Effects of MHD on variable viscosity and viscous dissipation in a third grade fluid: HAM solution

By

Arshad Riaz

Reg. No. 26-FBAS/MSMA/F08

Department of Mathematics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan

2010

1- Fluids Mechanics.
2. Variable viscosity.

Effects of MHD on variable viscosity and viscous dissipation in a third grade fluid: HAM solution

By

Arshad Riaz

Supervised by

Dr. Rahmat Ellahi

Department of Mathematics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2010

Effects of MHD on variable viscosity and viscous dissipation in a third grade fluid: HAM solution

By

Arshad Riaz

A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
In
MATHEMATICS

Supervised by

Dr. Rahmat Ellahi

Department of Mathematics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2010

Certificate

Effects of MHD on variable viscosity and viscous dissipation in a third grade fluid: HAM solution

By

Arshad Riaz

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE **MASTRER OF SCIENCE** IN **MATHEMATICS**

We accept this dissertation as conforming to the required standard.

1. ____________________
Dr. Rahmat Ellahi
(Chairman)

2. ____________________
Dr. Rahmat Ellahi
(Supervisor)

3. ____________________
Dr. Malik Muhammad Yousaf
(External Examiner)

4. ____________________
Dr. Ahmer Mehmood
(Internal Examiner)

Department of Mathematics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2010

Dedicated to

My Mamoo, my Ammy, my Abbu (late), my Nana Abbu (late) and all my family, whose prayers and invocations always made me successful and confident.

Acknowledgements

Foremost, I am always grateful to **Almighty ALLAH**, who made human being, the best creation of all the living species and made them understand to write with pen. He provided me the boldness and capability to achieve this task. I offer countless Darood and Salaams to my beloved **Holy Prophet Hazrat Muhammad** (PBUH), for Whom this universe has been manifested. **ALLAH** has shown His existence and oness by sending him as a messenger of Islam and born me as a Muslim.

I offer my most sincere gratitude to my affectionate, sincere, kind and most respected supervisor and The Chairman, Department of Mathematics, **Dr. Rahmat Ellahi**, whose kinetic supervision, admonition in a right inclination and inductance of hard work made my task easy and I completed my dissertation well with in time. His ideology and concepts have a remarkable impact on my research contrivances. He also arranged some suitable facilities, without which my objective might not be attained. I have learnt a lot from his ability.

I am grateful to all of my teachers. They always guided me sincerely and honestly throughout my course work as well as research work.

I offer special thanks to all my friends and my class fellows, who really helped me to their best throughout my research period. I also express special thanks to my close friend **Mr. Ahmad Zeeshan**, who helped me throughout my work, whenever I faced any difficulty relating my problem and consigned a lot of his precious time with me. I don't think that I would have been able to do it up to the caliber without his help.

I pay regards to my uncle **Hussain Ahmad Malik**, my Ammy, my nana Abbu **Hafiz Muhammad Hayat** (late), my naany Amman and all my family, whose sincere prayers, best wishes and loving memories always made me courageous and daring throughout my life. I also include the unseen guidance of my father **Muhammad Riaz** (late), who has always been the pivot of my thoughts.

Finally, I have really no piece of word to oblige the most respected, patron and caring person to me in this world, my mamoo **Prof. Hafiz Maqbool Hussain**, who made my life happy, bright and prosperous and nourished me when I was a child. He made me able to see my golden future now at this stage of my life. My best wishes are always with him and all his family. May he live long, aamin.

Arshad Riaz

Preface

In view of abundant applications of non-Newtonian fluids in industry and technology, the interest in the study of such fluids has been increased during the last few years. Mathematicians, modelers and computer scientist have been involved in carrying out the flow analysis of the non-Newtonian fluids under various aspects. Several constitutive expressions for these fluids have been suggested. These equations differ between the shear stress and rate of strain in view of different characteristics of the non-Newtonian fluids. As a consequence of these constitutive equations, the resulting equations of non-Newtonian fluids in general are more complicated and higher order in comparison to the Navier-Stokes equations. Considerable efforts have been devoted to study the non-Newtonian fluids through analytic and numerical treatments. Some progress on the topic can be mentioned in the studies [1-11]. In all these studies, constant viscosity fluid is used. Massoudi and Christie [12] numerically examined the pipe flow of a third grade fluid when viscosity depends upon the temperature. Ellahi and Afzal [13] reported such solutions when third grade fluid saturates the porous medium.

In chapter one, some basic definitions of fluids and homotopy are presented. Some basic thermo-dynamical laws and basic concept of magneto-hydrodynamics are introduced shortly.

Chapter two comprises the study of influence of variable viscosity and viscous dissipation on non-Newtonian flow. This chapter concerns with the effect of constant and variable viscosity on velocity and temperature distributions for a third grade fluid in a pipe and the review work of Hayat et al [14].

In chapter three, the motivation comes from a desire to understand the magnetic field effects on the pipe flow of a third grade fluid of ref. [14]. The viscosity here depends upon the space coordinate. The relevant equations for flow and temperature have been solved analytically by using homotopy analysis method [15-20]. Convergence of the obtained solutions is explicitly shown. The effects of the various parameters of interest on the velocity and temperature are pointed out. The present analysis is arranged as follows.

In sections 2 and 3, the governing nonlinear equation is modeled and solutions are developed, respectively for constant and variable viscosity by HAM. Convergence of the obtained solutions is shown in section 4. Graphical results and discussion are given in section 5 and 6, respectively.

Contents

Chapter 1

Some elementary and basic descriptions

1.1 Introduction

In this chapter, some basic definitions and concepts of various types of fluids are discussed. Basic equations governing the flow are described. The basic idea of homotopy and advantages of homotopy analysis method (HAM) are also explained in this chapter.

1.2 Basic definitions

1.2.1 Flow

A material undergoes a deformation when different forces act upon it. If the deformation increases continuously without limit, the phenomena is known as flow.

1.2.2 Fluid

A fluid is the substance (liquids and gases) that has tendency to flow under the action of applied shear stress, no matter how small it is. Some of the examples of liquid fluids are water, petrol, diesel, oil, mercury etc. Fluids include liquids, gases and plasmas.

1.2.3 Fluid mechanics

Fluid mechanics is the branch of physical sciences concerned with how fluids behave at rest or in motion. Fluid mechanics can be divided into fluid statics (the study of fluids at rest) and fluid dynamics (the study of fluids in motion).

1.2.4 Deformation

The relative change in position or length of the fluid particles is known as deformation (strain).

1.2.5 Shear stress

A shear stress, denoted by τ (tau) is defined as a stress which is applied parallel or tangential to a face of a material, as opposed to a normal stress which is applied normally (perpendicularly).

1.2.6 Pressure

Pressure is an effect which occurs when a force is applied on a surface per unit area.

Mathematically,

$$P = \frac{F}{A}, \tag{1.1}$$

where P is pressure, F is the normal force and A is the area. The SI unit for pressure is Pascal (Pa), equal to one newton per square meter (Nm^{-2} or $kgm^{-1}s^{-2}$).

1.2.7 Density

The density of a material is defined as its mass per unit volume. Symbolically density is denoted by ρ (the Greek letter rho). In mathematics it is written as

$$\rho = \frac{m}{V}, \tag{1.2}$$

where m is the mass and V is the volume.

1.2.8 Viscosity

Viscosity of the fluid is defined as the property of the fluid that tends to resist the movement of one layer of the fluid over adjacent layer of the fluid. Viscosity is very important property of the fluids. While considering the fluid for various applications it is crucial to consider the viscosity of the fluid. The top layer of the fluid flows at higher speeds, while the layers below it move at slightly lesser speed. Thus the layers of the fluid offer resistance to the flow of the adjoining layers. This property of the fluid is called as the viscosity of the fluid. Viscosity may be constant or depend upon some factors like pressure, temperature, space etc. It is denoted by the symbol μ and is given by

$$\mu = \frac{\text{shear stress}}{\text{rate of shear strain}}. \tag{1.3}$$

It is also termed as dynamic viscosity.

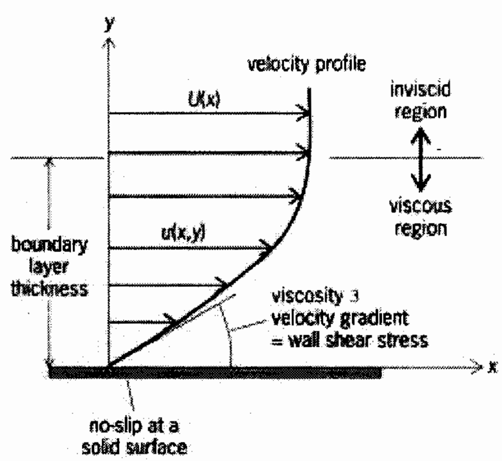

Fig.1.1 : The fluid flows in the form of various layers.

1.2.9 Kinematic viscosity

It is defined as the ratio of dynamic viscosity to fluid density and is denoted by ν.

Mathematically, it is defined as

$$\nu = \frac{\mu}{\rho}, \tag{1.4}$$

1.2.10 Viscous dissipation

An important consequence of the existence of shear viscosity is a loss of energy when fluid is sheared. This frictional energy loss is referred to as viscous dissipation. The general action of viscosity in a fluid flow is a tendency to convert the useful energy content of the fluid into heat. The useful energy lost appears as an increase in the internal energy of the fluid, corresponding to a rise in temperature. The rate of dissipation of energy per unit mass of fluid by the shear viscosity is given by the viscous dissipation ϕ. The viscous dissipation rate at any point in the flow is given by

$$\phi = \frac{2\mu}{\rho}(\frac{du}{dy})^2, \tag{1.5}$$

where du/dy is the rate of shear strain. Viscous dissipation has the dimension $\{L^2t^{-3}\}$ and is usually expressed in units of power per unit mass, i.e., $J/(skg)$ in SI.

1.2.11 Shear thinning effect

Shear thinning is an effect where viscosity decreases with increasing rate of shear stress. Materials that execute shear thinning are called pseudoplastic. There are certain complex solutions such as lava, ketchup, whipped cream, blood, paint and nail polish, which describe such effects.

1.2.12 Shear thickening effect

A shear thickening effect is one in which viscosity of a fluid increases with the rate of shear stress. Fluids which describe such effects are termed as dilatant. Mixture of cornstarch and water (sometimes called oobleck) can easily be seen to perform this effect.

1.3 Classification of fluid

1.3.1 Ideal fluid

A non-existent, assumed fluid without either viscosity or compressibility is called an ideal fluid or perfect fluid. In nature this type of fluid does not exist. Furthermore, a gas subject to *Boyle's-Charle's law* is called a perfect or an ideal gas. It is the hypothetical form of fluids. However, the fluid with negligible viscosity may be considered as an ideal fluid.

1.3.2 Real fluid

Real fluids are those in which fluid friction has significant effects on the fluid motion. In otherwords we can not neglect the viscosity effects on the motion. Real fluids are further classified into two classes on the basis of *Newton's law of viscosity*. According to this law

"shear stress is directly proportional to the rate of deformation". For one dimensional flow it can be written as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.6}$$

where τ_{yx} is the shear stress and du/dy is the rate of deformation.

Newtonian fluid

A Newtonian fluid (named after *Isaac Newton*) is a fluid whose stress versus strain (deformation) rate curve is linear and passes through the origin, i.e., Newtonian fluid obeys *Newton's law of viscosity*. Water, gasoline and mercury are some examples of Newtonian fluids.

Non-Newtonian fluid

A non-Newtonian fluid is a fluid whose flow properties are not described by a single constant value of viscosity, i.e., it does not satisfy *Newton's law of viscosity*. For non-Newtonian fluids

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n, \qquad n \neq 1 \tag{1.7}$$

or

$$\tau_{yx} = \eta\left(\frac{du}{dy}\right), \tag{1.8}$$

where

$$\eta = k\left(\frac{du}{dy}\right)^{n-1} \tag{1.9}$$

is the apparent viscosity. Examples of non-Newtonian fluids are tooth paste, ketchup, gel, shampoo, blood, soaps etc.

1.4 Types of flow

1.4.1 Steady flow

A flow for which the fluid properties (velocity, temperature etc.) remain independent of time. For such flow

$$\frac{\partial \mathbf{V}}{\partial t} = 0, \tag{1.10}$$

where $\mathbf{V}$ is the velocity field and t is the time.

1.4.2 Unsteady flow

A flow for which fluid velocity depends upon time, i.e.,

$$\frac{\partial \mathbf{V}}{\partial t} \neq 0. \tag{1.11}$$

1.4.3 Laminar flow

Fluid flow in which the fluid travels smoothly or in regular paths. The velocity, pressure and other flow properties at each point in the fluid remain constant. Laminar flow over a horizontal surface may be thought of as consisting of thin layers, all parallel to each other, that slide over each other. Examples include the flow of oil through a thin tube and blood flow through capillaries.

1.4.4 Turbulent flow

Turbulent flow is a type of fluid flow in which the fluid undergoes irregular fluctuations or mixing. Most kinds of fluid flow are turbulent, except for laminar flow at the leading edge of solids moving relative to fluids or extremely close to solid surfaces, such as the inside wall of a pipe, or in cases of fluids of high viscosity flowing slowly through small channels. Common examples of turbulent flow are lava flow, atmosphere and ocean currents, the flow through pumps and turbines and the flow in boat wakes and around aircraft-wing tips.

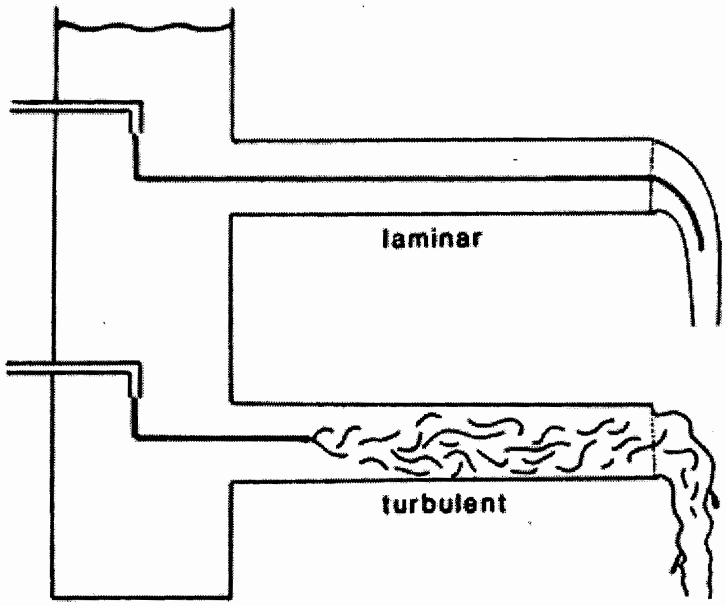

Fig. 1.2 : Laminar and turbulent flow.

1.4.5 Compressible and incompressible flow

All fluids are compressible to some extent, that is changes in pressure or temperature will result in changes in density. However, in many situations the changes in pressure and temperature are sufficiently small that the changes in density are negligible. In this case the flow can be modeled as an incompressible flow. Otherwise the more general compressible flow equations must be used. Mathematically, incompressibility is expressed by saying that the density ρ of a fluid particle does not change as it moves in the flow field, i.e.,

$$\frac{D\rho}{Dt} = 0, \qquad (1.12)$$

where D/Dt is the material derivative, which is the sum of local and convective derivatives.

1.5 Force

Force is a vector quantity, used to demonstrate an impression which causes a free body to undergo a change in velocity, i.e., acceleration.

1.6 Types of force

1.6.1 Surface force

Surface force is short-range force applying on a fluid element through physical contact between the element and its surroundings, e.g. force asserted by pressure or shear stress represents the surface force.

1.6.2 Body force

Body force is long-range force that acts on a small fluid element in such a way that the magnitude of the body force is proportional to the mass of the element. Since the mass is defined to be the product of density and volume, the magnitude of a body force is also proportional to the volume of a fluid element. Thus body force is expressed on a per-unit-volume basis in units such as newtons per cubic meter (Nm^{-3}) or pound-force per cubic foot ($lbfft^{-3}$) and on a per-unit-mass basis with units of acceleration. Gravitational and electromagnetic forces are the common examples of body force. It is usually denoted by the symbol $\mathbf{f}$.

1.7 Heat transfer

Heat transfer is that science which seeks to predict the energy transfer which may take place between material bodies as a result of temperature difference.

1.8 Fundamentals of heat transfer

1.8.1 Conduction

When a temperature gradient exists in a body, experience has shown that there is an energy transfer from higher temperature region to lower temperature region. We say that energy is transferred by conduction and that the heat transfer rate per unit area is proportional to the normal temperature gradient:

$$\frac{q}{A} \propto \frac{\partial \theta}{\partial x}. \tag{1.13}$$

When the proportionality constant is inserted, we reach

$$q = -kA\frac{\partial \theta}{\partial x}, \tag{1.14}$$

where q is the heat transfer rate and $\partial\theta/\partial x$ is the temperature gradient in the direction of heat flow. The positive constant k is called the thermal conductivity of the material.

1.8.2 Convection

It is well known that a hot plate of metal will cool faster when placed in front of a fan than when exposed to still air. We say that the heat is convected away and we call the process convection heat transfer.

1.8.3 Radiation

Radiation heat transfer is concerned with the exchange of thermal radiation energy between two or more bodies. No medium need exist between the two bodies for heat transfer to take place (as is needed by conduction and convection).

1.8.4 Specific heat

Specific heat is the amount of heat or thermal energy required to raise the temperature of a unit quantity of a body by one unit. It is denoted by the symbol c_p. For example, at a temperature of $15°C$, the heat required to raise the temperature of a water sample by $1K$ (equivalent to $1°C$) is 4.186 $kJkg^{-1}K^{-1}$.

1.8.5 Fourier's law of heat conduction

The Fourier's law states that the time rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area at right angles to that gradient through which the heat is flowing.

Mathematically, it is given by

$$\frac{dQ}{dt} = -kA\frac{d\theta}{dx}, \tag{1.15}$$

in which Q is the amount of heat transferred.

Differential form

The differential form of *Fourier's Law* of thermal conduction shows that the local heat flux $\mathbf{q}$ is equal to the product of thermal conductivity k and the negative local temperature gradient $\nabla\theta$. The heat flux is the amount of energy that flows through the surface per unit area per unit time, i.e.,

$$\mathbf{q} = -k\nabla\theta. \tag{1.16}$$

Integral form

If we integrate the differential form over the whole surface S, we get the integral form of *Fourier's law,* that is

$$\frac{\partial Q}{\partial t} = -k\oint_S \nabla\theta \cdot \mathbf{d}A, \tag{1.17}$$

where $\mathbf{d}A$ is total surface area.

1.8.6 Thermal conductivity

Thermal conductivity k is the property of a material that indicates its capability to conduct heat. It appears basically in *Fourier's Law* for heat conduction. Thermal conductivity is measured in watts per kelvin per metre $\left(WK^{-1}m^{-1}\right)$. The reciprocal of thermal conductivity is called thermal resistivity.

1.9 Some Basic laws

1.9.1 Law of conservation of mass

This law states that in any closed system, the mass is always invariant regardless of its changes in shape when external forces are absent or the principle that matter cannot be created or destroyed. In fluid mechanics, this law is also named as equation of continuity.

Mathematically, it is described as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{1.18}$$

1.9.2 Law of conservation of momentum

The law of conservation of momentum states that when some bodies constituting an isolated system act upon one another, the total momentum of the system remains same. It is also recognized as the Navier-Stokes equations derived by *Claude-Louis Navier* and *George Gabriel Stoke*, used to describe the motion of the fluid. In an inertial frame of reference, the general form of the equations of fluid motion is

$$\rho \frac{D\mathbf{V}}{Dt} = \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{f}, \tag{1.19}$$

where $\mathbf{T}$ is the Cauchy stress tensor which is different for different fluids.

1.9.3 Law of conservation of energy

Energy in a system may take on various forms (e.g. kinetic, potential, heat, light). The law of conservation of energy states that energy may neither be created nor destroyed. Therefore, the sum of all the energies in the system is a constant. The laws of conservation of energy which is also called the energy equation is described as

$$\rho c_p \frac{D\theta}{Dt} = \mathbf{T} \cdot \mathbf{L} - \boldsymbol{\nabla} \cdot \mathbf{q}, \tag{1.20}$$

in which

$$\mathbf{L} = \boldsymbol{\nabla} \mathbf{V}. \tag{1.21}$$

1.10 Magnetohydrodynamics (MHD)

Magnetohydrodynamics (MHD) is the academic procedure which studies the dynamics of electrically conducting fluids. Examples of such fluids include plasmas, liquid metals and salt water. The word magnetohydrodynamics (MHD) is derived from magneto- meaning magnetic field; hydro- meaning liquid and dynamics- meaning movement. The field of MHD was initiated by *Hannes Alfvén* for which he received the Nobel Prize in Physics in 1970.

The idea of MHD is that magnetic fields can induce currents in a moving conductive fluid, which create forces on the fluid and also change the magnetic field itself. The set of equations which describe MHD are a combination of the *Navier-Stokes equations* of fluid dynamics and *Maxwell's equations* of electromagnetism.

1.10.1 Lorentz force

The Lorentz force is the force on a point charge due to electromagnetic fields. It is given by the following equation in terms of the electric and magnetic fields

$$\mathbf{F} = q(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{1.22}$$

where $\mathbf{F}$ is the Lorentz force, q is the point charge, $\mathbf{E}$ is the electric field , $\mathbf{V}$ is the instantaneous velocity of the particle and $\mathbf{B}$ is the magnetic field. When there is no electric field present, we can write it as

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}, \tag{1.23}$$

where $\mathbf{J}$ (=$q\mathbf{V}$) is the total current density.

1.10.2 Magnetic field

Magnetic fields are produced by electric currents, which can be macroscopic currents in wires or microscopic currents associated with electrons in atomic orbits. The magnetic field $\mathbf{B}$ is defined in terms of force on moving charge in the *Lorentz force law.* The SI unit for magnetic field is Tesla. We can obtain the expression for magnetic force by neglecting electric field from Eq. (1.22), i.e.,

$$\mathbf{F}_m = q(\mathbf{V} \times \mathbf{B}), \tag{1.24}$$

where $\mathbf{F}_m$ is the magnetic force.

1.10.3 Maxwell's equations

Maxwell's equations are the set of four equations which relate the electric and magnetic field to their sources, charge density and current density. Individually, these equations are known as *Gauss's law, Gauss's law for magnetism, Faraday's law of induction* and *Ampere's law with Maxwell's correction.* These equations are described as

$$\nabla.\mathbf{E} = \frac{\varrho}{\epsilon_0}, \tag{1.25}$$

$$\nabla.\mathbf{B} = \mathbf{0}, \tag{1.26}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.27}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.28}$$

In the above equations ϵ_0 is the permittivity of the free space also called electric constant, μ_0 is the permeability of free space which is also called magnetic constant, ϱ is the total charge density and $\mathbf{J}$ is the total current density. The total magnetic field is $\mathbf{B}$ $(=B_0 + \mathbf{b})$, where $\mathbf{b}$ is induced magnetic field. By *Ohm's law* in generalized form we have

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right), \tag{1.29}$$

where σ is the electric conductivity of the fluid. In the present case there is no applied electric field, also the induced magnetic field is neglected due to the assumption of low magnetic Reynold number. Therefore, the Lorentz force in the direction of the flow in a pipe becomes

$$\left(\mathbf{J} \times \mathbf{B}\right)_z = -\sigma B_0^2 v, \tag{1.30}$$

where B_0 is the applied magnetic field and v is the velocity component normal to the magnetic field and parallel to the flow.

1.11 Dimensionless numbers

1.11.1 Prandtl number

It is the ratio of the product of dynamic viscosity and specific heat to the thermal conductivity. It is denoted by the symbol P_r and is given by

$$P_r = \frac{\mu c_p}{k}. \tag{1.31}$$

1.11.2 Eckert number

The ratio between square of the free stream velocity to the product of specific heat and temperature difference between the body and the surface is recognized as Eckert number and is denoted by E_c, i.e.,

$$E_c = \frac{v_0^2}{c_p(\theta_0 - \theta_\infty)}, \tag{1.32}$$

where v_0 is the free stream velocity and $\theta_0 - \theta_\infty$ represents the temperature difference.

1.12 Boundary layer

The flow region adjacent to the wall in which the viscous effects are significant is called boundary layer.

1.12.1 Boundary layer thickness

The boundary layer thickness, signified by δ is simply the thickness of the viscous boundary layer region. Because the main effect of viscosity is to slow the fluid near a wall, the edge of the viscous region is found at the point where the fluid velocity is essentially equal to the free-stream velocity.

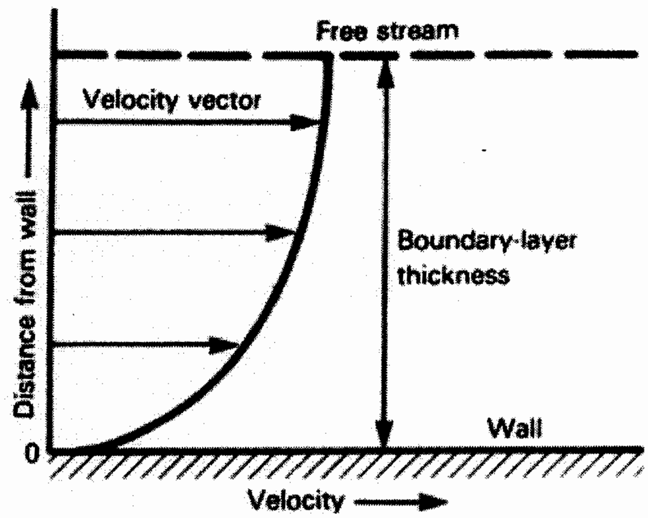

Fig 1.3 : Boundary layer thickness

1.13 Boundary conditions

The set of conditions specified for the behavior of the solution to a set of differential equations at the boundary of its domain. Boundary conditions are important in determining the mathematical solutions to many physical problems.

1.14 Homotopy

A homotopy between two functions f and g from a space X to a space Y is a continuous map H,

$$H : X \times [0,1] \longrightarrow Y, \tag{1.35}$$

such that

$$H(x, o) = f(x) \tag{1.36}$$

and

$$H(x, 1) = g(x), \tag{1.37}$$

where X denotes set pairing and $x \in X$. If we consider the second element in the set $X \times [0, 1]$, then we can say that at time $t = 0$, the function H equals f and at $t = 1$, H becomes g. Two mathematical objects are said to be homotopic if one can be continuously deformed into the other. The concept of homotopy was first formulated by *Poincaré* around 1900 (*Collins* 2004). When two functions f and g are homotopic, we relate them as

$$f \simeq g. \tag{1.38}$$

1.14.1 Homotopy analysis method (HAM)

HAM is the more efficient technique than the others to solve the non linear equations. *Liao* developed this analytical method which is derived from the basic concept of homotopy from topology.

1.14.2 The advantages of HAM

It is valid even if a given non-linear problem does not contain any small/large parameters at all. It provides us an appropriate way to control the convergence of approximation series and adjust convergence regions where essential. It can be employed efficiently to approximate a non-linear problem by selecting various sets of base functions and linear operators.

Chapter 2

The influence of variable viscosity and viscous dissipation on the non-Newtonian flow: An analytical solution

2.1 Introduction

This chapter concerns with the effect of constant and variable viscosity on velocity and temperature distributions for a third grade fluid in a pipe and the review work of Hayat et al [14]. The governing equations are formulated mathematically. The resulting equations are second order non-linear ODE's and solved analytically by well known homotopy analysis method (HAM) upto second order approximation. The impact of different material parameters occurring in the concerned equations is verified graphically.

2.2 Mathematical analysis of the problem

We analyze the steady flow of an incompressible, third grade fluid in a pipe. The $z-$axis is taken along the axis of the flow. The velocity field (in cylindrical coordinates) is given by

$$\mathbf{V} = [0, 0, v(r)]\,. \tag{2.1}$$

By definition of incompressible fluids, the continuity equation (1.18) is

$$\nabla \cdot \mathbf{V} = 0. \tag{2.2}$$

Using Eq. (1.16) in Eq. (1.20), we obtain

$$\rho c_p \frac{D\theta}{Dt} = \mathbf{T} \cdot \mathbf{L} + k\nabla^2 \theta. \tag{2.3}$$

For third grade fluid, we have

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \tag{2.4}$$

where p_1 is hydrostatic pressure, $\mathbf{I}$ is the identity tensor and $\alpha_i (i = 1, 2)$ and $\beta_j (j = 1, 2)$ are material constants. The first three *Rivlin-Ericksen* tensors ($\mathbf{A}_1 - \mathbf{A}_3$) are defined by the following general relations

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^t, \tag{2.5}$$

$$\mathbf{A_n} = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^t\mathbf{A}_{n-1}, \quad n > 1. \tag{2.6}$$

Thermodynamical limitations [21] comprise

$$\mu \geq 0,\ \alpha_1 \geq 0,\ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},\ \beta_1 = \beta_2 = 0,\ \beta_3 \geq 0. \tag{2.7}$$

Using Eq. (2.7) in Eq. (2.4), we have

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{2.8}$$

Using the velocity field given in Eq. (2.1), we obtain

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{dv}{dr} & 0 & 0 \end{bmatrix}, \qquad \mathbf{L}^t = \begin{bmatrix} 0 & 0 & \frac{dv}{dr} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.9}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^t = \begin{bmatrix} 0 & 0 & \frac{dv}{dr} \\ 0 & 0 & 0 \\ \frac{dv}{dr} & 0 & 0 \end{bmatrix}, \tag{2.10}$$

$$\mathbf{A}_1\mathbf{L} + \mathbf{L}^t\mathbf{A}_1 = \begin{bmatrix} 2\left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.11}$$

For steady flow

$$\frac{\partial \mathbf{A}_1}{\partial t} = 0, \tag{2.12}$$

so

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^t\mathbf{A}_1 = \begin{bmatrix} 2\left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.13}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{dv}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{dv}{dr}\right)^2 \end{bmatrix}, \tag{2.14}$$

$$tr(\mathbf{A}_1^2)\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 2\left(\frac{dv}{dr}\right)^3 \\ 0 & 0 & 0 \\ 2\left(\frac{dv}{dr}\right)^3 & 0 & 0 \end{bmatrix}, \tag{2.15}$$

$$\left.\begin{aligned} \tau_{rr} = -p_1 + (2\alpha_1 + \alpha_2)\left(\tfrac{dv}{dr}\right)^2, \ \tau_{r\theta} = 0 = \tau_{\theta r}, \ \tau_{rz} = \mu\tfrac{dv}{dr} + 2\beta_3\left(\tfrac{dv}{dr}\right)^3 = \tau_{zr}, \\ \tau_{\theta\theta} = -p_1, \ \tau_{\theta z} = 0 = \tau_{z\theta}, \ \tau_{zz} = -p_1 + \alpha_2\left(\tfrac{dv}{dr}\right)^2. \end{aligned}\right\} \tag{2.16}$$

Neglecting body forces and using cylindrical coordinates (for the flow in a pipe), the momentum equation (1.19) will be in simplified form as

$$\frac{1}{r}\frac{d}{dr}\left[r\mu\left(\frac{dv}{dr}\right)\right]+\frac{2\beta_3}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right]=\frac{\partial\hat{p}}{\partial z}, \tag{2.17}$$

subject to the boundary conditions

$$v(R)=0, \qquad \frac{dv}{dr}(0)=0, \tag{2.18}$$

where

$$\hat{p}=p_1-\alpha_2\left(\frac{dv}{dr}\right)^2 \tag{2.19}$$

is the modified pressure. Now using the definition of product of two tensors, we have

$$\mathbf{T.L}=tr(\mathbf{TL})=\tau_{zr}\frac{dv}{dr}, \tag{2.20}$$

so

$$\mathbf{T.L}=\mu\left(\frac{dv}{dr}\right)^2+2\beta_3\left(\frac{dv}{dr}\right)^4 \tag{2.21}$$

and

$$\nabla^2\theta=\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right). \tag{2.22}$$

The energy equation (2.3) becomes

$$\mu\left(\frac{dv}{dr}\right)^2+2\beta_3\left(\frac{dv}{dr}\right)^4+k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right]=0. \tag{2.23}$$

The relating boundary conditions are

$$\theta(R)=0, \qquad \frac{d\theta}{dr}(0)=0. \tag{2.24}$$

Using non-dimensionalization criteria, we set

$$v=\frac{\overline{v}}{v_0},\ r=\frac{\overline{r}}{R},\ \mu=\frac{\overline{\mu}}{\mu_0},\ \theta=\frac{\overline{\theta}-\theta_0}{\theta_1-\theta_0} \tag{2.25}$$

and introducing the dimensionless parameters

$$c_1 = \frac{\partial \hat{p}}{\partial z}, \; c = \frac{c_1 R^2}{v_0 \mu_0}, \; \Lambda = \frac{2\beta_3 v_0^2}{\mu_0 R^2}, \; \Gamma = \frac{\mu_0 v_0^2}{k(\theta_1 - \theta_0)}, \tag{2.26}$$

the boundary value problems consisting of Eqs. (2.17), (2.18), (2.23) and (2.24) become

$$\frac{1}{r}\frac{d}{dr}\left[r\mu\left(\frac{dv}{dr}\right)\right] + \frac{\Lambda}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right] = c, \tag{2.27}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right] = 0, \tag{2.28}$$

$$v(1) = \theta(1) = 0, \qquad \frac{dv}{dr}(0) = \frac{d\theta}{dr}(0) = 0, \tag{2.29}$$

in which R, v_0, μ_0, θ_0, $\overline{\theta}$ and θ_1 are the radius, reference velocity, reference viscosity, reference temperature, pipe and fluid temperatures, respectively. Also, c_1 is the axial pressure drop, Λ is third grade parameter and Γ is related to the Prandtl and Eckert numbers. For simplicity we have omitted the bar symbols.

2.3 Solution of the problem

We use homotopy analysis method (HAM) to solve the problem under consideration.

Case I: For the constant viscosity $\mu = 1$

When we use $\mu = 1$, the governing equations (2.27) and (2.28) in simplified form reduce to

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + 3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 = c \tag{2.30}$$

and

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2 + \Lambda\Gamma\left(\frac{dv}{dr}\right)^4 = 0, \tag{2.31}$$

respectively. We use the method of higher order differential mapping [22], to choose the linear operator $\mathcal{L}$, i.e.,

$$\mathcal{L} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}. \tag{2.32}$$

The above operator satisfies the following relation

$$\mathcal{L}[C_1 + C_2 \ln r] = 0. \tag{2.33}$$

Here C_1 and C_2 are the arbitrary constants. Integrating the linear part of Eq. (2.30), we get

$$v_0(r) = \frac{1}{4}c(r^2 - 1), \tag{2.34}$$

as the initial approximation of velocity v, which satisfies the linear operator $\mathcal{L}$ and boundary conditions too.

Zeroth order deformation equation

For non-zero auxiliary parameter $\hbar$ and an embedding parameter $p \in [0, 1]$, the zeroth order deformation equation in HAM is given by the following relation

$$(1-p)\mathcal{L}[v^*(r,p) - v_0(r)] = p\hbar\left[\frac{d^2v^*}{dr^2} + \frac{1}{r}\frac{dv^*}{dr} + 3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 - c\right], \tag{2.35}$$

subject to the following boundary conditions

$$v^*(1,p) = 0, \qquad \frac{dv^*}{dr}(0,p) = 0. \tag{2.36}$$

mth order deformation equation

If we differentiate $m-$times the zeroth order deformation equations (2.35) and (2.36) with respect to p, dividing by $m!$ and finally taking $p = 0$, we have the mth order deformation equation, of the following form

$$\mathcal{L}\left[v_m - \chi_m v_{m-1}\right] = \hbar R_m(r), \tag{2.37}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{2.38}$$

and

$$R_m(r) = v''_{m-1} + \frac{1}{r}v'_{m-1} + \Lambda\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{j=0}^{k} v'_{k-j}(\frac{1}{r}v'_j + 3v''_j) - c(1-\chi_m). \tag{2.39}$$

Corresponding boundary conditions take the following form

$$v'_m(0) = v_m(1) = 0, \tag{2.40}$$

where prime denotes the differentiation with respect to r. From Eq. (2.35) by setting $p = 0$, it can be shown that

$$v^*(r,p) = v_0(r). \tag{2.41}$$

By the definition of homotopy, as p varies from 0 to 1, $v^*(r,p)$ varies from initial guess $v_0(r)$ to the exact solution $v(r)$, that is for properly chosen $\hbar$, we get

$$v^*(r,p) = v(r) \qquad \text{for } p = 1. \tag{2.42}$$

Then employing the Taylor's theorem, we can write

$$v^*(r,p) = v_0(r) + \sum_{m=1}^{\infty} v_m(r)p^m, \tag{2.43}$$

where

$$v_m(r) = \frac{1}{m!}\frac{\partial^m v^*(r,p)}{\partial p^m}\bigg|_{p=0}. \tag{2.44}$$

Now using Eq. (2.42) in Eq. (2.43), we get

$$v(r) = v_0(r) + \sum_{m=1}^{\infty} v_m(r). \tag{2.45}$$

Differentiate Eq. (2.35) with respect to p and set $p = 0$, then after solving the resulting equation we obtain the following

$$v_1(r) = \frac{1}{32}\hbar\Lambda c^3(r^4 - 1). \tag{2.46}$$

Again differentiating Eq. (2.35) with respect to p, putting $p = 0$ and using the similar procedure, we get

$$v_2(r) = \frac{1}{32}\hbar\Lambda c^3(r^4 - 1)(\hbar + 1) + \frac{1}{64}\hbar^2\Lambda^2 c^5\left(r^6 - 1\right). \tag{2.47}$$

Now from Taylor series, we have the three terms solution as

$$v(r) = v_0(r) + v_1(r) + v_2(r). \tag{2.48}$$

Finally, inserting Eqs. (2.34), (2.46) and (2.47) in above relation, we get the expression for velocity as follows

$$v(r) = \frac{1}{4}c(r^2 - 1) + \frac{1}{32}\hbar\Lambda c^3(r^4 - 1)(\hbar + 2) + \frac{1}{64}\hbar^2\Lambda^2 c^5(r^6 - 1). \tag{2.49}$$

Now using Eqs. (2.31) and (2.49), with boundary conditions (2.29), we can find θ by using Cauchy-Euler equation and computer software, 'mathematica'. The result is given below

$$\begin{aligned} \theta(r) \;=\; & B_1(r^4 - 1) + B_2(r^6 - 1) + B_3(r^8 - 1) + B_4(r^{10} - 1) \\ & + B_5(r^{12} - 1) + B_6(r^{14} - 1) + B_7(r^{16} - 1) + B_8(r^{18} - 1) \\ & + B_9(r^{20} - 1) + B_{10}(r^{22} - 1). \end{aligned} \tag{2.50}$$

The calculated values of coefficients $B_i (i = 1, 2, ...10)$ are given in Appendix A.

Case II: For the variable viscosity $\mu = r$

Let us now assume that the viscosity is space dependent and choose $\mu = r$. From Eq. (2.27), we have

$$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) + \frac{\Lambda}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right] = c, \tag{2.51}$$

$$r\frac{d^2v}{dr^2} + 2\frac{dv}{dr} + 3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 = c, \tag{2.52}$$

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + \frac{3\Lambda}{r}\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} + \frac{\Lambda}{r^2}\left(\frac{dv}{dr}\right)^3 = \frac{c}{r}, \tag{2.53}$$

with boundary conditions (2.29). Similarly, Eq. (2.28) simplifies to

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\left(\frac{d\theta}{dr}\right) + \Gamma r\left(\frac{dv}{dr}\right)^2 + \Gamma\Lambda\left(\frac{dv}{dr}\right)^4 = 0, \tag{2.54}$$

which corresponds to the boundary conditions (2.29). The linear operator in this case will be

$$\mathcal{L}_1 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}, \tag{2.55}$$

which gives

$$\mathcal{L}_1\left[C_3 + \frac{C_4}{r}\right] = 0, \tag{2.56}$$

where C_3 and C_4 are constants of integration. Thus the initial approximation for the velocity v is

$$v_0\left(r\right) = \frac{1}{6}c(r^2 - 1). \tag{2.57}$$

With the use of Eq. (2.53), one can define the zeroth order deformation equation for v as

$$(1-p)\mathcal{L}_1[v^*(r,p) - v_0(r)] = p\hbar\left[\frac{d^2v^*}{dr^2} + \frac{2}{r}\frac{dv^*}{dr} + \frac{3\Lambda}{r}\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} + \frac{\Lambda}{r^2}\left(\frac{dv^*}{dr}\right)^3 - \frac{c}{r}\right] \tag{2.58}$$

and boundary conditions will be same as in Eq. (2.36). The expression for θ can also be defined in the same manner. The mth order deformation equation can be obtained by using similar procedure like that of given in case I. Following the same procedure, we find three terms series solution of v as follows

$$v(r) = \frac{1}{6}c(r^2-1) + \frac{1}{6}\hbar c(r^2-1)(\hbar+2) + \frac{2}{81}\hbar\Lambda c^3(2\hbar+1)(r^3-1) - \frac{1}{2}\hbar c(\hbar+2)(r-1) + \frac{1}{324}\hbar^2\Lambda^2 c^5(r^4-1) - \frac{1}{12}\hbar^2\Lambda c^3(r^2-1). \quad (2.59)$$

For finding the solution of temperature θ, we use 'mathematica' to solve the Cauchy-Euler equation. Then we obtain

$$\theta(r) = B_{11}(r^2-1) + B_{12}(r^3-1) + B_{13}(r^4-1) + B_{14}(r^5-1) + B_{15}(r^6-1) + B_{16}(r^7-1) + B_{17}(r^8-1) + B_{18}(r^9-1) + B_{19}(r^{10}-1) + B_{20}(r^{11}-1) + B_{21}(r^{12}-1) + B_{22}(r^{13}-1) + B_{23}(r^{14}-1), \quad (2.60)$$

where the coefficients $B_j(j = 11, 12, ...23)$ are given in Appendix A.

2.4 Graphical explanation

In this section, we discussed the results of velocity and temperature profiles for both constant and variable viscosity with the help of graphs.

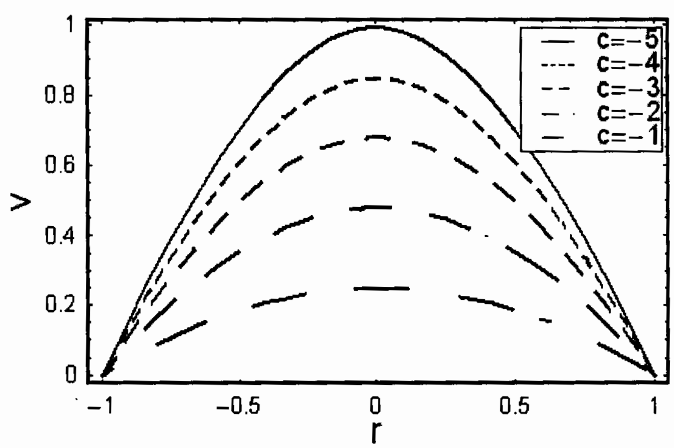

Fig. 2.1 : Influence of c on velocity when $\Lambda = 1$.

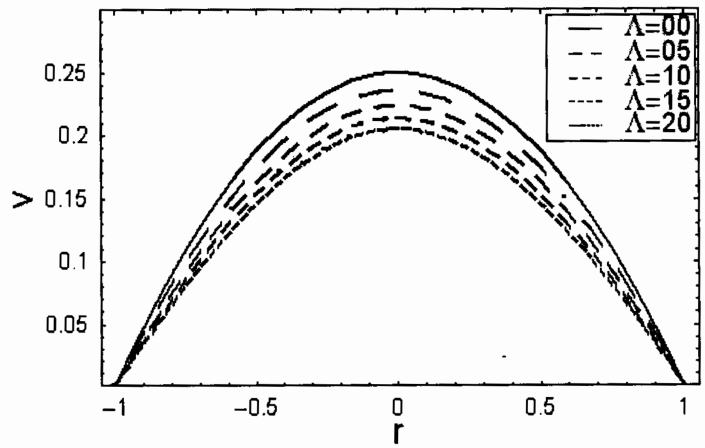

Fig. 2.2 : Influence of Λ on velocity when $c = -1$.

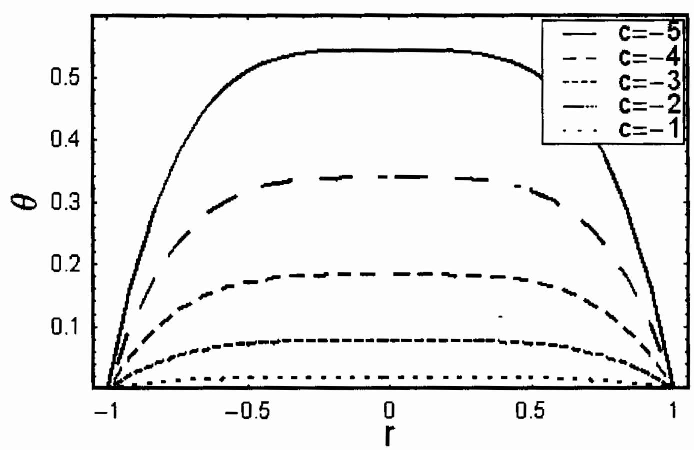

Fig. 2.3 : Influence of c on temperature when $\Gamma = 1.2,\ \Lambda = 1$.

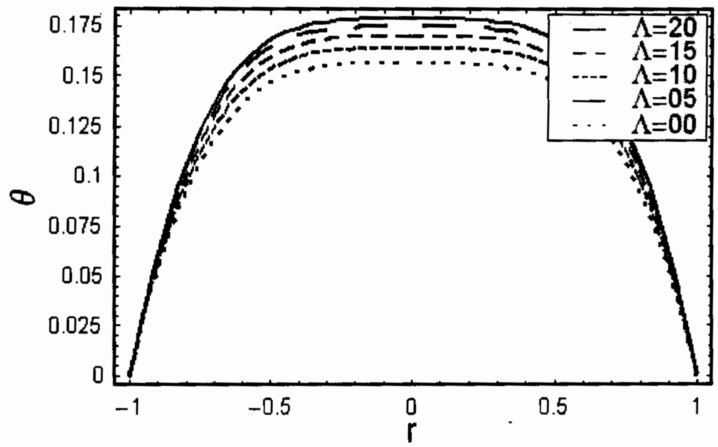

Fig. 2.4 : Influence of Λ on temperature when $c = -1, \ \Gamma = 10$.

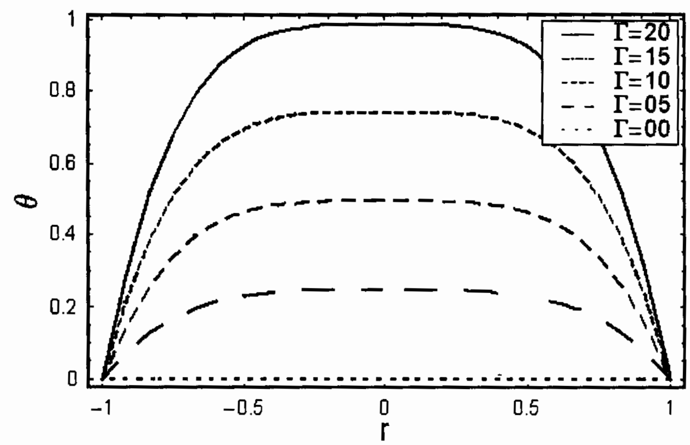

Fig. 2.5 : Influence of Γ on temperature when $c = -1.75, \ \Lambda = 1$.

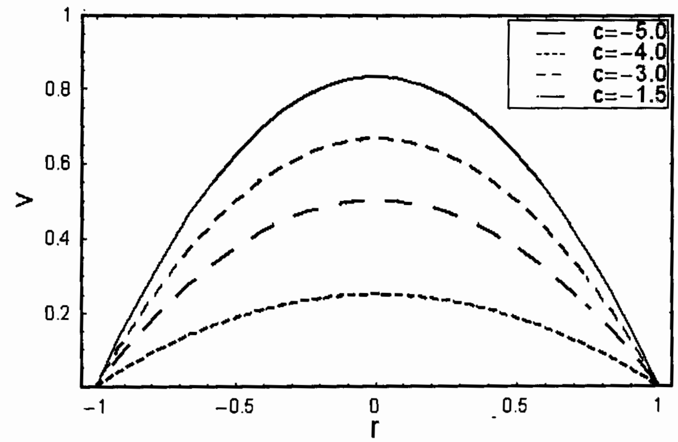

Fig. 2.6 : Influence of c on velocity when $\Lambda = 1$.

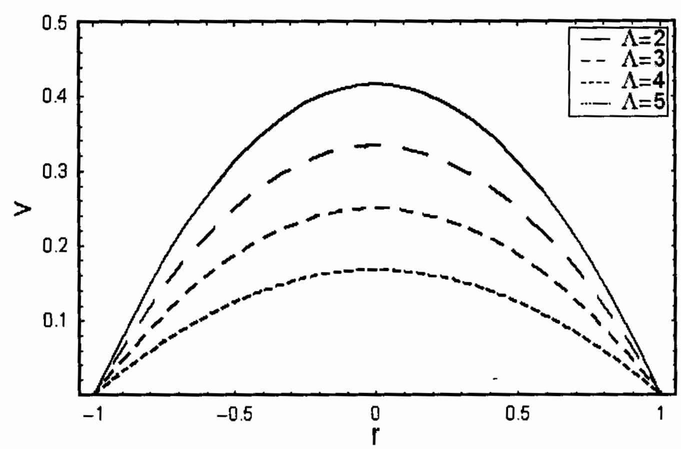

Fig. 2.7 : Influence of Λ on velocity when $c = -1$.

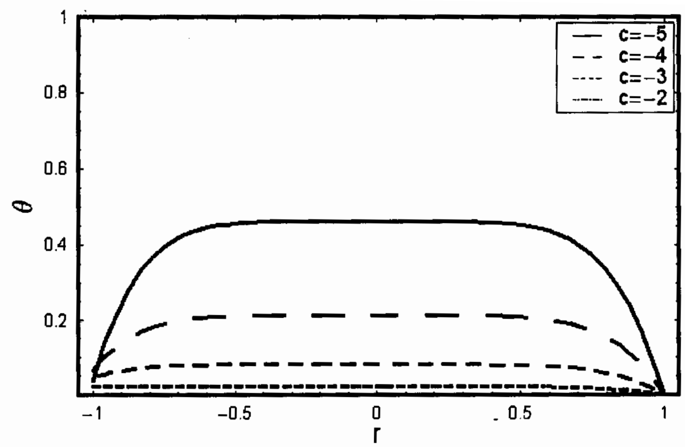

Fig. 2.8 : Influence of c on temperature when $\Gamma = 30,\ \Lambda = 3.35$.

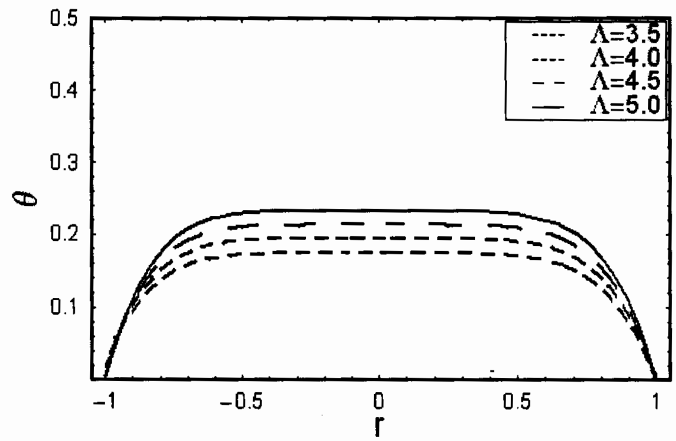

Fig. 2.9 : Influence of Λ on temperature when $\Gamma = 20,\ c = -5.4$.

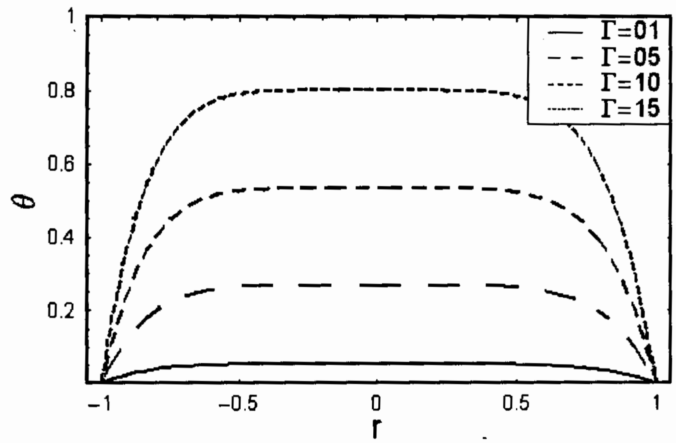

Fig. 2.10 : Influence of Γ on temperature when $c = -8.8$, $\Lambda = 1.35$.

2.5 Results and discussions

As mentioned above, the solution for the velocity and temperature distributions are plotted against the pipe radius. Figs. 2.1 to 2.5 show the variation of velocity and temperature profiles for constant viscosity case and for space dependent viscosity, Figs. 2.6 to 2.10 are presented. In these figures, the variation of the velocity v and temperature θ with the emerging parameters Λ, c and Γ is revealed.

In Fig. 2.1, the effect of pressure gradient c is depicted (when $\hbar$ is approximately equal to -0.05). It is clear that the velocity approaches its maximums at the center of the pipe and varies inversely with c. Also, the effect of c on θ (in Fig. 2.3) is similar to that of velocity. The effect of third grade parameter Λ on the velocity and temperature distributions are shown in Figs. 2.2 and 2.4, respectively. As expected, an increase in Λ results in a decrease in both velocity and temperature. However, the temperature profile is more flatter than the velocity profile for same values of Λ. Fig. 2.5 illustrates the effect of the parameter Γ on temperature distribution θ. It is concluded that θ increases with the increase of Γ and hence the thermal boundary layer thickness decreases.

So far, we disclosed the results of the velocity and temperature for constant viscosity model. Now we turn our consideration to the discussion of above mentioned parameters for space dependent viscosity. Figs. 2.6 to 2.10 represent the influence of all dealing parameters (c, Λ

and Γ) on both, velocity and temperature solutions when viscosity is depending upon space. From these figures, it is observed that the impact of c, Λ and Γ on v and θ (when $\hbar$ is nearly equal to -0.01) is similar to that of constant viscosity case.

Chapter 3

Effects of MHD on variable viscosity and viscous dissipation in a third grade fluid: HAM solution

3.1 Introduction

In this Chapter, we extend the work of the preceding chapter by imposing the MHD term in momentum equation. Analytic solutions have been obtained by homotopy analysis method (HAM). The influence of different parameters appeared in the resulting equations is described graphically and obtained consequences are discussed.

3.2 Mathematical structure

Consider the MHD steady flow of a third grade fluid in a pipe. The fluid is electrically conducting in the presence of an applied magnetic field B_0. The electric and induced magnetic field are neglected. The viscosity of the fluid is not constant. The flow is maintained due to constant pressure gradient. In the presence of MHD as a body force, the momentum equation (1.19) becomes

$$\rho\frac{D\mathbf{V}}{Dt} = \nabla \cdot \mathbf{T} + \mathbf{J} \times \mathbf{B}. \tag{3.1}$$

After simplification, the governing equations (3.1) and (2.3), respectively reduce to

$$\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dv}{dr}\right)+\frac{2\beta_3}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right]=\frac{\partial\widehat{p}}{\partial z}+\sigma B_0^2\, v, \tag{3.2}$$

$$\mu\left(\frac{dv}{dr}\right)^2+2\beta_3\left(\frac{dv}{dr}\right)^4+k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right]=0, \tag{3.3}$$

with respect to the boundary conditions (2.29).

Now using Eqs. (2.25), (2.26) and introducing a new parameter

$$M^2=\frac{\sigma B_0^2R^2}{\mu_0}, \tag{3.4}$$

the above Eqs. (3.2) and (3.3) are non-dimensionalized as

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\frac{dv}{dr}+\mu\frac{d^2v}{dr^2}+3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3=c+M^2v \tag{3.5}$$

and

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2\left[\mu+\Lambda\left(\frac{dv}{dr}\right)^2\right]=0, \tag{3.6}$$

respectively, where M is representing the MHD parameter.

3.3 Solution of the problem

Our interest in this section is to carry out the analysis for the homotopy solutions for two cases of viscosity and viscous dissipation.

Case I: For constant viscosity model $\mu=1$

The zeroth-order deformation problems become

$$(1-p)\mathcal{L}[v^*(r,p)-v_0(r)]=p\hbar\mathcal{N}_1[v^*(r,p),\theta^*(r,p)], \tag{3.7}$$

$$(1-p)\mathcal{L}[\theta^*(r,p)-\theta_0(r)]=p\hbar\mathcal{N}_2[v^*(r,p),\theta^*(r,p)], \tag{3.8}$$

$$v^*(1,p)=\theta^*(1,p)=0,\qquad \left.\frac{\partial v^*(r,p)}{\partial r}\right|_{r=0}=\left.\frac{\partial\theta^*(r,p)}{\partial r}\right|_{r=0}=0, \tag{3.9}$$

$$\mathcal{N}_1[v^*(r,p),\theta^*(r,p)] = \frac{1}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 + 3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} - M^2v^* - c, \qquad (3.10)$$

$$\mathcal{N}_2[v^*(r,p),\theta^*(r,p)] = \frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2} + \Gamma\left(\frac{dv^*}{dr}\right)^2 + \Gamma\Lambda\left(\frac{dv^*}{dr}\right)^4. \qquad (3.11)$$

For $p = 0$ and $p = 1$, we have

$$v^*(r,0) = v_0(r),\ \ \theta^*(r,0) = \theta_0(r) \text{ and } v^*(r,1) = v(r),\ \ \theta^*(r,1) = \theta(r). \qquad (3.12)$$

By Taylor's theorem, we have

$$v^*(r,p) = v_0(r) + \sum_{m=1}^{\infty} v_m(r)p^m, \qquad \theta^*(r,p) = \theta_0(r) + \sum_{m=1}^{\infty}\theta_m(r)p^m, \qquad (3.13)$$

where

$$v_m(r) = \frac{1}{m!}\left.\frac{\partial^m v^*(r,p)}{\partial p^m}\right|_{p=0}, \qquad \theta_m(r) = \frac{1}{m!}\left.\frac{\partial^m \theta^*(r,p)}{\partial p^m}\right|_{p=0}. \qquad (3.14)$$

The convergence of the series (3.13) depends upon $\hbar$. We choose $\hbar$ in such a way that the series (3.13) is convergent at $p = 1$, then due to Eq. (3.12) we have

$$v(r) = v_0(r) + \sum_{m=1}^{\infty} v_m(r), \qquad \theta(r) = \theta_0(r) + \sum_{m=1}^{\infty}\theta_m(r). \qquad (3.15)$$

The mth order deformation problems are

$$\mathcal{L}[v_m(r) - \chi_m v_{m-1}(r)] = \hbar\Re 1_m(r), \qquad (3.16)$$

$$\mathcal{L}[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar\Re 2_m(r), \qquad (3.17)$$

$$v_m(1) = \theta_m(1) = 0, \qquad v_m'(0) = \theta_m'(0) = 0, \qquad (3.18)$$

where

$$\begin{aligned}\Re 1_m(r) \ &= \ \frac{1}{r}\frac{dv_{m-1}}{dr} + \frac{d^2v_{m-1}}{dr^2} + \frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-i}}{dr}\frac{dv_i}{dr} + \\ &3\Lambda\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-i}}{dr}\frac{d^2v_i}{dr^2} - (1-\chi_m)\,c - M^2v_{m-1} \qquad (3.19)\end{aligned}$$

and

$$\Re 2_m(r) = \frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2} + \Gamma\sum_{k=0}^{m-1}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr} + \Lambda\Gamma\sum_{k=0}^{m-1}\sum_{j=0}^{k}\sum_{i=0}^{j}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr}. \quad (3.20)$$

Case II: For space dependent viscosity $\mu = r$

The *zerothħ*− and *mthħ*−order deformation problems are

$$(1-p)\mathcal{L}_1[v^*(r,p) - v_0(r)] = p\hbar\mathcal{N}_3[v^*(r,p), \theta^*(r,p)], \quad (3.21)$$

$$(1-p)\mathcal{L}[\theta^*(r,p) - \theta_0(r)] = p\hbar\mathcal{N}_4[v^*(r,p), \theta^*(r,p)], \quad (3.22)$$

$$\mathcal{L}_1[v_m(r) - \chi_m v_{m-1}(r)] = \hbar\Re 3_m(r), \quad (3.23)$$

$$\mathcal{L}[\theta_m(r) - \chi_m\theta_{m-1}(r)] = \hbar\Re 4_m(r). \quad (3.24)$$

Here

$$\mathcal{N}_3[v^*(r,p), \theta^*(r,p)] = \frac{2}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\Lambda}{r^2}\left(\frac{dv^*}{dr}\right)^3 + \frac{3\Lambda}{r}\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} - \frac{M^2v^*}{r} - \frac{c}{r}, \quad (3.25)$$

$$\mathcal{N}_4[v^*(r,p), \theta^*(r,p)] = \frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2} + \Gamma\left(\frac{dv^*}{dr}\right)^2 + \Gamma\Lambda\left(\frac{dv^*}{dr}\right)^4 + \Gamma r\left(\frac{dv^*}{dr}\right)^2, \quad (3.26)$$

$$\Re 3_m(r) = 2r\frac{dv_{m-1}}{dr} + r^2\frac{d^2v_{m-1}}{dr^2} + \Lambda\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-i}}{dr}\frac{dv_i}{dr} + 3\Lambda r\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-i}}{dr}\frac{d^2v_i}{dr^2} - (1-\chi_m)cr - M^2rv_{m-1} \quad (3.27)$$

and

$$\Re 4_m(r) = \frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2} + \Gamma r \sum_{k=0}^{m-1} \left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr}$$
$$+\Lambda\Gamma \sum_{k=0}^{m-1}\sum_{j=0}^{k}\sum_{i=0}^{j} \left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr}. \quad (3.28)$$

3.4 Convergence of the solution

It is noticed that the explicit, analytical expressions (3.16), (3.17), (3.23) and (3.24) contain the auxiliary parameter $\hbar$. As specified by *Liao* [23], the convergence region and rate of approximations given by the HAM are strongly dependent upon $\hbar$. Figs. 3.1 and 3.2 potray the $\hbar-$curves of velocity and temperature profiles, respectively just to find the range of $\hbar$ in case of constant viscosity. The range for admissible values of $\hbar$ for velocity in this case is $-2.4 \leq \hbar \leq 0.4$ and for temperature is $-2.2 \leq \hbar \leq 0.5$. Figs. 3.4 and 3.5 represent the $\hbar-$curves for variable viscosity when $\mu = r$. The admissible ranges for both velocity and temperature profiles are $-3 \leq \hbar \leq 0.4$ and $-2.8 \leq \hbar \leq 0.8$, respectively. In Figs. 3.3 and 3.6, the graphs of residual errors for constant and variable viscosity are plotted, respectively. The error norm 2 of two consecutive approximations over $[0, 1]$ with HAM by $10th$ order approximations are calculated by

$$E_2 = \sqrt{\frac{1}{11}\sum_{i=0}^{10}[v_{10}(i/10)]^2} = f \quad \text{(say)}. \quad (3.29)$$

It is seen that the error is minimum at $\hbar = -0.05$ for constant viscosity and $\hbar = -0.01$ for variable viscosity. These values of $\hbar$ also lie in the admissible range of $\hbar$.

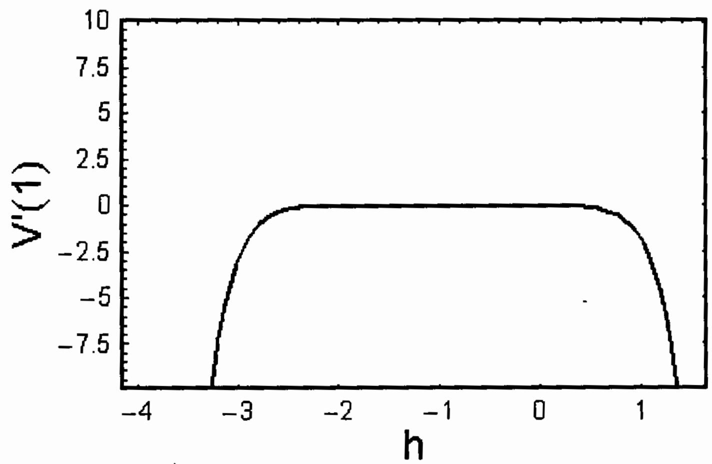

Fig. 3.1: $\hbar$−curve for velocity in case of constant viscosity at $10th$ order approximation when $c = -0.1$, $\Lambda = 0.1$, $\Gamma = 1$, $M = 0.2$.

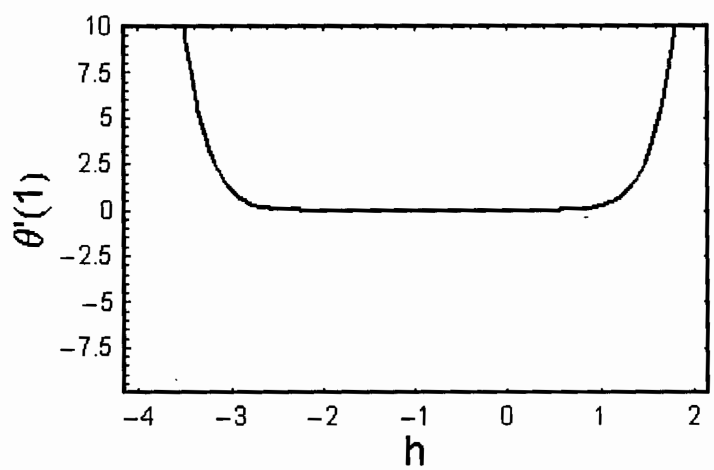

Fig. 3.2: $\hbar$−curve for temperature in case of constant viscosity at $10th$ order approximation when $c = -0.1$, $\Lambda = 0.1$, $\Gamma = 1$, $M = 0.2$.

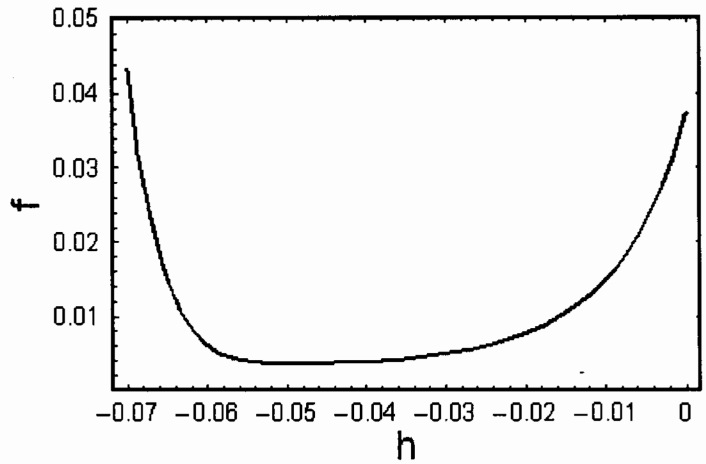

Fig. 3.3: Residual error curve for constant viscosity when $c = -1.3$, $\Lambda = 9$, $\Gamma = 1$, $M = 7$.

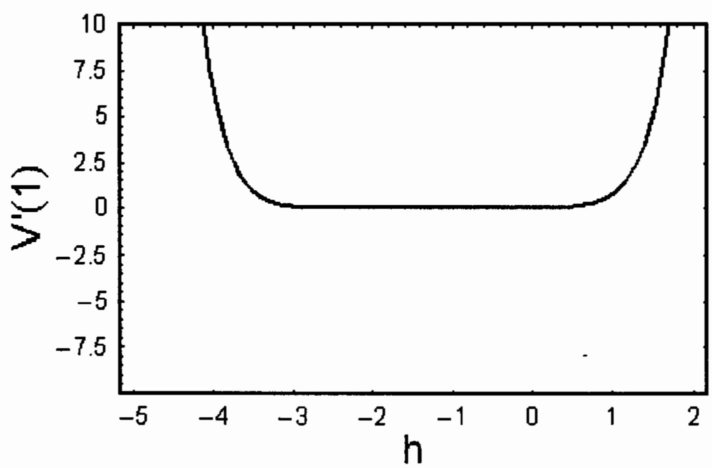

Fig. 3.4: $\hbar$−curve for velocity in case of variable viscosity at $10t\hbar$ order approximation when $c = -0.1$, $\Lambda = 0.1$, $\Gamma = 1$, $M = 0.2$.

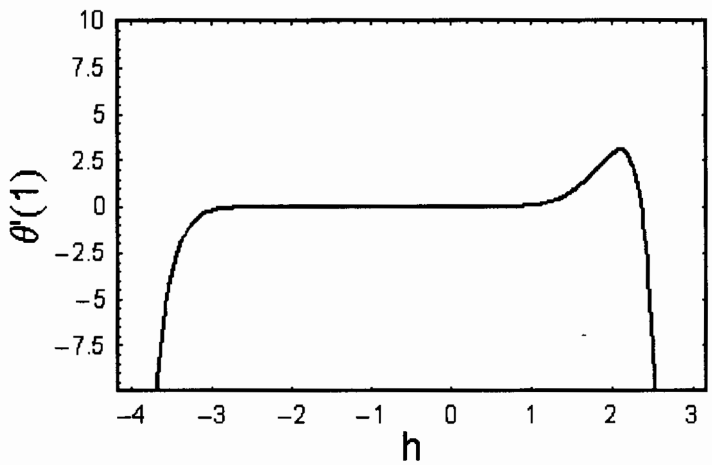

Fig. 3.5: $\hbar$−curve for temperature in case of variable viscosity at $10th$ order approximation when $c = -0.1$, $\Lambda = 0.1$, $\Gamma = 1$, $M = 0.2$.

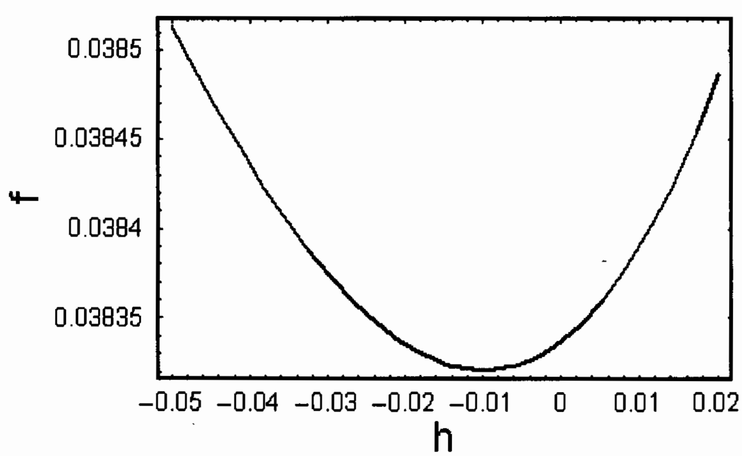

Fig. 3.6: Residual error curve for variable viscosity when $c = -2$, $\Lambda = 1.17$, $\Gamma = 1$, $M = 1$.

3.5 Graphical results

Here, the solutions of the velocity v and temperature θ distributions are plotted against the pipe radius. The velocity and temperature plots for constant viscosity are described in the Figs. 3.7 to 3.13 and Figs. 3.14 to 3.20 represent the graphs for the case of space dependent viscosity. In these figures, the variations for pressure gradient c, third grade parameter Λ, Γ which is related to the Prandtl and Eckert numbers and MHD parameter M are illustrated. From Figs. 3.7 to 3.9, It can be seen that velocity is inverse function of c and Λ. The effects of above mentioned parameters on temperature is similar to that of velocity which are shown in Figs. 3.10 and 3.11. In Figs. 3.9 and 3.13, it is clear that both v and θ are decreasing functions of M. From Fig. 3.12, one can easily observe that θ increases with increasing Γ. For variable viscosity, the effects of c, Λ, Γ and M are qualitatively same, but differ quantitatively as compared with the constant viscosity case (see Figs. 3.14 – 3.20).

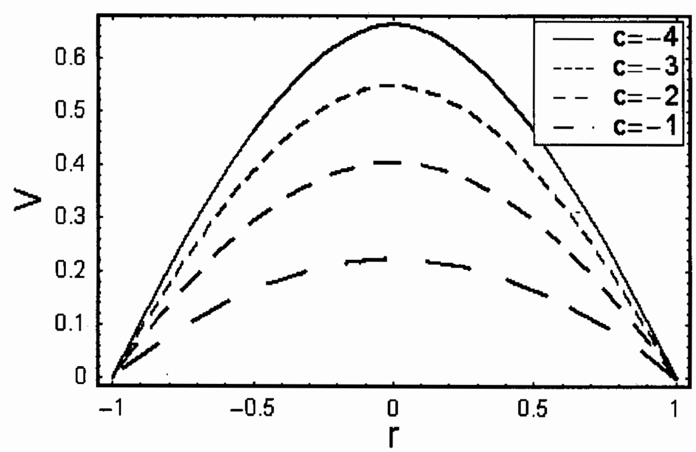

Fig. 3.7: Influence of c on velocity when $\Lambda = 1$, $M = 1$, $h = -0.05$.

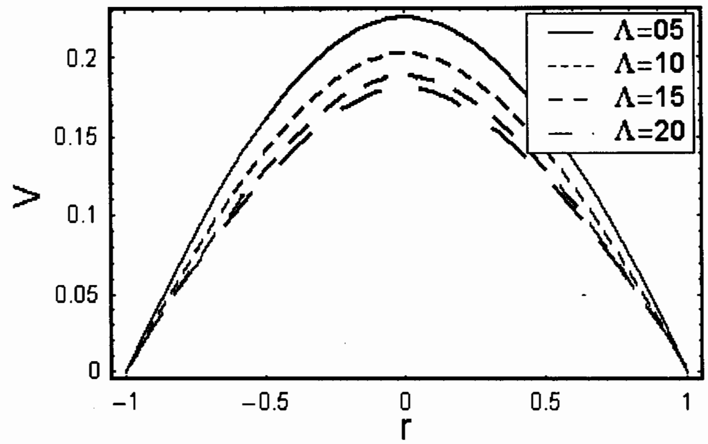

Fig. 3.8: Influence of Λ on velocity when $c = -1.2$, $M = 1$, $h = -0.05$.

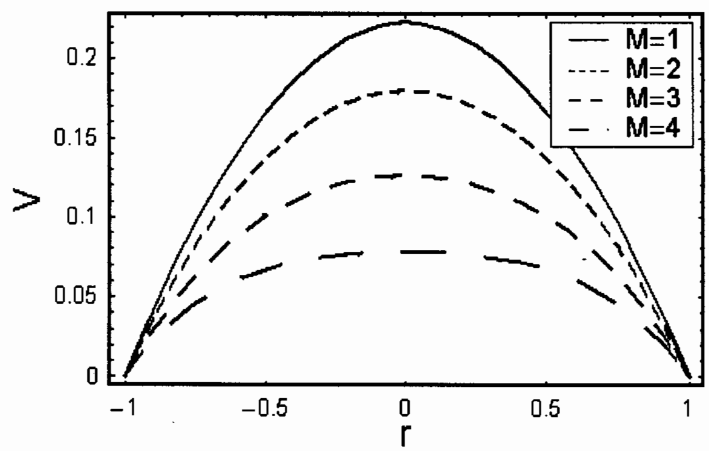

Fig. 3.9: Influence of M on velocity when $\Lambda = 1$, $c = -1$, $h = -0.05$.

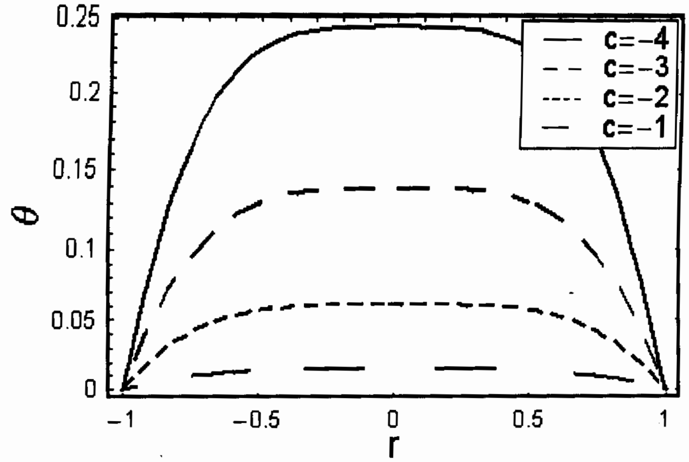

Fig. 3.10: Influence of c on temperature when $\Lambda = 1$, $\Gamma = 1$, $M = 1.5$, $h = -0.05$.

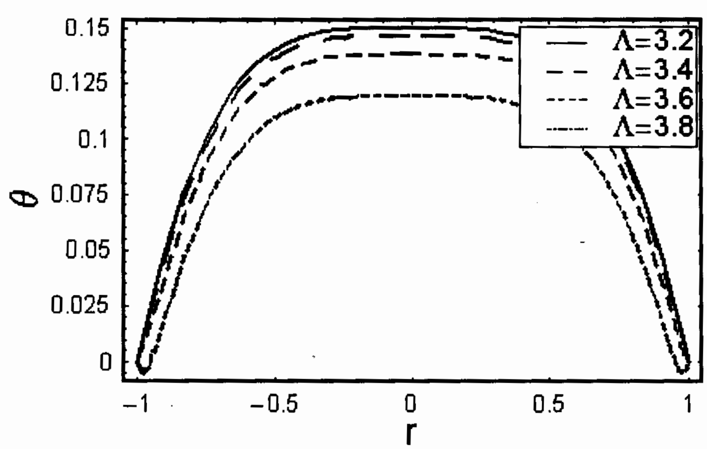

Fig. 3.11: Influence of Λ on temperature when $c = -3.1$, $\Gamma = 1$, $M = 1$, $h = -0.05$.

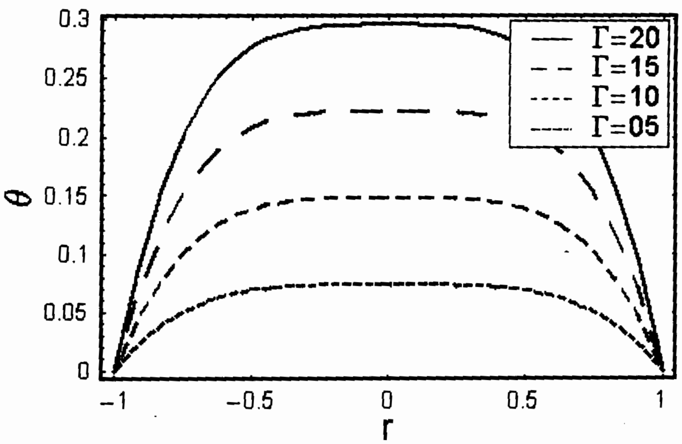

Fig. 3.12: Influence of Γ on temperature when $c = -1$, $\Lambda = 1$, $M = 1.5$, $h = -0.05$.

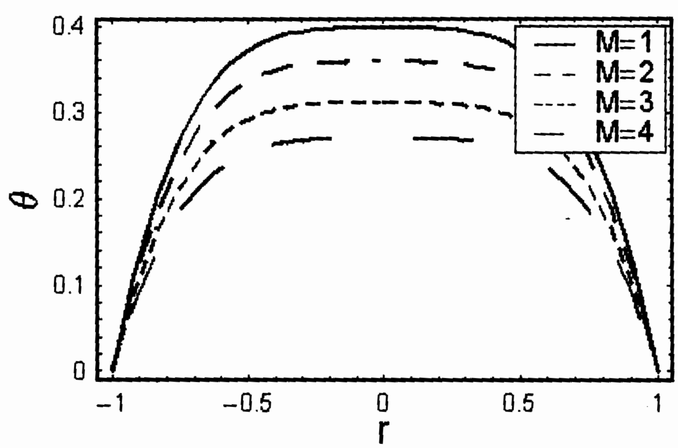

Fig. 3.13: Influence of M on temperature when $c = -5$, $\Lambda = 1$, $\Gamma = 1.5$, $h = -0.05$.

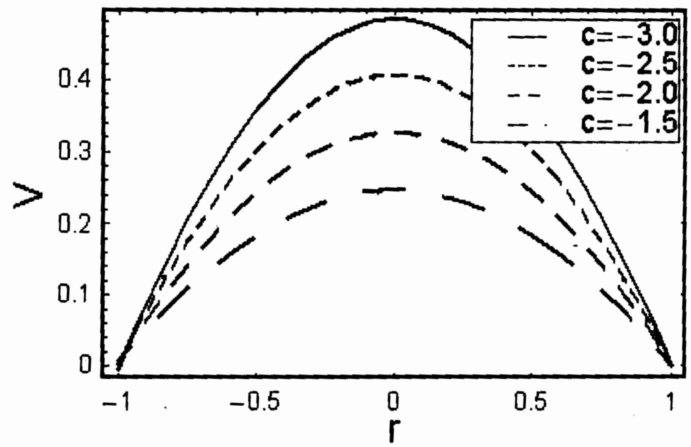

Fig. 3.14: Influence of c on velocity when $M = 2.5$, $\Lambda = 1$, $h = -0.01$.

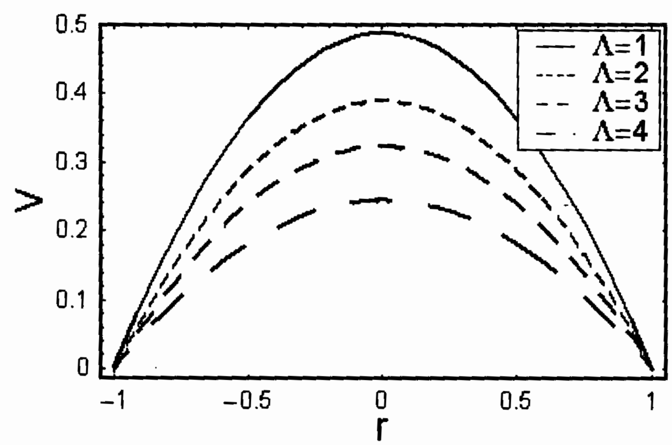

Fig. 3.15: Influence of Λ on velocity when $M = 2$, $h = -0.01$.

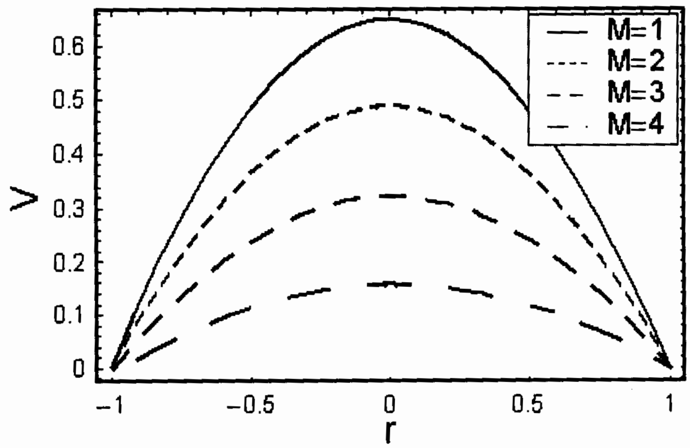

Fig. 3.16: Influence of M on velocity when $\Lambda = 1$, $h = -0.01$.

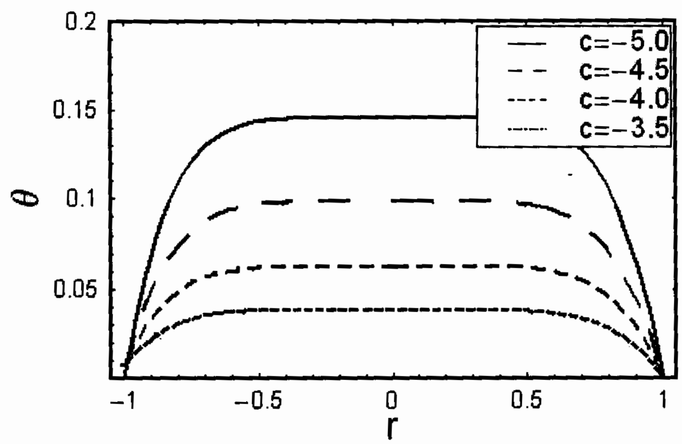

Fig. 3.17: Influence of c on temperature when $h = -0.01$, $\Gamma = 16$, $\Lambda = 8.5$, $M = 2$.

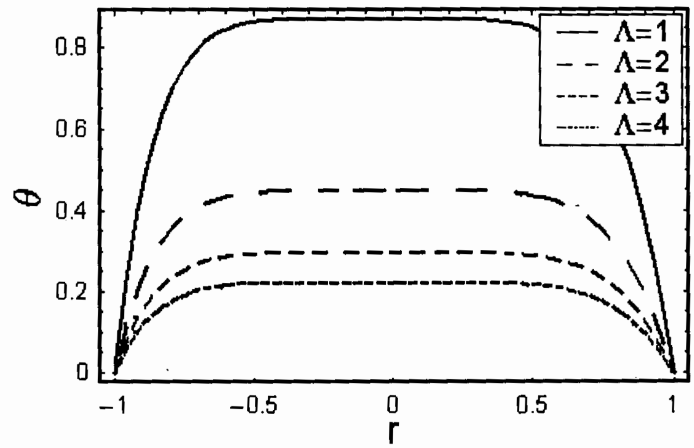

Fig. 3.18: Influence of Λ on temperature when $h = -0.01$, $\Gamma = 10$, $M = 2$.

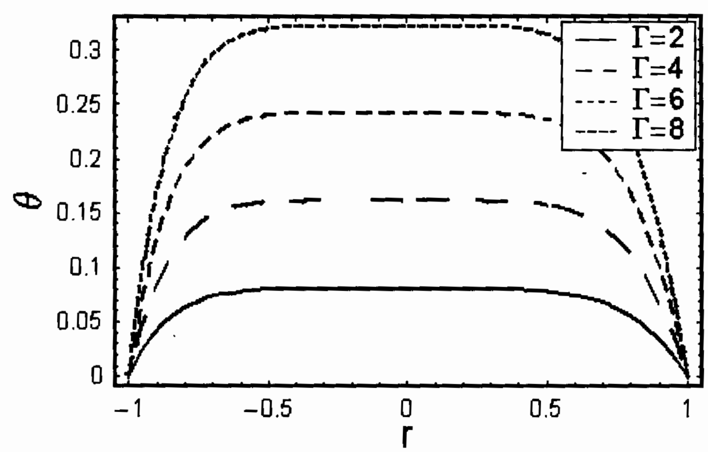

Fig. 3.19: Influence of Γ on temperature when $h = -0.01$, $\Lambda = 2$, $c = -5$, $M = 2.5$.

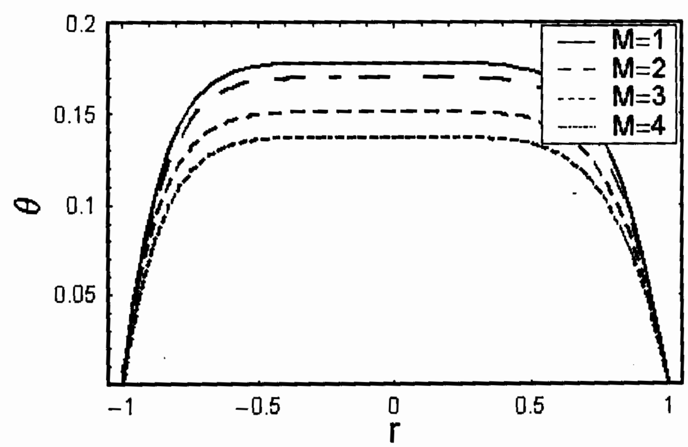

Fig. 3.20: Influence of M on temperature when $h = -0.01$, $c = -10$, $\Gamma = 1$.

3.6 Concluding remarks

In this study, we examine the influences of MHD on a constant and variable viscosity for steady flow of a third grade fluid in a pipe. Analytical solutions are computed. These solutions are valid not only for small but also for large values of all the emerging parameters. To the best of our knowledge, no such analysis is available in the literature which can describe the heat transfer and MHD effects simultaneously on variable viscosity. Convergence values and residual errors are also examined. The main results are listed below.

- An increase in pressure gradient c leads to a decrease in the velocity v.
- It is found that both are decreasing functions of Λ.
- It is observed that the profiles of velocity v and temperature θ decrease monotonically by increasing the MHD parameter M.
- It is seen that the temperature θ increases by increasing Γ.
- It is concluded that M suppresses the graphs of v and θ in both the cases.
- The residual error is found almost negligible (see Figs. 3.3 and 3.6).
- The results of [14] can be recovered by taking $M = 0$.

Appendix A

The related coefficients are given by

$$B_1 = \frac{1}{64}c^2\Gamma,$$

$$B_2 = \frac{1}{576}c^4\Gamma\Lambda + \frac{1}{96}c^4\hbar\Gamma\Lambda + \frac{1}{144}c^4\hbar^2\Gamma\Lambda,$$

$$B_3 = \frac{3}{1024}c^6\hbar\Gamma\Lambda^2 + \frac{29}{4096}c^6\hbar^2\Gamma\Lambda^2 + \frac{3}{1024}c^6\hbar^3\Gamma\Lambda^2 + \frac{1}{1024}c^6\hbar^4\Gamma\Lambda^2,$$

$$B_4 = \frac{39}{12800}c^8\hbar^2\Gamma\Lambda^3 + \frac{27}{6400}c^8\hbar^3\Gamma\Lambda^3 + \frac{3}{1600}c^8\hbar^4\Gamma\Lambda^3,$$

$$B_5 = \frac{9}{4096}c^{10}\hbar^3\Gamma\Lambda^4 + \frac{11}{4096}c^{10}\hbar^4\Gamma\Lambda^4 + \frac{1}{1024}c^{10}\hbar^5\Gamma\Lambda^4 + \frac{1}{4608}c^{10}\hbar^6\Gamma\Lambda^4,$$

$$B_6 = \frac{135}{114688}c^{12}\hbar^4\Gamma\Lambda^5 + \frac{135}{100352}c^{12}\hbar^5\Gamma\Lambda^5 + \frac{9}{14336}c^{12}\hbar^6\Gamma\Lambda^5 + \frac{3}{25088}c^{12}\hbar^7\Gamma\Lambda^5 + \frac{1}{50176}c^{12}\hbar^8\Gamma\Lambda^5,$$

$$B_7 = \frac{243}{524288}c^{14}\hbar^5\Gamma\Lambda^6 + \frac{135}{262144}c^{14}\hbar^6\Gamma\Lambda^6 + \frac{27}{131072}c^{14}\hbar^7\Gamma\Lambda^6 + \frac{3}{65536}c^{14}\hbar^8\Gamma\Lambda^6,$$

$$B_8 = \frac{13}{98304}c^{16}\hbar^6\Gamma\Lambda^7 + \frac{1}{8192}c^{16}\hbar^7\Gamma\Lambda^7 + \frac{1}{24576}c^{16}\hbar^8\Gamma\Lambda^7,$$

$$B_9 = \frac{81}{3276800}c^{18}\hbar^7\Gamma\Lambda^8 + \frac{27}{1638400}c^{18}\hbar^8\Gamma\Lambda^8,$$

$$B_{10} = \frac{81}{31719424}c^{20}\hbar^8\Gamma\Lambda^9,$$

$$B_{11} = \frac{1}{4}c^4\hbar^4\Lambda\Gamma + \frac{1}{2}c^4\hbar^5\Lambda\Gamma + \frac{3}{8}c^4\hbar^6\Lambda\Gamma + \frac{1}{8}c^4\hbar^7\Lambda\Gamma + \frac{1}{64}c^4\hbar^8\Lambda\Gamma,$$

$$\begin{aligned} B_{12} \;=\; & \frac{1}{9}c^2\hbar^2\Gamma + \frac{1}{9}c^2\hbar^3\Gamma + \frac{1}{36}c^2\hbar^4\Gamma - \frac{4}{27}c^4\hbar^3\Lambda\Gamma - \frac{14}{27}c^4\hbar^4\Lambda\Gamma - \frac{19}{27}c^4\hbar^5\Lambda\Gamma - \frac{25}{54}c^4\hbar^6\Lambda\Gamma - \\ & \frac{4}{27}c^4\hbar^7\Lambda\Gamma - \frac{1}{54}c^4\hbar^8\Lambda\Gamma + \frac{2}{27}c^6\hbar^5\Lambda^2\Gamma + \frac{1}{9}c^6\hbar^6\Lambda^2\Gamma + \frac{1}{18}c^6\hbar^7\Lambda^2\Gamma + \frac{1}{108}c^6\hbar^8\Lambda^2\Gamma, \end{aligned}$$

$$B_{13} = -\frac{1}{24}c^2\hbar\Gamma - \frac{5}{48}c^2\hbar^2\Gamma - \frac{1}{12}c^2\hbar^3\Gamma - \frac{1}{48}c^2\hbar^4\Gamma + \frac{1}{24}c^4\hbar^2\Lambda\Gamma + \frac{11}{48}c^4\hbar^3\Lambda\Gamma + \frac{7}{16}c^4\hbar^4\Lambda\Gamma + \frac{11}{24}c^4\hbar^5\Lambda\Gamma + \frac{13}{48}c^4\hbar^6\Lambda\Gamma + \frac{1}{12}c^4\hbar^7\Lambda\Gamma + \frac{1}{96}c^4\hbar^8\Lambda\Gamma - \frac{13}{216}c^6\hbar^4\Lambda^2\Gamma - \frac{41}{216}c^6\hbar^5\Lambda^2\Gamma - \frac{59}{288}c^6\hbar^6\Lambda^2\Gamma - \frac{5}{54}c^6\hbar^7\Lambda^2\Gamma - \frac{13}{864}c^6\hbar^8\Lambda^2\Gamma + \frac{1}{96}c^8\hbar^6\Lambda^3\Gamma + \frac{1}{96}c^8\hbar^7\Lambda^3\Gamma + \frac{1}{384}c^8\hbar^8\Lambda^3\Gamma,$$

$$B_{14} = \frac{1}{225}c^2\Gamma + \frac{4}{225}c^2\hbar\Gamma + \frac{2}{75}c^2\hbar^2\Gamma + \frac{4}{225}c^2\hbar^3\Gamma + \frac{1}{225}c^2\hbar^4\Gamma - \frac{4}{675}c^4\hbar\Lambda\Gamma - \frac{11}{225}c^4\hbar^2\Lambda\Gamma - \frac{88}{675}c^4\hbar^3\Lambda\Gamma - \frac{13}{75}c^4\hbar^4\Lambda\Gamma - \frac{4}{27}c^4\hbar^5\Lambda\Gamma - \frac{2}{25}c^4\hbar^6\Lambda\Gamma - \frac{16}{675}c^4\hbar^7\Lambda\Gamma - \frac{2}{675}c^4\hbar^8\Lambda\Gamma + \frac{14}{675}c^6\hbar^3\Lambda^2\Gamma + \frac{271}{2700}c^6\hbar^4\Lambda^2\Gamma + \frac{122}{675}c^6\hbar^5\Lambda^2\Gamma + \frac{106}{675}c^6\hbar^6\Lambda^2\Gamma + \frac{44}{675}c^6\hbar^7\Lambda^2\Gamma + \frac{7}{675}c^6\hbar^8\Lambda^2\Gamma - \frac{1}{81}c^8\hbar^5\Lambda^3\Gamma - \frac{43}{1350}c^8\hbar^6\Lambda^3\Gamma - \frac{16}{675}c^8\hbar^7\Lambda^3\Gamma - \frac{11}{2025}c^8\hbar^8\Lambda^3\Gamma + \frac{1}{1350}c^{10}\hbar^7\Lambda^4\Gamma + \frac{1}{2700}c^{10}\hbar^8\Lambda^4\Gamma,$$

$$B_{15} = \frac{1}{2916}c^4\Lambda\Gamma + \frac{1}{243}c^4\hbar\Lambda\Gamma + \frac{11}{729}c^4\hbar^2\Lambda\Gamma + \frac{19}{729}c^4\hbar^3\Lambda\Gamma + \frac{13}{486}c^4\hbar^4\Lambda\Gamma + \frac{14}{729}c^4\hbar^5\Lambda\Gamma + \frac{7}{729}c^4\hbar^6\Lambda\Gamma + \frac{2}{729}c^4\hbar^7\Lambda\Gamma + \frac{1}{2916}c^4\hbar^8\Lambda\Gamma - \frac{5}{1458}c^6\hbar^2\Lambda^2\Gamma - \frac{17}{729}c^6\hbar^3\Lambda^2\Gamma - \frac{19}{324}c^6\hbar^4\Lambda^2\Gamma - \frac{56}{729}c^6\hbar^5\Lambda^2\Gamma - \frac{83}{1458}c^6\hbar^6\Lambda^2\Gamma - \frac{16}{729}c^6\hbar^7\Lambda^2\Gamma - \frac{5}{1458}c^6\hbar^8\Lambda^2\Gamma + \frac{97}{17496}c^8\hbar^4\Lambda^3\Gamma + \frac{101}{4374}c^8\hbar^5\Lambda^3\Gamma + \frac{25}{729}c^8\hbar^6\Lambda^3\Gamma + \frac{46}{2187}c^8\hbar^7\Lambda^3\Gamma + \frac{79}{17496}c^8\hbar^8\Lambda^3\Gamma - \frac{1}{648}c^{10}\hbar^6\Lambda^4\Gamma - \frac{2}{729}c^{10}\hbar^7\Lambda^4\Gamma - \frac{1}{972}c^{10}\hbar^8\Lambda^4\Gamma + \frac{1}{46656}c^{12}\hbar^8\Lambda^5\Gamma,$$

$$B_{16} = \frac{8}{35721}c^6\hbar\Lambda^2\Gamma + \frac{74}{35721}c^6\hbar^2\Lambda^2\Gamma + \frac{244}{35721}c^6\hbar^3\Lambda^2\Gamma + \frac{422}{35721}c^6\hbar^4\Lambda^2\Gamma + \frac{440}{35721}c^6\hbar^5\Lambda^2\Gamma + \frac{32}{3969}c^6\hbar^6\Lambda^2\Gamma + \frac{104}{35721}c^6\hbar^7\Lambda^2\Gamma + \frac{16}{35721}c^6\hbar^8\Lambda^2\Gamma - \frac{40}{35721}c^8\hbar^3\Lambda^3\Gamma - \frac{233}{35721}c^8\hbar^4\Lambda^3\Gamma - \frac{520}{35721}c^8\hbar^5\Lambda^3\Gamma - \frac{572}{35721}c^8\hbar^6\Lambda^3\Gamma - \frac{304}{35721}c^8\hbar^7\Lambda^3\Gamma - \frac{62}{35721}c^8\hbar^8\Lambda^3\Gamma + \frac{34}{35721}c^{10}\hbar^5\Lambda^4\Gamma + \frac{38}{11907}c^{10}\hbar^6\Lambda^4\Gamma + \frac{40}{11907}c^{10}\hbar^7\Lambda^4\Gamma + \frac{38}{35721}c^{10}\hbar^8\Lambda^4\Gamma - \frac{4}{35721}c^{12}\hbar^7\Lambda^5\Gamma - \frac{1}{10206}c^{12}\hbar^8\Lambda^5\Gamma,$$

$$
\begin{aligned}
B_{17} \;=\;& \frac{1}{11664}c^8\hbar^2\Lambda^3\Gamma + \frac{23}{34992}c^8\hbar^3\Lambda^3\Gamma + \frac{23}{11664}c^8\hbar^4\Lambda^3\Gamma + \frac{1}{324}c^8\hbar^5\Lambda^3\Gamma + \frac{97}{34992}c^8\hbar^6\Lambda^3\Gamma + \\
& \frac{23}{17496}c^8\hbar^7\Lambda^3\Gamma + \frac{1}{3888}c^8\hbar^8\Lambda^3\Gamma - \frac{151}{629856}c^{10}\hbar^4\Lambda^4\Gamma - \frac{47}{39366}c^{10}\hbar^5\Lambda^4\Gamma - \frac{229}{104976}c^{10}\hbar^6\Lambda^4\Gamma - \\
& \frac{34}{19683}c^{10}\hbar^7\Lambda^4\Gamma - \frac{307}{629856}c^{10}\hbar^8\Lambda^4\Gamma + \frac{5}{46656}c^{12}\hbar^6\Lambda^5\Gamma + \frac{1}{3888}c^{12}\hbar^7\Lambda^5\Gamma + \frac{13}{93312}c^{12}\hbar^8\Lambda^5\Gamma - \\
& \frac{1}{279936}c^{14}\hbar^8\Lambda^6\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
B_{18} \;=\;& \frac{104}{4782969}c^{10}\hbar^3\Lambda^4\Gamma + \frac{697}{4782969}c^{10}\hbar^4\Lambda^4\Gamma + \frac{1808}{4782969}c^{10}\hbar^5\Lambda^4\Gamma + \frac{2368}{4782969}c^{10}\hbar^6\Lambda^4\Gamma + \\
& \frac{1544}{4782969}c^{10}\hbar^7\Lambda^4\Gamma + \frac{400}{4782969}c^{10}\hbar^8\Lambda^4\Gamma - \frac{172}{4782969}c^{12}\hbar^5\Lambda^5\Gamma - \frac{230}{1594323}c^{12}\hbar^6\Lambda^5\Gamma - \\
& \frac{304}{1594323}c^{12}\hbar^7\Lambda^5\Gamma - \frac{386}{4782969}c^{12}\hbar^8\Lambda^5\Gamma + \frac{4}{531441}c^{14}\hbar^7\Lambda^6\Gamma + \frac{5}{531441}c^{14}\hbar^8\Lambda^6\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
B_{19} \;=\;& \frac{107}{26572050}c^{12}\hbar^4\Lambda^5\Gamma + \frac{302}{13286025}c^{12}\hbar^5\Lambda^5\Gamma + \frac{43}{885735}c^{12}\hbar^6\Lambda^5\Gamma + \frac{614}{13286025}c^{12}\hbar^7\Lambda^5\Gamma + \\
& \frac{443}{26572050}c^{12}\hbar^8\Lambda^5\Gamma - \frac{11}{2952450}c^{14}\hbar^6\Lambda^6\Gamma - \frac{16}{1476225}c^{14}\hbar^7\Lambda^6\Gamma - \frac{23}{2952450}c^{14}\hbar^8\Lambda^6\Gamma + \\
& \frac{1}{3936600}c^{16}\hbar^8\Lambda^7\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
B_{20} \;=\;& \frac{104}{192913083}c^{14}\hbar^5\Lambda^6\Gamma + \frac{160}{64304361}c^{14}\hbar^6\Lambda^6\Gamma + \frac{248}{64304361}c^{14}\hbar^7\Lambda^6\Gamma + \frac{400}{192913083}c^{14}\hbar^8\Lambda^6\Gamma - \\
& \frac{16}{64304361}c^{16}\hbar^7\Lambda^7\Gamma - \frac{26}{64304361}c^{16}\hbar^8\Lambda^7\Gamma,
\end{aligned}
$$

$$
B_{21} = \frac{1}{19131876}c^{16}\hbar^6\Lambda^7\Gamma + \frac{5}{28697814}c^{16}\hbar^7\Lambda^7\Gamma + \frac{1}{6377292}c^{16}\hbar^8\Lambda^7\Gamma - \frac{1}{114791256}c^{18}\hbar^8\Lambda^8\Gamma,
$$

$$
B_{22} = \frac{8}{2424965283}c^{18}\hbar^7\Lambda^8\Gamma + \frac{16}{2424965283}c^{18}\hbar^8\Lambda^8\Gamma,
$$

$$
B_{23} = \frac{1}{8437157316}c^{20}\hbar^8\Lambda^9\Gamma,
$$

References

[1] C. Fetecau and C. Fetacau, Starting solutions for the motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder, Int. J. Eng. Sci. 44 (2006) 788-796.

[2] C. Fetecau and C. Fetecau, Starting solutions for some unsteady unidirectional flows of a second grade fluid, Int. J. Eng. Sci. 43 (2005) 781-789.

[3] C. Fetecau and C. Fetecau, On some axial Couette flows of non-Newtonian fluids, Z. Angew. Math. Phys. (ZAMP) 56 (2005) 1098-1106.

[4] W. C. Tan and T. Masuoka, Stokes' first problem for a second grade fluid in a porous half space with heated boundary, Int. J. Non-Linear Mech. 40 (2005) 512-522.

[5] W. C. Tan and T. Masuoka, Stability analysis of a Maxwell fluid in a porous medium heated from below, Physics Letters A 360 (2007) 454-460.

[6] T. Hayat and A. H. Kara, A variational analysis of a non-Newtonian flow in a rotating system, Int. J. Computational Fluid Dynamics 20 (2006) 157-162.

[7] T. Hayat and A. H. Kara, Couette flow of a third grade fluid with variable magnetic field, Math. Computer Modeling, 42 (2004) 132-137.

[8] T. Hayat and F. M. Mahomed, Note on an exact solution for the pipe flow of a third grade fluid,Acta Mech. 190 (2007) 233-236.

[9] R. Ellahi, T. Hayat, T. Javed and S. Asghar, On the analytic solution of non-linear flow problem involving Oldroyd 8-constant fluid. Mathematical and Computer Modelling, 48 (2008) 1191-1200.

[10] R. Ellahi and A. Zeeshan, A study of pressure distribution for a slider bearing lubricated with a second grade fluid, Numerical methods for partial differential equations DOI 10.1002 (in press).

[11] P. D. Ariel, Flow of a third grade fluid through a porous flat channel, Int. J. Eng. Sci. 41 (2003) 1267-1285.

[12] M. Massoudi and I. Christie, Effects of variable viscosity and viscous dissipation on the flow of a third grade fluid in a pipe, Int. J. Non-Linear Mech. 30 (1995) 687-699.

[13] R. Ellahi and S. Afzal, Effects of variable viscosity in a third grade fluid with porous medium: An analytic solution, Commun Nonlinear Sci Numer Simulation,14 (2009) 2056-2072.

[14] Hayat T, Ellahi R, Asghar S. The influence of variable viscosity and viscous dissipation on the non-Newtonian flow: An analytic solution, Communication in Non-linear science and Numerical Simulation 12 (2007) 300-313.

[15] S. J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PhD thesis, Shanghai Jiao Tong University, (1992).

[16] S. J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499-513.

[17] S. J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method. Chapman & Hall. Boca Raton, (2003).

[18] S. Abbasbandy, The application of homotopy analysis method to no linear equations arising in heat transfer, Phys. Lett. A 360 (2006) 109-113.

[19] Ellahi R, Effect of the slip boundary condition on non-Newtonian flows in a channel, Commun Nonlinear Sci Numer Simulation, 14 (2009) 1377-1384.

[20] T. Hayat, R. Ellahi, P. D. Ariel and S. Asghar, Homotopy solution for the channel flow of a third grade fluid, Non-linear Dynamics, 45 (2006), 55-64.

[21] R. L. Fosdick and K. R. Rajagopal, Thermodynamics and stability of fluids of third grade, Proc. R. Soc. Lond. A339 (1980) 351-377.

[22] R. A. Van Gorder, K.Vajravelu, On the selection of auxiliary functions, operators, and con-vergence control parameters in the application of the Homotopy Analysis Method to

nonlinear differential equations: A general approach, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 4078-4089.

[23] S. J. Liao, An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude. Int. J. Non-Linear Mech. 38 (2003) 1173-1183.