A New Tangent Topp Leone Generated Family of Distributions:Chracteristics and Applications

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A Thesis
Submitted in the Partial Fulfillment of the
Requirement of the Degree of
MASTER OF SCIENCE
In
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CERTIFICATE

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A dissertation submitted in the partial fulfillment of the requirements for the degree of Master of Science in STATISTICS We accept this dissertation as conforming to the required standard

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Declaration

I, Huda Urooj, solemnly declares that the work presented in this dissertation entitled *"A New Tangent Topp Leone Generated Family of Distributions:Chracteristics and Applications"* is original and otherwise acknowledged. This work has not been submitted as a whole or in part for any other degree to any other university in Pakistan or abroad.

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Thesis Completion Certificate From Supervisor

The thesis entitled "A New Tangent Topp Leone Generated Family of Distributions:Chracteristics and Applications" submitted by Huda Urooj, 193-FBAS/MSSt/F22 in partial fulfillment of MS Degree in Statistics has been completed under my guidance and supervision. I am satisfied with the quality of her work and allow her to submit this thesis for further process to graduate with Master of Science from the Department of Mathematics and Statistics, as per IIUI rules and regulation.

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This work is dedicated to

My Parents, My Brothers

AND

Respected Teachers

FOR THEIR UN-CONDITIONED LOVE, UN-ENDING SUPPORT AND SAINTLY PRESENCE IN MY LIFE

Acknowledgement

I am immensely appreciative of Allah Ta'ala, without whom I could not have completed my thesis. Subsequently, I express my gratitude to Dr. Maryam Siddiqa for her excellent supervision. Her guidance, insightful suggestions, and kind monitoring made it possible for me to complete this task. It has been an honor having her as my supervisor. When needed, she is always willing to impart her vast expertise.
I am grateful to the Department of Statistics Chairperson, Dr. Farzana Akhter Abbasi, for her guidance throughout coursework, her encouraging attitude during research projects, and for fostering an environment that is perfect for learning and research in the department.
I'm especially grateful to my loving parents and my supporting brothers, because their encouragement and support always motivated me to continue my struggle and helping me to finish this research project.
I also want to express my gratitude to Sahar Afsheen for all of her help and support with my research project.
A particular thanks to my best friend Sumyyah Khaliq for all of her support, love, encouragement, and care throughout my research.

Huda Urooj

Abstract

Lifespan distributions are crucial for describing real-world events that arise in many scientific domains, including engineering, computer science, social science, and survival analysis. Many distributions have been developed and regularly explored in order to address the problem with the current distributions.It is challenging for the well-known distributions to appropriately fit the real-world data due to their complexity. In order to enhance their efficacy and more precisely reflect the complex structure of the data.

The Gumbel distribution is one of these distributions that is well-known for its wide applicability in both theoretical and practical sectors. The distribution's results are useful for survival and dependability analyses. The current study explores a novel modification using various techniques to provide a new modified life-time distribution that is more flexible than the current extension of the gumbel distribution.

In this research endeavor, initially introduces a new, more adaptable generator i-e, "Logistic Cotangent Topp-Leone-G"(LCTL-G) family of distribution. Subsequently, a new extension of distribution is created by combining a new generator with the base line Gumbel distribution distribution.The resulting form of distribution create a new three-parameter probability model called the Logistic Cotangent Topp-Leone Gumbel (LCTL-Gu) distribution.The proposed model is more capable of representing the monotonic and nonmonotonic hazard rate functions when compared to other current distributions.

Numerous statistical features of the proposed distribution (LCTLGu) have been explored, including the hazard function, quantile function, survival function, and rth moments. A variety of probability models are compared to the proposed model, such as the Beta Gumbel, Kumaraswamy Gumbel, and the Exponentiated Generalized Gumbel.It has been shown that, in comparison to previous existing distributions, the recently established probability model is more suited and yields more flexible results.Entropy computation, mean residual life function and pararmeter estimation have

all been discussed in relation to certain attributes. Many statistical aspects have been examined by considering the two real data sets, including the quantile function, survival function, hazard function, and rth moments. The proposed model is contrasted with a number of existing probability models, and it is clear from the application of real data. The LCTLGu distribution is better than the other recommended distribution and It has been demonstrated that the proposed model is the ideal option for modeling.

Nomenclature

- AD Anderson-Darling
- AIC Akaike information criterion
- BIC Bayesian information criterion
- CAIC Consistent Akaike information criterion
- CDF Cumulative density function
- EL Exponential Lomax
- GL Gamma-Lomax
- HQIC Hannan-quinn information criteria
- HRF Hazard Rate Function
- HLLD Half-Logistic Lomax Distribution
- IPL Inverse Power Lomax
- KSG Kumaraswamy Generalized
- LL Log-Likelihood
- lD Lomax Distribution
- KS Kolomogrov-Smirnov
- McL McDonald Lomax
- MOL Marshall-Olkin Lomax
- MOAP Marshall-Olkin Alpha Power

- MSE Mean square error
- MLE Maximum likelihood Estimation
- MGF Moment generating function
- OS Order Statitics
- PDF Probability density function
- POLO Power Lomax
- SF Survival function
- TLL Topp-Leone Lomax
- TTL Tangent Topp-Leone
- TL-E Topp-Leone Exponential
- TTLB Tangent Topp-Leone Burr
- TIITLPL Type II Topp-Leone Power Lomax
- TTL-KS Tangent Topp-Leone Kumaraswamy
- TTL-KSL Tangent Topp-Leone Kumaraswamy Lomax
- TIIT-G TypeII Tangent Generalized
- TTL-KSE Tangent Topp-Leone Kumaraswamy Exponential
- TTL-KSB Tangent Topp-Leone Kumaraswamy Burr
- W Cramer-Von Mises
- WL Weibull-Lomax
- WIL Weibull Inverse Lomax
- LCTL Logistic Cotangent Topp-Leone
- LCTLGu Logistic Cotangent Topp-Leone Gumbel

- EGu Exponentiated Gumbel
- BGu Beta Gumbel
- KGu Kumaraswamy Gumbel

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Chapter 1

Introduction

1.1 Historical background

In survival analysis, probability distributions are mostly utilized for data modeling since they offer valuable insights into the characteristics of different functions and parameters, specially the failure rate function. Probability distributions serve as both the theoretical and practical basis for statistical techniques in a number of scientific domains, including survival analysis, inference, and reliability analysis. The collection of statistical probability distributions used in life data analysis and reliability engineering is called a lifetime distribution. Lifetime probability distributions are used to study a variety of topics including analyzing the length of unemployment spells and the factors affecting re-employment chances, estimating the time to claim occurrence (e.g., life insurance, health insurance),examining the survival times of species under different environmental conditions or threats, calculating the amount of time until certain occurrences, including death, a disease's recurrence, or recuperation etc. Correct lifetime data analysis can offer real-world solutions to a wide range of issues.

One of the more modern techniques for increasing the adaptability of current distributions is the employment of trigonometric functions, due to their numerous benefits, including their ability to improved the current distributions' qualities and adaptability.

1.1.1 Trignometric Function

One significant aspect of recent research on probabilistic distributions has been the invention of a basic class of trigonometric distributions, which can be more or less complicated.
A lot of interest has been shown in the general classes of trigonometric distributions. These classes are of great significance because they are generally quite good at fitting various kinds of real-world data sets, have easily comprehensible mathematical features, and are very easy to apply.
Recent advances in statistics may be found in publications by Kumar et al., (2015); Jamal and Chesneau (2019); and Souza et al., (2019). More precisely, Souza et al.,(2019) created a novel Sin-G distribution, of which the CDF is one of the most fundamental.

$$\sin_p(x) \quad = \quad \sin\{\frac{\pi}{2}p(x)\}, x\epsilon R$$

Where P(x) is baseline distribution function. They both are very simple trigonometric classes and have a remarkable degree of flexibility in statistical modeling.
Also Souza et al., (2019) introduced the Tan-G class, an entirely novel, straightforward category of trigonometric distributions with the characteristic of being centered on the tangent function. The next CDF defines it:

$$\tan_G(x) \quad = \quad \tan\{\frac{\pi}{4}G(x)\}, x\epsilon R$$

Tashkandy et al., (2023) devloped The Exponentiated Cotangent Generalized Distributions

$$F(X) \quad = \quad -Cot[\pi G(x)^{\alpha}], x\epsilon R$$

Mahmood et al.(2022) presented A New Family of Distributions Using a Trigonometric Function.

$$F(X) \quad = \quad -Cot[\pi G(x)], x\epsilon R$$

1.1.2 Topp-Leone Family of Distribution

A unique distribution family is the Topp-Leone family,which is likely one of the most well-known. The TL distribution is a simple bounded J-shape distribution that has drawn the attention of many statisticians as a potential replacement for the Beta distribution. Al-Shomrani et al. (2016) was developed a family of Topp leone distribution. It is a well-liked generator because of its numerous positive qualities, which include the validity of probabilistic functions and mathematical simplicity.Moreover, the said distribution have ambitions to manage the ability of the new TL generator, which is based on a single shape parameter. Many statistical trend proposed in the literature commonly incorporate several parameters to achieve flexibility.

1.1.3 Gumbel Distribution

Emil Julius Gumbel(1891-1966), a German mathematician, invented the Gumbel distribution in 1958. In extreme value theory, a distribution of greatest(or lowest) values from a sample or set of data has been studied. The Gumbel distribution is a continuous probability distribution has utilized in this study. There are two types of Gumbel distributions:

1. Gumbel Type I (or Gumbel Max):used to simulate how a sample of data's greatest value might be distributed.

2. Gumbel Type II (or Gumbel Min):used to simulate how a sample of data's smallest value might be distributed.

The Gumbel Distribution is also called the log-Weibull distribution and the double exponential distribution.

1.1.4 Gumbel Type I(Max)Distribution

Another name for the Gumbel Max distribution is the Gumbel distribution with the pdf.

$$g(z) \;=\; \frac{1}{\sigma}e^{-}[\frac{z-\mu}{\sigma} + e^{-}(\frac{z-\mu}{\sigma})]$$

here:

- μ is the location parameter

- σ is the scale parameter (β>0)

The CDF is:

$$G(z) \;=\; e^{-e^{-(\frac{z-\mu}{\sigma})}}$$

1.1.5 Gumbel Type II (Min)Distribution

The Gumbel Min distribution has the pdf:

$$g(z) \;=\; \frac{1}{\sigma} e^{(\frac{z-\mu}{\sigma})} e^{-e^{(\frac{z-\mu}{\sigma})}}$$

where:

- location parameter is μ
- scale parameter is σ and (σ is greater than or equal to 1)

The CDF is:

$$G(x) \;=\; 1 - e^{-e^{(\frac{x-\mu}{\beta})}}$$

1.2 Preliminaries

1.2.1 Distribution Function

A distribution function, sometimes referred to as a cumulative distribution function, or CDF, is basic idea in probability and statistics expresses the random variable will take on a value that is less or equal to certain value. Formally speaking a random variable X is a CDF, defined as:

$$F_X(x) \;=\; P(X \leq x)$$

Here:

- $F_X(x)$ is the CDF of the random variable X.
- The value x falls within the domain of X.
- $P(X \leq x)$ shows that the probability of X takes the value that is equal or less than x.

1.2.2 Properties of Distribution Function

- $F_X(x)$ is non decreasing function. This means that if $x_1 < x_2$ then $F(x_1) \leq F(x_2)$
- Is x approaches to minus infinity, then CDF approaches to 0 and if x approaches to plus infinity, then the CDF approaches to 1
- The values of $F(x)$ range between 0 and 1 for all x.
- The CDF exhibits right-continuous behavior.

1.3 Survival Function

The survival function denoted as $S(t)$, is a fundamental concept in survival analysis is the branch of statistics deals with analyzing time-to-event data. It represents the probability that an individual or an object survives beyond a certain time point t. It deals with failure in mechanical systems and death in natural organisms. In addition to being used in a variety of other fields, survival functions are also known, such as event history analysis in sociology, duration analysis or duration modeling in economics, and reliability theory (analysis) in the engineering field. Analysis of patient survival times using these functions is common in medical studies. Survival function is also known as the survivor function or survivorship function in biological survival problem.
Mathematically,

$$S(t) = P(T > t)$$

Here 'P' shows probability, 'T' (random variable) presenting the time until the event of interest occurs and t is the specific some point. The survival function can be increasing or decreasing and approaches to zero as time increases.

1.4 Hazard Function

The PDF f(t) divided by the survival reliability function s(t)which is the definition of the hazard function often called the failure rate.The hazard function is useful for understanding the underlying risk of experiencing an event at different time points. It is often employed in survival analysis to simulate the influence of factors on the event occurrence rate or to assess the relative risk of various groups. Mathematically Hazard function is:

$$\phi(t) = \frac{f(t)}{s(t)}$$

- The instant failure rate at any given time t is specified by hazard function $\phi(t)$.
- The probability of any event happening at time t is higher when $\phi(t)$ is high.
- The distribution of the event times might affect the hazard function's shape. For instance, depending on the underlying mechanism, it may be rising, decreasing, constant, or have more complex shapes.

1.5 Quantile Function

A statistical function that indicates the value below which a specific percentage of data falls is called the quantile function. It is the opposite of a random variable's CDF. In basic terms, the quantile function helps in locating the data point in a random variable's distribution that corresponds to a given probability.
Suppose that X is a random variable with CDF $F(x)$. Here, Quantile function $Q(p)$ is defined as:

$$Q(p) = F^{-1}(p), 0 \leq p \leq 1$$

1.5.1 Properties of Quantile Function

- There is no declining trend in the quantile function.
- The random variable's support is compared closely to the quantile function's range.
- The quantile function Q(p) will also be continuous if the CDF F(x) is continuous.

1.6 Moment Generating Function

In statistics and probability theory , the moment-generating function (MGF) is very a helpful concept since it gives a unique description of distribution of random variable. It offers an efficient way of obtaining a random variable's moments, such as its variance and mean.
For a random variableX, the moment generating function $M_x(t)$is defined as the expected value of e^{tx}

$$M_x(t) = E[e^{tx}]$$

1.6.1 Properties of Moment Generating Function

1. **Existence**:Not all values of t will have the MGF. If the expectation is finite, then it exists for values of t in some region around 0.
2. **Uniqueness**:The probability distribution of the random variable X can be uniquely determined by MGF if it exists in the interval around t=0.

3. **Moments**:The $n-th$ moment of X (i.e $E[X^n]$ can be obtained by differentiating the MGF n times with respect to t and evaluating at $t=0$.

1.7 Renyi Entropy

In the 1850s and 1860s, Rudolf Clausius introduced the entropy measure for the first time.The Renyi entropy which measures the degree of uncertainty or randomness in a probability distribution, is a generalization of the Shannon entropy. It offers a range of entropy measurements which is sensitive to distinct regions of the distribution.Distinct values of α correspond to distinct features of the distribution's spread and form.
For a continuous probability density function $f(x)$, Renyi entropy of order α defined as:

$$H_\alpha(x) = \frac{1}{1-\alpha} log \int_{-\infty}^{\infty} f^\alpha(x)dx$$

Where

- $H_\alpha(x)$ is a convex function of α.
- $H_\alpha(x)$ is non-negative for $\alpha > 0$

1.8 Order Statistic

Order statistics are utilized in many statistical studies and applications and are essential for knowing the distribution of a dataset. They offer insightful information about a sample's distribution, central tendency, and extremes.
Particularly in non-parametric statistics, order statistics are essential for building confidence intervals.Order statistics are values that result from sorting a sample of random variables in ascending or descending order. Given a sample $X_1, X_2, X_3, ..., X_n$ from a population, the order statistics are shown as $X_{(1)}X_{(2)}X_{(3)}, ..., X_{(n)}$

$$f_{1:N}(x) = \frac{f(x)}{\beta(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \left(\begin{array}{c} n-i \\ j \end{array} \right) F(x)^{i+j-1} \tag{1.1}$$

1.8.1 Properties of Order Statistic

1. **Ordering**: The order statistics are arranged in ascending order; $X_1, X_2, X_3, ..., X_n$
2. **Independence**: In general, order statistics are not independent. However, for some specific distributions (e.g., uniform distribution), certain order statistics may be independent.

1.9 Maximum Likelihood Estimation

The unknown parameters can be estimated using a variety of techniques. Maximum Likelihood Estimation (MLE) is a widely utilized technique in statistical model parameter estimation. The process of estimating maximum likelihood maximizes the agreement between the chosen model and the observed data, and gives us the appropriate estimations for the model's parameters.A maximum likelihood estimate is an analytical approach for maximizing data that may be applied to any type of data, whether it is censored or not.Given a probability distribution with parameters θ and it is a set of observed data $x_1, x_2, ..., x_n$ the likelihood function $L(\theta|x)$ is defined as the probability of observing the data given the parameters θ. For independent and identically distributed (i.i.d.) random variables likelihood function is a multiplication of probability density functions (pdf) evaluated at every data point:

$$L(\theta|x) = f(x_1; \theta) \times f(x_2; \theta) \times \times \times f(x_n; \theta)$$

where $f(x_i; \theta)$ is the pdf of the distribution evaluated at x_i with parameters θ.
It is more convenient to work with the log-likelihood function $l(\theta|x)$, which is a natural logarithm of the likelihood function:

$$l(\theta|x) = \log L(\theta|x)$$

To find the maximum likelihood estimates $\hat{\theta}$ of the parameters, one typically takes the derivative of the log-likelihood function with respect to θ and sets it equal to zero.

1.10 Criteria for the goodness of fit test

The methods used to determine how good a data set supports the idea that comes from specific distribution are known as goodness-of-fit tests. A few of these tests are employed to evaluate the probability distribution's quality of fit.In certain circumstances, researchers might gather data with the intention of applying a new model to it. The majority of the time, researchers are unable to select the best model. To assess how Several models are applied to the data, depending on how well it functions. Model selection is the process of deciding which model will eventually solve the problem. The effectiveness of these models is evaluated using the GOF criteria. Among them are HQIC, AIC, BIC, and CAIC.These parametric specifications are all based on the likelihood function. Generally speaking, these information requirements are as follows:

1.10.1 Akaike information criteria(AIC)

Akaike Information Criterion (AIC] (1971) introduced by Akaike. AIC is one of the most commonly used criterion in statistical modeling for model selection. It offers a way to contrast various models and choose the one that most effectively achieves a balance between model goodness and the model complexity of fit. AIC is defined as:

$$AIC = -2log(L) + 2k$$

Here is The maximum value of the likelihood function is L for the model and the number of estimated parameters in the model k is represented as

1.10.2 Bayesian Akaike Information Criterion (BAIC)

[Schwarz(1978)] provides Gideon E. Schwarz's Bayesian information criteria.Model selection among a limited number of models is based on the Bayesian Information Criterion (BIC). Simpler models are preferred over overfitting ones by the BIC, which is based on the likelihood function and adds a penalty term for the amount of parameters in the model.It can be expressed as:

$$BIC = -2log(L) + klog(n)$$

where L is the maximum value of the likelihood function for the model.The number of estimated parameters in the model is denoted by k. The number of observations is denoted by n.

1.10.3 Hannan-quinn Akaike Information Criterion (HQIC)

Hannan-quinn information criterion proposed by [Hannan and Quinn(1979)] Similar to the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC), the Hannan-Quinn Information Criterion (HQIC) is another criterion used for model selection. When it comes to punishing model complexity, HQIC offers a compromise between AIC and BIC. It is very helpful in econometrics and time series analysis.

$$HQIC = -2log(L) + 2klog(log(n))$$

1.10.4 Consistent Akaike Information Criterion (CAIC)

Bozdogan [1987] created the Consistent Akaike Information Criterion (CAIC), a novel model selection criterion.An increase in the penalty for model complexity is included in the Consistent Akaike Information Criterion (CAIC), which is an extension of the Akaike Information Criterion (AIC). Some of the shortcomings of AIC are intended to be addressed by the CAIC, also known as the Corrected Akaike Information Criterion. This is especially true when there are several parameters in relation to the sample size.CAIC is described as

$$CAIC = -2log(L) + k(log(n) + 1$$

where L is the maximum value of the likelihood function for the model.The number of estimated parameters in the model is denoted by k. The number of observations is denoted by n.

1.10.5 Anderson-Darling (A*)

The Anderson Darling (AD) statistic is an empirical distribution function-based goodness of fit test that was first presented by Anderson and Darling [1954].One statistical technique to determine whether a sample of data is representative of a given distribution is the Anderson-Darling test. Compared to other goodness-of-fit tests like the Kolmogorov-Smirnov test, it is especially helpful for small sample sizes and is more sensitive to deviations in the distribution's tails.

$$A_n^2 = n\int_{-\infty}^{\infty}[F_n(x) - F(x)]^2\psi(F(x))dF(x)$$

The empirical distribution function is $F_n(x)$, the particular cumulative distribution function is $F(x)$, and n is the sample size. $\psi(F(x))$ is a non-negative weight function.

1.10.6 Cramer-von Mises (W*)

Anderson (1962) proposed a set of criteria for estimating the minimal distance.An additional statistical test for evaluating the fit between an empirical distribution function (EDF) produced from sample data and a theoretical cumulative distribution function (CDF) is the Cramer-von Mises criterion. This test, which is well-known for its ease of use and broad applicability, focuses on the variations between the observed and expected cumulative distributions.W_n^2 is mathematically defined as:

$$W_n^2 = n\int_{-\infty}^{\infty}[F_n(x) - F(x)]^2\psi(F(x))dF(x)$$

here n is the sample size, $F_n(x)$ is the distribution function which is empirically observed, $F(x)$ is the cumulative distribution function and $\psi(F(x))$ is a weight function with $\psi(F(x)) = 1$.

1.10.7 Kolmogorov-Smirnov (KS)

Massey Jr. created the Kolmogorov-Smirnov (KS) goodness of fit test in 1951, which is one of the most popular nonparametric tests. They claimed that the largest difference between an empirical distribution function and the hypothesised distribution function is where KS lies. The KS test is represented by Dn, and the KS statistic is expressed mathematically as

$$D_n = max|F^*(x) - S_n(x)|$$

$S_n(x)$ denotes the sample-based distribution function, $F(x)$ is the distribution function of a particular population with partially defined parameters, and n is the sample size.

1.11 Purpose of the study

The aim of this study is,

1. To suggest a newly created Logistic Cotangent Topp-Leone generator.
2. To create a new, more flexible Gumbel distribution extension by using a unique trigonometric generator.
3. To study the statistical aspects of the Proposed distribution.

4. To get parameter estimates by applying the MLE approach.

5. Utilizing real life applications by using two datasets, ascertain the effectiveness of the distribution that is being examined.

1.12 Outline of the study

In chapter 1, we include the preliminary information about Topp-Leone generator as well as trignometic function also discuss about our baseline Gumbel distriuibution.
In chapter 2, we discussed the existed published work on Topp-leone generator, different trignometric function and many existed form of Gumbel distribution.
To create new, modified versions of the current probability distribution, we have proposed a new generator and applied it to the current probability model in accordance with the first approach, which is discussed in Chapter 3.
We modified the well-known Gumbel distributions as part of our second strategy by increasing their parameters or substituting one function for another. Many statistical aspects of proposed distribution (LCTGu) have been investigated. Bias and MSE have been employed to evaluate the consistency of the parameters. The three-parameter Logistic Cotangent Topp-Leone Gumbel (LCTLGu) probability distribution introduced in Chapter 4.
In chapter 5,a variety of probability models are compared to the proposed model, such as the Beta Gumbel, Kumaraswamy Gumbel, and the Exponentiated Generalized Gumbel.The proposed model is contrasted with a number of existing probability models.

Chapter 2

Literature Review

Most of these statistical distributions have limited ability to model different kinds of datasets. Certain datasets exhibit heavy tails, reversed J shapes, significant skewness and kurtosis levels, multimodality, and other characteristics.Modifying the flexibility of existing statistical distributions is therefore becoming more and more interesting when modeling large datasets. In order to maximize flexibility while modeling datasets, it has been common practice to use families of distributions, or generators, to change previous distributions in recent years. These generators provide control over the datasets' characteristics.In this chapter we will talk about various modified versions of the suggested Gumbel distributions.

Saralees Nadarajah and Samuel Kotz (2004) Presented the most often used statistical distribution for engineering challenges is the Gumbel distribution. A generalization created from logit of a beta random variable is known as the beta Gumbel distribution. They offered an extensive analysis of this novel distribution's mathematical characteristics. Along with graphical representations, we construct the mathematical figures of associated probability density function and the hazard rate function. We compute an asymptotic distribution of the extreme order statistics and the thenth moment expressions. We examine how the skewness and kurtosis metrics change over time. They also go over the maximum likelihood approach of estimate. We anticipate that this generalization will have greater engineering applicability. [8]

Saralees Nadarajah and Arjun K. Gupta (2007) descried the most widely used models for hydrological processes are gamma distributions. In this study, They explained a very flexible family that encompasses the gamma distribution as a specific example. In the same manner that the exponentiated exponential distribution generalizes the standard exponential distribution, it does the same for the standard gamma distribution. Deriving expressions for various specific cases and bounds, as well as the analytical figures of the corresponding probability density function and the hazard rate function, the moment-generating function, the nth moment, skewness, kurtosis, mean deviation, Shannon entropy, and the asymptotic distribution of the extreme order statistics, provide a thorough treatment of the mathematical properties. An equation for a Fisher information matrix and an estimation process using the maximum likelihood method are also given. Lastly, an illustration is shown using drought statistics from the State of Nebraska. [9]

Zografos and Balakrishnan (2009) addressed the Jones's general family of univariate distributions produced by beta random variables. This distribution family fits both skewed and symmetric models with different tail weights and remarkable adaptability. Similarly, they describe a family of univariate distributions here that are produced by the generalized gamma variables proposed by Stacy. They examine the maximum entropy characterizations for these two families of univariate distributions under appropriate constraints. An expected ratio of quantile densities is suggested for the purpose of differentiating between the members of these two large families of distributions based on these characterizations. They then highlight a few exceptional situations with these results. A different approach to the standard moment method is also suggested for parameter estimation. [13]

In order to describe univariate data, Cooray (2010) presented a variation of the Gumbel distribution that addresses a wide area of skewness in the density function. This generalization achieved by consideration of logarithmic transformation of an odd Weibull random variable. It makes a generalized Gumbel distribution helpful for modeling and fitting a variety of data sets which are not suitable for modeling by popular distributions. Gumbel and reverse-Gumbel distributions' goodness-of-fit as sub-models can also be evaluated with its aid. The generalized Gumbel distribution's skewness and kurtosis forms are demonstrated by building the Moor's kurtosis plane and Galton's skewness plane. There are two different methods to determine parameters using the maximum likelihood method because the reverse translation of the proposed distribution does not change its density function. The flexibility of this generalization is demonstrated by analyzing a data set of the wave and the upsurge heights and comparing the fitness to both Gumbel and the generalized extreme value distribution-

s. [2]

Cordeiro et al., (2011) conducted a study on four-parameter extension of generalized gamma distribution, which can be used to describe a hazard rate function with a bathtub shape. This distribution is also beautiful and significant since it can replicate both the monotonous and non-monotonous failure rate functions commonly encountered in the lifetime data analysis and dependability. There are several popular lifetime-specific sub-models of the new distribution, include the exponentiated generalized half-normal, exponentiated gamma, exponentiated Weibull and generalized Rayleigh to mention a few. For its moments, we develop two infinite sum representations. We compute the order statistics' density as well as two expansions for their moments. The best likely approach is used to figure out the model parameters after obtaining the observed information matrix. Lastly, a number of medical data sets are examined. [10]

Cordeiro et al., (2012) presented a generalization known as the Kumaraswamy Gumbel distribution and gave a thorough explanation of its structural characteristics. The density and hazard rate functions' analytical forms were obtained. For generating function and moments, they compute explicit expressions. As well as the asymptotic distribution of the extreme values, the variation of kurtosis and skewness measurements was analyzed. For the moments of order statistics, explicit formulations are also generated. Best possibility, parametric bootstrap and Bayesian processes are used to find the model parameters. We get the information matrix that was anticipated. [11]

Al-Aqtash (2013) developed the Gumbel-Lomax distribution, which comes from Gumbel-X generator. A novel four-parameter model is presented. In addition to a right-skewed and reversed J-shaped hazard rate, its density function can have a decreasing and upside-down bathtub-shaped hazard rate. An Explicit expressions for quantile function, incomplete and ordinary moments, Lorenz and Bonferroni curves, mean residual lifetime, mean waiting time, probability weighted moments, generating function, and Shannon entropy are among the structural properties of new distribution that are obtained. Additionally, they offer the order statistics' density function. Further characterizations of the new distribution are provided, based on conditional expectations of certain functions of a random variable. After determining an observed information matrix, model parameters calculated by using the maximum appropiate technique. Pair of real-time data sets have been used to show versatility of new model. [6]

Vidal,I (2014) established a bayesian estimation technique for the generalized extreme value type I distribution (Gumbel distribution). An uninformative prior distribution for the location parameter, μ, and three different prior distributions for the scale parameter, s, were considered in order to forecast the maximum annual rainfall intensities in different geographical zones of Chile. We were able to acquire the posterior distribution of (μ, s) along with summary statistics related to it, including quantiles, expected values, modes, and credibility intervals, under these conditions. They acquired the posterior distribution, anticipated value, quantiles, and credibility intervals of future observations for forecast and find return periods. To generate some of these posterior summary metrics, both Laplace and numerical approximations were required. They also estimate the intensity-duration-frequency and return period curves. [1]

Thiago Andrade (2015)recently proposed the exponentiated generalized class is a class of univariate distributions. Within this class, a four-parameter model called the exponentiated generalized Gumbel distribution has explained. We talk about the density function's forms and derive precise expressions for mean deviations, generating and quantile functions, Renyi entropy, Bonferroni and Lorenz curves and ordinary moments. The order statistic's density function is obtained. The model parameters are calculated by using the greatest appropiate method. The observed information matrix is ascertained. To demonstrate a significance of new model, we offer two uses to actual data as well as the Monte Carlo simulation analysis to assess the highest likelihood estimates of the model parameters. [12]

Okorie et al., (2016) presented the exponentiated Gumbel (EG) type-2 distribution, a three-parameter distribution. The new distribution has a unimodel, monotonically growing shape. The Gumbel type-2 (G type-2) distribution, the Exponentiated Frechet (EF) distribution, and the Frechet distribution are the special examples of the new distribution. They deduced several mathematical characteristics of the suggested model, including the order statistics, pth Quantile function, kth crude moment, and Renyi entropy. Model parameters were found using the greatest appropiate method. Ultimately, they fitted a genuine data set to the distributions of EG type-2, G type-2, EF, Frechet, Log Normal (LN), and Weibull and found that the Exponentiated Gumbel (EG) type-2 model fits the data better. [24]

Al-Shomrani et al., (2016) proposed the straightforward J-shaped alterna-

tive for the Beta distribution, the Topp-Leone distribution has captured the interest of many statisticians. A new family of distribution called Topp-Leone family of distribution has been provided by [7]
Brito et al., (2017)expanded on the one-parameter distribution developed by Topp-Leone [A family of J-shaped frequency functions] by introducing a new class of continuous distributions called the Topp-Leone odd log-logistic family. They described two exceptional situations and looked at some of its mathematical characteristics. Moreover, regression model grounded in novel Topp-Leone odd log-logistic Weibull distribution was put forth. Three real data sets are used to demonstrate the stretchiness and usefulness of proposed family. [14]

Alizadeh et al., (2019) developed an odd log-logistic exponentiated Gumbel distribution which is a versatile expansion of the Gumbel distribution. The study includes a brief tutorial on using the GAMLSS package of R software, which was used to develop the novel model. They give a thorough analysis of its general mathematical characteristics. In addition, They provide a fresh extended regression model that takes four regression structures into account. They talk about estimating techniques using both uncensored and censored data. The utility of proposed approach is demonstrated by using four real data sets and two simulated experiments. [4]

Hassan et al., (2019) invented Type II generalized Topp-Leone-G (TIIGTL-G) family of distributions the Topp-Leone distribution that develops a bathtub-shaped hazard function, it is a desirable model for life testing and reliability investigations. Its hazard rates are growing, decreasing, upside-down, J, and reversed-J, and its density function can be unimodel, left, right or reversed-J-shaped. A few unique models are exhibited. They examine a few of its statistical attributes. The quantile and generating functions, Renyi entropy, order statistics, ordinary and incomplete moments are all given explicit expressions. Estimating the model parameters was done by using maximum likelihood approach. Through the use of two actual data sets, significance of one particular model-the Type II generalized Topp-Leone exponential was demonstrated. [15]

Jamal et al., (2019) developed the Topp Leone Weibull-Lomax distribution, a new four-parameter lifespan distribution. The quantile function, ordinary and incomplete moments, probability-weighted moments, conditional moments, order statistics, stochastic ordering, and stress strength reliability parameter are among the mathematical characteristics of the new distribution that are examined. Additionally examined are the regression model and the residual analysis for the new model. A simulation study is conducted to investigate the behavior of the estimates of the model parameters, which are obtained by applying the maximum likelihood criterion. We provide empirical evidence for the new distribution's significance and adaptability in modeling four data sets. [16]

Chesneau et al., (2019) provided the novel class of (probability) distributions, derived by the compounding a baseline distribution with sine and cosine functions and based on a cosine-sine transformation. A few of its characteristics are examined. More specifically, a particular cosine-sine transformation is examined, with the exponential distribution serving as the baseline. It is possible to estimate the parameters of the certain sine-cosine exponential distribution using the highest likelihood estimation approach. Simulation research looks into how well these estimations perform. Applications are provided for four real data sets, demonstrating the superior via goodness-of-fit tests compared to several existing distributions. [23]

Al-Marzouki et al., (2019) presented new four-parameter lifespan distribution based on power Lomax distribution and type II Topp-Leone-G family. A new distribution is distinguished from existing ones by having extremely stretchable probability functions increasing, decreasing, J and reverse J shapes are noted for the hazard rate and probability density functions, providing early indications of the related model's potential for adaptability. In light of this notion, a thorough analysis of the new distribution is conducted, encompassing both theoretical and applied aspects. Following an explanation of its primary mathematical characteristics, the associated model was examined and its parameters are estimated by maximum likelihood technique. We implemented it on two real-world datasets, one of which is the widely used airplane windshield data. [17]

Bantan, R.A et al., (2020) introduced and investigated the type II power Topp-Leone-G family of continuous distributions. This way is so-called type II Topp-Leone-G family is naturally extended. They ascertained the principal characteristics of the novel family, demonstrating their dependence on the conditions at stake. Investigated topics include quantile functions, certain mixed representations, moments and derivations, stochastic ordering, order statistics, dependability, and forms and asymptotes of several significant functions. Subsequently, an inverse exponential distribution-based family model was presented. [18]

Tung Yen Liang et al., (2020) presented The Arcsine-X family of distributions is the name given to the proposed family that uses the trigonometric function. For demonstration, the Arcsine-Weibull distribution, a particular sub-model of the suggested family, is taken into consideration. To find the parameters of Arcsine-X distributions, the best estimation approach has used. Thorough the investigation using Monte Carlo simulation was conducted to evaluate the resulting estimators. To illustrate the Arcsine-Weibull, two insurance data sets are analysed. The Arcsine-Weibull model was compared to its familiar competitors with two and four parameters. Weibull, Lomax, Burr-XII and beta Weibull models are among the competitive models.The utility of the Arcsine-Weibull and other discussed models is evaluated by using several goodness of fit metrics. [19]

Ibrahim Alkhairy et al., (2021)studied the new family of distributions based on the arctangent function, an inverse trigonometric function. Heavy-tailed probability distributions are very useful in the field of actuarial science and are crucial for data set modeling. In order to get an outstanding fit to complex economic and actuarial data sets, actuaries are dedicated to searching for such distributions. In this work, a well-liked technique for creating new distributions that are great fits for handling heavy-tailed data is reviewed. [20]

Zafar Mahmood et al., (2022) created a new G-class disbursing cotangent function based on logistics and suggest a trigonometric generalizer/generator of distributions using quantile function with a modified Cauchy distribution. Notable mathematical properties and unique models are obtained. Furthermore suggested are expanded models and new mathematical transformations. We design and analyze a logistic cotangent Weibull (LCW) model with two parameters. The hazard rate of the suggested model displays both monotonous and non-monotonous figures,while Unimodal and bimodal (symmetrical, right-skewed, and decreasing) forms are displayed by density. This is what makes the model beautiful and significant. The maximum likelihood approach is employed for parametric estimation, and simulation analysis is carried out to guarantee the asymptoticness of the estimates. The significance of suggested trigonometric generalizer, G class and model is demonstrated by two applications that concentrate on survival and failure datasets. The outcomes of these applications attest to the model's considerable superiority over other, well-known competing models in terms of fit, flexibility, and capability. The authors believed that specialists in the fields of lifetime data and analysis, actuarial and financial sciences, and reliability analysis would find greater interest in the proposed classes and models. [21]

Fayomi et al., (2022) suggested a strong Gumbel distribution generalization. The T-X paradigm is the foundation of this family of distributions. This article also mentions three distinct models that have evolved from a list of special distributions as a consequence of this family. Density of a new family can be characterized using the linear combination of generalized exponential distributions, which is important for evaluating some of the family's characteristics.This family's statistical characteristics are identified, including precise expressions for the generating function, order statistics, quantile function, and ordinary and incomplete moments. The model parameters are estimated using the greatest likelihood method. One of the special models has also undergone a thorough investigation. The skewness measure is quantified using MacGillivray skewness in addition to traditional skewness measures. We can also identify several crucial risk indicators using the new probability distribution, both graphically and mathematically. To verify the validity and superiority of the proposed model, we employ three real-world data sets and a simulated evaluation of the proposed distribution. [5]

Tashkandy et al., (2023)produced the numerous algebraic generalized families and classes of statistical distributions. Objective of this study is to develop the novel distributions generator with support on real line that is cotangent, exponentiated, and generalized. The cotangent exponentiated generalized (CE-G) family and the logistic cotangent exponentiated generalized (LCE-G) family are two new families of distributions that incorporate the cotangent function that are proposed. This presents a thorough examination of the mathematical and structural characteristics of Burr-based innovative model (LCEB) and the recently proposed G-family. In Monte Carlo simulation studies, the maximum likelihood approach assesses the performance of model and predicts its parameters. To complete these tasks, the maximum likelihood technique is applied. The waiting and survival times data sets are statistically analyzed, and results validate the effectiveness, superiority, and usefulness of the proposed generator. G-family, and innovative distribution in comparison to other well-known Burr-based models are similar and competitive. [22]

Qingyang Liu et al., (2024) develops a new single-modal distribution family that is defined by the mode and three additional parameters that are obtained by combining a minimal Gumbel distribution with a maximum Gumbel distribution. The study delves into the characteristics of the suggested distribution, such as its capacity to identify heavy-tailed data and adapt to varying skewness directions across a broad spectrum. To infer parameters in the new distribution, frequentist and Bayesian approaches are devised.Simulation tests are carried out to show that both approaches perform satisfactorily. As shown in the analysis of simulated data under the proportions of skewness and kurtosis and the analysis of the hydrology application data, the flexible distribution has high performance for data having far tails at both directions. In this case, the mode plays the function of the location parameter. It can easily create a regression model for the mode of a response given the variables by using the proposed unimodal distribution. We use this methodology to uncover intriguing data elements hidden by outliers in criminology application data. [3]

Chapter 3

A new Cotangent Topp-Leone g family of distribution

3.1 Introduction

This chapter's main goal is to introduce a novel generator and add to the body of knowledge already available on probability models.The current probability distribution is converted into a new one by applying the general functions referred to as generators.The Topp-Leone generator of distribution presented by Sangsanit and Bodhisuwan (2016) as well as the Tangent Topp-Leone family of distribution recommended by Nanga et al., (2022) and Mahmood et al., (2022) developed a novel cotangent function-based trigonometric generator.
Further details regarding the contributions to the probability models has been reviwed in Chapter 2.This chapter proposes a new family of distributions that have significant implications for many scientific fields, such as survival analysis and biology. Because of the patterns in the survival function and risk function, modified families of distributions can be viewed as meaningful lifetime distributions.Numerous statistical features and parameter estimates have been examined using the maximum likelihood technique. The suggested model has the best match for the two real data sets when compared to other available probability models.

3.2 Logistic Cotangent Topp-leone Generalized Family of Distribution

3.2.1 Topp leone Family of Distribution

Topp-Leone generator of distribution was established by Sangsanit and Bodhisuwan (2016) as mentioned in the introduction. Next, for values of random variable z where $z\epsilon R$, the CDF and PDF of Topp-Leone generator of distribution are defined as
The cdf of Topp-Leone G family of distribution:

$$\begin{aligned} H(z) = F_{TL-G}(z) &= [G(z)]^{\alpha}[2 - G(z)]^{\alpha} \\ &= [1 - \bar{G}(z)]^{\alpha}[2 - \{1 - \bar{G}(z)\}]^{\alpha} \\ &= [1 - \bar{G(z)}]^{\alpha}[2 - 1 + \bar{G(z)}^{\alpha}] \\ &= [1 - \bar{G(z)}]^{\alpha}[1 + \bar{G(z)}]^{\alpha} \\ &= [1 - \bar{G(z)}^{2}]\alpha \\ &= [1 - (1 - G(z))^{2}]^{\alpha} \end{aligned} \tag{3.1}$$

The pdf of Topp-Leone G family of distribution:

$$\begin{aligned} f_{TL-G}(z) &= \alpha[1 - (1 - G(z))^{2}]^{\alpha-1}. - 2(1 - G(z)). - g(x) \\ &= 2\alpha g(z)(1 - G(z))[1 - (1 - G(z))^{2}] \end{aligned} \tag{3.2}$$

3.2.2 Cotangent Topp-Leone Family of Distribution

Mahmood et al. (2022) created a unique trigonometric generator based on cotangent functions.
Cdf of Cot G family of distribution:

$$F(z) = -\cot[\pi G(z)], x\epsilon R$$

By putting the (3.1) in above expression, we get

$$F(z) = -\cot[\pi\{1 - (1 - G(z))^{2}\}] \tag{3.3}$$

Suppose T represent a logistic random variable with CDF

$$R(t) = (1 + e^{-t})^{-1} \tag{3.4}$$

3.3 Logistic Cotangent Topp-Leone Family of Distribution

By putting the (3.3) in (3.4)
cdf of Logistic Cotangent Topp-Leone family of distribution is:

$$F_{CLTL-G}(z) \quad = \quad [1 + e^{\cot[\pi[1-(1-G(z))^2]^\alpha]}]^{-1} \tag{3.5}$$

The pdf of Logistic Cotangent Topp-Leone $cotG$ family of distribution:

$$\begin{aligned} f_{CLTL-G}(z) \quad = \quad & 2\pi\alpha g(z)[1 - G(z)][1 - \{1 - G(z)\}^2]^{\alpha-1} \\ & \csc^2[\pi\{1 - [1 - G(z)]^2\}^\alpha]e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha} \\ & [1 + e^{\cot[\pi\{1-(1-G(z))^2\}^\alpha]}]^{-2} \end{aligned} \tag{3.6}$$

3.3.1 Survival Function

Survival function of $LCTL - G$ family of distribution gain as>

$$S_{CLTL-G}(z) = 1 - F(z) \tag{3.7}$$

By substituting (3.5) into (3.7),we get

$$= \quad 1 - [1 + e^{\cot\{\pi[1-(1-G(z))^2]^\alpha\}}]^{-1} \tag{3.8}$$

3.3.2 Hazard Function

The definition of hazard rate function is the ratio between the PDF and survival function (SF), also referred to as the failure rate function of an object. The hazard rate function of $LCTL - G$ is given:

$$h_{CLTL-G}(z) = \frac{f_{CLTL-G}(z)}{S_{CLTL-G}(z)}$$

By putting 3.6 and 3.8 in above expression:

$$\begin{aligned} h_{CLTL-G}(z) \quad = \quad & 2\alpha g(z)[1 - G(z)][1 - \{1 - G(z)\}^2]^{\alpha-1} \\ & \times \csc^2[\pi\{1 - [1 - G(z)]^2\}^\alpha]e^{\cot[\pi\{1-(1-G(z))^2\}]^\alpha} \\ & \times[1 + e^{\cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-2} \\ & \times\frac{1}{1 - [1 + e^{cot[\pi[1-(1-G(z))^2]^\alpha]}]^{-1}}, (\alpha > 0), z\epsilon R \end{aligned} \tag{3.9}$$

3.3.3 Quantile function

Data creation using the distribution of $LCTL-G$ Since CDF has a closed form, it is easier to understand. To replicate data from this distribution, we must first create random integers with the typical uniform distribution. Therefore, next step is to convert these random values into the inverse CDF. The $LCTL-G$ family of distribution's quantile function is:

$$Q = F^{-1}(z)$$

using 3.5

$$\begin{aligned}
Q &= \left[1+e^{\cot\{\pi[1-\{1-G(z)\}^2]^\alpha\}}\right]^{-1} \\
Q &= \frac{1}{1+e^{\cot\{\pi[1-\{1-G(z)\}^2]^\alpha\}}} \\
1+e^{\cot\{\pi[1-\{1-G(z)\}^2]^\alpha\}} &= \frac{1}{Q} \\
e^{\cot\{\pi[1-\{1-G(z)\}^2]^\alpha\}} &= \frac{1}{Q}-1 \\
\cot\{\pi[1-\{1-G(z)\}^2]^\alpha\} &= \ln[\frac{1}{Q}-1] \\
\pi[1-\{1-G(z)\}^2]^\alpha &= \cot^{-1}\ln[\frac{1}{Q}-1] \\
\{1-[1-G(z)]^2\}^\alpha &= \frac{1}{\pi}\cot^{-1}\ln[\frac{1}{Q}-1]
\end{aligned}$$

Taking power $1/\alpha$ on both sides

$$\begin{aligned}
1-[1-G(z)]^2 &= \{\frac{1}{\pi}\cot^{-1}\ln[\frac{1}{Q}-1]\}^\alpha \\
\{1-G(z)\}^2 &= 1-\{\frac{1}{\pi}\cot^{-1}\ln[\frac{1}{Q}-1]\}^\alpha
\end{aligned}$$

Taking square root on both side

$$\begin{aligned}
1-G(z) &= \sqrt{1-\{\frac{1}{\pi}\cot^{-1}\ln[\frac{1}{Q}-1]\}^\alpha} \\
G(z) &= 1-\sqrt{1-\{\frac{1}{\pi}\cot^{-1}\ln[\frac{1}{Q}-1]\}^\alpha} \\
z &= G^{-1}\left[1-\sqrt{1-\{\frac{1}{\pi}\cot^{-1}\ln[\frac{1}{Q}-1]\}^\alpha}\right] \qquad (3.10)
\end{aligned}$$

3.3.4 Linear Representation of New C.D.F

Mahmood et al.,(2022) use the linear Representation of New C.D.F [25]

$$F(z) = [1 + e^{-\cot[\pi G(z)]}]$$

Using the Logistic function,

$$(1 + e^{-t})^{-1} = 1 - (1 + e^{-t})^{-1}$$

F(z) can be easily calculated as:

$$F(z) \quad = \quad 1 - [1 + e^{cot[\pi G(z)]}]^{-1}$$

As

$$(1 + x)^{-1} \quad = \quad \sum_{i=0}^{\infty} (-1)^i x^i$$

so

$$[1 + e^{cot[\pi G(z)]}]^{-1} \quad = \quad \sum_{i=0}^{\infty} (-1)^i \left[e^{\cot[\pi G(z)]}\right]^i \tag{3.11}$$

Now applying exponential series

$$e^y \quad = \quad \sum_{j=0}^{\infty} \frac{y^j}{j!} \tag{3.12}$$

equation (3.11) becomes

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j+1} (i)^j \frac{\cot[\pi G(z)]^j}{j!} \tag{3.13}$$

Expanding by power series

$$(\cot(x))^s \quad = \quad \sum_{k=0}^{\infty} a_k(s)(x)^{2k-s} \tag{3.14}$$

equation (3.13) becomes

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j+1} (i)^j \frac{1}{j!} \sum_{k=0}^{\infty} a_k(j)(\pi G(z))^{2k-j} \tag{3.15}$$

Now putting the Cdf of Topp-leone G family of distribution in equation (3.15), we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+1}(i)^j\frac{1}{j!}\sum_{k=0}^{\infty}a_k(j)\left[\pi\{1-[1-G(z)]^2\}^{\alpha(2k-j)}\right] \tag{3.16}$$

Now taking

$$\{1-[1-G(z)]^2\}^{\alpha(2k-j)}$$

From binomial series [Yahaya and Doguwa(2021)] is:

$$(1-x)^n = \sum_{i=0}^{\infty}(-1)^i x^i \binom{n}{i}$$

$$\sum_{l=0}^{\infty}(-1)^l \binom{\alpha(2k-j)}{l} [1-G(z)]^{2l}$$

Again apply binomial series

$$\sum_{l=0}^{\infty}(-1)^l \binom{\alpha(2k-j)}{l} \sum_{m=0}^{\infty}(-1)^m \binom{2l}{m}$$

equation (3.16) becomes

$$\begin{aligned} F(z) &= \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}(-1)^{i+j+l+m+1}(i)^j\frac{1}{j!}a_k(j)\pi^{2k-j}\binom{\alpha(2k-j)}{l} \\ &\quad \sum_{m=0}^{\infty}\binom{2l}{m}(G(z))^m \end{aligned} \tag{3.17}$$

$$F(z) = \sum_{m=0}^{\infty}W(i,j,k)H_m(z) \tag{3.18}$$

where

$$\begin{aligned} W(i,j,k) &= \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}\sum_{m=1}^{\infty}(-1)^{i+j+l+m+1}(i)^j\frac{1}{j!}a_k(j)\pi^{2k-j} \\ &\quad \binom{\alpha(2k-j)}{l}\binom{2l}{m} \end{aligned}$$

And

$$H_m(z) = (G(z))^m$$

$$f(z) = \sum_{m=1}^{\infty}\sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}(-1)^{i+j+l+m+1}(i)^j\frac{1}{j!}a_k(j)\pi^{2k-j} \binom{\alpha(2k-j)}{l}\sum_{m=1}^{\infty}\binom{2l}{m} mG(z)^{m-1}.g(z) \quad (3.19)$$

$$f(z) = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}V_{i,j,k,l}h_m(z) \quad (3.20)$$

where

$$V_{i,j,k,l} = \sum_{m=1}^{\infty}(-1)^{i+j+l+m+1}(i)^j\frac{1}{j!}a_k(j)\pi^{2k-j} \binom{\alpha(2k-j)}{l}\binom{2l}{m}$$

and

$$h_m(z) = mG(z)^{m-1}.g(z)$$

3.3.5 Moment Generating Function

To correctly know the structure of the data being investigated, unique moments are often calculated using MGF. MGF can be defined as:

$$M_zt = E[exp^{tz}] = \int_R exp^{tz}fzdz$$

and

$$exp^{tz} = \sum_{r=0}^{\infty}\frac{tz^r}{r!}$$

$$M_zt = \sum_{r=0}^{\infty}\frac{t^r\mu'_r}{r!}$$

using equation (3.19)

$$\begin{aligned} M_z t &= \sum_{m=1}^{\infty}\sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}(-1)^{i+j+l+m+1}(i)^j\frac{1}{j!}a_k(j)\pi^{2k-j} \\ &\left(\begin{array}{c}\alpha(2k-j)\\ l\end{array}\right)\sum_{m=1}^{\infty}\left(\begin{array}{c}2l\\ m\end{array}\right)\frac{t^r\mu'_r}{r!} \\ &\int_R mG(z)^{m-1}.g(z)dz \end{aligned}$$

$$M_z t = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}V_{i,j,k,l}\frac{t^r\mu'_r}{r!}\int_R mG(z)^{m-1}.g(z)dz \tag{3.21}$$

3.3.6 nth Moments

$$E(Z^n) = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}V_{i,j,k,l}E[{Y_m}^n]$$

$$E(Z^n) = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}V_{i,j,k,l}\int_{-\infty}^{\infty}z^n fzdz$$

using equation 3.19

$$\begin{aligned} E(Z^n) &= \sum_{m=1}^{\infty}\sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}(-1)^{i+j+l+m+1}(i)^j\frac{1}{j!}a_k(j)\pi^{2k-j} \\ &\left(\begin{array}{c}\alpha(2k-j)\\ l\end{array}\right) sum_{m=1}^{\infty}\left(\begin{array}{c}2l\\ m\end{array}\right)\int_{-\infty}^{\infty}z^n mG(z)^{m-1}.g(z)dz \end{aligned}$$

$$E(Z^n) = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}V_{i,j,k,l}\int_{-\infty}^{\infty}z^n mG(z)^{m-1}.g(z)dz$$

$$E(Z^n) = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}V_{i,j,k,l}(m)\int_{-\infty}^{\infty}z^n G(z)^{m-1}.g(z)dz \tag{3.22}$$

$$E(Z^n) = \sum_{m=0}^{\infty}d_{m-1}\tau_{n,m-1}$$

where

$$d_{m-1} = \sum_{i,l=1}^{\infty} \sum_{j,k=0}^{\infty} V_{i,j,k,l}(m)$$

and

$$\tau_{n,m-1} = \int_{-\infty}^{\infty} z^n G(z)^{m-1}.g(z)dz$$

By putting n=1,2,3... we can obtain 1st,2nd,3rd... ordinary moments of distribution

3.3.7 Mean Residual life

An estimated remaining life duration of the unit that survives at the time t, given as:

$$m(t) = \frac{1}{1-F(t)}[U - \int_{-\infty}^{t} zf(z)dz] - t \tag{3.23}$$

Mean residual life of LCTL-G family of distribution by using 3.19

$$\begin{aligned} m(t) = & \frac{1}{1-F(t)}\Big[U - \sum_{m=0}^{\infty} \sum_{i,l=1}^{\infty} \sum_{j,k=0}^{\infty} (-1)^{i+j+l+m+1} (i)^j \\ & \frac{1}{j!} a_k(j) \pi^{2k-j} \binom{\alpha(2k-j)}{l} \binom{2l}{m} \\ & \frac{t^r}{r!} \int_{-\infty}^{t} mG(z)^{m-1}.g(z)\Big] - t \end{aligned} \tag{3.24}$$

3.3.8 Renyi Entropy

As stated by Renyi (1961), the Renyi entropy is:

$$\begin{aligned}
I_R(\delta) &= \frac{1}{1-\delta} log \int_0^\infty f^\delta(z)dz, \delta \neq 1, \delta > 0 \\
&= \frac{1}{1-\delta} log \int_0^\infty \Bigg\{ 2\pi\alpha g(z)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\
& \csc^2[\pi\{1-[1-G(z)]^2\}^\alpha] e^{cot[\pi\{1-(1-G(z))^2\}^\alpha]} \\
& [1+e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-2} \Bigg\}^\delta \\
&= \frac{1}{1-\delta} log \int_0^\infty (2\pi\alpha)^\delta (g(z))^\delta [1-G(z)]^\delta [1-\{1-G(z)\}^2]^{(\alpha-1)\delta} \\
& \csc^2[\pi\{1-[1-G(z)]^2\}^\alpha]^\delta e^{cot[\pi\{1-(1-G(z))^2\}^\alpha]^\delta} \\
& [1+e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-2\delta} \qquad (3.25)
\end{aligned}$$

Let

$$[1+e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-2\delta} \qquad (3.26)$$

By using the binomail series

$$(1+x)^n = \sum_{i=0}^\infty x^i \begin{pmatrix} n \\ i \end{pmatrix}$$

$$= \sum_{i=0}^\infty \begin{pmatrix} -2\delta \\ i \end{pmatrix} e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{\delta+i} \qquad (3.27)$$

using exponential series

$$e^y = \sum_{j=0}^\infty \frac{y^j}{j!} \qquad (3.28)$$

Using (3.28) in (3.27) and substitute in eq(3.26)

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty \begin{pmatrix} -2\delta \\ i \end{pmatrix}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+1}(\delta+i)^{j}\frac{[cot\pi\{1-(1-G(z))^{2}\}^{\alpha}]^{j}}{j!} \tag{3.29}$$

By putting eq(3.29) in eq (3.25)

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta}log\sum_{i,j=0}^{\infty}D_{i,j,\delta}\int_{-\infty}^{\infty}[g(z)]^{\delta}[1-G(z)]^{\delta}[1-\{1-G(z)\}^{2}]^{\delta(\alpha-1)} \\ & \csc^{2\delta}[\pi\{1-[1-G(z)]^{2}\}^{\alpha}][cot\pi\{1-(1-G(z))^{2}\}^{\alpha}]^{\delta j} \end{aligned} \tag{3.30}$$

Where

$$D_{i,j,\delta} = (2\pi\alpha)^{\delta}\begin{pmatrix} -2\delta \\ i \end{pmatrix}\frac{(\delta+i)^{j}}{j!}$$

Expanding by power series, we get

$$(cot(x))^{s} = \sum_{k=0}^{\infty}a_k(s)(x)^{2k-s}$$

$$\begin{aligned} [cot\pi\{1-(1-G(z))^{2}\}^{\alpha}]^{\delta j} &= \sum_{m=0}^{\infty}a_m(\delta j)\pi^{2m-\delta j} \\ & [1-\{1-G(z)\}^{2}]^{\alpha(2m-\delta j)} \end{aligned} \tag{3.31}$$

Here, we will use the series of

$$\csc^{2}[\pi(G(x))] = \sum_{l=0}^{\infty}c_l(2)(x)^{2l-2}$$

$$\csc^{2\delta}[\pi\{1-[1-G(z)]^{2}\}^{\alpha}] = \sum_{l=0}^{\infty}c_l(2\delta)[\pi\{1-[1-G(z)]^{2}\}^{\alpha(2l-2\delta)} \tag{3.32}$$

By using (3.32) and (3.31)in equation (3.30).
The equation becomes:

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g(z)^{\delta}[1-G(z)]^{\delta}[1-\{1-G(z)\}^{2}]^{\delta(\alpha-1)} \sum_{l=0}^{\infty} c_l(2\delta)\pi^{2l-2\delta}[1-\{1-G(z)\}^{2}]^{\alpha(2l-2\delta)} \\
&\quad \sum_{m=0}^{\infty} a_m(\delta j)\pi^{2m-\delta j}[1-\{1-G(z)\}^{2}]^{\alpha(2m-\delta j)} \\
&= \int_{-\infty}^{\infty} g(z)^{\delta}[1-G(z)]]^{\delta} \sum_{l=0}^{\infty} c_l(2) \sum_{m=0}^{\infty} a_m(\delta j)\pi^{2(l+m)-\delta(2+j)}[1-\{1-G(z)\}^{2}]^{\alpha-1} \\
&\quad [1-\{1-G(z)\}^{2}]^{\alpha(2l-2\delta)}[1-\{1-G(z)\}^{2}]^{\alpha(2m-\delta j)} \\
&= \int_{-\infty}^{\infty} g(z)^{\delta}[1-G(z)]^{\delta} \sum_{l=0}^{\infty} c_l(2\delta) \sum_{m=0}^{\infty} a_m(\delta j)\pi^{2(l+m)-\delta(2+j)} \\
&\quad [1-\{1-G(z)\}^{2}]^{\alpha[2(l+m)-\delta(j+2)+1]-1}
\end{aligned}
$$

As the binomial series Yahaya and Doguwa(2021) is:

$$
(1-x)^{n-1} = \sum_{i=0}^{\infty}(-1)^{i}x^{i}\binom{n-1}{i}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g(z)^{\delta}[1-G(z)]^{\delta} \sum_{l=0}^{\infty} c_l(2\delta) \sum_{m=0}^{\infty} a_m(\delta j)\pi^{2(l+m)-\delta(2+j)} \\
&\quad \sum_{h=0}^{\infty}(-1)^{h}[1-G(z)]^{2h}\binom{\alpha[2(l+m)-\delta(j+2)+1]-1}{h}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g(z)^{\delta} \sum_{l=0}^{\infty} c_l(2\delta) \sum_{m=0}^{\infty} a_m(j\delta)\pi^{2(l+m)-\delta(2+j)} \\
&\quad \sum_{h=0}^{\infty}(-1)^{h}[1-G(z)]^{2h+1}\binom{\alpha[2(l+m)-\delta(j+2)+1]-1}{h}
\end{aligned}
$$

$$
\begin{aligned}
I_R(\delta) &= \frac{1}{1-\delta} log \sum_{i,j=0}^{\infty} D_{i,j,\delta} \int_{-\infty}^{\infty} g(z)^{\delta} \sum_{l=0}^{\infty} c_l(2\delta) \sum_{m=0}^{\infty} a_m(\delta j)\pi^{2(l+m)-\delta(2+j)} \\
&\quad \sum_{h=0}^{\infty}(-1)^{h}[1-G(z)]^{2h+1}\binom{\alpha[2(l+m)-\delta(j+2)+1]-1}{h}
\end{aligned}
$$

$$I_R(\delta) = \frac{1}{1-\delta} log \sum_{i,j=0}^{\infty} D_{i,j,\delta} \sum_{l=0}^{\infty} c_l(2\delta) \sum_{m=0}^{\infty} a_m(\delta j)\pi^{2(l+m)-\delta(2+j)}$$
$$\sum_{h=0}^{\infty}(-1)^h \left(\begin{array}{c} \alpha[2(l+m)-\delta(j+2)+1]-1 \\ h \end{array} \right) \int_{-\infty}^{\infty} g(z)^{\delta}[1-G(z)]^{2h+1}$$

using Binomial series

$$(1-x)^n = \sum_{i=0}^{\infty}(-1)^i x^i \left(\begin{array}{c} n \\ i \end{array} \right)$$

$$I_R(\delta) = \frac{1}{1-\delta} log \sum_{i,j=0}^{\infty} D_{i,j,\delta} \sum_{l=0}^{\infty} c_l(2\delta) \sum_{m=0}^{\infty} a_m(\delta j)\pi^{2(l+m)-\delta(2+j)}$$
$$\sum_{h=0}^{\infty}(-1)^h \left(\begin{array}{c} \alpha[2(l+m)-\delta(j+2)+1]-1 \\ h \end{array} \right)$$
$$\int_{-\infty}^{\infty} g(z)^{\delta} \sum_{k=0}^{\infty}(-1)^k G(z)^k \left(\begin{array}{c} 2h+1 \\ k \end{array} \right)$$

$$I_R(\delta) = \frac{1}{1-\delta} log \sum_{i,j,l,m,h=0}^{\infty} A_{i,j,l,m,h,\delta} \int_{-\infty}^{\infty} g(z)[1-G(z)]^k \quad (3.33)$$

where

$$A_{i,j,l,m,h,\delta} = \sum_{i,j=0}^{\infty} D_{i,j,\delta} \sum_{l=0}^{\infty} c_l(2) \sum_{m=0}^{\infty} a_m(j)\pi^{2m-j}$$
$$\sum_{h=0}^{\infty}(-1)^h \left(\begin{array}{c} \alpha[2(l+m)-(j+1)]-1 \\ h \end{array} \right)$$
$$\left(\begin{array}{c} 2h+1 \\ k \end{array} \right) \quad (3.34)$$

3.3.9 Order Statistic

We will concentrate on one of the most significant attributes in this section that is order statistics. Ith order can be write with ease and arrange the density's function of statistics as follows:

$$f_{1:N}(z) = \frac{f(z)}{\beta(i, n-i+1)} \sum_{j=0}^{n-i}(-1)^j \left(\begin{array}{c} n-i \\ j \end{array} \right) F(z)^{i+j-1}$$

$$\frac{1}{\beta(i,n-i+1)} = \frac{n!}{(i-1)!(n-i!)!}$$

$$F(z)^{i+j-1} = \left[\{1+e^{Cot[\pi[1-(1-G(z))^2]^\alpha]}\}^{-1}\right]^{i+j-1}$$

$$F(z)^{i+j-1} = \{1+e^{Cot[\pi[1-(1-G(z))^2]^\alpha]}\}^{-1(i+j-1)}$$

$$\begin{aligned} f(z).F(z)^{i+j-1} = {} & 2\pi\alpha g(z)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\ & \times \csc^2[\pi\{1-[1-G(z)]^2\}^\alpha]e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha} \\ & \times[1+e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-2} \\ & \times[1+e^{Cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{(-i-j+1)} \end{aligned}$$

$$\begin{aligned} f(z).F(z)^{i+j-1} = {} & 2\pi\alpha g(z)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\ & \times \csc^2[\pi\{1-[1-G(z)]^2\}^\alpha]e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha} \\ & \times[1+e^{Cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-1(i+j+1)} \end{aligned}$$

Let

$$[1+e^{Cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-1(i+j+1)}$$

By Binomial expansion in above equation, we get

$$(1+x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k$$

We get

$$\sum_{l=0}^{\infty}\binom{-(j+i+1)}{l}\left[e^{Cot[\pi\{1-(1-G(z))^2\}]^\alpha}\right]^l$$

$$\begin{aligned} f(z).F(z)^{i+j-1} = {} & 2\pi\alpha g(z)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\ & \csc^2[\pi\{1-[1-G(z)]^2\}^\alpha]e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha} \\ & \sum_{l=0}^{\infty}\binom{-(j+i+1)}{l}\left[e^{Cot[\pi\{1-(1-G(z))^2\}]^\alpha}\right]^l \end{aligned}$$

$$
\begin{aligned}
&= 2\pi\alpha \sum_{l=0}^{\infty} \begin{pmatrix} -(j+i+1) \\ l \end{pmatrix} g(z)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\
&\quad \times \csc^2[\pi\{1-[1-G(z)]^2\}^{\alpha}] \left[e^{Cot[\pi\{1-(1-G(z))^2\}]^{\alpha}}\right]^{l+1}
\end{aligned}
$$

After using exponential series ,we get

$$
\left[e^{Cot[\pi\{1-(1-G(z))^2\}]^{\alpha}}\right]^{l+1} = \sum_{m=0}^{\infty} \frac{cot\,[\pi\{1-[1-G(z)]^2\}^{\alpha}.(l+1)]^m}{m!}
$$

$$
\begin{aligned}
&= 2\pi\alpha \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \begin{pmatrix} -(j+i+1) \\ l \end{pmatrix} \frac{(l+1)^m}{m!} g(z)[1-G(z)] \\
&\quad [1-\{1-G(z)\}^2]^{\alpha-1} \csc^2[\pi\{1-[1-G(z)]^2\}^{\alpha}] \\
&\quad cot\left[\pi\{1-[1-G(z)]^2\}^{\alpha}\right]^m
\end{aligned}
$$

Using cotangent series

$$
\begin{aligned}
cot\left[\pi\{1-[1-G(z)]^2\}^{\alpha}\right]^m &= \sum_{p=0}^{\infty} a_p(m)\left[\pi\{1-[1-G(z)]^2\}^{\alpha}\right]^{2p-m} \\
&= \sum_{p=0}^{\infty} a_p(m)\pi^{2p-m}\left[\{1-[1-G(z)]^2\}^{\alpha}\right]^{2p-m}
\end{aligned}
$$

using (3.32)

$$
\begin{aligned}
\csc^2[\pi\{1-[1-G(z)]^2\}^{\alpha}] &= \sum_{N=0}^{\infty} c_l(2)\left[\pi\{1-[1-G(z)]^2\}^{\alpha}\right]^{2N-2} \\
&= \sum_{N=0}^{\infty} c_l(2)\pi^{2N-2}\left[\{1-[1-G(z)]^2\}^{\alpha}\right]^{2N-2}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi\alpha \sum_{N=0}^{\infty}\sum_{p=0}^{\infty} a_p(m)c_l(2)\pi^{2N-2}\pi^{2p-m} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \begin{pmatrix} -(j+i+1) \\ l \end{pmatrix} \\
&\quad \frac{(l+1)^m}{m!} g(x)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\
&\quad \left[\{1-[1-G(z)]^2\}^{\alpha}\right]^{2N-2}\left[\{1-[1-G(z)]^2\}^{\alpha}\right]^{2p-m}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi^{2(N+p)-(m+1)}\alpha g(z)\sum_{N=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}c_l(2)a_p(m)\begin{pmatrix} -(j+i+1) \\ l \end{pmatrix} \\
&\quad\times\frac{(l+1)^m}{m!}[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\
&\quad\times\left[\{1-[1-G(z)]^2\}^{\alpha}\right]^{2N-2}\left[\{1-[1-G(z)]^2\}^{\alpha}\right]^{2p-m} \\
&= 2\pi^{2(N+p)-(m+1)}\alpha g(z)\sum_{N=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}c_l(2)a_p(m)\begin{pmatrix} -(j+i+1) \\ l \end{pmatrix} \\
&\quad\times\frac{(l+1)^m}{m!}[1-G(z)]\{1-[1-G(z)]^2\}^{2\alpha(N+p)-\alpha(m+1)-1}
\end{aligned}
$$

using binomial series

$$
(1-x)^{n-1} = \sum_{k=0}^{\infty}\begin{pmatrix} n-1 \\ k \end{pmatrix}(-1)^k x^k
$$

$$
\{1-[1-G(z)]^2\}^{2\alpha(N+p)-\alpha(m+1)-1} = (-1)^q[1-G(z)]^{2q} \sum_{q=0}^{\infty}\begin{pmatrix} 2\alpha(N+p)-\alpha(m+1)-1 \\ q \end{pmatrix}
$$

$$
\begin{aligned}
&= 2\pi^{2(N+p)-(m+1)}\alpha g(z)\sum_{N=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}c_l(2)a_p(m)\begin{pmatrix} -(j+i+1) \\ l \end{pmatrix} \\
&\quad\times\frac{(l+1)^m}{m!}[1-G(z)]\sum_{q=0}^{\infty}\begin{pmatrix} 2\alpha(N+p)-\alpha(m+1)-1 \\ q \end{pmatrix} \\
&\quad\times(-1)^q[1-G(z)]^{2q} \\
&= 2\pi^{2(N+p)-(m+1)}\alpha(-1)^q\sum_{N=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}c_l(2)a_p(m)\begin{pmatrix} -(j+i+1) \\ l \end{pmatrix} \\
&\quad\times\frac{(l+1)^m}{m!}\sum_{q=0}^{\infty}\begin{pmatrix} 2\alpha(N+p)-\alpha(m+1)-1 \\ q \end{pmatrix}g(z)[1-G(z)]^{2q+1}
\end{aligned}
$$

Again using Binomial series

$$
[1-G(z)]^{2j+1} = \sum_{h=0}^{\infty}\begin{pmatrix} 2q+1 \\ h \end{pmatrix}(-1)^h G(z)^h
$$

$$
\begin{aligned}
&= 2\pi^{2(N+p)-(m+1)}\alpha(-1)^q \sum_{N=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} c_l(2)a_p(m)\frac{(l+1)^m}{m!} \\
&\quad \binom{-(j+i+1)}{l} \sum_{q=0}^{\infty} \binom{2\alpha(N+p)-\alpha(m+1)-1}{q} \\
&\quad \sum_{h=0}^{\infty} \binom{2q+1}{h} (-1)^h G(z)^h g(z)
\end{aligned}
$$

$$
\begin{aligned}
f(z).F(z)^{i+j-1} &= 2\pi^{2(N+p)-(m+1)}\alpha(-1)^{h+q} \sum_{m,N,p=0}^{\infty}\sum_{q,l,h=0}^{\infty} c_l(2)a_p(m) \\
&\quad \frac{(l+1)^m}{m!}\binom{-(j+i+1)}{l}\binom{2q+1}{h} \\
&\quad \binom{2\alpha(N+p)-\alpha(m+1)-1}{q} g(z)G(z)^h
\end{aligned}
$$

$$
\begin{aligned}
f_{1:N}(z) &= \frac{n!}{(i-1)!(n-i!)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}(-1)^{h+q}2\alpha\pi^{2(N+p)-(m+1)} \\
&\quad \sum_{m,N,p=0}^{\infty}\sum_{q,l,h=0}^{\infty} c_l(2)a_p(m)\frac{(l+1)^m}{m!}\binom{-(j+i+1)}{l}\binom{2q+1}{h} \\
&\quad \binom{2\alpha(N+p)-\alpha(m+1)-1}{q} g(z)G(z)^h
\end{aligned} \tag{3.35}
$$

3.3.10 Parameter Estimation

Since the parameters of the probability model are unknown, it is through the sample data that they will have to be estimated. In this section, arameters of probability model will be estimated using with the help of maximum likelihood estimation.

Suppose, x_1, x_2, x_3,...,x_n are independent random sample having size n taken from the $LCTL-G$ distributions family. The likelihood function (L) is defined as

Takaing the pdf of LCTL-G from 3.6:

$$
\begin{aligned}
f(z) &= 2\pi\alpha g(z)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\
&\quad \csc^2[\pi\{1-[1-G(z)]^2\}^\alpha]e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha} \\
&\quad [1+e^{cot[\pi\{1-(1-G(z))^2\}]^\alpha}]^{-2}
\end{aligned}
$$

$$
\begin{aligned}
L &= \prod_{i=1}^{n} f(z,\theta) \\
L &= \prod_{i=1}^{n} \Big\{ 2\pi\alpha g(z)[1-G(z)][1-\{1-G(z)\}^2]^{\alpha-1} \\
&\quad \times \csc^2[\pi\{1-[1-G(z)]^2\}^{\alpha}] e^{cot[\pi\{1-(1-G(z))^2\}]^{\alpha}} \\
&\quad \times [1+e^{cot[\pi\{1-(1-G(z))^2\}]^{\alpha}}]^{-2} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\log L &= n\log(2) + n\log\pi + n\log\alpha + \log\sum_{i=1}^{n} g(z) \\
&\quad + \log\sum_{i=1}^{n}[1-G(z)] + (\alpha-1)\log\sum_{i=1}^{n}\{1-[1-G(z)]^2\} \\
&\quad + 2log\csc^2[\pi\{1-[1-G(z)]^2\}^{\alpha}] + \sum_{i=1}^{n} cot[\pi\{1-[1-G(z)]^2\}^{\alpha}] \\
&\quad + 2\log\sum_{i=1}^{n}\left[1+e^{cot[\pi\{1-[1-G(z)]^2\}^{\alpha}]}\right] \qquad (3.36)
\end{aligned}
$$

3.3.11 Some special cases

LCTL Inverted Rayleigh Distribution

The inverse Rayleigh distribution was presented by Trayer (1964) as a way to model the survival and accuracy of data sets.
The c.d.f & p.d.f of Inverted Rayleigh distribution are

$$F(z) = e^{\frac{-\lambda}{z^2}} \qquad (3.37)$$

$$f(z) = e^{\frac{-\lambda}{z}} \frac{2\lambda}{x^3} \qquad (3.38)$$

where $z, \lambda > 0$ Putting the vaue of 3.37 in 3.5 and 3.38 in 3.6:

$$
\begin{aligned}
T_{LCTL-IR}(z) &= [1+e^{cot\{\pi[1-(1-e^{\frac{-\lambda}{z^2}})^2]^{\alpha}\}}]^{-1} \\
t_{LCTL-IR}(z) &= 2\pi\alpha\lambda e^{\frac{-\lambda}{z^2}} \frac{2\lambda}{z^3}[1-e^{\frac{-\lambda}{z^2}}]\left[1-\{1-e^{\frac{-\lambda}{z^2}}\}\right]^{\alpha-1} \\
&\quad \times \csc^2\left[\pi[1-\{1-e^{\frac{-\lambda}{z^2}}\}^2]^{\alpha}\right] e^{cot[\pi\{1-(1-e^{\frac{-\lambda}{z^2}})^2\}]^{\alpha}} \\
&\quad \times [1+e^{cot[\pi\{1-(1-e^{\frac{-\lambda}{z^2}})^2\}]^{\alpha}}]^{-2}
\end{aligned}
$$

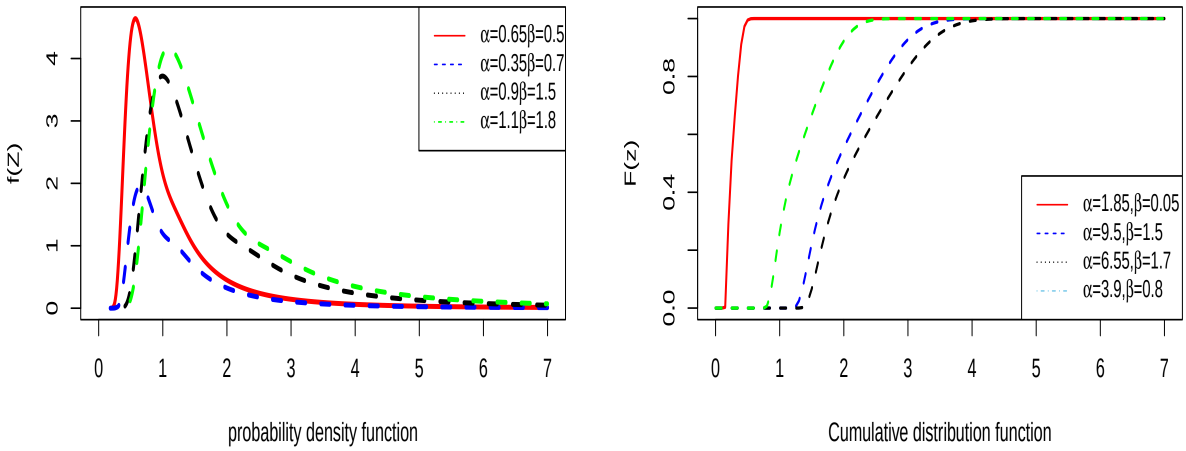

Figure 3.1: Shapes of the PDF & CDF of LCTL-IR

The survival and Hazard rate function are:

$$
\begin{aligned}
S_{LCTL-IR}(z) &= 1-[1+e^{cot\{\pi[1-(1-e^{\frac{-\lambda}{z^2}})^2]^\alpha\}}] \\
h_{LCTL-IR}(z) &= 2\pi\lambda e^{\frac{-\lambda}{z}}\frac{2\lambda}{z^3}[1-e^{\frac{-\lambda}{z^2}}] \\
&\times\left[1-\{1-e^{\frac{-\lambda}{z^2}}\}\right]^{\alpha-1} csc^2\left[\pi[1-\{1-e^{\frac{-\lambda}{z^2}}\}]^\alpha\right] \\
&\times e^{cot[\pi\{1-(1-e^{\frac{-\lambda}{z^2}})^2\}]^\alpha}[1+e^{cot[\pi\{1-(1-e^{\frac{-\lambda}{z^2}})^2\}]^\alpha}]^{-2} \\
&\times\frac{1}{1-[1+e^{cot\{\pi[1-(1-e^{\frac{-\lambda}{z^2}})^2]^\alpha\}}]}
\end{aligned}
$$

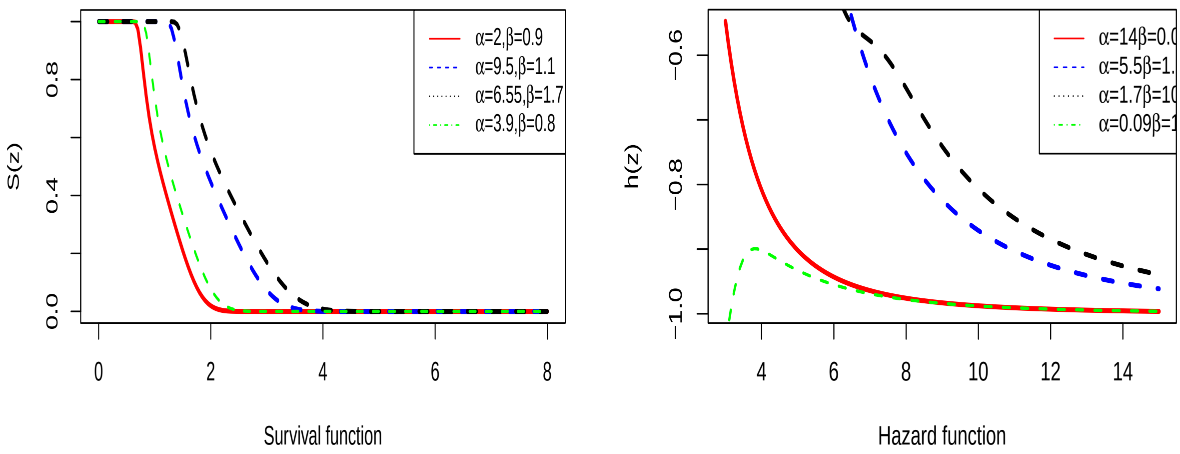

Figure 3.2: Shapes of Survival and Hazard Functions of LCTL-IR

LCTL Pareto distribution

It holds the name Vilfredo Pareto in honor of the Italian economist $(1848-1923)$, who created the distribution model in the $1890s$ to explain how wealth is distributed in society. The CDF & PDF of Pareto distribution are:

$$F(z) = \frac{ak^a}{z^{a+1}} \quad (3.39)$$

$$f(z) = (\frac{k}{z})^a \quad (3.40)$$

where$k > 0, a > 0$
by using equation (3.39) in (3.5) and (3.40) in (3.6):

$$\begin{aligned} T_{LCTL-P}(x) &= [1 + e^{Cot[\pi\{1-[1-\frac{ak^a}{z^{a+1}}]^2\}^\alpha]}]^{-1} \\ t_{LCTL-P}(x) &= 2\pi\alpha(\frac{k}{z})^a[1-\frac{ak^a}{z^{a+1}}][1-\{1-\frac{ak^a}{z^{a+1}}\}^2]^{\alpha-1} \\ &\times \csc^2[\pi\{1-[1-\frac{ak^a}{z^{a+1}}]^2\}^\alpha]e^{cot[\pi\{1-(1-\frac{ak^a}{z^{a+1}})^2\}]^\alpha} \\ &\times[1+e^{cot[\pi\{1-(1-\frac{ak^a}{z^{a+1}})^2\}]^\alpha}]^{-2} \end{aligned}$$

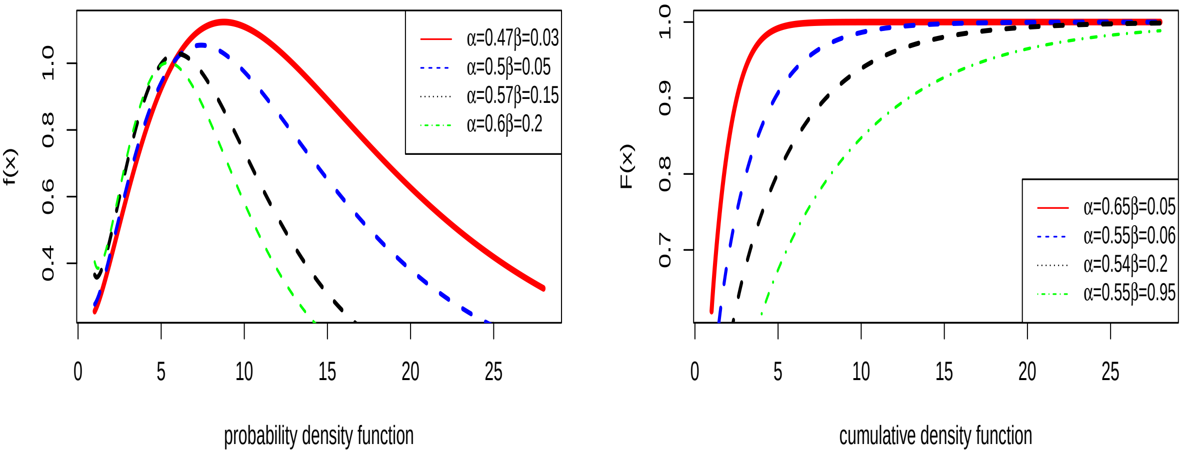

Figure 3.3: Shapes of the PDF & CDF of LCTL-P

The survival and Hazard rate function are:

$$
\begin{aligned}
S_{LCTL-P}(x) &= 1-[1+e^{Cot[\pi[1-\frac{ak^a}{z^{a+1}})^2]^\alpha}]^{-1} \\
h_{LCTL-P}(x) &= 2\pi\alpha(\frac{k}{z})^a[1-\frac{ak^a}{z^{a+1}}][1-\{1-\frac{ak^a}{z^{a+1}}\}^2]^{\alpha-1} \\
&\quad \times \csc^2[\pi\{1-[1-\frac{ak^a}{z^{a+1}}]^2\}^\alpha]e^{cot[\pi\{1-(1-\frac{ak^a}{z^{a+1}})^2\}]^\alpha} \\
&\quad \times[1+e^{cot[\pi\{1-(1-\frac{ak^a}{z^{a+1}})^2\}]^\alpha}]^{-2} \\
&\quad \times\frac{1}{1-[1+e^{cot[\pi\{1-(1-\frac{ak^a}{z^{a+1}})^2\}]^\alpha}]^{-1}}
\end{aligned}
$$

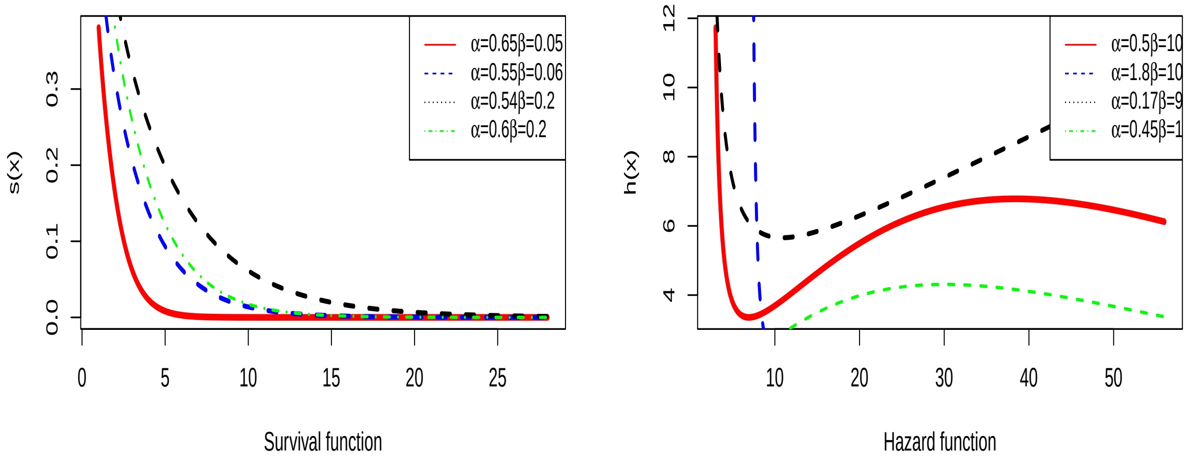

Figure 3.4: Shapes of Survival and Hazard Functions of LCTL-P

Chapter 4

Logistic Cotangent Topp-Leone Gumbel (LCTLGu) Distribution

4.1 Introduction

Chapter 3 describes the Logistic Cotangent Topp-Leone-G (LCTL-G) family of distributions, a novel method for generating lifetime probability distributions. The Cotangent Logistic Topp-Leone Gumbel distribution (LCTL) is one example of a LCTL-G distribution that has been examined, with the Gumbel distribution serving as the baseline distribution.
We look at its quantile function, moments, order statistics (OS), and entropies, among other statistical properties. There are closed-form formulations for entropies, moment-generating functions, and order statistics. Parameter estimation frequently makes use of the MLE technique.
Furthermore, a simulation study would be carried out. The value of the new family of distributions has been demonstrated through the application of real-world data collection.To compare, goodness of fit criteria such as AIC, CAIC, BIC, & others are employed. This findings demonstrate that, when applied to the identical data set, the suggested distribution, i.e., CLTL-GU, performs significantly better.

4.2 LCTLGu Distribution

The German mathematician and statistician Emil Julius Gumbel(1891-1966) is honored with the Gumbel distribution. Around the 1930s, he was the one who first established the distribution in the early 20th century. Emil Gumbel made important contributions to probability theory and statistics.

The CDF & PDF of Gumbel distribution are:

$$G(z) = e^{-e^{-(\frac{z-\mu}{\beta})}} \tag{4.1}$$
$$g(z) = \frac{1}{\beta}e^{-[\frac{z-\mu}{\beta} + e^{-(\frac{z-\mu}{\beta})}]} \tag{4.2}$$

By putting the eq (4.1) in eq (3.5) and eq (4.2) in eq (3.6)
The CDF & PDF of Logistic Cotangent Topp Leone Gumbel distribution are:

$$T_{LCTLGu} = [1 + e^{Cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]}]^{-1} \tag{4.3}$$
$$\begin{aligned} t_{LCTLGu} = {} & 2\pi\alpha\frac{1}{\beta}e^{-[\frac{z-\mu}{\beta} + e^{-(\frac{z-\mu}{\beta})}]}\left[1 - e^{-e^{-(\frac{z-\mu}{\beta})}}\right] \\ & \left[1 - \{1 - e^{-e^{-(\frac{z-\mu}{\beta})}}\}^2\right]^{\alpha-1} \csc^2\left[\pi\{1 - [1 - e^{-e^{-(\frac{z-\mu}{\beta})}}]^2\}^\alpha\right] \\ & e^{cot\left[\pi\{1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2\}\right]^\alpha}\left[1 + e^{cot[\pi\{1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2\}]^\alpha}\right]^{-2} \end{aligned} \tag{4.4}$$

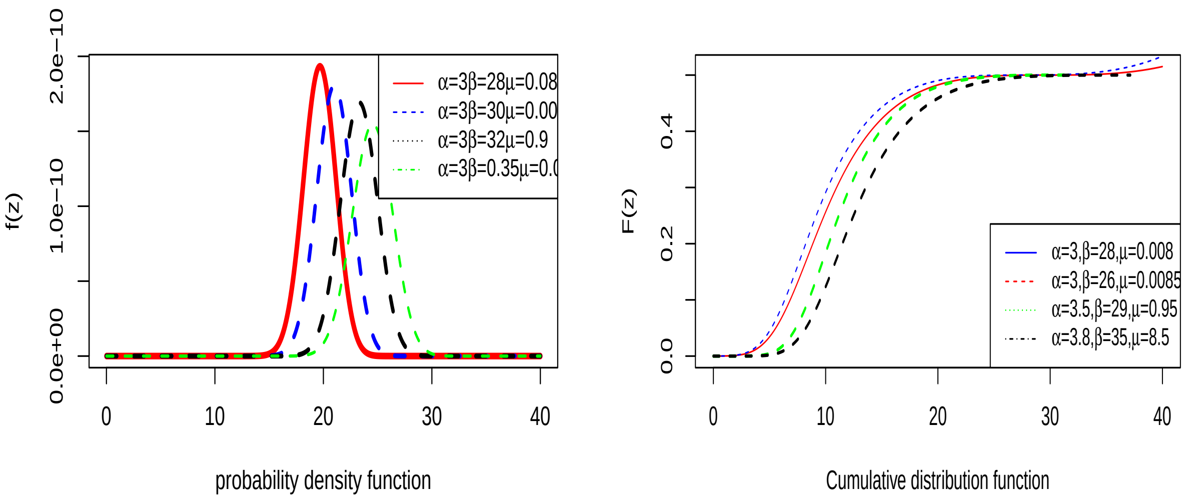

Figure 4.1: PDF & CDF of LCTLGu

4.2.1 Validity of LCTLGu distribution

Total area under the normal curve is unity

$$\int_{-\infty}^{\infty} f(z)dz = 1$$

Using 4.4 the Pdf of Logistic Cotangent Topp-Leone Gumbel Distribution:

$$\int_{-\infty}^{\infty} 2\pi\alpha\frac{1}{\beta}e^{-}[\frac{z-\mu}{\beta}+e^{-}(\frac{z-\mu}{\beta})]\left[1-e^{-e^{-}(\frac{z-\mu}{\beta})}\right]\left[1-\{1-e^{-e^{-}\frac{z-\mu}{\beta}}\}^{2}\right]^{\alpha-1}$$
$$\times\csc^{2}\left[\pi\{1-[1-e^{-e^{-}(\frac{z-\mu}{\beta})}]^{2}\}^{\alpha}\right]e^{cot\left[\pi\{1-(1-e^{-e^{-}(\frac{z-\mu}{\beta})})^{2}\}\right]^{\alpha}}$$
$$\times\left[1+e^{cot[\pi\{1-(1-e^{-e^{-}(\frac{z-\mu}{\beta})})^{2}\}]^{\alpha}}\right]^{-2}$$

Let

$$u = 1+e^{cot[\pi\{1-(1-e^{-e^{-}(\frac{z-\mu}{\beta})})^{2}\}^{\alpha}]}dz$$

When

$$z\to-\infty \quad then \quad u\to 1$$

When

$$z\to\infty \quad then \quad u\to\infty$$

$$du = 2\pi\alpha\frac{1}{\beta}e^{-}[\frac{z-\mu}{\beta}+e^{-}(\frac{z-\mu}{\beta})]\left[1-e^{-e^{-}(\frac{z-\mu}{\beta})}\right]\left[1-\{1-e^{-e^{-}(\frac{z-\mu}{\beta})}\}^{2}\right]^{\alpha-1}$$
$$\times\csc^{2}\left[\pi\{1-[1-e^{-e^{-}(\frac{z-\mu}{\beta})}]^{2}\}^{\alpha}\right]e^{cot\left[\pi\{1-(1-e^{-e^{-}(\frac{z-\mu}{\beta})})^{2}\}^{\alpha}\right]}$$

so,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(z)dz &= \int_{1}^{\infty} u^{-2}du \\
\int_{1}^{\infty} u^{-2}du &= -\frac{1}{u} \\
&= -\frac{1}{\infty}+\frac{1}{1} \\
&= -0+1 \\
&= 1
\end{aligned}$$

4.3 Statistical Properties:

4.3.1 Survival Function

The following is the Logistic Cotangent Topp Leone Gumbel distribution's survival function:
using the CDF from equation (3.33):

$$S_{LCTL-GU} = 1 - [1 + e^{Cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]}]^{-1} \tag{4.5}$$

4.3.2 Hazard rate Function

By using equation (4.4) and (4.5)

$$\begin{aligned} h_{CTL-GU} &= 2\pi\alpha\frac{1}{\beta}e^{-}[\frac{z-\mu}{\beta} + e^{-}(\frac{z-\mu}{\beta})] \left[1 - e^{-e^{-(\frac{z-\mu}{\beta})}}\right] \\ &\times \left[1 - \{1 - e^{-e^{-(\frac{z-\mu}{\beta})}}\}^2\right]^{\alpha-1} \csc^2\left[\pi\{1 - [1 - e^{-e^{-(\frac{z-\mu}{\beta})}}]^2\}^\alpha\right] \\ &\times e^{cot\left[\pi\{1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2\}\right]^\alpha} \left[1 + e^{cot[\pi\{1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2\}]^\alpha}\right]^{-2} \\ &\times \frac{1}{1 - [1 + e^{Cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]}]^{-1}} \end{aligned} \tag{4.6}$$

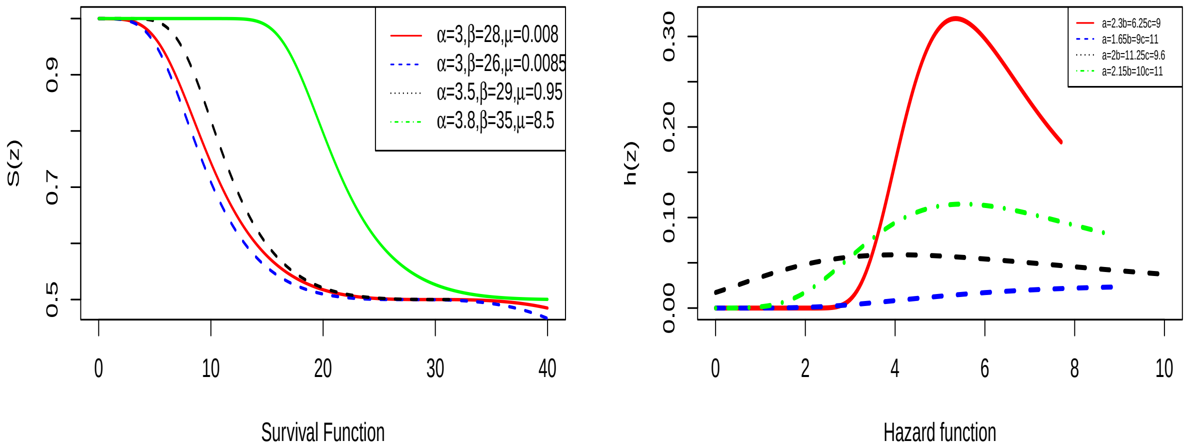

Figure 4.2: Survival & Hazard Functions of LCTLGu

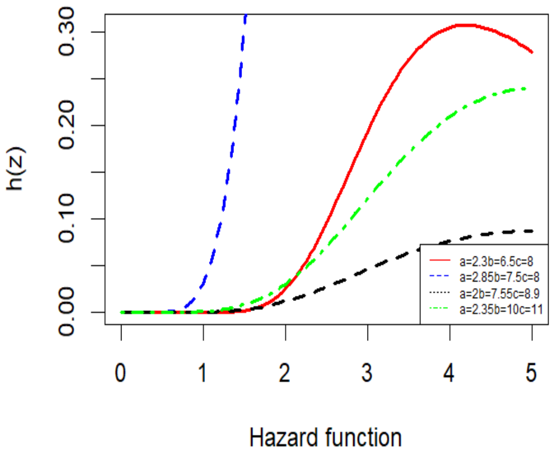

Figure 4.3: Hazard Functions of LCTLGu

4.3.3 Quantile function

The Quantile function of Logistic Cotangent Topp-Leone Gumbel distributuon is:

$$\begin{aligned}
q &= F^{-1}(z)\\
q &= [1+e^{\cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]}]^{-1}\\
q &= \frac{1}{[1+e^{\cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]}]}\\
\frac{1}{q} &= [1+e^{\cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]}]\\
\frac{1}{q}-1 &= e^{\cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]}\\
\ln(\frac{1}{q}-1) &= \cot[\pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha]\\
\cot^{-1}[\ln(\frac{1}{q}-1)] &= \pi[1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha\\
\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)] &= [1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]^\alpha\\
\left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]\right]^{\frac{1}{\alpha}} &= [1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2]\\
1-(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2 &= \left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]^{\frac{1}{\alpha}}\right]\\
(1-e^{-e^{-(\frac{z-\mu}{\beta})}})^2 &= 1-\left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]^{\frac{1}{\alpha}}\right]\\
(1-e^{-e^{-(\frac{z-\mu}{\beta})}}) &= \sqrt{1-\left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]^{\frac{1}{\alpha}}\right]}\\
e^{-e^{-(\frac{z-\mu}{\beta})}} &= 1-\sqrt{1-\left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]^{\frac{1}{\alpha}}\right]}\\
-e^{-}(\frac{z-\mu}{\beta}) &= \ln\left(1-\sqrt{1-\left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]^{\frac{1}{\alpha}}\right]}\right)\\
e^{-}(\frac{z-\mu}{\beta}) &= -\ln\left(1-\sqrt{1-\left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]^{\frac{1}{\alpha}}\right]}\right)\\
-(\frac{z-\mu}{\beta}) &= \ln\frac{1}{\left(\ln\left[1-\sqrt{1-\left[\frac{1}{\pi}\cot^{-1}[\ln(\frac{1}{q}-1)]^{\frac{1}{\alpha}}\right]}\right]\right)}
\end{aligned}$$

$$
\begin{aligned}
-z+\mu &= \beta \ln \frac{1}{\left(\ln \left[1 - \sqrt{1 - \left[\frac{1}{\pi} \cot^{-1}[\ln(\frac{1}{q} - 1)]^{\frac{1}{\alpha}} \right]} \right] \right)} \\
z &= \mu - \beta \ln \left(\ln \left[1 - \sqrt{1 - \{ \frac{1}{\pi} \cot^{-1}[\ln(\frac{1}{q} - 1)]^{\frac{1}{\alpha}} \}} \right] \right) \quad (4.7)
\end{aligned}
$$

4.3.4 Moments

As the nth moments of LCTL-G family of distribution from eq(3.22):

$$
\mu_n' = \sum_{i,l=1}^{\infty} \sum_{j,k=0}^{\infty} V_{i,j,k,l}(m) \int_{-\infty}^{\infty} z^n G(z)^{m-1}.g(z)dz
$$

where

$$
V_{i,j,k,l} = \sum_{m=1}^{\infty} \sum_{i,l=1}^{\infty} \sum_{j,k=0}^{\infty} (-1)^{i+j+l+m+1} (i)^j \frac{1}{j!} a_k(j) \pi^{2k-j} \begin{pmatrix} \alpha(2k-j) \\ l \end{pmatrix} \begin{pmatrix} 2l \\ m \end{pmatrix}
$$

consider,

$$
m \int_{-\infty}^{\infty} z^n G(z)^{m-1}.g(z)dz
$$

By using the PDF & CDF of Gumbel distribution from eq (4.1)and(4.2):

$$
m \int_{-\infty}^{\infty} z^n [e^{-e^{-(\frac{z-\mu}{\beta})}}]^{m-1} \frac{1}{\beta} e^{-}(\frac{z-\mu}{\beta}) e^{-e^{-(\frac{z-\mu}{\beta})}}
$$

Let

$$
\begin{aligned}
e^{-}(\frac{z-\mu}{\beta}) &= \theta \\
-\frac{1}{\beta} e^{-}(\frac{z-\mu}{\beta}) &= d\theta \\
(\mu - \beta \ln \theta) &= z
\end{aligned}
$$

When

$$
z \to -\infty \quad then \quad \theta \to \infty
$$

When

$$
z \to \infty \quad then \quad \theta \to 0
$$

$$m\int_{\infty}^{0}(\mu-\beta\ln\theta)^{n}[e^{-\theta}]^{m-1}e^{-\theta}-d\theta$$
$$m\int_{\infty}^{0}(\mu-\beta\ln\theta)^{n}[e^{-\theta}]^{m}-d\theta$$
$$m\int_{0}^{\infty}(\mu-\beta\ln\theta)^{n}e^{-\theta m}d\theta$$
$$m\int_{0}^{\infty}(\mu-\beta\ln\theta)^{n}[e^{-\theta}]^{m}d\theta$$

Using the binomial expansion for

$$(\mu-\beta\ln\theta)^{n}=\sum_{h=0}^{n}(-\beta)^{h}\mu^{n-h}[log(\theta)]^{h}$$
$$m\sum_{h=0}^{n}(-\beta)^{h}\mu^{n-h}\int_{0}^{\infty}[log(\theta)]^{h}e^{-\theta m}d\theta$$

Using a result by [4]

$$\begin{aligned} I_{i,j} &= \int_{0}^{\infty}[log(u)]^{i}e^{-u(j+1)}du \\ I_{i,j} &= [\frac{\partial}{\partial a}]^{i}[(j+1)^{a}\Gamma(a)]|_{a=1} \end{aligned}$$

$$\begin{aligned} \sum_{h=0}^{n}m(-\beta)^{h}\mu^{n-h}\int_{0}^{\infty}[log(\theta)]^{h}e^{-\theta m}d\theta &= \sum_{h=0}^{n}m(-\beta)^{h}\mu^{n-h} \\ & [\frac{\partial}{\partial a}]^{h}[(m)^{a}\Gamma(a)]|_{a=1} \end{aligned}$$

$$\begin{aligned} \mu_{n}' &= \sum_{m=1}^{\infty}\sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}(-1)^{i+j+l+m+1}(i)^{j}\frac{1}{j!}a_{k}(j)\pi^{2k-j}\begin{pmatrix}\alpha(2k-j)\\ l\end{pmatrix} \\ & \begin{pmatrix}2l\\ m\end{pmatrix}\sum_{h=0}^{n}m(-\beta)^{h}\mu^{n-h}[\frac{\partial}{\partial a}]^{h}[(m)^{a}\Gamma(a)]|_{a=1} \end{aligned} \tag{4.8}$$

$$\mu_{n}' = V_{i,j,k,l}\sum_{h=0}^{n}m(-\beta)^{h}\mu^{n-h}[\frac{\partial}{\partial a}]^{h}[(m)^{a}\Gamma(a)]|_{a=1} \tag{4.9}$$

4.3.5 Renyi Entropy

Using the LCTL-G's generic form of Renyi entropy from eq (3.33):

$$I_R(\delta) = \frac{1}{1-\delta} log \sum_{i,j,l,m,h=0}^{\infty} A_{i,j,l,m,h,\delta} \int_{-\infty}^{\infty} g(z)[1-G(z)]^k \tag{4.10}$$

Putting eq (4.1) (4.2) in above equation

$$I_R(\delta) = \frac{1}{1-\delta} log \sum_{i,j,l,m,h=0}^{\infty} A_{i,j,l,m,h,\delta} \int_{-\infty}^{\infty} \frac{1}{\beta} e^{-[(\frac{z-\mu}{\beta})+e^{-(\frac{z-\mu}{\beta})}]} \left[1-e^{-e^{-(\frac{z-\mu}{\beta})}}\right]^k$$

Using the Binomial Series in above expression

$$(1-x)^n = \sum_{i=0}^{\infty} (-1)^i x^i \binom{n}{i}$$

$$I_R(\delta) = \frac{1}{1-\delta} log \sum_{i,j,l,m,h=0}^{\infty} A_{i,j,l,m,h,\delta} \int_{-\infty}^{\infty} \frac{1}{\beta} e^{-[(\frac{z-\mu}{\beta})+e^{-(\frac{z-\mu}{\beta})}]} \sum_{p=0}^{\infty} (-1)^p \left[e^{-e^{-(\frac{z-\mu}{\beta})}}\right]^p \binom{k}{p}$$

$$I_R(\delta) = \frac{1}{1-\delta} log \sum_{i,j,l,m,h=0}^{\infty} \sum_{p=0}^{\infty} (-1)^p A_{i,j,l,m,h,\delta} \binom{k}{p} \frac{1}{\beta} \int_{-\infty}^{\infty} e^{-(\frac{z-\mu}{\beta})} \times \left[e^{-e^{-(\frac{z-\mu}{\beta})}}\right]^{p+1} \tag{4.11}$$

4.3.6 Order Statistic

The LCTL Order statistics in its general format from eq(3.35):

$$f_{1:N}(z) = \frac{n!}{(i-1)!(n-i!)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (-1)^{h+q} 2\alpha\pi^{2(N+p)-(m+1)} \sum_{m,N,p=0}^{\infty} \sum_{q,l,h=0}^{\infty} c_l(2) a_p(m) \frac{(l+1)^m}{m!} \binom{-(j+i+1)}{l} \binom{2q+1}{h} \binom{2\alpha(N+p)-\alpha(m+1)-1}{q} g(z)G(z)^h \tag{4.12}$$

By using PDF and CDF from eq (4.1)and eq (4.2)in above equation

$$\begin{aligned} f_{1:N}(z) &= \frac{n!}{(i-1)!(n-i!)!}(-1)^{h+q+j}\sum_{j=0}^{n-i}\sum_{m,N,p=0}^{\infty}\sum_{q,l,h=0}^{\infty}\binom{n-i}{j} \\ & 2\alpha\pi^{2(N+p)-(m+1)}\binom{-(j+i+1)}{l}\binom{2q+1}{h} \\ & \binom{2\alpha(N+p)-\alpha(m+1)-1}{q}c_l(2)a_p(m)\frac{(l+1)^m}{m!} \\ & \frac{1}{\beta}e^{-}[\frac{z-\mu}{\beta}+e^{-}(\frac{z-\mu}{\beta})][e^{-e^{-\frac{z-\mu}{\beta}}}]^h \end{aligned} \tag{4.13}$$

4.3.7 Parameter Estimation

Taking eq (3.36) after putting the pdf and cdf of Gumbel distribution from eq (4.1) & (4.2)

$$\begin{aligned} \log L &= n\log(2)+n\log\pi+n\log\alpha+\log\sum_{i=1}^{n}\frac{1}{\beta}e^{-}[\frac{z_i-\mu}{\beta}+e^{-}(\frac{z_i-\mu}{\beta})] \\ & +\log\sum_{i=1}^{n}\{1-[e^{-e^{-}(\frac{z_i-\mu}{\beta})}]\}+(\alpha-1)\log\sum_{i=1}^{n}\left[1-\{1-[e^{-e^{-}(\frac{z_i-\mu}{\beta})}]\}^2\right] \\ & +2log\csc^2\left[\pi\{1-[1-\left(e^{-e^{-}(\frac{z_i-\mu}{\beta})}\right)]^2\}^{\alpha}\right] \\ & +\sum_{i=1}^{n}cot[\pi\{1-[1-(e^{-e^{-}(\frac{z_i-\mu}{\beta})})]^2\}^{\alpha}] \\ & +2\log\sum_{i=1}^{n}\left[1+e^{cot\left[\pi\{1-[1-(e^{-e^{-}(\frac{z_i-\mu}{\beta})})]^2\}^{\alpha}\right]}\right] \end{aligned} \tag{4.14}$$

Taking the derivative of (eq 4.14) w.r.t to α:

$$\begin{aligned}
\frac{\partial \log L}{\partial \alpha} &= \frac{1}{\alpha} - 4\cot\left[\pi\{1-[1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}]^2\}^\alpha\right] \times \\
&\left[\pi\{1-[1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}]^2\}^\alpha\right] \times \log\left[1-\{1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}^2\right] \\
&-\pi\sum_{i=1}^{n}\csc^2\left[\pi\{1-[1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}]^2\}^\alpha\right] \\
&\times\left[1-\{1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}\}^2\right]^\alpha \log\left[1-\{1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}^2\right] \\
&-\frac{2\pi\sum_{i=1}^{n} e^{\cot\left[\pi\{1-[1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}]^2\}^\alpha\right]}}{\sum_{i=1}^{n}\left[1+\cot\left[\pi\{1-[1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}]^2\}^\alpha\right]\right]} \\
&\times\csc^2\left[\pi\{1-[1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}]^2\}^\alpha\right]\left[1-\{1-e^{-e^{-(\frac{z_i-\mu}{\beta})}}\}^2\right]^\alpha \\
&\times\log\left[1-\{1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}^2\right] \qquad (4.15)
\end{aligned}$$

Taking the derivative of eq (4.14) w.r.t to β:

$$
\begin{aligned}
\frac{\partial \log L}{\partial \beta} \quad = \quad & \frac{1}{\sum_{i=1}^{n} \frac{1}{\beta} e^{-\left[\frac{z_i-\mu}{\beta} + e^{-(\frac{z_i-\mu}{\beta})}\right]}} \\
& \times \sum_{i=1}^{n} \left[-\frac{1}{\beta^2} e^{-[\frac{z_i-\mu}{\beta} + e^{-(\frac{z_i-\mu}{\beta})}]} \times [\frac{z_i-\mu}{\beta} + e^{-(\frac{z_i-\mu}{\beta})}] \right] \\
& + \sum_{i=1}^{n} \left[- e^{-e^{-(\frac{z_i-\mu}{\beta})}} \times e^{-(\frac{z_i-\mu}{\beta})} \times \frac{z_i-\mu}{\beta^2} \right] \\
& \times \frac{1}{\sum_{i=1}^{n} \left[1 - \{ e^{-e^{-(\frac{z_i-\mu}{\beta})}} \} \right]} + (\alpha - 1) \frac{1}{\sum_{i=1}^{n} \left[1 - \{ 1 - [e^{-e^{-(\frac{z_i-\mu}{\beta})}}] \}^2 \right]} \\
& \times \left[2[1 - \{1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}] \times e^{-e^{-(\frac{z_i-\mu}{\beta})}} \times e^{-(\frac{z_i-\mu}{\beta})} \times \frac{z_i-\mu}{\beta^2} \right] \\
& + 8\pi\alpha(\frac{z_i-\mu}{\beta^2}) \left[1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}}) \right] \times e^{-e^{-(\frac{z_i-\mu}{\beta})}} \times e^{-(\frac{z_i-\mu}{\beta})} \\
& \times \left[1 - \{1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}^2 \right]^{\alpha-1} \times cot \left[\pi[1 - \{1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}^2]^{\alpha} \right] \\
& + \sum_{i=1}^{n} 2\pi\alpha \csc^2 \left[\pi[1 - \{1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}^2]^{\alpha} \right] \\
& \times \left[1 - \{1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})\}^2 \right]^{\alpha-1} \left[\{1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})\} \right. \\
& \left. \times e^{-e^{-(\frac{z_i-\mu}{\beta})}} \times e^{-(\frac{z_i-\mu}{\beta})} \times \frac{z_i-\mu}{\beta^2} \right] \\
& + \frac{4\alpha\pi}{\sum_{i=1}^{n} \left[1 + e^{cot\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right]} \right]} \times \sum_{i=1}^{n} e^{cot\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right]} \\
& \times \csc^2 \left[\pi\{1 - [1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha} \right] \\
& \times \left[\{1 - (e^{-e^{-(\frac{z_i-\mu}{\beta})}})\} \times e^{-e^{-(\frac{z_i-\mu}{\beta})}} \times e^{-(\frac{z_i-\mu}{\beta})} \times \frac{z_i-\mu}{\beta^2} \right] \qquad (4.16)
\end{aligned}
$$

Taking the derivative of eq (4.14) w.r.t to μ:

$$
\begin{aligned}
\frac{\partial \log L}{\partial \mu} &= \sum_{i=1}^{n}(-\frac{1}{\beta^2})\frac{e^{\left[-(\frac{z_i-\mu}{\beta})+e^{-(\frac{z_i-\mu}{\beta})}\right]}}{\sum_{i=1}^{n}\frac{1}{\beta}e^{\left[-(\frac{z_i-\mu}{\beta})+e^{-(\frac{z_i-\mu}{\beta})}\right]}} \\
&\times\left[1+e^{-(\frac{z_i-\mu}{\beta})}\right]+\frac{1}{\sum_{i=1}^{n}\{1-[e^{-e^{-(\frac{z_i-\mu}{\beta})}}]\}} \\
&\times\frac{1}{\beta}\sum_{i=1}^{n}e^{-}(\frac{z_i-\mu}{\beta})\times e^{-}(\frac{z_i-\mu}{\beta})-2(\alpha-1) \\
&\times\frac{1}{\sum_{i=1}^{n}\left[1-\{1-[e^{-e^{-(\frac{z_i-\mu}{\beta})}}]\}^2\right]}\times\sum_{i=1}^{n}\{1-[e^{-e^{-(\frac{z_i-\mu}{\beta})}}]\} \\
&\times[e^{-e^{-(\frac{z_i-\mu}{\beta})}}]\times e^{-}(\frac{z_i-\mu}{\beta})\times(\frac{z_i-\mu}{\beta^2})-8\pi\alpha\frac{1}{\beta} \\
&\times\cot\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right] \\
&\times\csc^2\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right] \\
&\times\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha-1}\times[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})] \\
&\times(e^{-e^{-(\frac{z_i-\mu}{\beta})}})\times e^{-}(\frac{z_i-\mu}{\beta})\times-2\pi\alpha\frac{1}{\beta} \\
&\sum_{i=1}^{n}\csc^2\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right] \\
&\times\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha-1}\times[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})] \\
&\times(e^{-e^{-(\frac{z_i-\mu}{\beta})}})\times e^{-}(\frac{z_i-\mu}{\beta})\times-4\pi\alpha \\
&\times\frac{1}{\beta\sum_{i=1}^{n}\left[1+e^{cot\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right]}\right]} \\
&\times\sum_{i=1}^{n}e^{cot\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right]} \\
&\times\csc^2\left[\pi\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha}\right] \\
&\times\{1-[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})]^2\}^{\alpha-1}\times[1-(e^{-e^{-(\frac{z_i-\mu}{\beta})}})] \\
&\times(e^{-e^{-(\frac{z_i-\mu}{\beta})}})\times e^{-}(\frac{z_i-\mu}{\beta}) \qquad (4.17)
\end{aligned}
$$

Equations don't have a closed form. It appears challenging to only compute the parameter values as a result. In order to derive MLEs, we apply an iterative procedure similar to the mathematical Newton-Raphson technique.

Chapter 5

Application

Two distinct applications of two real datasets to demonstrate how beneficial the suggested Gumbel distribution (LCTLGu) are presented here. The goodness of fit criteria demonstrated that it can be substituted for numerous other models, including well-known two, three, and four parameter models. The R script Adequacy Model is used for all computations. Additionally, the suggested CLTL-GU is contrasted with the Exponentiated Generalized Gumbel (EGGU) [Andrade et al. 2015] and the Kumaraswamy Gumbel (KumGU) [Cordeiro et al. 2012] and the four-parameter Beta Gumbel (BGU) Nadarajah and Kotz (2004).

5.0.8 Application 1:

The first data set which came from [26]. The dataset is readily accessible in [26]. Data contains 30 successive values of March precipitation which is: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05 (in inches) in Minneapolis/St Paul. The following are the dataset's summary statistics. Table 5.1 provides a number of descriptive statistics, including central tendency statistics, for these data. Table 5.2 displays the MLE for the CLTL-GU distribution. Table 5.3 gives the Information criteria Akaike information criterion (AIC), Consistent information criterion (CAIC),Bayesian Akaike information criterion (BA-IC),Hannan Quinn informatio (HQIC) ,W*,A* and p-value of fitted model.

Table 5.1: Summary of the data set.

Min.	Max.	Q_1	Median	Mean	Q_3
0.320	4.750	0.915	1.470	1.675	1.773

Table 5.2: MLEs of the data sets' considered distributional parameters.

	α	β	μ	σ
LCTL-GU	**-15.4823**	**20.85592**	**21.19220**	**-**
$EGGu$	0.0358	1.3392	2.5508	0.0078
$B-GU$	2.0008	0.4855	1.5573	0.3628
$KUM-GU$	0.2094	0.4428	0.3800	0.3151

Table 5.2 displays the MLE for the LCTLGu distribution.

In both its theoretical basis and empirical density and distribution function versions, the Tangent Topp-Leone Kumara Swamy distribution for strength data is seen in Figure 5.1. Figure 5.2 displays the data's Q-Q and P-P plots.Table 5.1 displays a summary of the dataset.

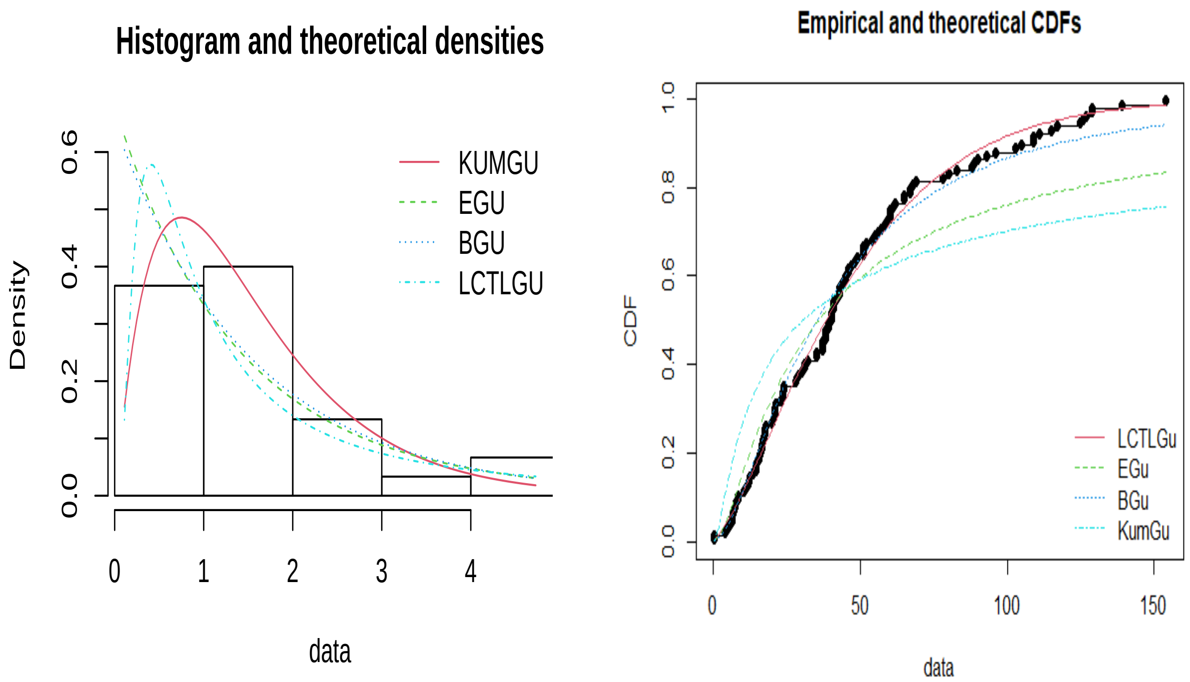

Figure 5.1: LCTL's empirical and theoretical CDF and PDF

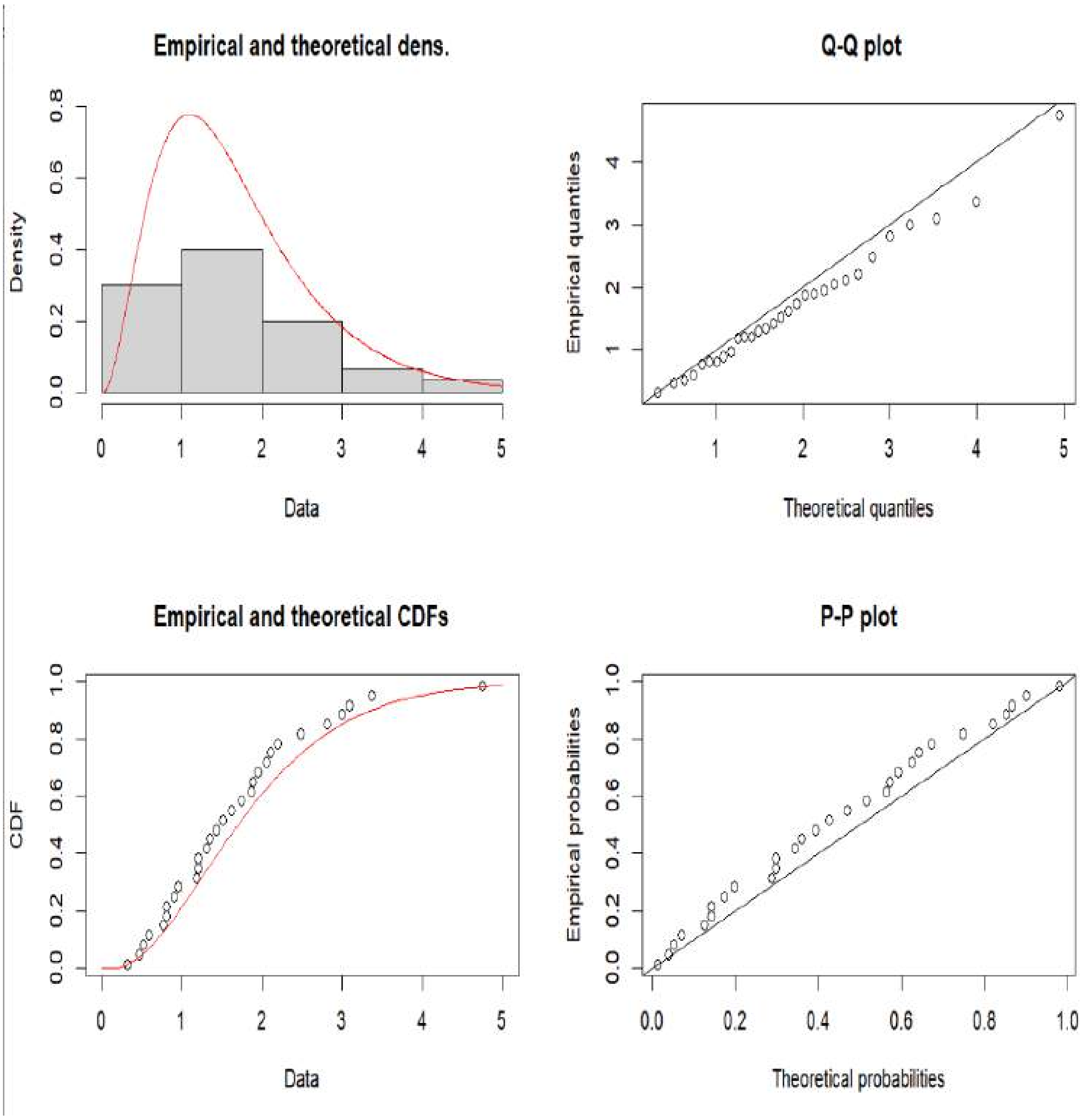

Figure 5.2: Empirical and theory-based CDF and PDF of LCTLGu with Q-Q and P-Plots

5.0.9 Application 2:

The second set of data is provided by [27]. The data refer the time between failures for repairable item: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82,

Table 5.3: Goodness of fit tests.

	AIC	CAIC	BIC	HQIC	W*	A*	P-value
LCTL-GU	**62.05802**	**62.98109**	**66.26161**	**63.40278**	**0.0122**	**0.1154**	**0.9941**
$EG-GU$	84.1	89.7	85.7	-	0.0151	0.1169	0.9932
$B-GU$	84.7	90.3	86.3	-	0.0205	0.1606	0.9611
$KUM-GU$	84.7	90.3	86.3	-	0.0193	0.1520	0.9718

2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.
The dataset has the following summary statistics.For these data, Table 5.4 offers several descriptive statistics, including statistics on central tendency.The MLE for the CLTL-GU distribution are shown in Table 5.5. The information criteria and the values of W*, A* and p-value of the fitted model are listed in Table 5.6
Table 5.5

Table 5.4: Summary of the data set.

Min.	Max.	Q_1	Median	Mean	Q_3
0.1100	4.7300	0.7175	1.2350	1.5427	1.9425

Table 5.5: MLEs of the data sets' considered distributional parameters.

	α	β	μ	σ
LCTL-GU	**-10.80047**	**20.733212**	**20.37893**	**-**
$EGGu$	0.2914	1.3294	0.3146	0.3004
$B-GU$	7.7144	0.2089	-0.2351	0.2600
$KUM-GU$	2.4766	0.2749	0.1804	0.3115

Table 5.4 displays the MLE for the LCTLGu distribution.
In both its theoretical basis and empirical density and distribution function versions, the Tangent Topp-Leone Kumara Swamy distribution for strength data is seen in Figure 5.3. Fig. 5.4 displays the data's Q-Q and P-P plots.Table 5.3 displays a summary of the dataset.

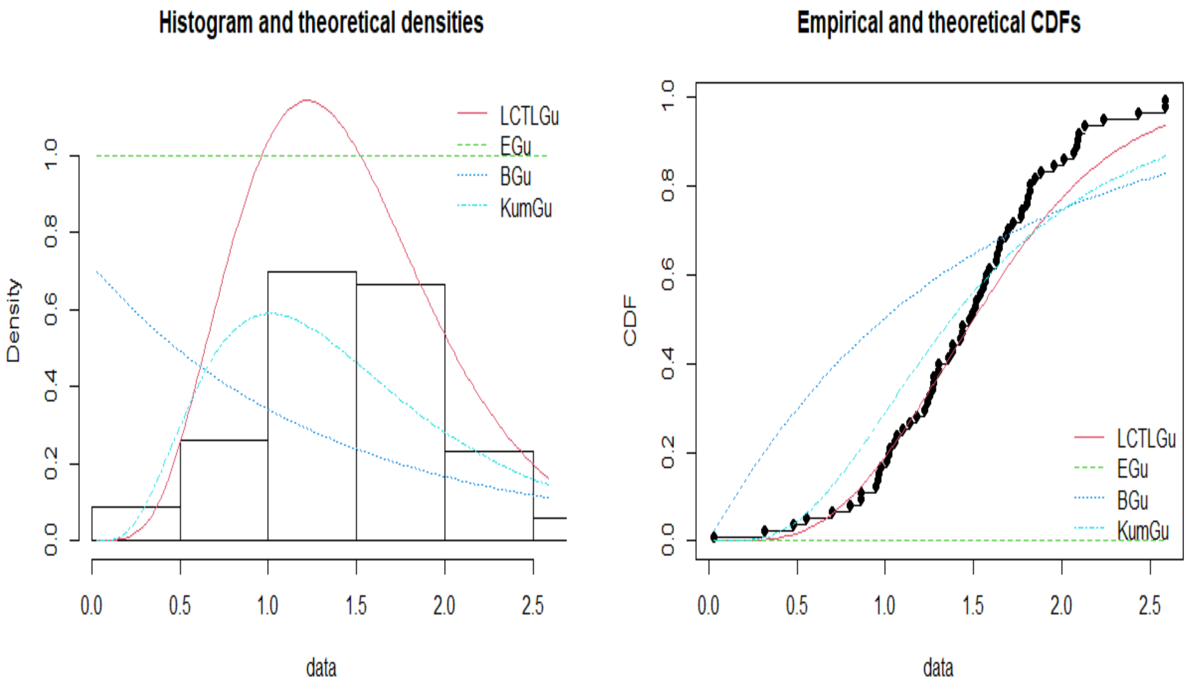

Figure 5.3: LCTL's empirical and theoretical CDF and PDF

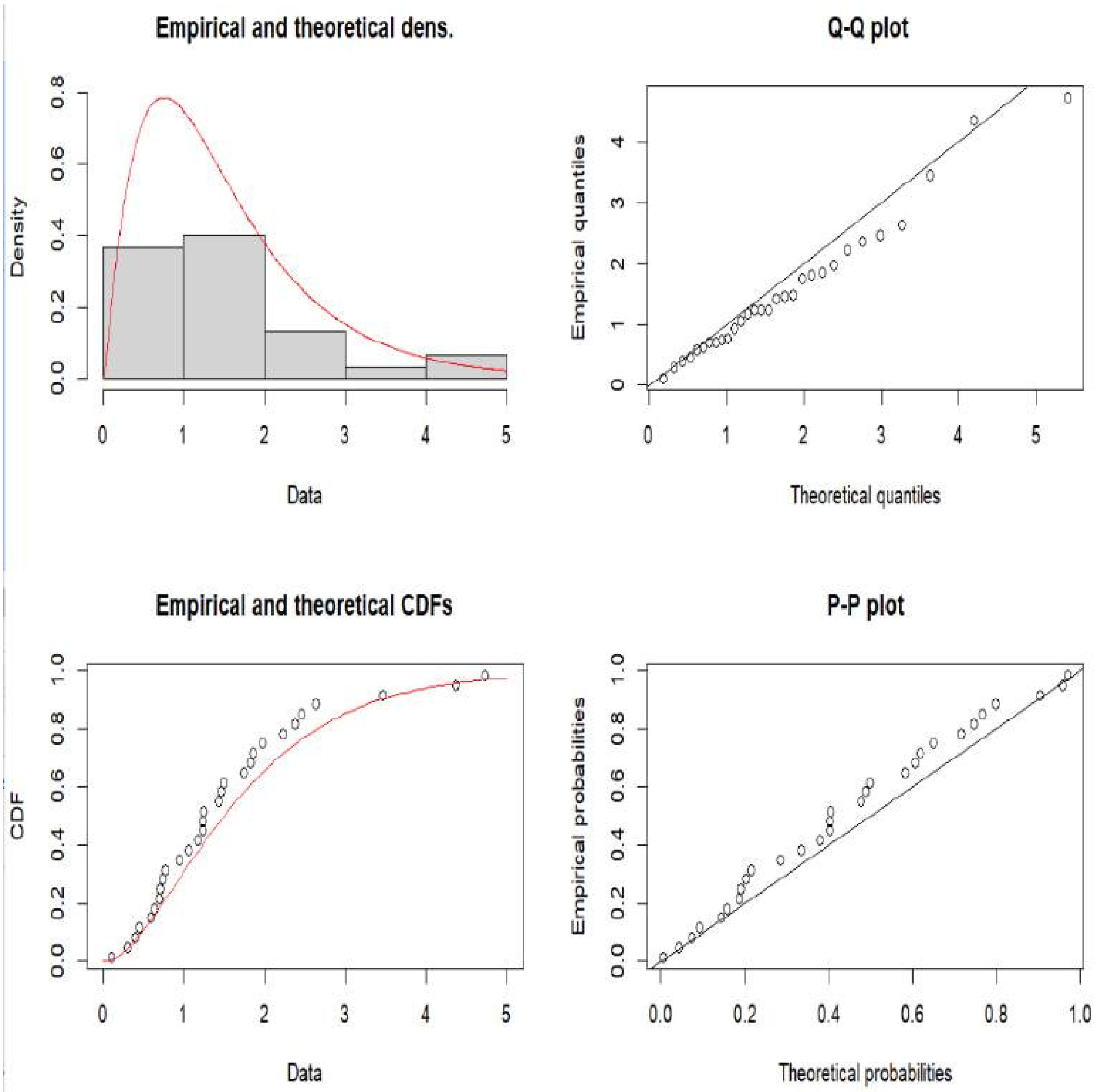

Figure 5.4: Empirical and theory-based CDF and PDF of LCTLGu with Q-Q and P-Plots

Having the lowest BIC, AIIC, CAIC, and HQIC values, the distribution that best fits the data is found in Table 5.3 and Table 5.7
The Cotangent ToppLeone Gumbel distribution clearly has lower AIC, BIC, CAIC, and HQIC values than other distributions (see Table 5.3)and(see Table 5.7). Therefore, compared to the Exponentiated Generalized Gumbel(EGGU), Kumaraswamy Gumbel(KumGU) and the Beta Gumbel(BGU),

Table 5.6: Goodness of fit tests.

	AIC	CAIC	BIC	HQIC	W*	A*	P-value
LCTL-GU	**86.16066**	**87.08374**	**90.36426**	**87.50543**	**0.0162**	**0.1195**	**0.9902**
$EG-GU$	87.55	89.15	93.16	-	0.0168	0.1198	0.9885
$B-GU$	87.82	89.42	93.43	-	0.0181	0.1231	0.9821
$KUM-GU$	87.55	89.15	93.16	-	0.0176	0.1204	0.9848

The CLTL-GU distribution provides a superior fit for the strength data.

5.1 Conclusion

In this research work, we introduced the newest distribution i-e The Logistic Cotangent Topp-Leone Gumbel (LCTLGu) Distribution. To create new modified versions of the current probability distribution, we have proposed a new generator and applied it to the current probability model. After that we modified the well-known Gumbel distributions by increasing their parameters or substituting one function for another. Bias and MSE have been employed to evaluate the consistency of the parameters. The three-parameter Logistic Cotangent Topp-Leone Gumbel (CLTLGu) probability distribution was introduced. This distribution is derived from a generator that is Logistic Cotangent Topp-Leone. The PDF and CDF of LCTLGu are in closed form, and several of its statistical properties such as Hazard Functions, Survival Functions, Order Statistic, Renyi entropy, mean residual life, and MGF are also given. The MLE method is used to calculate the parameters. Furthermore, we examined the various hazard function forms and found that this distribution (LCTLGu) might bridge a gap in the literature by representing real data in either a monotonic manner. We have also shown that this probability distribution provides a very good fit when compared to the current probability models. The strength data is used to evaluate the LCTLGu goodness of fit to established lifespan distributions, including the Exponentiated Generalized Gumbel, Kumaraswamy Gumbel, and Beta Gumbel distributions. By contrasting the values of criterion BIC, HQIC, AIC, CAIC, AD, KS, and W statistics, it is evident that the LCTLGu values are significantly lower than those of other distributions currently in use. It is possible to conclude that LCTLGu provides a better match than other distributions.

5.2 Recommendations

This section discusses the future recommendation on the Logistic Cotangent Topp-Leone G family (LCTL-G) distribution. The suggestions are grounded in several statistical fields, including quality control and mathematical statistics. New studies based on other distributions such as the Inverted Rayleigh distribution, the Frechet distributions family and the Pareto distributions family are listed, can be conducted by employing the PDF & CDF of proposed (LCTL-G) distributions family. Additionally, the parameter's estimation study using various techniques such as parametric bootstrapping, Bayesian estimation, weighted least squares, and ordinary least squares can be undertaken. Examining the proposed LCTLGu distribution control charts is advised for those working in the quality control industry. By employing the current generators, further families of distributions utilizing tangent,cotangent or other trigonometric functions can be created to suggest more adaptable versions of the baseline distributions that are now in use.

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