FLOW OF VISCOELASTIC FLUID IN A CHANNEL INDUCED BY PERISTALTIC WAVES

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2010.

1- Viscoelasticity

2. Fluid dynamics.

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
IN
MATHEMATICS

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dedicated To

my mother, father, family members and friends specially (Basharat Ullah)

Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTRER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

1. Dr. Rahmat Ellahi (Chairman)
2. Dr. Nasir Ali (Supervisor)
3. Prof. Dr. Muhammad Ayub (External Examiner)
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Acknowledgements

To begin with the name of Almighty ***ALLAH*,** Who inculcated the consecration upon me to fulfill the requirement for this dissertation I offer my humblest Darood to the Holy Prophet ***Muhammad*** (Peace be upon Him) who is forever a torch of guidance for humanity.

I express my gratitude to all my teachers whose teachings have brought me to this stage of academic zenith, in particular, I wish to express my profound gratitude to my kind natured, eminent, affectionate, candid and devoted supervisor, *Dr. Nasir Ali*, who aided me with many inspirational discussions. His many valuable comments and suggestions were most welcome and instructive and greatly improved the clarity of this document. I would never have been able to do it up to the standard without his help. I am placing my earnest thanks to *Dr. Nasir Ali.*

I would also like to thank all my friends and others who directly or indirectly helped me during my research period.

Finally, it would have been impossible for me to complete this work without a great support and understanding of my family. I am placing regards to them.

Date: August 19, 2010 **Hakeem Ullah**

Preface

Flow of fluid induced by propagation of waves along the flexible walls of the channel, also known as peristaltic flow is of vital importance and subject of recent interest due to its occurrence in physiology and industry. Specifically, in physiology peristaltic flows occur in transport of urine from kidney to the bladder, movement of chyme in gastro-intestinal tract, vasomotion of small blood vessels and the flows in many glandular ducts. In industry peristaltic mechanism is exploited for transport of corrosive fluids and in manufacturing of peristaltic pumps. Bio-medical instruments such as heart-lung machine also operate according to peristaltic mechanism.

Pioneering works on peristaltic flows were made by Latham [1], Shapiro et al. [2], Fung and Yih [3] and many others [4-6]. In all these studies the considered fluid obeys the Newton's law of viscosity. However, it is well known that many physiological and industrial fluids are non-Newtonian in nature and cannot be understood using Newton's law of viscosity. Raju and Devanathan [8] first time analyzed the peristaltic flow by considering fluid to be non-Newtonian. Later on several researchers investigated interaction of peristalsis with rheologically complex fluids [7-17].

Amongst many non-Newtonian fluids Oldroyd-B fluid is quite popular. This model is capable of predicting viscoelastic effects such as stress relaxation and retardation. Flows of Oldroyd-B fluids were studied extensively in the literature [18-20]. However, this fluid model does not exhibit viscoelastic effects when peristaltic flow under long wavelength approximation is considered. The simplest non-Newtonian model which can predict rheological effects under long wavelength assumption is Oldroyd 4-constant model. Ali et al. [21] discussed the peristaltic motion of Oldroyd 4-consatant fluid in a planner channel. However, there analysis is only valid for hydrodynamic fluid. The study of peristaltic flow with magnetohydrodynamic (MHD) effects fall in the area of biomagnetic fluid dynamics (BFD). Flows of MHD biological fluids are quite important in bioengineering and medical sciences. These fluids are extensively found in living creatures and there flows are greatly influenced by magnetic field. Blood, urine, chyme etc. are examples of biofluids. Further, MHD peristaltic flows of biofluids are useful in problems of conductive physiological fluids for example the blood and blood pump machines and peristaltic MHD compressor. Motivated by these facts the purpose of this dissertation is to extended the analysis of Ali et al. [21] for a magnetohydrodynamic fluid.

The layout of the dissertation is as follow:
Basic definitions and concepts in fluid mechanics are given in chapter one. Chapter two present a detail review of work done by Ali et al. [21]. In chapter three the work of ref. [21] is extended for MHD fluid. Finite difference method with an iterative scheme is used for the solution of the problem. Various interesting features of the flow problem under consideration are analyzed and discussed through graphs.

Contents

Chapter 1

Preliminaries

The purpose of this chapter is to provide the reader a brief overview of the fundamentals of fluid mechanics and peristaltic flow phenomenon.

1.1 Basic Definitions and concepts in fluid mechanics

1.1.1 Fluid

Fluid is a material that flows under the action of applied shear stress.

1.1.2 Flow

A flow is the phenomena of continuous deformation which increases without limit when different forces act upon it.

1.1.3 Definition of fluid mechanics

It is that branch of science in which we study the fluid at rest and motion.

1.1.4 System

A system is defined as a fixed, identifiable quantity of mass. The system boundaries separate the system from surroundings. The boundaries of the system may be fixed or movable.

1.1.5 Density

The density of the fluid is the mass of unit volume of the fluid at given temperature and pressure. If the density of the fluid varies throughout the system, then the density at a point is defined to be the limiting value

$$\rho = \lim_{\delta V \to \delta V'} (\frac{\delta m}{\delta V'}), \tag{1.1}$$

where $\delta V'$ is the infinitesimal volume over which the substance can be considered continuum.

1.1.6 Pressure

Pressure at a point P in the fluid is defined as

$$P = \lim_{\delta A \to 0} (\frac{\delta F}{\delta A}), \tag{1.2}$$

where δF is the normal force acting on small element of area δA enclosing the point P.

1.1.7 Velocity field

The flow velocity of a fluid is a vector field

$$\mathbf{V} = \mathbf{V}(\mathbf{X}, t), \tag{1.3}$$

which gives the velocity of an element of fluid at a position $\mathbf{X}$ and time t. In components form we can write

$$\mathbf{V} = u(\mathbf{X}, t)\mathbf{i} + v(\mathbf{X}, t)\mathbf{j} + w(\mathbf{X}, t)\mathbf{k}. \tag{1.4}$$

1.1.8 Stress

The stress is defined as the force per unit area of the surface on which it acts. The stress at any point in the fluid is defined as

$$\text{Stress at any point } p = \lim_{\delta s \to 0} (\frac{\delta F}{\delta S}), \tag{1.5}$$

where δF is the force acting on element of surface area δS enclosing the point P.

1.1.9 Viscosity

The viscosity of the fluid is one of the important properties in the analysis of the fluid behavior and a measure of fluid resistance to flow. Therefore, it plays vital role in every constitutive equation of the fluid behavior. Although a fluid offers no resistance to change of shape, it does inhibit resistance to the rate of change of shape. The property producing this resistance is called the viscosity.

Mathematically, we have

$$\mu = \frac{\text{shear stress}}{\text{rate of shear strain}}. \tag{1.6}$$

Here μ is called dynamic viscosity. The dimension of μ is $[M/LT]$.

1.1.10 Kinematic viscosity

The kinematic viscosity is the ratio of dynamic viscosity to density. It is denoted by ν and is defined as

$$\nu = \frac{\mu}{\rho}. \tag{1.7}$$

1.1.11 Newton's law of viscosity

The Newton's law of viscosity state that the shear stress is directly and linearly proportional to the rate of deformation. For the one dimensional flow it can be written as

$$\tau_{yx} = \mu\frac{du}{dy}, \tag{1.8}$$

where τ_{yx} is the shear stress and du/dy is the deformation rate.

1.1.12 Scaler field

If to each point (x, y, z) of some region D in space there corresponds a scaler $\phi(x, y, z)$. Then we say that scaler field ϕ has been defined in D and ϕ is called a scaler point function of position. Examples: The Temperature $T = T(x, y, z)$ and the density $\rho = \rho(x, y, z)$ at any point within the fluid.

1.1.13 Vector field

If to each point (x, y, z) of some region D in space there corresponds a vector $\mathbf{v}(x, y, z)$. Then we say that vector field $\mathbf{v}$ has been defined in D and $\mathbf{v}$ is called a vector function of position. Example: The velocity $\mathbf{V}$ at any point (x, y, z) within the moving fluid.

1.1.14 Gradient of scaler field

Let $\phi(x, y, z)$ be a differentiable scaler field. Then the gradient of ϕ denoted by $\nabla\phi$, is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}. \tag{1.9}$$

It is evident from above expression that gradient of a scaler field is a vector field.

1.1.15 Gradient of vector field

Let $\mathbf{V} = \mathbf{V}(x, y, z)$ be a differentiable vector field. Then

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{1.10}$$

defines the gradient of $\mathbf{V}$. The gradient of vector field is a tensor field.

1.2 Types of flow

Steady flow

A flow is said to be steady when every fluid property is invariant with respect to time at each point of the space. Mathematically,

$$\frac{d\zeta}{dt} = 0, \tag{1.11}$$

where ζ represents any fluid property.

Unsteady flow

A flow whose flow properties (such as velocity, pressure, density etc.) at any position change with time is called unsteady flow. In unsteady flow

$$\frac{d\zeta}{dt} \neq 0. \tag{1.12}$$

Two-dimensional flow

A flow in which the velocity field depends upon two space variables is called a two-dimensional flow.

Uniform flow

A flow in which the velocities of fluid particles are equal at all sections of the flow.

Non-uniform flow

The flow in which the velocities of the fluid particles are not the same at all sections of the flow.

Compressible flow

A flow for which the density is not constant is called compressible flow. Flow of gases is treated as compressible.

Incompressible flow

The flow of an incompressible fluid (i.e. for which density remains constant throughout the fluid) is said to be an incompressible flow.

1.3 Types of forces

Inertial forces

The product of the mass and acceleration of the body particle in inertial frame of reference is called inertial force.

Body force

A body force is the force whose magnitude is proportional to the volume of the fluid element. Gravitational and magnetic forces are the examples of body forces.

Surface force

The surface force is the force that act on the boundaries of a medium through direct contact. Pressure force is an example of surface force.

1.4 Basic equations

1.4.1 Equation of continuity

The law of conservation of mass asserts that matter cannot be created or destroyed. In particular this principle may be applied to a moving fluid and its mathematical formulation is known as the equation of continuity of the fluid. Let S be a closed surface drawn entirely within a moving fluid, and let V be the volume it encloses. If $\mathbf{n}$ is the unit normal vector to a surface element dS, its sense being outwards from V, then the local volumetric rate of flow through dS due to $\mathbf{V}$ is

$$\mathbf{n}.\mathbf{V}dS,$$

where $\mathbf{V}$ is the local velocity through dS. The local mass flow rate is given by

$$\rho(\mathbf{n}.\mathbf{V})dS.$$

The net rate of outward mass flow is then calculated by integrating the above expression over the entire surface S that bounds V, as follow

$$\int_S \mathbf{n}.(\rho\mathbf{V})dS.$$

As the total mass of the fluid within S at time t is $\int_V \rho dV$, the total rate of increase of fluid mass within S is

$$\frac{d}{dt}\int_V \rho dV.$$

Now suppose that within S there is a continuous distribution of sources of fluid creation of density τ per unit mass. Then the distribution per unit volume is $\rho\tau$ and so, the total rate of mass creation of the fluid is

$$4\pi \int_V \rho\tau dV.$$

In virtue of the law of conservation of mass, this rate of mass creation of fluid within S through the distributed sources must balance rate of increases of fluid mass within volume V together with the rate of mass transportation to the boundary S. Hence we have

$$4\pi \int_V \rho\tau dV = \frac{d}{dt}\int_V \rho dV + \int_S \mathbf{n}.(\rho\mathbf{V})dS.$$

Using Leibnitz's formula and the Gauss-Ostrogradskii divergence theorem we get

$$4\pi \int_V \rho\tau dV = \int_{V(t)} \frac{\partial\rho}{\partial t}dV + \int_{S(t)} \rho(\mathbf{V}_{\text{surface}}.\mathbf{n})dS + \int_V \nabla.(\rho\mathbf{V})dV,$$

where $\mathbf{V}_{surface}$ is the velocity of the surface element dS (if the surface is moving). If the volume is fixed in space, the second term goes to zero because $\mathbf{V}_{surface}$ is zero. Hence we obtained the above equation of the form

$$4\pi \int_V \rho\tau dV = \int_{V(t)} \frac{\partial\rho}{\partial t}dV + \int_V \nabla.(\rho\mathbf{V})dV.$$

The equation is valid for any considered volume V of the moving fluid. It follows that at each point of the fluid we must have

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V}) = 4\pi\rho\tau,$$

which is the general form of the equation of continuity. Whenever there are no distributed point sources within S then $\tau = 0$ and we obtain the above equation as

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V}) = 0, \tag{1.13}$$

this equation is known as the continuity equation, and it expresses the physical law that mass is conserved. For incompressible flow Eq. (1.13) reduces to

$$\nabla.\mathbf{V} = 0. \tag{1.14}$$

1.4.2 Momentum equation

The law of conservation of momentum asserts that the total momentum of a system remains constant, if no external force act on the system. The application of this principle to a moving fluid results the equation of motion /momentum equation. The general form of the equation of fluid motion is

$$\rho\frac{D\mathbf{V}}{Dt} = \nabla.\mathbf{T} + \rho\mathbf{b}, \tag{1.15}$$

where $\mathbf{T}$ is called the Cauchy stress tensor or total stress tensor and is different for different fluids and $\mathbf{b}$ the body force vector.

1.5 Classification of fluids

1.5.1 Newtonian fluids

A fluid that obeys the Newton's law of viscosity (1.8) is called Newtonian fluid. There are two major contribution to the total stress tensor $\mathbf{T}$: the thermodynamic pressure p and a second portion that originates due to the deformation of fluid. We can write

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1.16}$$

where $\mathbf{I}$ is the identity tensor and $\mathbf{S}$ is the extra stress tensor which contains the contribution to the stress that results from fluid deformation. When the fluid is at rest $\mathbf{T}$ becomes $-p\mathbf{I}$.

An equation that specifies $\mathbf{S}$ for a fluid is called a stress constitutive relation for that fluid. The constitutive relation expresses the molecular stresses generated in the flow in terms of kinetic variables such as velocities, strains and derivatives of velocities and strains. Once a constitutive relation is decided it may be inserted into the equation for $\mathbf{T}$ and subsequently into the equation of motion. The equation may be solved, along with the continuity equation, for the unknown flow variables.

Newtonian constitutive equation for compressible fluids

The Newtonian constitutive equation for a compressible fluid is

$$\mathbf{S} = -\mu(\nabla\mathbf{V} + (\nabla\mathbf{V})^T) + (\frac{2}{3}\mu - \kappa)(\nabla.\mathbf{V})\mathbf{I}, \tag{1.17}$$

where κ is the dilational viscosity. The shear viscosity describes the resistance of a fluid to sliding motion and the dilational viscosity describes an isotropic contribution to the stress that is generated when the density of a fluid changes upon deformation.

Newtonian constitutive equation for incompressible fluids

Using continuity equation (1.14) into Eq. (1.17). We get the following constitutive equation for incompressible fluid.

$$\mathbf{S} = -\mu\left(\nabla\mathbf{V} + (\nabla\mathbf{V})^T\right). \tag{1.18}$$

1.5.2 Non-Newtonain fluids

A fluid whose behavior cannot be predicted by Newtonian constitutive equation is known as non-Newtonain fluid. For such fluid shear stress is directly and nonlinearly proportional to the rate of deformation. Mathematically,

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n, \qquad n \neq 1 \tag{1.19}$$

or

$$\tau_{yx} = \eta\left(\frac{du}{dy}\right),$$

where

$$\eta = \left(\frac{du}{dy}\right)^{n-1},$$

is the apparent viscosity. Examples of non-Newtonain fluids are tooth paste, ketchup, gel, shampoo, blood and soaps, etc.

1.5.3 Viscoelastic fluids

A fluid that behaves as solid as well as liquid is called viscoelastic fluid. It has elastic nature. It will regain back its shape partially when applied stress is removed.

1.5.4 Retardation time

Retardation time refers to a time scale for the build up of stress in a fluid.

1.5.5 Relaxation time

Relaxation time refers to a time scale for the relaxation of stress in a fluid. It varies widely among materials.

1.6 Constitutive equation for the Oldroyd 4-constant fluids

The constitutive equation for the Oldroyd 4-constant fluid is given by

$$\mathbf{S} + \lambda_1 \frac{\mathbf{DS}}{\mathbf{D}t} + \lambda_3 \mathrm{tr}(\mathbf{S})\mathbf{A}_1 = \mu(1 + \lambda_2 \frac{\mathbf{D}}{\mathbf{D}t})\mathbf{A}_1, \tag{1.20}$$

where λ_1 and λ_3 are the relaxation times, λ_2 is the retardation time, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{1.21}$$

L is the velocity gradient and

$$\frac{\mathbf{DS}}{\mathbf{D}t} = \frac{\mathbf{dS}}{\mathbf{d}t} - \mathbf{LS} - \mathbf{SL}^T, \tag{1.22}$$

is the upper-convected time derivative. It should be noted that the model (1.20) includes the Oldroyd 3-consatnt model (for $\lambda_3 = 0$), the Maxwell model (for $\lambda_2 = \lambda_3 = 0$), the viscous fluid model (for $\lambda_1 = \lambda_2 = \lambda_3 = 0$) and the second grade fluid (if $\lambda_1 = \lambda_3 = 0$) as the limiting cases.

1.7 Peristalsis

The word peristaltic comes from a Greek word "Peristaltikos" which means clasping and compressing. Peristalsis is a normal function of the body to move fluid from one place to another.

It is an automatic and vital process that moves food through the digestive tract, urine from the kidneys through the ureters into the bladder, blood in the arteries and veins, and bile from the gallbladder into the duodenum. Technical roller and finger pumps also operate according to this principle.

1.8 Dimensionless parameters

1.8.1 Reynold number

The Reynold number is the representative of the ratio of force of inertia to the viscous force. It is denoted by Re. Mathematically,

$$\mathrm{Re} = \frac{ul}{\nu}, \tag{1.23}$$

where l is the characteristic length scale and u is a typical velocity.

1.8.2 Hartman number

The Hartman number is the ratio of electromagnetic force to the viscous force. It is defined by

$$M = Bl\sqrt{\sigma/\rho\nu}, \tag{1.24}$$

where B is the magnitude of magnetic field and σ is the electrical conductivity.

1.8.3 Wave number

The wave number refers to the ratio of radius (width) of tube (channel) to the wave length of the wave.

1.8.4 Amplitude ratio

It is the ratio of the amplitude of the peristaltic wave to the radius (width) of the tube (channel).

1.9 Magnetohydrodynamic (MHD)

Magnetohydrodynamic (MHD) is the study of the electrically conducting fluid in motion. This field was first initiated by Hannes Alfven and for this he received the Nobel Prize in Physics in 1970. Examples of such fluids are liquid metals and salt water.

1.10 Governing equation for MHD fluid

MHD is the study of motion of fluid in the presence of a magnetic field. The situation is essentially one of mutual interaction between the final velocity field and the electromagnetic field: electric current induced in the fluid as a result of its motion modify the field; at the same time their flow in the magnetic field produces mechanical forces which modify the motion. For MHD flow there is an extra term due to Lorentz force $\mathbf{J} \times \mathbf{B}$ in the momentum equation. Here $\mathbf{J}$ is the electric current density and $\mathbf{B}$ the magnetic flux. The expression of $\mathbf{J}$ or the generalized Ohm's law is

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{1.25}$$

where σ is the electrical conductivity and the Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}, \tag{1.26}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.27}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.28}$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.29}$$

These equations are known as Ohm's law, Gauss's law, Gauss's law of magnetism, Faraday's law of induction and Ampere's law with Maxwell's correction respectively. In the above equations ϵ_0 is the permittivity of the free space also called electric constant, μ_0 is the permeability of free space which is also called magnetic constant, ρ_c is the total charge density and $\mathbf{J}$ is the total current density.

For a linear medium, Maxwell's Eqs. (1.26)-(1.29) with no charge density and electric displacement $\mathbf{D} = \epsilon_0 \mathbf{E}$ reduces to the following

$$\nabla \cdot \mathbf{E} = 0, \tag{1.30}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.31}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.32}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{1.33}$$

From Ohm's law (1.25) and Maxwell's Eqs. (1.30)-(1.33), an evaluation equation for the magnetic flux $\mathbf{B}$ can be easily derived, which is known as the magnetic induction equation and suggests that the motion of an electrically conducting fluid in an applied magnetic field induces a magnetic field in the medium. The total field is the sum of the applied and induced magnetic fields $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}^*$, where $\mathbf{b}^*$ is the induced magnetic field.

Chapter 2

Peristaltic flow of an Oldroyd 4-constant fluid in a planar channel

2.1 Introduction

In this chapter an attempt is made to investigate the peristaltic flow of an Oldroyd 4-constant fluid in a planar channel. The flow problem is modeled under long wavelength and low Reynolds number assumptions. The governing equations are transformed by introducing stream function into a single nonlinear ordinary differential equation. The finite difference scheme and an iterative method are used to solve this nonlinear equation together with appropriate boundary conditions. The effects of non-Newtonian parameters on longitudinal velocity, stream function, longitudinal pressure gradient and pressure rise per wavelength are illustrated graphically. The content of this chapter have been published by N.Ali et al [21]. Here we have presented the details of mathematical modeling and solution procedure.

2.2 Governing Equations

Consider a two dimensional channel of uniform thickness $2a$. Let it be filled with a homogenous incompressible Oldroyd 4-constant fluid. The walls of the channel are assumed flexible. Assume two symmetric infinite wave trains traveling with velocity c along the walls. If $\overline{X}$ and $\overline{Y}$ are the longitudinal and transverse coordinates, respectively, then the wall surface is mathematically defined as

$$\overline{h}(\overline{X},\overline{t}) = a + b\cos\left[\frac{2\pi}{\lambda}\left(\overline{X} - c\overline{t}\right)\right]. \tag{2.1}$$

Here b is the wave amplitude, λ is the wavelength and $\overline{t}$ is the time. A further assumption is that there is no motion of the wall in the longitudinal direction. This assumption implies that for the no-slip condition i.e., longitudinal velocity is zero at the wall.

For the flow under consideration the velocity field is given by

$$\overline{\mathbf{V}} = \left[\overline{U}(\overline{X},\overline{Y},\overline{t}), \overline{V}(\overline{X},\overline{Y},\overline{t}), 0\right], \tag{2.2}$$

where $\overline{U}$ and $\overline{V}$ are the longitudinal and transverse velocity components, respectively.

Substituting Eq. (2.2) in Eqs. (1.14) and (1.15) yield the following scalar equations

$$\frac{\partial\overline{U}}{\partial\overline{X}} + \frac{\partial\overline{V}}{\partial\overline{Y}} = 0, \tag{2.3}$$

$$\rho\left(\frac{\partial}{\partial\overline{t}} + \overline{U}\frac{\partial}{\partial\overline{X}} + \overline{V}\frac{\partial}{\partial\overline{Y}}\right)\overline{U} = -\frac{\partial\overline{p}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{XX}}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{XY}}}{\partial\overline{Y}}, \tag{2.4}$$

$$\rho\left(\frac{\partial}{\partial\overline{t}} + \overline{U}\frac{\partial}{\partial\overline{X}} + \overline{V}\frac{\partial}{\partial\overline{Y}}\right)\overline{V} = -\frac{\partial\overline{p}}{\partial\overline{Y}} + \frac{\partial\overline{S}_{\overline{XY}}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{YY}}}{\partial\overline{Y}}. \tag{2.5}$$

Now we proceed to calculate the components of stress tensor using the relation (1.20).

$$\overline{\mathbf{L}} = \begin{pmatrix} \frac{\partial\overline{U}}{\partial\overline{X}} & \frac{\partial\overline{U}}{\partial\overline{Y}} \\ \frac{\partial\overline{V}}{\partial\overline{X}} & \frac{\partial\overline{V}}{\partial\overline{Y}} \end{pmatrix}, \tag{2.6}$$

and

$$\overline{\mathbf{L}}^T = \begin{pmatrix} \frac{\partial\overline{U}}{\partial\overline{X}} & \frac{\partial\overline{V}}{\partial\overline{X}} \\ \frac{\partial\overline{U}}{\partial\overline{Y}} & \frac{\partial\overline{V}}{\partial\overline{Y}} \end{pmatrix}. \tag{2.7}$$

Then

$$\overline{\mathbf{A}}_1 = \overline{\mathbf{L}} + \overline{\mathbf{L}}^T = \begin{pmatrix} 2\frac{\partial\overline{U}}{\partial\overline{X}} & \frac{\partial\overline{U}}{\partial\overline{Y}} + \frac{\partial\overline{V}}{\partial\overline{X}} \\ \frac{\partial\overline{U}}{\partial\overline{Y}} + \frac{\partial\overline{V}}{\partial\overline{X}} & 2\frac{\partial\overline{V}}{\partial\overline{Y}} \end{pmatrix}, \tag{2.8}$$

for two-dimensional flow we can define

$$\overline{\mathbf{S}} = \begin{pmatrix} \overline{S}_{XX} & \overline{S}_{XY} \\ \overline{S}_{XY} & \overline{S}_{YY} \end{pmatrix}. \tag{2.9}$$

With the help of Eq. (2.9) we have

$$\frac{D\overline{\mathbf{S}}}{D\overline{t}} = \begin{pmatrix} (\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{S}_{XX} & (\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{S}_{XY} \\ -2\overline{S}_{XX}\frac{\partial \overline{U}}{\partial \overline{X}} - 2\overline{S}_{XY}\frac{\partial \overline{U}}{\partial \overline{Y}} & -\overline{S}_{XX}\frac{\partial \overline{V}}{\partial \overline{X}} - \overline{S}_{YY}\frac{\partial \overline{V}}{\partial \overline{Y}} \\ (\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{S}_{XY} & (\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{S}_{YY} \\ -\overline{S}_{XX}\frac{\partial \overline{V}}{\partial \overline{X}} - \overline{S}_{YY}\frac{\partial \overline{V}}{\partial \overline{Y}} & -2\overline{S}_{XY}\frac{\partial \overline{V}}{\partial \overline{X}} - 2\overline{S}_{YY}\frac{\partial \overline{V}}{\partial \overline{Y}} \end{pmatrix}. \tag{2.10}$$

Similarly

$$\frac{D\overline{\mathbf{A}}_1}{D\overline{t}} = \begin{pmatrix} 2(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\frac{\partial \overline{U}}{\partial \overline{X}} & (\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) \\ -4(\frac{\partial \overline{U}}{\partial \overline{X}})^2 - 2\frac{\partial \overline{U}}{\partial \overline{Y}}(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) & -2\frac{\partial \overline{U}}{\partial \overline{X}}\frac{\partial \overline{V}}{\partial \overline{X}} - 2\frac{\partial \overline{U}}{\partial \overline{Y}}\frac{\partial \overline{V}}{\partial \overline{Y}} \\ (\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) & 2(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\frac{\partial \overline{V}}{\partial \overline{Y}} \\ -2\frac{\partial \overline{U}}{\partial \overline{X}}\frac{\partial \overline{V}}{\partial \overline{X}} - 2\frac{\partial \overline{U}}{\partial \overline{Y}}\frac{\partial \overline{V}}{\partial \overline{Y}} & -4(\frac{\partial \overline{V}}{\partial \overline{Y}})^2 - 2\frac{\partial \overline{V}}{\partial \overline{Y}}(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) \end{pmatrix}, \tag{2.11}$$

and

$$tr(\overline{\mathbf{S}})\overline{\mathbf{A}}_1 = \begin{pmatrix} 2(\overline{S}_{XX} + \overline{S}_{YY})\frac{\partial \overline{U}}{\partial \overline{X}} & (\overline{S}_{XX} + \overline{S}_{YY})(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) \\ (\overline{S}_{XX} + \overline{S}_{YY})(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) & 2(\overline{S}_{XX} + \overline{S}_{YY})\frac{\partial \overline{V}}{\partial \overline{Y}} \end{pmatrix}. \tag{2.12}$$

Inserting Eqs. (2.8)-(2.12) into relation (1.20) and equating the corresponding components on the both sides yield the following equations

$$\begin{aligned} &\overline{S}_{XX} + \overline{\lambda}_1\left[\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{S}_{XX} - 2\frac{\partial \overline{U}}{\partial \overline{X}}\overline{S}_{XX} - 2\frac{\partial \overline{U}}{\partial \overline{Y}}\overline{S}_{XY}\right] \\ &\qquad + 2\overline{\lambda}_3(\overline{S}_{XX} + \overline{S}_{YY})\frac{\partial \overline{U}}{\partial \overline{X}} \\ &= 2\mu\frac{\partial \overline{U}}{\partial \overline{X}} + 2\mu\overline{\lambda}_2\left[\begin{array}{c} \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\frac{\partial \overline{U}}{\partial \overline{X}} - 2(\frac{\partial \overline{U}}{\partial \overline{X}})^2 \\ -\frac{\partial \overline{U}}{\partial \overline{Y}}(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) \end{array}\right], \end{aligned} \tag{2.13}$$

$$\overline{S}_{\overline{XY}} + \overline{\lambda}_1 \left[\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}} \right) \overline{S}_{\overline{XY}} - 2\frac{\partial \overline{V}}{\partial \overline{X}}\overline{S}_{\overline{XX}} - 2\frac{\partial \overline{U}}{\partial \overline{Y}}\overline{S}_{\overline{YY}} \right. \tag{2.14}$$
$$\left. + \overline{\lambda}_3(\overline{S}_{\overline{XX}} + \overline{S}_{\overline{YY}})(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) \right]$$
$$= \mu(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) + \mu\overline{\lambda}_2 \left[\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}} \right) (\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) \right.$$
$$\left. -2\left(\frac{\partial \overline{U}}{\partial \overline{X}}\frac{\partial \overline{V}}{\partial \overline{X}} + \frac{\partial \overline{U}}{\partial \overline{Y}}\frac{\partial \overline{V}}{\partial \overline{Y}} \right) \right],$$

$$\overline{S}_{\overline{YY}} + \overline{\lambda}_1 \left[\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}} \right) \overline{S}_{\overline{YY}} - 2\frac{\partial \overline{V}}{\partial \overline{X}}\overline{S}_{\overline{XX}} - 2\frac{\partial \overline{V}}{\partial \overline{Y}}\overline{S}_{\overline{YY}} \right] \tag{2.15}$$
$$+ 2\overline{\lambda}_3(\overline{S}_{\overline{XX}} + \overline{S}_{\overline{YY}})\frac{\partial \overline{V}}{\partial \overline{Y}}$$
$$= 2\mu\frac{\partial \overline{V}}{\partial \overline{Y}} + 2\mu\overline{\lambda}_2 \left[\begin{array}{c} \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}} \right) \frac{\partial \overline{V}}{\partial \overline{Y}} - 2(\frac{\partial \overline{V}}{\partial \overline{Y}})^2 \\ -\frac{\partial \overline{V}}{\partial \overline{X}}(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}) \end{array} \right].$$

In the laboratory frame $(\overline{X}, \overline{Y})$, the flow in a channel is unsteady. However, it can be treated as steady in a coordinates system $(\overline{x}, \overline{y})$ moving at the wave speed (wave frame). The transformations relating coordinates and velocities in two frames are given by

$$\overline{x} = \overline{X} - c\overline{t}, \qquad \overline{y} = \overline{Y}, \qquad \overline{u} = \overline{U} - c, \qquad \overline{v} = \overline{V}, \tag{2.16}$$

where $\overline{u}$ and $\overline{v}$ are respectively the dimensional velocity components parallel to $\overline{x}$ and $\overline{y}$ in the wave frame. With the help of Eq. (2.16), Eqs. (2.3)-(2.5) and (2.13)-(2.15) become

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \tag{2.17}$$

$$\rho(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}})\overline{u} = -\frac{\partial \overline{p}}{\partial \overline{x}} + \frac{\partial \overline{S}_{\overline{xx}}}{\partial \overline{x}} + \frac{\partial \overline{S}_{\overline{xy}}}{\partial \overline{y}}, \tag{2.18}$$

$$\rho(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}})\overline{v} = -\frac{\partial \overline{p}}{\partial \overline{y}} + \frac{\partial \overline{S}_{\overline{xy}}}{\partial \overline{x}} + \frac{\partial \overline{S}_{\overline{yy}}}{\partial \overline{y}}, \tag{2.19}$$

$$\overline{S}_{\overline{xx}} + \overline{\lambda}_1 \left[\left(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}} \right) \overline{S}_{\overline{xx}} - 2\frac{\partial \overline{u}}{\partial \overline{x}}\overline{S}_{\overline{xx}} - 2\frac{\partial \overline{u}}{\partial \overline{y}}\overline{S}_{\overline{xy}} \right] + 2\overline{\lambda}_3(\overline{S}_{\overline{xx}} + \overline{S}_{\overline{yy}})\frac{\partial \overline{u}}{\partial \overline{x}} \tag{2.20}$$
$$= 2\mu\frac{\partial \overline{u}}{\partial \overline{x}} + 2\mu\overline{\lambda}_2 \left[\left(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}} \right) \frac{\partial \overline{u}}{\partial \overline{x}} - 2(\frac{\partial \overline{u}}{\partial \overline{x}})^2 - \frac{\partial \overline{u}}{\partial \overline{y}}(\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}) \right],$$

$$\overline{S}_{\overline{x}\overline{y}} + \overline{\lambda}_1 \left[\left(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}} \right) \overline{S}_{\overline{x}\overline{y}} - 2\frac{\partial \overline{v}}{\partial \overline{x}}\overline{S}_{\overline{x}\overline{x}} - 2\frac{\partial \overline{u}}{\partial \overline{y}}\overline{S}_{\overline{y}\overline{y}} + \overline{\lambda}_3(\overline{S}_{\overline{x}\overline{x}} + \overline{S}_{\overline{y}\overline{y}})(\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}) \right] = \mu(\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}) + \mu\overline{\lambda}_2 \left[\left(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}} \right) (\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}) - 2\left(\frac{\partial \overline{u}}{\partial \overline{x}}\frac{\partial \overline{v}}{\partial \overline{x}} + \frac{\partial \overline{u}}{\partial \overline{y}}\frac{\partial \overline{v}}{\partial \overline{y}} \right) \right], \quad (2.21)$$

$$\overline{S}_{\overline{y}\overline{y}} + \overline{\lambda}_1 \left[\left(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}} \right) \overline{S}_{\overline{y}\overline{y}} - 2\frac{\partial \overline{v}}{\partial \overline{x}}\overline{S}_{\overline{x}\overline{y}} - 2\frac{\partial \overline{v}}{\partial \overline{y}}\overline{S}_{\overline{y}\overline{y}} \right] + 2\overline{\lambda}_3(\overline{S}_{\overline{x}\overline{x}} + \overline{S}_{\overline{y}\overline{y}})\frac{\partial \overline{v}}{\partial \overline{y}} = 2\mu\frac{\partial \overline{v}}{\partial \overline{y}} + 2\mu\overline{\lambda}_2 \left[\left(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}} \right) \frac{\partial \overline{v}}{\partial \overline{y}} - 2(\frac{\partial \overline{v}}{\partial \overline{y}})^2 - \frac{\partial \overline{v}}{\partial \overline{x}}(\frac{\partial \overline{v}}{\partial \overline{x}} + \frac{\partial \overline{u}}{\partial \overline{y}}) \right]. \quad (2.22)$$

In order to non-dimensionlize the governing Eqs. (2.17)-(2.22), we introduce the following variables and parameters.

$$\overline{x} = \frac{\lambda x}{2\pi}, \quad \overline{y} = ay, \quad \overline{u} = cu, \quad \overline{v} = cv, \quad \mathrm{Re} = \frac{\rho c a}{\mu}, \quad (2.23)$$
$$\overline{S} = \frac{\mu c}{a}S, \quad \overline{p} = \frac{\lambda \mu c}{2\pi a^2}p, \quad \overline{h} = ah, \quad \delta = \frac{2\pi a}{\lambda},$$
$$\lambda_1 = \frac{\overline{\lambda_1} c}{a}, \quad \lambda_2 = \frac{\overline{\lambda_2} c}{a}, \quad \lambda_3 = \frac{\overline{\lambda_3} c}{a}.$$

Using these variables and parameters and defining the stream function by the relation

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\delta\frac{\partial \psi}{\partial x}, \quad (2.24)$$

the continuity equation (2.17) is identically satisfied and the Eqs. (2.18)-(2.22) take the following form

$$\delta\,\mathrm{Re}\left[\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} \right) \frac{\partial \psi}{\partial y} \right] = -\frac{\partial p}{\partial x} + \delta\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y}, \quad (2.25)$$

$$-\delta^3\,\mathrm{Re}\left[\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} \right) \frac{\partial \psi}{\partial x} \right] = -\frac{\partial p}{\partial x} + \delta^2\frac{\partial S_{xy}}{\partial x} + \delta\frac{\partial S_{yy}}{\partial y}, \quad (2.26)$$

$$\delta\,\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi}{\partial y^2}+\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right]=\delta\frac{\partial^2\left(S_{xx}-S_{yy}\right)}{\partial x\partial y}+\left(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial^2}{\partial x^2}\right)S_{xy}, \quad (2.27)$$

$$\begin{aligned} & S_{xx}+\lambda_1\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)S_{xx}-2\delta\frac{\partial^2\psi}{\partial x\partial y}S_{xx}-2\frac{\partial^2\psi}{\partial y^2}S_{xy}\right] \qquad (2.28)\\ & \quad +2\delta\lambda_3\left(S_{xx}+S_{yy}\right)\frac{\partial^2\psi}{\partial x\partial y}\\ = \; & 2\delta\frac{\partial^2\psi}{\partial x\partial y}+2\lambda_2\left[\delta^2\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\psi}{\partial x\partial y}-2\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right.\\ & \left.\quad -\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial^2}{\partial x^2}\right)\right], \end{aligned}$$

$$\begin{aligned} & S_{xy}+\lambda_1\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)S_{xy}+\delta^2\frac{\partial^2\psi}{\partial x^2}S_{xx}-\frac{\partial^2\psi}{\partial y^2}S_{yy}\right] \qquad (2.29)\\ & \quad +\lambda_3\left(S_{xx}+S_{yy}\right)\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\\ = \; & \left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)+\lambda_2\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right.\\ & \left.\quad +2\delta\left(\delta^2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\psi}{\partial x\partial y}\right)\right], \end{aligned}$$

$$\begin{aligned} & S_{yy}+\lambda_1\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)S_{yy}+2\delta^2\frac{\partial^2\psi}{\partial x^2}S_{xy}+2\delta\frac{\partial^2\psi}{\partial x\partial y}S_{yy}\right] \qquad (2.30)\\ & \quad -2\delta\lambda_3\left(S_{xx}+S_{yy}\right)\frac{\partial^2\psi}{\partial x\partial y}\\ = \; & -2\delta\frac{\partial^2\psi}{\partial x\partial y}+2\lambda_2\left[-\delta^2\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\psi}{\partial x\partial y}-2\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right.\\ & \left.\quad +\delta^2\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right]. \end{aligned}$$

In above equations Re is the Reynolds number, δ is the Wave number and λ_{1-3} are the Wessenberg numbers.

It is formidable task to solve the Eqs. (2.25)-(2.30) in their current form. Fortunately, many physiological processes, where peristalsis is involved, the wavelength of the wave is large as compared to the radius of the vessel or organ. This assumption amount to assume that δ

$\simeq 0$ and known in the literature as long wavelength approximation. Therefore, under the long wavelength and low Reynolds number assumptions, Eqs. (2.25)-(2.30) reduces to

$$\frac{\partial S_{xy}}{\partial y} = \frac{\partial p}{\partial x}, \tag{2.31}$$

$$\frac{\partial p}{\partial y} = 0, \tag{2.32}$$

$$\frac{\partial^2 S_{xy}}{\partial y^2} = 0, \tag{2.33}$$

$$S_{xx} - 2\lambda_1 \frac{\partial^2 \psi}{\partial y^2} S_{xy} = -2\lambda_2 \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2, \tag{2.34}$$

$$S_{xy} + \lambda_3 S_{xx} \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial y^2}, \tag{2.35}$$

$$S_{yy} = 0. \tag{2.36}$$

Solving Eq. (2.34) and Eq. (2.35) for S_{xy} we get

$$S_{xy} = \left\{ \frac{1 + 2\alpha_1 \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2}{1 + 2\alpha_2 \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2} \right\} \frac{\partial^2 \psi}{\partial y^2}, \tag{2.37}$$

where $\alpha_1 = \lambda_2 \lambda_3$, $\alpha_2 = \lambda_1 \lambda_3$. It follows from (2.32) that $p \neq p(y)$, This implies that $p = p(x)$. By substituting the value of S_{xy} from Eq. (2.37) into Eq. (2.31) and Eq. (2.33), we have

$$\frac{\partial}{\partial y} \left[\left\{ \frac{1 + 2\alpha_1 \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2}{1 + 2\alpha_2 \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2} \right\} \frac{\partial^2 \psi}{\partial y^2} \right] = \frac{dp}{dx}, \tag{2.38}$$

$$\frac{\partial^2}{\partial y^2} \left[\left\{ \frac{1 + 2\alpha_1 \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2}{1 + 2\alpha_2 \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2} \right\} \frac{\partial^2 \psi}{\partial y^2} \right] = 0. \tag{2.39}$$

We observe that when $\alpha_1 = \alpha_2$, Eqs. (2.38) and (2.39) reduce to corresponding equations of the viscous fluid [2]. The dimensionless pressure rise per one wavelength can be calculated via following expression

$$\Delta p = \int_0^{2\pi} \frac{dp}{dx} dx. \tag{2.40}$$

Due to the symmetric nature of the solution of the flow problem is obtained only in the half flow domain $y \in [0, h]$.

2.3 Rate of volume flow and boundary conditions

The instantaneous volume flow rate in the fixed frame is given by

$$Q = \int_0^{\overline{h}} \overline{U}(\overline{X}, \overline{Y}, \overline{t}) d\overline{Y}, \tag{2.41}$$

where $\overline{h}$ is a function of $\overline{X}$ and $\overline{t}$.

The rate of volume flow in the wave frame is given by

$$q = \int_0^{\overline{h}} \overline{u}(\overline{x}, \overline{y}) d\overline{y}, \tag{2.42}$$

where $\overline{h}$ is a function of $\overline{x}$ alone. If we substitute Eq. (2.16) into Eq. (2.41) and make use of Eq. (2.23), we find that the two rates of volume flow are related by

$$Q = q + c\overline{h}. \tag{2.43}$$

The time mean flow over a period T at a fixed position $\overline{X}$ is defined as

$$\overline{Q} = \frac{1}{T} \int_0^T Q dt. \tag{2.44}$$

Substituting Eq. (2.43) into Eq. (2.44) and integrating we get

$$\overline{Q} = q + ac. \tag{2.45}$$

On defining the dimensionless time mean flows Θ and F respectively in the fixed and wave frame as

$$\Theta = \frac{\overline{Q}}{ac}, \text{ and } F = \frac{\overline{F}}{ac}, \tag{2.46}$$

one finds that Eq. (2.45) can be written as

$$\Theta = F + 1, \tag{2.47}$$

where

$$F = \int_0^h \frac{\partial \psi}{\partial y} dy = \psi(h) - \psi(0) \tag{2.48}$$

and h represents the dimensionless form of the surface of the peristaltic wall, i.e.

$$h(x) = 1 + \phi \cos x. \tag{2.49}$$

Here $\phi = b/a$ is the amplitude ratio or the occlusion.

If we select the zero value of the streamline at the centerline $(y = 0)$. we have

$$\psi(0) = 0, \tag{2.50}$$

then the wall $(y = h)$ is a streamline of value

$$\psi(h) = F. \tag{2.51}$$

The appropriate boundary conditions for the dimensionless stream function in the wave frame are

$$\begin{aligned} \psi &= 0, \quad \frac{\partial^2 \psi}{\partial y^2} = 0, \quad \text{at } y = 0, \\ \psi &= F, \quad \frac{\partial \psi}{\partial y} = -1, \quad \text{at } y = 1 + \phi \cos x. \end{aligned} \tag{2.52}$$

2.4 Numerical method

In this section we proceed to find direct numerical solution of the differential Eq. (2.39) subject to boundary conditions (2.52) by means of a suitable numerical technique. The differential Eq. (2.39) is nonlinear in ψ and cannot be solved by the direct finite difference method. In solving such a nonlinear equations, iterative methods are commonly used. We can now construct an iterative procedure in the following form

$$\left(\frac{1+2\alpha_1\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}\right)\frac{\partial^4\psi^{(n+1)}}{\partial y^4}+2\frac{\partial}{\partial y}\left(\frac{1+2\alpha_1\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}\right)\frac{\partial^3\psi^{(n+1)}}{\partial y^3} \tag{2.53}$$
$$+\frac{\partial^2}{\partial y^2}\left(\frac{1+2\alpha_1\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}\right)\frac{\partial^2\psi^{(n+1)}}{\partial y^2}=0,$$

$$\begin{aligned}\psi=0,\quad &\frac{\partial^2\psi^{(n+1)}}{\partial y^2}=0,\quad \text{at } y=0,\\ \psi=F,\quad &\frac{\partial\psi^{(n+1)}}{\partial y}=-1,\quad \text{at } y=h,\end{aligned} \tag{2.54}$$

where the index (n) indicates the iterative step. It is easy to confirm that if the indices (n) and $(n+1)$ are withdrawn, the Eq. (2.53) is consistent with the original differential Eq. (2.39). Equation (2.53) and the boundary conditions (2.54) define a linear differential boundary value problem for $\psi^{(n+1)}$. By means of the finite difference method a linear algebraic equation system can be deduced and solved for each iterative step $(n+1)$. Therefore, a sequence of functions $\psi^{(0)}(x,y),\psi^{(1)}(x,y),\psi^{(2)}(x,y),...$ is determined in the following manner: if an initial estimated $\psi^{(0)}(x,y)$ is given, then $\psi^{(1)}(x,y),\psi^{(2)}(x,y),...$ are calculated successively as the solutions of the boundary value problem (2.53) and (2.54). Unfortunately, such an iteration is often divergent, especially when the initial estimated $\psi^{(0)}(x,y)$ is not given carefully and suitably. Usually, in order to achieve a better convergence, the so called method of successive under-relaxtion is used. We solve the boundary value problem (2.53) and (2.54) for the iterative step $(n+1)$ to obtain an estimated value of $\psi^{(n+1)}$ and $\widetilde{\psi}^{(n+1)}$, then $\psi^{(n+1)}$ is defined by the formula

$$\psi^{(n+1)}=\psi^{(n)}+\tau^*\left(\widetilde{\psi}^{(n+1)}-\psi^{(n)}\right),\quad \tau^*\in(0,1] \tag{2.55}$$

where τ^* is under-relaxation parameter. We should choose τ^* so small that convergent iteration is reached. In our simulation we choose an initial guess of $\psi^{(0)}(x,y)=Fy/h$ which fulfils the first and third boundary conditions in (2.54). Of course, some other choices are also possible. The iteration should be carried out until the relative differences of the computed $\psi^{(n+1)}$ and

$\psi^{(n)}$ between two iterative steps are smaller than a given error chosen to be 10^{-8}.

2.5 Results and discussion

The governing equations involves two non-Newtonian parameters α_1 and α_2. In this section we analyze the influence of these parameters on different flow features of the investigated peristaltic motion. A closer examination of Eq. (2.38) shows that the coefficient of $\partial^2\psi\ /\partial y^2$ in the braces equals unity when $\alpha_1 = \alpha_2$. The case of $\alpha_1 > \alpha_2$ $(\lambda_2 > \lambda_1)$ is physically unacceptable since the retardation time cannot be larger than the relaxation time. When $\alpha_1 < \alpha_2$ the coefficient in the braces of Eq. (2.38) lies between zero and one. So we expect that longitudinal pressure gradient for an Oldroyd 4-consatnt fluid should be less in magnitude as compared to a Newtonian fluid. In the forthcoming discussion we will see that the numerical results will confirm our observation.

Fig. 2.1(a) is made to see the effects of α_1 and α_2 on longitudinal velocity u plotted against y at a fixed position $x = -\pi$.

We observe that for the small values of α_1 strong non-Newtonian effects near the boundary, i.e. a thin boundary layer is formed. However, as we move away from the boundary the velocity profile becomes linear. When $\alpha_1 \to \alpha_2$ the boundary layer becomes thick and the velocity profile tends to become Newtonian. The magnitude of velocity near the centerline increases as α_1 decreases from 0.5 (the value of α_2) to 0. It means that the magnitude of the velocity for an Oldroyd 4-constant fluid is greater in comparison to that of a Newtonian fluid near the centre of the channel. This is due to the smaller (flow-dependent) viscosity of the Oldroyd 4-constant fluid.

Fig. 2.1(b) presents the variation of longitudinal velocity u with y for various values of α_2. This figure reveals that with the increase of α_2, or more strictly speaking, with the deviation of α_2 from α_1, the profile near the centerline becomes flatter, similar to flug-like flow i.e. there is no rapid change in velocity near the centerline for an Oldroyd 4-constant fluid as compared to a Newtonian fluid. The influence of α_1 on the stream function ψ against y is shown in Fig. 2.1(c). It can be seen from Fig. 2.1(c) that the profile of ψ for $\alpha_1 = 0$ and 0.01 are nearly linear over the whole width of the channel. However, the gradient changes slightly near the boundary. As α_1 increases from 0.05 to 0.5 the profile become nonlinear and their gradient increases from negative near the wall to positive near the centerline. Further, these profiles do nearly coincide near the boundary of the channel. Similarly, from Fig. 2.1(d) it is seen that an increase in α_2 changes the profiles of ψ near the centre of the channel and the effects are not so obvious near the boundary wall.

The variation of longitudinal pressure gradient dp/dx over one wavelength for different values of α_1 and α_2 and for two values of Θ, $\Theta = 0.8$ ($F = -0.2$) and $\Theta = 0.2$ ($F = -0.8$), can be seen in Fig. 2.2.

It is observed that dp/dx decreases as α_1/α_2 deviates from unity i.e. the longitudinal pressure gradient for a Newtonian fluid is greater in magnitude as compared to that of an Oldroyd 4-constant fluid. This observation confirms our expectation, as mentioned in the beginning of this section. Further, we note that dp/dx is negative over the whole wavelength for $\Theta = 0.8$ both for Newtonian and Oldroyd 4-constant fluid. It means that for larger values of Θ the longitudinal pressure gradient assists the flow over the whole wavelength (both in narrow and wider parts of the channel). When Θ equals 0.2 the longitudinal pressure gradient is positive near the cross section $x = \pm\pi$, whilst it is negative near $x = 0$. It means that in the narrow part of the channel the pressure gradient resists the flow while it assists the flow in the wider part of the channel. It is also noted that the magnitude of resistance or assistance from the pressure gradient is less for an Oldroyd 4-constant fluid as compared to a Newtonian fluid.

One of the features of peristalsis is pumping against the pressure rise. To demonstrate the effects of α_1 and α_2 on pumping against pressure rise Δp, we have plotted Fig.2.3.

The maximum pressure rise against which the peristalsis works as a pump i.e. Δp for $\Theta = 0$ is denoted by P_0. When $\Delta p > P_0$ then there is a negative flux. The value of Θ corresponding to $\Delta p = 0$ (which is known as free pumping) is denoted by Θ_0. When $\Delta p < P_0$, the pressure assists the flow and this is known as co-pumping. Figure 2.3(a) and (b) illustrates the relation between pressure rise per wavelength Δp and flow rate Θ for various values of α_1 and α_2, respectively. It is observed that P_0 increases by increasing α_1 (Fig. 2.3(a)). However, it decreases with increasing α_2 (Fig. 2.3 (b)). This means that peristalsis has to work against greater pressure rise for a Newtonian fluid in comparison with an Oldroyd 4-constant fluid. Further, the peristaltic pumping rate and free pumping rate decrease when the fluid deviates from Newtonian fluid to an Oldroyd 4-constant fluid. In the case of co-pumping, for small values of Δp, the pumping rate decreases from Newtonian to the Oldroyd 4-constant fluid. However, for large values of Δp the pumping rate for the Oldroyd 4-constant fluid is greater than for the Newtonian fluid.

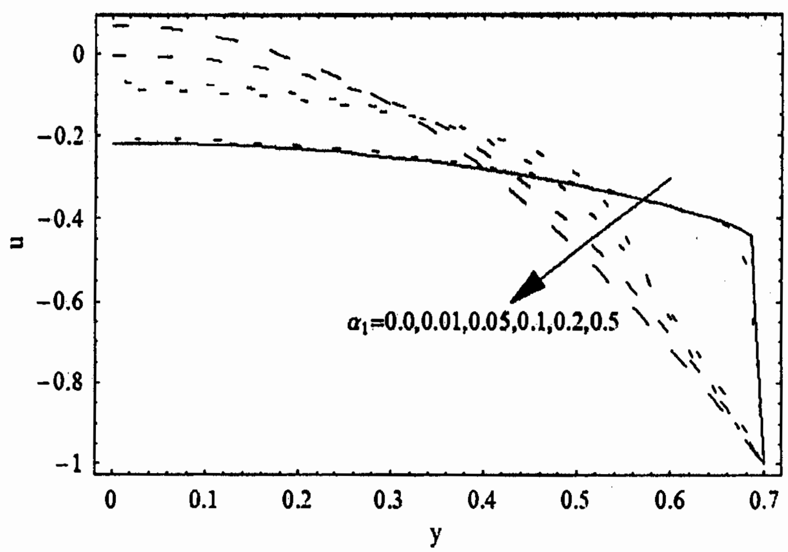

Fig. 2.1(a): Plot of the longitudinal velocity u for various values of $\alpha_1(\alpha_2 = 0.5)$. The other parameters chosen are $F = -0.2$ and $\phi = 0.3$.

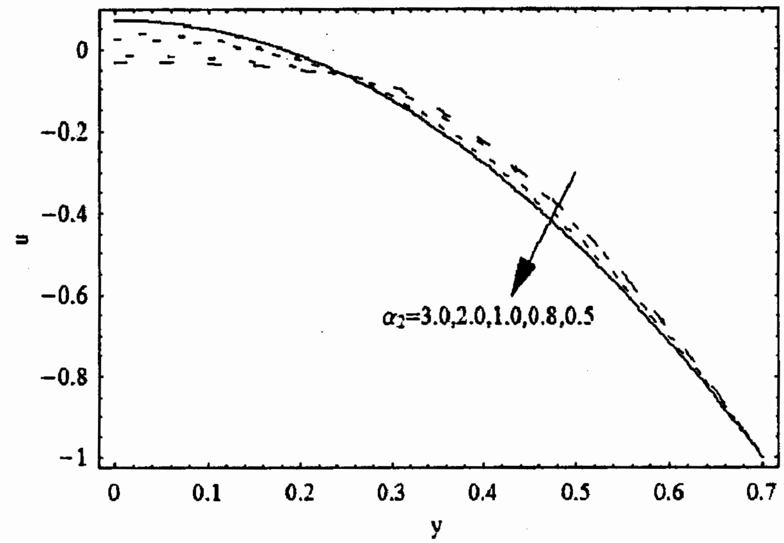

Fig. 2.1(b): Plot of the longitudinal velocity u of $\alpha_2(\alpha_1 = 0.5)$. The other parameters chosen are $F = -0.2$ and $\phi = 0.3$.

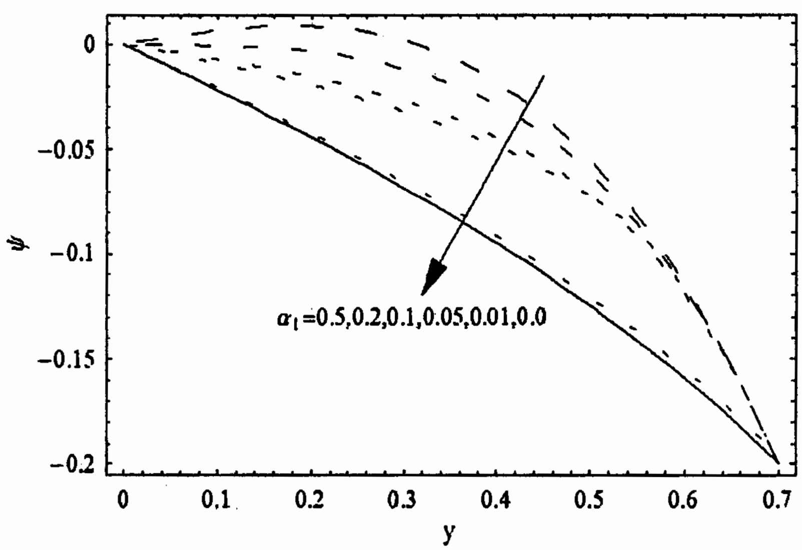

Fig. 2.1(c): Plot of the stream function ψ for various values of $\alpha_1(\alpha_2 = 0.5)$. The other parameters chosen are $F = -0.2$ and $\phi = 0.3$.

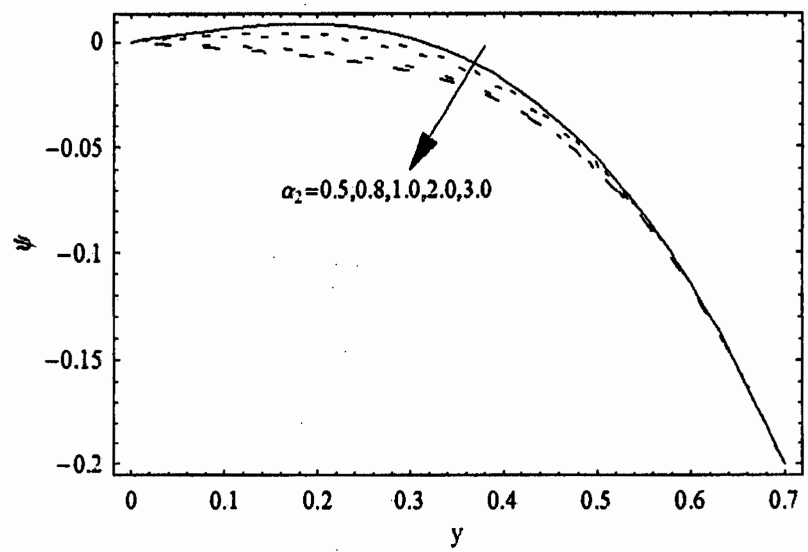

Fig. 2.1(d): Plot of the stream function ψ for various values of $\alpha_2 (\alpha_1 = 0.5)$. The other parameters chosen are $F = -0.2$ and $\phi = 0.3$.

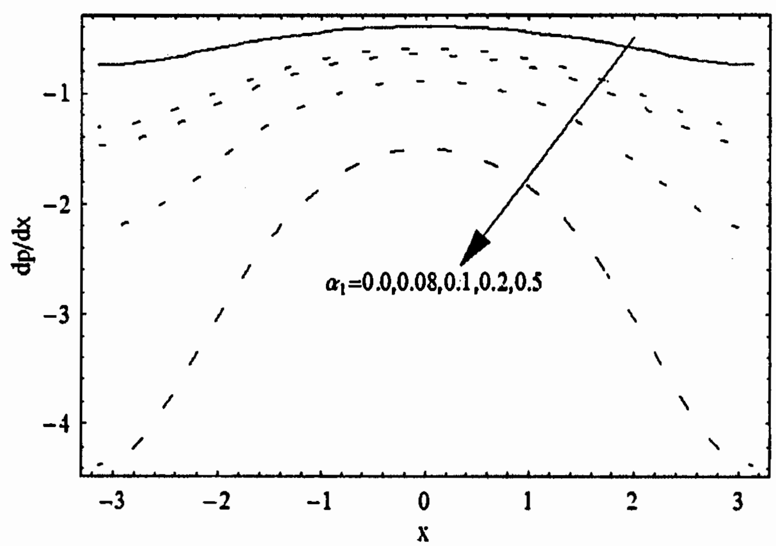

Fig. 2.2(a). Plot of the longitudinal pressure gradient dp/dx for various values of $\alpha_1 \, (\alpha_2 = 0.5)$. The other parameters chosen are $\Theta = 0.8$ and $\phi = 0.3$.

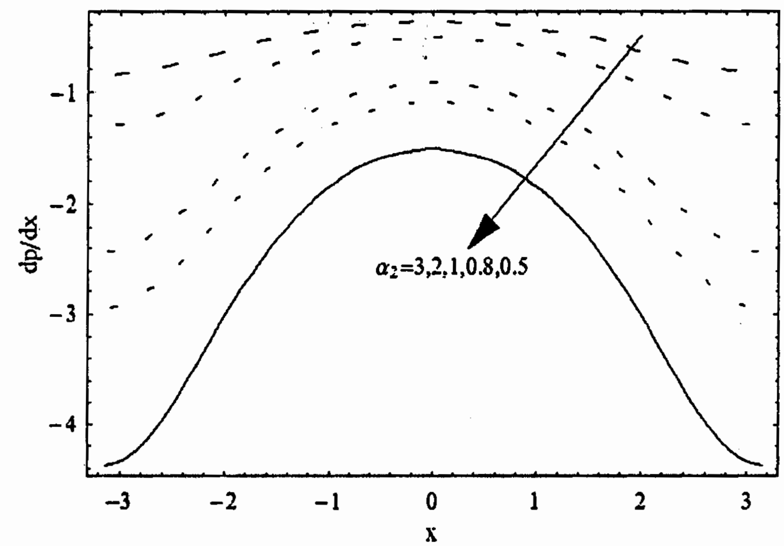

Fig. 2.2(b). Plot of the longitudinal pressure gradient dp/dx for various values of $\alpha_2\,(\alpha_1 = 0.5)$. The other parameters chosen are $\Theta = 0.8$ and $\phi = 0.3$.

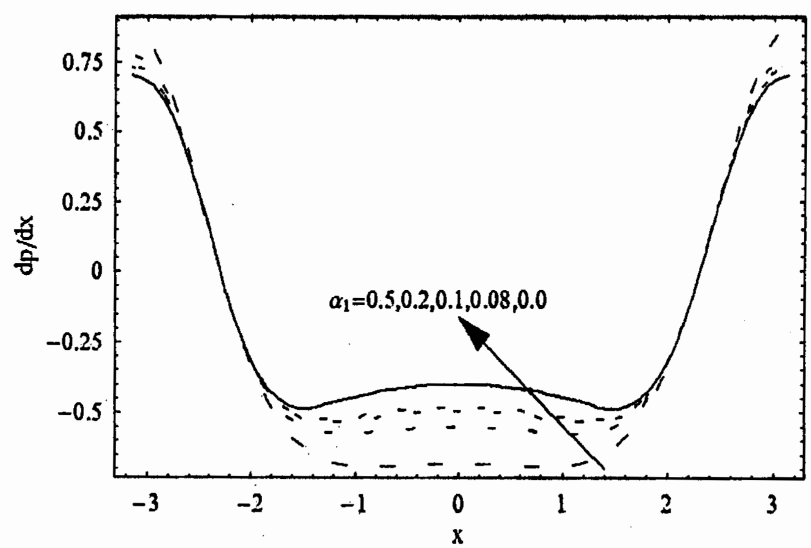

Fig. 2.2(c). Plot of the longitudinal pressure gradient dp/dx for various values of $\alpha_1\,(\alpha_2 = 0.5)$. The other parameters chosen are $\Theta = 0.2$ and $\phi = 0.3$.

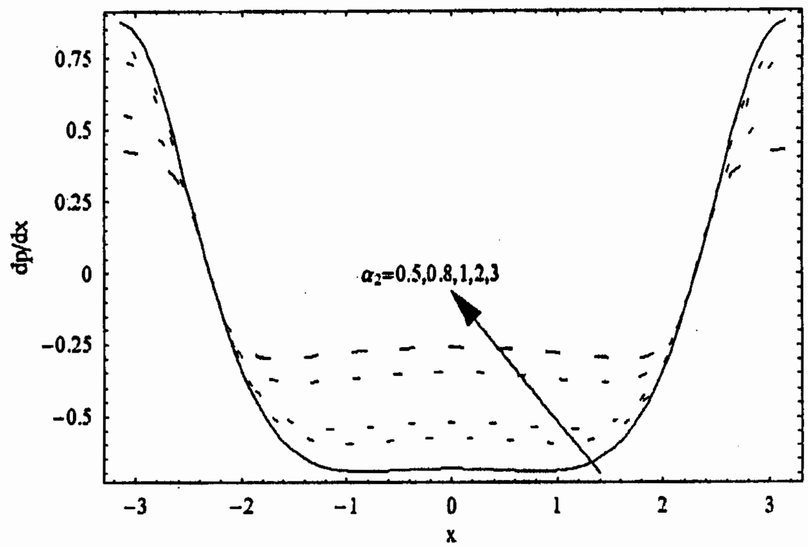

Fig. 2.2(d). Plot of the longitudinal pressure gradient dp/dx for various values of $\alpha_2\,(\alpha_1 = 0.5)$. The other parameters chosen are $\Theta = 0.2$ and $\phi = 0.3$.

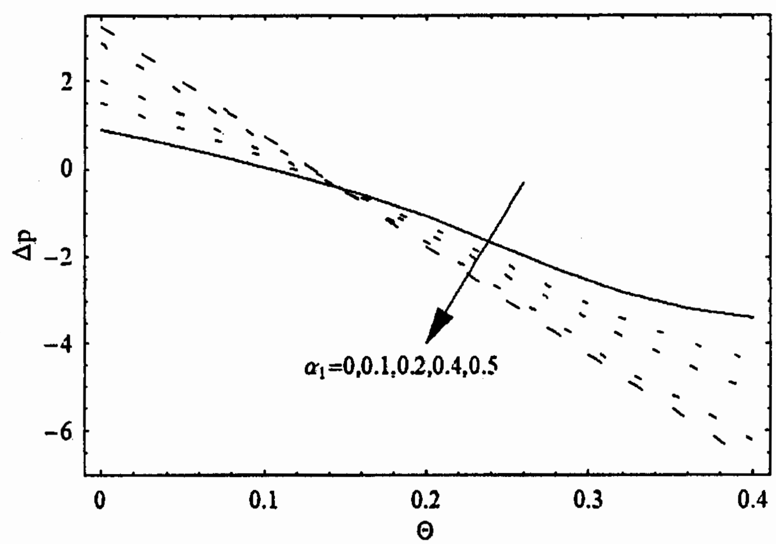

Fig. 2.3(a). Profile of pressure rise per wavelength Δp versus flow rate Θ for various values of $\alpha_1(\alpha_2 = 0.5)$ and $\phi = 0.3$.

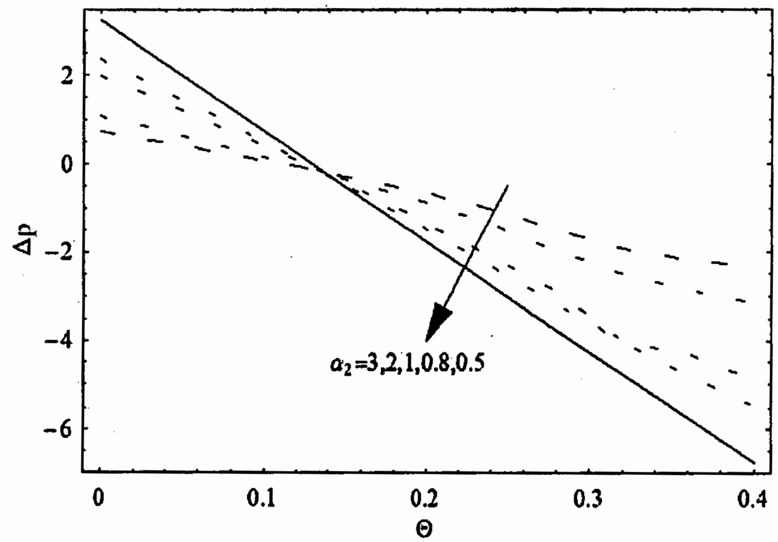

Fig. 2.3(b). Profile of pressure rise per wavelength Δp versus flow rate Θ for various values of $\alpha_2(\alpha_1 = 0.5)$ and $\phi = 0.3$.

Chapter 3

Peristaltic flow of a magnetohydrodynamic Oldroyd 4-constant fluid in a planar channel

3.1 Introduction

The purpose of this chapter is to generalize the flow problem considered in chapter two for magnetohydrodynamic fluid. The problem is first formulated in the form of a differential equation by taking into account the effects of magnetic field and then solved numerically for various values of material parameters. Here, in addition to the non-Newtonian parameters, non-dimensional Hartman number also comes into play. The effects of Hartman number on longitudinal velocity, stream function, longitudinal pressure gradient and pressure rise per wavelength are discussed with the help of graphs.

3.2 Formulation of the problem

The geometry of the problem is same as considered in chapter 2. However, the fluid considered here is electrically conducting. A uniform magnetic field $\mathbf{B_0}$ is applied perpendicular to the flow. The total magnetic field is

$$\overline{\mathbf{B}} = \overline{\mathbf{B}}_0 + \overline{\mathbf{b}}, \tag{3.1}$$

where $\overline{b}$ is induced magnetic field. However, under the assumption of small magnetic Reynold number the induced magnetic field can be neglected.

The governing equations, taking into account the effect of magnetic field in the laboratory frame are

$$\frac{\partial \overline{U}}{\partial \overline{X}} + \frac{\partial \overline{V}}{\partial \overline{Y}} = 0, \tag{3.2}$$

$$\rho(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{U} = -\frac{\partial \overline{p}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XX}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XY}}}{\partial \overline{Y}} + (\mathbf{J}\times\mathbf{B})_{\overline{X}}, \tag{3.3}$$

$$\rho(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{V} = -\frac{\partial \overline{p}}{\partial \overline{Y}} + \frac{\partial \overline{S}_{\overline{XY}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{YY}}}{\partial \overline{Y}} + (\mathbf{J}\times\mathbf{B})_{\overline{Y}}. \tag{3.4}$$

To calculate the additional term appearing in Eqs. (3.3) and (3.4), we make use of Maxwell equations with generalized Ohm's law given in chapter 1. Using Eqs. (1.25) and (1.30)-(1.33) we find that

$$\mathbf{J}\times\mathbf{B} = [-\sigma B_0^2\overline{U}, -\sigma B_0^2\overline{V}]. \tag{3.5}$$

Thus, the governing Eqs. (3.2)-(3.4) become

$$\frac{\partial \overline{U}}{\partial \overline{X}} + \frac{\partial \overline{V}}{\partial \overline{Y}} = 0, \tag{3.6}$$

$$\rho(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{U} = -\frac{\partial \overline{p}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XX}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XY}}}{\partial \overline{Y}} - \sigma B_0^2\overline{U}, \tag{3.7}$$

$$\rho(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}})\overline{V} = -\frac{\partial \overline{p}}{\partial \overline{Y}} + \frac{\partial \overline{S}_{\overline{XY}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{YY}}}{\partial \overline{Y}} - \sigma B_0^2\overline{V}. \tag{3.8}$$

Upon making use of the transformation (2.16), Eqs. (3.6)-(3.8) can be casted in the wave frame as

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \tag{3.9}$$

$$\rho(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}})\overline{u} = -\frac{\partial \overline{p}}{\partial \overline{x}} + \frac{\partial \overline{S}_{\overline{xx}}}{\partial \overline{x}} + \frac{\partial \overline{S}_{\overline{xy}}}{\partial \overline{y}} - \sigma B_0^2(\overline{u} + c), \tag{3.10}$$

$$\rho(\overline{u}\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{y}})\overline{v} = -\frac{\partial \overline{p}}{\partial \overline{y}} + \frac{\partial \overline{S}_{\overline{xy}}}{\partial \overline{x}} + \frac{\partial \overline{S}_{\overline{yy}}}{\partial \overline{y}} - \sigma B_0^2\overline{v}. \tag{3.11}$$

With the help of dimensionless parameters defined by Eq. (2.23) and stream function given by Eq. (2.24), we obtain the following dimensionless equations

$$\delta\,\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\psi}{\partial y}\right]=-\frac{\partial p}{\partial x}+\delta\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}-M^2\left(\frac{\partial\psi}{\partial y}+1\right), \tag{3.12}$$

$$-\delta^3\,\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\psi}{\partial x}\right]=-\frac{\partial p}{\partial y}+\delta^2\frac{\partial S_{xy}}{\partial x}+\delta\frac{\partial S_{yy}}{\partial y}-M^2c\delta^2\frac{\partial\psi}{\partial x}, \tag{3.13}$$

$$\begin{aligned}\delta\,\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi}{\partial y^2}+\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right]&=\delta\frac{\partial^2(S_{xx}-S_{yy})}{\partial x\partial y}+\left(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial^2}{\partial x^2}\right)S_{xy}\\&-M^2\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial\psi}{\partial y}+1\right)\right]+M^2c\delta\frac{\partial^2\psi}{\partial y^2}.\end{aligned} \tag{3.14}$$

Where the compatibility equation (3.14) is obtained by cross differentiating Eqs. (3.12) and (3.13) and then adding them. Here $M=\sqrt{\sigma B_0^2a^2/\mu}$ is the non-dimensional Hartman number.

Considering the long wavelength and low Reynolds number approximations, Eqs. (3.12)-(3.14) can be written in the following form

$$\frac{\partial S_{xy}}{\partial y}=\frac{\partial p}{\partial x}+M^2\left(\frac{\partial\psi}{\partial y}+1\right), \tag{3.15}$$

$$\frac{\partial p}{\partial y}=0, \tag{3.16}$$

$$\frac{\partial^2 S_{xy}}{\partial y^2}=-M^2\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial\psi}{\partial y}+1\right)\right]. \tag{3.17}$$

Substituting S_{xy} from Eq. (2.37) we get

$$\frac{\partial}{\partial y}\left[\left\{\frac{1+2\alpha_1\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}\right\}\frac{\partial^2\psi}{\partial y^2}\right]=\frac{dp}{dx}+M^2\left(\frac{\partial\psi}{\partial y}+1\right), \tag{3.18}$$

and

$$\frac{\partial^2}{\partial y^2}\left[\left\{\frac{1+2\alpha_1\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}\right\}\frac{\partial^2\psi}{\partial y^2}\right]-M^2\frac{\partial^2\psi}{\partial y^2}=0. \tag{3.19}$$

It is to be noted that when $M=0$, Eqs. (3.18) and (3.19) reduce to corresponding equations of hydrodynamic fluid.

The Eq. (3.19) is subjected to the same boundary conditions as given in chapter 2 i.e.

$$\begin{aligned} \psi &= 0, \quad \frac{\partial^2 \psi}{\partial y^2} = 0, \quad \text{at } y = 0, \\ \psi &= F, \quad \frac{\partial \psi}{\partial y} = -1, \quad \text{at } y = 1 + \phi \cos x. \end{aligned} \tag{3.20}$$

3.3 Solution methodology

The solution here is obtained by the same method as described briefly in chapter 2. The form of iterative procedure here is

$$\begin{aligned} &\left(\frac{1 + 2\alpha_1 \left(\frac{\partial^2 \psi^{(n)}}{\partial y^2}\right)^2}{1 + 2\alpha_2 \left(\frac{\partial^2 \psi^{(n)}}{\partial y^2}\right)^2} \right) \frac{\partial^4 \psi^{(n+1)}}{\partial y^4} + 2\frac{\partial}{\partial y} \left(\frac{1 + 2\alpha_1 \left(\frac{\partial^2 \psi^{(n)}}{\partial y^2}\right)^2}{1 + 2\alpha_2 \left(\frac{\partial^2 \psi^{(n)}}{\partial y^2}\right)^2} \right) \frac{\partial^3 \psi^{(n+1)}}{\partial y^3} \\ &+ \frac{\partial^2}{\partial y^2} \left(\frac{1 + 2\alpha_1 \left(\frac{\partial^2 \psi^{(n)}}{\partial y^2}\right)^2}{1 + 2\alpha_2 \left(\frac{\partial^2 \psi^{(n)}}{\partial y^2}\right)^2} \right) \frac{\partial^2 \psi^{(n+1)}}{\partial y^2} - M^2 \frac{\partial^2 \psi^{(n+1)}}{\partial y^2} = 0, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \psi &= 0, \quad \frac{\partial^2 \psi^{(n+1)}}{\partial y^2} = 0, \quad \text{at } y = 0, \\ \psi &= F, \quad \frac{\partial \psi^{(n+1)}}{\partial y} = -1, \quad \text{at } y = h. \end{aligned} \tag{3.22}$$

Where the index (n) indicates the iterative step.

3.4 Results and discussion

To see the effect of Hartman number on various features of the peristaltic motion we have plotted Figs. 3.1-3.4.

In Fig. 3.1(a) the longitudinal velocity u is plotted against y for different values of α_1 at a fixed position $x = -\pi$ with non-zero values of M.

Fig. 3.1(b) is made to see the effects of α_2 on u for MHD fluid. The profiles of stream function for MHD fluid for different values of α_1 and α_2 are shown in Figs. 3.1(c) and 3.1(d). We observe from these figures the similar behavior as observed for hydrodynamics fluid. However, Figs. 3.2(a)-(d) reveal some interesting results. These are summarized below.

• An increase in M increases the velocity near the boundary. However, near the centerline the situation is reversed.

• The values of stream function decreases in going from hydrodynamic to magnetohydrodynamic fluid.

The variation of longitudinal pressure gradient dp/dx over one wavelength for different values of M is shown in Figs. 3.3(a)-(d). In Figs. 3.3(a) and 3.3(b) $\Theta = 0.8$, while in Figs. 3.3(c) and 3.3(d) $\Theta = 0.2$. The following results are worth mentioning.

• The magnitude of longitudinal pressure gradient increases with an increase in M.

• The longitudinal pressure gradient resists/assist the flow in the narrow/wider of the channel for the small values of Θ. For the large values of Θ longitudinal pressure gradient become favorable over the whole width of the channel.

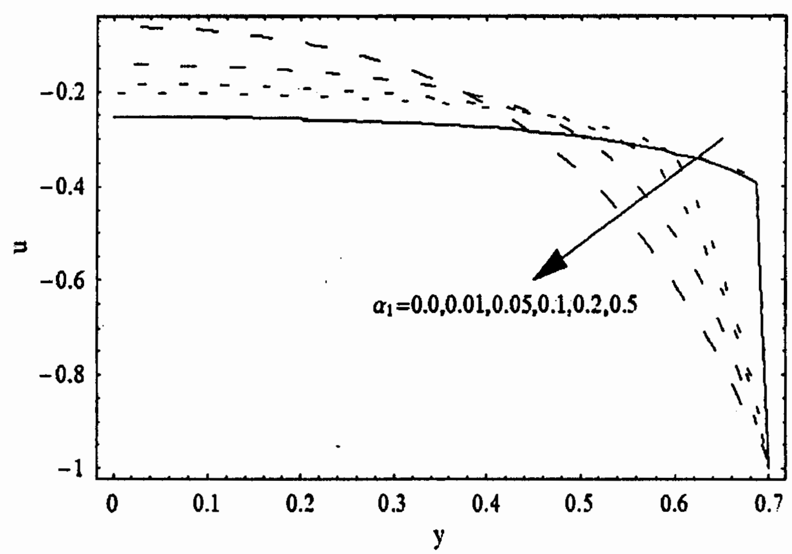

Fig. 3.1(a): Plot of the longitudinal velocity u for various values of $\alpha_1 (\alpha_2 = 0.5)$. The other parameters chosen are $M = 5$, $F = -0.2$ and $\phi = 0.3$.

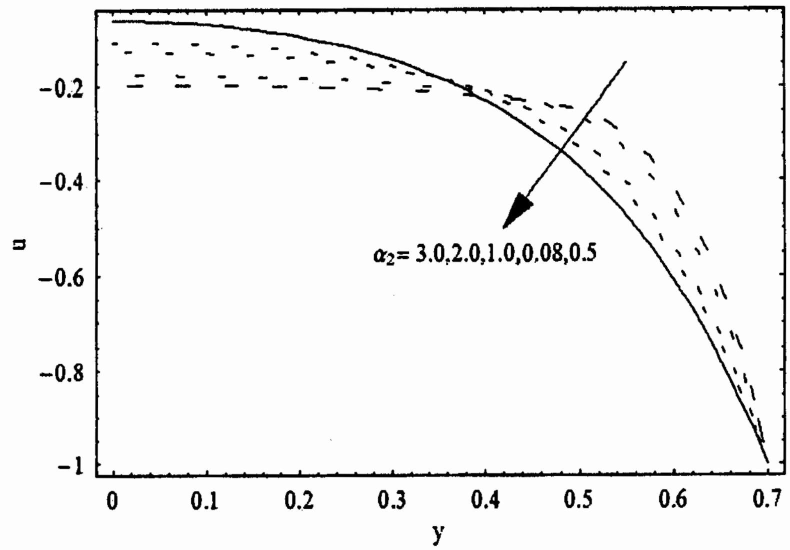

Fig. 3.1(b): Plot of the longitudinal velocity u for various values of $\alpha_2(\alpha_1 = 0.5)$. The other parameters chosen are $M = 5$, $F = -0.2$ and $\phi = 0.3$.

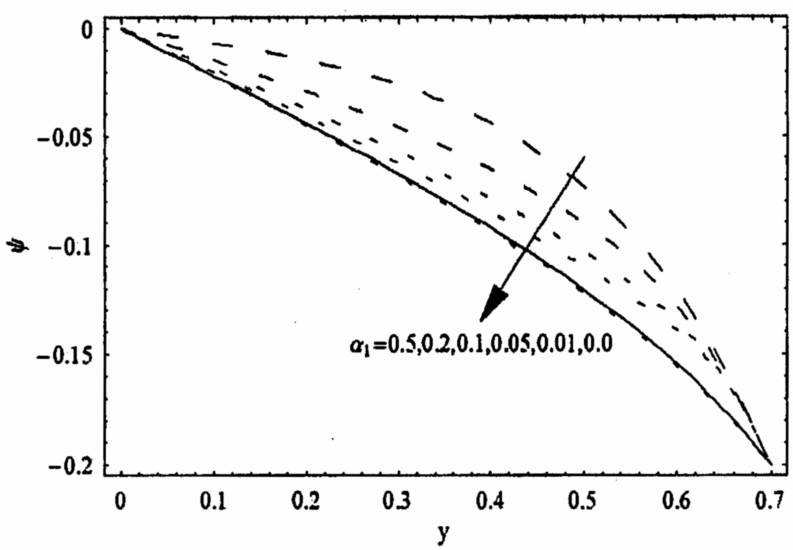

Fig. 3.1(c): Plot of the stream function ψ for various values of $\alpha_1(\alpha_2 = 0.5)$. The other parameters chosen are $M = 5, F = -0.2$ and $\phi = 0.3$.

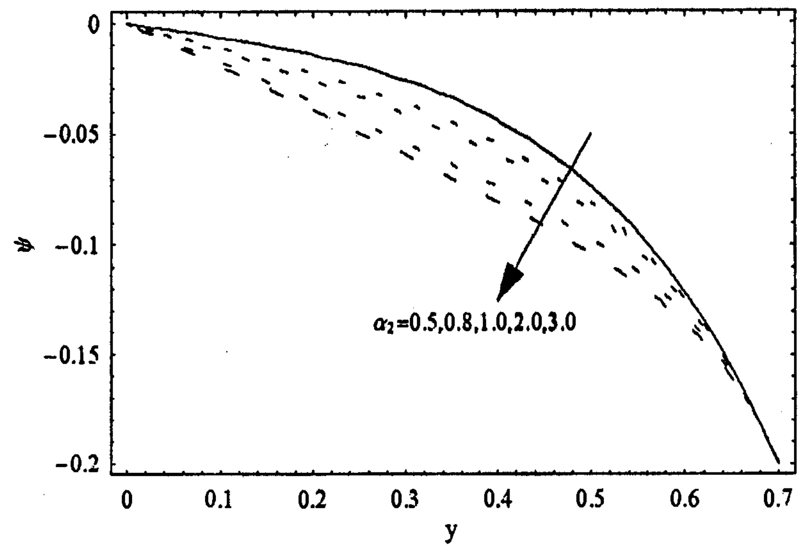

Fig. 3.1(d): Plot of the stream function ψ for various values of $\alpha_2(\alpha_1 = 0.5)$. The other parameters chosen are $M = 5$, $F = -0.2$ and $\phi = 0.3$.

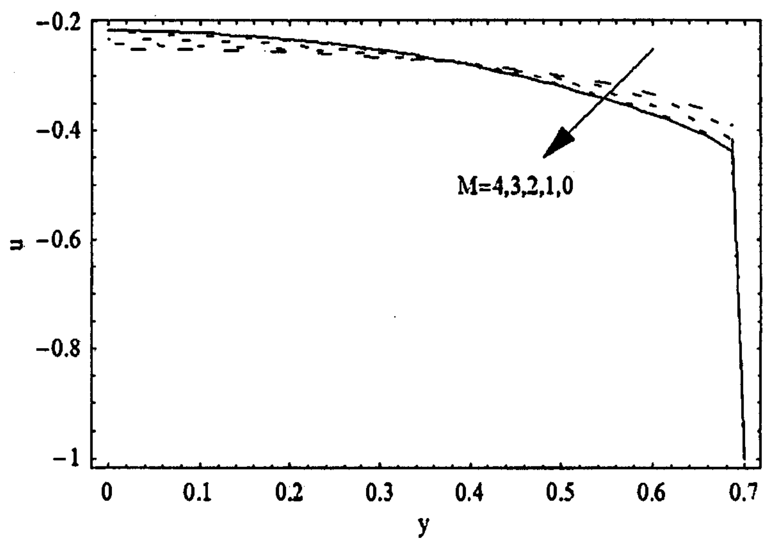

Fig. 3.2(a): Plot of the longitudinal velocity u for various values of M. The other parameters chosen are $\alpha_1 = 0, \alpha_2 = 0.5, F = -0.2$ and $\phi = 0.3$.

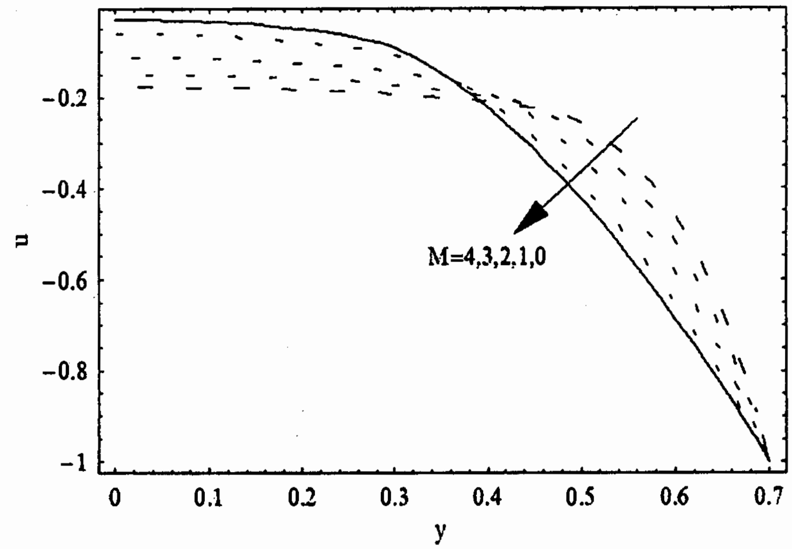

Fig. 3.2(b): Plot of the longitudinal velocity u for various values of M. The other parameters chosen are $\alpha_1 = 0.5, \alpha_2 = 3, F = -0.2$ and $\phi = 0.3$.

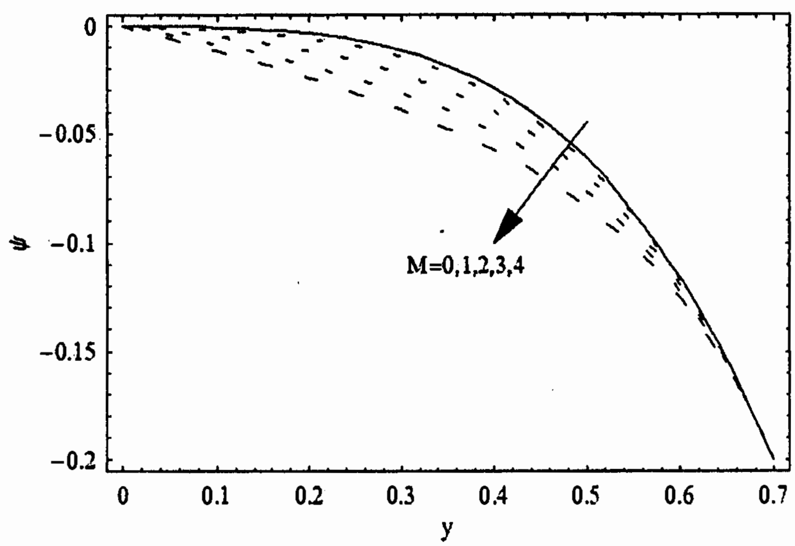

Fig. 3.2(c): Plot of the stream function ψ for various values of M. The other parameters chosen are $\alpha_1 = 0.5, \alpha_2 = 0.5, F = -0.2$ and $\phi = 0.3$.

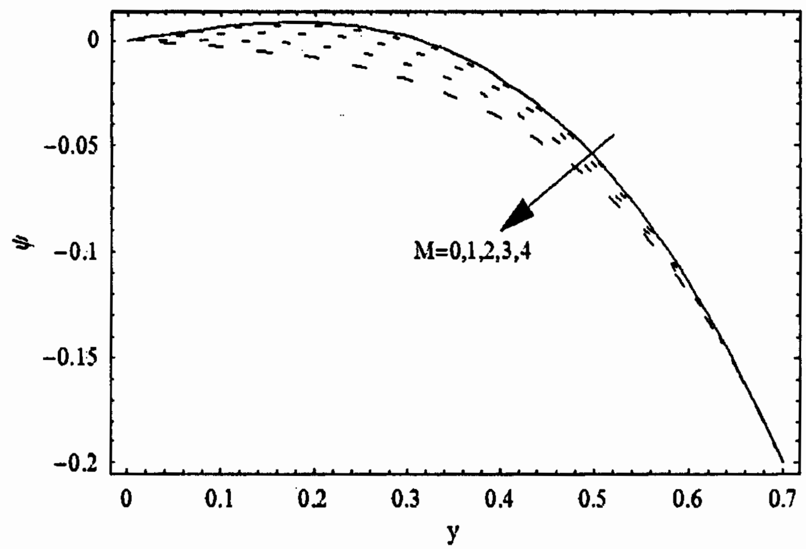

Fig. 3.2(d): Plot of the stream function ψ for various values of M. The other parameters chosen are $\alpha_1 = 0.5, \alpha_2 = 1, F = -0.2$ and $\phi = 0.3$.

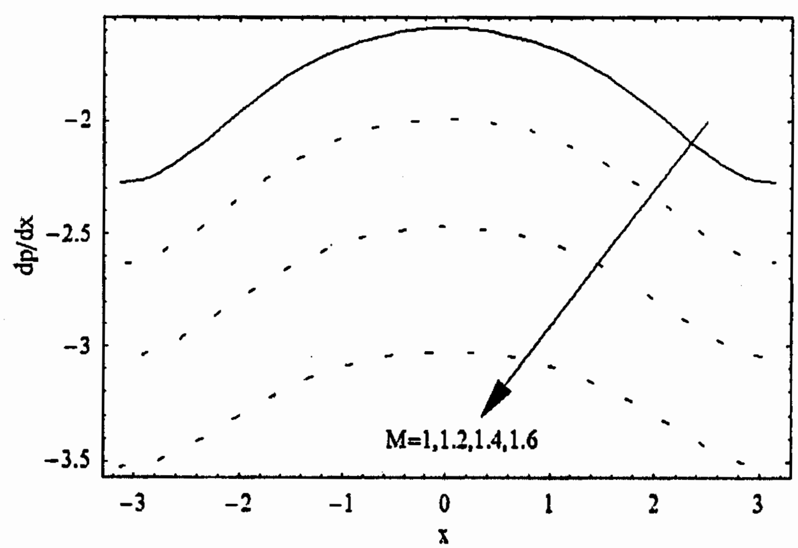

Fig. 3.3(a). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.1, \alpha_2 = 0.5, \Theta = 0.8$ and $\phi = 0.3$.

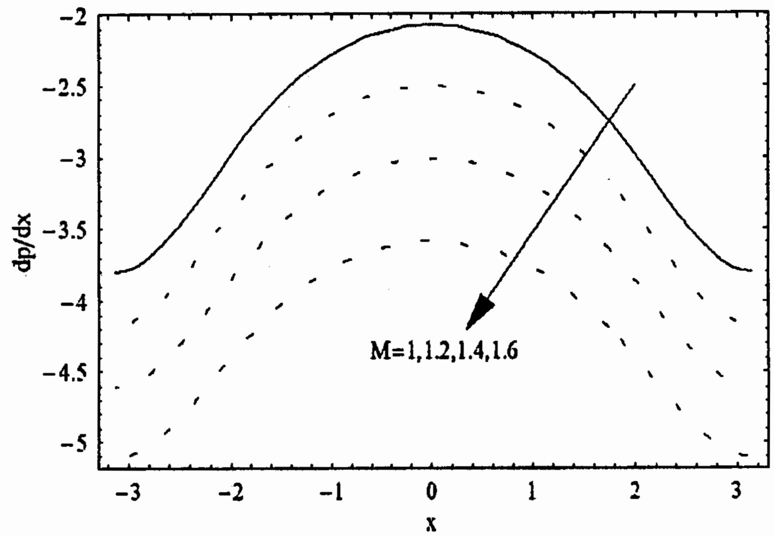

Fig. 3.3(b). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.5, \alpha_2 = 0.8, \Theta = 0.8$ and $\phi = 0.3$.

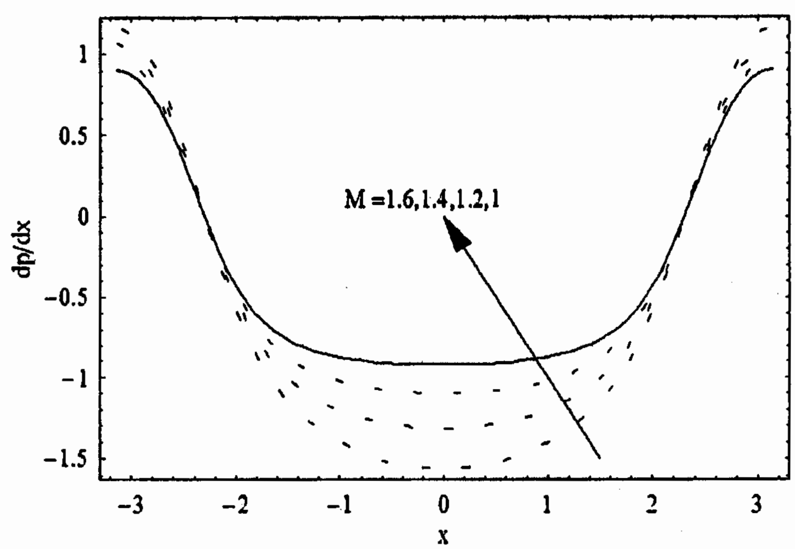

Fig. 3.3(c). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.1, \alpha_2 = 0.5, \Theta = 0.2$ and $\phi = 0.3$.

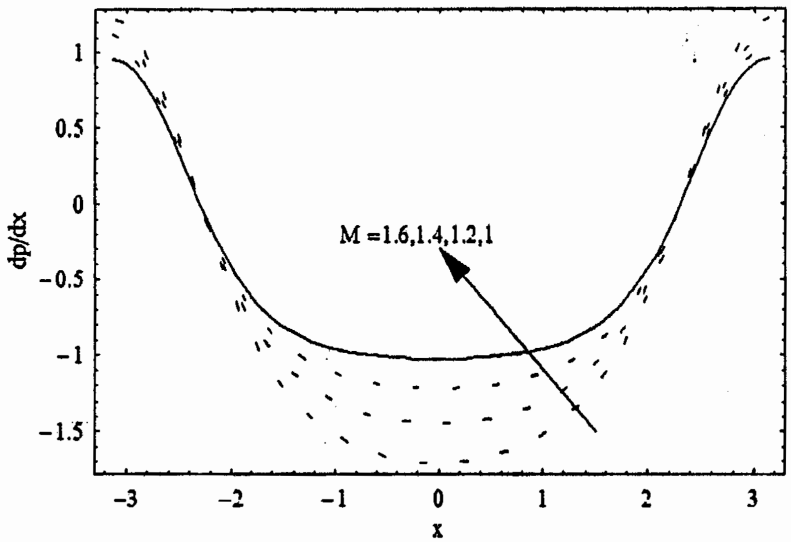

Fig. 3.3(d). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.5, \alpha_2 = 0.8, \Theta = 0.2$ and $\phi = 0.3$.

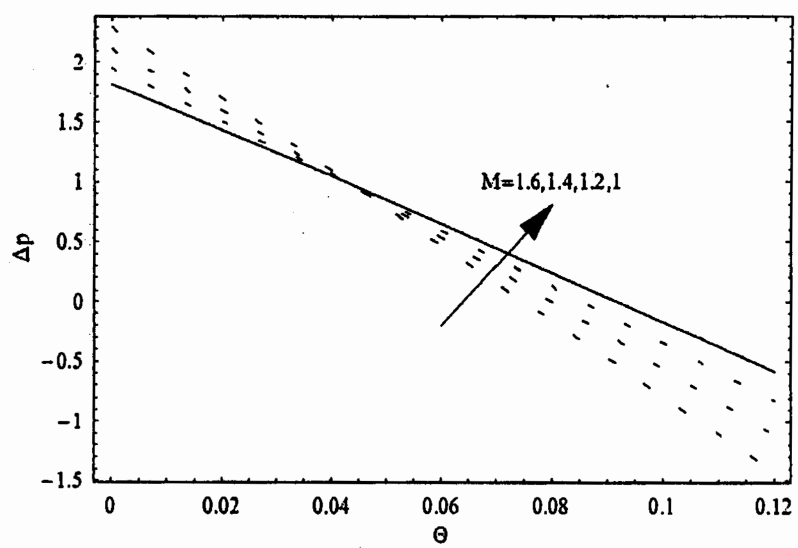

Fig. 3.4. Profile of pressure rise per wavelength Δp versus flow rate Θ for various values of M. The other parameters chosen are $\alpha_1 = 0, 0.1, 0.2, 0.4, 0.5, \alpha_2 = 0.5$ and $\phi = 0.3$.

M	$\Theta = 0$	$\Theta = 0.04$	$\Theta = 0.08$	$\Theta = 0.12$	$\Theta = 0.16$	$\Theta = 0.2$	$\Theta = 0.24$
0	0.88432	0.55901	0.212003	-0.15553	-0.56624	-1.06249	-1.66624
1	1.80726	1.04089	0.246788	-0.58384	-1.46769	-2.42249	-3.43287
1.2	1.94645	1.05173	0.130933	-0.82452	-1.83173	-2.91073	-4.04737
1.4	2.11167	1.06628	-0.00318	-1.10535	-2.25792	-3.48343	-4.76919
1.6	2.30321	1.08524	-0.15463	-1.42541	-2.74548	-4.13974	-5.59735

Table 3.1: Values of Δp for different values of M and Θ.

Figure 3.4 and Table 3.1 illustrates the relation between pressure rise per wavelength ΔP and flow rate Θ for the various values of M. It is observed that P_0 increases by increasing M. This means that the peristalsis has to do work against greater pressure rise for MHD fluid as compared to hydrodynamic fluid. Moreover, in co-pumping the pumping rate decreases for large values of M.

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