بسم الله الرحمن الرحيم

In the name of Allah,
The most Gracious, The most merciful.

Boundary Layer Flow
Due to Uniform Stretching of a Circular Cylinder

By

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1. Cylinders
2. Fluid mechanics

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A Dissertation
Submitted in the Partial Fulfillment of the
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IN
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Certificate

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We accept this dissertation as conforming to the required standard.

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Declaration

I hereby, declare, that this thesis neither as a whole nor as a part thereof has been copied out from any source It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning

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Dedication

To my parents.

The reason of what I become today.

Thanks for your great support and continuous care.

To my loving brothers, sister, students and my loving friends.

I am really grateful to both of you.

You have been my inspiration and my soul mates.

This humble work is a sign of my love to you!

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My Deepest gratitude to my Mother and Father who are the real pillars of my life They always encouraged me and showed their everlasting love, care and support throughout my life The continuous encouragement and humble prayers support (both financially and moral) from my father and mother is unforgettable The love from elder brothers is priceless

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Preface

The study of convective transport and heat transfer phenomenon in boundary layer flow due to stretching flat plates or cylinders are of practical importance in fiber technology and extrusion processes and of theoretical interest as well The production of polymer sheets and plastic films is based on this technology There are number of examples which include the cooling of an infinite metallic plate in a cooling bath, the boundary layer along material handling conveyers, the aerodynamic extrusion of plastic sheets, the boundary layer along a liquid film in condensation processes, paper production, glass blowing, metal spinning and drawing plastic films, and polymer extrusion. The quality of the final product depends largely on the rate of heat transfer at the stretching surface

Being inspired by these studies we intend to investigate the heat transfer phenomenon in unsteady boundary layer flow due to stretching cylinder Two cases have been considered namely, the impulsively started cylinder and the oscillatory stretching of cylinder This dissertation comprises of three chapters The first chapter includes the preliminaries. In second the numerical solution by using finite difference scheme is presented The effects of physical parameter like strouhal number, amplitude of oscillation on velocity profile and skin friction coefficient have also been discussed

In chapter three unsteady boundary layer flows due to uniform stretching of an impulsively started circular cylinder has been considered. The impact of the curvature parameters and the time variable has been investigated on the coefficient of skin friction and the velocity profile

Chapter 1

Basics Definitions

1.1 Fluid Mechanics

Fluid mechanics is the branch of science which deals with the behaviour of fluid i.e. (liquid/gases).

1.1.1 Branches of Fluid Mechanics

Fluid mechanics usually sub divided into three main branches. (1) Fluid Statics (2) Fluid Kinematics (3) Fluid Dynamics

1.1.2 Fluid Statics

It is the study of fluids at rest.

1.1.3 Fluid Kinematics

It is the study of fluids in motion neglecting pressure forces.

1.1.4 Fluid Dynamics

It is the study of fluids in motion considering pressure forces. It has several sub disciplines such as Aerodynamics (the study of air and other gases in motion) and Hydrodynamics (the study of liquids in motion) etc.

1.2 Fluid

A substance that has no fixed shape and yields easily to external pressure, a gas or (especially) a liquid

1.2.1 Types of Fluids

There are mainly two types of fluids (1) Ideal fluids (2) Real fluids

1.2.2 Ideal Fluid

A fluid whose density ρ is constant and viscosity μ is zero

$$\rho = \text{const},$$

$$\mu = 0 \text{ (No friction force)}$$

1.2.3 Real Fluid

All liquids for which viscosity is not to zero ($\mu \neq 0$) are called viscous or real fluids There are further two classes of real fluids (1) Newtonian fluids (2) Non-Newtonian fluids

1.2.4 Newtonian Fluid

The fluids that obey the Newton's law of viscosity are called Newtonian fluids Linear relationship holds between shear stress and the rate of deformation

$$\tau_{yx} = \mu \frac{du}{dy}$$

1.2.5 Non-Newtonian Fluid

The fluids which doesn't comply with the Newton's law of viscosity are called Non-Newtonian fluids There are further three sub classes of Non-Newtonian fluids (1) Dilant fluid (Shear thickening) (2) Pseudoplastic fluid (Shear thining) (3) Viscoplastic fluid (Bingham)

Dilant Fluid

For higher rate of shear strain higher viscosity is reflected i e (quick sand, corn, starch etc)

Pseudoplastic Fluid

For higher rate of shear strain low viscosity is reflected i e (paints blood ketchup etc)

Viscoplastic Fluid

Solid upto certain stress and after that they start to flow i e (maynese, tooth paste, etc)

1.3 Flow

A material goes under distortion when certain forces are applied on it if the deformation continuously increases without limit then the phenomenon is known as flow

1.3.1 Incompressible Flow

A flow in which the volume and thus the density of fluid does not change during the flow All liquids are generally considered to have incompressible flow

1.3.2 Compressible Flow

A flow in which the volume and thus the density of fluid changes during the flow All gasses are generally considered to have compressible flow

1.3.3 Uniform Flow

A flow in which the velocity of fluid particles at all areas of channels are equivalent

1.3.4 Non-Uniform Flow

A flow in which the velocity of fluid particles at all sections of channels are not equivalent

1.3.5 Laminar Flow

A flow in which every fluid particle has a definite path and the path of individual particle doesn't cross each other

1.3.6 Turbulent Flow

A flow in which every fluid particle doesn t has a definite path and the path of individual particles are also cross each other

1.3.7 Steady Flow

A flow whose flow state expressed by velocity, pressure, density etc at any position doesn't change with time is called a steady flow

$$\frac{\partial \eta}{\partial t} = 0,$$

where η represent any fluid property

1.3.8 Unsteady Flow

A flow whose flow state expressed by velocity, pressure, density etc at any position does change with time is called a unsteady flow At whatever point water comes up short on a tap while the handle is being turned the flow is an unsteady On the other hand, when water runs out while the handle is stationary, leaving the opening consistent, the flow is steady

1.3.9 Rotational Flow

A flow in which the fluid particles rotate about their own axis during the flow i e fluid particles have some angular velocity e g in a rotational stream if a match stick (bit of which) is tossed on the surface of the moving liquid it will pivot about its axis

$$\nabla \times \mathbf{V} \neq 0 \text{ or } \operatorname{curl} \mathbf{V} \neq 0$$

1.3.10 Irrotational Flow

A flow in which the fluid particles do not rotate about their own axis and retain their original orientation e g in an irrotational flow if a match stick thrown on the surface of the moving fluid it does not rotate about the axis

$$\nabla \times \mathbf{V} = 0 \text{ or curl } \mathbf{V} = 0$$

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \quad \text{and} \quad \mathbf{V} = ui + vj + wk$$

1.4 Properties and Physical Parameters of Fluid

There are following properties and some physical parameters of fluid

1.4.1 Viscosity

Viscosity is the resistance of fluid to its motion, it is denoted by μ It is also known as kinematic viscosity Viscosity is a physical quantity of fluid which can be mathematically defined as "ratio of shear stress to rate of shear strain" SI unit for μ is $\frac{Ns}{m^2}$

$$\mu = \frac{\text{shear stress}}{\text{shear strain}},$$

$$\mu = \frac{\tau}{\frac{du}{dy}}$$

1.4.2 Density

The mass per unit volume at constant pressure and temperature is called density It is denoted by ρ and defined as

$$\rho = \frac{m}{V}$$

SI unit for ρ is $\frac{m^2}{s}$ or stokes

1.4.3 Dynamic Viscosity

The ratio of absolute viscosity μ to the density ρ is called dynamic viscosity It is denoted by ν and is defined as

$$\nu = \frac{\mu}{\rho}$$

1.4.4 Reynolds Number

Reynolds number is defined as

$$\text{Re} = \frac{\delta \nu d}{\mu},$$

where ν is the dynamic viscosity d is the diameter δ is the density and μ be the viscosity A leminar flow turns to the turbulent flow when the value of non dimensional quantity $\text{Re} = \frac{\delta \nu d}{\mu}$ reaches a certain amount and whatever the values of the average velocity v

1.4.5 Strouhal Number

The Strouhal number St is a dimensionless number describing oscillating flow mechanism The parameter is named after Vincenc Strouhal a Czech Physicist The Strouhal number is an integral part of the fundamentals of fluid mechanics The Strouhal number is often given as

$$St = \frac{fL}{U},$$

where f is the frequency, L is the characteristics length and U is the flow velocity

1.5 Velocity Vector

A velocity vector defined as change of position of object with respect to time We know that velocity is a vector quantity, the magnitude of velocity vector gives the speed of object while the vector direction give its direction

1.6 Streamline

A velocity vector forming a curve of each particles of fluid at a certain time is called streamline In other words, the curve where the tangent at each point indicates the direction of fluid flow

1.7 Velocity Profile

The variation in velocity along a line at right angles to the general direction of flow

1.8 Curvature

The curvature measures how fast a curve is changing direction at a given point There are several formulas for determing the curvature of a curve

1.9 Stream Function

The flow velocity components can be expressed as the derivatives of the scalar stream function The stream function can be used to plot streamlines which represents the trajectories of particles in a steady flow The two dimensional lagrange stream function was introduced by Joseph Louis Lagrange in 1781

1.10 Inertial Force

An inertial force is a force that resists a change in velocity of an object It is equal and opposite direction of an applied force as well as a resistive force The concept is based on Newton law of motion including the law of inertia and action-reaction law

1.11 Viscous Force

Viscous force is the force between a body and a fluid (liquid or gases) moving past it, in a direction so as to oppose the flow of fluid past the object

1.12 Boundary Layer Thickness

The thickness of the velocity boundary layer is normally defined as the distance from the solid body at which the viscous flow velocity is 99% of the freestream velocity

1.13 Wall Shear Stress

The wall shear stress τ_w is given by

$$\tau_u = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

where μ is the dynamics viscosity, u is the flow velocity parallel to the wall and y is the distance to the wall The SI unit of shear stress is pascal which is identical to $\frac{kg}{ms^2}$

1.14 Surface Tension

It is defined as a tensile force acting on the surface of liquid in contact with air, gas or between two immiscible liquids Surface tension is denoted by σ Its unit is $\frac{N}{m}$

1.15 Types of Forces

There are two main categories of forces (1) Contact Forces (2) Non-contact Forces

1.15.1 Contact Forces

There are further types of contact forces (1) Frictional force (2) Tension force (3) Normal force (4) Air resistance force (5) Applied force (6) Spring force

1.15.2 Non-Contact Forces

It can be further divided three types (1) Gravitational force (2) Electrical force (3) Megnetic force

1.15.3 Centrifugal Force

The tendency of an object following a curved path to fly away from the centre of curvature is called centrifugal force It might be described as "lack of centripetal force Its direction is along the radius of the circle from the centre towards the object Mud flying of a tire is an example of centrifugal force

1.15.4 Centripetal Force

The force that keeps an object moving with a uniform speed along a circular path Its direction is along the radius of the circle from the object towards the centre Satellite orbiting a planet is an example of centripetal force

$$F_c = \frac{mv^2}{r}$$

1.15.5 Applied Force

An applied force is a force that is applied to an object by a person or another object If a person is pushing a desk across the room then there is an applied force acting upon the object

1.15.6 Gravity Force

The force of gravity is the force with which the earth, moon or other massively large object attracts another object towards itself

$$F_{grav} = mg$$

1.15.7 Normal Force

The normal force is the support force exerted upon an object that is in contact with another stable object

1.15.8 Friction Force

The friction force is the force exerted by a surface as an object moves across it or makes an efforts to move across it

$$F_{frict} = uF_{norm}$$

1.15.9 Air Resistance Force

The air resistance is a special type of frictional force that acts upon objects as they travel through the air

1.15.10 Tension Force

The tension force is the force that is transmitted through a string, rope cable or wire when it is pulled tight by force acting from opposite ends

1.15.11 Spring Force

The spring force is the force exerted by a compressed or stretched spring upon any object that is attached to it

1.16 Continuity Equation or Law of Conservation of Mass

1.16.1 Statement

Mass in any classical system can neither be created nor be destroyed, i e (mass remains conserved) In steady flow, the mass flow per unit time passing through each section does not change, even if the pipe diameter changes (M= constant)

$$\frac{dM}{dt} = 0, \tag{1 1}$$

$$dM = \int \rho dV,$$

$$M = \int dM = \int_{v} \rho dV,$$

$$M = \int_{\bar{v}} \rho dV$$

$$\frac{dM}{dt} = \frac{d}{dt} \int_{\bar{v}} \rho dV,$$

According to law of conservation of mass

$$\frac{dM}{dt} = 0,$$

Eq (1 1) becomes

$$0 = \frac{d}{dt} \int_{\bar{v}} \rho dV,$$

$$\frac{d}{dt} \int_{\bar{v}} \rho dV = 0, \tag{1 2}$$

$$\int_{\bar{v}} \frac{\partial \rho}{\partial t} dV + \int_{\bar{s}} \rho V n ds = 0$$

according to Gauss Divergence Theorem on second integral

$$\int_{\bar{v}} \frac{\partial \rho}{\partial t} dV + \int_{\bar{v}} \mathrm{div}(\rho \mathbf{V}) dV = 0,$$

$$\int_{\bar{v}} [\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{V})] dV = 0$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{V}) = 0, \tag{1 3}$$

which is the required continuity equations for incompressible flow

$$\rho = \mathrm{const},$$

$$\frac{\partial \rho}{\partial t} = 0,$$

Therefore, Eq (1 3) becomes

$$\mathrm{div}(\rho \mathbf{V}) = 0,$$

$$\rho \operatorname{div} \mathbf{V} = 0$$

$$\operatorname{div} \mathbf{V} = 0$$

$$\nabla \cdot \mathbf{V} = 0 \tag{1.4}$$

$$\mathbf{V} = [u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)]$$

Therefore, Eq (1.4) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.4*}$$

which is the required continuity equation for incompressible flow

1.17 Newton's Law of Viscosity

1.17.1 Statement

The Newton law states that 'The shear stress acting on a fluid is proportional to the rate of shear strain'

$$\tau_{yx} \propto \text{Rate of Deformation}$$

$$\tau_{yx} = k \text{ (Rate of Deformation)},$$

OR

$$\text{Rate of Deformation} = \frac{d\theta}{dt} = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} \tag{1.5}$$

$$\delta l = (\delta y)(\delta\theta)$$

$$\delta l = (\text{velocity})(\text{time}),$$

$$\delta l = (\delta u)(\delta t)$$

$$\delta u \delta t = \delta y \delta\theta$$

$$\frac{\delta\theta}{\delta t} = \frac{\delta u}{\delta y}$$

$$\lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{du}{dy},$$

$$\frac{d\theta}{dt} = \frac{du}{dy},$$

$$\text{Rate of Deformation} = \frac{du}{dy}$$

$$\tau_{yx} \propto \frac{du}{dy},$$

$$\tau_{yx} = (\text{constant of proportionality})\frac{du}{dy},$$

$$\tau_{yx} = \mu\frac{du}{dy} \tag{1 6}$$

1.18 Equation of Motion for Viscous Fluid

For viscous fluid following forces are significant

(a) Pressure force per unit volume

$$\vec{F}_p = -\nabla P, \tag{1 7}$$

where p is pressure

(b) Body force per unit volume

$$\vec{F}_b = \rho B, \tag{1 8}$$

where ρ is density, B is body force per unit mass

(c) Viscous force (friction force)

$$\vec{F}_v = \nabla\ \boldsymbol{\tau} \tag{1 9}$$

The result out of $\vec{F}_p$, $\vec{F}_b$, and $\vec{F}_v$ is

$$\vec{F} = \vec{F}_p + \vec{F}_b + \vec{F}_v,$$

$$F = -\nabla P + \rho\mathbf{B} + \text{div}\ \mathbf{V} \tag{1 10}$$

$$m\vec{a} = \vec{F},$$

$$\frac{m}{v}\vec{a} = \frac{\vec{F}}{v},$$

$$\rho\vec{a} = \bar{F}, \tag{1.11}$$

From (1.9) and (1.11) we have

$$\rho\frac{dV}{dt} = -\nabla P - \rho\mathbf{B} + \operatorname{div}\mathbf{V} \tag{1.12}$$

which is the required equation of motion for viscous fluid

Case (1), when fluid is at rest

$$\frac{dV}{dt} = 0,$$

Eq (1.12) becomes

$$-\nabla P + \rho\mathbf{B} = 0,$$

$$\nabla P = \rho\mathbf{B} \tag{1.13}$$

which is the describing equation for static fluid

Case (2) when fluid is inviscous

$$\operatorname{div}\delta = 0.$$

due to which Eq (1.12) becomes

$$\rho\frac{dV}{dt} = -\nabla P + \rho\mathbf{B}, \tag{1.14}$$

which is the equation of motion for inviscid fluid

Case (3) when

$$\frac{d}{dt} = \text{Material derivative},$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V\nabla$$

Eq (1.12) becomes

$$\frac{\partial v}{\partial t} + (V\nabla)V = -\frac{1}{\rho}\nabla P + \frac{1}{\rho}\operatorname{div}\mathbf{V} + \mathbf{B} \tag{1.15}$$

Case (4) when

$$\nabla P = \frac{\partial p}{\partial x}i + \frac{\partial p}{\partial y}j + \frac{\partial p}{\partial z}k$$

$$\mathbf{B} = Bx\imath + By\jmath + Bzk$$

$$\mathbf{V} = u\imath + v\jmath + wk.$$

$$\frac{d\mathbf{V}}{dt} = \frac{\partial u}{\partial t}\imath + \frac{\partial v}{\partial t}\jmath + \frac{\partial w}{\partial t}k.$$

$$\nabla = \imath\frac{\partial}{\partial x} + \jmath\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\mathbf{V}\ \nabla = (u\imath + v\jmath + wk)\ (\imath\frac{\partial}{\partial x} + \jmath\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}).$$

$$\mathbf{V}\ \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$(\mathbf{V}\ \nabla)\mathbf{V} = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})(u\imath + v\jmath + wk)$$

$$= \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]\imath + \left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]\jmath + \left[u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right]k.$$

x-component of momentum equation (1 12)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + B_x \qquad (1\ 16)$$

y-component of (1 12)

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + B_y \qquad (1\ 17)$$

z-component of (1 12)

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + B_z \qquad (1\ 18)$$

Eqs (1 16) (1 18) are valid for unsteady viscous flow flow

1.19 Navier Stoke's Equation for a Viscous Incompressible flow in Cylindrical Coordinates

(u_r, u_θ, u_z) denotes the velocity components in (r, θ, z) directions respectively, then the system (1.16) – (1.18) in cylindrical system of coordinates read as

$$
\begin{aligned}
& \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \\
= \; & F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} \right]
\end{aligned}
\tag{1.19}
$$

$$
\begin{aligned}
& \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \\
= \; & F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu \left[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} \right],
\end{aligned}
\tag{1.20}
$$

$$
\begin{aligned}
& \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \\
= \; & F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right],
\end{aligned}
\tag{1.21}
$$

Eq (1.19) – (1.21) are the Navier Stokes' equations in cylindrical polar coordinates

Chapter 2

Unsteady Flow of Viscous Fluid over the Vacillate Stretching Cylinder

2.1 Introduction

The boundary layer flow due to stretching of elastic sheets or wires is very important practically in engineering and industrial processes For example, generally raw material in liquified state passes through an extrusion die for manufacturing of metallic and polymeric sheets even under high temperature, material elongation and linear stretching is usually observed at this stage According to Vleggaar [1], the velocity of material is propotional to the distance so, stretching plates, wires, rods or cylinders are such kind of system which are modeled mathematically The flow over moving surfaces was hypothetical worked for the first time by Sakiadis [2] Crane [3] was the first who studied the two dimensional steady state flow of viscous fluid because of stretching sheet Later, the work was extended by Wang [4], considering three dimensional flow and obtained a numerical solution Exploration of flow over stretching surfaces under different interesting material, great work of Crane [3] inspired various authers later, which has features such as porous medium and porous sheets, electrically conducting fluid, nonlinear stretching and non-Newtonian nature of fluids The flow caused by periodical oscillating streching sheet was previously considered by Wang [4] who calculated the perturbation solution by taking large frequency and small amplitude of oscillation Oscillatory motion of magneto-hydrodynamic viscoelastic fluid was studied by Rajagopal [5] over a porous sheet in porous medium Wang's

[4] work was extended by Abbas [6] by investigating the slip effects of viscous fluid over an oscillatory stretching flat plate

To the best of our knowledge the Wang [4] was the first who discussed the effects of stretching cylinder over a stationary fluid and examine the solution analytically by using the technique of asymptotic expansion and numerically by shooting method Afterwards, many authors investigated various aspects of this idea and obtained similarity solution near a stretching circular cylinder The exact solution for axisymmetric motion of fluid was studied by Burde [7], Ishak [8] and Nazar [9] by discussing the flow and heat transfer over a stretching cylinder and obtained a numerical solution by Keller Box method Ishak [8] and Nazar [9] examined the effect of suction in electrically conducting fluid because of stretching cylinder Matroberardino and Paullet [10] proved the existence of solution for all values of parameters by examining of flow over a permeable stretching cylinder According to concept of Sparrow and Yu [11] the flow may be considered as axisymmetric, if the order of radius of cylinder and boundary layer thickness is same The governing equations of such kind of flow contain the terms of transverse curvature because of which the boundary layer is effected considerable

Our current study is the time depended flow caused by oscillatory stretching of cylinder is considered The solution of this problem is determined by using finite difference strategy Initially we transform semi infinite spatial domain into finite domain by using appropriate transformation and after that we discretize the spatial derivatives by using difference quotients The resulting albebraic equations are solved by using Gaussian elimination method with differet time steps Many authors like Abbas [6], Wang [4] have already obtained numerical solution due to the same strategy

2.2 Mathematical Modeling of the Problem

Consider an unsteady axisymmetric flow of an incompressible viscous fluid due to an oscillatory stretching of a uniform cylinder Initially, at the $\bar{t} = 0$ the cylinder and the fluid are assumed to be at rest At $\bar{t} > 0$ the cylinder is subjected to an oscillatory stretching given by

$$U_u = a\overline{r}(1 + \epsilon \cos \omega \bar{t})$$

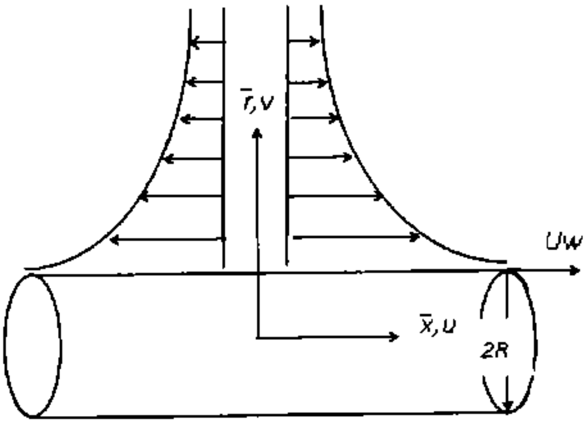

Figure 2-1 Geometry of flow phenomenon

In cylindrical coordinates described in Fig 2 1 the law of conservation of mass can be expressed in the form

$$\frac{\partial(\bar{r}u)}{\partial\bar{x}} + \frac{\partial(\bar{r}v)}{\partial\bar{r}} + \frac{\partial(\bar{r}w)}{\partial\bar{z}} = 0 \tag{2 1}$$

where $\frac{\partial(\bar{r}u)}{\partial\bar{z}} = 0$, because no angular motion, where u & v denote the components of velocity in $\bar{x}$ & $\bar{r}$ direction respectively The momentum equations for axisymmetric flow in cylindrical system read as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial\bar{x}} + v\frac{\partial u}{\partial\bar{r}} + w\frac{\partial u}{\partial\bar{z}} = -\frac{1}{\rho}\frac{\partial P}{\partial\bar{x}} + \nu\left(\bar{x}\frac{\partial^2 u}{\partial\bar{x}^2} + \bar{r}\frac{\partial^2 u}{\partial\bar{r}^2} + \bar{z}\frac{\partial^2 u}{\partial\bar{z}^2}\right) + B_{\bar{x}}, \tag{2 2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial\bar{x}} + v\frac{\partial v}{\partial\bar{r}} + w\frac{\partial v}{\partial\bar{z}} = -\frac{1}{\rho}\frac{\partial P}{\partial\bar{r}} + \nu\left(\bar{x}\frac{\partial^2 v}{\partial\bar{x}^2} + \bar{r}\frac{\partial^2 v}{\partial\bar{r}^2} + \bar{z}\frac{\partial^2 v}{\partial\bar{z}^2}\right) + B_{\bar{r}} \tag{2 3}$$

Because of the absence of any potential flow the pressure gradiant is zero in the boundary layer Under the boundary layer assumptions and in the absence of body force the above system reduces to the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial\bar{x}} + v\frac{\partial u}{\partial\bar{r}} = \frac{\nu}{\bar{r}}\frac{\partial}{\partial\bar{r}}(\bar{r}\frac{\partial u}{\partial\bar{r}}) \tag{2 4}$$

The appropriate boundary conditions for $\bar{t} > 0$ are given by

$$u(\bar{x}, \bar{r}, \bar{t}) = a\bar{x}(1 + \epsilon \cos \omega t) = U_w, \tag{2 5}$$

$$u(\bar{x}, \infty, \bar{t}) = 0 \text{ when } \bar{r} \to \infty, \tag{2 6}$$

$$u(\bar{x}, R\ \bar{t}) = 0 \text{ when } \bar{r} \to R \tag{2 7}$$

and the initial conditions at $\bar{t} = 0$ read as

$$u(\bar{x}, \bar{r}, 0) = 0, \tag{2 8}$$

$$v(\bar{x}, \bar{r}, 0) = 0 \tag{2 9}$$

here R is the radius of the cylinder, a is the stretching rate, ω be the oscillations frequency and ϵ be the amplitude of oscillation with $\epsilon < 1$ The velocity components defined in terms of stream function and stream function are given by

$$u = \frac{1}{\bar{r}}\frac{\partial \Psi}{\partial \bar{r}} \qquad v = \frac{1}{\bar{r}}\frac{\partial \Psi}{\partial \bar{x}} \ . \tag{2 10}$$

Above system of equations is transformed to dimensionless form by using following transformations

$$\begin{aligned} u &= 2a\bar{x}f'(r,t),\ v = -\frac{aR}{\sqrt{1+r}}f(r,t),\ \tau = \omega\bar{t},\ \Psi = R^2 a\bar{x}f(r\ t) \\ r &= (\frac{\bar{r}}{R})^2 - 1, \end{aligned} \tag{2 11}$$

due to which Eq (2 1) is satisfied identically and Eq (2 4) takes the form

$$\mathrm{Re}\left[St\frac{\partial f'}{\partial t} + (f')^2 - ff''\right] = f'' + (1+r)f''' \tag{2 12}$$

Subjected to the boundary conditions

$$\begin{aligned} f(r,t) &= 0 \ \text{ when } r=0 \qquad f'(r,t) = 1+\epsilon\cos(t) \ \text{ also when } r=0, \\ f'(r\ t) &= 0 \ \text{ when } r\to\infty \end{aligned} \tag{2 13}$$

where Re is the Reynolds number define by $\mathrm{Re} = \frac{aR^2}{2\upsilon}$ and St denotes Strouhal number define by $St = \frac{\omega}{2a}$ and the prime denote by derivative with respect to r Strouhal number is the ratio of oscillating frequency to the constant stretching rate The values $St < 1$ correspond the small frequency where as $St > 1$ correspond to high frequency At the surface of cylinder the shear stress can be represented by

$$\tau_w = \mu\left(\frac{\partial u}{\partial \overline{r}}\right) \quad \text{at} \quad \overline{r} = R$$

Putting the value of $\frac{\partial u}{\partial \overline{r}}$, we get

$$\tau_u = \frac{4a\overline{x}\mu}{R} f''(0,t) \tag{2 14}$$

where μ is dynamic viscosity, here we also define the coefficients of skin friction C_f which comes out of the form

$$C_f = \frac{\tau_w}{\rho(a\overline{x})^2}$$

Putting the value of τ_w we get

$$C_f = \frac{2}{\mathrm{Re}_x} f''(0,t) \tag{2 15}$$

2.3 Numerical Solution

To find the numerical solution of above equations (2 12 & 2 13), we apply the finite difference scheme Before doing so we transform the semi infinite domain $r \in [0,\infty)$ into finite domain $\eta \in [0\ ,1]$ as

$$\eta = \frac{1}{1+r} \tag{2 16}$$

due to which the governing system modifies as

$$\eta^3 f_{\eta\eta\eta} + 5\eta^2 f_{\eta\eta} + 4\eta f_\eta - \mathrm{Re}\, St\frac{\partial^2 f}{\partial t \partial \eta} = \mathrm{Re}\left[\eta^2 f f_{\eta\eta} + 2\eta f f_\eta - (\eta f_\eta)^2\right] \tag{2 17}$$

and

$$\frac{\partial f}{\partial \eta}(0,t) = 0 \quad f(\eta,t) = 0\ ,\ \text{when } \eta = 1\ ,\quad \frac{\partial^2 f}{\partial \eta^2}(1,t) = -1 - \epsilon\cos(t) \tag{2 18}$$

Eqs (2 17 & 2 18) are solved by using finite difference scheme We replace the linear terms at advanced time step $k+1$ and the nonlinear terms at the previous time step k The time derivative $\frac{\partial}{\partial t}$ is approximated by forward difference By doing so the Eqs takes the form

$$\begin{aligned} &\eta^3 f_{\eta\eta\eta}^{(k+1)} + 5\eta^2 f_{\eta\eta}^{(k+1)} + 4\eta f_{\eta}^{(k+1)} - \mathrm{Re}\, St\left[\frac{\partial f^{(k+1)}}{\partial \eta} - \frac{\partial f^{(k)}}{\partial \eta}\right] \\ = \ & \mathrm{Re}\left[\eta^2 f^{(k)} f_{\eta\eta}^{(k+1)} + 2\eta f^{(k)} f_{\eta}^{(k)} - \eta^2 (f_{\eta}^{(k)})^2\right] \end{aligned} \tag{2 19}$$

Initial condition at $t = 0$ is described as

$$f(\eta, 0) = 0 = f_i^0 \tag{2 20}$$

Furthermore, the equations (2 19 & 2 20) are discretize to the linear equations as $M+2$ uniformly distributed grid points as $(0 = \eta_0\ \eta_1, \quad , \eta_M\ \eta_{M+1} = 1)$ with a grid size of $\Delta\eta = \frac{1}{M+1}$ Finite difference approximation has been used to discretize spatial derivatives For our situation we used first two derivatives for second order central difference approximation

$$f_\eta = \frac{f_{i+1} - f_{i-1}}{2\Delta\eta} + O(2\Delta\eta^2) \tag{2 21}$$

$$f_{\eta\eta} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta\eta^2} + O(\Delta\eta^2) \tag{2 22}$$

because of one boundary condition at $\eta = 0$ and two boundary condition at $\eta = 1$ The third order derivative is approximated by

$$f_{\eta\eta\eta} = \frac{f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}}{(\Delta\eta)^3}, \tag{2 23}$$

where f_i is the value of f at $\eta_i = i\Delta\eta$ for $i = 0, 1, 2, \quad , M-1$ $(m-1)$ linear system of equations as produced by using above mentioned finite difference scheme for $f_i^{(k+1)}$ as follow

$$w_i f_{i-1}^{(k+1)} + x_i f_i^{(k+1)} + y_i f_{i+1}^{(k+1)} + z_i f_{i+2}^{(k+1)} = B_i^{(k)} \quad \text{for} \quad i = 1, 2, \quad , M-1, \tag{2 24}$$

where w_i, x_i, y_i, z_i are the known algebraic coefficients of $f_{i-1}^{(k+1)}, f_i^{(k+1)}, f_{i+1}^{(k+1)}, f_{i+2}^{(k+1)}$ respectively, and $B_i^{(k)}$ makes the right side of algebraic system (2 24) The mesh term including $\eta_0, \eta_1, \ldots, \eta_M, \eta_{M+1}$ in equation (2 24) can be found by discretizing boundary date mentioned in (2 18) Equation (2 24) formed a quad diagonal matrix of dimension $M + 2$ The system of algebraic equations can easily be solved by using Gaussian elimination method for different time levels with time step size $t^{(k)} = k\Delta t$ for $k = 0, 1,$

2.4 Results and Discussion

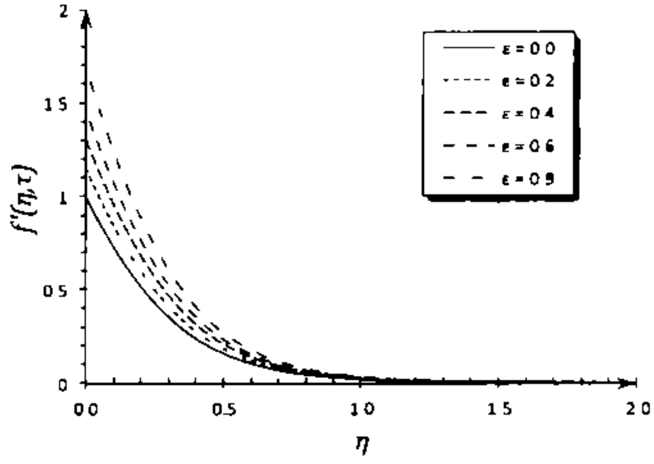

Figure 2-2 Effect of ϵ on the velocity profile $f'(\eta\ \tau)$ against η

In this section we discuss the effects of some physical parameters e g strouhal number, amplitude of oscillation and Reynolds number on skin friction coefficients and velocity profile It is founded in Figure 2 2 that as ϵ increases there is no effect occur on the boundary layer thickness but near the surface of cylinder velocity profile enhances In Figure 2 3 the effect of Strouhal number on velocity profile is investigated It is seen that boundary layer thickness decreases when Strouhal number increases This is a consequence of the increased frequency of oscillations The velocity profile at different time values has been plotted in Fig 2 4 The velocity curves exhibit an overall asymptotic behavior having different values at the cylinder's

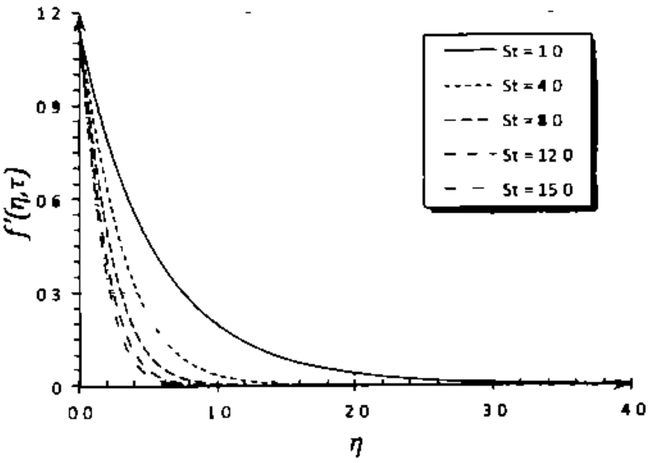

Figure 2-3 Effect of St on the velocity profile $f'(\eta, \tau)$ against η

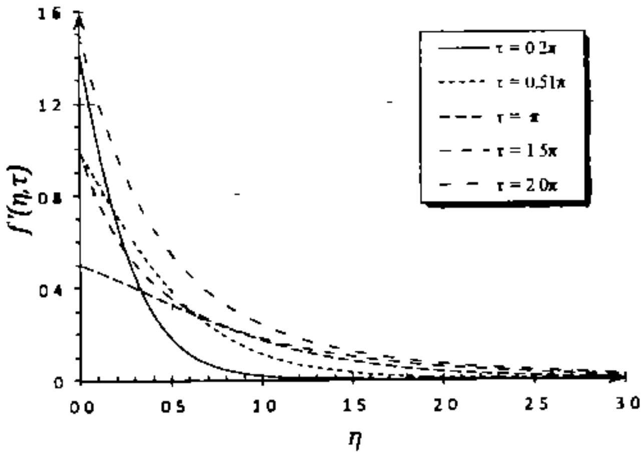

Figure 2-4 Effect of time on the velocity profile $f'(\eta, \tau)$ with in a time-period against η

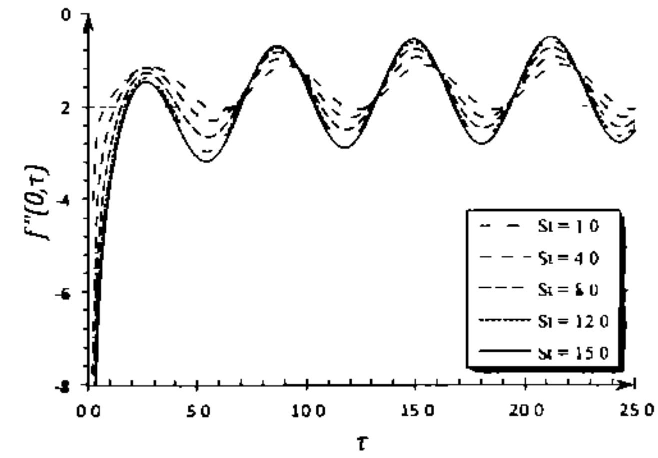

Figure 2-5 Effect of St on the skin friction coefficient $f''(0,\tau)$ against τ

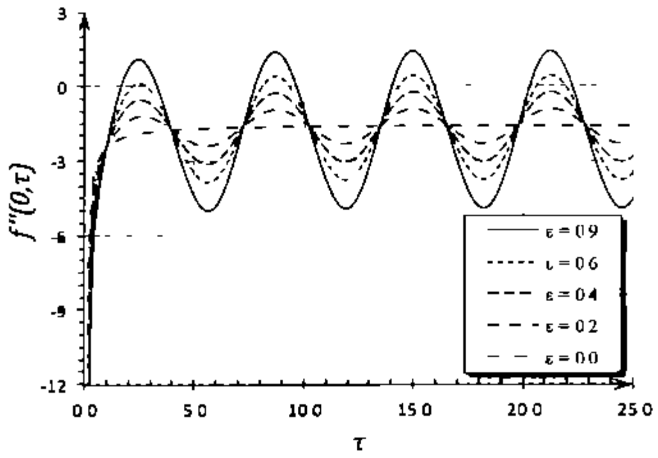

Figure 2-6 Effect of ϵ on the skin friction coefficient $f''(0,\tau)$ against τ

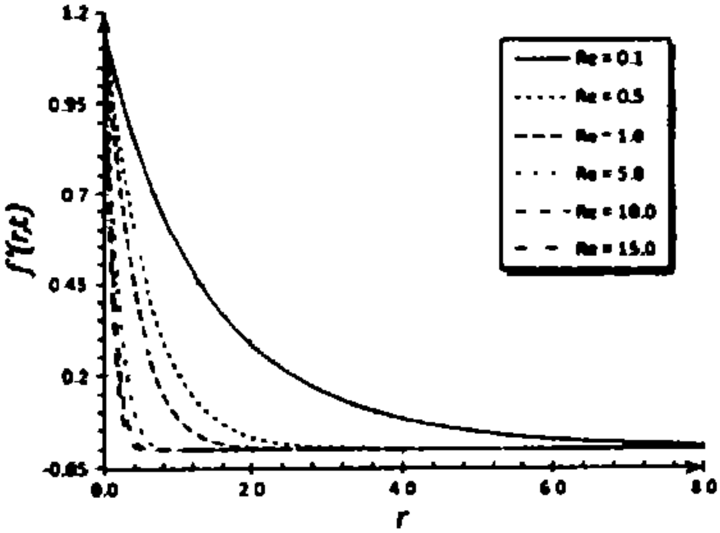

Figure 2-7 Effect of Re on the velocity profile $f'(\eta\ \tau)$ against τ

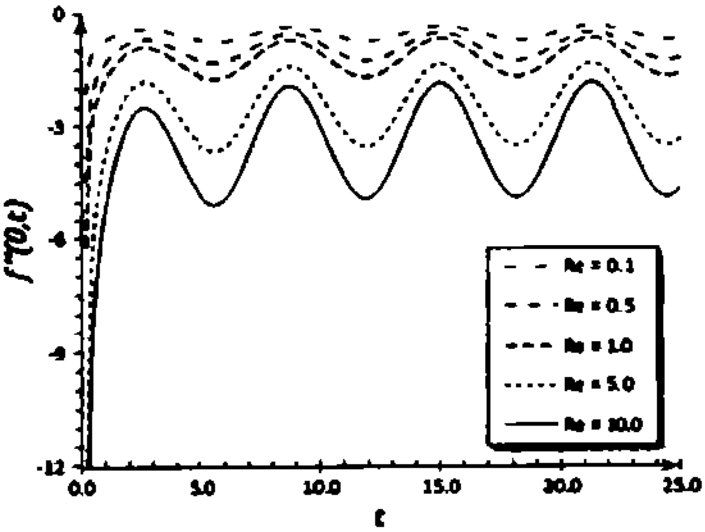

Figure 2-8 Effect of Re on the skin friction coefficient $f''(0\ \tau)$ against τ

surface This is because of the oscillatory nature of the cylinder's surface The variation of coefficient of skin friction due to the Strouhal number is depicted in Fig 2 5 The amplitude of oscillation is seen increasing upon increasing the value of St This is because of the reason that increased values of St correspond to high frequency which in terms enhances the momentum transfer in the boundary layer The role of amplitude parameters ϵ is some what trivial, upon increasing the values of ϵ the amplitude of the fluctuations in the coefficient of skin friction also increases as depicted in Fig 2 6 In both the figures 2 5 and 2 6, it can be seen that for small time values there is a great variations in the coefficient of skin friction but after certain time the curves attain the steady fluctuation state The role of Re on velocity profile and skin friction coefficient is depicted in Fig 2 7 and 2 8 respectively It is founded in Fig 2 7 that as Re increases the velocity profile decreases It is due to the fact that inertial force becomes more stronger than viscous force as Re increases From Fig 2 8 it is observed that the skin friction coefficient decreases as time increases This is because of the reason that initially fluid offer great resistance to the motion but with the passage of time, such kind of resistive forces decreases and motion of fluid becomes steady It is also seen that in Fig 2 8 that as Re increases the amplitude of oscillation in skin friction coefficient increases This is because of the reason that decreasing curvature which results increasing surface area that is in direct contact with the fluid

2.5 Conclusions

Unsteady boundary layer flow due to an oscillatory stretching of the cylinder has been considered in this chapter The impact of physical parameters such as the amplitude of oscillations the strouhal number and the curvature parameters have been observed on the velocity and skin friction graphs It is noted that the flow enhances in the boundary layer upon increasing the strouhal number and amplitude parameter ϵ

Chapter 3

Unsteady Boundary Layer Flow due to Uniform Stretching of a Circular Cylinder

3.1 Introduction

In this chapter, a study has been caried out to investigate unsteady boundary layer flow due to uniform stretching of a circular cylinder The problem is modeled mathematically and is solved numerically by using finite difference scheme explained in chapter 2 The effect of physical parameters like curvature parameter and dimensionless time on the velocity profile are examined through graph Moreover the results have been compared to the flate stretching surface having some values of curvature It is observed that the boundary layer thickness has more effect in case of cylinder having curvature parameter $\kappa = 1$ as compared to the flate strectching surface $\kappa = 0$

3.2 Mathematical Modeling of the Problem

Consider a long slim cylinder of infinite length continuously form of a slit and passes through the fluid It is assumed that the cylinder is being impulsively started to be stretching uniformly in the axial direction at $t > 0$ At $t = 0$ the surrounding fluid are assumed to be at rest The velocity of the stretching surface is described by

$$U_u = ax$$

In view of the above assumptions the governing system arises of the equation of continuity and momentum equation In cylindrical coordinates the continuity equation is given by

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 \tag{3 1}$$

and the momentum equation after the implications of boundary layer assumption read as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) \tag{3 2}$$

where u, v denotes the velocity component along x and y axes The appropriate boundary conditions for $t > 0$ are given by

$$u(x, r, t) = ax \text{ at } r = R, \tag{3 3}$$

$$v(x\ r, t) = 0 \text{ at } r = R, \tag{3 4}$$

$$u(x, r\ t) = 0 \text{ as } r \to \infty \tag{3 5}$$

The initial conditions defined at $t = 0$ read as

$$u(x, r, 0) = 0, \tag{3 6}$$

$$v(x, r, 0) = 0 \tag{3 7}$$

where R is the radius of the cylinder, a is the stretching rate, here the velocity components defined in terms of stream function are given by

$$u = \frac{1}{r}\frac{\partial\Psi}{\partial r}, \quad v = \frac{1}{r}\frac{\partial\Psi}{\partial x} \tag{3 8}$$

Above system of equations are transformed to dimensionless form by using following transformations

$$u = axf'(\eta),\ v = -\frac{1}{r}(a\nu)^{\frac{1}{2}}\xi^{\frac{1}{2}}Rf(\eta),\ \tau = at,\ \xi = 1 - e^{-\tau},\ \Psi = (ax^2\nu)^{\frac{1}{2}}R\xi^{\frac{1}{2}}f(\eta),$$
$$\eta = \frac{r^2 - R^2}{2R}\left(\frac{a}{\nu}\right)^{\frac{1}{2}}\xi^{-(\frac{1}{2})} \quad (3.9)$$

Our governing equations are transformed by using above mentioned transformations (3.8 & 3.9) into following nonlinear partial differential equations

$$\left(1 + 2\kappa\sqrt{\xi}\eta\right) f_{\eta\eta\eta} + [\frac{1}{2}\eta(1-\xi) + 2\kappa\sqrt{\xi}]f_{\eta\eta} + \xi[ff_{\eta\eta} - (f_\eta)^2] = \xi(1-\xi)\frac{\partial^2 f}{\partial\eta\partial\xi}, \quad (3.10)$$

where $\kappa = \sqrt{\frac{\nu}{aR^2}}$ denotes the curvature parameter. According to the boundary conditions also transform to the form

$$f(\eta) = 0 \text{ when } \eta = 0,\ \ f'(\eta) = 1 \text{ when } \eta = 0,\ f'(\eta) \to 0 \text{ when } \eta \to \infty \quad (3.11)$$

The governing equation show a good agreement between Rayleigh and Crane type of equation. It is a Rayleigh type of equation for large time and it is Crane type of equation for a small time.

3.3 Results and Discussion

The governing equations (3.10 and 3.11) have been solved by the finite difference scheme described in the previous chapter. The impact of the curvature parameter and the time variable has been observed and show graphically in Figure 3.1 and 3.2. In Fig 3.1 the velocity profile for different values of the time variable ξ have been plotted. Obviously, for small time values the flow develops in the boundary layer and establishes continuously with passage of time. For sufficiently large values of ξ the flow has fully been established and no further developments are observed for large values of time. Initially the surface skin friction is large enough which decreases with the passage of time and stabilizes for large value of time. This fact can be confirmed from Fig 3.2. Furthermore, the effect of curvature parameter κ on the coefficient of skin friction is also shown in Fig 3.2. Upon increasing the values of κ the skin friction coefficient

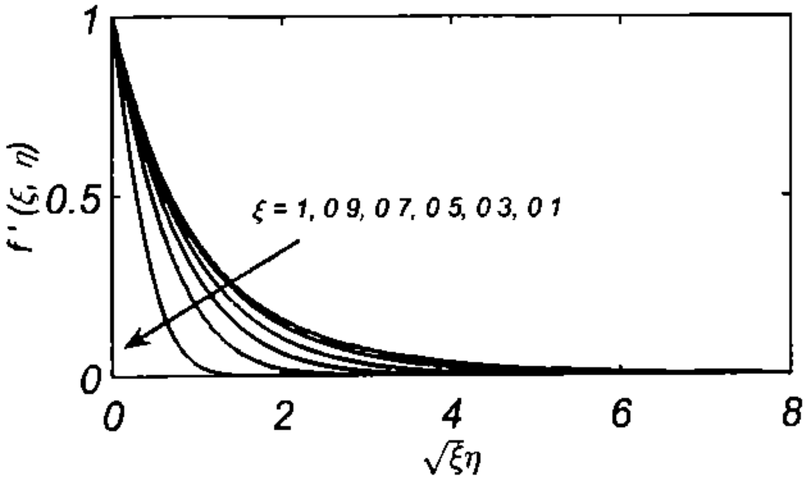

Figure 3-1 Effect of ξ on velocity profile $f'(\xi,\eta)$ against $\sqrt{\xi}\eta$

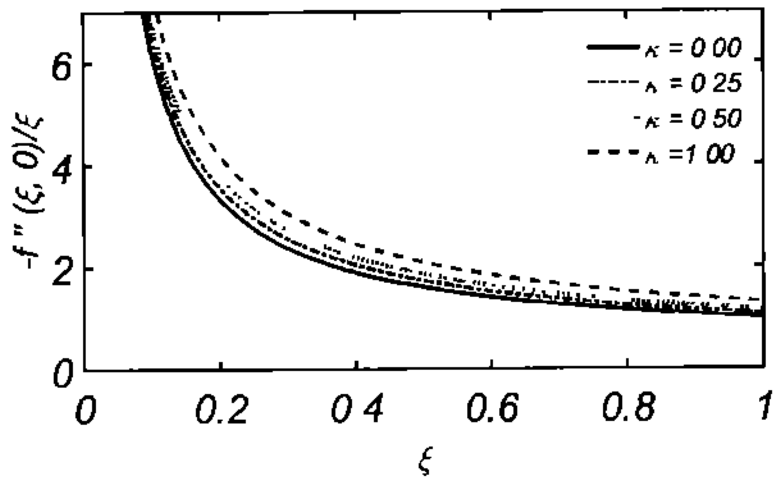

Figure 3-2 Effect of κ on the skin friction coefficient $-f''(\xi,0)/\xi$ against ξ

increases is depicted in Fig 3 2

3.4 Conclusions

Numerical solutions to the governing dimensionless partial differential equation and the associated boundary conditions have been obtained The results have been depicted graphically It is observed that the flow stables to develop for small time value and get established for large time values The coefficient of skin friction is high for a small times and decreases quickly at the initial stages Finally, when the steady state is achieved the coefficient of skin friction also gets stabilized for sufficiently large values of time

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