FIXED POINTS OF GENERALIZED CONTRACTIONS IN METRIC SPACES

By

Eskandar Ameer Abdullah Ahmed

DEPARTMENT OF MATHEMATICS & STATISTICS

INTERNATIONAL ISLAMIC UNIVERSITY

ISLAMABAD, PAKISTAN

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• Fixed point theory
• Metric spaces.

بسم الله الرحمن الرحيم

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ESKANDAR AMEER ABDULLAH AHMED

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***MASTER OF PHILOSOPHY IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS & STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD.

SUPERVISED BY

DR. MUHAMMAD ARSHAD ZIA

DEPARTMENT OF MATHEMATICS & STATISTICS

INTERNATIONAL ISLAMIC UNIVERSITYISLAMABAD, PAKISTAN

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Certificate

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We accept this dissertation as conforming to the required standard.

1. Prof. Dr. Moiz Ud Din
(*External Examiner*)

2. Dr. Tahir Mehmood
(*Internal Examiner*)

3. Dr. Muhammad Arshad Zia
(Supervisor)

4. Dr. Rahmat Ellah
(Chairman)
8/8/15

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2015

DECLARATION

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No portion of the work presented in this thesis has been submitted in support of an application for other degree or qualification of this or any other University or Institute of learning.

Eskandar Ameer Abdullah Ahmed
(154-FBAS/MSMA/F13)

DEDICATED TO....

"My Parents, Wife and Teachers".

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First and Foremost, all praise to '**ALLAH**', lord of the world, the almighty, who gave me the courage and patience to accomplish this task, and peace be upon his beloved apostle **Muhammad (S.A.W)**. May **ALLAH** guide us and the whole humanity to the right path.

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Eskandar Ameer Abdullah Ahmed
154-FBAS/MSMA/F13

CONTENTS

PREFACE

Fixed point theorems deal with the assurance that a mapping T on a set X has one or more fixed points, i.e., the functional equation $x = Tx$ has one or more solutions. A large variety of the problems of analysis and applied mathematics relate to finding solutions of nonlinear functional equations which can be formulated in terms of finding the fixed point of a nonlinear mappings. In fact, fixed point theorems are extremely substantial tools for proving the existence and uniqueness of the solutions to various mathematical models (differential, integral and partial differential equations and variational inequalities etc.) existing phenomena arising in broad spectrum of fields, such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, functional analysis, epidemics, biomedical research and flow of fluids etc.

The Banach fixed point theorem is commonly known as Banach contraction principle, which states that if X is a complete metric space and T a single-valued contraction self mapping on X, then T has a unique fixed point in X. This theorem looks simple but plays a fundamental role in the field of fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. Subsequently many authors generalized the Banach fixed point theorem in different way (see[1-20,22-61]) and the references therein. .

Following the Banach contraction principle Nadler [47] introduced the concept of set valued contractions and established that a set valued contraction possesses a fixed point in a complete metric space.

Jachymski et al. [31] established a result which generalized the Banach contraction principle for graphs, Beg et al. [14],[15] extended some results of [31] by defining G-contraction for multi-valued mappings. Kirk et al. [5] proved some remarks which was based on the idea of a metric transform and extended Nadler's theorem.

In 2000, Branciari [18] introduced the concept of generalized metric spaces, where the triangle inequality is replaced by the inequality $d(x,y) \le d(x,u) + d(u,v) + d(v,y)$ for all pairwise distinct points $x, y, u, v \in X$. Various fixed point results were established on such spaces, see ([3],[20],[33],[34],[43],[44],[45],[46],[58],[60]) and the references therein.

In 2012 Wardowski [61] introduced a new type of contraction called F-contraction and prove a new fixed point theorem concerning F-contraction.

Secelean [57] showed that the condition $(F2)$ in definition of F-contraction introduced by Wardowski [61] can be replaced by condition $(F2')$ or $(F2'')$, Piri et al. [50] described a large class of functions by replacing condition $(F3')$ instead of the condition $(F3)$ in the definition of F-contraction introduced by Wardowski [61]. Cosentino et al. [19] presented some fixed point results for F-contraction of Hardy-Rogers-type for single-valued mappings on complete metric spaces, Sgroi et al. [56] established fixed point theorems for multi-valued F-contractions of Hardy-Rogers-type for multi-valued mappings on complete metric spaces.

More recently Hussain et. al. [25], introduced α-GF-contractions and obtained fixed point results in metric spaces and partially ordered metric spaces. They also established Suzuki type results for such GF-contractions.

The thesis is divided into four chapters.

Chapter 1, is essentially an introduction, where we fix notations and terminologies to be used. It is a survey aimed at recalling some basic definitions and facts. While some of the classical and recent results about fixed point existence are also presented in this chapter.

Chapter 2, deals with some new fixed point theorems concerning metric transforms for uniform local multivalued graph contractions in complete metric spaces with a graph.

Chapter 3, is devoted to the study of Hardy-Rogers-Type fixed point theorems for generalized F-contractions in complete metric spaces.

Chapter 4, concerned with the study of fixed point results of generalized contractions on generalized metric space to extend the idea of Jleli et al. [33],[34].

Chapter 1

Preliminaries

The aim of this chapter is to present basic concepts and to explain the terminology used through out this dissertation. Some previously known results are given without proof in order to keep the chapter with reasonable length. Section 1.1 deals with some basic concepts. In section 1.2, we present the notion of Hausdorff metric on the family of non-empty closed bounded subsets of a metric space. Section 1.3 concerns with the concept of metric transform and fixed point results concerning metric transforms. In section 1.4 the terminology of graphs and related notions are given. In section 1.5 we present the concept of generalized metric space which is a generalization of metric space and recall some fixed point theorems on generalized metric spaces in the related literature. In section 1.6 the notion of F-contraction and fixed point theorems concerning F-contractions.

1.1 Some basic concepts

Throughout the thesis we shall denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers, by $\mathbb{N}$ the set of all positive integers. For a nonempty set X, we shall denote by $N(X)$ the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $B(X)$ the class of all non empty bounded subsets of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X.

Definition1.1.1 [39] Let (X,d) be a metric space. A point $x \in X$ is said to be a fixed point of mapping $T: X \to X$ if $x = Tx$.

In 1922, Banach gave the following useful definition of contraction.

Theorem 1.1.2 [16] Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be a contraction mapping (i.e $\forall x, y \in X,\ d(Tx,Ty) \leq kd(x,y)$, where $k \in (0,1)$), then T has a unique fixed point.

Definition 1.1.3 [55] Let $T: X \to X$ and $\alpha: X \times X \to [0, +\infty)$. We say that T is α-admissible if $x, y \in X$, $\alpha(x,y) \geq 1$ implies that $\alpha(Tx,Ty) \geq 1$.

Definition 1.1.4 [54] Let $T: X \to X$ and $\alpha, \eta: X \times X \to [0, +\infty)$ two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x,y) \geq \eta(x,y)$ implies that $\alpha(Tx,Ty) \geq \eta(Tx,Ty)$.

If $\eta(x,y) = 1$, then above Definition reduces to Definition 1.1.3. If $\alpha(x,y) = 1$, then T is called an η-subadmissible mapping.

Definition 1.1.5 [28] Let (X,d) be a metric space. Let $T: X \to X$ and $\alpha, \eta: X \times X \to [0, +\infty)$ be two functions. We say that T is $\alpha - \eta$-continuous mapping on (X,d) if for given $x \in X$, and sequence $\{x_n\}$ with

$$x_n \to x \text{ as } n \to \infty,\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow Tx_n \to Tx.$$

Definition 1.1.6 [29] Let (X,d) be a metric space, $T: X \to CL(X)$ be a given closed-valued multifunction and $\alpha: X \times X \longrightarrow [0, +\infty)$. We say that T is called α_*-admissible whenever $\alpha(x,y) \geq 1$ implies that $\alpha_*(Tx,Ty) \geq 1$.

Definition 1.1.7 [30] Let $T: X \to CL(X)$ be a multifunction, $\alpha, \eta: X \times X \to [0, +\infty)$ be two functions where η is bounded. We say that T is α_*-admissible mapping with respect to η if $\alpha(x,y) \geq$

$\eta(x,y)$ implies $\alpha_*(Tx,Ty) \geq \eta_*(Tx,Ty)$, $x,y \in X$, where $\alpha_*(A,B) = \inf\{\alpha(x,y) : x \in A,\ y \in B\}$ and $\eta_*(A,B) = \sup\{\eta(x,y) : x \in A,\ y \in B\}$.

If $\eta(x,y) = 1$ for all $x,y \in X$, then this Definition reduces to Definition 1.1.6. In the case $\alpha(x,y) = 1$ for all $x,y \in X$, T is called η_*-subadmissible mapping.

Definition 1.1.8 [2] Let (X,d) be a metric space. Let $T : X \to CL(X)$ and $\alpha : X \times X \to [0,+\infty)$ be two functions. We say that T is α-continuous multivalued mapping on $(CL(X),H)$ if for given $x \in X$, and sequence $\{x_n\}$ with $\lim\limits_{n\longrightarrow\infty} d(x_n,x) = 0$, $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \Longrightarrow \lim\limits_{n\longrightarrow\infty} H(Tx_n,Tx) = 0$.

Theorem 1.1.9 [22] Let (X,d) be a metric space and $T : X \to X$ be a self mapping. Assume that

$$d(Tx,Ty) < d(x,y), \text{ holds for all } x,y \in X \text{ with } x \neq y.$$

Then T has a unique fixed point in X.

Definition 1.1.10 [8] Let X be a non-empty set, T be a self- mapping on X, and $\alpha,\beta : X \longrightarrow [0,\infty)$ be two mappings. We say that T is a cyclic (α,β)-admissible mapping if

$$x \in X,\ \ \alpha(x) \geq 1 \Longrightarrow \beta(Tx) \geq 1,$$

and

$$x \in X,\ \ \beta(x) \geq 1 \Longrightarrow \alpha(Tx) \geq 1.$$

1.2 Hausdorff metric

Hausdorff metric is a measure of the resemblance of two sets (of geometric points). Let (X,d) be a metric space. For $x \in X$ and $A,B \subseteq X$, we denote $\rho(A,B) = \sup\limits_{x\in A} D(x,B)$ and $D(x,A) = \inf\{d(x,y) : y \in A\}$, Let H be the Hausdorff metric induced by the metric d on X, that is

$$H(A,B) = \max\{\rho(A,B),\rho(B,A)\}, \quad \text{for } A,B \in CB(X).$$

A point $x \in X$ is said to be a fixed point of mapping $T: X \to CB(X)$ if $x \in Tx$.

Definition 1.2.1 [47] A mapping $T: X \longrightarrow CB(X)$ is called a multivalued contraction mapping if there exists a number $k \in (0,1)$ such that

$$H(Tx, Ty) \leqslant kd(x,y), \qquad x, y \in X.$$

Theorem 1.2.2 [47] Let (X,d) be a complete metric space and suppose $T: X \longrightarrow CB(X)$ be a multivalued contraction mapping. Then T has a fixed point.

Definition 1.2.3 [47] A metric space (X,d) is called a ϵ-chainable metric space for some $\epsilon > 0$ if given $x, y \in X$, there is $n \in \mathbb{N}$ and a sequence $(x_i)_{i=0}^{n}$ such that

$$x_0 = x, \quad x_n = y \text{ and } d\left(x_{i-1}, x_i\right) < \epsilon \ \text{ for } i = 1, 2, ..., n.$$

We shall require the following well known facts due to definition of H.

Lemma 1.2.4 [47] Let $A, B \in CB(X)$ with $a \in A$. If $\epsilon > 0$ then there exists an element $b \in B$ such that $d(a,b) \leqslant H(A,B) + \epsilon$.

Lemma 1.2.5 [6] Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim\limits_{n \longrightarrow \infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$and $\lim\limits_{n \longrightarrow \infty} d(x_n, x) = 0$, then $x \in A$.

One example of a metric on $CB(X)$ which is metrically equivalent to the Hausdorff metric H is the metric H^+. Which was introduced in [36]. The metric H^+ is defined by setting

$$H^+(A,B) = \frac{1}{2}\left(\rho(A,B) + \rho(B,A)\right), \quad \text{for } A, B \in CB(X).$$

Definition 1.2.6 [37] Let (X,d) be a metric space. A multivalued mapping $T: X \longrightarrow CB(X)$ is called H^+- type multivalued weak contractive if

(1) there exists $k \in (0,1)$ such that

$$H^{+}(Tx,Ty) \leqslant k \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

for all $x, y \in X$,

(2) if for every x in X, y in Tx, $\epsilon > 0$, there exists z in Ty such that

$$d(y,z) \leqslant H^{+}(Ty,Tx) + \epsilon.$$

Theorem 1.2.7 [37] Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ an H^{+}-type multivalued weak contractive mapping. Then T has a fixed point.

1.3 Metric transform

Blumenthal [12],[13] introduced the concept of metric transforms.

Definition 1.3.1 Astrictly increasing concave function $\phi : [0,\infty) \to \mathbb{R}$ for which $\phi(0) = 0$ is called a metric transform.

Remark [12] If (X,d) is a metric space and if $\rho(x,y) = \phi(d(x,y))$ for each $x, y \in X$, where ϕ is a metric transform, then (X,ρ) is also a metric space.

Definition 1.3.2 [47] A mapping $T : X \longrightarrow CB(X)$ is said to be an (ϵ,k)-uniform local multivalued contraction(where $\epsilon > 0$ and $k \in (0,1)$) if for

$$x, y \in X,\ d(x,y) < \epsilon \Longrightarrow H(Tx,Ty) \leqslant kd(x,y).$$

Recently Kirk et al. [5] proved some remarks which was based on the idea of a metric transform and extended Nadler's theorem as follows.

Theorem 1.3.3 [5] Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$. Suppose there exists a metric transform ϕ on X and $k \in (0, 1)$ such that the following conditions hold:

$a)$ for each $x, y \in X$,

$$\phi(H(Tx, Ty)) \leqslant kd(x, y),$$

$b)$ there exists $c \in (0, 1)$ such that for $t > 0$ sufficiently small,

$$kt \leqslant \phi(ct).$$

Then for $\epsilon > 0$ sufficiently small, T is an (ϵ, c)-uniform local multivalued contraction on (X, d).

Theorem 1.3.4 [5] Let (X, d) be a complete and connected metric space. If $T : X \longrightarrow CB(X)$ is an (ϵ, k)-uniform local multivalued contraction, then T has a fixed point.

1.4 Metric spaces endowed with a graph

Consider a directed graph G such that the set of its vertices coincides with X (i.e, $V(G) = X$) and the set of its edge $E(G) = \{(x, y) \in X \times X, x \neq y\}$ .We assume that G has no parallel edge and weighted graph by assigning to each edge the distance between the vertices. For details about definitions in graph theory, see([21]) . We can identify G as $(V(G), E(G))$. G^{-1} denotes the conversion of a graph G, the graph obtained from G by reversing the direction of its edges. $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edge of G. We consider $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, thus we have

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Definition 1.4.1 A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and for any edge $(x, y) \in E(H)$, $x, y \in V(H)$. The number of edge in G constituting the path is called the length of the path.

Definition 1.4.2 A graph G is connected if there is a path between any two vertices of G. If a graph G is not connected, then it is called disconnected. Moreover, G is weakly connected if $\widetilde{G}$ is connected. Assume that G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices, which are contained in some path in G beginning at x, is called the component of G containing x. In this case the equivalence class $[x]_G$ defined on $V(G)$ by the rule $R(uRv$ if there is a path from u to $v)$ is such that $V(G_x) = [x]_G$.

Definition 1.4.3 Let x and y be vertices in a graph G. A path in G from x to y of length $n(n \in \mathbb{N} \cup \{0\})$ is a sequence $(x_i)_{i=0}^n$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, ..., n$.

Jachymski proved the following well known Banach contraction principle for graphs.

Theorem 1.4.4 [31] We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, i.e.,

$$\forall x\ , y \in X((x,y) \in E(G) \Rightarrow (T(x),T(y)) \in E(G))$$

and T decreases weights of edges of G in the following way:

$$\exists k \in (0,1), \forall x\ , y \in X((x,y) \in E(G) \Rightarrow d(T(x),T(y)) \leq kd(x,y)).$$

Definition 1.4.5 [31] A mapping $T : X \to X$ is called G-continuous, if given $x \in X$ and sequence $\{x_n\}$,

$$x_n \to x \,\text{as}\, n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \,\text{for all}\, n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$

Property A [31]: For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \longrightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$.

Property B [32]: For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \longrightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Beg et al. [14],[15] obtained sufficient condition for the existence of a fixed point of a multivalued graph contraction mapping and common fixed points for multivalued graph contractive mappings in metric spaces endowed with a graph G.

Definition 1.4.6 [14] The mapping $T : X \longrightarrow CB(X)$ is said to be a graph contraction(G-contraction) if there exists a $k \in (0,1)$ such that

$$H(Tx, Ty) \leqslant kd(x,y) \quad \text{for all } (x,y) \in E(G),$$

and if $u \in Tx$ and $v \in Ty$ are such that

$$d(u,v) \leqslant kd(x,y) + \alpha \quad \text{for each } \alpha > 0$$

then $(u,v) \in E(G)$.

Theorem 1.4.7 [14] Let (X,d) be a complete metric space and suppose that the triple (X,d,G) has the property A. Let $T : X \longrightarrow X$ be a G-contraction and $X_T := \{x \in X : (x, Tx) \in E(G)\}$. Then the following statements hold:

1.for any $x \in X_T, T \mid [x]_{\tilde{G}}$ has a fixed point,

2.if $X_T \neq \emptyset$ and G is weakly connected, then T has a fixed point,

3.if $X' = \cup \left\{[x]_{\tilde{G}} : x \in X_T\right\}$, then$T \mid X'$ has a fixed point,

4.if $T \subseteq E(G)$ then T has a fixed point,

5.$Fix\,(T) \neq \emptyset$ if and only if $X_T \neq \emptyset$.

1.5 Generalized metric space

Definition 1.5.1 [18] Let X be a non-empty set and $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0 \Longleftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then (X, d) is called a generalized metric space(or for short g.m.s).

Definition 1.5.2 Let (X, d) be a g.m.s, $\{x_n\}$ be a sequence in X and $x \in X$, we say that $\{x_n\}$ is convergent to x if and only if $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$. We denote this by $x_n \longrightarrow x$.

Definition 1.5.3 Let (X, d) be a g.m.s and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Definition 1.5.4 Let (X, d) be a g.m.s. We say that (X, d) is complete if and only if every Cauchy sequence in X converges to some element in X.

Lemma 1.5.5 [3] Let (X, d) be a g.m.s and $\{x_n\}$ be a Cauchy sequence in (X, d) such that $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for some $x \in X$. Then $d(x_n, y) \longrightarrow d(x, y)$ as $n \longrightarrow \infty$ for all $y \in X$. In particular, $\{x_n\}$ does not converge to y if $y \neq x$.

Lemma 1.5.6 [35] Let (X, d) be a g.m.s and $\{x_n\}$ be a Cauchy sequence in (X, d) and $x, y \in X$. Suppose that there exists a positive integer N such that

(i) $x_n \neq x_m$ for all $n, m > N$;

(ii) x_n and x are distinct points in X for all $n > N$;

(iii) x_n and y are distinct points in X for all $n > N$;

(iv) $\lim\limits_{n \longrightarrow \infty} d(x_n, x) = \lim\limits_{n \longrightarrow \infty} d(x_n, y)$.

Then we have $x = y$.

We denote by Θ the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

(Θ1) θ is non-decreasing,

(Θ2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim\limits_{n \longrightarrow \infty} \theta(t_n) = 1$ if and only if $\lim\limits_{n \longrightarrow \infty} t_n = 0^+$,

(Θ3) there exists $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim\limits_{t \longrightarrow 0+} \frac{\theta(t)-1}{t^r} = \ell$.

Theorem 1.5.7 [33] Let (X, d) be a complete g.m.s and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k .$$

Then T has a unique fixed point.

Example 1.5.8 [33] The following functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ are elements of Θ :

(1) $\theta(t) = e^{\sqrt{t}}$,

(2) $\theta(t) = e^{\sqrt{te^t}}$,

(3) $\theta(t) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\gamma}\right), \;\; 0 < \gamma < 1, \; t > 0.$

Theorem 1.5.9 [34] Let (X, d) be a complete g.m.s and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ is continuous and $k \in (0, 1)$ such that

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^k ,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\} .$$

Then T has a unique fixed point.

1.6 F-contractions

Definition 1.6.1 [61] Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall x,y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F\left(d(Tx,Ty)\right) \leq F\left(d(x,y)\right), \tag{1.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote by $\mathcal{F}$, the set of all functions satisfying the conditions (F1)-(F3).

Example 1.6.2 [61] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfied (F1)-(F3) ((F3) for any $k \in (0,1)$. Each mapping $T : X \to X$ satisfying (1.1) is an F-contraction such that

$$d(Tx,Ty) \leq e^{-\tau} d(x,y), \text{ for all } x,y \in X,\ Tx \neq Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \leq e^{-\tau} d(x,y)$, also holds, i.e. T is a Banach contraction.

Example 1.6.3 [61] If $F(r) = \ln r + r$, $r > 0$ then F satisfies (F1)-(F3) and the condition (1.1) is of the form

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau}, \text{ for all } x,y \in X,\ Tx \neq Ty.$$

Remark 1.6.4 [61] From (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

Theorem 1.6.5 [61] Let (X,d) be a complete metric space and let $T : X \to X$ be an F- contraction.

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.

Definition 1.6.6 [19] Let (X,d) be a metric space. a mapping $T : X \longrightarrow X$ is called an F-contraction of Hardy-Rogers-type if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F\left(d(Tx,Ty)\right) \leq F\left(\kappa d\left(x,y\right) + \beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right) + \delta d\left(x,Ty\right) + Ld(y,Tx)\right), \qquad (1.2)$$

for all $x, y \in X$ with $d\left(Tx,Ty\right) > 0$, where $\kappa, \beta, \gamma, \delta, L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

Theorem 1.6.7 [19] Let (X,d) be a complete metric space and let $T : X \longrightarrow X$. Assume there exists $F \in \mathcal{F}$ and $\tau > 0$ such that T is an F-contraction of Hardy-Rogers-type, that is

$$\tau + F\left(d(Tx,Ty)\right) \leq F\left(\kappa d\left(x,y\right) + \beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right) + \delta d\left(x,Ty\right) + Ld(y,Tx)\right),$$

for all $x, y \in X$ with $d\left(Tx,Ty\right) > 0$, where $\kappa, \beta, \gamma, \delta, L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then T has a fixed point. Moreover, if $\kappa + \delta + L \leq 1$, then the fixed point of T is unique.

Theorem 1.6.8 [56] Let (X,d) be a complete metric space and let $T : X \longrightarrow CB(X)$. Assume there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$2\tau + F\left(H(Tx,Ty)\right) \leq F\left(\begin{array}{c} \kappa d\left(x,y\right) + \beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right) + \\ \delta d\left(x,Ty\right) + Ld(y,Tx) \end{array}\right), \qquad (1.3)$$

for all $x, y \in X$ with $Tx \neq Ty$, where $\kappa, \beta, \gamma, \delta, L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then T has a fixed point.

Hussain et al. [26] introduced a family of functions as follows.

Let Δ_G denotes the set of all functions $G : \mathbb{R}^{+4} \to \mathbb{R}^+$ satisfying:

(G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 t_2 t_3 t_4 = 0$ there exists $\tau > 0$ such that

$G(t_1, t_2, t_3, t_4) = \tau$.

Example 1.6.9 [25] If $G(t_1,t_2,t_3,t_4) = \tau e^{v\min\{t_1,t_2,t_3,t_4\}}$ where $v \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Delta_G$.

Definition 1.6.10 [25] Let (X,d) be a metric space and T be a self mapping on X. Also suppose that $\alpha,\eta : X \times X \to [0,+\infty)$ be two functions. We say that T is α-η-GF-contraction if for $x,y \in X$, with $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$ we have

$$G(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)) + F\left(d(Tx,Ty)\right) \leq F\left(d(x,y)\right), \tag{1.4}$$

where $G \in \Delta_G$ and $F \in \mathcal{F}$.

On the other hand Secelean [57] proved the following lemma.

Lemma 1.6.11 [57] Let $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be an increasing map and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:

(a) if $\lim\limits_{n\longrightarrow\infty} F(\alpha_n) = -\infty$ then $\lim\limits_{n\longrightarrow\infty} \alpha_n = 0$;

(b) if $\inf F = -\infty$ and $\lim\limits_{n\longrightarrow\infty} \alpha_n = 0$, then $\lim\limits_{n\longrightarrow\infty} F(\alpha_n) = -\infty$.

By proving Lemma 1.6.11, Secelean [57] showed that the condition $(F2)$ in Definition 1.6.1 can be replaced by an equivalent but a more simple condition,

$(F2')$ $\inf F = -\infty$

or, also, by

$(F2'')$ there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim\limits_{n\longrightarrow\infty} F(\alpha_n) = -\infty$.

Recently Piri [50] replaced the following condition instead of the condition $(F3)$ in Defintion 1.6.1.

$(F3')$ F is continuous on $(0,\infty)$.

We denote by $\Delta_{\mathcal{F}}$ the set of all functions satisfying the conditions $(F1)$, $(F2')$ and $(F3')$.

For $p \geq 1$, $F(\alpha) = -\frac{1}{\alpha^p}$ satisfies in $(F1)$ and $(F2)$ but it does not apply in $(F3)$ while satisfy conditions $(F1)$, $(F2)$ and $(F3')$. Therefore $\Delta_{\mathcal{F}} \nsubseteq \mathcal{F}$. Again, for $a > 1$, $t \in \left(0,\frac{1}{a}\right)$, $F(\alpha) = \frac{-1}{(\alpha+[\alpha])^t}$, where $[\alpha]$ denotes the integral part of α, satisfies the condition $(F1)$ and $(F2)$ but it does not satisfy

$(F3')$, while it satisfies the condition $(F3)$ for any $k \in \left(\frac{1}{a}, 1\right)$. Therefore $\mathcal{F} \not\subseteq \Delta_{\mathcal{F}}$. Also, if we take $F(\alpha) = \ln \alpha$, then $F \in \mathcal{F}$ and $F \in \Delta_{\mathcal{F}}$. Therefore, $\Delta_{\mathcal{F}} \cap \mathcal{F} \neq \emptyset$.

Theorem 1.6.12 [50] Let T be a self-mapping of a complete metric space X into itself. Suppose $F \in \Delta_{\mathcal{F}}$ and there exists $\tau > 0$ such that

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right).$$

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.

Chapter 2

Fixed point in metric spaces with a graph

Jachymski et al. [31] established a result which generalized the Banach contraction principle for graphs, Beg et al. [14],[15] extended results of Jachymski et al. [31] by defining G-contraction for multi-valued mappings. Kirk et al. [5] proved some remarks which was based on the idea of a metric transform and extended Nadler's theorem.

In this chapter, we extend some results of Kirk et al.[5] on a metric space endowed with a graph.

2.1 Fixed point results on a metric space with a graph

In this section we introduce the notion of an uniform local multivalued graph contractions on a metric space endowed with a directed graph G. we also prove some new fixed point theorems concerning metric transforms for such contractions.

We start this section with the definition of an (ϵ, k)-uniform local multivalued graph contraction.

Definition 2.1.1 Let (X, d) be a metric space with a graph G, a mapping $T : X \longrightarrow CB(X)$ is said to be an (ϵ, k)-uniform local multivalued graph contraction (where $\epsilon > 0$ and $k \in (0, 1)$) if for every $(x,y) \in E(G)$,

$$d(x,y) < \epsilon \Longrightarrow H(Tx,Ty) \leqslant kd(x,y),$$

and if $u \in Tx$ and $v \in Ty$ are such that

$$d(u,v) \leq kd(x,y) + \alpha \text{ for each } \alpha > 0.$$

Then $(u,v) \in E(G)$.

Theorem 2.1.2 Let (X,d) be a complete metric space with graph G such that G is weakly connected, the triple (X,d,G) has the property A and $T : X \longrightarrow CB(X)$ be an (ϵ,k)-uniform local multivalued graph contraction on (X,d) and $X_T = \{x \in X : (x,Tx) \in E(G)\ \} \neq \emptyset$. Then T has fixed point.

Proof. Let $x_0 \in X_T$, then there exists $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$. Since T is an $(\epsilon,k)-$uniform local multivalued graph contraction on (X,d), so there exists $\epsilon > 0$, $k \in (0,1)$ and for $d(x_0,x_1) < \epsilon$, we have

$$H(Tx_0,Tx_1) \leqslant kd(x_0,x_1).$$

Using Lemma 1.2.4, we have $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1,x_2) &\leqslant H(Tx_0,Tx_1) + k \\ &\leqslant kd(x_0,x_1) + k. \end{aligned}$$

Again because of T is an $(\epsilon,k)-$uniform local multivalued graph contraction, $(x_1,x_2) \in E(G)$, $d(x_1,x_2) < \epsilon$, we have

$$H(Tx_1,Tx_2) \leqslant kd(x_1,x_2).$$

Lemma 1.2.4 gives the existence of an $x_3 \in Tx_2$ such that,

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) + k^2 \\ &\leq kd(x_1, x_2) + k^2 \\ &\leq k^2 d(x_0, x_1) + 2k^2. \end{aligned}$$

Continuing in this manner, we have $x_{n+1} \in Tx_n$ such that $(x_n, x_{n+1}) \in E(G)$, $d(x_n, x_{n+1}) < \epsilon$, and

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) + nk^n.$$

Now for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots\dots\dots\dots + d(x_{m-1}, x_m) \\ &\leq d(x_0, x_1) \sum_{i=n}^{m-1} k^i + \sum_{i=n}^{m-1} ik^i. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X and X is complete, so $\{x_n\}$ converges to a point x in X. Now, we prove x is fixed point of T. By using property A, we deduce

$$(x_n, x) \in E(G) \text{ for } n \in \mathbb{N}.$$

Now since T is an(ϵ, k)-uniform local multivalued graph contraction, for $n \in \mathbb{N}$, $d(x_n, x) < \epsilon$, we have,

$$H(Tx_n, Tx) \leqslant kd(x_n, x).$$

Since $x_{n+1} \in Tx_n$ and $x_n \longrightarrow x$. Therefore by Lemma 1.2.5, $x \in Tx$. Next as $(x_n, x) \in E(G)$ for $n \in \mathbb{N}$, G is weakly connected, we infer that $(x_0, x_1, x_2, \dots, x_n, x)$ is a path in G and so $x \in X = [x_0]_{\tilde{G}}$.

Now we present simple condition in terms of metric transforms which implies that a mapping

$T : X \longrightarrow CB(X)$ is an (ϵ, k)-uniform local multivalued graph contraction on (X, d). Notice that if ϕ is taken to be the identity mapping, the following result reduces to the definition of an (ϵ, k)-uniform local multivalued graph contraction.

Theorem 2.1.3 Let (X, d) be a metric space and endowed with graph G and $T : X \longrightarrow CB(X)$. Set $X_T := \{x \in X : (x, Tx) \in E(G)\} \neq \emptyset$. Suppose there exists a metric transform ϕ on X and $k \in (0,1)$ such that the following conditions hold:

$a)$ for each $(x, y) \in E(G)$,

$$\phi(H(Tx, Ty)) \leqslant kd(x, y),$$

$b)$ there exists $c \in (0,1)$ such that for $t > 0$ sufficiently small,

$$kt \leqslant \phi(ct),$$

$c)$ for $u \in Tx$ and $v \in Ty$ if

$$d(u, v) \leq kd(x, y) + \alpha \quad \text{for each } \alpha > 0.$$

Then $(u, v) \in E(G)$.

Then, for $\epsilon > 0$ sufficiently small, T is an (ϵ, c)-uniform local multivalued graph contraction on (X, d).

Proof. Let $x \in X_T$, then there exists $y \in Tx$ such that $(x, y) \in E(G)$, from (a) we observe that

$$\phi(H(Tx, Ty)) \leqslant kd(x, y).$$

Suppose there exists $c \in (0,1)$, such that for t sufficiently small,we have form (b)

$$kt \leq \phi(ct).$$

Then for $d(x,y)$ sufficiently small,

$$\phi(H(Tx,Ty)) \leq kd(x,y) \leq \phi(cd(x,y)).$$

Since ϕ is strictly increasing. This implies that

$$H(Tx,Ty) \leqslant cd(x,y).$$

Thus from condition (c) and previous inequality, for $\epsilon > 0$ sufficiently small, T is an (ϵ, c)-uniform local multivalued graph contraction on (X,d).

Theorem 2.1.4 If, in addition to the assumptions of theorem 2.1.3, X is complete, G is weakly connected and the triple (X,d,G) has the property A, then T has a fixed point.

Example 2.1.5 Consider $X = \left\{0, \frac{1}{2}, 1\right\} = V(G)$ to be a subset of $\mathbf{R}$ with the usual metric defined as $d(x,y) = |x-y|$, so that (X,d) is a complete metric space and $E(G) = \left\{(1,\frac{1}{2}),(0,\frac{1}{2})\right\}$ is such that $\Delta \subseteq E(G)$ and let $T : X \longrightarrow CB\,(X)$ defined as

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ \{0, \frac{1}{2}\} & \text{if } x = \frac{1}{2} \\ \{\frac{1}{2}\} & \text{if } x = 1. \end{cases}$$

Also consider a metric transform

$$\phi(t) = \frac{t}{t+1}, \ \ t \in [0, \infty).$$

Since $1 \in X$ is such that there exists $\frac{1}{2} \in T\,(1)$ with $\left(1, \frac{1}{2}\right) \in E\,(G)$, then $X_T \neq \emptyset$. We see that for each $(x,y) \in E\,(G)\,,\ \phi\,(H\,(Tx,Ty)) < kd\,(x,y)\,.$ Indeed, if $(x,y) = (1, \frac{1}{2})$, we have

$$H\left(T(1), T\left(\frac{1}{2}\right)\right) = H\left(\left\{\frac{1}{2}\right\}, \left\{0, \frac{1}{2}\right\}\right) = \frac{1}{2}.$$

This implies

$$\phi\left(H\left(T(1),T\left(\frac{1}{2}\right)\right)\right)=\phi\left(\frac{1}{2}\right)=0.66d\left(1,\frac{1}{2}\right)\leq kd\left(1,\frac{1}{2}\right),\text{ where } k=0.66.$$

Next if $(x,y)=(0,\frac{1}{2})$, we have

$$H\left(T(0),T\left(\frac{1}{2}\right)\right)=H\left(\{0\},\left\{0,\frac{1}{2}\right\}\right)=\frac{1}{2}.$$

This implies

$$\phi\left(H\left(T(0),T\left(\frac{1}{2}\right)\right)\right)=\phi\left(\frac{1}{2}\right)=0.66d\left(0,\frac{1}{2}\right)\leq kd\left(0,\frac{1}{2}\right),\text{ where } k=0.66.$$

Thus the condition (a) is satisfied. Let $k\in(0,1)$ and select $c\in(k,1)$. Then

$$kt\leq\phi\left(ct\right)\Longleftrightarrow t\leq\frac{\frac{ct}{1+ct}}{k}\Longleftrightarrow kt\leq\frac{ct}{1+ct}\Longleftrightarrow k\leq\frac{c}{1+ct}\Longleftrightarrow t\leq\frac{c-k}{ck}.$$

Since $c>k$, then condition (b) is also satisfied. It is easy to check that condition (c) is satisfied. Therefore all assumptions of Theorem 2.1.4 are satisfied and clearly 0 and $\frac{1}{2}$ are fixed point of T.

Remark 2.1.6 If we assume G is such that $E(G)=X\times X$, then clearly Theorem 2.1.4 gives Kirk's result [5](Theorem 1.3.4).

We now introduce the concept of H^{+}-type multivalued weak graph contraction mappings in metric space endowed with a graph G.

Definition 2.1.7 Let (X,d) be a metric space with graph G. A multivalued mapping $T:X\longrightarrow CB(X)$ is called H^{+}-type multivalued weak graph contraction if

(1) there exists $k\in(0,1)$ such that

$$H^{+}(Tx,Ty)\leq k\max\left\{d\left(x,y\right),d\left(x,Tx\right),d\left(y,Ty\right),\frac{d\left(x,Ty\right)+d\left(y,Tx\right)}{2}\right\}$$

for all $(x,y) \in E(G)$,

(2) if for every x in X, y in Tx, $\epsilon > 0$ there exists z in Ty such that

$$d(y,z) \leqslant H^{+}(Ty,Tx) + \epsilon, \text{ then } (y,z) \in E(G).$$

Theorem 2.1.8 Let (X,d) be a complete metric space with graph G such that G is weakly connected, the triple (X,d,G) has the property A and $T : X \longrightarrow CB(X)$ be an H^{+}-type multivalued weak graph contraction mapping,

$$X_T = \{x \in X : (x,u) \in E(G) \text{ for some } u \in Tx\} \neq \emptyset.$$

Then T has a fixed point.

Proof. Let $\epsilon > 0$ be given, let $x_0 \in X_T$. Fix an element $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$. Since T is H^{+}-type multivalued weak graph contraction, we have

$$H^{+}(Tx_0,Tx_1) \leq k \max\left\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Tx_1), \frac{d(x_0,Tx_1)+d(x_1,Tx_0)}{2}\right\}.$$

We can select $x_2 \in Tx_1$ such that

$$\begin{aligned}
d(x_1,x_2) &\leqslant H^{+}(Tx_0,Tx_1) + \epsilon \\
&\leq k \max\left\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Tx_1), \frac{d(x_0,Tx_1)+d(x_1,Tx_0)}{2}\right\} + \epsilon \\
&= k \max\left\{d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_2)}{2}\right\} + \epsilon \\
&\leq k \ \max\left\{d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}\right\} + \epsilon \\
&= k \max\{d(x_0,x_1), d(x_1,x_2)\} + \epsilon.
\end{aligned}$$

If $\max\{d(x_0,x_1), d(x_1,x_2)\} = d(x_1,x_2)$, then it is a contradiction. Therefore

$$\max\{d(x_0,x_1), d(x_1,x_2)\} = d(x_0,x_1),$$

which implies,

$$d(x_1,x_2) \le kd(x_0,x_1) + \epsilon.$$

This implies $(x_1,x_2) \in E(G)$, we have

$$\begin{aligned} H^+(Tx_1,Tx_2) &\le k\max\left\{d(x_1,x_2), d(x_1,Tx_1), d(x_2,Tx_2), \frac{d(x_1,Tx_2)+d(x_2,Tx_1)}{2}\right\} \\ &= kd(x_1,x_2). \end{aligned}$$

Similarly there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\leqslant H^+(Tx_1,Tx_2) + \epsilon \\ &\le kd(x_1,x_2) + \epsilon \\ &\le k^2 d(x_0,x_1) + \epsilon. \end{aligned}$$

Continuing in this way, we have $x_{n+1} \in Tx_n$ such that $(x_n,x_{n+1}) \in E(G)$,

$$d(x_n,x_{n+1}) \leqslant k^n d(x_0,x_1) + \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Set $\epsilon = (k^{\frac{n}{2}} - k^n)d(x_0,x_1)$.Then from previous inequality it follows that

$$\begin{aligned} d(x_n,x_{n+1}) &\leqslant k^n d(x_0,x_1) + (k^{\frac{n}{2}} - k^n)d(x_0,x_1) \\ &= k^{\frac{n}{2}} d(x_0,x_1). \end{aligned}$$

Which implies

$$d(x_n, x_{n+1}) \leqslant k^{\frac{n}{2}} d(x_0, x_1).$$

It is clear that $\{x_n\}$ is bounded. In deed, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_0, x_n) &\leqslant \sum_{i=0}^{i=n} d(x_i, x_{i+1}) \\ &\leq \left(1 + k^{\frac{1}{2}} + k^{\frac{2}{2}} + k^{\frac{3}{2}} + \dots + k^{\frac{n}{2}}\right) d(x_0, x_1) \\ &< \left(1 + k^{\frac{1}{2}} + k^{\frac{2}{2}} + k^{\frac{3}{2}} + \dots\right) d(x_0, x_1) \\ &= \frac{1}{1 - k^{\frac{1}{2}}} d(x_0, x_1) < \infty. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. since X is complete, there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} x_n = x$. Now we prove that x is a fixed point of the mapping T. Assume that $d(x, Tx) > 0$. By using the property A and the fact of T being a H^+-type multivalued weak graph contraction, we have $(x_n, x) \in E(G)$,

$$\begin{aligned} \frac{1}{2}\{\rho(Tx_n, Tx) + \rho(Tx, Tx_n)\} &= H^+(Tx_n, Tx) \\ &\leq k \max\left\{d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2}\right\} \\ &\leq k \max\left\{d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x_n, Tx) + d(x, x_{n+1})}{2}\right\}, \end{aligned}$$

it follows that

$$\frac{1}{2} \lim_{n \longrightarrow \infty} \inf\{\rho(Tx_n, Tx) + \rho(Tx, Tx_n)\} \leq k d(x, Tx).$$

Since $\lim_{n \longrightarrow \infty} \inf d(x_{n+1}, x) = 0$ exists, and

$$d(x, Tx) = \frac{1}{2}(d(x, Tx) + d(Tx, x)) \leq \frac{1}{2}(\rho(Tx_n, Tx) + \rho(Tx, Tx_n)) + d(x_{n+1}, x),$$

it follows that

$$\begin{aligned} d(x,Tx) &\leqslant \frac{1}{2}\lim_{n\longrightarrow\infty}\inf\left(\rho(Tx_n,Tx)+\rho(Tx,Tx_n)\right)+\lim_{n\longrightarrow\infty}\inf d(x_{n+1},x) \\ &\leqslant kd\left(x,Tx\right)+\lim_{n\longrightarrow\infty}d(x_{n+1},x)=kd\left(x,Tx\right)<d\left(x,Tx\right), \end{aligned}$$

a contradiction. This implies that $d(x,Tx)=0$ and Tx is closed. Hence $x\in Tx$. Next as $(x_n,x)\in E(G)$ for $n\in\mathbb{N}$, G is weakly connected, we infer that $(x_0,x_1,x_2,.....,x_n,x)$ is a path in G and so $x\in X=[x_0]_{\widetilde{G}}$.

Chapter 3

Hardy-Rogers-Type fixed point theorems for generalized F-contractions

In 2012, Wardowski [61] introduced a new type of contraction called F-contraction and prove a new fixed point theorem concerning F-contraction, Piri et al. [50] described a large class of functions by replacing condition $(F3')$ instead of the condition $(F3)$ in the definition of F-contraction introduced by Wardowski [61]. Cosentino et al. [19] presented some fixed point results for F-contraction of Hardy-Rogers-type for single-valued mappings on complete metric spaces. Sgroi et al. [56] established fixed point theorems for multi-valued F-contractions of Hardy-Rogers-type for multi-valued mappings on complete metric spaces. More recently Hussain et. al.[25], introduced α-η-GF-contractions and obtained fixed point results in metric spaces and partially ordered metric spaces. They also established Suzuki type results for such GF-contractions.

The aim of this chapter is to extend the concept of F-contraction into an α-η-GF-contraction of Hardy-Rogers-type for single-valued, multi-valued mappings. We also establish some new Hardy-Rogers-Type fixed point results for α-η-GF-contraction, multi-valued α-η-GF-contraction in complete metric spaces.

3.1 Hardy-Rogers-Type fixed point results for α-GF-Contractions

In this section we establish fixed point theorems for α-η-GF-contraction of Hardy-Rogers-type for single-valued mappings in a complete metric space. We start this section with the definition of α-η-GF-contraction of Hardy-Rogers-type.

Definition 3.1.1 Let (X,d) be a metric space and T be a self mapping on X. Also suppose that $\alpha,\eta : X \times X \to [0,+\infty)$ be two functions. We say that T is an α-η-GF-contraction of Hardy-Rogers-type if for $x,y \in X$, with $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$, we have

$$\begin{aligned} & G\left(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right) + F\left(d(Tx,Ty)\right) \\ \leq \quad & F\left(\kappa d\left(x,y\right) + \beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right) + \delta d\left(x,Ty\right) + L d\left(y,Tx\right)\right), \end{aligned} \tag{3.1}$$

where $G \in \Delta_G$, $F \in \Delta_{\mathcal{F}}$, $\kappa, \beta, \gamma, \delta, L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

Theorem 3.1.2: Let (X,d) be a complete metric space. Let T be a self mapping satisfying the following assertions:

(i) T is an α-admissible mapping with respect to η;

(ii) T is an α-η-GF-contraction of Hardy-Rogers-type;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$;

(iv) T is $\alpha - \eta$-continuous.

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x,y \in Fix(T)$ and $\kappa + \delta + L \leq 1$.

proof. Let $x_0 \in X$, such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$. Continuing this process, $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now since, T is an α-admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, Tx_{n_0}) = 0$, then x_{n_0} is fixed point of T and, there is nothing to prove. So, we assume that

$$x_n \neq x_{n+1} \text{ or } d(Tx_{n-1}, Tx_n) > 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Since, T is an α-η-GF-contraction of Hardy-Rogers-type, we have

$$\begin{aligned} & G\begin{pmatrix} d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}) \end{pmatrix} + F\left(d(Tx_{n-1}, Tx_n)\right) \\ \leq \; & F\begin{pmatrix} \kappa d(x_{n-1}, x_n) + \beta d\left(x_{n-1}, Tx_{n-1}\right) + \gamma d\left(x_n, Tx_n\right) + \\ \delta d\left(x_{n-1}, Tx_n\right) + Ld\left(x_n, Tx_{n-1}\right) \end{pmatrix} \end{aligned}$$

which implies

$$\begin{aligned} & G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) \\ & +F\left(d(Tx_{n-1}, Tx_n)\right) \\ \leq \; & F\begin{pmatrix} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, Tx_{n-1}\right) + \gamma d\left(x_n, Tx_n\right) + \\ \delta d\left(x_{n-1}, Tx_n\right) + Ld\left(x_n, Tx_{n-1}\right) \end{pmatrix}. \end{aligned} \tag{3.4}$$

Now since, $d(x_{n-1}, x_n).d(x_n, x_{n+1}).d(x_{n-1}, x_{n+1}).0 = 0$, so from (G) there exists $\tau > 0$ such that,

$$G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) = \tau.$$

From (3.4), we deduce that

$$\begin{aligned} & \tau + F\left(d\left(Tx_{n-1}, Tx_n\right)\right) \\ \leq \; & F\left(\begin{array}{c} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, Tx_{n-1}\right) + \gamma d\left(x_n, Tx_n\right) + \\ \delta d\left(x_{n-1}, Tx_n\right) + Ld\left(x_n, Tx_{n-1}\right) \end{array} \right). \end{aligned}$$

This implies

$$\begin{aligned} & F\left(d\left(Tx_{n-1}, Tx_n\right)\right) \\ \leq \; & F\left(\begin{array}{c} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, Tx_{n-1}\right) + \gamma d\left(x_n, Tx_n\right) + \\ \delta d\left(x_{n-1}, Tx_n\right) + Ld\left(x_n, Tx_{n-1}\right) \end{array} \right) - \tau \\ = \; & F\left(\begin{array}{c} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, x_n\right) + \gamma d\left(x_n, x_{n+1}\right) + \\ \delta d\left(x_{n-1}, x_{n+1}\right) + Ld\left(x_n, x_n\right) \end{array} \right) - \tau \\ \leq \; & F\left(\begin{array}{c} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, x_n\right) + \gamma d\left(x_n, x_{n+1}\right) + \\ \delta d\left(x_{n-1}, x_n\right) + \delta d\left(x_n, x_{n+1}\right) \end{array} \right) - \tau \\ = \; & F\left(\left(\kappa + \beta + \delta\right) d\left(x_{n-1}, x_n\right) + \left(\gamma + \delta\right) d\left(x_n, x_{n+1}\right)\right) - \tau \end{aligned}$$

and hence

$$F\left(d\left(Tx_{n-1}, Tx_n\right)\right) < F\left(\left(\kappa + \beta + \delta\right) d\left(x_{n-1}, x_n\right) + \left(\gamma + \delta\right) d\left(x_n, x_{n+1}\right)\right).$$

Since F is strictly increasing, we get

$$d\left(Tx_{n-1}, Tx_n\right) < \left(\kappa + \beta + \delta\right) d\left(x_{n-1}, x_n\right) + \left(\gamma + \delta\right) d\left(x_n, x_{n+1}\right).$$

This implies

$$\left(1 - \gamma - \delta\right) d\left(Tx_{n-1}, Tx_n\right) < \left(\kappa + \beta + \delta\right) d\left(x_{n-1}, x_n\right), \text{ for all } n \in \mathbb{N}.$$

From $\kappa+\beta+\gamma+2\delta=1$ and $\gamma\neq 1$, we deduce that $1-\gamma-\delta>0$ and so

$$d\left(Tx_{n-1},Tx_n\right)<\frac{(\kappa+\beta+\delta)}{(1-\gamma-\delta)}d\left(x_{n-1},x_n\right)=d\left(x_{n-1},x_n\right),\text{ for all } n\in\mathbb{N}.$$

Consequently

$$F\left(d\left(Tx_{n-1},Tx_n\right)\right)\leq F\left(d(x_{n-1},x_n)\right)-\tau. \tag{3.5}$$

Continuing this process, we get

$$\begin{aligned}
F\left(d\left(Tx_{n-1},Tx_n\right)\right) &\leq F\left(d(x_{n-1},x_n)\right)-\tau\\
&= F\left(d(Tx_{n-2},Tx_{n-1})\right)-\tau\\
&\leq F\left(d(x_{n-2},x_{n-1})\right)-2\tau\\
&= F\left(d(Tx_{n-3},Tx_{n-2})\right)-2\tau\\
&\leq F\left(d(x_{n-3},x_{n-2})\right)-3\tau\\
&\;\;\vdots\\
&\leq F\left(d(x_0,x_1)\right)-n\tau.
\end{aligned}$$

This implies that

$$F\left(d\left(Tx_{n-1},Tx_n\right)\right)\leq F\left(d(x_0,x_1)\right)-n\tau. \tag{3.6}$$

And so $\lim_{n\longrightarrow\infty}F\left(d\left(Tx_{n-1},Tx_n\right)\right)=-\infty$, which together with $(F2')$ and Lemma 1.6.11 gives that

$$\lim_{n\to\infty}d(x_n,Tx_n)=0. \tag{3.7}$$

Now , we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we have that there exists $\epsilon > 0$ and sequence $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n,\ d(x_{p(n)}, x_{q(n)}) \geq \epsilon,\ d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \text{ for all } n \in \mathbb{N}. \tag{3.8}$$

So, we have

$$\begin{aligned} \epsilon &\leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &\leq d(x_{p(n)}, x_{p(n)-1}) + \epsilon \\ &= d(x_{p(n)-1}, Tx_{p(n)-1}) + \epsilon. \end{aligned} \tag{3.9}$$

Letting $n \longrightarrow \infty$ in (3.9) and using (3.7), we obtain

$$\lim_{n \longrightarrow \infty} d(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{3.10}$$

Also, from (3.7) there exists a natural number $n_1 \in \mathbb{N}$ such that

$$d(x_{p(n)}, Tx_{p(n)}) < \frac{\epsilon}{4} \text{ and } d(x_{q(n)}, Tx_{q(n)}) < \frac{\epsilon}{4}, \text{ for all } n \geq n_1. \tag{3.11}$$

Next, we claim that

$$d(Tx_{p(n)}, Tx_{q(n)}) = d(x_{p(n)+1}, x_{q(n)+1}) > 0, \text{ for all } n \geq n_1. \tag{3.12}$$

Arguing by contradiction, there exists $m \geq n_1$ such that

$$d(x_{p(m)+1}, x_{q(m)+1}) = 0. \tag{3.13}$$

It follows from (3.8), (3.11) and (3.13) that

$$\begin{aligned}
\epsilon &\leq d(x_{p(m)}, x_{q(m)}) \leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)}) \\
&\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, x_{q(m)}) \\
&= d(x_{p(m)}, Tx_{p(m)}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)}, Tx_{q(m)}) \\
&< \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4}.
\end{aligned}$$

This contradiction establishes the relation (3.12). Hence, it follows from (3.12) and (3.1) that

$$\begin{aligned}
& G\begin{pmatrix} d\left(x_{p(n)}, Tx_{p(n)}\right), d\left(x_{q(n)}, Tx_{q(n)}\right), \\ d\left(x_{p(n)}, Tx_{q(n)}\right), d\left(x_{q(n)}, Tx_{p(n)}\right) \end{pmatrix} + F\left(d\left(Tx_{P(n)}, Tx_{q(n)}\right)\right) \\
\leq\ & F\begin{pmatrix} \kappa d\left(x_{p(n)}, x_{q(n)}\right) + \beta d\left(x_{p(n)}, Tx_{p(n)}\right) + \gamma d\left(x_{q(n)}, Tx_{q(n)}\right) + \\ \delta d\left(x_{p(n)}, Tx_{q(n)}\right) + Ld\left(x_{q(n)}, Tx_{p(n)}\right) \end{pmatrix},
\end{aligned}$$

for all $n \geq n_1$. Now since, $0.d\left(x_{q(n)}, Tx_{q(n)}\right).d\left(x_{p(n)}, Tx_{q(n)}\right).d\left(x_{q(n)}, Tx_{p(n)}\right) = 0$, so from (G) there exists $\tau > 0$ such that,

$$G(0, d\left(x_{q(n)}, Tx_{q(n)}\right), d\left(x_{p(n)}, Tx_{q(n)}\right), d\left(x_{q(n)}, Tx_{p(n)}\right)) = \tau.$$

Therefore,

$$\begin{aligned}
& \tau + F\left(d\left(Tx_{P(n)}, Tx_{q(n)}\right)\right) \\
\leq\ & F\begin{pmatrix} \kappa d\left(x_{p(n)}, x_{q(n)}\right) + \beta d\left(x_{p(n)}, Tx_{p(n)}\right) + \gamma d\left(x_{q(n)}, Tx_{q(n)}\right) + \\ \delta d\left(x_{p(n)}, Tx_{q(n)}\right) + Ld\left(x_{q(n)}, Tx_{p(n)}\right) \end{pmatrix}
\end{aligned} \tag{3.14}$$

By using $(F3')$, (3.7), (3.10) and (3.14), we have

$$\tau + F(\epsilon) \leq F((\kappa + \delta + L)\epsilon) = F(\epsilon).$$

This contradiction show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$

converges to some point x in X. Since T is an α-η-continuous and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$, then $x_{n+1} = Tx_n \to Tx$ as $n \to \infty$. That is, $x = Tx$. Hence x is a fixed point of T. Let $x, y \in Fix(T)$ where $x \neq y$, then from

$$\begin{aligned} & G(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) + F(d(Tx,Ty)) \\ \leq \; & F(\kappa d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx)) \\ = \; & F((\kappa + \delta + L) d(x,y)) \end{aligned}$$

we get,

$$\tau + F(d(x,y)) \leq F((\kappa + \delta + L) d(x,y)),$$

which is a contradiction, if $\kappa + \delta + L \leq 1$ and hence $x = y$.

Theorem 3.1.3 Let (X, d) be a complete metric space. Let T be a self mapping on X satisfying the following assertions:

(i) T is an α-admissible mapping with respect to η;

(ii) T is an α-η-GF-contraction of Hardy-Rogers-Type;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n),$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$ and $\kappa + \delta + L \leq 1$.

proof. As similar lines of the Theorem 3.1.2, we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \to x \text{ as } n \to \infty$$

where $Tx_n = x_{n+1}$. By (iv), either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n),$$

holds for all $n \in \mathbb{N}$. This implies

$$\alpha(x_{n+1}, x) \geq \eta(x_{n+1}, x_{n+2}) \text{ or } \alpha(x_{n+2}, x) \geq \eta(x_{n+2}, x_{n+3}).$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x)$$

and from (3.1), we deduce that

$$\begin{aligned} & G(d(x_{n_k}, Tx_{n_k}), d(x, Tx), d(x_{n_k}, Tx), d(x, Tx_{n_k})) + F\left(d(Tx_{n_k}, Tx)\right) \\ \leq \quad & F\left(\kappa d\left(x_{n_k}, x\right) + \beta d\left(x_{n_k}, Tx_{n_k}\right) + \gamma d\left(x, Tx\right) + \delta d\left(x_{n_k}, Tx\right) + Ld(x, Tx_{n_k})\right). \end{aligned}$$

This implies

$$F\left(d(Tx_{n_k}, Tx)\right) < F\begin{pmatrix} \kappa d\left(x_{n_k}, x\right) + \beta d\left(x_{n_k}, x_{n_k+1}\right) + \\ \gamma d\left(x, Tx\right) + \delta d\left(x_{n_k}, Tx\right) + Ld(x, x_{n_k+1}) \end{pmatrix}. \tag{3.15}$$

From $(F1)$ we have

$$\begin{aligned} & d(x_{n_k+1}, Tx) < \\ & \kappa d\left(x_{n_k}, x\right) + \beta d\left(x_{n_k}, x_{n_k+1}\right) + \gamma d\left(x, Tx\right) + \delta d\left(x_{n_k}, Tx\right) + Ld(x, x_{n_k+1}). \end{aligned} \tag{3.16}$$

By taking the limit as $k \to \infty$ in (3.16), we obtain

$$d(x, Tx) < (\gamma + \delta)\, d(x, Tx) < d(x, Tx), \tag{3.17}$$

Which implies $d(x, Tx) = 0$, thus x is a fixed point of T. Uniqueness follows similarly as in theorem 3.1.2.

Theorem 3.1.4 Let (X, d) be a complete metric space and T be a continuous selfmapping on X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have

$$\begin{aligned} & G(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) + F\left(d(Tx,Ty)\right) \\ \leq \quad & F\left(\kappa d\left(x,y\right) + \beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right) + \delta d\left(x,Ty\right) + L d(y,Tx)\right), \end{aligned}$$

where $G \in \Delta_G$, $F \in \Delta_{\mathcal{F}}$, $\kappa, \beta, \gamma, \delta, L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$, $\kappa + \delta + L \leq 1$ and $\gamma \neq 1$. Then T has a unique fixed point.

proof. Let us define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = d(x, y) \text{ and } \eta(x, y) = d(x, y) \text{ for all } x, y \in X.$$

Now, since $d(x, y) \leq d(x, y)$ for all $x, y \in X$, so $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 3.1.2 hold true. Since T is continuous, so T is α-η-continuous. Let $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have $d(x, Tx) \leq d(x, y)$ with $d(Tx, Ty) > 0$, then

$$\begin{aligned} & G(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) + F\left(d(Tx,Ty)\right) \\ \leq \quad & F\left(\kappa d\left(x,y\right) + \beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right) + \delta d\left(x,Ty\right) + L d(y,Tx)\right). \end{aligned}$$

That is, T is an α-η-GF-contraction of Hardy-Rogers-type. Hence, all conditions of Theorem 3.1.2 satisfied and T has a unique fixed point.

Corollary 3.1.5 Let (X,d) be a complete metric space and T be a continuous selfmapping on X. If for $x,y\in X$ with $d(x,Tx)\leq d(x,y)$ and $d(Tx,Ty)>0$, we have

$$\tau+F\left(d(Tx,Ty)\right)\leq F\left(\kappa d\left(x,y\right)+\beta d\left(x,Tx\right)+\gamma d\left(y,Ty\right)+\delta d\left(x,Ty\right)+Ld(y,Tx)\right),$$

where $\tau>0$, $\kappa,\beta,\gamma,\delta,L\geq 0$, $\kappa+\beta+\gamma+2\delta=1$, $\kappa+\delta+L\leq 1$, $\gamma\neq 1$ and $F\in\Delta_{\mathcal{F}}$. Then T has a unique fixed point.

Corollary 3.1.6 Let (X,d) be a complete metric space and T be a continuous selfmapping on X. If for $x,y\in X$ with $d(x,Tx)\leq d(x,y)$ and $d(Tx,Ty)>0$, we have

$$\begin{aligned}&\tau e^{v\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}}+F\left(d(Tx,Ty)\right)\\ \leq\;& F\left(\left(\kappa d\left(x,y\right)+\beta d\left(x,Tx\right)+\gamma d\left(y,Ty\right)+\delta d\left(x,Ty\right)+Ld(y,Tx)\right)\right),\end{aligned}$$

where $\tau>0$, $\kappa,\beta,\gamma,\delta,L,v\geq 0$, $\kappa+\beta+\gamma+2\delta=1$, $\kappa+\delta+L\leq 1$, $\gamma\neq 1$ and $F\in\Delta_{\mathcal{F}}$. Then T has a unique fixed point.

Example 3.1.7 Let $S_n=\frac{n(n+1)(n+2)}{3}$, $n\in\mathbb{N}$, $X=\{S_n:n\in\mathbb{N}\}$ and $d(x,y)=|x-y|$. Then (X,d) is a complete metric space. Define the mapping $T:X\longrightarrow X$ by, $T(S_1)=S_1$ and $T(S_n)=S_{n-1}$, for all $n>1$and $\alpha(x,y)=1$, $\eta(x,y)=\frac{1}{2}$, $G(t_1,t_2,t_3,t_4)=\tau$ where $\tau=\frac{7}{2}>0$. Since $\lim\limits_{n\longrightarrow\infty}\frac{d(T(S_n),T(S_1))}{d(S_n,S_1)}=\lim\limits_{n\longrightarrow\infty}\frac{S_{n-1}-2}{S_n-2}=\frac{(n-1)n(n+1)-6}{n(n+1)(n+2)-6}=1$, T is not Banach contraction. Clearly $\alpha(S_1,T(S_1))\geq\eta(S_1,T(S_1))$ and T is an α-η-continuous. Let $\alpha(S_m,S_n)\geq\eta(S_m,S_n)$ for all $m,n\in\mathbb{N}$, then $\alpha(TS_m,TS_n)\geq\eta(TS_m,TS_n)$. That is, T is an α-admissible mapping with respect to η. On the other hand taking $F(r)=\frac{-1}{r}+r\in\Delta_{\mathcal{F}}$, we obtain the result that T is an α-η-GF-contraction of Hardy-Rogers-type with $\kappa=\beta=\frac{1}{3}$, $\gamma=\frac{1}{6}$, $\delta=\frac{1}{12}$ and $L=\frac{7}{12}$. To see this, let us consider the following calculation. We conclude the following three cases:

Case 1:

for every $m \in \mathbb{N}, m > n = 1$, we have

$$\begin{aligned}
|T(S_m) - T(S_1)| &= |S_1 - T(S_m)| = |S_{m-1} - S_1| = 2 \times 3 + 3 \times 4 + ... + (m-1)m, \\
|S_m - S_1| &= 2 \times 3 + 3 \times 4 + ... + m(m+1), \\
|S_m - T(S_m)| &= |S_m - S_{m-1}| = m(m+1), \\
|S_1 - T(S_1)| &= |S_1 - S_1| = 0.
\end{aligned}$$

Since $m > 1$ and

$$\begin{aligned}
&\frac{-1}{2 \times 3 + ... + (m-1)m} \\
< \; &\frac{-1}{\left[\begin{array}{c} \frac{1}{3}(2 \times 3 + ... + m(m+1)) + \frac{1}{3}m(m+1) + \\ \frac{1}{12}(2 \times 3 + ... + m(m+1)) + \frac{7}{12}(2 \times 3 + ... + (m-1)m) \end{array}\right]},
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{7}{2} - \frac{1}{2 \times 3 + 3 \times 4 + ... + (m-1)m} + [2 \times 3 + 3 \times 4 + ... + (m-1)m] \\
< \; &\frac{7}{2} - \frac{1}{\left[\begin{array}{c} \frac{1}{3}(2 \times 3 + ... + m(m+1)) + \frac{1}{3}m(m+1) + \\ \frac{1}{12}(2 \times 3 + ... + m(m+1)) + \frac{7}{12}(2 \times 3 + ... + (m-1)m) \end{array}\right]} + \\
&[2 \times 3 + 3 \times 4 + ... + (m-1)m] \\
\leq \; &-\frac{1}{\left[\begin{array}{c} \frac{1}{3}(2 \times 3 + .. + m(m+1)) + \frac{1}{3}m(m+1) + \\ \frac{1}{12}(2 \times 3 + ... + m(m+1)) + \frac{7}{12}(2 \times 3 + ... + (m-1)m) \end{array}\right]} + \\
&\left[\begin{array}{c} \frac{1}{3}(2 \times 3 + ... + m(m+1)) + \frac{1}{3}m(m+1) + \\ \frac{1}{12}(2 \times 3 + ... + m(m+1)) + \frac{7}{12}(2 \times 3 + ... + (m-1)m) \end{array}\right].
\end{aligned}$$

So, we get

$$
\begin{aligned}
& \frac{7}{2}-\frac{1}{|T(S_m)-T(S_1)|}+|T(S_m)-T(S_1)| \\
< \; & -\frac{1}{\frac{1}{3}|S_m-S_1|+\frac{1}{3}|S_m-T(S_m)|+\frac{1}{6}|S_1-T(S_1)|+\frac{1}{12}|S_m-T(S_1)|+\frac{7}{12}|S_1-T(S_m)|}+ \\
& \left[\frac{1}{3}|S_m-S_1|+\frac{1}{3}|S_m-T(S_m)|+\frac{1}{6}|S_1-T(S_1)|+\frac{1}{12}|S_m-T(S_1)|+\frac{7}{12}|S_1-T(S_m)|\right].
\end{aligned}
$$

Case 2:

for $1 \leq m < n$, similar to case 1.

Case 3:

for $m > n > 1$, we have

$$
\begin{aligned}
|T(S_m)-T(S_n)| &= n\times(n+1)+(n+1)(n+2)+...+(m-1)m, \\
|S_m-S_n| &= (n+1)(n+2)+(n+2)(n+3)+...+m(m+1), \\
|S_m-T(S_m)| &= |S_m-S_{m-1}| = m(m+1), \\
|S_n-T(S_n)| &= |S_n-S_{n-1}| = n(n+1), \\
|S_m-T(S_n)| &= |S_m-S_{n-1}| = n(n+1)+...+m(m+1), \\
|S_n-T(S_m)| &= |S_n-S_{m-1}| = (n+1)(n+2)+...+(m-1)m.
\end{aligned}
$$

Since $m > n > 1$, and

$$
\begin{aligned}
& \frac{-1}{n\times(n+1)+(n+1)(n+2)+...+(m-1)m} \\
< \; & \frac{-1}{\left[\begin{array}{c}\frac{1}{3}((n+1)(n+2)+...+m(m+1))+\frac{1}{3}m(m+1)+\frac{1}{6}n(n+1)+ \\ \frac{1}{12}(n(n+1)+...+(m-1)m)+\frac{7}{12}((n+1)(n+2)+...+(m-1)m)\end{array}\right]}.
\end{aligned}
$$

Therefore

$$\frac{7}{2}-\frac{1}{n\times(n+1)+(n+1)(n+2)+...+(m-1)m}+$$
$$[n\times(n+1)+(n+1)(n+2)+...+(m-1)m]$$

$$<\frac{7}{2}-\frac{1}{\left[\begin{array}{c}\frac{1}{3}((n+1)(n+2)+...+m(m+1))+\frac{1}{3}m(m+1)+\frac{1}{6}n(n+1)+\\ \frac{1}{12}(n(n+1)+...+(m-1)m)+\frac{7}{12}((n+1)(n+2)+...+(m-1)m)\end{array}\right]}+$$

$$[n\times(n+1)+(n+1)(n+2)+...+(m-1)m]$$

$$\leq\ -\frac{1}{\left[\begin{array}{c}\frac{1}{3}((n+1)(n+2)+...+m(m+1))+\frac{1}{3}m(m+1)+\frac{1}{6}n(n+1)+\\ \frac{1}{12}(n(n+1)+...+(m-1)m)+\frac{7}{12}((n+1)(n+2)+...+(m-1)m)\end{array}\right]}+$$
$$\left[\begin{array}{c}\frac{1}{3}((n+1)(n+2)+...+m(m+1))+\frac{1}{3}m(m+1)+\\ \frac{1}{6}n(n+1)+\frac{1}{12}(n(n+1)+...+(m-1)m)+\frac{7}{12}((n+1)(n+2)+...+(m-1)m)\end{array}\right].$$

So, we get

$$\frac{7}{2}-\frac{1}{|T(S_m)-T(S_n)|}+|T(S_m)-T(S_n)|$$
$$<\ -\frac{1}{\frac{1}{3}|S_m-S_n|+\frac{1}{3}|S_m-T(S_m)|+\frac{1}{6}|S_n-T(S_n)|+\frac{1}{12}|S_m-T(S_n)|+\frac{7}{12}|S_n-T(S_m)|}+$$
$$\left[\frac{1}{3}|S_m-S_n|+\frac{1}{3}|S_m-T(S_m)|+\frac{1}{6}|S_n-T(S_n)|+\frac{1}{12}|S_m-T(S_n)|+\frac{7}{12}|S_n-T(S_m)|\right].$$

Therefore

$$\frac{7}{2}+F(d(T(S_m),T(S_n)))$$
$$\leq\ F\left(\frac{1}{3}d(S_m,S_n)+\frac{1}{3}d(S_m,T(S_m))+\frac{1}{6}d(S_n,T(S_n))+\frac{1}{12}d(S_m,T(S_n))+\frac{7}{12}d(S_n,T(S_m))\right).$$

for all $m,n\in\mathbb{N}$. Hence all condition of Theorem 3.1.2 are satisfied, T has a unique fixed point (here, S_1 is fixed point of T).

3.2 Hardy-Rogers-Type fixed point results for multivalued α-GF-Contractions

We start this section with the following definitions.

Definition 3.2.1 Let (X, d) be a metric space, $T : X \to CB(X)$ be a given multifunction and $\alpha : X \times X \longrightarrow [0, +\infty)$. We say that T is called α_*-admissible whenever $\alpha(x, y) \geq 1$ implies that $\alpha_*(Tx, Ty) \geq 1$.

Definition 3.2.2 Let $T : X \to CB(X)$ be a multifunction, $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions where η is bounded. We say that T is α_*-admissible mapping with respect to η if $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty)$, $x, y \in X$, where $\alpha_*(A, B) = \inf\{\alpha(x, y) : x \in A,\ y \in B\}$ and $\eta_*(A, B) = \sup\{\eta(x, y) : x \in A,\ y \in B\}$.

If $\eta(x, y) = 1$ for all $x, y \in X$, then this Definition reduces to Definition 3.2.1.

Definition 3.2.3 Let (X, d) be a metric space. Let $T : X \to CB(X)$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is $\alpha - \eta$-continuous multivalued mapping on $(CB(X), H)$ if for given $x \in X$, and sequence $\{x_n\}$ with $\lim\limits_{n \longrightarrow \infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \Longrightarrow \lim\limits_{n \longrightarrow \infty} H(Tx_n, Tx) = 0$.

Definition 3.2.4 Let (X, d) be a metric space and $T{:}X \longrightarrow CB(X)$. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is a multivalued α-η-GF-contraction of Hardy-Rogers-type if for $x, y \in X$, with $\eta(x, y) \leq \alpha(x, y)$ and $Tx \neq Ty$ we have

$$\begin{aligned} & 2G\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right) + F\left(H(Tx, Ty)\right) \qquad (3.18) \\ \leq\ & F\left(\kappa d\left(x, y\right) + \beta d\left(x, Tx\right) + \gamma d\left(y, Ty\right) + \delta d\left(x, Ty\right) + L d\left(y, Tx\right)\right), \end{aligned}$$

where $G \in \Delta_G$, $F \in \mathcal{F}$, $\kappa, \beta, \gamma, \delta, L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

Theorem 3.2.5 Let (X, d) be a complete metric space. Let $T : X \longrightarrow CB(X)$ satisfying the following assertions:

(i) T is an α_*-admissible mapping with respect to η;

(ii) T is a multivalued α-η-GF-contraction of Hardy-Rogers-type;

(iii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;

(iv) T is $\alpha - \eta$-continuous multivalued mapping.

Then T has a fixed point in X.

proof. Let $x_0 \in X$, and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. Since T is an α_*-admissible mapping with respect to η, so we have

$$\alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1). \tag{3.19}$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T and thus, we have nothing to prove. So, we assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$\begin{aligned} F\left(hH\left(Tx_0, Tx_1\right)\right) &< F\left(H\left(Tx_0, Tx_1\right)\right) + \\ &\quad G\left(\begin{array}{c} d(x_0, Tx_0), d(x_1, Tx_1), \\ d(x_0, Tx_1), d(x_1, Tx_0) \end{array}\right). \end{aligned}$$

Now from $d\left(x_1, Tx_1\right) < hH\left(Tx_0, Tx_1\right)$, we deduce that there exists $x_2 \in Tx_1$ such that

$$d\left(x_1, x_2\right) \leq hH\left(Tx_0, Tx_1\right).$$

Consequently, we obtain

$$\begin{aligned} F(d(x_1, x_2)) &\leq F(hH(Tx_0, Tx_1)) \\ &< F(H(Tx_0, Tx_1)) + \\ &\quad G\left(\begin{array}{c} d(x_0, Tx_0), d(x_1, Tx_1), \\ d(x_0, Tx_1), d(x_1, Tx_0) \end{array}\right). \end{aligned}$$

Which implies

$$
\begin{aligned}
& 2G\begin{pmatrix} d(x_0,Tx_0),d(x_1,Tx_1), \\ d(x_0,Tx_1),d(x_1,Tx_0) \end{pmatrix} + F(d(x_1,x_2)) \\
\leq\ & 2G\begin{pmatrix} d(x_0,Tx_0),d(x_1,Tx_1), \\ d(x_0,Tx_1),d(x_1,Tx_0) \end{pmatrix} + F\left(H(Tx_0,Tx_1)\right) + \\
& G\begin{pmatrix} d(x_0,Tx_0),d(x_1,Tx_1), \\ d(x_0,Tx_1),d(x_1,Tx_0) \end{pmatrix} \\
\leq\ & F\begin{pmatrix} \kappa d\left(x_0,x_1\right)+\beta d\left(x_0,Tx_0\right)+\gamma d\left(x_1,Tx_1\right)+ \\ \delta d\left(x_0,Tx_1\right)+Ld\left(x_1,Tx_0\right) \end{pmatrix} + \\
& G\begin{pmatrix} d(x_0,Tx_0),d(x_1,Tx_1), \\ d(x_0,Tx_1),d(x_1,Tx_0) \end{pmatrix}
\end{aligned}
$$

we get

$$
\begin{aligned}
& G\begin{pmatrix} d(x_0,x_1),d(x_1,x_2), \\ d(x_0,x_2),d(x_1,x_1) \end{pmatrix} + F(d(x_1,x_2)) \\
\leq\ & F\begin{pmatrix} \kappa d\left(x_0,x_1\right)+\beta d\left(x_0,Tx_0\right)+\gamma d\left(x_1,Tx_1\right)+ \\ \delta d\left(x_0,Tx_1\right)+Ld\left(x_1,Tx_0\right) \end{pmatrix}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
& G(d(x_0,x_1),d(x_1,x_2),d(x_0,x_2),0) + F\left(d(x_1,x_2)\right) \\
\leq\ & F\begin{pmatrix} \kappa d\left(x_0,x_1\right)+\beta d\left(x_0,Tx_0\right)+\gamma d\left(x_1,Tx_1\right)+ \\ \delta d\left(x_0,Tx_1\right)+Ld\left(x_1,Tx_0\right) \end{pmatrix}.
\end{aligned}
\tag{3.20}
$$

Now since, $d(x_0,x_1).d(x_1,x_2).d(x_0,x_2).0 = 0$, so from (G) there exists $\tau > 0$ such that,

$$G(d(x_0,x_1),d(x_1,x_2),d(x_0,x_2),0) = \tau.$$

Therefore from (3.20) we deduce that

$$
\begin{aligned}
& \tau + F\left(d\left(x_1, x_2\right)\right) \\
\leq\ & F\left(\begin{array}{c}\kappa d\left(x_0, x_1\right)+\beta d\left(x_0, T x_0\right)+\gamma d\left(x_1, T x_1\right)+ \\ \delta d\left(x_0, T x_1\right)+L d\left(x_1, T x_0\right)\end{array}\right) \\
\leq\ & F\left(\begin{array}{c}\kappa d\left(x_0, x_1\right)+\beta d\left(x_0, x_1\right)+\gamma d\left(x_1, x_2\right)+ \\ \delta d\left(x_0, x_2\right)\end{array}\right) \\
\leq\ & F\left(\begin{array}{c}\kappa d\left(x_0, x_1\right)+\beta d\left(x_0, x_1\right)+\gamma d\left(x_1, x_2\right)+ \\ \delta d\left(x_0, x_1\right)+\delta d\left(x_1, x_2\right)\end{array}\right) \\
\leq\ & F\left((\kappa+\beta+\delta) d\left(x_0, x_1\right)+(\gamma+\delta) d\left(x_1, x_2\right)\right).
\end{aligned}
$$

From $(F1)$, we deduce

$$
d\left(x_1, x_2\right)<(\kappa+\beta+\delta) d\left(x_0, x_1\right)+(\gamma+\delta) d\left(x_1, x_2\right).
$$

This implies

$$
(1-\gamma-\delta) d\left(x_1, x_2\right)<(\kappa+\beta+\delta) d\left(x_0, x_1\right).
$$

From $\kappa+\beta+\gamma+2\delta=1$ and $\gamma \neq 1$, we deduce that $1-\gamma-\delta>0$ and so

$$
d\left(x_1, x_2\right)<\frac{(\kappa+\beta+\delta)}{(1-\gamma-\delta)} d\left(x_0, x_1\right)=d\left(x_0, x_1\right).
$$

Consequently

$$
F\left(d(x_1, x_2)\right) \leq F\left(d(x_0, x_1)\right)-\tau.
$$

Note that $x_1 \neq x_2$(since $x_1 \notin Tx_1$). Also, since $\alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1)$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, then $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$. So $\alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2)$. Again since F is continuous from the right, there exists a real number $h > 1$ such that

$$F(hH(Tx_1, Tx_2)) < F(H(Tx_1, Tx_2)) + G\begin{pmatrix} d(x_1, Tx_1), d(x_2, Tx_2), \\ d(x_1, Tx_2), d(x_2, Tx_1) \end{pmatrix}.$$

Now from $d(x_2, Tx_2) < hH(Tx_1, Tx_2)$, we deduce that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq hH(Tx_1, Tx_2).$$

Consequently, we obtain

$$\begin{aligned} F(d(x_2, x_3)) &\leq F(hH(Tx_1, Tx_2)) \\ &< F(H(Tx_1, Tx_2)) + \\ &\quad G\begin{pmatrix} d(x_1, Tx_1), d(x_2, Tx_2), \\ d(x_1, Tx_2), d(x_2, Tx_1) \end{pmatrix}. \end{aligned}$$

Which implies

$$\begin{aligned} & 2G\begin{pmatrix} d(x_1, Tx_1), d(x_2, Tx_2), \\ d(x_1, Tx_2), d(x_2, Tx_1) \end{pmatrix} + F(d(x_2, x_3)) \\ \leq\ & 2G\begin{pmatrix} d(x_1, Tx_1), d(x_2, Tx_2), \\ d(x_1, Tx_2), d(x_2, Tx_1) \end{pmatrix} + F(H(Tx_1, Tx_2)) + \\ & G\begin{pmatrix} d(x_1, Tx_1), d(x_2, Tx_2), \\ d(x_1, Tx_2), d(x_2, Tx_1) \end{pmatrix} \\ \leq\ & F\begin{pmatrix} \kappa d(x_1, x_2) + \beta d(x_1, Tx_1) + \gamma d(x_2, Tx_2) + \\ \delta d(x_1, Tx_2) + Ld(x_2, Tx_1) \end{pmatrix} + \\ & G\begin{pmatrix} d(x_1, Tx_1), d(x_2, Tx_2), \\ d(x_1, Tx_2), d(x_2, Tx_1) \end{pmatrix} \end{aligned}$$

we get

$$
\begin{aligned}
& G\begin{pmatrix} d(x_1,x_2),d(x_2,x_3), \\ d(x_1,x_3),d(x_2,x_2) \end{pmatrix} + F(d(x_2,x_3)) \\
\leq \; & F\begin{pmatrix} \kappa d(x_1,x_2)+\beta d(x_1,Tx_1)+\gamma d(x_2,Tx_2)+ \\ \delta d(x_1,Tx_2)+Ld(x_2,Tx_1) \end{pmatrix}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
& G(d(x_1,x_2),d(x_2,x_3),d(x_1,x_3),0)+F(d(x_2,x_3)) \\
\leq \; & F\begin{pmatrix} \kappa d(x_1,x_2)+\beta d(x_1,Tx_1)+\gamma d(x_2,Tx_2)+ \\ \delta d(x_1,Tx_2)+Ld(x_2,Tx_1) \end{pmatrix}.
\end{aligned} \tag{3.21}
$$

Now since, $d(x_1,x_2).d(x_2,x_3).d(x_1,x_3).0=0$, so from (G) there exists $\tau>0$ such that,

$$G(d(x_1,x_2),d(x_2,x_3),d(x_1,x_3),0)=\tau.$$

Therefore from (3.21) we deduce that

$$
\begin{aligned}
& \tau+F(d(x_2,x_3)) \\
\leq \; & F\begin{pmatrix} \kappa d(x_1,x_2)+\beta d(x_1,Tx_1)+\gamma d(x_2,Tx_2)+ \\ \delta d(x_1,Tx_2)+Ld(x_2,Tx_1) \end{pmatrix} \\
\leq \; & F\begin{pmatrix} \kappa d(x_1,x_2)+\beta d(x_1,x_2)+\gamma d(x_2,x_3)+ \\ \delta d(x_1,x_3) \end{pmatrix} \\
\leq \; & F\begin{pmatrix} \kappa d(x_1,x_2)+\beta d(x_1,x_2)+\gamma d(x_2,x_3)+ \\ \delta d(x_1,x_2)+\delta d(x_2,x_3) \end{pmatrix} \\
\leq \; & F((\kappa+\beta+\delta)d(x_1,x_2)+(\gamma+\delta)d(x_2,x_3)).
\end{aligned}
$$

From $(F1)$, we deduce

$$d(x_2,x_3)<(\kappa+\beta+\delta)d(x_1,x_2)+(\gamma+\delta)d(x_2,x_3).$$

This implies

$$(1-\gamma-\delta)\, d\,(x_2, x_3) < (\kappa+\beta+\delta)\, d\,(x_1, x_2)\,.$$

From $\kappa+\beta+\gamma+2\delta = 1$ and $\gamma \neq 1$, we deduce that $1-\gamma-\delta > 0$ and so

$$d\,(x_2, x_3) < \frac{(\kappa+\beta+\delta)}{(1-\gamma-\delta)} d\,(x_1, x_2) = d\,(x_1, x_2)\,.$$

Consequently

$$F\,(d(x_2, x_3)) \leq F\,(d(x_1, x_2)) - \tau.$$

Continuing in this way, we can define a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$,

$$\eta_*(Tx_{n-1}, Tx_n) \leq \alpha_*(Tx_{n-1}, Tx_n) \tag{3.22}$$

and

$$F\,(d(x_n, x_{n+1})) \leq F\,(d(x_{n-1}, x_n)) - \tau, \tag{3.23}$$

for all $n \in \mathbb{N}$. By (3.23), we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \qquad (3.24) \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \leq \dots \leq F(d(x_0, x_1)) - n\tau \end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \longrightarrow \infty$ in (3.24), we deduce

$$\lim_{n \longrightarrow \infty} F(d\,(x_n, x_{n+1})) = -\infty.$$

By using $(F2)$, we have

$$\lim_{n \longrightarrow \infty} d\,(x_n, x_{n+1}) = 0. \tag{3.25}$$

Now from (F_3), there exists $0 < k < 1$ such that

$$\lim_{n \longrightarrow \infty} [d(x_n, x_{n+1})]^k F(d(x_n, x_{n+1})) = 0. \tag{3.26}$$

By (3.24), we have

$$\begin{aligned} & d(x_n, x_{n+1})^k F(d(x_n, x_{n+1})) - d(x_n, x_{n+1})^k F(d(x_0, x_1)) \\ \leq \; & -n\tau [d(x_n, x_{n+1})]^k \leq 0. \end{aligned} \tag{3.27}$$

Letting $n \longrightarrow \infty$ in (3.27) and applying (3.25) and (3.26), we have,

$$\lim_{n \longrightarrow \infty} n [d(x_n, x_{n+1})]^k = 0. \tag{3.28}$$

It follows from (3.28) that there exists $n_1 \in \mathbb{N}$ such that $n(d(x_n, x_{n+1}))^k \leq 1$ for all $n \geq n_1$, this implies

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}. \tag{3.29}$$

For all $m > n > n_1$ by using (3.29) and the triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &< \sum_{i=n}^{\infty} d(x_n, x_{n+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{3.30}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, taking limit as $n \longrightarrow \infty$ in (3.30), we get

$$\lim_{n,m \longrightarrow \infty} d(x_n, x_m) = 0.$$

This shows that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} d(x_n, x) = 0$. As $\eta_*(Tx_{n-1}, Tx_n) \leq \alpha_*(Tx_{n-1}, Tx_n)$ for all $n \in \mathbb{N}$, we have $\eta(x_n, x_{n+1}) \leq$

$\alpha(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By α-η-continuity of the multivalued mapping T, we get

$$\lim_{n \longrightarrow \infty} H(Tx_n, Tx) = 0.$$

Now we obtain

$$d(x, Tx) = \lim_{n \longrightarrow \infty} d(x_{n+1}, Tx) \leq \lim_{n \longrightarrow \infty} H(Tx_n, Tx) = 0.$$

Therefore, $x \in Tx$ and hence T has a fixed point.

Theorem 3.2.6 Let (X, d) be a complete metric space. Let $T{:}X \longrightarrow CB(X)$ satisfying the following assertions:

(i) T is an α_*-admissible mapping with respect to η;

(ii) T is a multivalued α-η-GF-contraction of Hardy-Rogers-Type;

(iii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$ then either

$$\alpha_*(Tx_n, x) \geq \eta_*(Tx_n, T^2x_n) \text{ or } \alpha_*(T^2x_n, x) \geq \eta_*(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

proof. As similar lines of the Theorem 3.2.5, we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \to x \text{ as } n \to \infty.$$

Since, by (iv), either

$$\alpha_*(Tx_n, x) \geq \eta_*(Tx_n, T^2x_n) \text{ or } \alpha_*(T^2x_n, x) \geq \eta_*(T^2x_n, T^3x_n),$$

holds for all $n \in \mathbb{N}$. Since $x_{n+1} \in Tx_n$, so we have

$$\alpha(x_{n+1}, x) \geq \eta(x_{n+1}, x_{n+2}) \text{ or } \alpha(x_{n+2}, x) \geq \eta(x_{n+2}, x_{n+3}).$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_{n_k+1} \in Tx_{n_k}$ such that

$$\eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x)$$

and so from (3.18) we deduce that

$$\begin{aligned} & 2G\begin{pmatrix} d(x_{n_k}, Tx_{n_k}), d(x, Tx), \\ d(x_{n_k}, Tx), d(x, Tx_{n_k}) \end{pmatrix} + F\left(H(Tx_{n_k}, Tx)\right) \\ \leq\ & F\begin{pmatrix} \kappa d\left(x_{n_k}, x\right) + \beta d\left(x_{n_k}, Tx_{n_k}\right) + \gamma d\left(x, Tx\right) + \\ \delta d\left(x_{n_k}, Tx_{n_k}\right) + Ld\left(x, Tx_{n_k}\right) \end{pmatrix}. \end{aligned}$$

Which implies

$$\begin{aligned} & 2G\begin{pmatrix} d(x_{n_k}, Tx_{n_k}), d(x, Tx), \\ d(x_{n_k}, Tx), d(x, Tx_{n_k}) \end{pmatrix} + F\left(d(x_{n_k+1}, Tx)\right) \\ \leq\ & 2G\begin{pmatrix} d(x_{n_k}, Tx_{n_k}), d(x, Tx), \\ d(x_{n_k}, Tx), d(x, Tx_{n_k}) \end{pmatrix} + F\left(H(Tx_{n_k}, Tx)\right) \\ \leq\ & F\begin{pmatrix} \kappa d\left(x_{n_k}, x\right) + \beta d\left(x_{n_k}, Tx_{n_k}\right) + \gamma d\left(x, Tx\right) + \\ \delta d\left(x_{n_k}, Tx\right) + Ld(x, Tx_{n_k}) \end{pmatrix} \\ \leq\ & F\begin{pmatrix} \kappa d\left(x_{n_k}, x\right) + \beta d\left(x_{n_k}, x_{n_k+1}\right) + \gamma d\left(x, Tx\right) + \\ \delta d\left(x_{n_k}, Tx\right) + Ld(x, x_{n_k+1}) \end{pmatrix}. \end{aligned}$$

We get

$$2\tau + F\left(d(x_{n_k+1}, Tx)\right) \leq F\begin{pmatrix} \kappa d\left(x_{n_k}, x\right) + \beta d\left(x_{n_k}, x_{n_k+1}\right) + \gamma d\left(x, Tx\right) + \\ \delta d\left(x_{n_k}, Tx\right) + Ld(x, x_{n_k+1}) \end{pmatrix}.$$

Since F is strictly increasing, we have

$$d(x_{n_k+1}, Tx) < \tag{3.31}$$
$$\kappa d(x_{n_k}, x) + \beta d(x_{n_k}, x_{n_k+1}) + \gamma d(x, Tx) + \delta d(x_{n_k}, Tx) + Ld(x, x_{n_k+1}).$$

By taking the limit as $k \to \infty$ in (3.31), as $\gamma + \delta < 1$ we obtain

$$d(x, Tx) < (\gamma + \delta) d(x, Tx) < d(x, Tx). \tag{3.32}$$

Which implies $d(x, Tx) = 0$. Thus $x \in Tx$, implies x is a fixed point of T.

Corollary 3.2.7 Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a continuous multivalued mapping. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $Tx \neq Ty$, we have

$$2\tau + F(H(Tx, Ty)) \leq F(\kappa d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)),$$

where $\tau > 0$, $\kappa, \beta, \gamma, \delta, L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$ and $F \in \mathcal{F}$. Then T has a fixed point in X.

Corollary 3.2.8 Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a continuous multivalued mapping. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $Tx \neq Ty$, we have

$$\begin{aligned} & 2\tau e^{v \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}} + F(H(Tx, Ty)) \\ \leq \; & F((\kappa d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)), \end{aligned}$$

where $\tau > 0$, $\kappa, \beta, \gamma, \delta, L, v \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $F \in \mathcal{F}$. Then T has a fixed point in X.

Example 3.2.9 Let $X = [0, 1]$, $T : X \to CB(X)$ be defined as $Tx = [0, \frac{x}{2}]$ and d be the usual metric on X. Define $\alpha, \eta : X \times X \longrightarrow [0, \infty)$, $G : \mathbb{R}^{+^4} \longrightarrow \mathbb{R}^+$ and $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by $\alpha(x, y) = \frac{1}{2}$, $\eta(x, y) = \frac{1}{4}$, $G(t_1, t_2, t_3, t_4) = \tau$ where $\tau = \ln(\sqrt{4})$ and $F(t) = \ln(t) + t \in \mathcal{F}$ for all $t > 0$. It is easy to check that conditions (i), (iii) and (iv)of Theorem 3.2.5 hold. Now for all $x, y \in X$, $Tx \neq Ty$, we obtain

$$
\begin{aligned}
& 2G\left(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right) + F(H(Tx,Ty)) \\
= \; & 2\tau + F(H(Tx,Ty)) \\
= \; & \ln(4) + \ln(H(Tx,Ty)) + H(Tx,Ty) \\
= \; & \ln(4) + \ln(\frac{1}{2}|y-x|) + \frac{1}{2}|y-x| \\
\leq \; & \ln(4) + \ln(\frac{3}{8}|y-x|) + \frac{3}{2}|y-x| \\
\leq \; & \ln(4) + \ln(\frac{1}{4}) + \ln(\frac{3}{2}|y-x|) + \frac{3}{2}|y-x|
\end{aligned}
$$

$$
\begin{aligned}
= \; & \ln\left(\frac{1}{2}|x-y| + \frac{1}{2}|y-x| + \frac{1}{2}|y-x|\right) + \left(\frac{1}{2}|x-y| + \frac{1}{2}|y-x| + \frac{1}{2}|y-x|\right) \\
\leq \; & \ln\left(\frac{1}{2}|x-y| + \frac{1}{4}\left|x-\frac{x}{2}\right| + \frac{1}{8}\left|y-\frac{y}{2}\right| + \frac{1}{16}\left|x-\frac{y}{2}\right| + \frac{35}{16}\left|y-\frac{x}{2}\right|\right) + \\
& \left(\frac{1}{2}|x-y| + \frac{1}{4}\left|x-\frac{x}{2}\right| + \frac{1}{8}\left|y-\frac{y}{2}\right| + \frac{1}{16}\left|x-\frac{y}{2}\right| + \frac{35}{16}\left|y-\frac{x}{2}\right|\right) \\
= \; & F\left(\kappa d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + L d(y,Tx)\right).
\end{aligned}
$$

Hence T is a multivalued α-η-GF-contraction of Hardy-Rogers-Type with $\kappa = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{8}$, $\delta = \frac{1}{16}$ and $L = \frac{35}{16}$ (that is, condition (ii) of Theorem 3.2.5 holds). Therefore all conditions of Theorems 3.2.5 are satisfied and T has a fixed point

Chapter 4

Fixed points of generalized contractions in generalized metric spaces

Very recently Jleli et al. [33],[34] established new fixed point theorems in the setting of Branciari metric spaces.

In this chapter, we extend the results given in [33],[34] by using the concept of cyclic (α, β)-admissible mappings obtained in [8]. As an application, we apply our main results for proving fixed point theorems involving a cyclic mapping.

4.1 Fixed point results of generalized contractions with cyclic(α, β)-admissible mapping in generalized metric spaces

Now, we state and prove our main results in this section.

Theorem 4.1.1 Let (X, d) be a complete g.m.s, $T : X \longrightarrow X$ be a given map and let $\alpha, \beta : X \longrightarrow [0, \infty)$ be two mappings. Suppose that the following conditions hold:

(1) there exists $\theta \in \Theta$ and $k \in (0,1)$ such that

$$x, y \in X,\ d(Tx, Ty) \neq 0 \Longrightarrow \alpha(x)\beta(y).\theta(d(Tx, Ty)) \leq [\theta(d(x,y))]^k,$$

(2) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$,

(3) T is a cyclic (α, β)-admissible mapping,

(4)one of the following conditions holds:

(4.1) T is continuous,

(4.2) if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $\beta(x) \geq 1$.

Then T has a fixed point. Furthermore, if $\alpha(x) \geq 1$and $\beta(x) \geq 1$ for every fixed point $x \in X$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$. We define the iterative sequence $\{x_n\}$ in X by the rule $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Obviously, if there exists $n_0 \in \mathbb{N}$ for which $T^{n_0}x_0 = T^{n_0+1}x_0$ then $T^{n_0}x_0$ shall be a fixed point of T. Thus, we suppose that $d(T^n x_0, T^{n+1}x_0) > 0$ for every $n \in \mathbb{N}$. Now from conditions (2) and (3), we get that

$$\alpha(x_0) \geq 1 \Longrightarrow \beta(x_1) = \beta(Tx_0) \geq 1$$

and

$$\beta(x_0) \geq 1 \Longrightarrow \alpha(x_1) = \alpha(Tx_0) \geq 1.$$

By a similar way, we get

$$\alpha(T^n x_0) \geq 1 \text{ and } \beta(T^n x_0) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Which implies

$$\alpha\left(T^{n-1}x_0\right)\beta\left(T^{n}x_0\right)\geq 1 \text{ for all } n\in\mathbb{N}, \tag{4.1}$$

also

$$\alpha\left(T^{n-1}x_0\right)\beta\left(T^{n+1}x_0\right)\geq 1 \text{ for all n}\in\mathbb{N}. \tag{4.2}$$

From condition (1) and inequality (4.1), then for every $n\in\mathbb{N}$, we write

$$\begin{aligned}&\theta\left(d\left(T^{n}x_0,T^{n+1}x_0\right)\right)\\ \leq\ &\alpha\left(T^{n-1}x_0\right)\beta\left(T^{n}x_0\right).\theta\left(d\left(T^{n}x_0,T^{n+1}x_0\right)\right).\end{aligned}$$

$$\begin{aligned}\leq\ &\left[\theta\left(d\left(T^{n-1}x_0,T^{n}x_0\right)\right)\right]^{k}\leq\left[\theta\left(d\left(T^{n-2}x_0,T^{n-1}x_0\right)\right)\right]^{k^2} \\ \leq\ &...\leq\left[\theta\left(d\left(x_0,Tx_0\right)\right)\right]^{k^n}.\end{aligned} \tag{4.3}$$

Thus we have

$$1\leq\theta\left(d\left(T^{n}x_0,T^{n+1}x_0\right)\right)\leq\left[\theta\left(d(x_0,Tx_0)\right)\right]^{k^n} \text{ for all } n\in\mathbb{N}. \tag{4.4}$$

Letting $n\longrightarrow\infty$, we obtain

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(T^{n}x_0,T^{n+1}x_0\right)\right)=1, \tag{4.5}$$

that together with $(\Theta 2)$ gives as

$$\lim_{n\longrightarrow\infty}d\left(T^{n}x_0,T^{n+1}x_0\right)=0.$$

From condition $(\Theta 3)$, there exist $r\in(0,1)$ and $\ell\in(0,\infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\theta\left(d\left(T^{n}x_0,T^{n+1}x_0\right)\right)-1}{\left[d\left(T^{n}x_0,T^{n+1}x_0\right)\right]^{r}}=\ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta\left(d\left(T^n x_0, T^{n+1} x_0\right)\right) - 1}{\left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r} - \ell \right| \leq B \text{ for all } n \geq n_0.$$

This implies

$$\frac{\theta\left(d\left(T^n x_0, T^{n+1} x_0\right)\right) - 1}{\left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r} \geq \ell - B = B \text{ for all } n \geq n_0.$$

Then

$$n\left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^n x_0, T^{n+1} x_0\right)\right) - 1\right] \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta\left(d\left(T^n x_0, T^{n+1} x_0\right)\right) - 1}{\left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r} \geq B \text{ for all } n \geq n_0.$$

Which implies

$$n\left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^n x_0, T^{n+1} x_0\right)\right) - 1\right] \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^n x_0, T^{n+1} x_0\right)\right) - 1\right] \text{ for all } n \geq n_0.$$

By using (4.4), we get

$$n\left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r \leq An\left(\left[\theta\left(d(x_0, Tx_0)\right)\right]^{k^n} - 1\right) \text{ for all } n \geq n_0. \tag{4.6}$$

Letting $n \longrightarrow \infty$ in the inequality (4.6), we obtain

$$\lim_{n \longrightarrow \infty} n \left[d\left(T^n x_0, T^{n+1} x_0\right)\right]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_1. \tag{4.7}$$

Now, we will prove that T has a periodic point. Suppose that it is not the case, then $T^n x_0 \neq T^m x_0$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using condition (1) and inequality (4.2), we get

$$\begin{aligned} & \theta\left(d\left(T^n x_0, T^{n+2} x_0\right)\right) \\ \leq \quad & \alpha\left(T^{n-1} x_0\right) \beta\left(T^{n+1} x_0\right) . \theta\left(d\left(T^n x_0, T^{n+2} x_0\right)\right). \end{aligned}$$

$$\begin{aligned} \leq \quad & \left[\theta\left(d\left(T^{n-1} x_0, T^{n+1} x_0\right)\right)\right]^k \leq \left[\theta\left(d\left(T^{n-2} x_0, T^n x_0\right)\right)\right]^{k^2} \\ \leq \quad & \ldots \leq \left[\theta\left(d\left(x_0, T^2 x_0\right)\right)\right]^{k^n}. \end{aligned} \tag{4.8}$$

Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n \longrightarrow \infty} \theta\left(d\left(T^n x_0, T^{n+2} x_0\right)\right) = 1. \tag{4.9}$$

By using $(\Theta 2)$, we have

$$\lim_{n \longrightarrow \infty} d\left(T^n x_0, T^{n+2} x_0\right) = 0.$$

Similarly from $(\Theta 3)$ there exists $n_2 \in \mathbb{N}$ such that

$$d\left(T^n x_0, T^{n+2} x_0\right) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_2. \tag{4.10}$$

Let $h = \max\{n_0, n_1\}$. we consider two cases.

Case 1: If $m > 2$ is odd, then writing $m = 2L+1, L \geq 1$, using (4.7), for all $n \geq h$, we obtain

$$\begin{aligned} d\left(T^n x_0, T^{n+m} x_0\right) &\leq d\left(T^n x_0, T^{n+1} x_0\right) + d\left(T^{n+1} x_0, T^{n+2} x_0\right) \\ &\quad + \ldots + d\left(T^{n+2L} x_0, T^{n+2L+1} x_0\right) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+1)^{\frac{1}{r}}} + \ldots + \frac{1}{(n+2L)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Case 2: If $m > 2$ is even, then writing $m = 2L, L \geq 2$, using (4.7) and (4.10), for all $n \geq h$, we have

$$\begin{aligned} d\left(T^n x_0, T^{n+m} x_0\right) &\leq d\left(T^n x_0, T^{n+2} x_0\right) + d\left(T^{n+2} x_0, T^{n+3} x_0\right) \\ &\quad + \ldots + d\left(T^{n+2L-1} x_0, T^{n+2L} x_0\right) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+2)^{\frac{1}{r}}} + \ldots + \frac{1}{(n+2L-1)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Thus, combining all cases, we have

$$d\left(T^n x_0, T^{n+m} x_0\right) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}} \text{ for all } n \geq h,\ m \in \mathbb{N}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ is convergent (since $\frac{1}{r} > 1$), we deduce that $\{T^n x_0\}$ is a Cauchy sequence. From the completeness of X, there $z \in X$ such that $T^n x_0 \longrightarrow z$ as $n \longrightarrow \infty$ (that is, $\lim_{n \longrightarrow \infty} d(T^n x_0, z) = 0$).

Now, we assume that T is continuous. Hence, we have

$$z = \lim_{n \longrightarrow \infty} T^{n+1} x_0 = \lim_{n \longrightarrow \infty} T\left(T^n x_0\right) = T\left(\lim_{n \longrightarrow \infty} T^n x_0\right) = Tz.$$

Next, we will assume that condition (4.2) holds. Hence $\beta(z) \geq 1$. we can suppose that $T^{n+1} x_0 \neq Tz$

for all n (or for n large enough). Using condition (1), we have

$$\begin{aligned} & \theta\left(d\left(T^{n+1}x_0, Tz\right)\right) \\ \leq \ & \alpha\left(T^n x_0\right)\beta\left(z\right).\theta\left(d\left(T^{n+1}x_0, Tz\right)\right) \\ \leq \ & \left[\theta\left(d\left(T^n x_0, z\right)\right)\right]^k. \end{aligned}$$

Which implies

$$\begin{aligned} \theta\left(d\left(T^{n+1}x_0, Tz\right)\right) & \leq \left[\theta\left(d\left(T^n x_0, z\right)\right)\right]^k \\ \ln\left[\theta\left(d\left(T^{n+1}x_0, Tz\right)\right)\right] & \leq k\ln\left[\theta\left(d\left(T^n x_0, z\right)\right)\right] \leq \ln\left[\theta\left(d\left(T^n x_0, z\right)\right)\right]. \end{aligned}$$

This implies from $(\Theta 1)$ that

$$d\left(T^{n+1}x_0, Tz\right) \leq d\left(T^n x_0, z\right)$$

Letting $n \longrightarrow \infty$ in the above inequality, we get $T^{n+1}x_0 \longrightarrow Tz$. From Lemma 1.5.6, we obtain $z = Tz$, which is a contradiction with the assumption: T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \text{ and } d\left(z, Tz\right) > 0.$$

By using condition (1) and inequality (4.1), we get

$$\begin{aligned} \theta\left(d\left(z, Tz\right)\right) & = \theta\left(d\left(T^q z, T^{q+1}z\right)\right) \leq \alpha\left(T^{q-1}z\right)\beta\left(T^q z\right).\theta\left(d\left(T^q z, T^{q+1}z\right)\right) \\ & \leq \left[\theta\left(d\left(z, Tz\right)\right)\right]^{k^q} < \theta\left(d\left(z, Tz\right)\right), \end{aligned}$$

which is a contradiction. Thus the set of fixed points of T is non-empty (that is, T has at least one fixed point). Now we suppose that $z, u \in X$ are two fixed points of T such that $d\left(z, u\right) = d\left(Tz, Tu\right) > 0$.

From the hypothesis, we find that $\alpha(z) \geq 1$ and $\beta(z) \geq 1$. Using condition (1), we obtain

$$\begin{aligned}\theta(d(z,u)) &= \theta(d(Tz,Tu)) \leq \alpha(z)\beta(z).\theta(d(Tz,Tu)) \\ &\leq [\theta(d(z,u))]^k < \theta(d(z,u)),\end{aligned}$$

it is a contradiction. Therefore T has a unique fixed point.

Example 4.1.2 Let $X = \{0,1,2,3,4\}$. Define $d : X \times X \longrightarrow \mathbb{R}$ as follows

$$\begin{aligned}d(x,x) &= 0, \text{ for all } x \in X, \\ d(1,2) &= d(2,1) = 3, \\ d(2,3) &= d(3,2) = d(1,3) = d(3,1) = 1, \\ d(x,y) &= |x-y|, \qquad \text{otherwise.}\end{aligned}$$

It is clear that (X,d) is a complete g.m.s, but it is not metric space because d does not satisfy triangle inequality on X. Indeed,

$$3 = d(1,2) > d(1,3) + d(3,2) = 1 + 1 = 2.$$

Let $T : X \longrightarrow X$ be the mapping defined by

$$T(x) = \begin{cases} 2 & \text{if } x \in \{0,1,2,3\} \\ 0 & \text{if } x = 4. \end{cases}$$

Define

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in \{0,1,2,3\}, \\ 0 & \text{otherwise.} \end{cases},$$

and

$$\beta(x) = \begin{cases} 1 & \text{if } x \in \{0,1,2,3\}, \\ 0 & \text{otherwise.} \end{cases}.$$

Also define $\theta : (0, \infty) \longrightarrow (1, \infty)$ by

$$\theta(t) = e^{\sqrt{t}}.$$

It is not difficult to show that $\theta \in \Theta$ and T is a cyclic (α, β)-admissible mapping. We shall prove that the hypotheses of Theorem 4.1.1 are satisfied by T. Now if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Therefore, $x_n \in \{0, 1, 2, 3\}$. Hence $x \in \{0, 1, 2, 3\}$, that is $\beta(x) \geq 1$. Next for $x \in \{0, 1, 2, 3\}, y = 4$, we have

$$\begin{aligned} \alpha(x)\beta(4).\theta(d(T(x), T(4))) &= \alpha(x)\beta(4).\theta(d(2, 0)) \\ &\leq [\theta(d(x, 4))]^k, \end{aligned}$$

for all $k \in (0, 1)$. So the hypotheses of Theorem 4.1.1 hold and hence, T has a unique fixed point. But the result of Jleli et al. [33] (the hypotheses of Theorem 1.5.7) can not applied to T. In deed, for $x = 2, y = 4$, we get

$$\begin{aligned} \theta(d(T(2), T(4))) &= \theta(d(2, 0)) = \theta(2) \\ &= e^{\sqrt{2}} \nleq \left[e^{\sqrt{2}}\right]^k = [\theta(d(2, 4))]^k, \end{aligned}$$

for all $k \in (0, 1)$.

Since a metric space is a generalized metric space, we can obtain the following result from Theorem 4.1.1.

Corollary 4.1.3 Let (X, d) be a complete metric space, $T : X \longrightarrow X$ be a given map and let $\alpha, \beta : X \longrightarrow [0, \infty)$ be two mappings. Suppose that the following conditions hold:

(1) there exists $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$x, y \in X, \ d(Tx, Ty) \neq 0 \Longrightarrow \alpha(x)\beta(y).\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

(2) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$,

(3) T is a cyclic (α, β)-admissible mapping,

(4)one of the following conditions holds:

(4.1) T is continuous,

(4.2) if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $\beta(x) \geq 1$.

Then T has a fixed point. Furthermore, if $\alpha(x) \geq 1$and $\beta(x) \geq 1$ for every fixed point $x \in X$, then T has a unique fixed point.

Example 4.1.4 Let $X = [0, 1]$ and $d : X \times X \longrightarrow \mathbb{R}$ given by $d(x, y) = |x - y|$ for all $x, y \in X$. It is easy to show that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be the mapping defined by

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, 1) \\ 0 & \text{if } \ x = 1 \end{cases},$$

and $\alpha, \beta : X \longrightarrow [0, \infty)$ be given by

$$\alpha(x) = \beta(x) = \begin{cases} 1 & \text{if } x \in [0, 1) \\ 0 & \text{otherwise} \end{cases}.$$

Also define $\theta : (0, \infty) \longrightarrow (1, \infty)$ by

$$\theta(t) = e^{\sqrt{t}}.$$

It is not difficult to show that $\theta \in \Theta$ and T is a cyclic (α, β)-admissible mapping. We shall prove that the hypotheses of Theorem 4.1.1 (or Corollary 4.1.3) are satisfied by T. Moreover, the result of Jleli et al. [33] can not applied to T. Now if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ and $x_n \longrightarrow x$ as

$n \longrightarrow \infty$. Therefore, $x_n \in [0,1)$. Hence $x \in [0,1)$, that is $\beta(x) \geq 1$. Next for $x \in [0,1), y = 1$, we have

$$\begin{aligned} \alpha(x)\beta(1).\theta(d(T(x),T(1))) &= \alpha(x)\beta(1).\theta\left(d\left(\frac{1}{2},0\right)\right) \\ &\leq [\theta(d(x,1))]^k, \text{ for all } k \in (0,1). \end{aligned}$$

So the hypotheses of Theorem 4.1.1 (or Corollary 4.1.3) hold and hence, T has a fixed point. But the hypotheses of Theorem 1.5.7 can not applied to T. In deed, for $x = \frac{1}{2}, y = 1$, we get

$$\begin{aligned} \theta\left(d\left(T\left(\frac{1}{2}\right),T(1)\right)\right) &= \theta\left(d\left(\frac{1}{2},0\right)\right) = \theta\left(\frac{1}{2}\right) \\ &= e^{\sqrt{\frac{1}{2}}} \nleq \left[e^{\sqrt{\frac{1}{2}}}\right]^k = \left[\theta\left(d\left(\frac{1}{2},1\right)\right)\right]^k, \end{aligned}$$

for all $k \in (0,1)$.

Corollary 4.1.5 [33] Let (X,d) be a complete g.m.s and $T : X \longrightarrow X$ be a given map. Suppose that there exists $\theta \in \Theta$ and $k \in (0,1)$ such that

$$x, y \in X,\ d(x,y) \neq 0 \Longrightarrow \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k.$$

Then T has a unique fixed point.

Proof. Setting $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$ in Theorem 4.1.1, we get this result.

Theorem 4.1.6 Let (X,d) be a complete g.m.s, $T : X \longrightarrow X$ be a given map and let $\alpha, \beta : X \longrightarrow [0,\infty)$ be two mappings. Suppose that the following conditions hold:

(1) there exists $\theta \in \Theta$ is continuous and $k \in (0,1)$ such that

$$x, y \in X,\ d(Tx,Ty) \neq 0 \Longrightarrow \alpha(x)\beta(y).\theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^k,$$

where

$$R(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\},$$

(2) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$,

(3) T is a cyclic (α, β)-admissible mapping,

(4)one of the following conditions holds:

(4.1) T is continuous,

(4.2) if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $\beta(x) \geq 1$.

Then T has a fixed point. Furthermore, if $\alpha(x) \geq 1$and $\beta(x) \geq 1$ for every fixed point $x \in X$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$. We define the iterative sequence $\{x_n\}$ in X by the rule $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Obviously, if there exists $n_0 \in \mathbb{N}$ for which $T^{n_0}x_0 = T^{n_0+1}x_0$ then $T^{n_0}x_0$ shall be a fixed point of T. Thus, we suppose that $d(T^n x_0, T^{n+1}x_0) > 0$, for every $n \in \mathbb{N}$. Now from (2) and (3), we get that

$$\alpha(x_0) \geq 1 \Longrightarrow \beta(x_1) = \beta(Tx_0) \geq 1$$

and

$$\beta(x_0) \geq 1 \Longrightarrow \alpha(x_1) = \alpha(Tx_0) \geq 1.$$

By a similar way, we get

$$\alpha(T^n x_0) \geq 1 \text{ and } \beta(T^n x_0) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Which implies

$$\alpha\left(T^{n-1}x_0\right)\beta\left(T^n x_0\right) \geq 1 \text{ for all } n \in \mathbb{N}, \tag{4.11}$$

also

$$\alpha\left(T^{n-1}x_0\right)\beta\left(T^{n+1}x_0\right) \geq 1 \text{ for all n} \in \mathbb{N}. \tag{4.12}$$

From condition (1) and inequality (4.11), then for every $n \in \mathbb{N}$, we write

$$\begin{aligned} & \theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) \\ \leq \;& \alpha\left(T^{n-1}x_0\right)\beta\left(T^n x_0\right).\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right). \end{aligned}$$

$$\begin{aligned} \leq \;& \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0, T^n x_0\right), d\left(T^{n-1}x_0, TT^{n-1}x_0\right), \\ d\left(T^n x_0, TT^n x_0\right), \frac{d\left(T^{n-1}x_0, TT^{n-1}x_0\right)d\left(T^n x_0, TT^n x_0\right)}{1+d\left(T^{n-1}x_0, T^n x_0\right)} \end{array}\right\}\right)\right]^k \\ = \;& \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0, T^n x_0\right), d\left(T^n x_0, T^{n+1}x_0\right), \\ \frac{d\left(T^{n-1}x_0, T^n x_0\right)d\left(T^n x_0, T^{n+1}x_0\right)}{1+d\left(T^{n-1}x_0, T^n x_0\right)} \end{array}\right\}\right)\right]^k \\ = \;& \left[\theta\left(\max\left\{d\left(T^{n-1}x_0, T^n x_0\right), d\left(T^n x_0, T^{n+1}x_0\right)\right\}\right)\right]^k. \end{aligned} \tag{4.13}$$

If there exists $n \in \mathbb{N}$ such that $\max\left\{d\left(T^{n-1}x_0, T^n x_0\right), d\left(T^n x_0, T^{n+1}x_0\right)\right\} = d\left(T^n x_0, T^{n+1}x_0\right)$, then inequality (4.13) turns into

$$\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) \leq \left[\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right)\right]^k,$$

this implies

$$\ln\left[\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right)\right] \leq k \ln\left[\theta\left(d(T^n x_0, T^{n+1}x_0)\right)\right],$$

that is a contradiction with $k \in (0,1)$. Therefore $\max\left\{d\left(T^{n-1}x_0, T^n x_0\right), d\left(T^n x_0, T^{n+1}x_0\right)\right\} =$

$d\left(T^{n-1}x_0, T^n x_0\right)$ for all $n \in \mathbb{N}$. Thus, from (4.13), we have

$$\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) \leq \left[\theta\left(d\left(T^{n-1}x_0, T^n x_0\right)\right)\right]^k \text{ for all } n \in \mathbb{N}.$$

Which implies

$$\begin{aligned} \theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) &\leq \left[\theta\left(d\left(T^{n-1}x_0, T^n x_0\right)\right)\right]^k \\ &\leq \left[\theta\left(d\left(T^{n-2}x_0, T^{n-1}x_0\right)\right)\right]^{k^2} \leq \ldots \leq \left[\theta\left(d(x_0, Tx_0)\right)\right]^{k^n}. \end{aligned}$$

Thus we have

$$1 \leq \theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) \leq \left[\theta\left(d(x_0, Tx_0)\right)\right]^{k^n} \text{ for all } n \in \mathbb{N}. \tag{4.14}$$

Letting $n \longrightarrow \infty$, we obtain

$$\lim_{n \longrightarrow \infty} \theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) = 1, \tag{4.15}$$

that together with $(\Theta 2)$ gives as

$$\lim_{n \longrightarrow \infty} d\left(T^n x_0, T^{n+1}x_0\right) = 0.$$

From condition $(\Theta 3)$, there exist $r \in (0,1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n \longrightarrow \infty} \frac{\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) - 1}{\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) - 1}{\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r} - \ell \right| \leq B \text{ for all } n \geq n_0.$$

This implies

$$\frac{\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) - 1}{\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r} \geq \ell - B = B \text{ for all } n \geq n_0.$$

Then

$$n\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) - 1\right] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) - 1}{\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r} \geq B \quad \text{for all } n \geq n_0.$$

Which implies

$$n\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) - 1\right] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) - 1\right] \quad \text{for all } n \geq n_0.$$

By using (4.14), we get

$$n\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r \leq An\left(\left[\theta\left(d(x_0, Tx_0)\right)\right]^{k^n} - 1\right) \quad \text{for all } n \geq n_0. \tag{4.16}$$

Letting $n \longrightarrow \infty$ in the inequality (4.16), we obtain

$$\lim_{n \longrightarrow \infty} n\left[d\left(T^n x_0, T^{n+1}x_0\right)\right]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d\left(T^n x_0, T^{n+1}x_0\right) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_1. \tag{4.17}$$

Now, we will prove that T has a periodic point. Suppose that it is not the case, then $T^n x_0 \neq T^m x_0$

for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using condition (1) and inequality (4.12), we get

$$\begin{aligned}
&\theta\left(d\left(T^{n}x_0, T^{n+2}x_0\right)\right) \\
\leq\ & \alpha\left(T^{n-1}x_0\right)\beta\left(T^{n+1}x_0\right).\theta\left(d\left(T^{n}x_0, T^{n+2}x_0\right)\right).
\end{aligned}$$

$$\begin{aligned}
\leq\ & \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0, T^{n+1}x_0\right), d\left(T^{n-1}x_0, TT^{n-1}x_0\right), \\ d\left(T^{n+1}x_0, TT^{n+1}x_0\right), \frac{d\left(T^{n-1}x_0, TT^{n-1}x_0\right)d\left(T^{n+1}x_0, TT^{n+1}x_0\right)}{1+d\left(T^{n-1}x_0, T^{n+1}x_0\right)} \end{array}\right\}\right)\right]^k \\
=\ & \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0, T^{n+1}x_0\right), d\left(T^{n-1}x_0, T^{n}x_0\right), \\ d\left(T^{n+1}x_0, T^{n+2}x_0\right), \frac{d\left(T^{n-1}x_0, T^{n}x_0\right)d\left(T^{n+1}x_0, T^{n+2}x_0\right)}{1+d\left(T^{n-1}x_0, T^{n+1}x_0\right)} \end{array}\right\}\right)\right]^k
\end{aligned}$$

$$= \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0, T^{n+1}x_0\right), d\left(T^{n-1}x_0, T^{n}x_0\right), \\ d\left(T^{n+1}x_0, T^{n+2}x_0\right) \end{array}\right\}\right)\right]^k. \tag{4.18}$$

Since θ is non-decreasing, we obtain from (4.18)

$$\theta\left(d\left(T^{n}x_0, T^{n+2}x_0\right)\right) \leq \left[\max\left\{\begin{array}{c} \theta\left(d\left(T^{n-1}x_0, T^{n+1}x_0\right)\right), \theta\left(d\left(T^{n-1}x_0, T^{n}x_0\right)\right), \\ \theta\left(d\left(T^{n+1}x_0, T^{n+2}x_0\right)\right) \end{array}\right\}\right]^k. \tag{4.19}$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned}
u_n &= \max\left\{\theta\left(d\left(T^{n-1}x_0, T^{n+1}x_0\right)\right), \theta\left(d\left(T^{n-1}x_0, T^{n}x_0\right)\right), \theta\left(d\left(T^{n+1}x_0, T^{n+2}x_0\right)\right)\right\} \\
&= \theta\left(d\left(T^{n-1}x_0, T^{n+1}x_0\right)\right).
\end{aligned}$$

If $|I| < \infty$ then there $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\begin{aligned}
&\max\left\{\theta\left(d\left(T^{n-1}x_0, T^{n+1}x_0\right)\right), \theta\left(d\left(T^{n-1}x_0, T^{n}x_0\right)\right), \theta\left(d\left(T^{n+1}x_0, T^{n+2}x_0\right)\right)\right\} \\
=\ & \max\left\{\theta\left(d\left(T^{n-1}x_0, T^{n}x_0\right)\right), \theta\left(d\left(T^{n+1}x_0, T^{n+2}x_0\right)\right)\right\}.
\end{aligned}$$

In this case, we get from (4.19)

$$\begin{aligned} 1 &\leq \theta\left(d\left(T^n x_0, T^{n+2} x_0\right)\right) \\ &\leq \left[\max\left\{\theta\left(d\left(T^{n-1} x_0, T^n x_0\right)\right), \theta\left(d\left(T^{n+1} x_0, T^{n+2} x_0\right)\right)\right\}\right]^k \end{aligned}$$

for all $n \geq N$. Letting $n \longrightarrow \infty$ in the above inequality and using (4.15), we obtain

$$\lim_{n \longrightarrow \infty} \theta\left(d\left(T^n x_0, T^{n+2} x_0\right)\right) = 1.$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, then we denote also by $\{u_n\}$, such that

$$u_n = \theta\left(d\left(T^{n-1} x_0, T^{n+1} x_0\right)\right) \quad \text{for } n \text{ large enough.}$$

In this case, we obtain from (4.19)

$$\begin{aligned} 1 &\leq \theta\left(d\left(T^n x_0, T^{n+2} x_0\right)\right) \leq \left[\theta\left(d\left(T^{n-1} x_0, T^{n+1} x_0\right)\right)\right]^k \\ &\leq \left[\theta\left(d\left(T^{n-2} x_0, T^n x_0\right)\right)\right]^{k^2} \leq \ldots \leq \left[\theta\left(d\left(x_0, T^2 x_0\right)\right)\right]^{k^n} \end{aligned}$$

for n large. Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n \longrightarrow \infty} \theta\left(d\left(T^n x_0, T^{n+2} x_0\right)\right) = 1. \tag{4.20}$$

Then in all cases, (4.20) holds. Using (4.20) and the property ($\Theta 2$), we have

$$\lim_{n \longrightarrow \infty} d\left(T^n x_0, T^{n+2} x_0\right) = 0.$$

Similarly from ($\Theta 3$) there exists $n_2 \in \mathbb{N}$ such that

$$d\left(T^n x_0, T^{n+2} x_0\right) \leq \frac{1}{n^{\frac{1}{r}}} \quad \text{for all } n \geq n_2. \tag{4.21}$$

Let $h = \max\{n_0, n_1\}$. we consider two cases.

Case 1: If $m > 2$ is odd, then writing $m = 2L+1, L \geq 1$, using (4.17), for all $n \geq h$, we obtain

$$\begin{aligned} d\left(T^n x_0, T^{n+m} x_0\right) &\leq d\left(T^n x_0, T^{n+1} x_0\right) + d\left(T^{n+1} x_0, T^{n+2} x_0\right) \\ &\quad + ... + d\left(T^{n+2L} x_0, T^{n+2L+1} x_0\right) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+1)^{\frac{1}{r}}} + ... + \frac{1}{(n+2L)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Case 2: If $m > 2$ is even, then writing $m = 2L, L \geq 2$, using (4.17) and (4.21), for all $n \geq h$, we have

$$\begin{aligned} d\left(T^n x_0, T^{n+m} x_0\right) &\leq d\left(T^n x_0, T^{n+2} x_0\right) + d\left(T^{n+2} x_0, T^{n+3} x_0\right) \\ &\quad + ... + d\left(T^{n+2L-1} x_0, T^{n+2L} x_0\right) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+2)^{\frac{1}{r}}} + ... + \frac{1}{(n+2L-1)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Thus, combining all cases, we have

$$d\left(T^n x_0, T^{n+m} x_0\right) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}} \text{ for all } n \geq h,\ m \in \mathbb{N}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ is convergent (since $\frac{1}{r} > 1$), we deduce that $\{T^n x_0\}$ is a Cauchy sequence. From the completeness of X, there $z \in X$ such that $T^n x_0 \longrightarrow z$ as $n \longrightarrow \infty$. Now, we assume that T is continuous. Hence, we have

$$z = \lim_{n \longrightarrow \infty} T^{n+1} x_0 = \lim_{n \longrightarrow \infty} T\left(T^n x_0\right) = T\left(\lim_{n \longrightarrow \infty} T^n x_0\right) = Tz.$$

Next, we will assume that condition (4.2) holds. Hence $\beta(z) \geq 1$. Without restriction of the generality, we can suppose that $T^n x_0 \neq z$ for all n (or for n large enough). Suppose that $d(z, Tz) > 0$, using condition (1), we have

$$\begin{aligned} & \theta\left(d\left(T^{n+1}x_0, Tz\right)\right) \\ \leq \; & \alpha\left(T^n x_0\right)\beta(z).\theta\left(d\left(T^{n+1}x_0, Tz\right)\right). \end{aligned}$$

$$\begin{aligned} & \leq \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^n x_0, z\right), d\left(T^n x_0, T^{n+1}x_0\right), \\ d(z, Tz), \frac{d\left(T^n x_0, T^{n+1}x_0\right)d(z,Tz)}{1+d(T^n x_0, z)} \end{array}\right\}\right)\right]^k \\ & = \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^n x_0, z\right), d\left(T^n x_0, T^{n+1}x_0\right), \\ d(z, Tz) \end{array}\right\}\right)\right]^k. \end{aligned}$$

Which implies

$$\theta\left(d\left(T^{n+1}x_0, Tz\right)\right) \leq \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^n x_0, z\right), d\left(T^n x_0, T^{n+1}x_0\right), \\ d(z, Tz) \end{array}\right\}\right)\right]^k.$$

Letting $n \longrightarrow \infty$ in the above inequality, using the continuity of θ and Lemma 1.5.5, we obtain

$$\theta(d(z, Tz)) \leq [\theta(d(z, Tz))]^k < \theta(d(z, Tz)),$$

which is a contradiction. Thus we have $z = Tz$, which is also a contradiction with the assumption: T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \text{ and } d(z, Tz) > 0.$$

By using condition (1) and inequality (4.11), we get

$$\begin{aligned}\theta(d(z,Tz)) &= \theta\left(d\left(T^q z, T^{q+1}z\right)\right) \leq \alpha\left(T^{q-1}z\right)\beta\left(T^q z\right).\theta\left(d\left(T^q z, T^{q+1}z\right)\right) \\ &\leq [\theta(d(z,Tz))]^{k^q} < \theta(d(z,Tz)),\end{aligned}$$

which is a contradiction. Thus the set of fixed points of T is non-empty (that is, T has at least one fixed point). Now we suppose that $z, u \in X$ are two fixed points of T such that $d(z,u) = d(Tz,Tu) > 0$. From the hypothesis, we find that $\alpha(z) \geq 1$ and $\beta(z) \geq 1$. Using condition (1), we obtain

$$\begin{aligned}\theta(d(z,u)) &= \theta(d(Tz,Tu)) \leq \alpha(z)\beta(z).\theta(d(Tz,Tu)) \\ &\leq [\theta(d(z,u))]^k < \theta(d(z,u)),\end{aligned}$$

it is a contradiction. Therefore T has a unique fixed point.

Also we can obtain the following corollaries from Theorem 4.1.6.

Corollary 4.1.7 Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a given map and let $\alpha, \beta : X \longrightarrow [0,\infty)$ be two mappings. Suppose that the following conditions hold:

(1) there exists $\theta \in \Theta$ is continuous and $k \in (0,1)$ such that

$$x, y \in X,\ d(Tx,Ty) \neq 0 \Longrightarrow \alpha(x)\beta(y).\theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^k,$$

where

$$R(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\},$$

(2) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$,

(3) T is a cyclic (α,β)-admissible mapping,

(4)one of the following conditions holds:

(4.1) T is continuous,

(4.2) if $\{x_n\}$ is a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $\beta(x) \geq 1$.

Then T has a fixed point. Furthermore, if $\alpha(x) \geq 1$and $\beta(x) \geq 1$ for every fixed point $x \in X$, then T has a unique fixed point.

Corollary 4.1.8 Let (X, d) be a complete g.m.s and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ is continuous and $k \in (0,1)$ such that

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(R(x,y))]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}.$$

Then T has a unique fixed point.

Proof. Setting $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$ in Theorem 4.1.6, we get this result.

Corollary 4.1.9 [34] Let (X, d) be a complete g.m.s and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ is continuous and $k \in (0,1)$ such that

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(M(x,y))]^k,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Then T has a unique fixed point.

4.2 Some cyclic contractions via cyclic (α, β)-admissible mapping

In 2003, Kirk et al. [4] introduced the concept of cyclic mappings and cyclic contractions as follows.

Definition 4.2.1 [4] Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \longrightarrow A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Definition 4.2.2 [4] Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called a cyclic contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$.

Notice that although a Banach-contraction is continuous, a cyclic contraction need not to be. This is one of the important gains of fixed point results for cyclic mappings, see ([9], [10], [11], [41], [42], [48], [49], [51], [52], [53], [59]).

In this section, we apply Theorem 4.1.1 for proving fixed point theorems involving a cyclic mapping in generalized metric spaces.

Definition 4.2.3 Let A and B be nonempty subsets of a g.m.s (X, d). A mapping $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Theorem 4.2.4 Let A and B be two closed subsets of a complete g.m.s (X, d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \longrightarrow A \cup B$ be a cyclic mapping and $\theta \in \Theta$. Assume that

$$\theta\left(d\left(Tx, Ty\right)\right) \leq \left[\theta\left(d(x, y)\right)\right]^{k},$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

Proof. Define mappings $\alpha, \beta : X \longrightarrow [0, \infty)$ by

$$\alpha\left(x\right) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases} \text{ and } \beta\left(x\right) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{otherwise} \end{cases}.$$

For $x \in A$ and $y \in B$, we get

$$\alpha(x)\beta(y).\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k.$$

Therefore condition (1) of Theorem 4.1.1 holds. It is easy to see that T is a cyclic (α,β)-admissible mapping. Since $A \cap B \neq \emptyset$, there exists $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$. Next, we show that condition (4.2) in Theorem 4.1.1 holds. Let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $x_n \in B$ for all $n \in \mathbb{N}$. Therefore $x \in B$. Which implies $\beta(x) \geq 1$. Now, the conditions (1), (2), (3), and (4.2) of Theorem 4.1.1 hold. So, T has a unique fixed point in $A \cup B$, say z. If $z \in A$, then $z = Tz \in B$. Similarly, if $z \in B$, then we have $z \in A$. Therefore $z \in A \cap B$.

Corollary 4.2.5 Let A and B be two closed subsets of a complete metric space (X,d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \longrightarrow A \cup B$ be a cyclic mapping and $\theta \in \Theta$. Assume that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k,$$

for all $x \in A$ and $y \in B$, where $k \in (0,1)$. Then T has a unique fixed point in $A \cap B$.

Example 4.2.6 Let $X = \mathbb{R}$ endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $T : A \cup B \longrightarrow A \cup B$ be defined by $Tx = -\frac{x}{6}$ where $A = [-1,0]$ and $B = [0,1]$. Also define $\theta : (0,\infty) \longrightarrow (1,\infty)$ by $\theta(t) = e^t$. Then T has a unique fixed point. Indeed, for all $x \in A$ and all $y \in B$, we have

$$\begin{aligned}\theta(d(Tx,Ty)) &= e^{|Tx-Ty|} = e^{\frac{|x-y|}{6}} = \left[e^{|x-y|}\right]^{\frac{1}{6}} \\ &\leq \left[e^{|x-y|}\right]^k, \text{ where } k \in \left[\frac{1}{6},1\right) \\ &= [\theta(d(x,y))]^k.\end{aligned}$$

Therefore, the conditions of Theorem 4.2.4 (or Corollary 4.2.5)hold with $k \in \left[\frac{1}{6}, 1\right)$and T has a unique fixed point (here, $x = 0$ is a unique fixed point of T).

Similarly, we can prove the following theorem.

Theorem 4.2.7 Let A and B be two closed subsets of a complete g.m.s (X, d) such that $A \cap B \neq \emptyset$, $T : A \cup B \longrightarrow A \cup B$ be a cyclic mapping and $\theta \in \Theta$ is continuous. Assume that

$$\theta\left(d\left(Tx, Ty\right)\right) \leq \left[\theta\left(R(x, y)\right)\right]^k;$$

$$R(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\},$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

Corollary 4.2.8 Let A and B be two closed subsets of a complete metric space (X, d) such that $A \cap B$ $\neq \emptyset$, $T : A \cup B \longrightarrow A \cup B$ be a cyclic mapping and $\theta \in \Theta$ is continuous. Assume that

$$\theta\left(d\left(Tx, Ty\right)\right) \leq \left[\theta\left(R(x, y)\right)\right]^k;$$

$$R(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\},$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

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