Stretching a surface non-linearly in a rotating frame

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1. Linear programming

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A Dissertation
Submitted in the Partial Fulfillment of the Requirements for the Degree of MASTER OF SCIENCE IN MATHEMATICS

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Dedicated to

My mother, father
& my jan

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1- Linear programming.

Acknowledgement

All praise is due only to Allah, the Almighty Who solely deserves to be worshipped, praised and glorified; Who bestows His infinite bounties upon His humble creation; is always kind and merciful. He created Man and blessed him with knowledge and enjoined upon mankind to ponder over the majesty of the Creator manifested through the creation. I humbly thank the Almighty for enabling me to complete my dissertation.

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Irfan Mustafa

Preface

Boundary layer flows induced by stretching sheet has gained significant importance for the last three decades. It is only due to prominent applications of such flows in manufacturing processes, in industry such as the aerodynamic extrusion of the plastic plate, cooling of an infinite metallic plate in a cooling bath, boundary layer along a liquid thin film and condensation process and many more. Large number of mathematicians, physicists, modeler, and engineers are attracted by theses applications and are investigating such flows in many different ways for the last so many years. The primeval researches on boundary layer flow by a continuously moving solid surface with constant velocity have been done by Sakiadis [1, 2]. After this Crane [3] investigated the continuously moving solid surface with linear velocity. Later, C. Y. Wang discussed the steady three dimensional flows due to stretching of the sheet [4] and then liquid film on an unsteady stretching sheet [5]. Till now, extensive literature is available on the linear stretching flow disused by different authors [6-14]. But unfortunately, a very little attention has been given to the nonlinear stretching flows. Vajravelu [15] discussed the fluid flow over a non-linear stretching sheet first time and then Vajravelu and Cannon [16] studied the existence and behavior of solutions of different equations arising in viscous flow over a nonlinear stretching sheet. In addition to the stretching surfaces, the heat and mass flow has driven many industrial applications due to thermal diffusion, concentration difference and due to chemical reaction. Thereafter, the viscous flow and heat transfer characteristics over a nonlinear stretching sheet have been discussed by Cortell [17]. In recent, analytic solution has been obtained for flow and diffusion of chemically reactive species over a nonlinear stretching sheet immersed in a porous medium by Ziabakhsh [18]. Motivated by the above facts, the aim of this dissertation is to investigate the effects of porous medium on the fluid flow over a nonlinear rotating stretching sheet. The dissertation is arranged as follows:

Chapter 1 includes some basic definitions, concept of boundary layer and its equations [19], equation of motion of fluid in rotating frame [20] for the convenience and better understanding of the reader. The contents of chapter 2 are based on the work of Ziabakhsh et al [18]. All the results are reproduced by shooting method [21] and by famous implicit finite difference scheme, Keller-Box Scheme [22]. In chapter 3 the work of reference [18] is generalized by taking the whole in rigid body rotation immersed in a porous medium. The similarity transformations for the rotating frame are introduced to the partial differential equations. As a result governing nonlinear ordinary differential equations are then solved by two well known methods namely shooting method [21] and by Keller-Box method [22]. Results obtained by both the methods are compared for different values of emerging parameters and found in excellent agreement. The influence of rotation and porous medium parameter are analyzed through graphs.

Contents

Chapter 1

Some basic definitions and equations

Some basic definitions, concept of boundary layer, derivation of the equations of motion of fluid in rotating frame are discussed in this chapter, which will be used in the subsequent chapters.

1.1 Definitions

1.1.1 Velocity field

In dealing with fluids in motion, we shall necessarily be concerned with the description of a velocity field. If we define a fluid particle as a small mass of fluid of fixed identity of volume δv, then the velocity at point C is defined as the instantaneous velocity of the fluid particle which, at a given instant, is passing through point C. At a given instant the velocity field, $\mathbf{V}$, is a function of the space coordinates x, y and z. The velocity at any point in the flow field might vary from one instant to another. Thus the complete representation of velocity is given by

$$\mathbf{V} = \mathbf{V}(x, y, z, t).$$

1.1.2 Flow

A material in which the that deformation become continuous under the action of any force upon it. If the deformation continuously increases without limit, then such phenomenon is called flow.

1.1.3 Fluid

Any substance that deforms continuously under the action of applied forces is called fluid.

1.1.4 Viscosity

The internal friction of a fluid produced by the movement of its molecules against each other is called viscosity of the fluid. Viscosity causes the fluid to resist flowing.

$$Viscosity = \frac{Shear\ stress}{rate\ of\ shear\ strain}.$$

1.1.5 Density

Density of a fluid is defined as the mass per unit volume. Mathematically, the density ρ at a point C may be defined as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v},$$

where δv is the total volume element around the point C and δm is the mass of the fluid within δv.

1.1.6 Kinematic viscosity

It is the ratio of absolute viscosity μ to the density. It is denoted by ν, and is defined as

$$\nu = \frac{\mu}{\rho}.$$

1.1.7 Fluid rotation

A fluid particle moving in a general three-dimensional flow field may rotate about all three coordinate axes. Thus particle rotation is a vector quantity and, in general

$$\omega = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z,$$

where ω_x, ω_y and ω_z are the rotation about the x, y and z-axis respectively. The rotation of the fluid element about the z-axis is the average angular velocity of the two mutually perpendicular line elements, in the xy - plane.

1.2 Types of fluid

1.2.1 Ideal fluid

An ideal fluid is one that possesses no viscosity. Ideal fluid do not actually exist but sometimes it is useful to consider what would happen to an ideal fluid in a particular fluid flow situation in order to simplify the problem.

1.2.2 Real fluid

A fluid which possess some viscosity is known as a real fluid. All the fluids in actual practice are real fluids. Real fluids are further subdivided into two main classes.

(1) Newtonian fluid

(2) Non-Newtonian fluid

1.2.3 Newtonian fluid

A Newtonian fluid is a fluid whose stress is proportional to strain rate linearly and constant of proportionality is the viscosity. Mathematically

$$\tau_{yx} = \mu \frac{du}{dy},$$

where τ_{yx} is the shear stress,du/dy is the velocity gradient perpendicular to the direction of shear stress.

1.2.4 Non-Newtonian fluid

A Non-Newtonian fluid is a fluid whose flow properties are not described by a single constant value of viscosity. Many polymer solutions and molten polymers are non-Newtonian fluids. Many commonly found substances such as ketchup, paint, blood and shampoo, etc. are examples of non-Newtonian fluids. In a non-Newtonian fluid, the relation between the shear stress and the strain rate is proportional but non linearly. Mathematically

$$\tau_{yx} = \eta \left(\frac{du}{dy}\right)^n, \; n \neq 1$$

where n is consistency index and η represent behavior.

1.3 Types of flow

1.3.1 Uniform flow

If the velocity of the fluid is same in magnitude and direction at every point in the fluid flow then this flow is consider as uniform flow.

1.3.2 Non-uniform flow

If at a given instant, the velocity is not same at each point of the flow, it is known as non-uniform flow.

1.3.3 Laminar flow

The smooth flow of a fluid in which adjacent layers of the fluid, flow parallel to each other is called a laminar flow. In simple words, during this flow, all the fluid particles move in distinct and separate layers, i.e. there is no mixing occurs between two adjacent layers.

1.3.4 Turbulent flow

The flow of a fluid in which the motion of particles at any point varies rapidly in both magnitude and direction is called turbulent flow. It is characterized by mixing of adjacent fluid layers.

1.3.5 Steady flow

If some properties of the fluid does not change with time during the flow, such a fluid flow is called steady flow.

1.3.6 Unsteady flow

A flow in which time effect some properties of fluid during the flow is said to be unsteady flow.

1.3.7 Compressible flow

All those flows in which density of the flowing fluid changes then such fluid flow is called compressible flow.

1.3.8 Incompressible flow

If the flow is not compressible then it is called an incompressible flow. Flow of liquids are generally lie into the category of incompressible flow.

1.4 Stretching flow

The flow in which the sheet is stretched in its own plane with the velocity proportional to the distance from a fixed point is known as the stretching flows. There can be different types of stretching flow, e.g. linear stretching, non-linear stretching and exponential stretching etc.

1.5 Porous medium

The porous medium is a material that contain interconnected pores (voids) in it. Due to interconnectedness of the voids (pores), the fluid may travel from the material.

1.6 Mass transfer

It is the phenomenon in which mass is transfer from high concentration to low concentration.

1.6.1 Diffusion

Diffusion is a process by which molecules spread from areas of high concentration to areas of low concentration.

1.6.2 Mass convection

Mass convection is the movement of molecules within fluids.

1.6.3 Fick's law of diffusion

According to Fick's law of diffusion, the rate of mass diffusion $\hat{m}_{diff}$ of a chemical species A in a stationary medium in the x direction is proportional to the concentration gradient dC/dx in that direction. It is mathematically expressed as

$$\hat{m}_{diff} = -D_{AB}A\frac{dC_A}{dx},$$

where D_{AB} is diffusivity of the species and C_A is the concentration of the species in the mixture at that location.

1.6.4 Schmidt number

Schmidt number is a dimensionless number defined as the ratio of momentum diffusivity (viscosity) and mass diffusivity. It is denoted by Sc and defined as

$$Sc = \frac{\mu}{\rho D} = \frac{\text{viscous diffusion rate}}{\text{molecular diffusion rate}}.$$

1.7 Equation of continuity

Let us consider the control volume chosen as an infinitesimal cube with sides of length dx, dy and dz in rectangular coordinate system. The density of the control volume is ρ and the velocity there is $\mathbf{V} = (u, v, w)$.

The net rate of mass flux coming out of control surface is given by

$$\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dxdydz.$$

Since the mass inside the control volume at any instant is the product of the mass per unit volume ρ, and its volume $dxdydz$. Thus the rate of change of mass inside the control volume is given by

$$\frac{\partial \rho}{\partial t} dxdydz.$$

It is noted that fixed control volume $dxdydz$ is independent of time. Combination of these two expressions is given by

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0. \tag{1.1}$$

As

$$\nabla\cdot\rho\mathbf{V}=\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z},$$

therefore Eq. (1.1) becomes

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\mathbf{V}=0, \tag{1.2}$$

which is known as equation of continuity.

For incompressible flows, density is neither a function of space coordinate nor a function of time. This imply that $\rho=$ constant, then Eq. (1.2) is simplified

$$\nabla\cdot\mathbf{V}=0. \tag{1.3}$$

1.8 The Momentum Equation

The equation of motion in vector form is

$$\rho\frac{\partial \mathbf{V}}{\partial t}=\rho\mathbf{b}+\nabla\cdot\mathbf{T}, \tag{1.4}$$

in which $\mathbf{b}$ are the body forces per unit mass. Here Cauchy stress $\mathbf{T}$ is

$$\mathbf{T}=\begin{pmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{pmatrix},$$

where τ_{xx}, τ_{yy} and τ_{zz} are the normal stresses and τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, and τ_{zy} are called shear stresses.

1.9 Equation of motion in rotating system

Let us consider two coordinate systems in which simed coordinate system is rotating with respect to an unsimed system, which is fixed. The instantaneous angular speed of the rotation is designated by ω. The product, $\omega\mathbf{n}$, is the angular velocity of the rotating system

$$\boldsymbol{\omega} = \omega\mathbf{n}.$$

The position of any point P in space can be designated by the vector $\mathbf{r}$ in the fixed, unsimed system and by the vector $\mathbf{r}^{\sim}$ in the rotating system. These vectors are equal, that is,

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{r}^{\sim} = \mathbf{i}^{\sim}x^{\sim} + \mathbf{j}^{\sim}y^{\sim} + \mathbf{k}^{\sim}z^{\sim}. \tag{1.5}$$

Differentiating with respect to time where $\mathbf{i}^{\sim}$, $\mathbf{j}^{\sim}$ and $\mathbf{k}^{\sim}$ in the rotating system are not constant. Thus we can write

$$\mathbf{v} = \mathbf{v}^{\sim} + x^{\sim}\frac{d\mathbf{i}^{\sim}}{dt} + y^{\sim}\frac{d\mathbf{j}^{\sim}}{dt} + z^{\sim}\frac{d\mathbf{k}^{\sim}}{dt}. \tag{1.6}$$

Where $\mathbf{v}$ and $\mathbf{v}^{\sim}$ are the velocity vectors in the fixed and rotating coordinate system. To find the time derivatives we show the change $\Delta\mathbf{i}^{\sim}$ in the unit vector $\mathbf{i}^{\sim}$ due to a small rotation $\Delta\theta$ about the axis of rotation. From the figure we see that the magnitude of $\Delta\mathbf{i}^{\sim}$ is given by the approximate relation

$$|\Delta\mathbf{i}^{\sim}| \approx (|\mathbf{i}^{\sim}|\sin\phi)\,\Delta\theta = (\sin\phi)\Delta\theta,$$

where ϕ is the angle between $\mathbf{i}^{\sim}$ and $\boldsymbol{\omega}$. Let Δt be the time interval for this change, then we can write

$$\left|\frac{d\mathbf{i}^{\sim}}{dt}\right| = \lim_{\Delta t \to 0}\left|\frac{\Delta\mathbf{i}^{\sim}}{\Delta t}\right| = \sin\phi\frac{d\theta}{dt} = (\sin\phi)\omega. \tag{1.7}$$

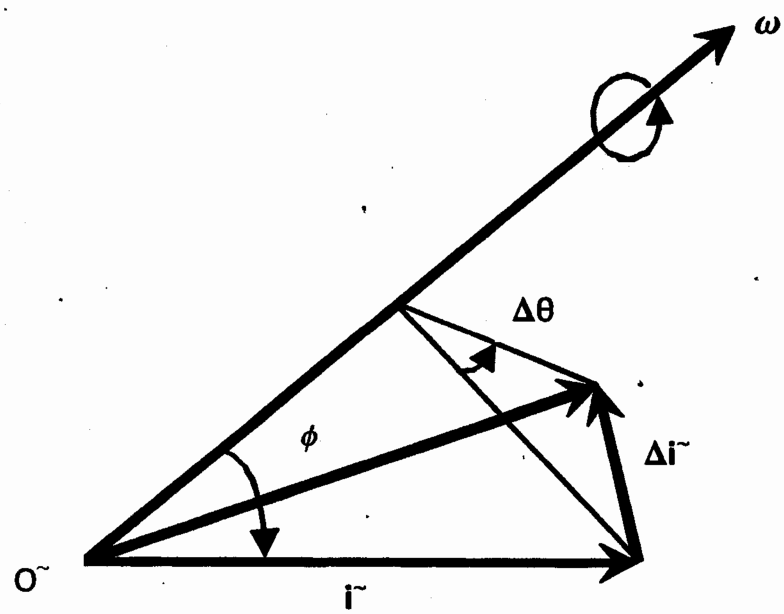

Fig. 1.1: Change in the unit vector i~ produced by a small rotation $\Delta\theta$.

Now the direction of Δi~ is perpendicular to both ω and i~; consequently, form the definition of the cross product, we can write Equation (1.7) in vector form as

$$\frac{d\mathbf{i}^{\sim}}{dt} = \omega \times \mathbf{i}^{\sim}. \tag{1.8}$$

Similarly, we find $\frac{d\mathbf{j}^{\sim}}{dt} = \omega \times \mathbf{j}^{\sim}$, and $\frac{d\mathbf{k}^{\sim}}{dt} = \omega \times \mathbf{k}^{\sim}$

Using the last three terms in Equation (1.6), we can write

$$\mathbf{v} = \mathbf{v}^{\sim} + \omega \times \mathbf{r}^{\sim}, \tag{1.9}$$

or, more explicitly

$$\left(\frac{d\mathbf{r}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{r}^{\sim}}{dt}\right)_{rot} + \omega \times \mathbf{r}^{\sim} = \left[\left(\frac{d}{dt}\right)_{rot} + \omega\times\right]\mathbf{r}^{\sim}. \tag{1.10}$$

In particular, if that vector is the velocity, then we have

$$\left(\frac{d\mathbf{v}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{v}}{dt}\right)_{rot} + \omega \times \mathbf{v}. \tag{1.11}$$

Using Eq. (1.9) in Eq. (1.11), then we can express the final result as follows

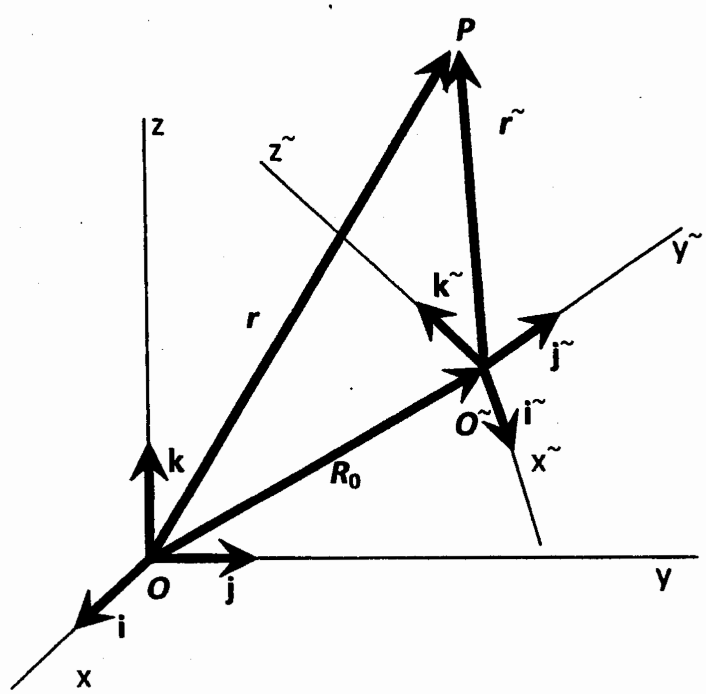

Fig. 1.2: Geometry for the general case of translation and rotation of the moving coordinate system.

$$\mathbf{a} = \mathbf{a}^{\sim} + \dot{\omega} \times \mathbf{r}^{\sim} + 2\omega \times \mathbf{v}^{\sim} + \omega \times (\omega \times \mathbf{r}^{\sim}). \tag{1.12}$$

Where $\left(\frac{d\omega}{dt}\right)_{rot} = \left(\frac{d\omega}{dt}\right)_{fixed} = \dot{\omega}$, $\mathbf{v}^{\sim} = \left(\frac{d\mathbf{r}^{\sim}}{dt}\right)_{rot}$ and $\mathbf{a}^{\sim} = \left(\frac{d\mathbf{v}^{\sim}}{dt}\right)_{rot}$, The simed system is undergoing both translation and rotation, so we must add the velocity of translation $\mathbf{V}_0$ in Eq. (1.9) and the acceleration $\mathbf{A}_0$ of the moving system in Eq. (1.12). This gives the general equations for transforming from a fixed system to a moving and rotating system:

$$\mathbf{v} = \mathbf{v}^{\sim} + \omega \times \mathbf{r}^{\sim} + \mathbf{V}_0,$$

$$\mathbf{a} = \mathbf{a}^{\sim} + \dot{\omega} \times \mathbf{r}^{\sim} + 2\omega \times \mathbf{v}^{\sim} + \omega \times (\omega \times \mathbf{r}^{\sim}) + \mathbf{A}_0. \tag{1.13}$$

The term $2\omega \times \mathbf{v}^{\sim}$ is known as the Coriolis acceleration, and the term $\omega \times (\omega \times \mathbf{r}^{\sim})$ is called the centripetal acceleration. The term $\dot{\omega} \times \mathbf{r}^{\sim}$ is called the transverse acceleration.

For equations of the rotating system, the simed coordinate system is fixed with respect to the rotating system, it does not translate, it does not undergo translational acceleration, and the rotation is constant, so the equation of motion (1.4) for rotating system becomes after neglecting simes

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + 2\omega \times \mathbf{v} + \omega \times (\omega \times \mathbf{r}) \right] = \nabla \cdot \mathbf{T} + \rho \mathbf{b}. \tag{1.14}$$

1.10 Boundary layer

In 1904, Ludwig Prandtl noted that, far away from a solid wall, viscous interactions are not significant in determining the flow field. However, in a thin region near a solid boundary, the viscous interactions have a significant effect on fluid motion. Prandtl's idea of a boundary layer made tractable the flow calculations that take viscosity into account. A boundary layer is a buffer region between the wall and the inviscid free stream above. Mathematically, its main purpose is to allow inviscid flow solutions to satisfy the no-slip condition at the wall.

1.10.1 Boundary layer equation

For developing a mathematical theory of boundary layers [19], the first step is to show the existence, as the Reynolds number Re tends to infinity, or kinematic viscosity ν tend to zero of a limiting form of equations of motion, different from that obtained by putting $\nu = 0$ in the first place. A solution of these limiting equations may thus reasonably be expected to describe approximately the flow in a laminar boundary layer for which Re is large, but not infinite. This is the basis of the classical theory of laminar boundary layers.

The full equations of motion for steady two dimensional flow are [19]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{1.15}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \tag{1.16}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.17}$$

where ν is the kinematic viscosity, x and y variables are horizontal and vertical coordinates and u and v are respectively, the horizontal and vertical fluid velocities. A wall is located in the plane $y = 0$, we consider following non-dimensional variables

$$\begin{gathered} x^* = \frac{x}{L},\ y^* = \frac{y}{\delta},\ u^* = \frac{u}{U}, \\ v^* = \frac{v}{U}\frac{L}{\delta},\ p^* = \frac{p}{\rho U^2}, \end{gathered} \tag{1.18}$$

where, L is the horizontal length scale, δ is the boundary layer thickness at $x = L$, which is known. We will obtain an estimate for it in terms of Reynolds number Re. U is the fluid velocity which is aligned in the x-direction parallel to the solid boundary. The non-dimensional form of the governing equations are

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{UL}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\nu}{UL}\left(\frac{L}{\delta}\right)^2\frac{\partial^2 u^*}{\partial y^{*2}}, \tag{1.19}$$

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\left(\frac{L}{\delta}\right)^2\frac{\partial p^*}{\partial y^*} + \frac{\nu}{UL}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\nu}{UL}\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v^*}{\partial y^{*2}}, \tag{1.20}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{1.21}$$

where the Reynold number for this problem is

$$\text{Re} = \frac{UL}{\nu}. \tag{1.22}$$

Inside the boundary layer, the viscous forces balance inertial and pressure gradient forces. In other words, inertial and viscous forces are of the same order, so

$$\frac{\nu}{UL}(\frac{L}{\delta})^2 = O(1), \tag{1.23}$$

which gives

$$\delta = O(R^{-1/2}L). \tag{1.24}$$

Now we drop the asteriks from the non-dimensional governing equations and with Eq. (1.24), we have

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1.25}$$

$$\frac{1}{R}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{1}{R^2}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{1.26}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1.27}$$

In the limit $R \to \infty$, the above equation reduces to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}, \tag{1.28}$$

$$-\frac{\partial p}{\partial y}=0, \tag{1.29}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.30}$$

where from Eq. (1.29), the pressure is constant across the boundary layer. In terms of dimensional variable, the system of the above equations assume the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1.31}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0, \tag{1.32}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1.33}$$

1.11 Numerical scheme

The mathematical formulation of most of the physical problems in science that involve rate of change with respect to two or more independent variables representing time, length or angle leads either to a partial differential equations or to a set of such equations. For such problems it is normally impossible to find the exact analytic solutions of such partial differential equations. We are only left with the numerical scheme to find the solution of such problems whether they are linear or not. Amongst the numerical approximation methods available for solving differential equations, finite difference and finite element methods are more frequently used.

1.11.1 Finite difference method

Finite difference method is an approximate method in the sense that derivative at a point are approximated by difference quotient over a small interval. It was first utilized by Euler, probably in 1768.

Assume that a function F and its derivatives are single-valued, finite and continuous functions of z, then by Taylor's theorem:

$$F(z+h) = F(z) + hF'(z) + \frac{h^2}{2}F''(z) + \frac{h^3}{6}F'''(z) + O(h^4), \tag{1.34}$$

and

$$F(z-h) = F(z) - hF'(z) + \frac{h^2}{2}F''(z) - \frac{h^3}{6}F'''(z) + O(h^4), \tag{1.35}$$

where $O(h^4)$ denotes terms containing fourth and higher powers of h. These expansions give

$$\left(\frac{dF}{dz}\right)_{z=z} \simeq \frac{F(z+h) - F(z)}{h}, \tag{1.36}$$

$$\left(\frac{dF}{dz}\right)_{z=z} \simeq \frac{F(z) - F(z-h)}{h}, \tag{1.37}$$

with an error of order h. We assume that the containing second and higher powers of h are negligible. Eqs. (1.36) and (1.37) are called forward and backward difference formulae respectively.

Subtraction of the Eq. (1.35) from (1.34) gives

$$\left(\frac{dF}{dz}\right)_{z=z} \simeq \frac{F(z+h) - F(z-h)}{h}, \tag{1.38}$$

with a leading error is of order h^2. This approximation is called a central difference formula. Similarly we can find the approximation for second and third order derivatives.

There are several numerical methods for solving the boundary layer equations in differential form but here in this thesis we used implicit finite difference scheme known as Keller-Box method.

1.11.2 Keller - Box method

Keller - Box Method is a two point finite-difference scheme, which is going to be used extensively by the researchers of this field. Here higher order differential equations are reduced to system of first-order differential equations by introducing new functions. The first-order differential system is approximated on an arbitrary rectangular net with "centered-difference" derivatives and averages at the midpoints of the net rectangle difference equations. As a result, the system of first order differential equation is reduced to system of linear/nonlinear algebraic equations. The resulting system of equations which is if nonlinear then linearized by Newton's method and then solved by the block-elimination method.

The main features of this method are

1. Only slightly more arithmetic to solve than the other implicit methods.
2. Second-order accuracy with arbitrary (nonuniform) x and y spacings.
3. Allows very rapid x variations.
4. Allows easy programming of the solution of large numbers of coupled equations.

Chapter 2

Flow and diffusion of chemically reactive species over a non-linearly stretching sheet immersed in a porous medium

2.1 Introduction

This chapter deals with the flow and diffusion of chemically reactive species over a non-linearly stretching sheet immersed in a porous medium. The governing system of coupled ordinary differential equations are solved by an implicit finite difference scheme. The numerical values of the velocity and mass transfer coefficients have been computed. The effect of various emerging parameters on velocity and concentration distributions have been discussed through graphs. This chapter is a review of the paper by Z. Ziabakhsh et. al [18]. However, detail of mathematical calculations of used numerical scheme is incorporated in this chapter.

2.2 Mathematical Formulation

Let us consider a steady, two-dimensional, incompressible fluid flow over a semi-infinite sheet immersed in a porous medium, which is stretching nonlinearly. The x-axis is chosen along the sheet and y-axis perpendicular to it. The stretching is kinematically imposed and the motion of the fluid in porous medium is generated due to the stretching of the sheet. The sheet is stretched with a nonlinear velocity by applying equal and opposite forces along the x-direction, while the reactive species is emitted from the sheet and diffuses species is destroyed. The concentration of the reactant at the wall is maintained at a constant value Cw and is assumed to be negligible at the distance far away from wall.

The equations that govern the flow and mass transfer analysis are

$$\nabla \cdot \mathbf{V} = 0, \tag{2.1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T} - \frac{\mu}{K}\mathbf{V}, \tag{2.2}$$

$$\frac{dC}{dt} = D\nabla^2 C + r, \tag{2.3}$$

where $\mathbf{V}$ the velocity, $\mathbf{T}$ the Cauchy stress tensor, K the permeability of the porous medium, C the species concentration, D the mass diffussivity and r the species production flux (reaction term). For viscous fluid the Cauchy stress tensor $\mathbf{T}$ is defined as

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1, \tag{2.4}$$

in which p is pressure, $\mathbf{I}$ is unit tensor, μ is the viscosity and $\mathbf{A}_1$ is the first Rivlin Ericksen tensor defined as

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{2.5}$$

where $\mathbf{L} = \text{grad}\mathbf{V}$ and T represent the matrix transpose. Reaction term r is defined as

$$r = -k_1 C, \tag{2.6}$$

where k_1 is the rate of chemical reaction. The subscript index 1 indicates that it concerns a first order homogeneous chemical reaction. The reaction term r is negative because species is consumed in the reaction. For steady two-dimensional flows, we define velocity and concentration field of the form:

$$\mathbf{V} = \mathbf{V}(u(x,y), v(x,y), 0), \quad C = C(x,y). \tag{2.7}$$

In view of Eqs. (2.4) - (2.7), we can write Eqs. (2.1) - (2.3) in the following forms:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.8}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\nu}{K}u, \tag{2.9}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\nu}{K}v, \tag{2.10}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)-k_1 C. \tag{2.11}$$

Under the usual boundary-layer approximations, the governing equations for the momentum and concentration fields are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\nu}{K}u, \tag{2.12}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-k_1 C. \tag{2.13}$$

Since the fluid flow is driven by the stretching of the sheet, we may assume the pressure gradient to be negligible. The boundary conditions applicable to the present flow are

$$u(x,0)=ax+cx^2,\quad v(x,0)=0,\quad C(x,0)=C_w, \tag{2.14}$$

$$u\to 0,\quad C\to 0,\quad as\ \ y\to\infty, \tag{2.15}$$

where a and c are constants verses linear and quadratic stretching of the sheet and the subscript w denotes concentration at the wall. Introducing the following similarity transformation and parameters

$$\eta=\sqrt{\frac{a}{\nu}}y,\qquad u=axF'(\eta)+cx^2G'(\eta), \tag{2.16}$$

$$v=-\sqrt{a\nu}F(\eta)-\frac{2cx}{\sqrt{\frac{a}{\nu}}}G(\eta), \tag{2.17}$$

$$C=C_w\left[H(\eta)+\frac{2cx}{a}S(\eta)\right], \tag{2.18}$$

$$\beta = \frac{k_1 Sc}{a}, \quad k = \frac{\nu}{K}, \quad Sc = \frac{\nu}{D}. \tag{2.19}$$

After using the transformation Eqs. (2.16) - (2.19), the governing Eqs (2.8) , (2.12) and (2.13) take the following form

$$F''' + FF'' - F'^2 - kF' = 0, \tag{2.20}$$

$$G''' + FG'' - 3F'G' + 2F''G - kG' = 0, \tag{2.21}$$

$$H'' + ScFH' - \beta H = 0, \tag{2.22}$$

$$S'' - ScF'S + ScGH' + ScFS' - \beta S = 0, \tag{2.23}$$

with boundary conditions

$$F(0) = 0, \qquad F'(0) = 1, \qquad F'(\infty) = 0, \tag{2.24}$$

$$G(0) = 0, \qquad G'(0) = 1, \qquad G'(\infty) = 0, \tag{2.25}$$

$$H(0) = 1, \qquad H(\infty) = 0, \tag{2.26}$$

$$S(0) = 0, \qquad S(\infty) = 0, \tag{2.27}$$

where dimensionless similarity functions F and G are functions associated with velocity field, H and S represent concentration distribution, k is the permeability parameter, Sc is the Schmidt number and β is the reaction rate parameter. The primes denotes differentiation with respect to dimensionless variable η.

Since the above system of equations is nonlinear, it is not easy to find their exact solution. Therefore, In order to solve the system of Eqs. (2.20) - (2.23) subject to the boundary conditions (2.24) - (2.27), a numerical implicit finite difference scheme is used.

2.3 Numerical solution

Here all the highest order ordinary differential equations are converted to the system of first order ODE by introducing new functions as

$$F' = U, \quad U' = V, \quad G' = W, \quad W' = M, \quad H' = L, \quad S' = P. \tag{2.28}$$

Here the prime denote differentiation w.r.t η. Next all first order derivatives are approximated by central difference gradients and average centered at the mid-points of the net defined by

$$\eta_0 = 0, \quad \eta_j = \eta_{j-1} + h, \quad \eta_J = \eta_\infty, \ \ j = 1, 2, ...J - 1. \tag{2.29}$$

Central difference and average centered at the midpoint $\eta_{j-1/2}$ are defined as

$$F' = \frac{F_j - F_{j-1}}{h}, \quad F = \frac{F_j + F_{j-1}}{2}.$$

After discretization, system of ordinary differential Eqs. (2.20) - (2.23) become

$$V_j - V_{j-1} + \frac{h}{4}(F_j + F_{j-1})(V_j + V_{j-1}) - \frac{h}{4}(U_j + U_{j-1})^2 - \frac{kh}{2}(U_j + U_{j-1}) = 0, \tag{2.30}$$

$$\begin{aligned} M_j - M_{j-1} + \frac{h}{4}(F_j + F_{j-1})(M_j + M_{j-1}) + \frac{h}{2}(G_j + G_{j-1})(V_j + V_{j-1}) - \\ \frac{3h}{4}(U_j + U_{j-1})(W_j + W_{j-1}) - \frac{kh}{2}(W_j + W_{j-1}) = 0, \end{aligned} \tag{2.31}$$

$$L_j - L_{j-1} + \frac{Sch}{4}(F_j + F_{j-1})(L_j + L_{j-1}) - \frac{\beta h}{2}(H_j + H_{j-1}) = 0, \tag{2.32}$$

$$\begin{aligned} P_j - P_{j-1} - \frac{Sch}{4}(U_j + U_{j-1})(S_j + S_{j-1}) + \frac{Sch}{4}(G_j + G_{j-1})(L_j + L_{j-1}) \\ + \frac{Sch}{4}(F_j + F_{j-1})(P_j + P_{j-1}) - \frac{\beta h}{2}(S_j + S_{j-1}) = 0. \end{aligned} \tag{2.33}$$

Equations (2.28) become

$$F_j - F_{j-1} = \frac{h}{2}(U_j + U_{j-1}), \tag{2.34}$$

$$U_j - U_{j-1} = \frac{h}{2}(V_j + V_{j-1}), \tag{2.35}$$

$$G_j - G_{j-1} = \frac{h}{2}(W_j + W_{j-1}), \tag{2.36}$$

$$W_j - W_{j-1} = \frac{h}{2}(M_j + M_{j-1}), \tag{2.37}$$

$$H_j - H_{j-1} = \frac{h}{2}(L_j + L_{j-1}), \tag{2.38}$$

$$S_j - S_{j-1} = \frac{h}{2}(P_j + P_{j-1}). \tag{2.39}$$

Since equations (2.30 - 2.33) are non linear algebraic equations and therefore, linearized by using Newton's Method. We introduce the $(i+1)$th iterates as

$$F_j^{(i+1)} = F_j^{(i)} + \delta F_j^{(i)}.$$

Similarly for all other dependent variables. By substituting the above expression into the equations (2.30 - 2.39) and dropping the quadratic and higher order terms in $\delta F_j^{(i)}$, $\delta U_j^{(i)}$, $\delta V_j^{(i)}$,...etc. We arrived at

$$\delta F_j - \delta F_{j-1} - \frac{h}{2}(\delta U_j + \delta U_{j-1}) = (r_1)_j, \tag{2.40}$$

$$\delta G_j - \delta G_{j-1} - \frac{h}{2}(\delta W_j + \delta W_{j-1}) = (r_2)_j, \tag{2.41}$$

$$(\xi_1)_j\delta V_j + (\xi_2)_j\delta V_{j-1} + (\xi_3)_j\delta F_j + (\xi_4)_j\delta F_{j-1} + (\xi_5)_j\delta U_j + (\xi_6)_j\delta U_{j-1} = (r_3)_j, \tag{2.42}$$

$$\begin{aligned}(\xi_1)_j\delta M_j + (\xi_2)_j\delta M_{j-1} + (\xi_7)_j\delta F_j + (\xi_8)_j\delta F_{j-1} + (\xi_9)_j\delta W_j + (\xi_{10})_j\delta W_{j-1} +\\ (\xi_{11})_j\delta U_j + (\xi_{12})_j\delta U_{j-1} + (\xi_{13})_j\delta G_j + (\xi_{14})_j\delta G_{j-1} + (\xi_{15})_j\delta V_j + (\xi_{16})_j\delta V_{j-1} = (r_4)_j,\end{aligned} \tag{2.43}$$

$$(\xi_{17})_j\delta L_j + (\xi_{18})_j\delta L_{j-1} + (\xi_{19})_j\delta F_j + (\xi_{20})_j\delta F_{j-1} + (\xi_{21})_j\delta H_j + (\xi_{22})_j\delta H_{j-1} = (r_5)_j, \tag{2.44}$$

$$\begin{aligned}(\xi_{17})_j\delta P_j + (\xi_{18})_j\delta P_{j-1} + (\xi_{23})_j\delta F_j + (\xi_{24})_j\delta F_{j-1} + (\xi_{25})_j\delta U_j + (\xi_{26})_j\delta U_{j-1} +\\ (\xi_{27})_j\delta S_j + (\xi_{28})_j\delta S_{j-1} + (\xi_{29})_j\delta L_j + (\xi_{30})_j\delta L_{j-1} + (\xi_{19})_j\delta G_j + (\xi_{20})_j\delta G_{j-1} = (r_6)_j,\end{aligned} \tag{2.45}$$

$$\delta U_j - \delta U_{j-1} - \frac{h}{2}(\delta V_j + \delta V_{j-1}) = (r_7)_j, \tag{2.46}$$

$$\delta W_j - \delta W_{j-1} - \frac{h}{2}(\delta M_j + \delta M_{j-1}) = (r_8)_j, \tag{2.47}$$

$$\delta H_j - \delta H_{j-1} - \frac{h}{2}(\delta L_j + \delta L_{j-1}) = (r_9)_j, \tag{2.48}$$

$$\delta S_j - \delta S_{j-1} - \frac{h}{2}(\delta P_j + \delta P_{j-1}) = (r_{10})_j, \tag{2.49}$$

where the coefficients of unknown and non homogeneous terms in above linear system of algebraic Eqs. (2.40 - 2.49) are defined as

$$\left.\begin{array}{c}
(\xi_1)_j = 1 + \frac{h}{4}(F_j + F_{j-1}),\ (\xi_2)_j = (\xi_1)_j - 2 \\
(\xi_3)_j = \frac{h}{4}(V_j + V_{j-1}) = (\xi_4)_j \\
(\xi_5)_j = -\frac{h}{2}(k + U_j + U_{j-1}) = (\xi_6)_j \\
(\xi_7)_j = \frac{h}{4}(M_j + M_{j-1}) = (\xi_8)_j \\
(\xi_9)_j = -\frac{h}{2}\{\frac{3}{2}(U_j + U_{j-1}) + k\} = (\xi_{10})_j \\
(\xi_{11})_j = -\frac{3h}{4}(W_j + W_{j-1}) = (\xi_{12})_j \\
(\xi_{13})_j = 2(\xi_3)_j = (\xi_{14})_j \\
(\xi_{15})_j = \frac{h}{2}(G_j + G_{j-1}) = (\xi_{16})_j \\
(\xi_{17})_j = 1 + \frac{Sch}{4}(F_j + F_{j-1}),\ (\xi_{18})_j = (\xi_{17})_j - 2 \\
(\xi_{19})_j = \frac{Sch}{4}(L_j + L_{j-1}) = (\xi_{20})_j \\
(\xi_{21})_j = -\frac{\beta h}{2} = (\xi_{22})_j \\
(\xi_{23})_j = \frac{Sch}{4}(P_j + P_{j-1}) = (\xi_{24})_j \\
(\xi_{25})_j = -\frac{Sch}{4}(S_j + S_{j-1}) = (\xi_{26})_j \\
(\xi_{27})_j = \frac{-h}{2}\{\frac{Sc}{2}(U_j + U_{j-1}) + \beta\} = (\xi_{28})_j \\
(\xi_{29})_j = \frac{Sch}{4}(G_j + G_{j-1}) = (\xi_{30})_j
\end{array}\right\}, \tag{2.50}$$

$$\left.\begin{aligned}
(r_1)_j &= F_{j-1} - F_j + \frac{h}{2}(U_j + U_{j-1}) \\
(r_2)_j &= G_{j-1} - G_j + \frac{h}{2}(W_j + W_{j-1}) \\
(r_3)_j &= V_{j-1} - V_j - \frac{h}{4}(F_j + F_{j-1})(V_j + V_{j-1}) + \frac{h}{4}(U_j + U_{j-1})^2 + \frac{kh}{2}(U_j + U_{j-1}) \\
(r_4)_j &= M_{j-1} - M_j - \frac{h}{4}(F_j + F_{j-1})(M_j + M_{j-1}) + \frac{3h}{4}(U_j + U_{j-1})(W_j + W_{j-1}) \\
&\quad - \frac{h}{2}(V_j + V_{j-1})(G_j + G_{j-1}) + \frac{kh}{2}(W_j + W_{j-1}) \\
(r_5)_j &= L_{j-1} - L_j - \frac{Sch}{4}(F_j + F_{j-1})(L_j + L_{j-1}) + \frac{\beta h}{2}(H_j + H_{j-1}) \\
(r_6)_j &= P_{j-1} - P_j - \frac{Sch}{4}(F_j + F_{j-1})(P_j + P_{j-1}) + \frac{Sch}{4}(U_j + U_{j-1})(S_j + S_{j-1}) + \\
&\quad \frac{\beta h}{2}(S_j + S_{j-1}) - \frac{Sch}{4}(G_j + G_{j-1})(L_j + L_{j-1}) \\
(r_7)_j &= U_{j-1} - U_j + \frac{h}{2}(V_j + V_{j-1}) \\
(r_8)_j &= W_{j-1} - W_j + \frac{h}{2}(M_j + M_{j-1}) \\
(r_9)_j &= H_{j-1} - H_j + \frac{h}{2}(L_j + L_{j-1}) \\
(r_{10})_j &= S_{j-1} - S_j + \frac{h}{2}(P_j + P_{j-1})
\end{aligned}\right\}, \tag{2.51}$$

Boundary conditions can be rewritten as

$$\begin{array}{lllll}
\delta F_0 = 0 & \delta U_0 = 0 & \delta G_0 = 0 & \delta W_0 = 0 & \delta H_0 = 0 \\
\delta S_0 = 0 & \delta U_J = 0 & \delta W_J = 0 & \delta H_J = 0 & \delta S_J = 0
\end{array}$$

Now the linear algebraic system of Eqs. (2.40 - 2.49) subject to the boundary condition can be written in matrix-vector form as

$$\bar{\mathbf{A}}\bar{\delta} = \bar{\mathbf{r}}, \tag{2.52}$$

where

$$\bar{\mathbf{A}} = \begin{pmatrix} A_0 & C_0 & & & & & & \\ B_1 & A_1 & C_1 & & & & & \\ & B_2 & A_2 & C_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & B_{J-1} & A_{J-1} & C_{J-1} \\ & & & & & & B_J & A_J \end{pmatrix}, \; \bar{\boldsymbol{\delta}} = \begin{pmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \delta_{J-1} \\ \delta_J \end{pmatrix}, \; \bar{\mathbf{r}} = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ r_{J-1} \\ r_J \end{pmatrix}, \tag{2.53}$$

$$\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (r_7)_1 \\ (r_8)_1 \\ (r_9)_1 \\ (r_{10})_1 \end{pmatrix}, \mathbf{r}_j = \begin{pmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \\ (r_6)_j \\ (r_7)_{j+1} \\ (r_8)_{j+1} \\ (r_9)_{j+1} \\ (r_{10})_{j+1} \end{pmatrix}, \; \mathbf{r}_J = \begin{pmatrix} (r_1)_J \\ (r_2)_J \\ (r_3)_J \\ (r_4)_J \\ (r_5)_J \\ (r_6)_J \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{\delta}_j = \begin{pmatrix} \delta F_j \\ \delta U_j \\ \delta V_j \\ \delta G_j \\ \delta W_j \\ \delta M_j \\ \delta H_j \\ \delta L_j \\ \delta S_j \\ \delta P_j \end{pmatrix}, \; j = 0, 1, 2, \ldots J \,, \tag{2.54}$$

and A_j, B_j, C_j, are 10×10 matrices defined as

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -\frac{h}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{h}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{h}{2} \end{pmatrix}, \tag{2.55}$$

$$A = \begin{pmatrix} 1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 \\ (\xi_3)_j & (\xi_5)_j & (\xi_1)_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\xi_7)_j & (\xi_{11})_j & (\xi_{15})_j & (\xi_{13})_j & (\xi_9)_j & (\xi_1)_j & 0 & 0 & 0 & 0 \\ (\xi_{19})_j & 0 & 0 & 0 & 0 & 0 & (\xi_{21})_j & (\xi_{17})_j & 0 & 0 \\ (\xi_{23})_j & (\xi_{25})_j & 0 & (\xi_{19})_j & 0 & 0 & 0 & (\xi_{29})_j & (\xi_{27})_j & (\xi_{17})_j \\ 0 & -1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -\frac{h}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{h}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{h}{2} \end{pmatrix}, \tag{2.56}$$

$$j = 1, 2, \ldots. J - 1,$$

$$A_J=\begin{pmatrix}
1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 \\
(\xi_3)_J & (\xi_5)_J & (\xi_1)_J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\xi_7)_J & (\xi_{11})_J & (\xi_{15})_J & (\xi_{13})_J & (\xi_9)_J & (\xi_1)_J & 0 & 0 & 0 & 0 \\
(\xi_{19})_J & 0 & 0 & 0 & 0 & 0 & (\xi_{21})_J & (\xi_{17})_J & 0 & 0 \\
(\xi_{23})_J & (\xi_{25})_J & 0 & (\xi_{19})_J & 0 & 0 & 0 & (\xi_{29})_J & (\xi_{27})_J & (\xi_{17})_J \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \tag{2.57}$$

$$B_j=\begin{pmatrix}
-1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 \\
(\xi_4)_j & (\xi_6)_j & (\xi_2)_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\xi_8)_j & (\xi_{12})_j & (\xi_{16})_j & (\xi_{14})_j & (\xi_{10})_j & (\xi_2)_j & 0 & 0 & 0 & 0 \\
(\xi_{20})_j & 0 & 0 & 0 & 0 & 0 & (\xi_{22})_j & (\xi_{18})_j & 0 & 0 \\
(\xi_{24})_j & (\xi_{26})_j & 0 & (\xi_{20})_j & 0 & 0 & 0 & (\xi_{30})_j & (\xi_{28})_j & (\xi_{18})_j \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{2.58}$$

$$j = 1, 2, ...J,$$

$$C_j=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{h}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{h}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{h}{2} \end{pmatrix}, \tag{2.59}$$

$$j = 0, 1, 2, \ldots. J-1.$$

Where $\bar{\mathbf{A}}$ is a tridiagonal matrix and each entry of it is a 10×10 matrix. For solution of system of Eqs. (2.52), $\bar{\mathbf{A}}$ can be reduced to the form of $\bar{\mathbf{L}}\bar{\mathbf{U}}$ as

$$\bar{\mathbf{L}} = \begin{pmatrix} I & & & & & & & \\ \Gamma_1 & I & & & & & & \\ & \Gamma_2 & I & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & \ddots \\ & & & & & & \Gamma_{J-1} & I \\ & & & & & & & \Gamma_J \quad I \end{pmatrix}, \tag{2.60}$$

and

$$\bar{\mathbf{U}} = \begin{pmatrix} \Delta_0 & C_0 & & & & & & & \\ & \Delta_1 & C_1 & & & & & & \\ & & \Delta_2 & C_2 & & & & & \\ & & & \ddots & \ddots & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \Delta_{J-1} & C_{J-1} \\ & & & & & & & & \Delta_J \end{pmatrix}, \quad (2.61)$$

are lower and upper strip diagonal matrices and are solved by using block-elimination method. The block-elimination method consists of two sweeps

(1) Forward sweep

(2) Backward sweep

2.3.1 Forward sweep

In the forward sweep we compute Γ_j, Δ_j, and w_j from the recursion formulas given by

$$\Delta_0 = A_0, \quad (2.62)$$

$$\left.\begin{aligned} \Gamma_j \Delta_{j-1} &= B_j \\ \Delta_j &= A_j - \Gamma_j C_{j-1} \end{aligned}\right\}, \quad (2.63)$$

$$w_0 = r_0, \quad (2.64)$$

$$w_j = r_j - \Gamma_j w_{j-1} \quad (2.65)$$

where $1 \le j \le J$

where w_j are the elements of matrix $\bar{\mathbf{W}}$ which is defined as $\bar{\mathbf{W}} = \bar{\mathbf{U}}\bar{\boldsymbol{\delta}}$. The recursion formulas from Eqs. (2.62 - 2.65) are obtained from the system of equations $\bar{\mathbf{L}}\bar{\mathbf{W}} = \bar{\mathbf{r}}$.

2.3.2 Backward sweep

In backward sweep, compute δ_j from the recursion formulae given by

$$\Delta_J \delta_J = w_J, \tag{2.66}$$

$$\Delta_j \delta_j = w_j - C_j \delta_{j+1},\ j = J-1, J-2, ...0. \tag{2.67}$$

The recursion formulas from Eqs. (2.66) and (2.67) are obtained from the system of equations $\bar{\mathbf{U}}\bar{\boldsymbol{\delta}} = \bar{\mathbf{W}}$. After solving this system of equation, we obtain $\bar{\boldsymbol{\delta}}$ which is our unknown vector. Iteration is performed on $\bar{\boldsymbol{\delta}}$ so that the accuracy upto 10^{-10} is achieved.

	HAM [18]	SM [18]	KBM
k	$-F''(0)$	$-F''(0)$	$-F''(0)$
0.6	1.264934	1.264911	1.264911
0.8	1.341624	1.341640	1.341641
1.0	1.414017	1.414213	1.414214
1.5	1.5802313	1.581113	1.58114

Table 1: Comparison of the KBM solution with HAM solution and SM solutions [18] of velocity profile for various k when $\beta = 0.2$ and $Sc = 0.24$.

	HAM [18]	SM [18]	KBM	HAM [18]	SM [18]	KBM
k	$-H'(0)$	$-H'(0)$	$-H'(0)$	$-S'(0)$	$-S'(0)$	$-S'(0)$
0.6	0.505910	0.50590	0.505910	0.040063	0.04035	0.04037
0.8	0.503601	0.50380	0.50382	0.040320	0.04022	0.04027
1.0	0.501943	0.50191	0.501971	0.040070	0.04004	0.040081
1.5	0.498112	0.49813	0.49814	0.039380	0.03937	0.03936

Table 2: Comparison of the KBM solution with HAM and SM solutions [18] of concentration profile for various k when $\beta = 0.2$ and $Sc = 0.24$.

Sc	HAM [18] $-H'(0)$	SM [18] $-H'(0)$	KBM $-H'(0)$	HAM [18] $-S'(0)$	SM [18] $-S'(0)$	KBM $-S'(0)$
0.24	0.503601	0.50381	0.50382	0.0403201	0.04022	0.04027
0.4	0.544218	0.54429	0.544217	0.0673822	0.06795	0.06808
0.6	0.596620	0.59659	0.596607	0.1014366	0.10221	0.10232
0.8	0.650062	0.65001	0.65004	0.1338458	0.13469	0.13490

Table 3: Comparison of the KBM solution with HAM and SM solutions [18] of concentration profile for various Sc when $\beta = 0.2$ and $k = 0.8$.

β	HAM [18] $-H'(0)$	SM [18] $-H'(0)$	KBM $-H'(0)$	HAM [18] $-S'(0)$	SM [18] $-S'(0)$	KBM $-S'(0)$
0.2	0.503601	0.50380	0.50382	0.040320	0.040220	0.04027
0.8	0.933541	0.93358	0.93354	0.030445	0.030667	0.030677
1.0	1.036527	1.03652	1.03653	0.028971	0.029101	0.029110
1.2	1.129936	1.12992	1.129929	0.027672	0.027822	0.027832

Table 4: Comparison of the KBM solution with HAM and SM solutions [18] of concentration profile for various β when $Sc = 0.24$ and $k = 0.8$.

	$-F'(0)$	$-G''(0)$	$-H'(0)$	$-S'(0)$
SM [18]	1.3416	2.0238	0.5038	0.0402
$20th$ -order app.	1.3416	2.0235	0.5036	0.0403
$18th$ -order app	1.3415	2.0228	0.5022	0.0434
$15th$ -order app.	1.3414	2.0227	0.4979	0.0499
KBM	1.3416	2.0238	0.5038	0.0403

Table 5: Comparison of the KBM solution with different order of HAM and SM solutions [18] for concentration profile when $Sc = 0.24$, $\beta = 0.2$ and $k = 0.8$.

2.4 Graphical results

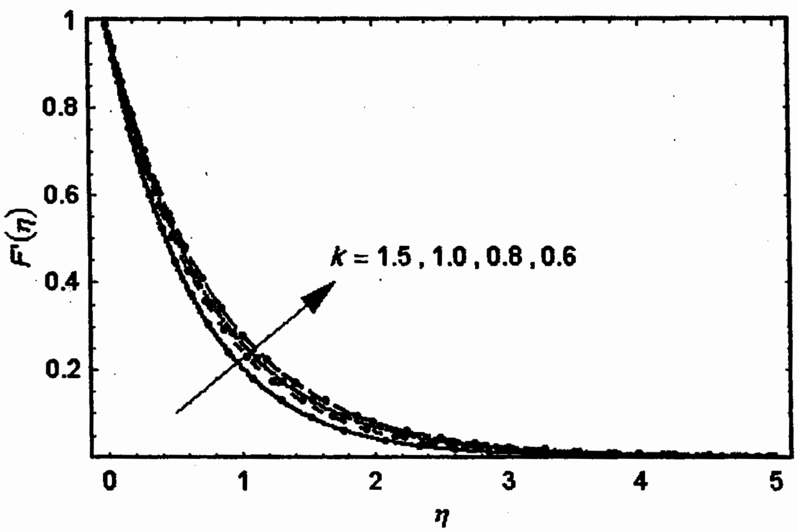

Fig. 2.1: Velocity profiles $F'(\eta)$ for different values of k when $Sc = 0.2$ and $\beta = 0.2$.

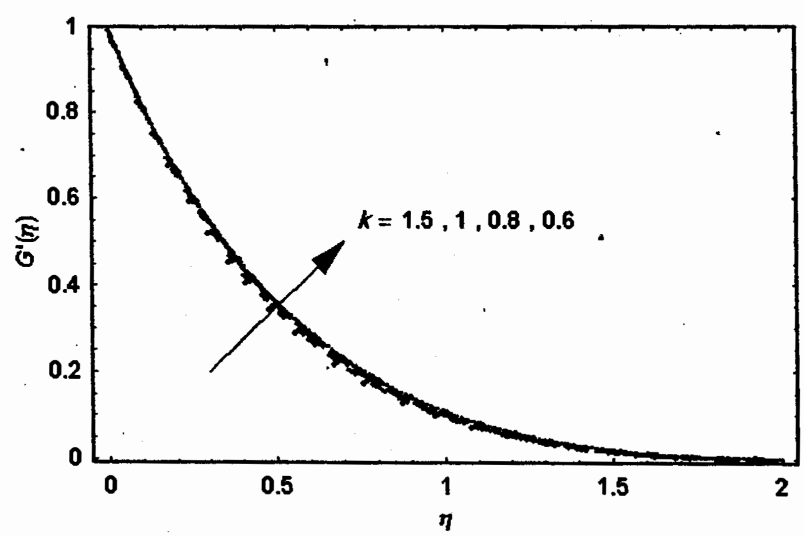

Fig. 2.2: Velocity profiles $G'(\eta)$ for different values of k when $Sc = 0.24$ and $\beta = 0.2$.

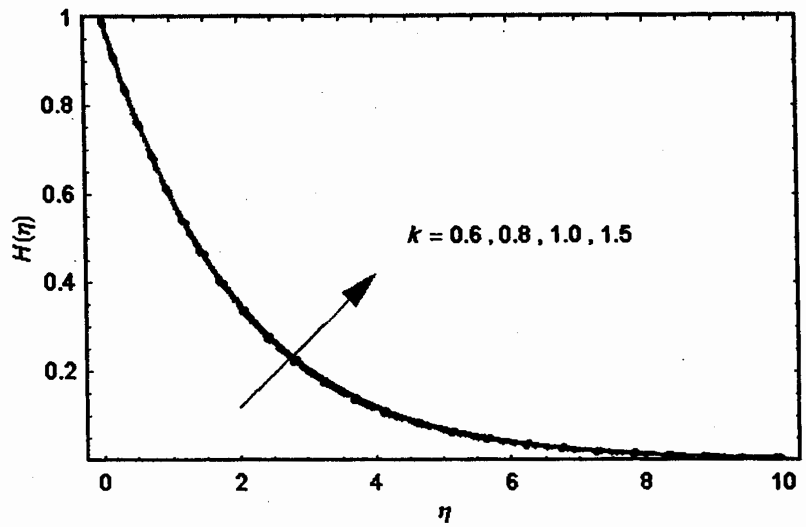

Fig. 2.3: Concentration profiles $H(\eta)$ for different values of k when $Sc = 0.24$ and $\beta = 0.2$.

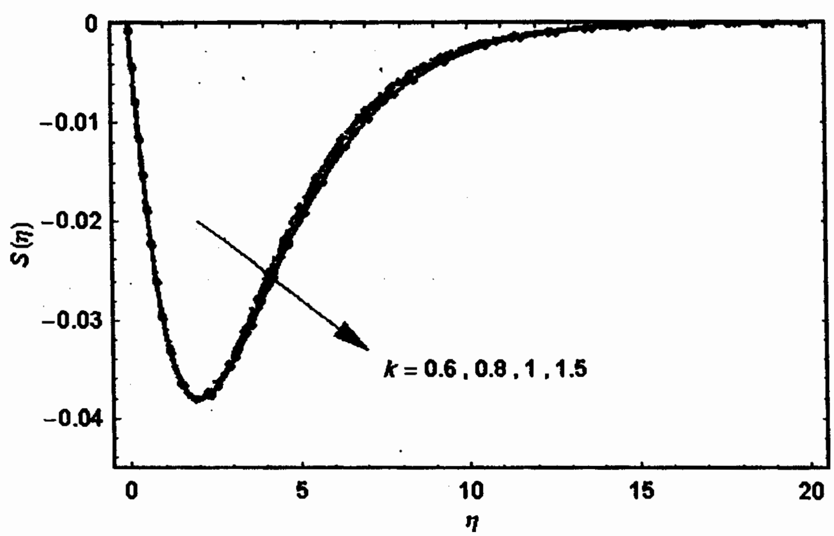

Fig. 2.4: Concentration profiles $S(\eta)$ for different values of k when $Sc = 0.24$ and $\beta = 0.2$.

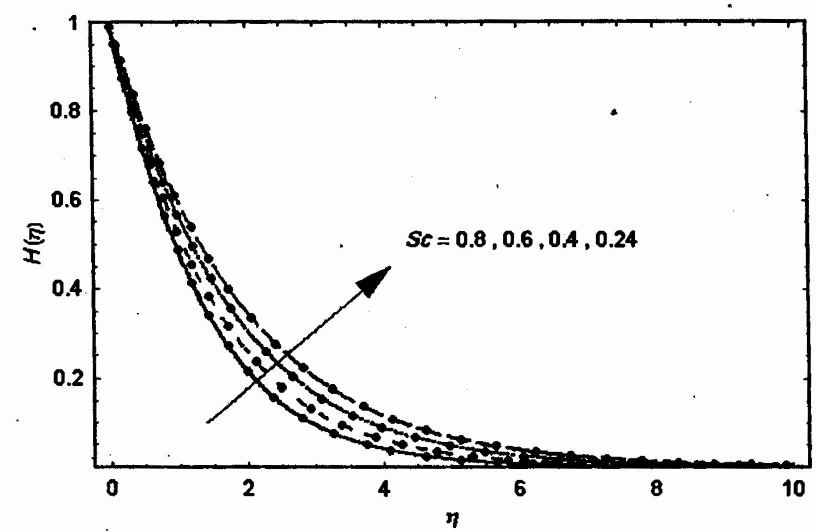

Fig. 2.5: Concentration profiles $H(\eta)$ for different values of Sc when $k = 0.8$ and $\beta = 0.2$.

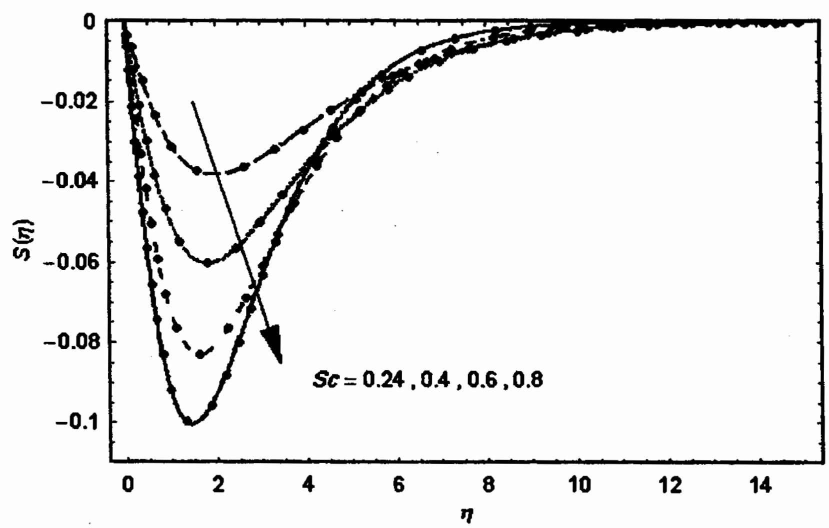

Fig. 2.6: Concentration profiles $S(\eta)$ for different value of Sc when $k = 0.8$ and $\beta = 0.2$.

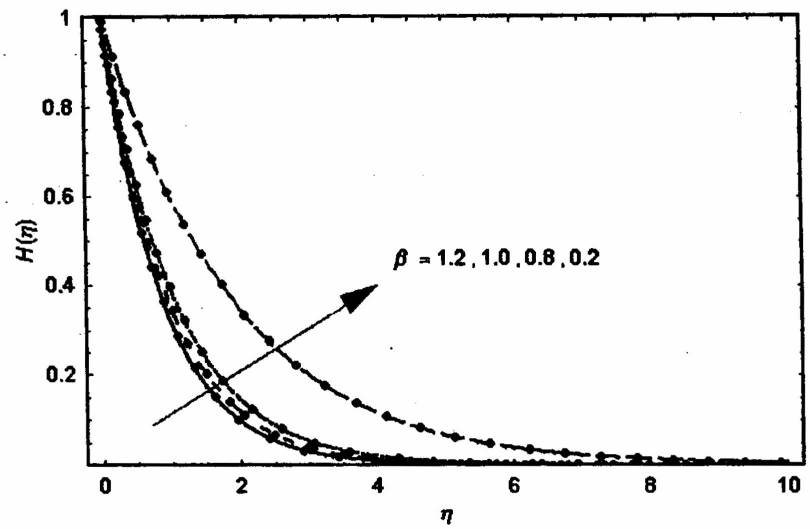

Fig. 2.7: Concentration profiles $H(\eta)$ for different values of β when $k = 0.8$ and $Sc = 0.24$.

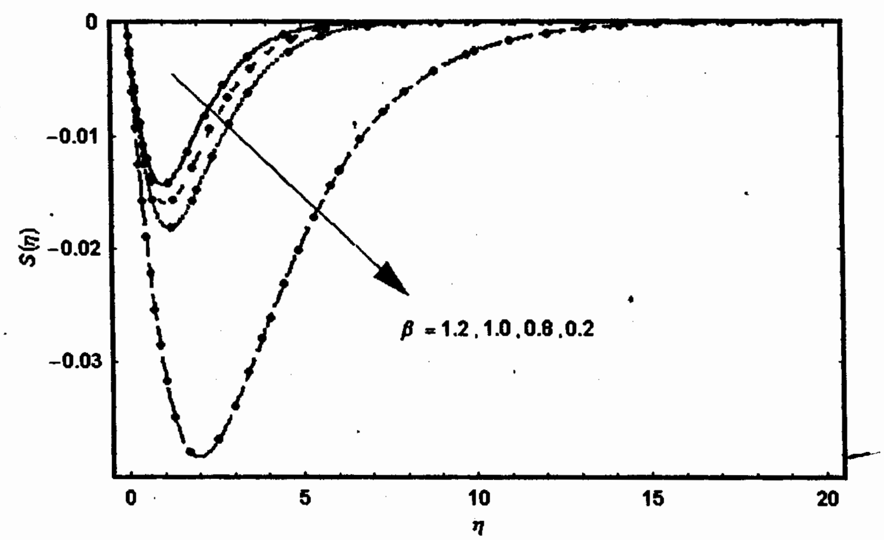

Fig. 2.8: Concentration profiles $S(\eta)$ for different values of β when $k = 0.8$ and $Sc = 0.24$.

2.5 Discussion

Boundary value problem (2.20) - (2.23) under the boundary conditions (2.24) - (2.27) are solved numerically for different values of the parameters, Darcy permeability (k), Schmidt number (Sc) and reaction rate parameter (β) using the Keller Box method described by Cebeci and Bradshaw [22]. It is pertinent to mention here that the value of η_{∞}, h is adjusted according to the value of the involved parameter. The effects of velocity and concentration profiles due to variation of parameters are shown in Figs. (2.1)$-$(2.8). Fig. 2.1 depicts the effects of the Darcy permeability parameter k on dimensionless stream function F' against η. It is observed that by increasing the permeability parameter, F' decreases. In the similar way boundary layer thickness is also reduced by increasing k. Dotted curve represents the solution obtained by shooting method [21] in which fourth order Runge-Kutta method is used to integrate the reduced initial value problem. It demonstrate the confidence on the implicit finite difference scheme, as it provide an excellent agreement with shooting method as shown in Figs (2.1) $-$ (2.8). Fig. 2.2 show the effects of permeability parameter k on the velocity profile G'. It is noted that the effect is similar to that of F' qualitatively but in quantitatively, these effects are minimum on G'. The variation of the concentration field H and S due to permeability parameter is shown in Fig. 2.3 and 2.4. It is interesting to see that the consumption of the low concentration H continues and reaches its least almost at the same distance of η from the wall. Through Fig. 2.4, it is seen that negative concentration profile is generated for permeability parameter k and it has no physical meaning. Fig. 2.5 and 2.6 shows the effects of Schmidt number on the concentration field H and S, it is observed that by increasing the Schmidt number H and S decreases rapidly. This is due to the fact that Schmidt number representing the ratio of the momentum diffusion and mass diffusion. And as soon as the schmidt number increases, mass diffusion become less significant and causes the decrement in the concentration field. Fig. 2.7 and 2.8 illustrate that increase in destructive chemical reaction rate parameter β ($\beta > 0$) reduces the concentration of H and made to increase of S contrarily very rapidly. This shows the fact that the diffusion rate can be significantly altered by chemical reaction rate. Table. 1-5 are prepared to compare our calculated results with that of [18]. It is observed that in general calculated results obtained by shooting method and implicit finite difference scheme are in good agreement with the solution obtained by homotopy analysis method and shooting method of order 2 by Ziabakhsh et al [18].

Chapter 3

Stretching a surface non-linearly in a rotating frame

3.1 Introduction

This chapter deals with the steady flow of an incompressible rotating boundary layer fluid over a non linear stretching surface. The similarity transformations are introduced for the rotating system. The obtained non-linear ordinary differential equations are solved numerically with the help of quite sufficient implicit Finite Difference scheme Keller Box scheme and with shooting method of order 4. The effect of various pertinent parameters on velocity profiles have been discussed through graphs. It is observed that the solution obtained by both the scheme agree excellently with each other.

3.2 Mathematical Formulation

We consider the steady three-dimensional incompressible rotating fluid flow over a nonlinearly semi-infinite stretching sheet in a porous medium. We consider a cartesian coordinate system rotating uniformly with an angular velocity Ω in the z-direction take positive in the vertically upward direction and the plate coinciding with the plane $z = 0$. The motion of the fluid in the porous medium is generated due to the stretching of the sheet. The sheet is stretched with a nonlinear (quadratic) velocity by applying equal and opposite forces along the x-direction.

For steady three dimensional flow, we define the velocity field of the form

$$\mathbf{V} = \mathbf{V}(u(x,z), v(x,z), w(x,z)), \tag{3.1}$$

in which u, v and w are the velocity components in x , y and z directions respectively.

The equations which govern the flow are Eq. (2.1) and

$$\rho\left[\frac{d\mathbf{V}}{dt} + 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})\right] = \nabla\cdot\mathbf{T} - \frac{\mu}{K}\mathbf{V}, \tag{3.2}$$

where $\mathbf{\Omega} = \Omega\mathbf{k}$, $\mathbf{k}$ is a unit vector parallel to the axis of rotation, $\mathbf{\Omega}$ is the angular velocity, d/dt denotes the material derivative, μ is viscosity and K is the permeability of the porous medium. $\rho(2\mathbf{\Omega}\times\mathbf{V})$ and $\rho(\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}))$ are the Coriolis and centripetal acceleration and $\mathbf{r}$ the radial coordinate given by $\mathbf{r}^2 = x^2 + y^2$. Pressure gradient is assumed to be negligible because of the flow is driven by the stretching sheet. Using Eq. (2.1) and (3.1) into the Eq. (3.2), and the governing equations for the momentum for the boundary layer flow become

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - 2v\Omega = \nu\frac{\partial^2 u}{\partial z^2} - \frac{\nu}{K}u, \tag{3.3}$$

$$u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + 2u\Omega = \nu\frac{\partial^2 v}{\partial z^2} - \frac{\nu}{K}v. \tag{3.4}$$

The boundary conditions representing flow are

$$u(x,0) = ax + cx^2, \quad v(x,0) = 0, \quad w(x,0) = 0, \tag{3.5}$$

$$u \to 0, \quad v \to 0, \quad \text{as } z \to \infty, \tag{3.6}$$

where a and c are constants related to the stretching of the sheet. Introducing the following similarity transformation and parameters

$$\eta = \sqrt{\frac{a}{\nu}}z, \qquad u = axF'(\eta) + cx^2G'(\eta), \tag{3.7}$$

$$v = axH(\eta) + cx^2M(\eta), \tag{3.8}$$

$$w = -\sqrt{a\nu}F(\eta) - \frac{2cx}{\sqrt{\frac{a}{\nu}}}G(\eta), \tag{3.9}$$

$$k = \frac{\nu}{aK}, \quad \omega = \frac{2\Omega}{a}. \tag{3.10}$$

After introducing the Eqs. (3.7 - 3.10), the Eqs. (3.3, 3.4) take the following form

$$F''' + FF'' - F'^2 - kF' + \omega H = 0, \tag{3.11}$$

$$G''' + FG'' - 3F'G' + 2F''G - kG' + \omega M = 0, \tag{3.12}$$

$$H'' + FH' - HF' - \omega F' - kH = 0, \tag{3.13}$$

$$M'' - 2F'M + 2GH' - HG' + FM' - \omega G' - kM = 0, \tag{3.14}$$

subject to the boundary conditions (2.24), (2.25) and

$$H(0) = 0, \qquad H(\infty) = 0, \tag{3.15}$$

$$M(0) = 0, \qquad M(\infty) = 0, \tag{3.16}$$

where k is the permeability parameter, ω is the dimensionless angular velocity parameter and primes denotes differentiation with respect to dimensionless variable η.

3.3 Numerical Solution

To solve the system of coupled ordinary differential Eqs. (3.11 - 3.14) subject to the boundary conditions (2.24, 2.25, 3.15, 3.16) numerically for several values of the parameters, k porosity parameter and ω, rotation parameter using the Keller Box as described in the book by Cebeci and Bradshaw [22] and by shooting method [21], the solution is obtained in generally four following steps by Keller-Box method as

1. Reduce Eqs. (3.11 - 3.14) to a system of first order differential equation.
2. Convert the differential system to the difference system using central differences formulae.

3. As the given system of Eqs. (3.11 - 3.14) is non-linear, so linearize the difference equations by using Newton's method and write them in matrix-vector form

4. Solve the obtained linear system by the block-tridiagonal-elimination scheme.

The step size of η, $\Delta\eta$, and the edge of the boundary layer η_{∞}, are adjusted with different values of the parameters involved. To conserve space, the details of the solution technique is presented only in chapter 2.

3.4 Graphical results

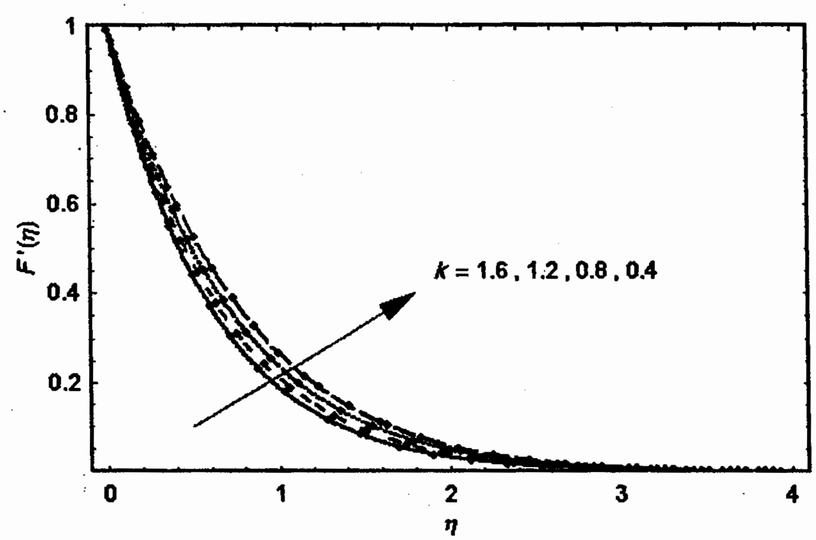

Fig.3.1 : Variation of velocity profile $F'(\eta)$ for different values of k when ω is fixed at 0.8.

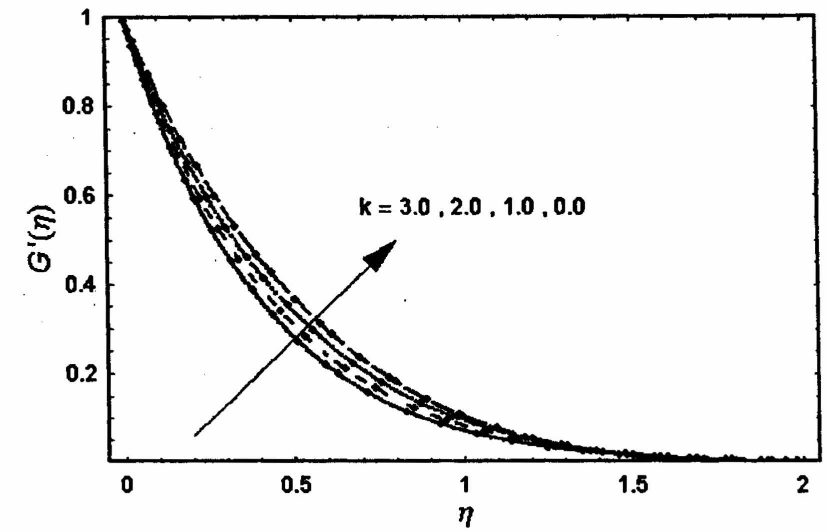

Fig.3.2 : Variation of velocity profile $G'(\eta)$ for different values of k when ω is fixed at 0.8

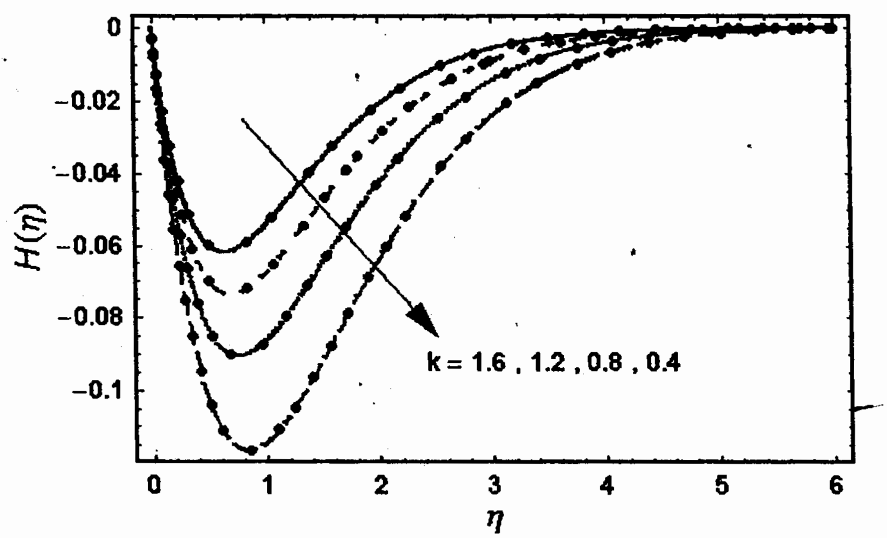

Fig.3.3 : Variation of velocity profile $H(\eta)$ for different values of k when ω is fixed at 0.8.

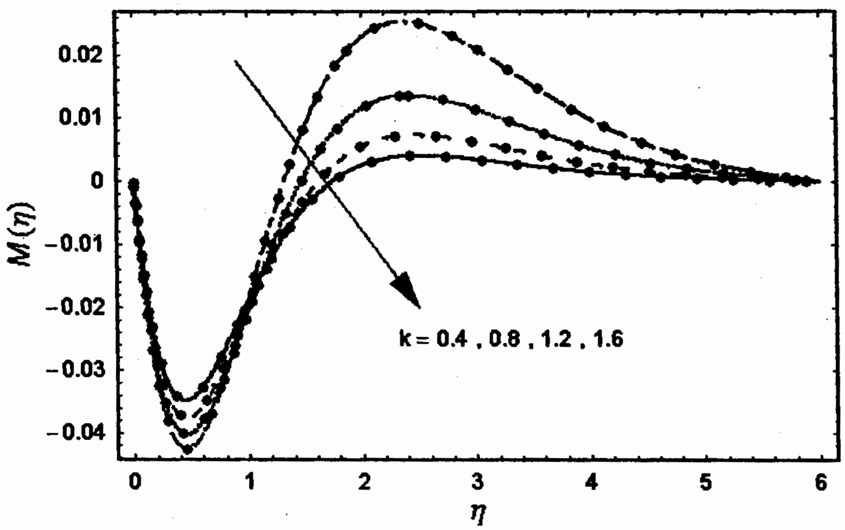

Fig.3.4 : Variation of velocity profile $M(\eta)$ for different values of k when ω is fixed at 0.8.

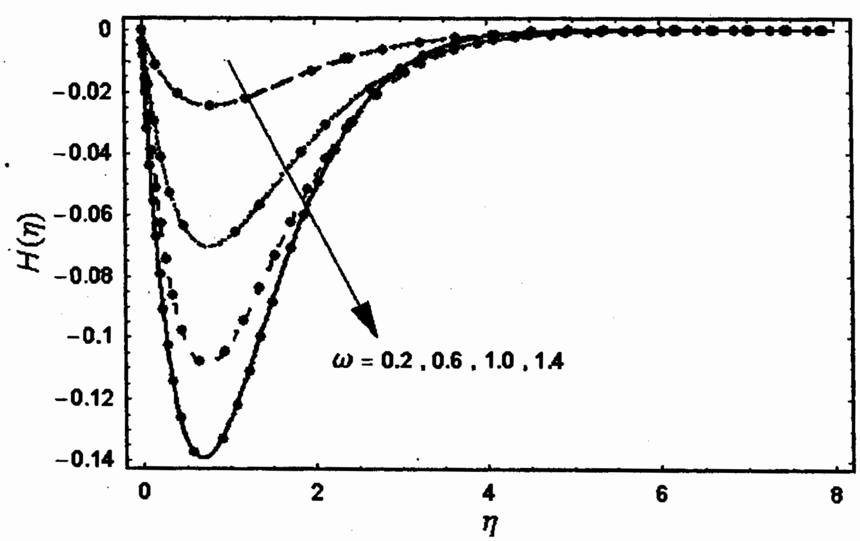

Fig.3.5 : Variation of velocity profile $H(\eta)$ for different values of ω when k is fixed at 0.8.

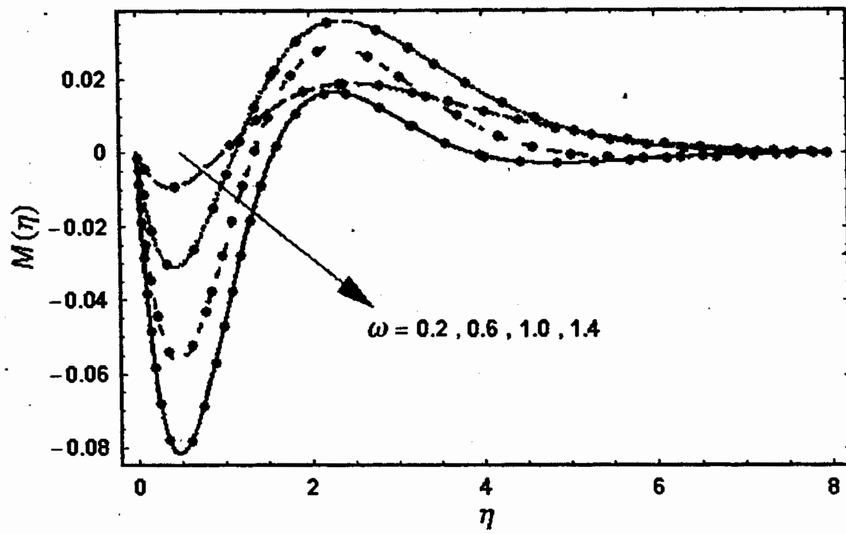

Fig.3.6 : Variation of velocity profile $M(\eta)$ for different values of ω when k is fixed at 0.2.

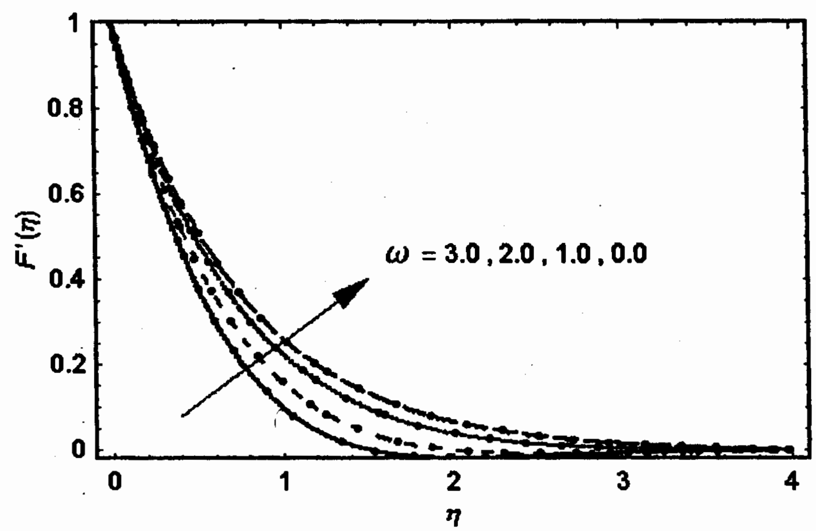

Fig.3.7 : Variation of velocity profiles $F'(\eta)$ for different values of ω when k is fixed at 0.8.

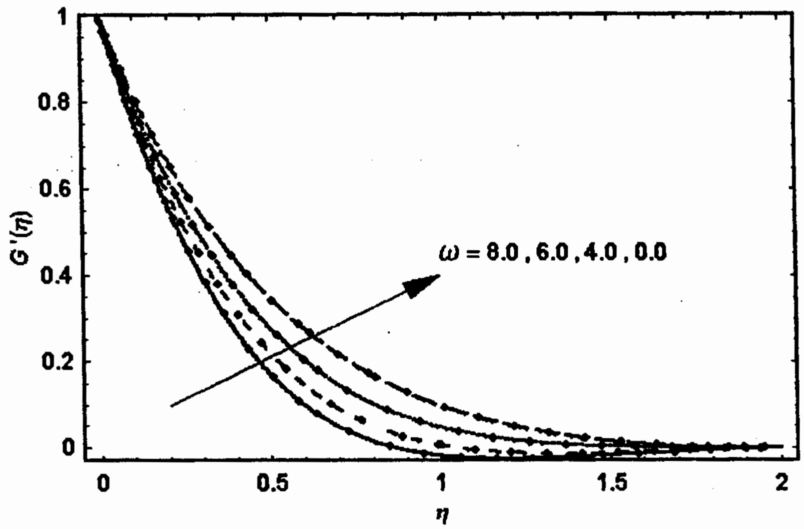

Fig.3.8 : Variation of velocity profiles $G'(\eta)$ for different values of ω when k is fixed at 0.8.

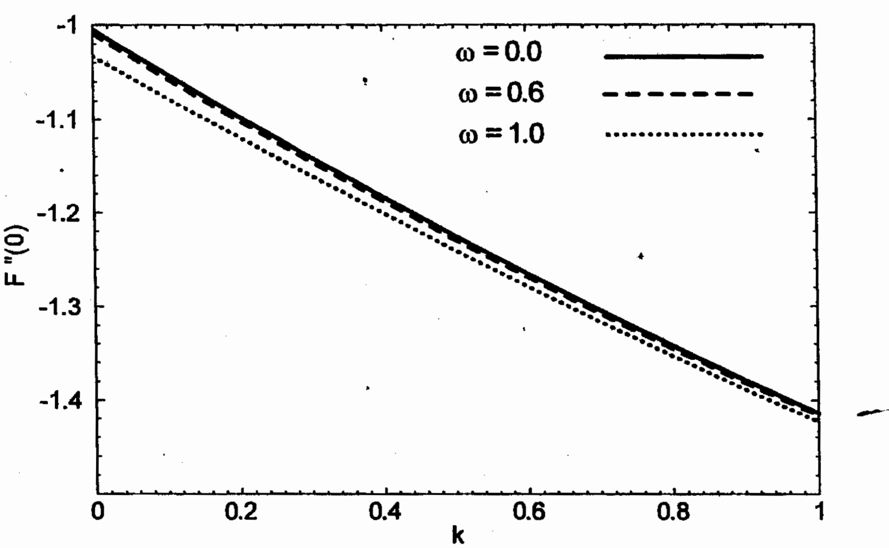

Fig.3.9 : Variation of the skin friction coefficient $F''(0)$ with k for various values of ω.

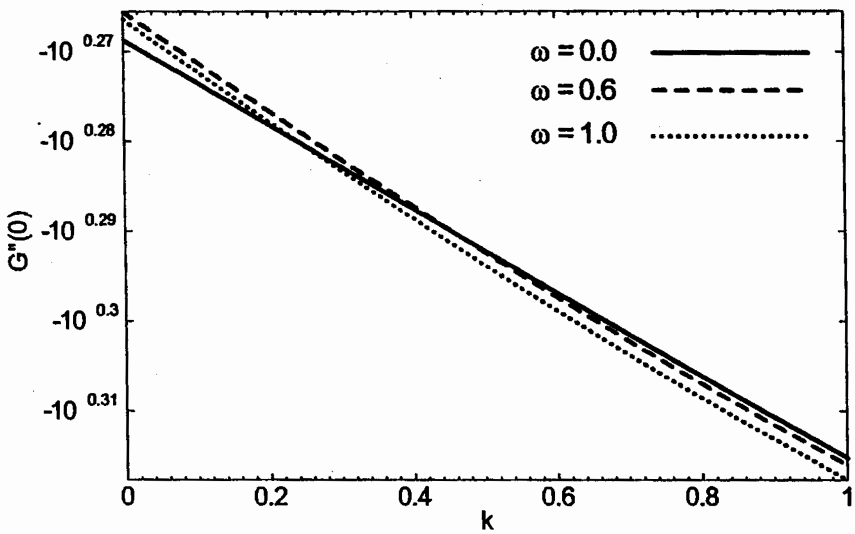

Fig.3.10 : Variation of the skin friction coefficient $G''(0)$ with k for various values of ω.

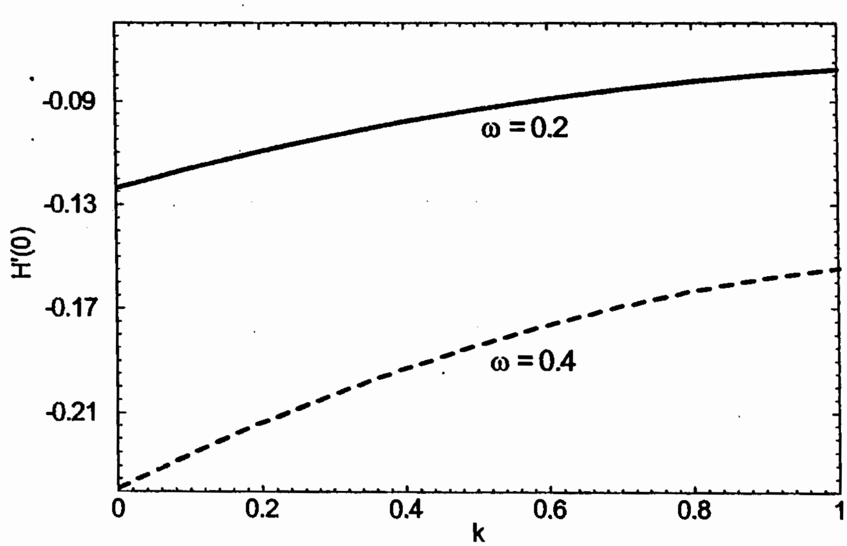

Fig.3.11 : Variation of the skin friction coefficient $H'(0)$ with k for various values of ω.

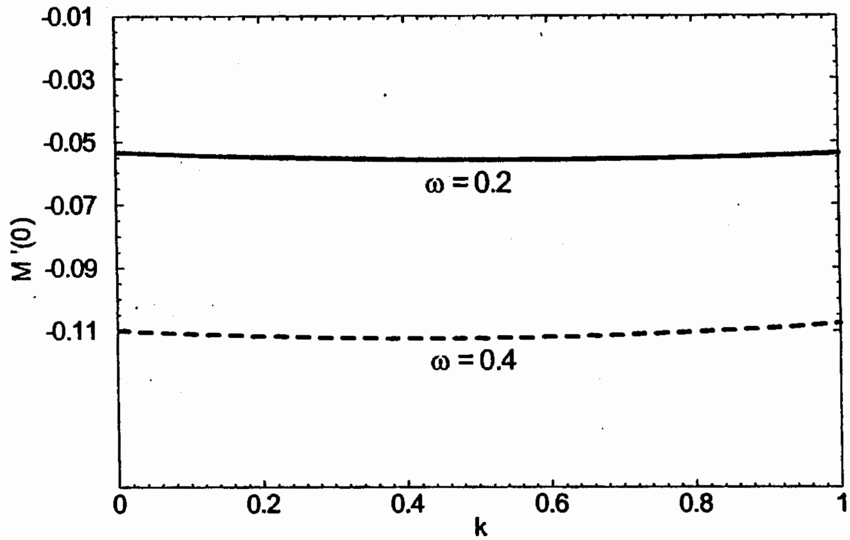

Fig.3.12 : Variation of the skin friction coefficient $M'(0)$ with k for various values of ω.

3.5 Discussion

The effects of rotation parameter (ω) and Darcy permeability (k) on the dimensionless stream function F', G', H and M against similarity variable η are shown through Figs. 3.1 – 3.8. It is observed through Fig. 3.1 that when the whole system in which the fluid is placed is rotating with angular velocity parameter $\omega = 0.8$, by increasing the porosity of the medium on the plate, the velocity of the fluid decrease and causes to reduce the boundary layer thickness as well. It is the universal observation of the porous medium whether the system is rotating or kept fixed. Similar effect are observed through Fig. 3.2 for the velocity field G' qualitatively as expected. But it is interesting here to see that the the profile of G' reaches to minimum at zero very shortly after the wall of the plate at $\eta = 0$ as compare to that of Fig. 3.1. The effects of the porosity parameter k on the dimensionless stream function H and M which are due to rotation, are shown in Figs. 3.3. and 3.4 respectively. It is noted that the magnitude of the velocity increase by decreasing the porosity parameter. Furthermore, boundary layer thickness is quite high and increases with the decrease in porosity parameter as shown in Fig. 3.3. Fig. 3.4 depicts the effects of the porosity parameter on M and it is observed that there is an oscillation in this profile. It oscillate just after the wall and attain its peak maximum by decreasing the parameter and then diffuse to minimum onward. This effect is only due to the rotation of the system. If the system is kept fixed with no rotation, then the profile become coincident with $\eta-$axis. It is observed through Fig. 3.5 and 3.6 that by increasing the rotation of the system the magnitude of the profile H and M increases, but in the negative phase. Oscillation is there in Fig. 3.6, because dimensionless stream function M is proportional to the nonlinear (quadratic) stretch of the sheet. However the velocity profiles F' and G' is decaying exponentially with the increase in the rotation parameter as shown in fig. 3.7 and 3.8. These profiles decreases oscillatory for larger values of the rotation parameter ω. Similar behavior is reported by Wang [4] and R. Nazar et al [10]. Since the values of the skin friction along x, $y-$directions are proportional to the linear combination of $F''(0)$ $G''(0)$, $H'(0)$ and $M'(0)$ respectively. It is noted from Fig. 3.9 that, by increasing the porosity of the porous medium skin friction coefficient is decreasing rapidly. However the effects on $G''(0)$ is observed minimum for all values of the rotation parameter as the log plot is shown along vertical in this Figure 3.10. On the other hand curves for all values of the rotation parameter ω remain almost parallel for $0 \leq k \leq 1$ as shown in Fig. 3.9. Significant contribution of skin coefficient along $y-$axis is observed through Fig. 3.11 and 3.12 for all values of ω. Furthermore, it is observed that the solution obtained by implicit finite difference scheme and shooting scheme (dotted curves) agree excellently as shown through Fig. $3.1 - 3.8$.

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