Effects of MHD and porosity in Reynolds' models and Vogel's models of variable viscosity in a third grade fluid

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1- Fluid dynamics

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Submitted in the Partial Fulfillment of the
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In
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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTRER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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2010

Dedicated to

The most dedicated one
Ch. Sada Hussain

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Preface

During the last few years, there is substantial progress in the steady and unsteady flows of non-Newtonian fluids. A huge amount of literature is now available on the topic (see some studies [1-10]). All real fluids are diverse in nature. Hence in view of rheological characteristics, all the non-Newtonian fluids cannot be explained by employing one constitutive equation. This is the striking difference between viscous and the non-Newtonian fluids. The rheological parameters appearing in the constitutive equations lead to a higher order and complicated governing equations than the Navier-Stokes equations. The simplest subclass of differential type fluids is called the second grade. In steady flow such fluids can predict the normal stress and does not show shear thinning and shear thickening behaviors. The third grade fluid puts forward the explanation of shear thinning and shear thickening properties. The present work aims to study the pipe flow of a third grade fluid. Some progress on the topic can be mentioned in the studies [11-12] and many refs there in. In all these studies, variable viscosity is used. Massoudi and Christie [12] numerically examined the pipe flow of a third grade fluid when viscosity depends upon temperature. Ellahi and Afzal [13] reported such solutions when third grade fluids saturate the porous medium.

In chapter one, the basic definitions of fluids and primary concept of homotopy is given. Maxwell's equations for electromagnetism, laws of conservation of mass and momentum are also discussed.

Chapter two is devoted to the study of effects of variable viscosity in a third grade fluid with porous medium. An analytical solution for the flow of third grade fluid in a pipe using HAM is studied. This chapter is the review work of Ellahi and Afzal [13].

In chapter three, we extend the analysis of ref. [13] and desire to understand the magnetic field effects on the pipe flow of third grade fluid in a porous medium by employing modified Darcy's law. Besides this Reynolds' model and Vogel's model of temperature dependent viscosity are considered. The relevant equations for flow and temperature have been solved analytically by using homotopy analysis method [14-19]. Convergence of the obtained solutions is explicitly shown. The effects of the various parameters of interest on the velocity and temperature are pointed out.

Contents

Chapter 1

Preliminaries

This chapter deals with some basic definitions and equations which are necessary for the better understanding of the chapter two and three.

1.1 Fluid

A fluid is the substance that deforms continuously under the application of shearing stress, no matter how small the shearing stress is.

1.2 Types of fluid

1.2.1 Thixotropic fluids

Such type of fluid shows a decrease in apparent viscosity η with time under constant applied shear stress.

1.2.2 Rheopectic

Fluids which shows an increase in apparent viscosity η with time.

1.2.3 Ideal fluids

All the fluids offer some resistance which opposes the flow. An ideal fluid is the one which has no such resistance i.e., the viscosity of an ideal fluid is zero.

1.2.4 Real fluids

All the fluids for which dynamic viscosity is not zero are known as the real fluids. All real fluids have finite viscosity but it may or may not be compressible.

1.3 Flow

A material undergoes a deformation when different forces act upon it. If the deformation continuously increases without limit, the phenomena is known as flow.

1.4 Deformation

It is a relative change in position or length of the fluid particles.

1.5 Types of flow

1.5.1 Uniform and non uniform flow

In uniform flow, all the particles in the fluid stream have same velocities both in magnitude and direction where as in case of non-uniform flow velocities of the particles are not same at all sections of uniform domains.

1.5.2 Compressible and incompressible flow

A flow is said to be incompressible if there is no change in the density. Liquids are treated as the incompressible fluids and if the density of the fluid changes then the flow is known as compressible. All gases are considered as compressible fluids. Mathematically for incompressible flow we have

$$\rho = \rho(x, y, z, t) = \text{cons}\tan\text{t}, \tag{1.1}$$

where as in case of compressible flow

$$\rho = \rho(x, y, z, t) \neq \text{cons}\tan\text{t}. \tag{1.2}$$

1.5.3 Viscous and inviscid flow

The flow in which the fluid friction have significant effect is called as viscous flow and those in which the inertial forces are more significant than fluid friction is termed as inviscid.

1.5.4 Steady and unsteady flow

In such flow velocity does not exhibit a change w.r.t. time i.e., for velocity we have

$$\frac{\partial \mathbf{V}}{\partial t} = 0 \tag{1.3}$$

and for the case of unsteady flow

$$\frac{\partial \mathbf{V}}{\partial t} \neq 0. \tag{1.4}$$

1.5.5 Newtonian and non-Newtonian flow

A Newtonian fluid is that in which the stress verses strain rate is linear and passes through the origin. The constant of proportionality is known as dynamic viscosity. i.e.,

$$\tau = \mu \frac{du}{dy}, \tag{1.5}$$

where τ is the shear stress exerted by the fluid, a constant of proportionality μ is the fluid viscosity and du/dy is the velocity gradient perpendicular to the direction of shear. Water, air and gasoline are the Newtonian fluids. A non-Newtonian fluid is that in which the shear stress verses strain rate is non linear. i.e.,

$$\tau = \eta(\frac{du}{dy}), \tag{1.6}$$

where η is an apparent viscosity which depends upon the deformation rate. i.e.,

$$\eta = (\frac{du}{dy})^{n-1}. \tag{1.7}$$

Ketchup, paint, blood, toothpaste and shampoo are the well known examples of non-Newtonian fluids.

1.5.6 Laminar and turbulent flows

A flow in which each fluid particle traces out a definite curve and the curve traced out by two different particles do not intersect is said to be laminar. On the other hand, the flow in which each fluid particle does not trace out a definite path and path traced out by different fluid particles intersects is known as the turbulent flow. Rising smoke at start is laminar flow but soon it becomes turbulent.

1.6 Basic characteristics of fluid

1.6.1 Density

Mass per unit volume is known as density. Mathematically the density ρ is defined by the following expression

$$\rho = \frac{m}{V}, \tag{1.8}$$

where m is the mass and V is the volume.

1.6.2 Kinematic viscosity

The ratio of the dynamic viscosity to the fluid density is called kinematic viscosity. It is defined by

$$\nu = \frac{\mu}{\rho}. \tag{1.9}$$

1.6.3 Pressure

Pressure is the magnitude of force per unit area applied in a direction perpendicular to the surface of an object. Mathematically

$$P = \frac{F}{A}, \tag{1.10}$$

in which F is the magnitude of force, A is the surface area and P is the pressure.

1.7 Forces

1.7.1 Surface force

The forces acting on the boundaries of a medium either internally or externally are known as surface force. Surface force can be divided in to two components: pressure and stress forces. Pressure forces acts normally over an area and stress forces act tangentially on an area.

1.7.2 Body force

The forces which acts throughout the volume of a body without any physical contact are known as the body forces. Gravity and electromagnetic forces are the examples of body forces.

1.8 Magnetohydrodynamics

1.8.1 Magnetism

The phenomena under which materials exert an attractive or repulsive force on other materials is known as magnetism.

1.8.2 Magnetic field

The region or space around a magnet in which the influence of a force exists by a magnet is called its magnetic field. The path of small magnet in a magnetic field is called magnetic field line. The SI, unit of magnetic field is tesla or weber per square meter.

1.8.3 Electric field

A region in which an electric charge experiences a force. That is force per unit charge situated at a point is the electric field at that point

$$\mathbf{E} = \frac{\mathbf{F}}{q}, \tag{1.11}$$

where $\mathbf{F}$ is the electric force experienced by the charge q and $\mathbf{E}$ is the electric field at the point where charge q is located. The SI, unit of electric field is volt per meter or newton per coulomb.

1.9 Maxwell's equations

A set of four partial differential equations that relate the electric and magnetic field with each other are known as Maxwell's equations. These are due to the name of James Clark Maxwell.

1.9.1 Gauss's law

This law relates that the electric flux through the enclosed surface is proportional to the electric charge density. i.e.,

$$\operatorname{div}\mathbf{E} = \frac{\rho_e}{\epsilon_o}. \tag{1.12}$$

1.9.2 Gauss's law for magnetism

It states that the total magnetic flux through a close surface is zero and that field whose flux is zero is known as solenoidal vector field. i.e.,

$$\operatorname{div}\mathbf{B} = 0. \tag{1.13}$$

1.9.3 Faraday's law of induction

This law relates the induced electromagnetic force to the change in magnetic flux that produces it. Mathematically, we have

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.14}$$

1.9.4 Ampere-Maxwell equation

This states that magnetic field can be produced from electric current and by changing electric field. This law allows that the electric current can produce the magnetic field;

$$\operatorname{curl}\mathbf{B} = \mu_o\mathbf{J} + \mu_o\epsilon_o\frac{\partial \mathbf{E}}{\partial t}, \tag{1.15}$$

where $\mathbf{B}$ is the total magnetic field, $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current density, ρ_e is the charge density, μ_o is the magnetic permeability and ϵ_o is permittivity of the free space.

1.10 Generalized Ohm's law

In 1827 George Simon Ohm states that the current through a conductor between two points is directly proportional to the voltage across the two points, and inversely proportional to the resistance. Mathematically

$$I = \frac{V}{R}, \tag{1.16}$$

where V_2 is the potential difference and I is the current flowing through resistance R.

The distribution of the flow of electric charge is known as current density $\mathbf{J}$ given as

$$\mathbf{J} = \sigma\mathbf{E}. \tag{1.17}$$

The electric field intensity $\mathbf{E}_m$ induced due to the motion of the conductor across the transverse magnetic field is given by

$$\mathbf{E}_m = \mathbf{V} \times \mathbf{B}. \tag{1.18}$$

The net electric field intensity is

$$\begin{aligned} \mathbf{E}_t &= \mathbf{E} + \mathbf{E}_m, \\ \mathbf{E}_t &= \mathbf{E} + (\mathbf{V} \times \mathbf{B}). \end{aligned} \tag{1.19}$$

Therefore,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}). \tag{1.20}$$

1.11 Fundamental laws

1.11.1 Equation of continuity

In fluid mechanics, conservation of mass is described by the continuity equation.

Mathematically, we can write

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho\mathbf{V}) = 0. \tag{1.21}$$

The above equation for an incompressible fluid becomes

$$\nabla . \mathbf{V} = 0. \tag{1.22}$$

1.11.2 Equation of motion

It represents the law of conservation of linear momentum. For an incompressible fluid, the equation of motion in vector form is

$$\rho \frac{d\mathbf{V}}{dt} = \nabla . \mathbf{T} + \rho \mathbf{b}. \tag{1.23}$$

where $\mathbf{V}$ is the velocity field, ρ is the fluid density, $\mathbf{T}$ is the Cauchy's stress tensor, $\mathbf{b}$ is the body force, p is the pressure and μ is the dynamic viscosity.

In scalar form Eq. (1.23) becomes

$$\begin{aligned}
\rho \frac{du}{dt} &= \frac{\partial (\sigma_{xx})}{\partial x} + \frac{\partial (T_{xy})}{\partial y} + \frac{\partial (T_{xz})}{\partial z} + \rho b_x, \\
\rho \frac{dv}{dt} &= \frac{\partial (T_{yx})}{\partial x} + \frac{\partial (\sigma_{yy})}{\partial y} + \frac{\partial (T_{yz})}{\partial z} + \rho b_y, \\
\rho \frac{dw}{dt} &= \frac{\partial (T_{zx})}{\partial x} + \frac{\partial (T_{zy})}{\partial y} + \frac{\partial (\sigma_{zz})}{\partial z} + \rho b_z,
\end{aligned} \tag{1.24}$$

where b_x, b_y and b_z are the body forces in x, y and $z-$ directions respectively; u, v and w are the velocity components in x, y and $z-$directions; σ_{xx}, σ_{yy} and σ_{zz} are the normal stresses; T_{xy}, T_{xz}, T_{yx}, T_{yz}, T_{zx} and T_{zy} are the shear stresses and d/dt is the material derivative.

1.11.3 Equation of energy

Energy in a system may take on various forms (e.g. kinetic, potential, heat, light). Equation of energy is represented by the law of conservation of energy. This law states that energy may neither be created nor destroyed. Therefore, the sum of all the energies in the system is a constant. The laws of conservation of energy is described as

$$\rho c_p \frac{D\theta}{Dt} = \mathbf{T} \cdot \mathbf{L} - \boldsymbol{\nabla} \cdot \mathbf{q},$$

where c_p is specific heat of the material and q is the heat flux.

1.12 Homotopy analysis method

In topology two functions are said to be homotopic if one function can transform continuously into the other.

Definition: Formally a homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function

$$H : X \times [0,1] \rightarrow Y, \tag{1.25}$$

from the product of the space X with the unit interval $[0,1]$ to Y such that for all point x in X and

$$H(x,0) = f(x) \ , \qquad H(x,1) = g(x). \tag{1.26}$$

If we consider second coordinate as time t then at time $t = 0$ we have a function f and at time $t = 1$ we have a function g.

In fluid mechanics sometimes governing differential equations become non-linear, which is very difficult to deal with. So we have no choice but to solve these numerically. Fortunately we have some method in which we can solve non-linear differential equations by HAM.

The zeroth order deformation is defined as

$$(1-p)\,\mathcal{L}[u(y\ ;p) - u_0] = p\hbar R_m, \tag{1.27}$$

in which $p \in [0,1]$ is the embedding parameter, $\hbar$ is auxiliary non-zero parameter and R_m is non-linear differential operator. For $p = 0$ and $p = 1$ we have

$$u(y,0) = u_0 \ , \ u(y,1) = u(y), \tag{1.28}$$

where u_0 is the initial guess and $u(y)$ is a solution of a given differential equation. By Taylor's theorem

$$u(y,p) = u_0(y) + \sum_{m=1}^{\infty} u_m(y)p^m, \tag{1.29}$$

where

$$u_m(y) = \frac{1}{m!}\frac{\partial^m u(y;p)}{\partial p^m} \qquad at \quad p = 0. \tag{1.30}$$

Note that convergence is depending upon auxiliary parameter $\hbar$. Thus we can write

$$u(y) = u_0(y) + \sum_{m=1}^{\infty} u_m(y), \tag{1.31}$$

which is a solution of given differential equation.

The advantages of HAM are

1. It is valid even if a given non-linear problem does not contain any small/large parameters at all.

2. It can provide us with a convenient way to control the convergence of approximation series and adjust convergence regions where necessary.

3. It can be employed to efficiently approximate a nonlinear problem by choosing different sets of base functions.

Chapter 2

Effects of variable viscosity in a third grade with porous medium

2.1 Introduction

This chapter is the review work of R. Ellahi and S. Afzal [13]. In this chapter we seek the effects of variable viscosity in a third grade fluid in a porous medium. The resulting equations are non linear and solved by homotopy analysis method (HAM). Different graphs have been drawn for different parameters.

2.2 Problem formulation

Let us consider an incompressible third grade fluid in a pipe. The fluid saturates the porous medium. The velocity filed is given by

$$\mathbf{V} = [0,0,v(r)], \tag{2.1}$$

and the governing equations are

$$\nabla.\mathbf{V} = 0, \tag{2.2}$$

$$\rho\frac{d\mathbf{V}}{dt} = div\mathbf{T}, \tag{2.3}$$

$$\rho c_p \frac{d\theta}{dt} = \mathbf{T} \cdot \mathbf{I} - \nabla^2 \theta, \tag{2.4}$$

where θ is the temperature and the Cauchy stress $\mathbf{T}$ is

$$\mathbf{T} = -p_1 \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_3 (tr \mathbf{A}_1^2) \mathbf{A}_1, \tag{2.5}$$

in which p_1 is hydrostatic pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity and $\alpha_i (i = 1, 2)$ and $\beta_j (j = 1 - 3)$ are material constants. The first three Rivlin – Ericksen tensors $(A_1 - A_3)$ are defined through the following expressions

$$A_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^t, \tag{2.6}$$

$$A_n = \frac{dA_{n-1}}{dt} + A_{n-1} \mathbf{L} + \mathbf{L}^t A_{n-1}, \quad n > 1, \tag{2.7}$$

$$\mathbf{L} = \nabla \mathbf{V} = grad \mathbf{V}, \tag{2.8}$$

in which ∇ is the gradient operator and $\mathbf{V}$ is the velocity field. For thermodynamic third grade fluid the coefficients μ, α_1, α_2, β_1, β_2,and β_3 satisfy the following conditions [20]

$$\mu \geq 0, \alpha_1 \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24 \mu \beta_3}, \beta_1 = \beta_2 = 0, \beta_3 \geq 0. \tag{2.9}$$

$$A_1^2 = \begin{pmatrix} \left(\frac{v_0}{r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2 & 0 & 0 \\ 0 & \left(\frac{v_0}{r}\right)^2 & 0 \\ 0 & -\frac{v_0}{r}\frac{\partial w}{\partial r} & \left(\frac{\partial w}{\partial r}\right)^2 \end{pmatrix},$$

$$A_2 = A_1 (\nabla \mathbf{V}) + (\nabla \mathbf{V})^t A_1,$$

$$A_2 = \begin{pmatrix} 2\left(\frac{\partial w}{\partial r}\right)^2 & 0 & 0 \\ 0 & 2\left(\frac{v_0}{r}\right)^2 & -\frac{v_0}{r}\frac{\partial w}{\partial r} \\ 0 & -\frac{v_0}{r}\frac{\partial w}{\partial r} & 0 \end{pmatrix},$$

$$Trace A_1^2 = 2 \left[\left(\frac{v_0}{r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2 \right].$$

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \mathbf{R} = -\frac{\mu}{k} \phi \left(1 + \lambda_r \frac{\partial}{\partial t}\right) \mathbf{V}, \tag{2.10}$$

where $\mathbf{R}$ is Darcy's resistance, λ and λ_r are the relaxation and retardation times, ϕ is porosity and k is the permeability.

Having in mind the above equation the $z-$components of $\mathbf{R}$ for steady flow of a third grade fluid is

$$R_z = -\frac{\phi}{k}\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right]v. \tag{2.11}$$

By the virtue of above set of equations, momentum and energy equations reduce to

$$\frac{1}{r}\frac{d}{dr}(r\mu\frac{dv}{dr}) + \frac{2\beta_3}{r}\frac{d}{dr}(r(\frac{dv}{dr})^3) - \frac{\phi}{k}\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right]v = -\frac{\partial\hat{p}}{\partial z}, \tag{2.12}$$

$$\mu(\frac{dv}{dr})^2 + 2\beta_3\left(\frac{dv}{dr}\right)^4 + k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right] = 0. \tag{2.13}$$

where k is the thermal conductivity and $\hat{p}$ is

$$\hat{p} = p_1 - \alpha_2\left(\frac{dv}{dr}\right)^2. \tag{2.14}$$

The boundary conditions are

$$v(R) = \theta(R) = 0, \qquad \frac{dv(0)}{dr} = \frac{d\theta(0)}{dr} = 0. \tag{2.15}$$

Equations (2.13), (2.14) and (2.16) in dimensionless variables become

$$\frac{d\mu}{dr}\frac{dv}{dr} + \frac{\mu}{r}\frac{dv}{dr} + \mu\frac{d^2v}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 + 3\Lambda\frac{d^2v}{dr^2}\left(\frac{dv}{dr}\right)^2 - P\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right]v = c, \tag{2.16}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2\left(\mu(r) + \Lambda\left(\frac{dv}{dr}\right)^2\right) = 0, \tag{2.17}$$

$$v(1) = \theta(1) = 0, \qquad \frac{dv(0)}{dr} = \frac{d\theta(0)}{dr} = 0, \tag{2.18}$$

where

$$\Lambda = \frac{2\beta_3 V_0^2}{R_0^2 v_0},\; c = \frac{\frac{\partial p_1}{\partial z} R^2}{\mu_0 V_0},\; \frac{\phi}{kR^2} = P,\; r = \frac{\overline{r}}{\overline{R}},\; v = \frac{\overline{v}}{\overline{v_0}},\; \mu = \frac{\overline{\mu}}{\mu_0} \tag{2.19}$$

$$\overline{\theta} = \frac{\theta - \theta_0}{\theta_1 - \theta_0},\; \Gamma = \frac{\mu_0 v_0^2}{k(\theta_1 - \theta_0)},$$

where R, v_0, μ_0, θ_0, θ and θ_1 are the radius, reference velocity, reference viscosity, reference temperature and pipe fluid temperatures respectively. Here c is the pressure drop, P is porous medium parameter and Γ is viscous dissipation and bars have been suppressed for simplicity.

2.3 Series solutions

In this section, Eqs. (2.17) and (2.18) with boundary conditions (2.19) will be solved by using HAM.

Case- I Reynold model

For Reynold model

$$\mu = e^{-B\theta}. \tag{2.21}$$

For HAM solutions we select

$$v_0(r) = \frac{c}{4}(r^2 - 1), \qquad \theta_0(r) = \frac{c}{2}(r^2 - 1) \tag{2.22}$$

as the initial approximations of v and θ. Further we choose the following auxiliary linear operators

$$\mathcal{L}_1 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \tag{2.23}$$

such that

$$\mathcal{L}_1 (C_1 + C_2 \ln r) = 0, \tag{2.24}$$

where C_1 and C_2 are arbitrary constants.

The zeroth – order problems are

$$(1-p)\mathcal{L}_1 [v^*(r,p) - v_0(r)] = p\hbar\mathcal{N}_1 [v^*(r,p), \theta^*(r,p)], \tag{2.25}$$

$$(1-p)\,\mathcal{L}_1\left[\theta^*(r,p)-\theta_0(r)\right]=p\hbar\mathcal{N}_2\left[v^*(r,p),\theta^*(r,p)\right], \tag{2.26}$$

$$v^*(1,p)=0,\quad \theta^*(1,p)=0,\quad \left.\frac{\partial v^*(r,p)}{\partial r}\right|_{r=0}=0,\qquad \left.\frac{\partial \theta^*(r,p)}{\partial r}\right|_{r=0}=0. \tag{2.27}$$

After the Taylor's series expansion of Eq. (2.21) and retrieving only first two components, we get

$$\mu\approx 1-\theta B. \tag{2.27a}$$

In view of Eq. (2.17) and 2.27a we can write

$$\mathcal{N}_1\left[v^*(r,p),\theta^*(r,p)\right]=\begin{array}{c}\left[\frac{1}{r}\frac{dv^*}{dr}+\frac{d^2v^*}{dr^2}-\frac{B}{r}\frac{dv^*}{dr}\theta^*-B\frac{d^2v^*}{dr^2}\theta+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3+3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}\right.\\ \left.+BPv\theta-Pv-P\Lambda\left(\frac{dv}{dr}\right)^2v-c\right],\end{array} \tag{2.28}$$

$$\mathcal{N}_2\left[v^*(r,p),\theta^*(r,p)\right]=\begin{array}{c}\left[\frac{1}{r}\frac{d\theta^*}{dr}+\frac{d^2\theta^*}{dr^2}-\Gamma B\left(\frac{dv^*}{dr}\right)^2\theta+\Gamma\left(\frac{dv}{dr}\right)^2\right.\\ \left.+\Lambda\Gamma\left(\frac{dv}{dr}\right)^4\right].\end{array} \tag{2.29}$$

For $p=0$ and $p=1$, we have

$$v^*(r;0)=v_0(r),\ \theta(r;0)=\theta_0(y)\ \text{ and }\ v^*(r;1)=v(r),\ \theta^*(r;1)=\theta(r). \tag{2.30}$$

When p increases from 0 to 1, $v^*(r,p)$, $\theta^*(r,p)$ varies from $v_0(r)$, $\theta_0(r)$ to $v(r)$ and $\theta(r)$ respectively. By Taylor's theorem and Eq. (2.30) one can write

$$v^*(r,p)=v_0(r)+\sum_{m=1}^{\infty}v_m(r)\ p^m,\ \theta^*(r,p)=\theta_0(r)+\sum_{m=1}^{\infty}\theta_m(r)\ p^m, \tag{2.31}$$

$$v_m(r)=\frac{1}{m!}\left.\frac{\partial^m v^*(r,p)}{\partial p^m}\right|_{p=0},\ \theta_m(r)=\frac{1}{m!}\left.\frac{\partial^m \theta^*(r,p)}{\partial p^m}\right|_{p=0}. \tag{2.32}$$

Clearly, the convergence of the series (2.31) depends upon $\hbar$. We choose $\hbar$ in such a way that the series (2.31) is convergent at $p=1$, then due to Eq. (2.30) we have

$$v(r)=v_0(r)+\sum_{m=1}^{\infty}v_m(r),\qquad \theta(r)=\theta_0(r)+\sum_{m=1}^{\infty}\theta_m(r). \tag{2.33}$$

The mth order deformation problems are

$$\mathcal{L}_1\left[v_m(r) - \chi_m v_{m-1}(r)\right] = \hbar \mathcal{R}1_m(r), \tag{2.34}$$

$$\mathcal{L}_1\left[\theta_m(r) - \chi_m \theta_{m-1}(r)\right] = \hbar \mathcal{R}2_m(r), \tag{2.35}$$

$$v_m(1) = 0, \quad v'_m(0) = 0, \quad \theta_m(1) = 0, \quad \theta'_m(0) = 0, \tag{2.36}$$

$$\begin{aligned}\mathcal{R}1_m(r) \quad = \quad & \Bigg[\tfrac{1}{r}\tfrac{dv_{m-1}}{dr} + \tfrac{d^2 v_{m-1}}{dr^2} - B\textstyle\sum_{k=0}^{m-1} \tfrac{dv_{m-1-k}}{dr}\tfrac{d\theta_k}{dr} - \tfrac{B}{r}\textstyle\sum_{k=0}^{m-1}\tfrac{dv_{m-1-k}}{dr}\theta_k \\ & \qquad -B\textstyle\sum_{k=0}^{m-1}\tfrac{dv_{m-1-k}}{dr^2}\theta_k + \\ & \frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-l}}{dr}\frac{dv_l}{dr} + 3\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{d^2 v_l}{dr^2} \\ & -P\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}v_l - (1-\chi_m)c + BP\sum_{k=0}^{m-1}v_{m-1-k}\theta_k - Pv_{m-1}\Bigg],\end{aligned} \tag{2.37}$$

$$\begin{aligned}\mathcal{R}2_m(r) \quad = \quad & \Bigg[\frac{1}{r}\frac{\partial\theta_{m-1}}{\partial r} + \frac{\partial^2\theta_{m-1}}{\partial r^2} + \Gamma\sum_{k=0}^{m-1}\frac{\partial v_{m-1-k}}{\partial r}\frac{\partial v_k}{\partial r} \\ & -\Gamma B\sum_{k=0}^{m-1}\frac{\partial v_{m-1-k}}{\partial r}\frac{\partial v_{k-l}}{\partial r}\theta_l \\ & +\Lambda\Gamma\sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{f=0}^{l}\frac{\partial v_{m-1-k}}{\partial r}\frac{\partial v_{k-l}}{\partial r}\frac{\partial v_{l-f}}{\partial r}\frac{\partial v_f}{\partial r}\Bigg].\end{aligned} \tag{2.38}$$

Case II Vogel model

Here

$$\mu = \mu_0 e^{\left(\frac{A}{B+\theta} - \theta\right)}. \tag{2.39}$$

The problems at the zeroth order are

$$(1-p)\,\mathcal{L}_1\left[v^*(r,p) - v_0(r)\right] = p\hbar\mathcal{N}_3\left[v^*(r,p), \theta^*(r,p)\right], \tag{2.40}$$

$$(1-p)\,\mathcal{L}_1\left[v^*(r,p) - v_0(r)\right] = p\hbar\mathcal{N}_4\left[v^*(r,p), \theta^*(r,p)\right], \tag{2.41}$$

$$v^*(1,p) = 0, \quad \theta^*(1,p) = 0, \quad \left.\frac{\partial v^*(r,p)}{\partial r}\right|_{r=0} = 0, \quad \left.\frac{\partial \theta^*(r,p)}{\partial r}\right|_{r=0} = 0. \tag{2.42}$$

By Taylor's series, Eq. (2.39) can be approximated as

$$\mu = \frac{c}{S}\left(1 - \frac{\theta A}{B^2}\right) \qquad \text{where} \quad S = \frac{c}{\mu_o e^{\left(\frac{A}{B} - w_0\right)}}. \tag{2.42a}$$

In view of Eq. (2.17) and 2.42a we can write

$$\mathcal{N}_3\left[v^*(r,p), \theta^*(r,p)\right] = \left[\begin{array}{c} \frac{c}{rS}\frac{dv^*}{dr} - \frac{Ac}{rSB^2}\frac{dv^*}{dr}\theta + \frac{c}{S}\frac{d^2v^*}{dr^2} - \frac{cA}{SB^2}\frac{d^2v^*}{dr^2}\theta + \frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 \\ +3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} - P\Lambda\left(\frac{dv^*}{dr}\right)^2 v - \frac{Pvc}{S} - \frac{PAc}{sB^2}v^*\theta^* - c \end{array}\right]. \tag{2.43}$$

$$\begin{aligned}\mathcal{N}_4\left[v^*(r,p), \theta^*(r,p)\right] \quad = \quad & \left[\frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2} - \frac{\Gamma c}{S}\left(\frac{dv^*}{dr}\right)^2 + \frac{\Gamma c A}{SB^2}\theta\left(\frac{dv^*}{dr}\right)^2\right. \\ & \left. + \Lambda\Gamma\left(\frac{dv^*}{dr}\right)^4\right].\end{aligned} \tag{2.44}$$

The mth order deformation problems are

$$\mathcal{L}_1\left[v_m(r) - \chi_m v_{m-1}(r)\right] = \hbar \mathcal{R}3_m(r), \tag{2.45}$$

$$\mathcal{L}_1\left[v_m(r) - \chi_m v_{m-1}(r)\right] = \hbar \mathcal{R}4_m(r), \tag{2.46}$$

$$v_m(1) = 0, \quad v_m'(0) = 0, \quad \theta_m(1) = 0, \quad \theta_m'(0) = 0, \tag{2.47}$$

$$\begin{aligned}\mathcal{R}3_m(r) \quad = \quad & \left[\frac{-AC}{B^2S}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{d\theta_k}{dr} + \frac{c}{rs}\frac{dv_{m-1}}{dr} - \frac{Ac}{rsB^2}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\theta_k + \frac{c}{s}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{dv_k}{dr} - \right. \\ & \frac{cA}{sB^2}\frac{d^2v_{m-1}}{dr^2}\theta + \frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{dv_l}{dr} + \\ & 3\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{d^2v_l}{dr^2} - P\Lambda\sum_{k=0}^{m-1}\frac{d^2v_{m-1-k}}{dr^2}\frac{dv_k}{dr} - \frac{Pc}{s}\frac{dv_{m-1}}{dr} - \\ & \left.\frac{PAc}{sB^2}\sum_{k=0}^{m-1}\sum_{l=0}^{k}v_{m-1-k}\theta_k - c(1-\chi_m)\right],\end{aligned} \tag{2.48}$$

$$\mathcal{R}4_m(r) = \begin{bmatrix} \frac{1}{r}\frac{\partial \theta_{m-1}}{\partial r} + \frac{\partial^2 \theta_{m-1}}{\partial r^2} + \Gamma \sum_{k=0}^{m-1} \frac{dv_{m-1-k}}{dr}\frac{dv_k}{dr} - \\ \Gamma B \sum_{k=0}^{m-1} \frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\theta_l \\ +\Lambda\Gamma \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p_2=0}^{l} \frac{dv_{m-1-k}}{dr}\frac{dv_{k-l}}{dr}\frac{dv_{l-p_2}}{dr}\frac{dv_{p_2}}{dr} \end{bmatrix}. \quad (2.49)$$

The above equation can be solved by Mathematica.

2.4 Convergence of the solution

The auxiliary parameter $\hbar$ is responsible for convergence region and rate of approximation given by the HAM as pointed out by Liao [21]. In order to determine the admissible values of $\hbar$, we plot the curves 2.1 and 2.2 , 2.3 and 2.4. It is noted from these Figs. that the range for the admissible values for $\hbar$ is $-1.4 \leq \hbar \leq -0.5$. The solution calculated finally converges for the whole region of r, when $\hbar$ is in the neighborhood of -1.

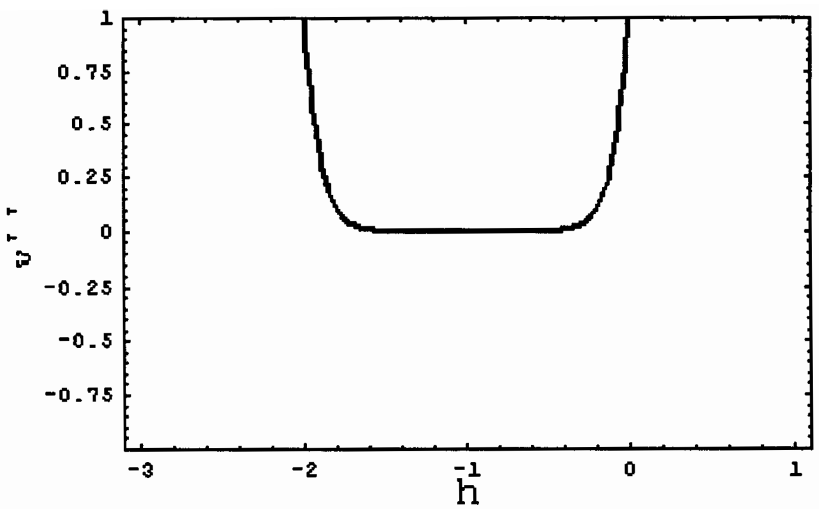

Fig. 2.1 $\hbar$ − curve for velocity of Reynolds model

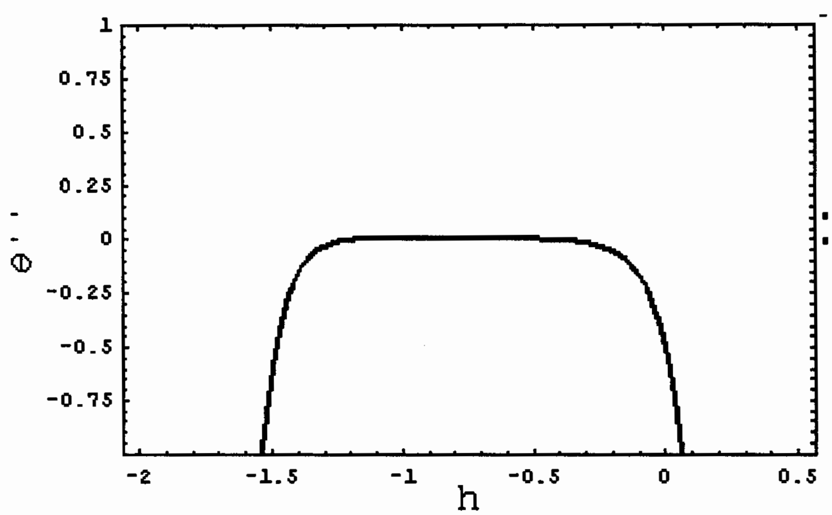

Fig. 2.2 $\hbar$ − curve for temperature of Reynolds model.

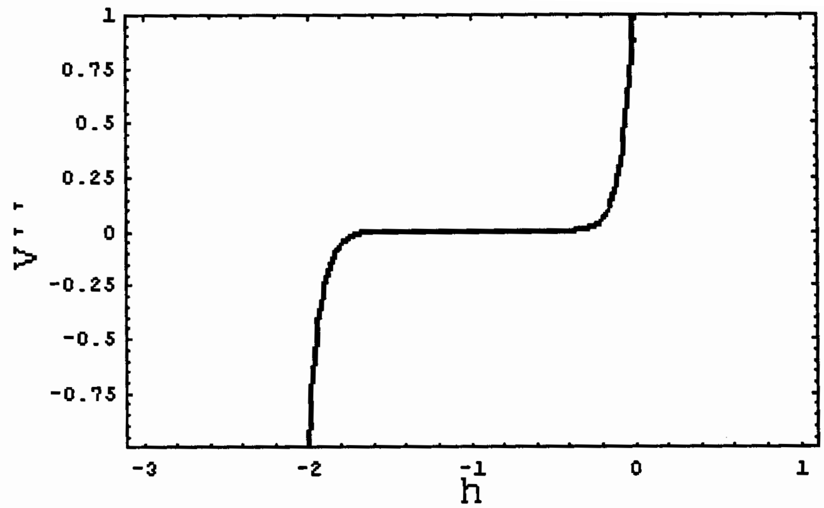

Fig. 2.3 $\hbar$ – curve for velocity of Vogel's model.

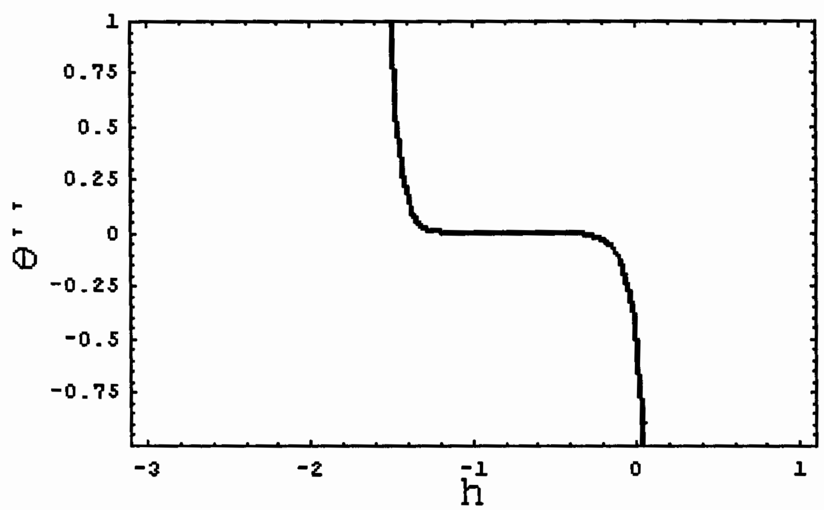

Fig. 2.4 $\hbar$ – curve for temperature of Vogel's mode.

2.5 Grapphs

2.5.1 Graphs for Reynolds model.

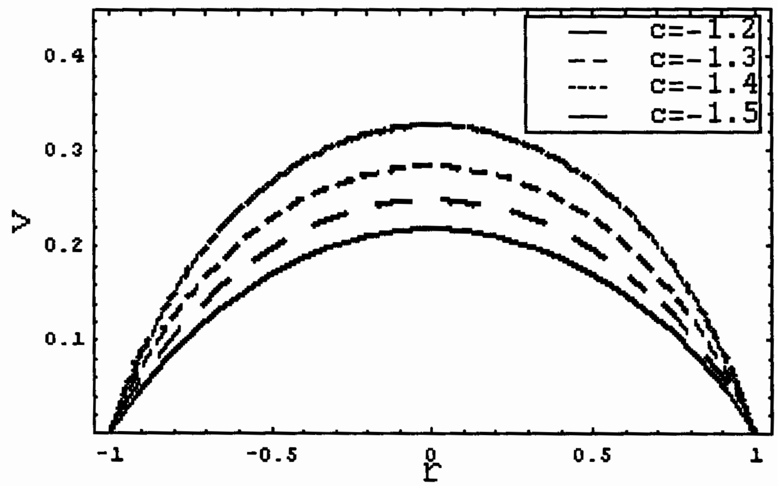

Fig. 2.5. Influence of c on the velocity.

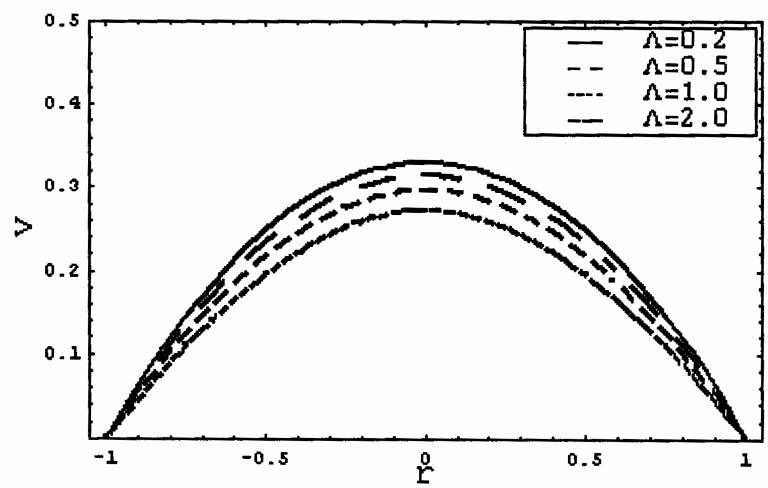

Fig. 2.6. Influence of Λ on the velocity.

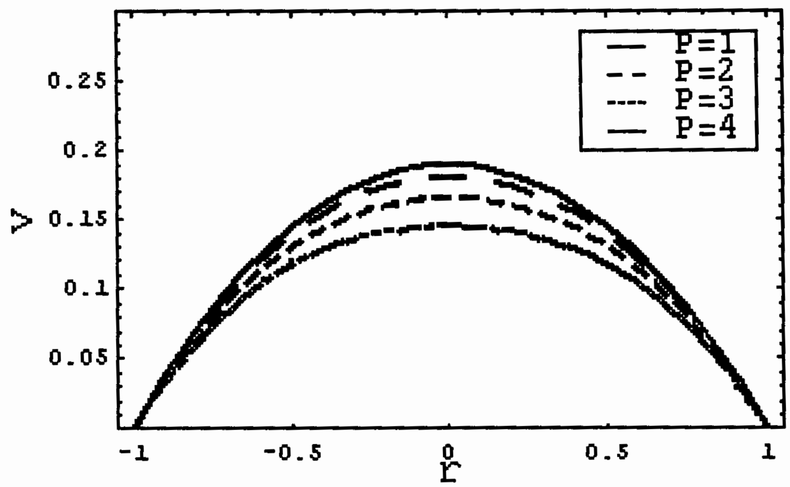

Fig. 2.7. Influence of P on the velocity.

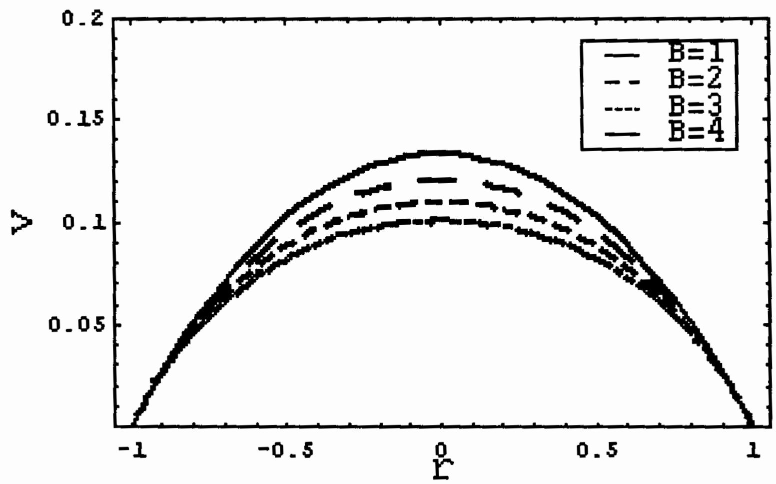

Fig. 2.8. Influence of B on the velocity.

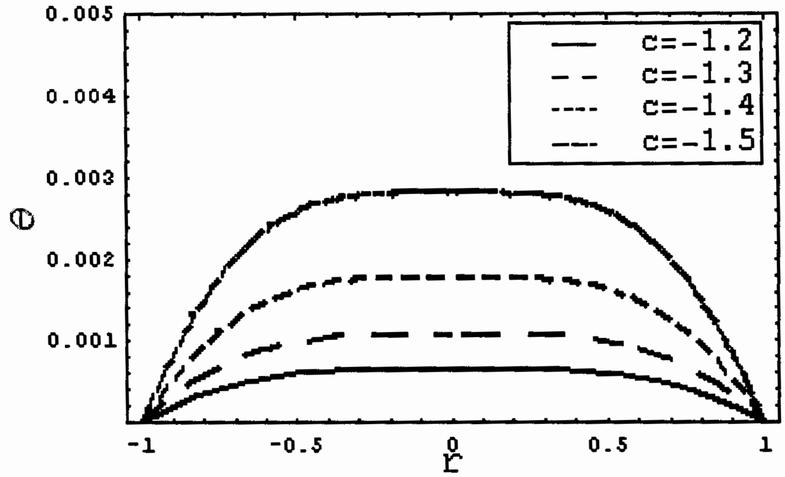

Fig. 2.9. Influence of c on the temperature.

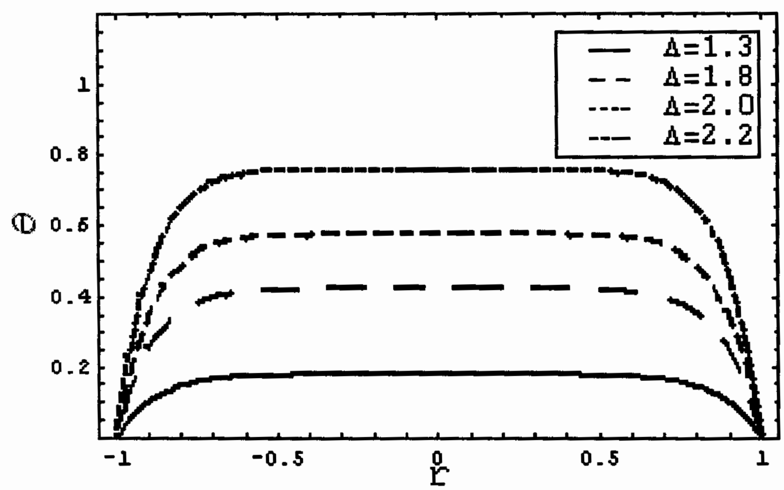

Fig. 2.10. Influence of Λ on the temperature.

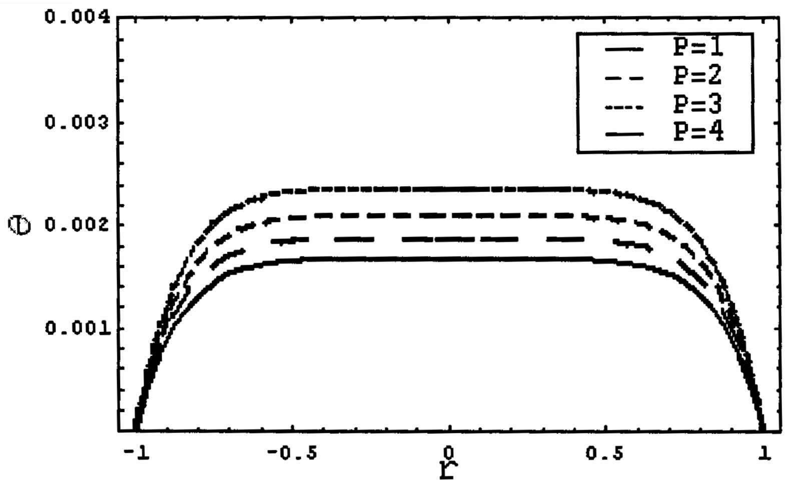

Fig. 2.11. Influence of P on the temperature.

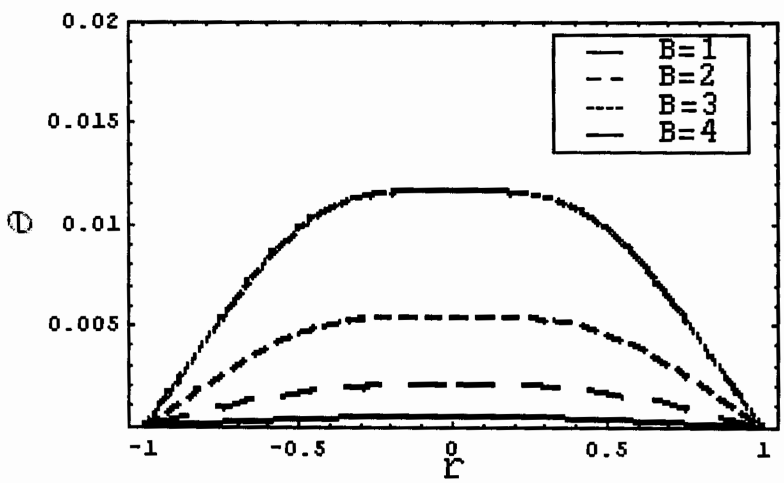

Fig. 2.12. Influence of B on the temperature.

2.5.2 Graphs for Vogel's model

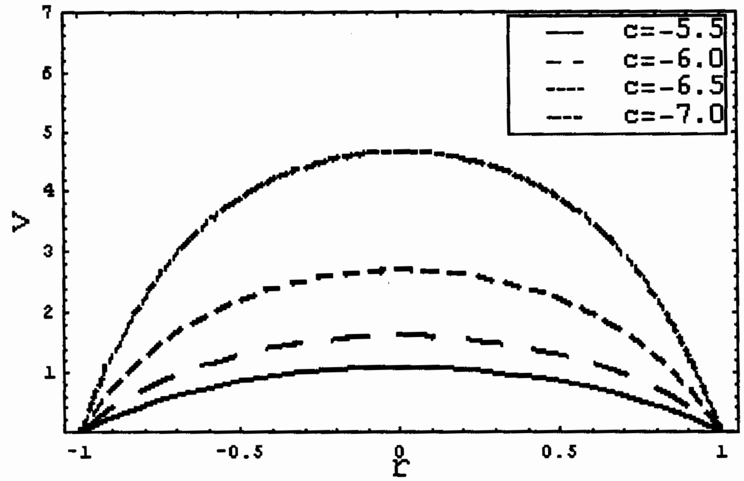

Fig. 2.13. Influence of c on the velocity.

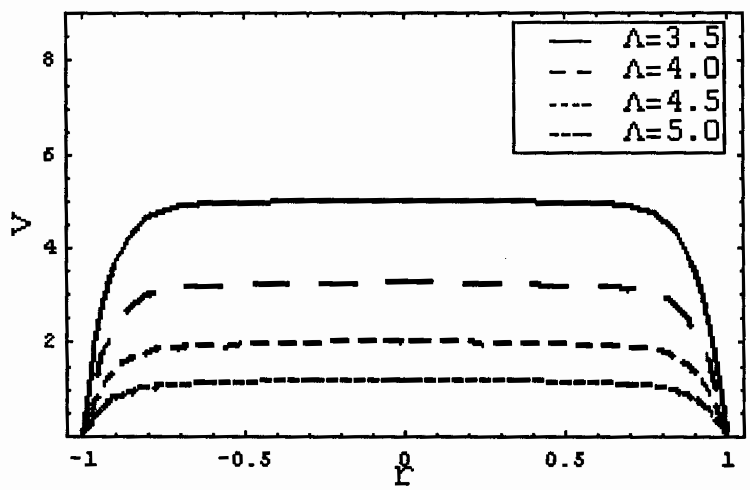

Fig. 2.14. Influence of Λ on the velocity.

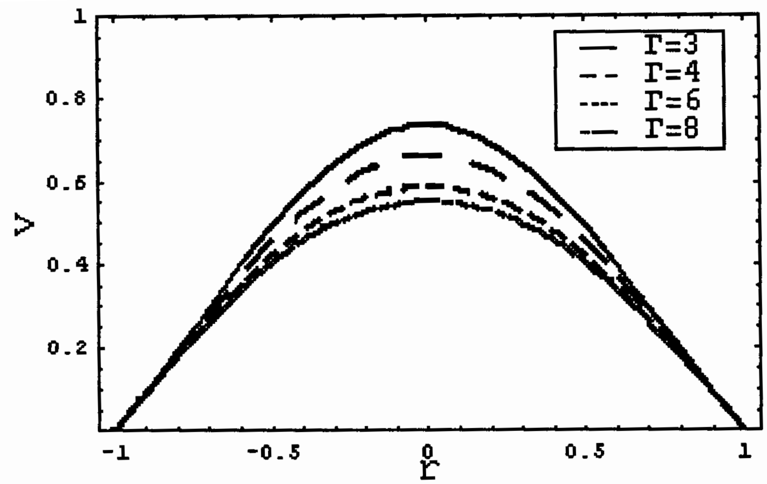

Fig. 2.15. Influence of Γ on the velocity

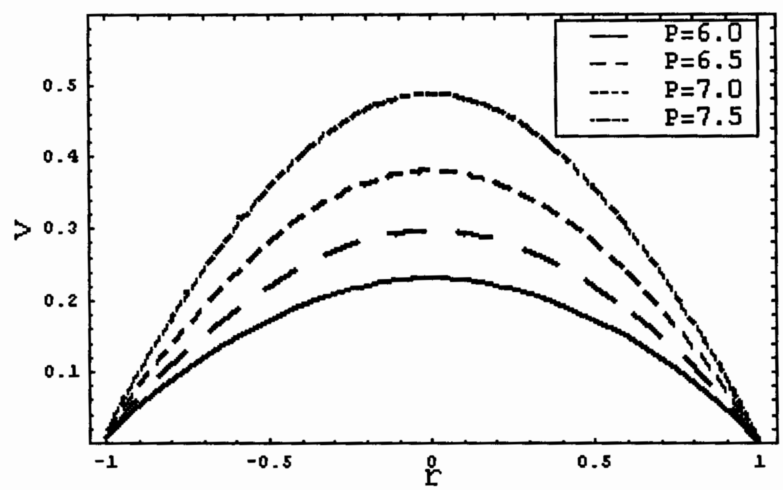

Fig. 2.16. Influence of P on the velocity.

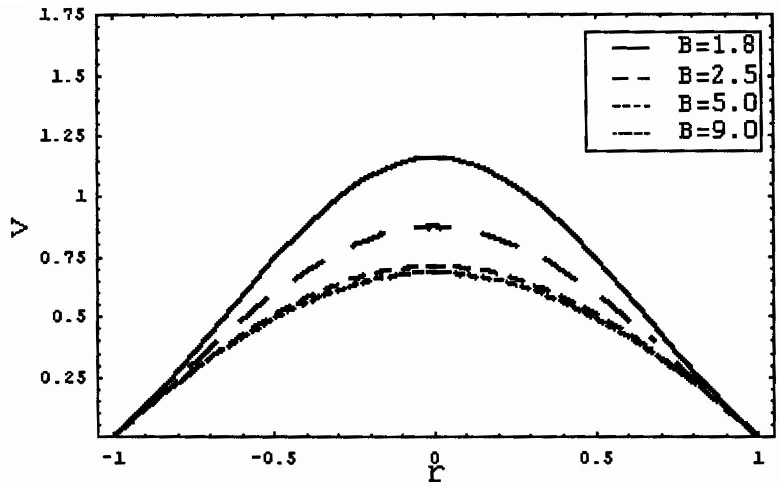

Fig. 2.17. Influence of B on the velocity

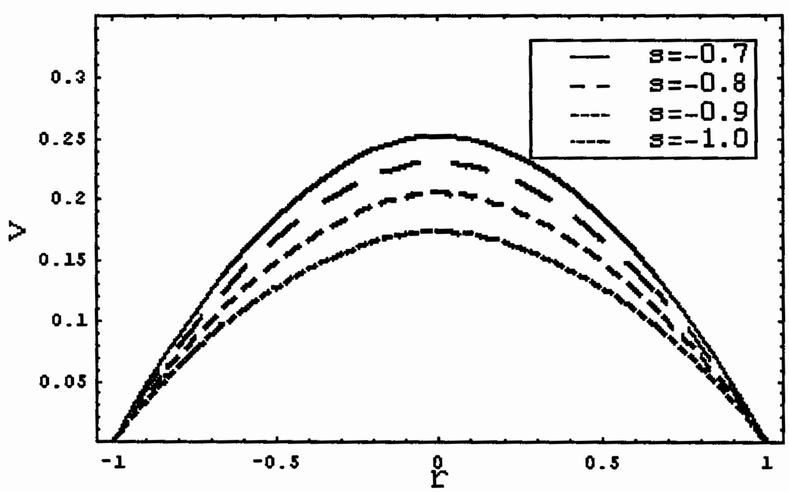

Fig. 2.18. Influence of s on the velocity.

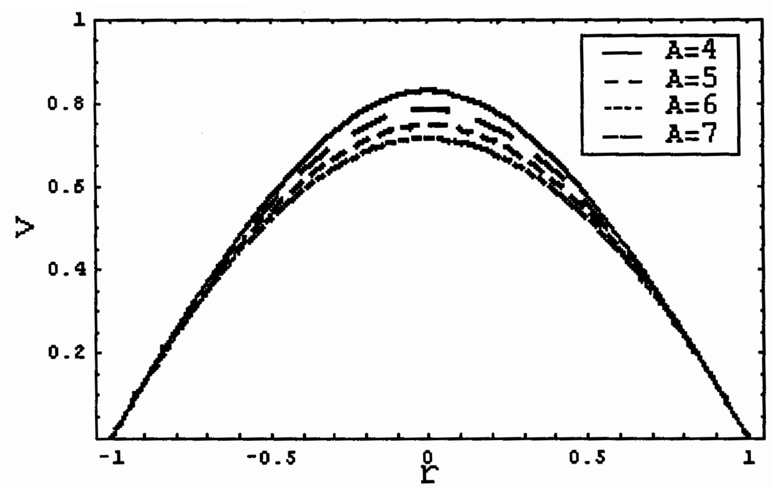

Fig. 2.19. Influence of A on the velocity.

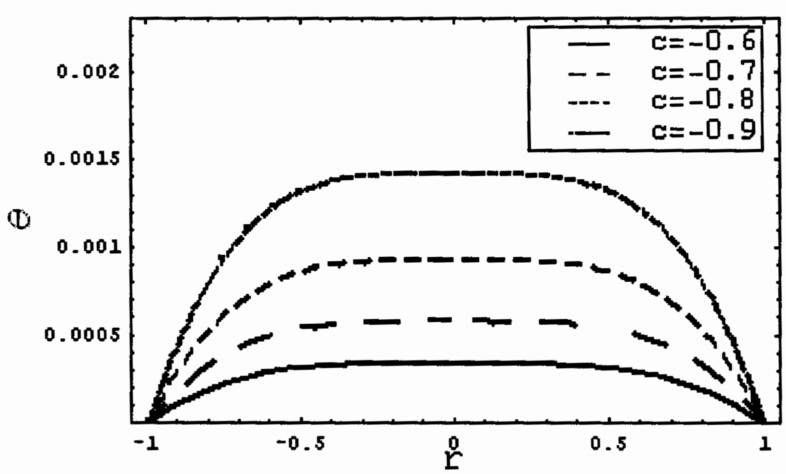

Fig. 2.20. Influence of c on the temperature.

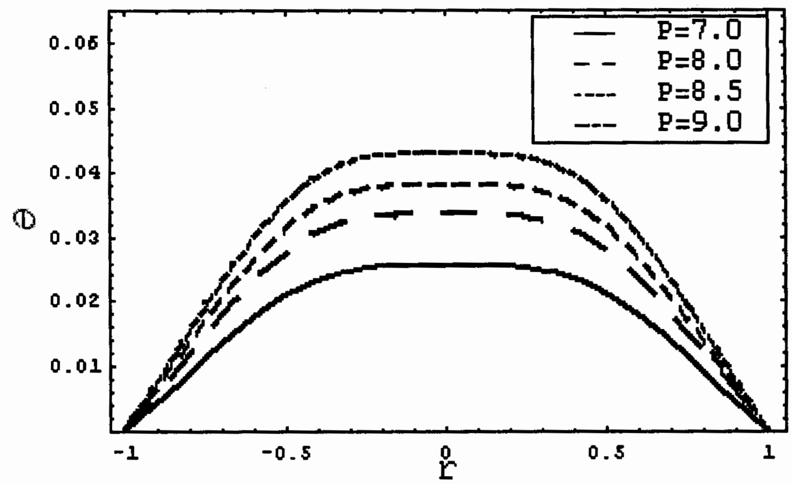

Fig. 2.23. Influence of P on the temperature.

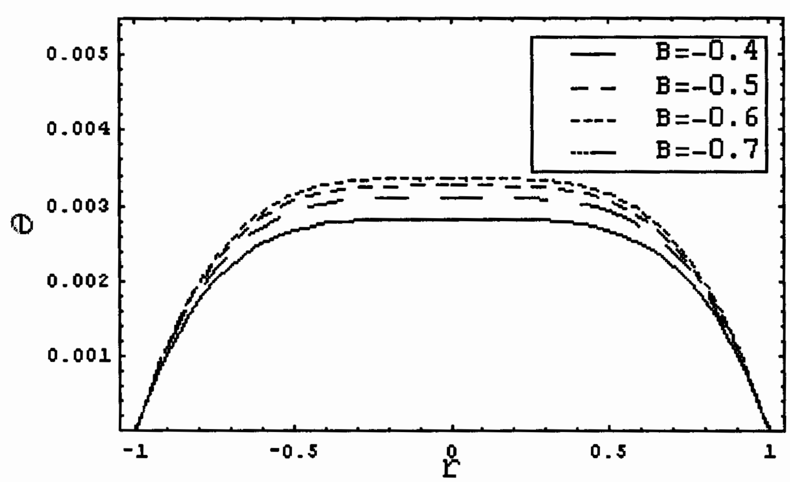

Fig. 2.24. Influence of B on the temperature.

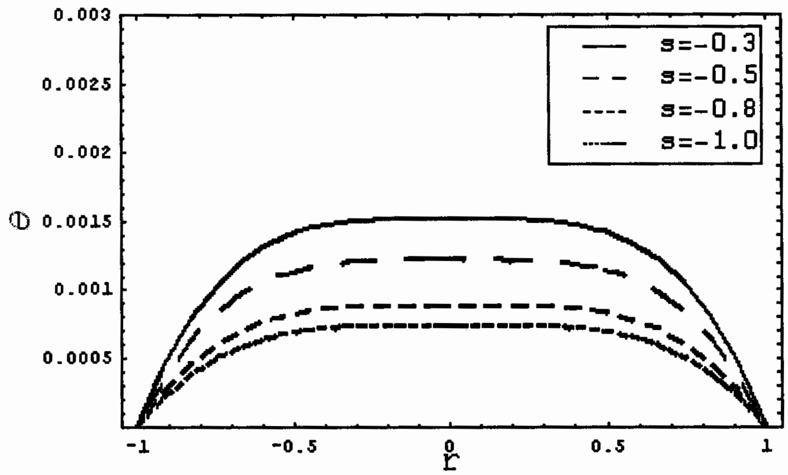

Fig. 2.25. Influence of s on the temperature.

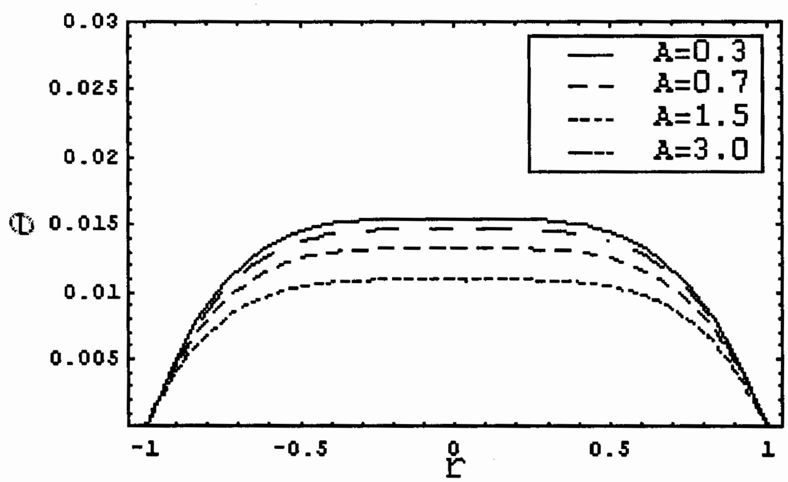

Fig. 2.26. Influence of A on the temperature.

Chapter 3

Effects of MHD and porosity in Reynold and Vogels' models of variable viscosity in third grade fluid

3.1 Introduction

This study extends the analysis of ref. [13] and desires to understand the magnetic field effects on the pipe flow of a third grade fluid in a porous medium by employing modified Darcy's law. Besides this Reynolds model and Vogel's model of temperature dependent viscosity are considered. The relevant equations for flow and temperature have been solved analytically by using homotopy analysis method [14-19]. Convergence of the obtained solutions is explicitly shown. The effects of the various parameters of interest on the velocity and temperature are pointed out.

3.2 Mathematical description

Let us consider an electrically conducting MHD steady flow of a third grade fluid in a pipe. No electric field is present. The induced magnetic field is neglected by considering the small magnetic Reynolds' number. The fluid is induced due to a constant pressure gradient. The appropriate velocity field is designated as follows.

$$\mathbf{V} = [0, 0, v(r)]\,. \tag{3.1}$$

The relevant equations governing the flow can be expressed as

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0, \tag{3.2}$$

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T} + \mathbf{J} \times \mathbf{B}, \tag{3.3}$$

$$\rho c_p \frac{d\theta}{dt} = \mathbf{T} \cdot \mathbf{L} - \boldsymbol{\nabla}^2 \theta, \tag{3.4}$$

$$\mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B}_0^2 v, \tag{3.5}$$

in which d/dt is the material derivative, c_p represents the specific heat, the temperature is denoted by θ, $\mathbf{J}$ indicates the electric current density, $\mathbf{B}_0$ is the applied magnetic field, $\mathbf{L}$ is the velocity gradient and σ the electrical conductivity of the fluid. The Cauchy stress tensor $\mathbf{T}$ is defined by

$$\mathbf{T} = -p_1 \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_3 (tr \mathbf{A}_1^2) \mathbf{A}_1, \tag{3.6}$$

where p_1 is hydrostatic pressure, μ the dynamic viscosity, $\mathbf{I}$ the identity tensor and $\alpha_i (i = 1, 2)$ and $\beta_j (j = 1 - 3)$ the material constants. The Rivlin−Ericksen tensors are defined as

$$\mathbf{A}_1 = \boldsymbol{\nabla} \mathbf{V} + (\boldsymbol{\nabla} \mathbf{V})^t, \tag{3.7}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1} \boldsymbol{\nabla} \mathbf{V} + (\boldsymbol{\nabla} \mathbf{V})^t A_{n-1}, \quad n > 1, \tag{3.8}$$

where $\boldsymbol{\nabla}$ is the gradient operator and the material parameters arising in Eq. (3.6) satisfy the condition given in Eq. 2.9

By [22] we have

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \boldsymbol{\nabla} p = -\frac{\mu}{k} \phi \left(1 + \lambda_r \frac{\partial}{\partial t}\right) \mathbf{V}, \tag{3.9}$$

in which λ and λ_r are the relaxation and retardations times respectively in an Oldroyed $-B$

fluid, ϕ and k are the porosity and permeability of the porous medium.

The Darcys resistance **R** is

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\mathbf{R}=-\frac{\mu}{k}\phi\left(1+\lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{3.10}$$

Following above expression we have in steady flow of a third grade fluid as follows

$$R_Z=-\frac{\phi}{k}\left[\mu+\Lambda\left(\frac{dv}{dt}\right)^2\right]\mathbf{V}, \tag{3.11}$$

Where R_z is the $z-$component of **R** and Λ is the third grade parameter. The Equations (3.3) and (3.4) reduce to

$$\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dv}{dr}\right)+\frac{2\beta_3}{r}\frac{d}{dr}\left(r\left(\frac{dv}{dr}\right)^3\right)-\frac{\phi}{k}\left[\mu+\wedge\left(\frac{dv}{dt}\right)^2\right]v=-\frac{\partial\widehat{p}}{\partial z}+\sigma B_0^2\,v, \tag{3.12}$$

$$\mu\left(\frac{dv}{dr}\right)^2+2\beta_3\left(\frac{dv}{dr}\right)^4+k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right]=0 \tag{3.13}$$

subject to the boundary conditions

$$v(R)=\theta(R)=0,\qquad \frac{dv}{dr}(0)=\frac{d\theta}{dr}(0)=0, \tag{3.14}$$

where k is the thermal conductivity and the modified pressure $\widehat{p}$ is

$$\widehat{p}=p_1-\alpha_2\left(\frac{dv}{dr}\right)^2, \tag{3.15}$$

where p_1 indicates the pressure and α_2 and $\beta_3(>0)$ are the material constants of third grade fluid.

Writing

$$M^2=\frac{\sigma B_0^2\,R^2}{\mu_0}. \tag{3.16}$$

The non–dimensional problems become

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\frac{dv}{dr}+\mu\frac{d^2v}{dr^2}+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3+3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}-P\left[\mu+\Lambda\left(\frac{dv}{dt}\right)^2\right]-M^2v=c, \quad (3.17)$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2\left[\mu(r)+\Lambda\left(\frac{dv}{dr}\right)^2\right]=0, \quad (3.18)$$

$$v(1)=\theta(1)=0,\ \ \frac{dv}{dr}(0)=\frac{d\theta}{dr}(0)=0, \quad (3.19)$$

3.3 Solution of the problem

We will solve Eqs. (3.18) – (3.20) by taking into account the two models of variable viscosity, namely the Reynolds' model and Vogel's model.

Case1:Reynolds' model

Here we have

$$\mu=\mathbf{e}^{-B\theta} \quad (3.20)$$

and for HAM solution we select the

$$v_0(r)=\frac{c}{4}(r^2-1), \qquad \theta_0=\frac{c(r^2-1)}{2}, \quad (3.21)$$

as the initial approximations of v and θ respectively, which satisfy the linear operator and corresponding boundary conditions. We use the method of higher order differential mapping [23], to choose the linear operator $\mathcal{L}_1$ which is given by

$$\mathcal{L}_1=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}, \quad (3.22)$$

such that

$$\mathcal{L}_1(C_1+C_2lnr)=0, \quad (3.23)$$

with C_1 and C_2 as the arbitrary constants.

$Zeroth-$ order problems are given by

$$(1-p)\mathcal{L}_1[v^*(r,p)-v_0(r)] = p\hbar\mathcal{N}_1[v^*(r,p),\theta^*(r,p)], \tag{3.24}$$

$$(1-p)\mathcal{L}_1[\theta^*(r,p)-\theta_0(r)] = p\hbar\mathcal{N}_2[v^*(r,p),\theta^*(r,p)], \tag{3.25}$$

$$v^*(1,p) = \theta^*(1,p) = 0, \quad \left.\frac{\partial v^*(r,p)}{\partial r}\right|_{r=0} = \left.\frac{\partial \theta^*(r,p)}{\partial r}\right|_{r=0} = 0. \tag{3.26}$$

Taking

$$\mu \approx 1-\theta B. \tag{3.27}$$

We have through Eqs. (3.18) to (3.27) as

$$\begin{aligned}\mathcal{N}_1[v^*(r,p),\theta^*(r,p)] &= \frac{1}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 + 3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} + BPv_0 \\ &\quad -Pv - P\wedge\left(\frac{dv}{dr}\right)^2 - M^2v^* - c,\end{aligned} \tag{3.28}$$

$$\mathcal{N}_2[v^*(r,p),\theta^*(r,p)] = \frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2} + \Gamma\left(\frac{dv^*}{dr}\right)^2 + \Gamma\Lambda\left(\frac{dv^*}{dr}\right)^4. \tag{3.29}$$

For $p=0$ and $p=1$, we have

$$v^*(r,0) = v_0(r),\ \theta^*(r,0) = \theta_0(r)\ and\ v^*(r,1) = v(r),\ \theta^*(r,1) = \theta(r). \tag{3.30}$$

When p increases from 0 to 1, $v^*(r,p)$, $\theta^*(r,p)$ varies from $v_0(r)$, $\theta_0(r)$ to $v(r)$, $\theta(r)$, respectively. By Taylor's theorem and Eq. (3.27) we can write

$$v^*(r,p) = v_0(r) + \sum_{m=1}^{\infty} v_m(r)p^m,\ \theta^*(r,p) = \theta_0(r) + \sum_{m=1}^{\infty} \theta_m(r)p^m, \tag{3.31}$$

$$v_m(r) = \frac{1}{m!}\left.\frac{\partial^m v^*(r,p)}{\partial p^m}\right|_{p=0}, \quad \theta_m(r) = \frac{1}{m!}\left.\frac{\partial^m \theta^*(r,p)}{\partial p^m}\right|_{p=0}. \tag{3.32}$$

The convergence of the series (3.31) depends upon $\hbar$. We choose $\hbar$ in such a way that the series

(3.31) is convergent at $p = 1$, then due to Eq. (3.30) we have

$$v(r) = v_0(r) + \sum_{m=1}^{\infty} v_m(r), \qquad \theta(r) = \theta_0(r) + \sum_{m=1}^{\infty} \theta_m(r). \tag{3.33}$$

The mth order deformation problems are

$$\mathcal{L}_1[v_m(r) - \chi_m v_{m-1}(r)] = \hbar \Re 1_m(r), \tag{3.34}$$

$$\mathcal{L}_1[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar \Re 2_m(r), \tag{3.35}$$

$$v_m(1) = \theta_m(1) = 0, \qquad v'_m(0) = \theta'_m(0) = 0. \tag{3.36}$$

$$\begin{aligned}
\Re 1_m(r) \;=\; & \frac{1}{r}\frac{dv_{m-1}}{dr} + \frac{d^2 v_{m-1}}{dr^2} - B\sum_{k=0}^{m-1} \frac{dv_{m-1-k}}{dr}\frac{d\theta_k}{dr} - \frac{B}{r}\sum_{k=0}^{m-1} \frac{dv_{m-1-k}}{dr}\theta_k \\
& - B\sum_{k=0}^{m-1} \frac{d^2 v_{m-1-k}}{dr^2}\theta_k + \frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{i=0}^{k} \left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr} \\
& + 3\Lambda \sum_{k=0}^{m-1}\sum_{i=0}^{k} \left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2 v_i}{dr^2} - P\Lambda \sum_{k=0}^{m-l}\sum_{i=0}^{k} \frac{dv_{m-l-k}}{dr}\frac{dv_{k-l}}{dr} v_l \\
& + BP\sum_{k=0}^{m-1} v_{m-1-k}\theta_k - Pv_{m-1} - (1-\chi_m)c - M^2 v_{m-1},
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
\Re 2_m(r) \;=\; & \frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2} + \Gamma \sum_{k=0}^{m-1} \left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr} - \Gamma B \sum_{k=0}^{m-1} \frac{dv_{m-l-k}}{dr} \\
& \frac{dv_{k-l}}{dr}\theta_l + \Lambda\Gamma \sum_{k=0}^{m-1}\sum_{j=0}^{k}\sum_{i=0}^{j} \left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr},
\end{aligned} \tag{3.38}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{3.39}$$

Case II Vogel's model

Here

$$\mu = \mu_0 e^{\left(\frac{A}{B-\theta} - \theta_0\right)} \tag{3.40}$$

which can be approximated as

$$\mu = \frac{c}{s}\left(1 - \frac{\theta A}{B^2}\right), \tag{3.41}$$

and

$$s = \mu_0 e^{\left(\frac{A}{B} - \theta_0\right)}.$$

The $zeroth-$ and $mth-$order order deformation problems are

$$(1-p)\mathcal{L}_1[v^*(r,p) - v_0(r)] = p\hbar\mathcal{N}_3[v^*(r,p), \theta^*(r,p)], \tag{3.42}$$

$$(1-p)\mathcal{L}_1[\theta^*(r,p) - \theta_0(r)] = p\hbar\mathcal{N}_4[v^*(r,p), \theta^*(r,p)], \tag{3.43}$$

$$v^*(1,p) = \theta^*(1,p) = 0, \qquad \left.\frac{\partial v^*(r,p)}{\partial r}\right|_{r=0} = \left.\frac{\partial \theta^*(r,p)}{\partial r}\right|_{r=0} = 0. \tag{3.44}$$

$$\mathcal{L}_1[v_m(r) - \chi_m v_{m-1}(r)] = \hbar\Re 3_m(r), \tag{3.45}$$

$$\mathcal{L}_1[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar\Re 4_m(r), \tag{3.46}$$

$$v_m(1) = \theta_m(1) = 0, \qquad v_m'(0) = \theta_m'(0) = 0. \tag{3.47}$$

$$\begin{aligned}\mathcal{N}_3[v^*(r,p), \theta^*(r,p)] \quad = \quad & \frac{c}{rs}\frac{dv^*}{dr} - \frac{Ac}{rsB^2}\frac{dv^*}{dr}\theta + \frac{c}{s}\frac{d^2v^*}{dr^2} - \frac{cA}{sB^2}\frac{d^2v^*}{dr^2}\theta \\ & + \frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 + 3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} - P\Lambda\left(\frac{dv^*}{dr}\right)^2 v \\ & - \frac{Pv^*c}{s} - \frac{PAc}{sB^2}v^*\theta - M^2v^* - c.\end{aligned} \tag{3.48}$$

$$\mathcal{N}_4[v^*(r,p), \theta^*(r,p)] = \frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2} - \frac{\Gamma c}{s}\left(\frac{dv^*}{dr}\right)^2 + \Gamma\Lambda\left(\frac{dv^*}{dr}\right)^4 + \frac{\Gamma cA}{sB^2}\theta\left(\frac{dv^*}{dr}\right)^2. \tag{3.49}$$

$$
\begin{aligned}
\Re 3_m(r) \;=\;& \frac{-Ac}{B^2 s}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{d\theta_k}{dt}+\frac{c}{rs}\frac{dv_{m-1}}{dr}-\frac{Ac}{rsB^2}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\theta_k \\
&+\frac{c}{s}\sum_{k=0}^{m-1}\frac{dv_{m-1-k}}{dr}\frac{dv_k}{dr}-\frac{cA}{sB^2}\frac{d^2v_{m-1}}{dr^2}\theta+\frac{\Lambda}{r}\sum_{i=0}^{k}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr} \\
&+3\Lambda\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2v_i}{dr^2}-P\Lambda\sum_{k=0}^{m-1}\frac{d^2v_{m-1-k}}{dr^2}\frac{dv_k}{dr}-\frac{Pc}{s}\frac{dv_{m-1}}{dr} \\
&-\frac{PAc}{sB^2}\sum_{k=0}^{m-k}\sum_{i=0}^{k}v_{m-1-k}\theta_k-(1-\chi_m)c-M^2v_{m-1}, \qquad (3.50)
\end{aligned}
$$

$$
\begin{aligned}
\Re 4_m(r) \;=\;& \frac{1}{r}\frac{d\theta_{m-1}}{dr}+\frac{d^2\theta_{m-1}}{dr^2}+\Gamma\sum_{k=0}^{m-1}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr} \\
&-\Gamma B\sum_{k=0}^{m-1}\sum_{l=0}^{k}\frac{dv_{m-l-k}}{dr}\frac{dv_{k-l}}{dr}\theta_l \\
&+\Lambda\Gamma\sum_{k=0}^{m-1}\sum_{j=0}^{k}\sum_{i=0}^{j}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr}. \qquad (3.51)
\end{aligned}
$$

3.4 Convergence of the solution

In order to determine the admissible values of $\hbar$, we plot the curves 3.1, 3.2, 3.3 and 3.4. It is noted from these Figs. that the range for the admissible values for $\hbar$ is $-1.5 \leq \hbar \leq 0$. The solution calculated finally converges for the whole region of r, when $\hbar$ is in the neighborhood of -1.

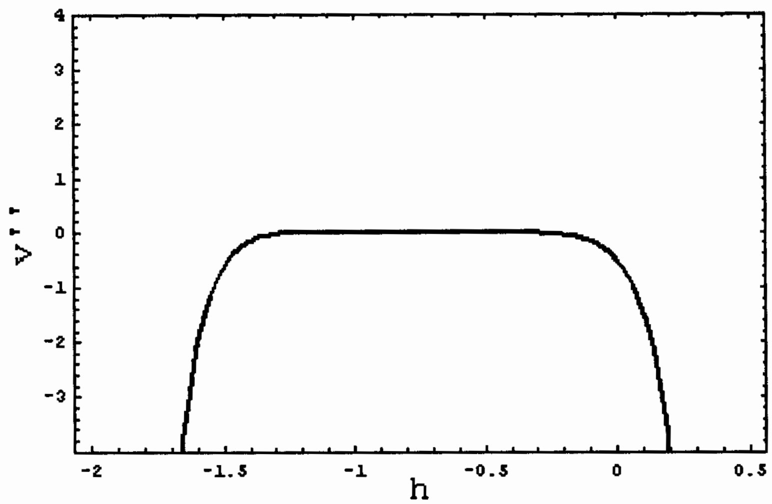

Fig.3.1. $\hbar$−curve for velocity in case of Reynolds' model at $14th$ order approximation.

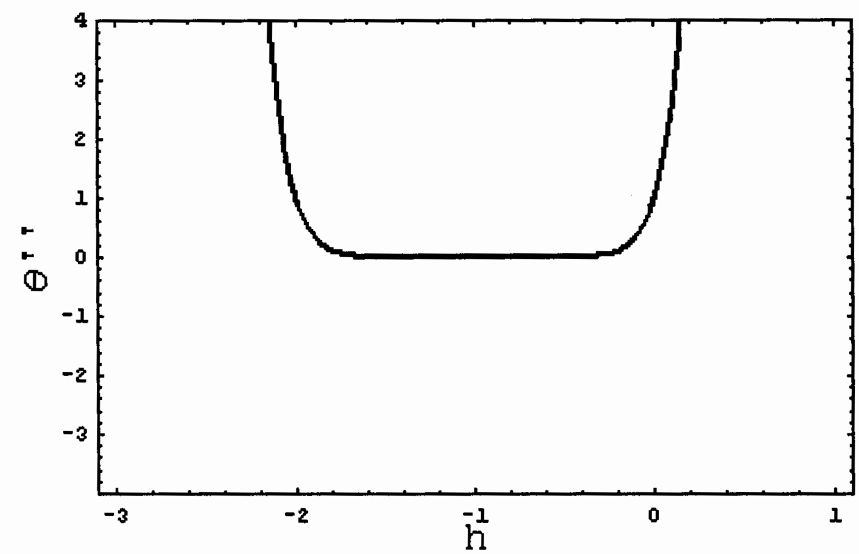

Fig.3.2 $\hbar$−curve for temperature in case of Reynolds' model at $14th$ order approximation.

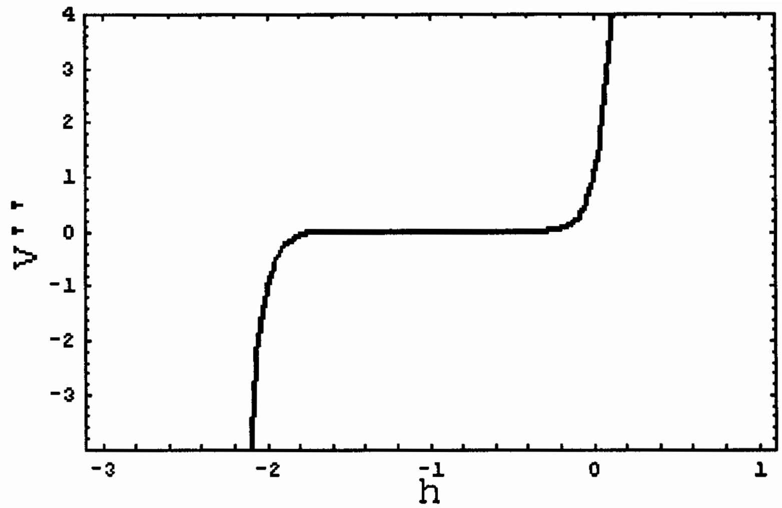

Fig.3.3. $h-$curve for velocity in case Vogel's model at $14th$ order approximation.

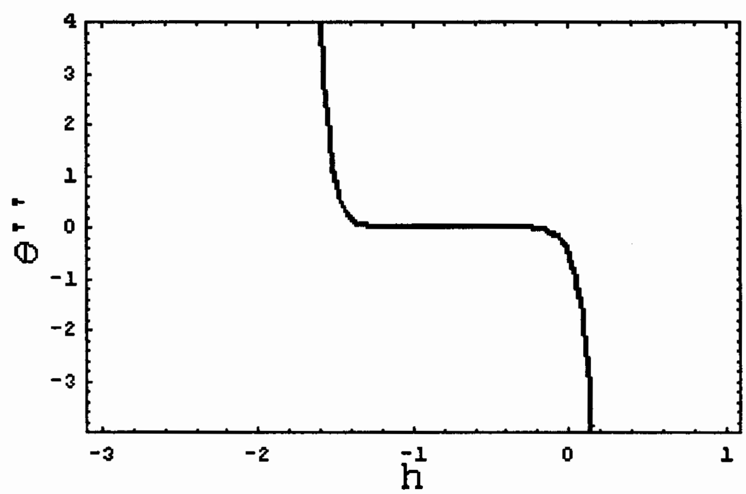

Fig.3.4. $\hbar-$curve for temperature in case of Vogel's model at $14th$ order approximation.

3.5 Graphs

Graphs of Reynolds model

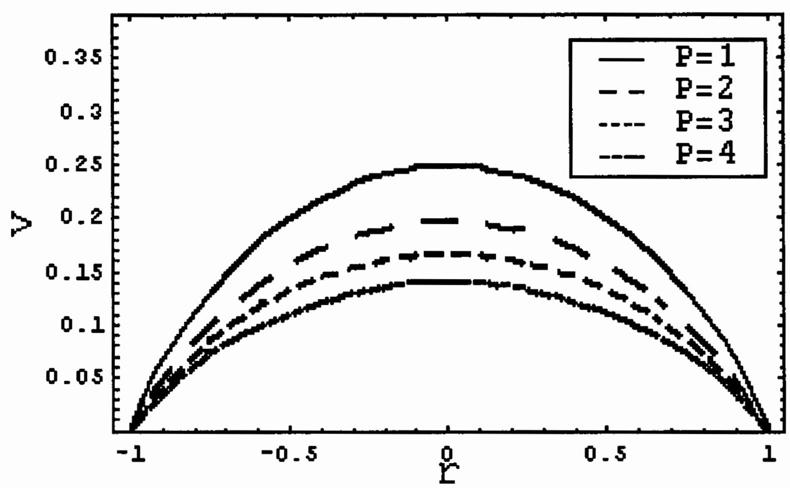

Fig.3.5. Influence of porosity parameter $\boldsymbol{P}$ on velocity.

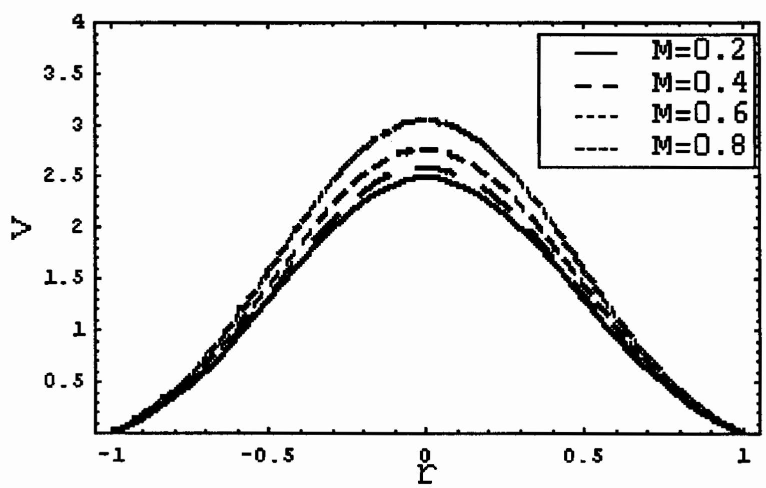

Fig.3.6. Influence of M on velocity.

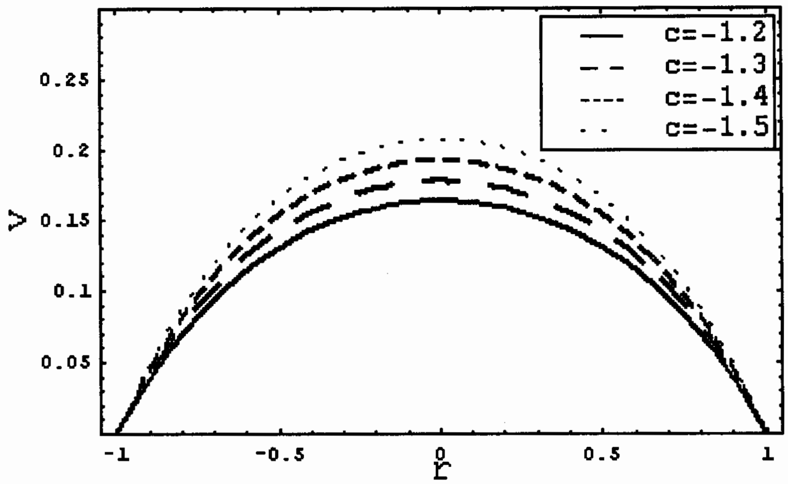

Fig.3.7. Influence of c on velocity.

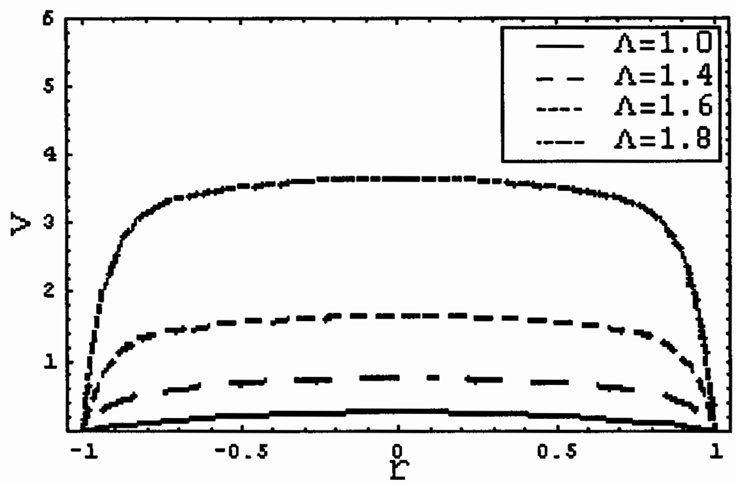

Fig.3.8. Influence of Λ on velocity.

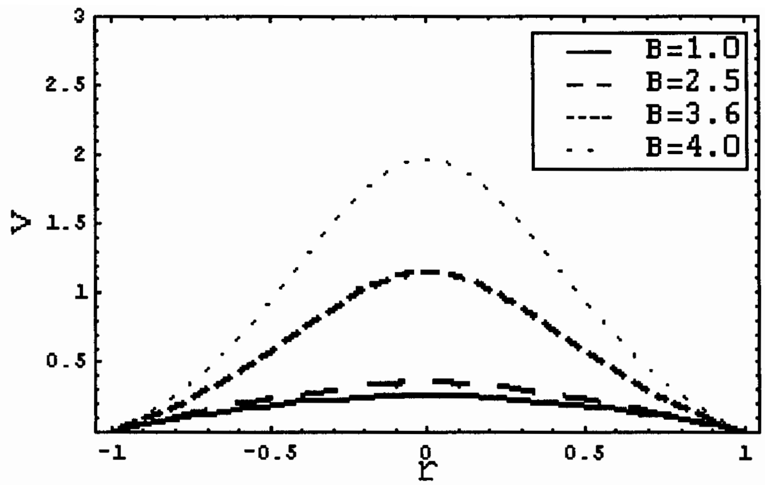

Fig.3.9. Influence of B on velocity.

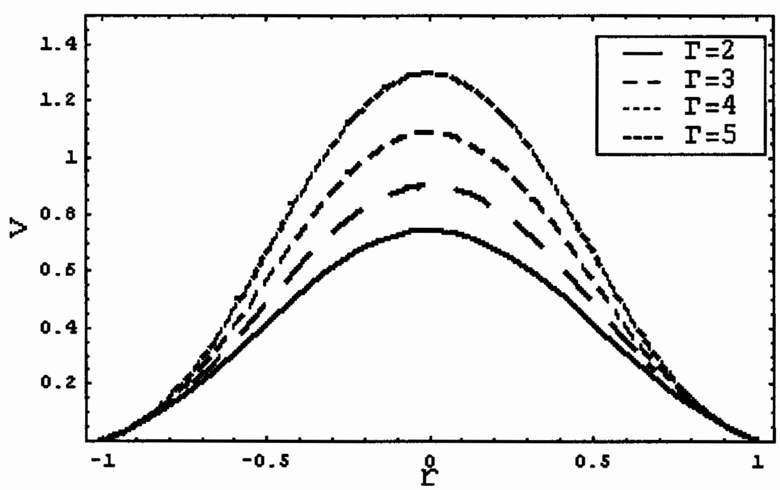

Fig.3.10. Influence of Γ on velocity.

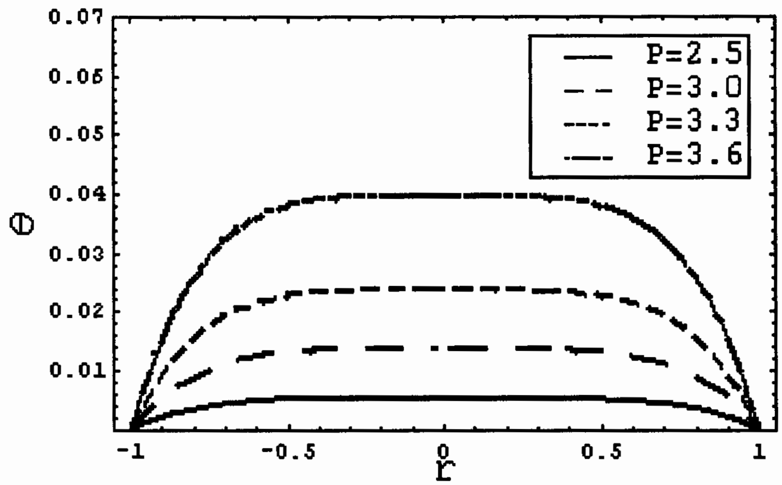

Fig.3.11. Influence of porocity parameter P on temperature.

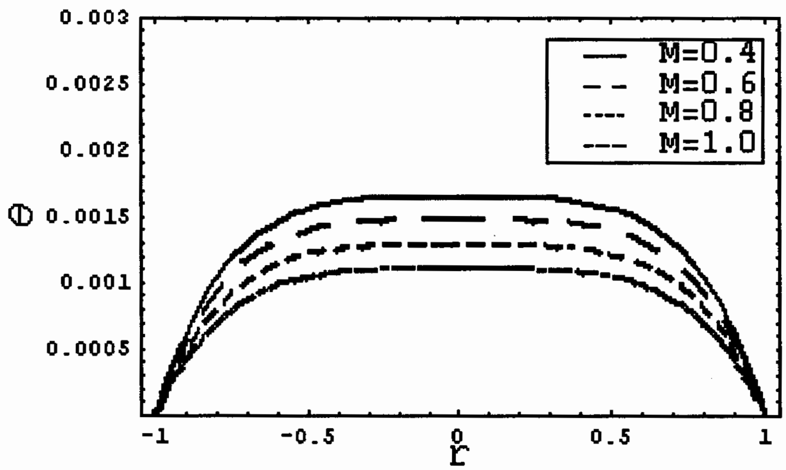

Fig.3.12. Influence of MHD parameter M on temperature.

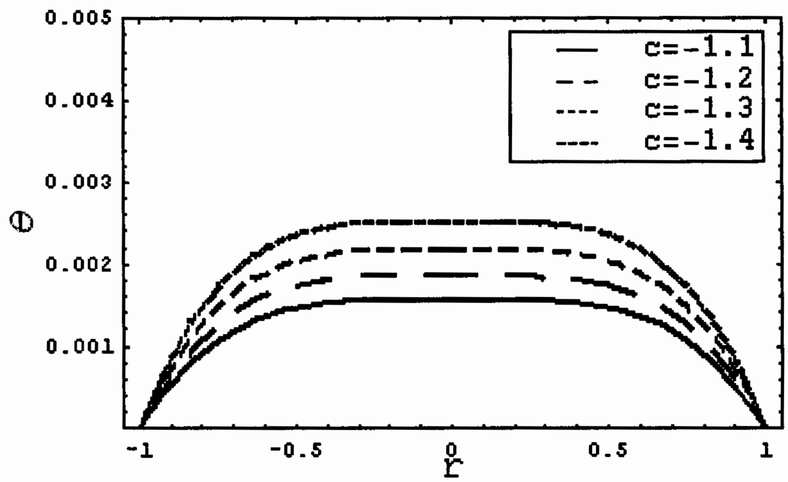

Fig.3.13. Influence of Λ on temperature.

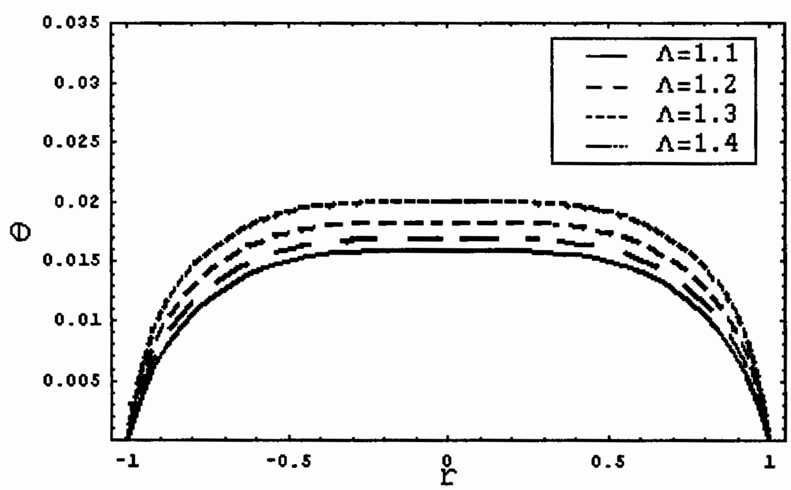

Fig.3.14. Influence of Λ on temperature.

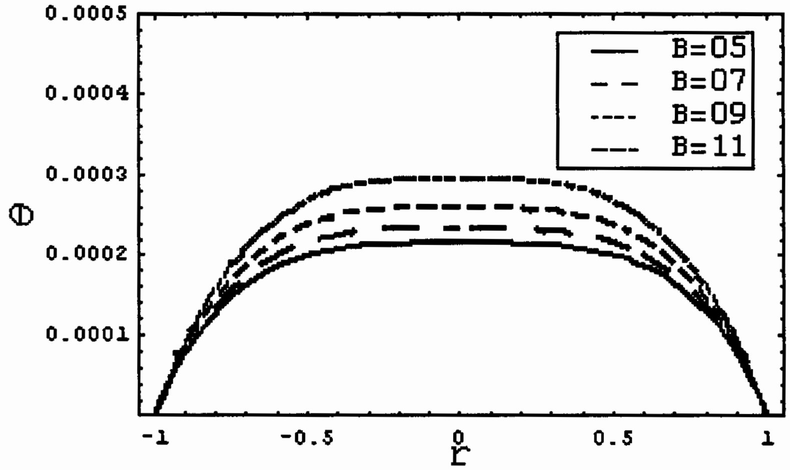

Fig.3.15. Influence of B on temperature.

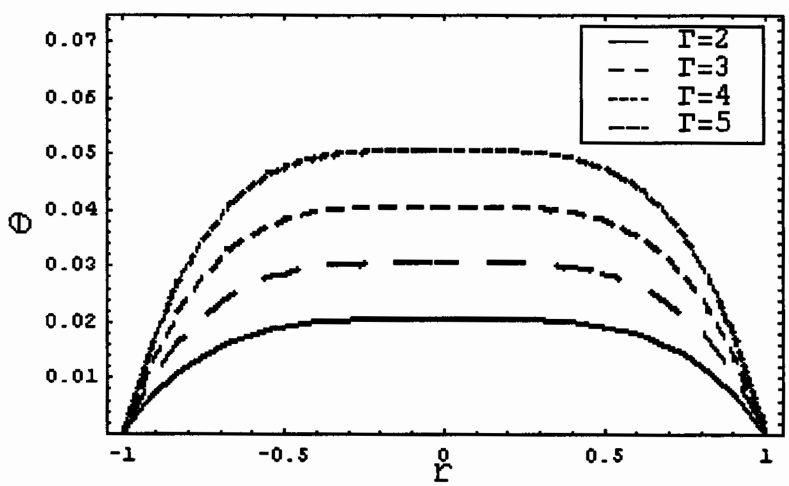

Fig.3.16. Influence of Γ on temperature.

3.5.1 Graphs of Vogel's model

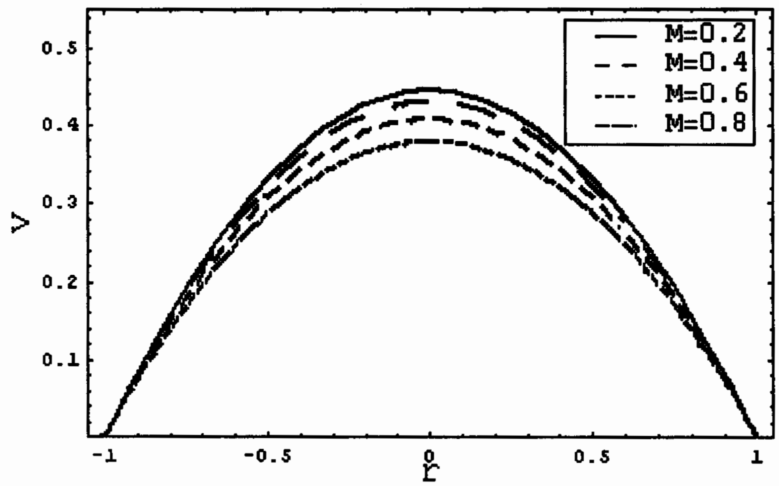

Fig.3.17. Influence of MHD parameter M on temperature.

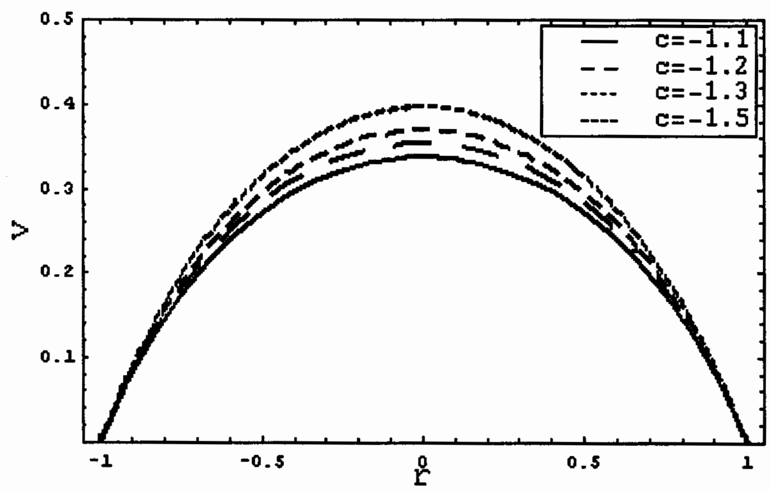

Fig.3.18. Influence of C on velocity.

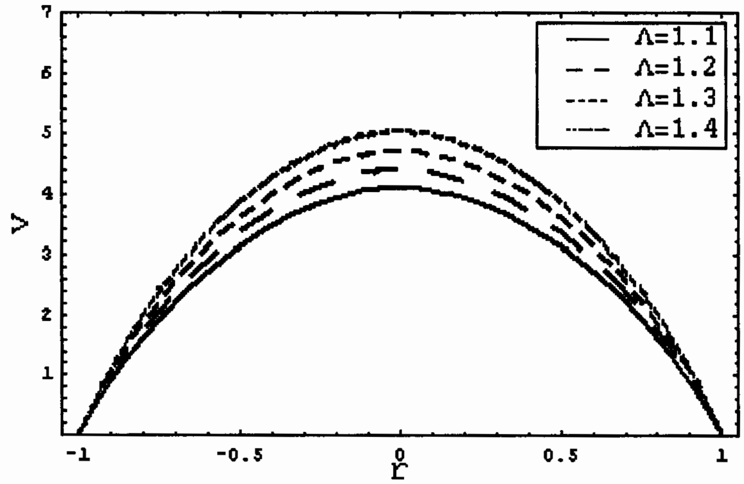

Fig.3.19. Influence of Λ on velocity.

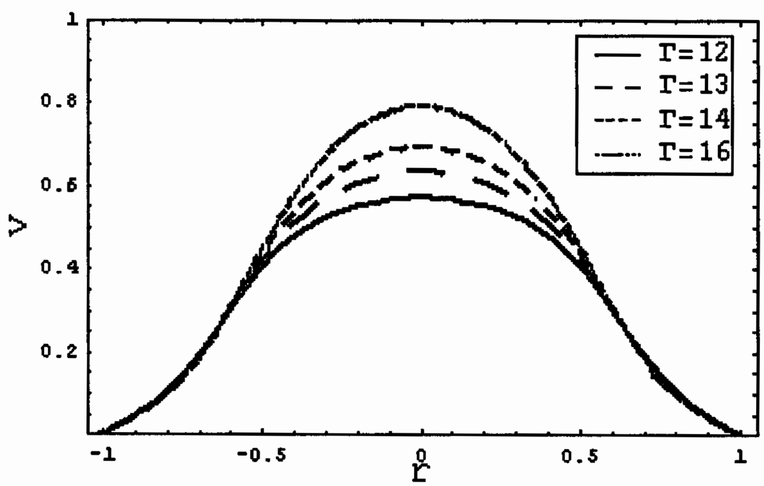

Fig.3.20. Influence of Γ on velocity.

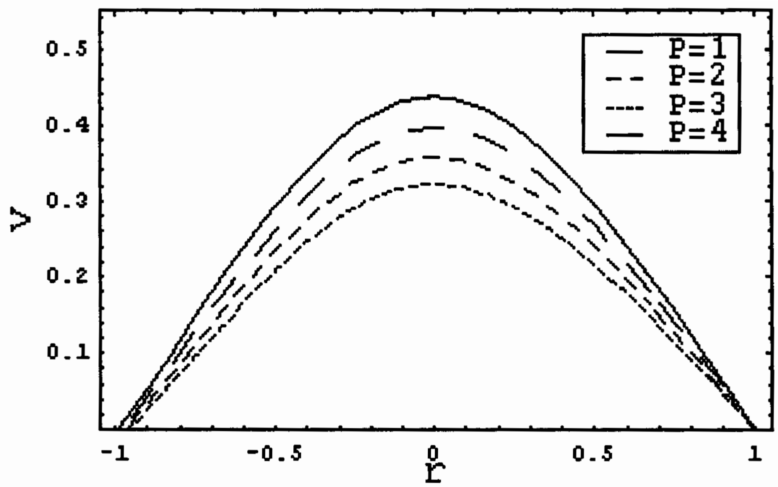

Fig.3.21. Influence of porocity parameter P on velocity.

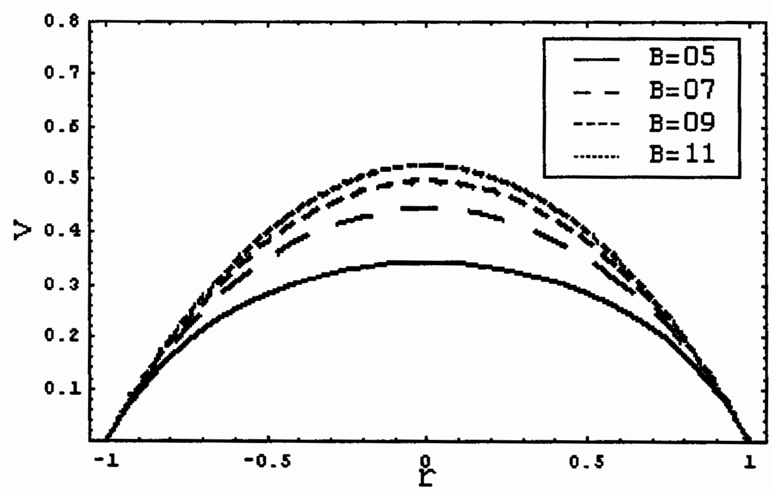

Fig.3.22. Influence of B on velocity.

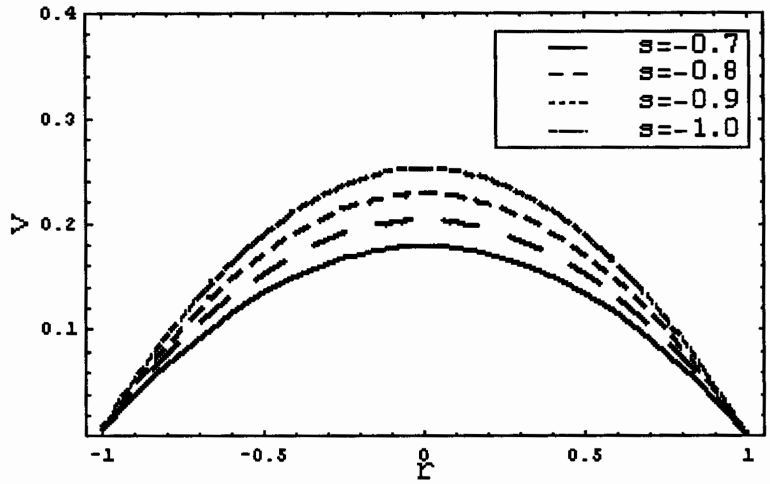

Fig.3.23. Influence of s on velocity.

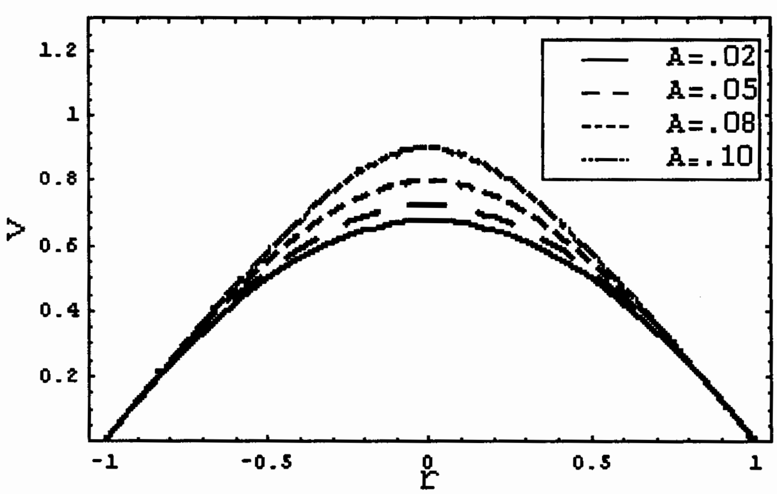

Fig.3.24. Influence of A on velocity.

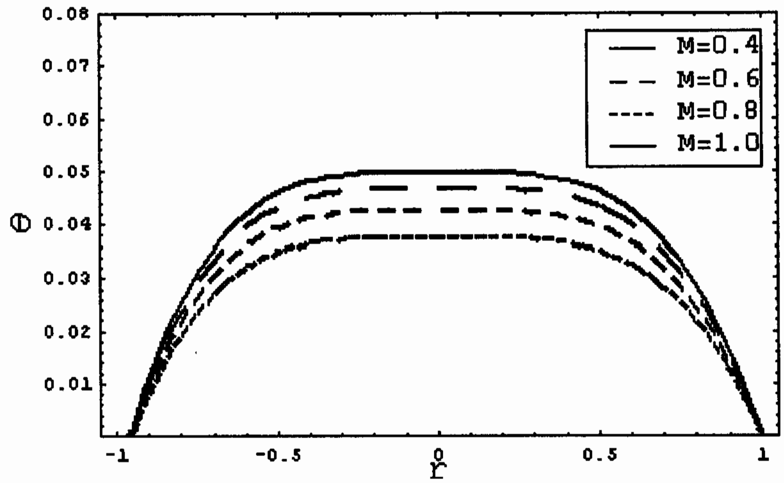

Fig.3.25. Influence of MHD parameter M on temperature.

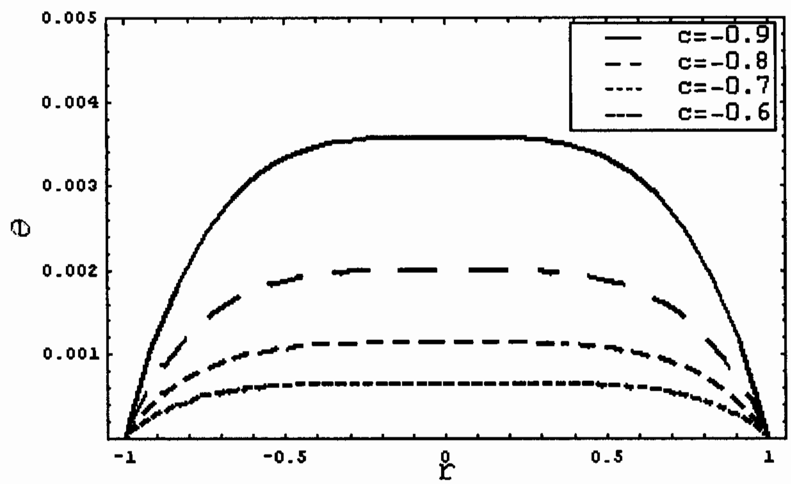

Fig.3.26. Influence of c on temperature.

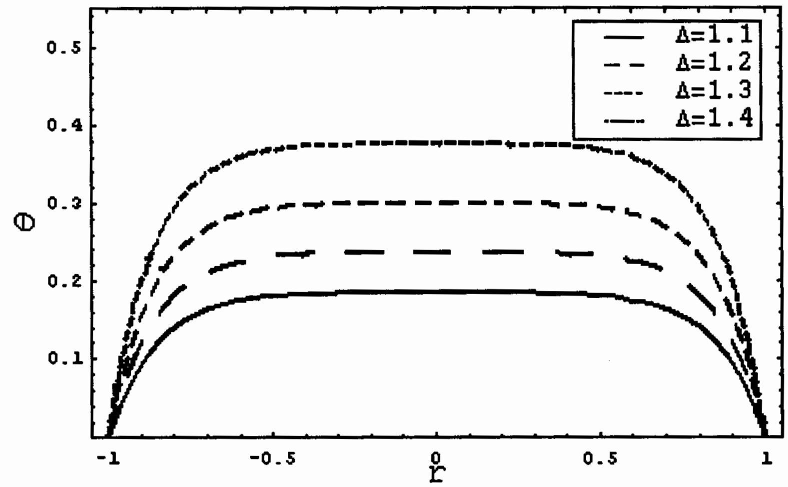

Fig.3.27. Influence of Λ on temperature.

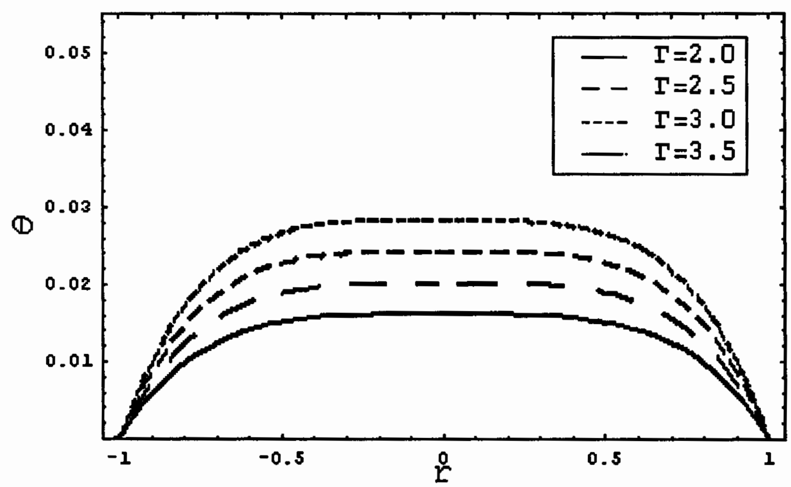

Fig.3.28. Influence of Γ on temperature.

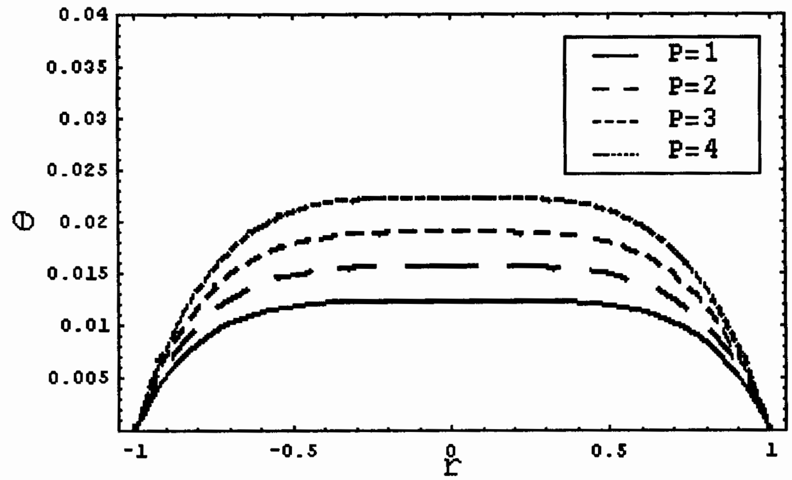

Fig.3.29. Influence of porocity parameter P on temperature.

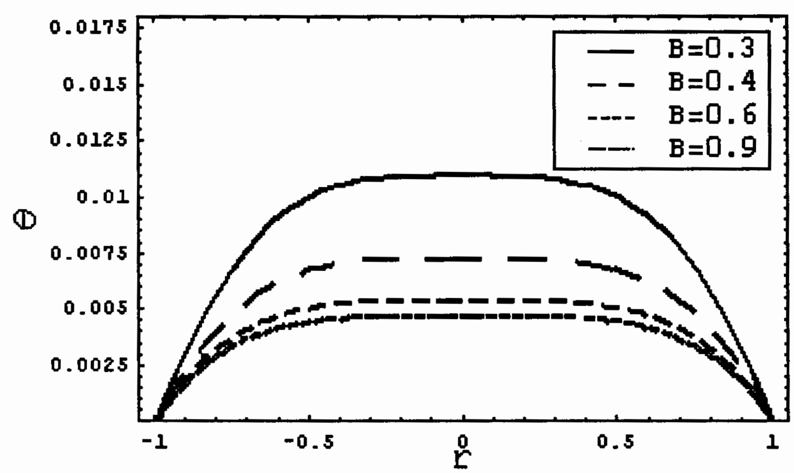

Fig.3.30. Influence of B on temperature.

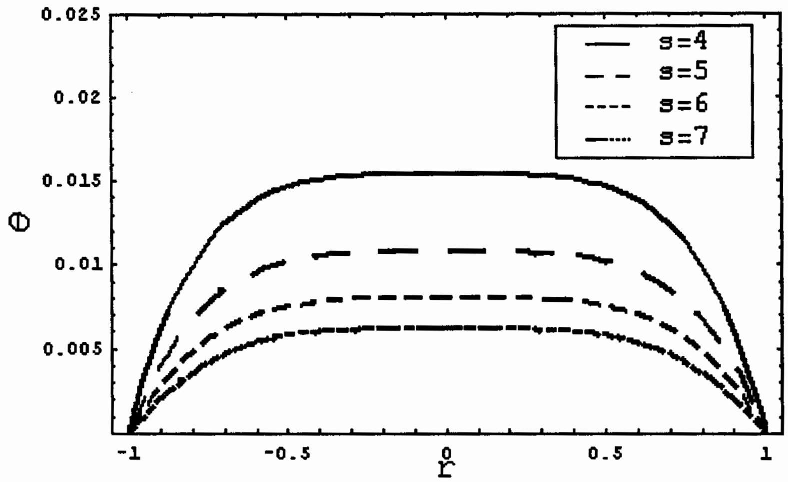

Fig.3.31. Influence of s on temperature.

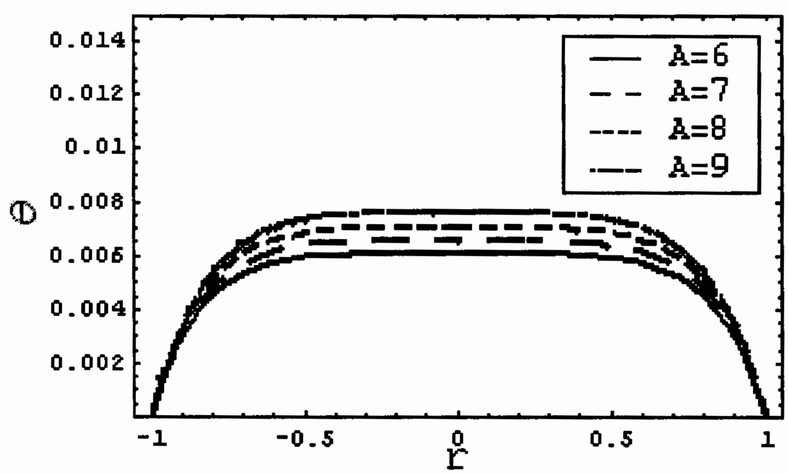

Fig.3.32. Influence of A on temperature.

3.6 Discussion

We have analyzed the effects of MHD and porosity in Reynolds and Vogel's model of variable viscosity in third grade fluid. Analytical solutions are computed. These solutions are valid not only for small but also for large values of all emerging parameters. In order to predict the salient features of the key parameters involved into the present analysis, we display Figs. 3.1 – 3.32. The convergence of the solutions are shown in Figs. 3.1 – 3.4. The dimensionless numbers which vary are P, M, Λ, Γ, A and B. The dimensionless number P is related to the porosity: M is magnetic parameter, Λ is measure of non newtonian behaviour of the fluid, Γ is related to viscous dissipation, B and A indicates how the viscosity of Reynolds model and vogel's model varies. Figs.3.5 – 3.16 show solutions obtained for velocity and temperature profiles for Reynolds vicosity model, while Figs. 3.17 – 3.32 are for Vogel's model. The effect of P on velocity for Reynolds model is shown in Fig. 3.5. It is found that the profile of velocity v decreases by increasing P. It is observed that the profiles of velocity v and temperature θ decrease monotonically by increasing the MHD parameter M (see Figs. 3.6 and 3.12) in Reynolds model.It means that electromagnetic forces provide some mechanism to control the boundry layer thickness. The effects of Λ, B and Γ on v and θ for Reynolds model are shown in Figs.3.8 – 3.10. and 3.14 – 3.16, respectively. It is noted that v and θ increase by increasing Λ, B and Γ. In Fig. 3.11, it is seen that the temperature θ increases by increasing P. It is also noted that the behaviour of velocity and temperatuer profiles for Reynolds model and Vogel's model are same but the penetration depth in Reynolds model mostly decreases more rapidly when compared with that of Vogel's model. Moreover, the obtained analytical solutions have also been compared with the previous studies in the literature which provides a confidence into the presented mathematical descriptions. For instance, the solutions with zero magnetic field and non porous plate obtained by [12] can be recovered by taking $P = M = 0$ and one can achieve the results of [13] when $M = 0$. To the best of our knowledge, no such analysis is available in the literature which can describe the porosity, heat transfer and MHD effects simultaneously on variable viscosity in Reynolds model and Vogel's model.

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