pohlhausen quartic profile gives much accurate results of the boundary layer with zero pressure gradient But it gives more accurate approximation when $n = 4, a_2 \neq 0$ are used in equation (2 45) This latter form of velocity profile was suggested by Schlichting and Ulrich which fails near the leading edge stagnation point

2.6 Timman's Method

In 1949 a new method was proposed by Timman [12] to calculate the laminar boundary layer He assumed a velocity profile of the form

$$\frac{u}{U} = 1 - \int_{\eta}^{\delta=\infty} \exp(-\eta^2)\left(a + c\eta^2 + \quad\right) d\,\eta - \exp(-\eta^2)\left(b + d\eta^2 + \quad\right), \tag{2 46}$$

where $\eta = \frac{y}{\delta}$ To obtain the value of unknown coefficients (a, b, c, d) only the boundary conditions at the wall were considered Substituting (2 37) in equation (2 46), one gets

$$\left.\begin{aligned} f(0) &= 1 - b - \frac{1}{2}\sqrt{\pi}\left(a + \frac{1}{2}c\right), \\ f''(0) &= 2(b - d) = -\frac{U}{\vartheta}\delta^2 = -A, \\ f'''(0) &= 2(c - a) = 0, \\ f''''(0) &= 12(2d - b) = \frac{U\delta^2}{\vartheta}a\left\{\frac{da}{dx} + a\left(\frac{U'}{U} - \frac{1}{\delta}\frac{da}{dx}\right)\right\} \end{aligned}\right\} \quad (2\ 47)$$

Initially first three conditions of (2 37) were considered by Timman and these boundary conditions determine a, b, c as a function of A, when d is kept zero From equation (2 46) we can obtain the momentum thickness $\frac{\delta_m}{\delta}$, the displacement thickness $\frac{\delta_D}{\delta}$ and the skin friction $\frac{\tau_w}{\rho U^2}$ as a function of A and also m is a function of A Therefore, we can get the functions l, H and L defined in equation (2 42) and (2 43) as a function of m from the momentum integral equation reducing to the form (2 44) Timman observed that the obtained solution yielded much satisfactory representation of flow near to a stagnation point, but not quite accurate in the region of retarded flow, as application to the Howarth's flow $\left[u = u_0\left(1 - \frac{x}{L}\right)\right]$ showed

Accordingly, he proposed that the condition $2d - b = 0$ should be used instead of $d = 0$ in the regions of adverse pressure gradients This assumption was made to satisfy the complicated fourth condition defined in (2 47)at the point where separation occurs When $f''''(0) = 0$ it follows from third and fourth condition of (2 47) that $b = 2d = -A$, so that the functions l, H and L can be calculated

2.7 Method of Thwaites

In 1949 Bryan Thwaites [6] proposed a new integral method to calculate the boundary layer thickness and the skin friction without assuming velocity profiles, as the methods previously described all do By suitably correlating the boundary layer characteristics l, H and L and shape parameter m we can find the shape parameter m as a function of x by numerical integration of equation (2 44) Then we can find the momentum thickness δ_m from equation (2 13), the displacement thickness δ_D and wall shear stress τ_w from equation (2 42) Since the different set of functions $H(m), l(m)$ and $L(m)$ are agreed with each accurate solution of the laminar boundary layer In similar manner an unambiguous set of functions correspond to an approximate method using a single-parameter family of profiles From here we conclude that an approximate method gives better approximation if and only if the values of the functions l, H *and* L of approximate and exact solution are very close to each other

B Thwaites used a similar approach to calculate the laminar boundary layer He examined and compared the set of functions $H(m), l(m)$ and $L(m)$ of all known exact and approximate solution of the laminar boundary layer equations He found that for the positive value of m, the solutions with respect to $H(m)$ and $l(m)$ are very close to each other For the negative value of m, the solutions differ considerably from each other, such that the values of m corresponding to the boundary-layer separation, that is the values for which $l(m)$=0, which were obtained ranging from $m = -0\ 068$ to $m = -0\ 157$ Luckily, at that time two most important and well known solutions, Howarth's solution and Hartree's solution of Schubauer's experimental results, were available to Thwaites These solutions were very close to each other, and the values of $H(m)$ and $l(m)$ chose by Thwaites were also quite close to these solutions

For different solutions, the values of $L(m)$ defined in equation (2 43) were much closer as compared to $H(m)$ and $l(m)$, for both positive and negative value of m Thwaites found that the function $L(m)$ was linear and approximated as follows

$$L(m) = 0\ 45 - 6m \tag{2 48}$$

He chose these coefficients as these were the best agreement with the available solutions at that time By using equation (2 48) in equation (2 44) we get

$$\delta_m^2 = \frac{0.45\nu}{U^6}\int_0^x U^5 dx \tag{2 49}$$

Having thus obtained δ_m^2, equation (2 42) reduces to

$$\left.\begin{aligned} \delta_D &= \delta_m H(m), \\ \tau_w &= \frac{\mu U_1}{\delta_m} l(m) \end{aligned}\right\} \tag{2 50}$$

Earlier, the function $L(m)$ was expressed as a linear function by Walz [10] and Tani [28], but Thwaites approximation is considered more acceptable because his approximation is not based on a single solution Curle and Skan [29] modified the Thwaites method by suggesting some improvements to Thwaites functions Thwaites obtained the values of $H(m)$ by finding the ratio of exact δ_D to exact δ_m Curle and Skan [29] proposed that the value of $H(m)$ would be determined by considering the ratio of exact δ_D to the approximate δ_m By applying similar argument to $l(m)$, Curle and Skan [29] modified the Thwaites functions in the region near separation where these functions differ considerably for individual solutions

Chapter 3

Calculation of the Boundary Layer Separation over a Sphere through Integral Method

3.1 Introduction

The problem of separation of flow is one of the most important phenomenon to be explored intensively, to find the solution of this problem Viscosity is the most important factor in the classical concept of flow separation, therefore, often it is given as "boundary layer flow separation" or "boundary layer separation" Adverse pressure gradient is the sufficient condition for the boundary layer separation Flow separation occurs when the boundary layer travels far enough against an adverse pressure gradient that the speed of the boundary layer relative to the object falls almost to zero The fluid flow becomes detached from the surface of the object and instead takes the forms of eddies and vortices The point of separation may be defined as the limit between the forward and reverse flow in the layer very close to the wall, i e , at the point of separation

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0,$$

This means that the shear stress at the wall is zero, i e ,

$$\tau_w = 0$$

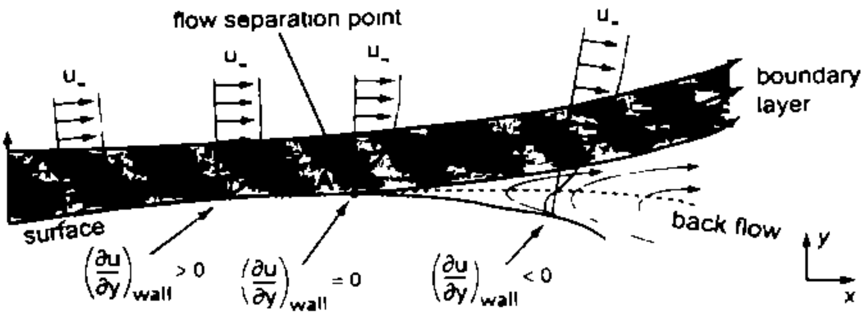

Fig 3.1 Velocity profile in a boundary layer in the vicinity of separation.

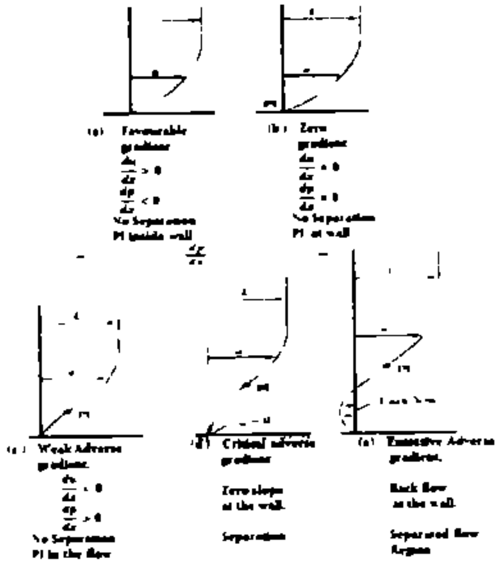

Fig. 3.2 Velocity profiles in a boundary layer subjected to a pressure rise

The velocity profiles exhibiting the separation phenomenon are shown diagrammatically in Figs 3 1 & 3 2

The problem of boundary layer separation is as old as that of the concept of boundary layer Ludwig Prandtl was the first one who investigated the flow separation before he started his work on the revolutionary theory of the boundary layer During his work at the "*Maschinen Fabrik Augsburg-Nurnberg*" (MAN), he observed that the computed pressure was not same as obtained in actual diffuser He worked for three years to investigate how and why the flow separation and pressure losses were occurred Finally, he was succeeded to solve this problem by giving his new concept of boundary layer [1] Ludwig Prandtl [21] provided the experimental evidences to show that the rise in pressure in the direction of the flow i e positive (or adverse) pressure gradient along the flow path, is the necessary condition for the boundary layer separation This argument is valid for both compressible and incompressible flows Fottinger [22] investigated that the adverse pressure gradient and viscosity are the two necessary conditions, responsible for the boundary layer separation Schlichting [23] proposed that the increase in pressure in the direction of the flow and the presence of the wall friction are two important factors due to which flow separates from the wall Gortler and Witting [24] obtained the separation of flow over a plate by using the series method For this problem they used the following velocity distribution

$$u_e(x) = u_0(1 - \frac{x}{L})^n, \qquad (x \geq 0)$$

Meksyn [25] explained the role of viscosity in the boundary layer separation Schubauer and Spangenberg [26] stated that the separation of flow occurs due to the relation of the pressure gradient along the wall to the velocity gradient along the normal distance to the wall Because of the great importance of boundary layer separation, Scientists have studied this phenomenon, extensively In this chapter we have used integral method (Method of Thwaites) to find the flow separation in Howarth's retarted flow and flow separation over a sphere

3.2 Method of Thwaites for the Calculation of the Boundary Layer Separation in Howarth's Retarted Flow.

Howarth [27] and Tani [28] explained a further family of solutions of the boundary layer equations These solutions relate to the following potential flow

$$U(x) = U_0(1 - \frac{ax}{U_0})^n, \quad (n = 1,2,3 \quad), \tag{3 1}$$

which, clearly, represents a generalized form of the flow along a flat plate and becomes identical with it when we put $a = 0$ Howarth studied the simplest case with $n = 1$, that is equation (3 1) is expressed in the form of $U(x) = U_0(1 - \frac{ax}{U_0})$ It can be inferred as representing the potential flow along a flat plate which starts at $x = 0$ and which adjoins to another infinite wall at right angles to it at $x = L$

3.2.1 Mathematical Modeling

The problem statement for Howarth's flow is given as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{3 2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3 3}$$

Subject to the following boundary conditions

$$
\left.\begin{aligned} u = v = 0 \; at \; y = 0, \\ u = U(x) = U_0\left(1 - \frac{ax}{U_0}\right) \; at \; y \to \infty \end{aligned}\right\} \tag{3 4}
$$

3.3 Integral solution (Method of Thwaites) of the problem.

In 1949 Bryan Thwaites [6] introduced a method to solve the boundary layer equations without assuming velocity profiles He used some suitable correlation between boundary layer characteristics H, l, L and the shape parameter m The momentum integral equation for two dimensional boundary layer flow in terms of shape factor H and momentum thickness δ_m is expressed as

$$
\frac{d}{dx}(\delta_m) + \frac{(H+2)}{U}\delta_m\frac{dU}{dx} = \frac{\tau_0}{\rho U^2} \tag{3 5}
$$

The above equation can be re- written as

$$
\frac{d}{dx}(\delta_m) = -(H+2)\delta_m\frac{U'}{U} + \frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{3 6}
$$

Using (3 4) in (3 2) we get

$$
UU' + \left(\nu\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = 0 \tag{3 7}
$$

To solve equation (3 5) we are required to know

$\left(\frac{\partial u}{\partial y}\right)_{y=0}$ and $\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$

Thwaites [6] considered all the known exact and approximate solutions available at that time for the boundary layer equations and defined by

$$
\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{U}{\delta_m}l(m) \; and \; \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{U}{\delta_m^2}m\,, \tag{3 8}
$$

where $l(m)$ and m are dimensionless functions or parameters These forms are chosen in order to ensure that l and m depend only on the shape of the velocity profile and do not depend upon the boundary layer thickness By using equation (3 8) in (3 7) we get

$$UU' + \nu \frac{U}{\delta_m^2} m = 0$$

This equation further can be expressed as

$$U' = \frac{\nu}{\delta_m^2} m, \tag{3 9}$$

or

$$m = -\frac{\delta_m^2 U'}{\nu} \tag{3 10}$$

Substituting equation (3 8) and (3 9) in equation(3 5), we get

$$\frac{U\delta_m}{\nu}\frac{d}{dx}(\delta_m^2) = [2 + H(m)]m + 2l(m),$$

or

$$\frac{U}{\nu}\frac{d}{dx}(\delta_m^2) = 2[2 + H(m)]m + 2l(m)$$

The above equation can be simplified as

$$\frac{U}{\nu}\frac{d}{dx}(\delta_m^2) = L(m), \tag{3 11}$$

where

$$L(m) = 2[(H + 2)]m + 2l(m) \tag{3 12}$$

Thwaites plotted $L(m)$ versus m and found that the relation was linear He suggested,

$$L(m) = 0\ 45 + 6m$$

By using this value in equation (3 11) we obtain

$$\frac{U}{\nu}\frac{d}{dx}(\delta_m^2) = 0\ 45 + 6m$$

This equation can be simplified as

$$\frac{d}{dx}(\delta_m^2) + 6\frac{U'}{U}\delta_m^2 = 0\ 45\frac{\nu}{U}, \tag{3 13}$$

which is linear in δ_m^2 and can be expressed as

$$\frac{d}{dx}(U^6\delta_m^2) = 0.45\nu U^5 \qquad (3\ 14)$$

By integrating equation (3 14) we obtain

$$\delta_m^2 = \frac{0\ 45\nu}{U^6}\int_0^x U^5 dx, \qquad (3\ 15)$$

where the constant of integration is zero because $\delta_m = 0$ at $x = 0$ In terms of dimensionless quantities equation (3 15) can be expressed as

$$\delta_m^2 = \frac{0\ 45\nu}{U^{*6}}\int_0^{x^*} U^{*5} dx^* \qquad (3\ 16)$$

Since

$$u = U(x) = U_0\left(1 - a\frac{x}{U_0}\right) = U_0\left(1 - \frac{a}{U_0}\frac{x}{L}\right),$$

which can also be expressed as

$$\frac{U}{U_0} = U^* = (1 - x^*)$$

By using the above value of U^* in equation (3 16) we have

$$\delta_m^2 = \frac{0\ 45\ \nu}{[(1 - x^*)]^6}\int_0^{x^*} [(1 - x^*)]^5\, dx^*,$$

Which upon integration reduces to

$$\delta_m^2 = 0\ 075\nu[(1 - x^*)^{-6} - 1] \qquad (3\ 17)$$

Since Thwaites's criterion of separation is

$$m = \frac{\delta_m^2\, U'}{\nu} \qquad (3\ 18)$$

Due to which we have

$$m = -0\ 075[(1 - x^*)^{-6} - 1] \qquad (3\ 19)$$

From the Newton's law of viscosity, we have

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0},$$

which in view of equation (3 8) takes the form

$$\tau_w = \mu \frac{U}{\delta_m} l(m)$$

Thwaites suggested correlations for $l(m)$ and $H(m)$ He proposed the correlation for the shear function as follows

$$l(m) = \frac{\tau_w \delta_m}{\mu U} \approx (m + 0\ 09)^{0\ 62}$$

As we know that separation occurs where the effects of viscosity are vanished and shear stress is zero, so

$$l(m) = \frac{\tau_w \delta_m}{\mu U} \approx (m + 0\ 09)^{0\ 62} = 0,$$

$$(m + 0\ 09)^{0\ 62} \approx 0,$$

$$m \approx -0\ 09$$

By using the value of m in equation(3 19), we have

$$-0\ 075[(1 - x^*)^{-6} - 1] = -0\ 09,$$

$$x^*_{sep} = 0\ 123 \qquad (3\ 20)$$

3.3.1 Results and Discussion

In this section we have implemented the method of Thwaites to determine the separation of Howarth's retarted flow The separation occurs at $x^* = 0\ 123$ It can be observed that our approximation is quite close to the Howarth's result which is $x^* = 0\ 120$, due to which we can claim that our result is acceptable The various properties of the boundary layer for several positions of x^* are described in Table 3 1

Table 3.1: Values of different boundary layer parameters correspond to different values of x^*

x^*	$-m$	δ_m	$H(m)$	δ_D	$l(m)$	$\frac{1}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}$
0 0125	0 0059	0 0767	2 6330	0 2019	0 2017	2 73
0 0250	0 0123	0 1109	2 6605	0 2951	0 2004	1 80
0 0375	0 0193	0 1390	2 6938	0 3745	0 1889	1 35
0 0500	0 0270	0 1644	2 7351	0 4497	0 1760	1 07
0 0625	0 0355	0 1883	2 7873	0 5249	0 1614	0 85
0 0750	0 0447	0 2115	2 8553	0 6039	0 1444	0 68
0 0875	0 0549	0 2343	2 9472	0 6906	0 1240	0 53
0 1000	0 0661	0 2571	3 0775	0 7914	0 0982	0 38
0 1125	0 0785	0 2801	3 2762	0 9178	0 0604	0 215
0 1170	0 0832	0 2885	3 3757	0 9738	0 0403	0 139
0 1200	0 0865	0 2941	3 4543	1 0159	0 0228	0 077
0 1230(x^*_{sep})	0 0899	0 2997	3 5453	1 062	0 0	0 00

3.4 Method of Thwaites to Investigate the Boundary Layer Separation over Non-Flat Surface

In this section we will apply the method of Thwaites to the non-flat surface, the sphere, to determine the boundary layer separation and as well as the values of other boundary layer parameters

3.4.1 Mathematical Modeling

Let us assume a steady two-dimensional flow past an axisymmetric body The coordinates (x, y) are chosen in such a manner that x coordinate is measured in the stream wise direction (in the direction of fluid flow) and y coordinate is taken normal to the surface of the body Figure 3 1 shows the physical coordinates, direction of free stream velocity and the flow lay out

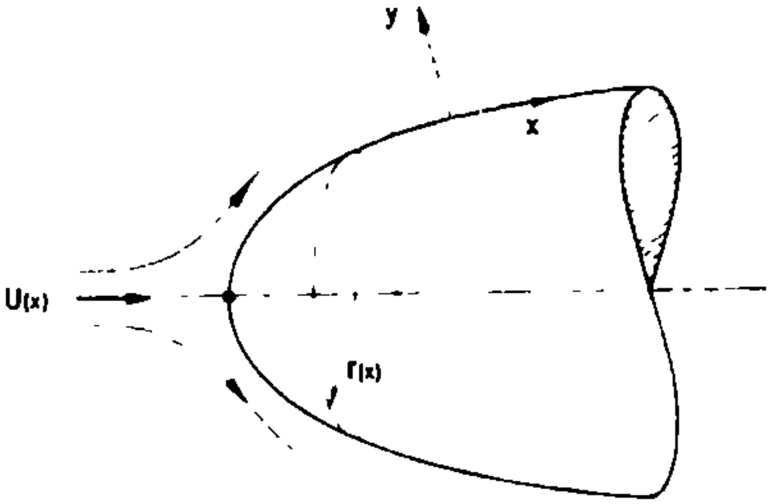

Fig 3.3. Physical model and coordinate system

The boundary layer equations for the steady two-dimensional, axisymmetric flow are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{3 21}$$

$$\frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial y} = 0 \tag{3 22}$$

where u, v are representing the fluid velocity components in the $x-$ direction and $y-$directions, respectively, ν is the kinematic viscosity coefficient and $r(x)$ is the radii of the contour of the body taken at right angles to the axis

The boundary conditions for this problem read as

$$u = v = 0 \ \text{at}\ y = 0, \tag{3 23}$$

$$u = U(x),\ v = 0\ \text{as}\ y \to \infty \tag{3 24}$$

3.5 Integral solution (Method of Thwaites) of the problem

To use method of Thwaites we will assume the momentum integral equation in terms of shape factor H and momentum thickness δ_m described in equation (2 30), which can be written as

$$\frac{d}{dx}(\delta_m) = -(2+H)\frac{\delta_m}{U}\frac{dU}{dx} - \frac{\delta_m}{r}\frac{dr}{dx} + \frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{3 25}$$

Now in view of (3 23) equation (3 21) becomes

$$0 = UU' + \left(\nu\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \tag{3 26}$$

To find the solution of equations(3 25) we need to know $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ and $\left(\nu\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$ The values of these parameters, suggested by Thwaites, are given in equation(3 8) By putting equation (3 8) and (3 9) in equation (3 25), we get

$$\frac{d}{dx}(\delta_m) = (2+H)\frac{\nu m}{U\delta_m} - \frac{\delta_m}{r}\frac{dr}{dx} + \frac{\nu}{U\delta_m}l(m) \tag{3 27}$$

Multiplying equation (3 27) by $\frac{U\delta_m}{\nu}$ on both sides, we obtain

$$\frac{U\delta_m}{\nu}\frac{d}{dx}(\delta_m) = (2+H)m - \frac{\delta_m^2}{r}\frac{dr}{dx}\,\frac{U}{\nu} + l(m)$$

This equation can be expressed as

$$\frac{U}{\nu}\frac{d}{dx}(\delta_m^2) = 2[(2+H)m + l(m)] - 2\frac{\delta_m^2}{r}\frac{dr}{dx}\,\frac{U}{\nu},$$

$$\frac{U}{\nu}\frac{d}{dx}(\delta_m^2) = L(m) - 2\frac{\delta_m^2}{r}\frac{dr}{dx}\ \frac{U}{\nu}, \tag{3 28}$$

where

$$L(m) = 2[(2 + H)m + l(m)]$$

Bryan Thwaites plotted $L(m)$ versus m and found that the relation was linear He chose to put

$$L(m) = 0\ 45 + 6m$$

By using this value in equation (3 28) we get

$$\frac{U}{\nu}\frac{d}{dx}(\delta_m^2) = 0\ 45 + 6m - 2\frac{\delta_m^2}{r}\frac{dr}{dx}\ \frac{U}{\nu} \tag{3 29}$$

In view of equation (3 10) the above equation can be written as

$$\frac{U}{\nu}\frac{d}{dx}(\delta_m^2) = 0\ 45 + 6\left(-\frac{\delta_m^2 U'}{\nu}\right) - 2\frac{\delta_m^2}{r}\frac{dr}{dx}\ \frac{U}{\nu},$$

Which further simplifies to

$$\frac{d}{dx}(\delta_m^2) + \left(6\frac{U'}{U} + 2\frac{1}{r}\frac{dr}{dx}\right)\delta_m^2 = 0\ 45\frac{\nu}{U} \tag{3 30}$$

Equation (3 30) is a first order linear differential equation Upon further manipulation equation (3 30) reduces to

$$\frac{d}{dx}(\delta_m^2 U^6 r^2) = 0\ 45\ \nu\ U^5 r^2 \tag{3 31}$$

The integration of equation (3 31) yields

$$\delta_m^2 = \frac{0\ 45\nu}{U^6 r^2}\int_0^x U^5 r^2 dx \tag{3 32}$$

In terms of dimensionless quantities equation (3 32) can be written as

$$\delta_m^2 = \frac{0\ 45\nu}{\overline{U}^6\overline{r}^2}\int_0^{\overline{x}} \overline{U}^5\overline{r}^2 d\overline{x} \tag{3 33}$$

Now we are ready to apply the method of Thwaites to the case of sphere We know that in the case of sphere of radius R, kept at rest in free stream velocity U, the ideal potential velocity is given by

$$U(x) = \frac{3}{2} U_0 sin \frac{x}{R},$$

or

$$\overline{U}(x) = \frac{3}{2} sin\overline{x} \qquad (3\ 34)$$

The schematic of the above equation can be seen in Fig 3 4 The $r(x)$ in the case of sphere is given by

$$r(x) = R sin \frac{x}{R},$$

or

$$\overline{r}(x) = sin\overline{x} \qquad (3\ 35)$$

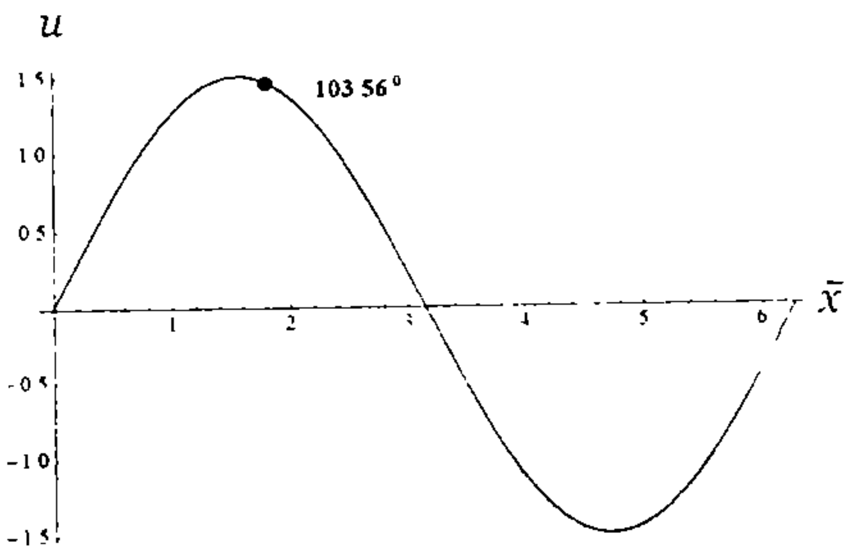

Fig 3.4. Free stream velocity of fluid

By putting equation (3 34) and (3 35) in equation (3 33), we obtain

$$\delta_m^2 = \frac{2}{3}\ \frac{0\ 45\nu}{\sin^8\overline{x}} \int_0^{\overline{x}} \sin^7\overline{x}\, d\overline{x} \tag{3 36}$$

To find the separation point we will use Thwaites criterion of separation which is given as

$$m = \frac{\delta_m^2 \overline{U}'}{\nu}$$

By invoking values of δ_m^2 and $\overline{U}'$ in above equation, we get

$$m = \frac{0\ 45\nu}{\sin^8\overline{x}}\ \cos\overline{x} \int_0^{\overline{x}} \sin^7\overline{x}\, d\overline{x}\,, \tag{3 37}$$

Which upon integration gives

$$m = \frac{0\ 45\nu}{\sin^8\overline{x}}\ \cos\overline{x}\left[\frac{16}{35} - \cos\overline{x} + \cos^3\overline{x} - \frac{3}{5}\cos^5\overline{x} + \frac{1}{7}\cos^7\overline{x}\right],$$

$$m\ \sin^8\overline{x} = 0\ 45\left[\frac{16}{35}\cos\overline{x} - \cos^2\overline{x} + \cos^4\overline{x} - \frac{3}{5}\cos^6\overline{x} + \frac{1}{7}\cos^8\overline{x}\right] \tag{3 38}$$

Since we know that in laminar flow, separation occurs when $m = -0\ 09$ Therefore, by using this value in above equation, we get

$$-0\ 09\ \sin^8\overline{x} = 0\ 45\left[\frac{16}{35}\cos\overline{x} - \cos^2\overline{x} + \cos^4\overline{x} - \frac{3}{5}\cos^6\overline{x} + \frac{1}{7}\cos^8\overline{x}\right]$$

This equation further simplifies to

$$\frac{12}{7}\cos^8\overline{x} - 7\cos^6\overline{x} + 11\cos^4\overline{x} - 9\cos^2\overline{x} + \frac{16}{7}\cos\overline{x} + 1 = 0 \tag{3 39}$$

Note that equation (3 39) is a polynomial equation in terms of $\cos\overline{x}$ and finding the separation point " $\overline{x}_{sep}$" is equivalent to finding the roots of this equation We utilize the famous 'Bisection Method' to find that

$$\overline{x}_{sep} = 1\ 8074 = 103\ 56^0 \tag{3 40}$$

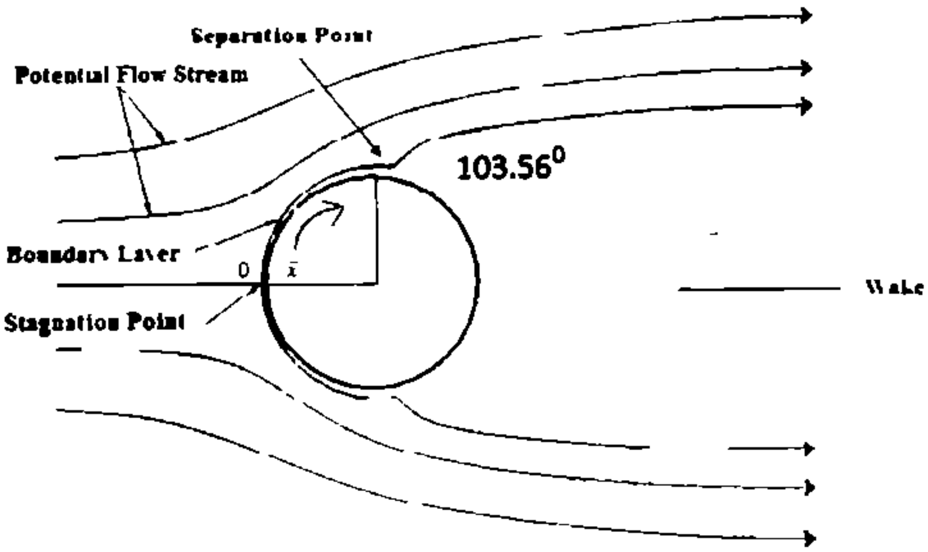

Fig 3.5 Flow separation over a sphere

3.5.1 Results and Discussion

We have applied Thwaites method to the axially symmetric body, which is a sphere The separation occurs at 103 56^0 (see Fig 3 5) In comparison to the Blasius series solution $\bar{x}_{sep}$ = 109 6^0 , the present result, obtained by Thwaites method, is quite acceptable The comparison of our result with other results which are available in literature is given in Table 3 2

Table 3.2: Comparison of calculated separation point with various methods

Present Method	CS Method	Smith-clutter	Series Method
103 56^0	107 5^0	105 9^0	109 6^0

The boundary layer parameters for different values of $\bar{x}$ are calculated by Thwaites correlation formulae and tabulated in Table 3.3

Table 3.3: Values of different boundary layer parameters correspond to different values of $\bar{x}$

$\bar{x}$	$-m$	δ_m	$H(m)$	δ_D	$l(m)$	$\frac{1}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}$
1.60	0.0064	0.3826	2.6352	1.0083	0.2099	0.5485
1.63	0.0139	0.3963	2.6679	1.0572	0.1978	0.4991
1.66	0.0226	0.4110	2.7105	1.1142	0.1835	0.4465
1.69	0.0320	0.4271	2.7683	1.1823	0.1665	0.3899
1.72	0.0441	0.4446	2.8501	1.2671	0.1456	0.3272
1.75	0.0575	0.4637	2.9739	1.3789	0.1185	0.2556
1.77	0.0676	0.4774	3.0983	1.4791	0.0942	0.1974
1.79	0.0789	0.4919	3.2851	1.6161	0.0587	0.1193
1.80	0.0851	0.4996	3.4184	1.7077	0.0310	0.0620
1.802	0.0863	0.5011	3.4497	1.7287	0.0239	0.0476
1.805	0.0882	0.5035	3.5003	1.7623	0.0116	0.0231
1.807	0.0895	0.5051	3.5366	1.7862	0.0022	0.0043
1.8074 $(\bar{x}_{sep})$	0.0898	0.5054	3.5441	1.7911	0.000	0.000

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