A Study of Soft Matrices

By

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the
Degree of MASTER OF SCIENCE
In MATHEMATICS

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بسم الله الرحمن الرحيم

Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF SCIENCE IN MATHEMATICS***

We accept this dissertation as conforming to the required standard.

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2016

Forwarding sheet by Research Supervisor

This thesis entitled "**A Study of Soft Matrices**" submitted by Azhar Rauf Khan (Reg. No. 175-FBAS/MSMA/S-14) is partial fulfillment of MS degree in Mathematics is completed under my guidance and supervision. I am satisfied with the quality of his research work and allow him to submit this thesis for further process to graduate with Master of Science degree from Department of Mathematics and Statistics, Faculty of Basic and Applied Sciences as per International Islamic University, Islamabad rules and regulations.

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DECLARATION

I hereby, declare, that this thesis neither as a whole nor as a part of it has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor **Dr. Tahir Mahmood**. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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Dedicated

To

My

Loving Parents,

Respectful Teachers

And My Friends.

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Azhar Rauf Khan

Contents

Preface

In this thesis, the center of discussion is some limitations of soft set matrices and its uses. The soft sets concept was expressed by Molodtsov in 1999 [17]. This concept is used to solve some complications in the fields of economics, engineering and environment because all these areas have some distinctive uncertainties regarding these problems. The concept of soft set is applied in fuzzy sets, intuitionistic fuzzy sets, vague set, interval mathematics and rough sets. In this thesis, some discussion is also done on matrices, which have a significant role in the vast field of engineering, economics and science. But, the old theory related to matrices is failed in solving the uncertainties, which are caused due to inaccurate circumstances. Matrices have different properties which include: commutative law, associative law and distributive law.

In the study, the idea of soft sets is described by linking an advantageous method with soft matrices. This study also involves the Naim Cagmanís and S. Enginoslu [5] research which highlights the usage of soft set theory in more precise manner. He describes the different dimensions of its applications. Initially, with the help of rough sets, he gave the theory of soft sets in decision making problems. Xiao et. all [27] had done a research highlighting business competitive capacity based on soft sets. Maji et al, [13] defined the fuzzy set, as the time passes a lot of work has done in fuzzy soft set. The definition of soft group was given by Aktas and Cagman [1]. They also made a comparison between soft sets to the rough soft sets and fuzzy soft sets. Subsequently, many other researchers have done a lot of work on this concept and gave many other theories related to the soft sets. Roy and maji [25] have also done some work on the applications and decision making problem. Majumdar [16] introduced the reduction of fuzzy soft set and then examine a decision making problem by fuzzy soft sets. The theory of the Rough sets is explained by Pawlak [23] for the analysis of the data possibly with inconsistent information. This theory has been used in many fields such as beauty contest, pattern recognition confliict analysis and switching circuits.

In the light of above mentioned facts, we indicates some limitations of the products of soft matrices given by Naim Cagman [5]. We pointed out that the products of soft matrices are not binary. It does not satisfy many laws which include Closure law, associative law and distributive law. Keeping in view this drawback in this thesis we have introduced new products of soft matrices, which are binary. We have also shown that accociative laws and distributive laws also holds.

Structure of the Thesis

The thesis is organized chapter wise as follows:

Chapter 1:

This chapter is introductory and sets up the background for the problems taken in the thesis. Semirings, Soft Sets, Soft-Union-Intersection Sum, Soft-Union-Intersection Product and related results are discussed.

Chapter 2:

In this chapter the article "Soft matrix theory and its decision making" is reviewed.

Chapter 3:

In this chapter, keeping in view the drawbacks and limitation such as the products of soft matrices defined in the paper reviewed are not binary and that associative and distributive laws are not satisfied, we improved the products of soft matrices and named them B-products of soft matrices. It is also shown that the defined products are binary. Further it is also shown that these products now satisfy the associative laws and distributive laws as well.

Chapter 1

Preliminaries

This chapter provides the essential definitions and preliminary results, which are useful for our subsequent chapters. For undefined terms and notions we refer to ([1], [2], [3], [4], [5], [8], [10], [14], [16], [15], [17], [23], [25], [27]).

1.1 Semigroups

Let S be a non-empty set and "$*$" be a binary operation on S. Then $(S, *)$ is called a *semigroup* if this operation is associative, that is

$$a * (b * c) = (a * b) * c \qquad \text{for all } a, b, c \in S.$$

A semigroup $(S, *)$ is called *commutative* if

$$a * b = b * a \qquad \text{for all } a, b \in S.$$

1.1.1 Definition

Let $(S, *)$ be a semigroup. If there exists an element $e \in S$ such that

$$a * e = e * a = a \qquad \text{for all } a \in S,$$

then e is called the *identity element* in S and $(S, *)$ is called a *monoid.*

An element $x \in S$ is called idempotent if $x * x = x$. If every element of S is idempotent then we say that S is idempotent.

Usually instead of writing $(S, *)$ we write S & instead of writing $x * y$ we write xy, for all $x,y \in S$.

1.1.2 Examples

1. $(\mathbb{N}, +)$ is a semigroup.

2. Let $S = \{a_1, a_2, a_3, ...\}$ such that $*$ be defined on S by $a_i * a_j = a_i$. Then $(S,*)$ is a semigroup.

3. $(\mathbb{N}_0, +)$ is a Monoid, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

4. $(\mathbb{Z}, \cdot)$ is a Monoid.

5. $\{0, 1\}$ is a monoid under "$\cdot$".

6. For any set X; $(P(X), \cup)$ and $(P(X), \cap)$ are monoids.

1.2 Semirings

A semiring is an algebraic system consisting of a non-empty set R together with two binary operations called "addition" and "multiplication" (denoted by "+" and "$\cdot$", respectively) such that $(R,+)$ and $(R,\cdot)$ are semigroups and multiplication distributes over addition from both sides, that is

$$a \cdot (b + c) = a \cdot b + a \cdot c, \text{ and } (b + c) \cdot a = b \cdot a + c \cdot a$$

for all $a, b, c \in R$.

1.3 Soft Sets

Soft set theory was introduced by D. Molodtsov [17]. It is a new approach for the real world problems in the field of economics, engineering, management etc. Molodtsov's soft set theory was proposed for dealing with ambiguity. He also defined some operations for soft set theory.

1.3.1 Definition [17]

Let U be an initial universe; E be the set of all possible parameters under consideration with respect to U and A be a subset of E. Then a pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$.

For $e \in A$, $F(e)$ may be considered as the set of $e-$approximate elements of the soft set (F, A).

Parameters are often attributes, characteristics, or properties of objects in soft sets. For example big, airy, tall, cool, hot, wooden, expensive, cheap etc.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of $e-$approximate elements of the soft set (F, A).

1.3.2 Definition [15]

For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a *soft subset* of (G, B) if

1. $A \subseteq B$ and
2. $F(e) \subseteq G(e)$ for all $e \in A$.

We write $(F, A)\tilde{\subset}(G, B)$.
In this case (G, B) is said to be a soft super set of (F, A).

1.3.3 Definition [15]

Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

1.3.4 Definition [2]

Let U be an initial universe set, E be the set of parameters, and $A \subseteq E$.

1. (F, A) is called a relative null soft set (with respect to the parameter set A), denoted by $\emptyset_A$, if $F(a) = \emptyset$ for all $a \in A$.
2. (G, A) is called a relative whole soft set (with respect to the parameter set A), denoted by $\mathfrak{U}_A$, if $G(a) = U$ for all $a \in A$..

The relative whole soft set with respect to the set of parameters E is called the *absolute soft set* over U and denoted by $\mathfrak{U}_E$. In a similar way, the relative null soft set with respect to E is called the *null soft set* over U and is denoted by $\emptyset_E$.

We shall denote by $\emptyset_\emptyset$ the unique soft set over U with an empty parameter set, which is called the *empty soft set* over U. Note that $\emptyset_\emptyset$ and $\emptyset_A$ are different soft sets over U and $\emptyset_\emptyset\tilde{\subset}\emptyset_A\tilde{\subset}(F, A)\tilde{\subset}\mathfrak{U}_A\tilde{\subset}\mathfrak{U}_E$ for all soft set (F, A) over U.

1.3.5 Definition [2]

Extended *union of two soft sets* (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e)=\begin{cases} F(e) & if\ e\in A-B \\ G(e) & if\ e\in B-A \\ F(e)\cup G(e) & if\ e\in A\cap B \end{cases}$$

We write $(F, A)\cup_{\mathcal{E}}(G, B)=(H, C)$.

1.3.6 Definition [2]

Let (F, A) and (G, B) be two soft sets over the same universe U, such that $A\cap B\neq\emptyset$. The *restricted union* of (F, A) and (G, B) is denoted by $(F, A)\cup_{\mathcal{R}}(G, B)$ and is defined as $(F, A)\cup_{\mathcal{R}}(G, B)=(H, C)$, where $C=A\cap B$ and for all $e\in C$, $H(e)=F(e)\cup G(e)$.

If $A\cap B=\emptyset$, then $(F, A)\cup_{\mathcal{R}}(G, B)=\emptyset_{\emptyset}$.

1.3.7 Definition [2]

The extended intersection of two soft sets (F, A) and (G, B) over a common universe U, is the soft set (H, C) where $C=A\cup B$ and for all $e\in C$,

$$H(e)=\begin{cases} F(e) & if\ e\in A-B \\ G(e) & if\ e\in B-A \\ F(e)\cap G(e) & if\ e\in A\cap B \end{cases}.$$

We write $(F, A)\cap_{\mathcal{E}}(G, B)=(H, C)$

1.3.8 Definition [2]

Let (F, A) and (G, B) be two soft sets over the same universe U such that $A\cap B\neq\emptyset$. The *restricted intersection* of (F, A) and (G, B) is denoted by $(F, A)\cap_{\mathcal{R}}(G, B)$ and is defined as $(F, A)\cap_{\mathcal{R}}(G, B)=(H, A\cap B)$, where $H(e)=F(e)\cap G(e)$ for all $e\in A\cap B$.

If $A\cap B=\emptyset$ then $(F, A)\cap_{\mathcal{R}}(G, B)=\emptyset_{\emptyset}$.

1.3.9 Definition [2]

Let (F, A) and (G, B) be two soft sets over the same universe U such that $A\cap B\neq\emptyset$. The *restricted difference* of (F, A) and (G, B) is denoted by $(F, A)\smile_{\mathcal{R}}(G, B)$ and is defined as $(F, A)\smile_{\mathcal{R}}(G, B)=(H, A\cap B)$, where $H(e)=F(e)-G(e)$ for all $e\in A\cap B$.

If $A\cap B=\emptyset$ then $(F, A)\smile_{\mathcal{R}}(G, B)=\emptyset_{\emptyset}$.

1.3.10 Definition [2]

The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c=(F^c, A)$ where $F^c: A\rightarrow P(U)$ is mapping given by $F^c(e)=U-F(e)$ for all $e\in A$

clearly $(F, A)^c = \mathfrak{U}_A \smile_{\mathcal{R}} (F, A)$ and $((F, A)^c)^c = (F, A)$.

1.3.11 Definition [15]

Let (F, A) and (G, B) be any two soft sets over a common universe U. Then the basic intersection of two soft sets (F, A) and (G, B) is defined as the soft set $(H, C) = (F, A) \wedge (G, B)$, where $C = A \times B$, and $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$.

1.3.12 Definition [15]

Let (F, A) and (G, B) be any two soft sets over a common universe U. Then the basic union of two soft sets (F, A) and (G, B) is defined as the soft set $(H, C) = (F, A) \vee (G, B)$, where $C = A \times B$, and $H(a, b) = F(a) \cup G(b)$ for all $(a, b) \in A \times B$.

1.3.13 Theorem

Let (F, A) and (G, B) be two soft sets over the same universe U such that $A \cap B \neq \emptyset$. Then

(1) $((F, A) \cup_{\mathcal{R}} (G, B))^c = (F, A)^c \cap_{\mathcal{R}} (G, B)^c$

(2) $((F, A) \cap_{\mathcal{R}} (G, B))^c = (F, A)^c \cup_{\mathcal{R}} (G, B)^c$

1.3.14 Distributive Laws for Soft Sets

In this section, we discuss distributive laws on the collection of soft sets. It is interesting to see that the equality does not hold in each and every case. We see the improperness in some assertions and counter example is given to show it. Let U be an initial universe and E be the set of parameters then we denote the collections of soft set as follows.

$\mathcal{SS}(U)^E$: The collection of all soft sets defined over U.

$\mathcal{SS}(U)_A$: The collection of all those soft sets defined over U with a fixed parameters set A.

1.3.15 Proposition [3]

Let (F, A) be a soft set over the universe set U.

(1) $(F, A)\alpha(F, A) = (F, A)$ for all $\alpha \in \{\cap_{\mathcal{R}}, \cup_{\mathcal{R}}\}$

(2) $(F, A) \cap_{\mathcal{R}} \emptyset_A = \emptyset_A$

(3) $(F, A) \cup_{\mathcal{R}} \emptyset_A = (F, A)$

(4) $(F, A) \cap_{\mathcal{R}} \mathfrak{U}_A = (F, A)$

(5) $(F, A) \cup_{\mathcal{R}} \mathfrak{U}_A = \mathfrak{U}_A$

Proof. Straightforward ■

1.3.16 Remark [3]

Let $\alpha, \beta \in \{\cup, \cap, \cup_\varepsilon, \cap_\varepsilon\}$. Then

$$(F,\ A)\,\alpha\,((G,\ B)\,\beta\,(H,\ C)) = ((F,\ A)\,\alpha\,(G,\ B))\,\beta\,((F,\ A)\,\alpha\,(H,\ C))$$

holds when we have 1 otherwise 0 in Table 2.

Table 2 shows that, if $\alpha, \beta \in \{\cup_\mathcal{R}, \cap_\mathcal{R}, \cup_\varepsilon, \cap_\varepsilon\}$, then there are sixteen combinations in all, there are four combinations in which $\alpha = \beta$ and for eight combination equality $(F,\ A)\,\alpha\,((G,\ B)\,\beta\,(H,\ C)) = ((F,\ A)\,\alpha\,(G,\ B))\,\beta\,((F,\ A)\,\alpha\,(H,\ C))$ will holds. Proofs in the case where equality holds can be followed by definitions of respective operations. For four remaining α and β this equality does not hold. To show this we have following example

	$\cup_\mathcal{R}$	$\cap_\mathcal{R}$	$\cup_\varepsilon$	$\cap_\varepsilon$
$\cup_\mathcal{R}$	1	1	1	1
$\cap_\mathcal{R}$	1	1	1	1
$\cup_\varepsilon$	0	1	1	0
$\cap_\varepsilon$	1	0	0	1

Table 2

1.3.17 Example [3]

Let U be the set of sample designs and$E$ be the set of available colors for dresses in a boutique,

$U = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$

$E = \{$Red, Green, Blue, Yellow, Black, White, Pink$\}$.

Suppose that

$A = \{$Red, Green, Blue, White$\}$, $B = \{$Green, Blue, Yellow, Black$\}$

and $C = \{$Blue, Yellow, White, Pink$\}$.

Let (F, A), (G, B) and (H, C) be the soft sets over U, which are defined as follows:

$F(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $F(\text{Green}) = \{S_3, S_4, S_5, S_6\}$;

$F(\text{Blue}) = \{S_1, S_2, S_4, S_7\}$; $F(\text{White}) = \{S_2, S_3, S_4\}$.

$G(\text{Green}) = \{S_4, S_5, S_6, S_8\}$; $G(\text{Blue}) = \{S_1, S_2, S_3, S_4\}$;

$G(\text{Yellow}) = \{S_4, S_5, S_6, S_7, S_8\}$; $G(\text{Black}) = \{S_1, S_2, S_4, S_7\}$

and

$H(\text{Blue}) = \{S_3, S_4, S_7, S_8\}$; $H(\text{Yellow}) = \{S_4, S_5, S_7\}$;

$H(\text{White}) = \{S_2, S_4, S_6, S_8\}$; $H(\text{Pink}) = \{S_2, S_3, S_5, S_7\}$.

Let

$(F, A) \cup_\varepsilon ((G, B) \cup_{\mathcal{R}} (H, C)) = (I, A \cup (B \cap C))$;
$((F, A) \cup_\varepsilon (G, B)) \cup_{\mathcal{R}} ((F, A) \cup_\varepsilon (H, C)) = (J, (A \cup B) \cap (A \cup C))$;
$(F, A) \cup_\varepsilon ((G, B) \cup_{\mathcal{R}} (H, C)) = (K, A \cup (B \cap C))$;
$((F, A) \cup_\varepsilon (G, B)) \cup_{\mathcal{R}} ((F, A) \cup_\varepsilon (H, C)) = (L, (A \cup B) \cap (A \cup C))$;
$(F, A) \cup_\varepsilon ((G, B) \cup_\varepsilon (H, C)) = (M, A \cup (B \cup C))$;
$((F, A) \cup_\varepsilon (G, B)) \cup_\varepsilon ((F, A) \cup_\varepsilon (H, C)) = (N, (A \cup B) \cup (B \cup C))$;
$(F, A) \cup_\varepsilon ((G, B) \cup_\varepsilon (H, C)) = (O, A \cup (B \cup C))$;
$((F, A) \cup_\varepsilon (G, B)) \cup_\varepsilon ((F, A) \cup_\varepsilon (H, C)) = (P, (A \cup B) \cup (B \cup C))$.

Then

$I(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $I(\text{Green}) = \{S_3, S_4, S_5, S_6\}$;
$I(\text{Blue}) = \{S_1, S_2, S_3, S_4, S_7, S_8\}$; $I(\text{Yellow}) = \{S_4, S_5, S_6, S_7, S_8\}$;
$I(\text{White}) = \{S_2, S_3, S_4\}$.
$J(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $J(\text{Green}) = \{S_3, S_4, S_5, S_6, S_8\}$;
$J(\text{Blue}) = \{S_1, S_2, S_3, S_4, S_7, S_8\}$; $J(\text{Yellow}) = \{S_4, S_5, S_6, S_7, S_8\}$;
$J(\text{White}) = \{S_2, S_3, S_4, S_6, S_8\}$.

Thus

$(F, A) \cup_\varepsilon ((G, B) \cup_{\mathcal{R}} (H, C)) \neq ((F, A) \cup_\varepsilon (G, B)) \cup_{\mathcal{R}} ((F, A) \cup_\varepsilon (H, C))$.

Now,

$K(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $K(\text{Green}) = \{S_3, S_4, S_5, S_6\}$;
$K(\text{Blue}) = \{S_4\}$;
$K(\text{Yellow}) = \{S_4, S_5, S_7\}$; $K(\text{White}) = \{S_2, S_3, S_4\}$.
$L(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $L(\text{Green}) = \{S_4, S_5, S_6\}$;
$L(\text{Blue}) = \{S_4\}$; $L(\text{Yellow}) = \{S_4, S_5, S_7\}$;
$L(\text{White}) = \{S_2, S_4\}$.

Thus

$(F, A) \cap_\varepsilon ((G, B) \cap_{\mathcal{R}} (H, C)) \neq ((F, A) \cap_\varepsilon (G, B)) \cap_{\mathcal{R}} ((F, A) \cap_\varepsilon (H, C))$.

Again, we see that

$M(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $M(\text{Green}) = \{S_3, S_4, S_5, S_6, S_8\}$;
$M(\text{Blue}) = \{S_1, S_2, S_3, S_4, S_7\}$; $M(\text{Yellow}) = \{S_4, S_5, S_7\}$;
$M(\text{Black}) = \{S_1, S_2, S_4, S_7\}$; $M(\text{White}) = \{S_2, S_3, S_4, S_6, S_8\}$;
$M(\text{Pink}) = \{S_2, S_3, S_5, S_7\}$

and

$N(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $N(\text{Green}) = \{S_3, S_4, S_5, S_6\}$;
$N(\text{Blue}) = \{S_1, S_2, S_3, S_4, S_7\}$; $N(\text{Yellow}) = \{S_4, S_5, S_7\}$;
$N(\text{Black}) = \{S_1, S_2, S_4, S_7\}$; $N(\text{White}) = \{S_2, S_3, S_4\}$;
$N(\text{Pink}) = \{S_2, S_3, S_5, S_7\}$.

Thus

$(F, A) \cup_\varepsilon ((G, B) \cap_\varepsilon (H, C)) \neq ((F, A) \cup_\varepsilon (G, B)) \cap_\varepsilon ((F, A) \cup_\varepsilon (H, C))$.

Now,

$O(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $O(\text{Green}) = \{S_4, S_5, S_6\}$;

$O(\text{Blue}) = \{S_1, S_2, S_4, S_7\}$; $O(\text{Yellow}) = \{S_4, S_5, S_6, S_7, S_8\}$;

$O(\text{Black}) = \{S_1, S_2, S_4, S_7\}$; $O(\text{White}) = \{S_2, S_4\}$;

$O(\text{Pink}) = \{S_2, S_3, S_5, S_7\}$

and

$P(\text{Red}) = \{S_1, S_2, S_3, S_4\}$; $P(\text{Green}) = \{S_3, S_4, S_5, S_6\}$;

$P(\text{Blue}) = \{S_1, S_2, S_4, S_7\}$; $P(\text{Yellow}) = \{S_4, S_5, S_6, S_7, S_8\}$;

$P(\text{Black}) = \{S_1, S_2, S_4, S_7\}$; $P(\text{White}) = \{S_2, S_3, S_4\}$;

$P(\text{Pink}) = \{S_2, S_3, S_5, S_7\}$.

Thus

$(F, A) \cap_\varepsilon ((G, B) \cup_\varepsilon (H, C)) \neq ((F, A) \cap_\varepsilon (G, B)) \cup_\varepsilon ((F, A) \cap_\varepsilon (H, C))$.

1.3.18 Definition[15]

Let U be an initial universal, $P(U)$ be the power set of U, E be the set of all parameter and $A, B \subseteq E$,

Let (F, A) and (G, B) be the two soft sets over a common universe U.

Then the basic intersection of the two soft sets (F, A) and (G, B) is define as the soft set

$(H, C) = (F, A) \wedge (G, B)$

where $C = A \times B$ such that

$H(e_1, e_2) = F(e_1) \cap G(e_2)\ \forall\ (e_1, e_2) \in A \times B$.

1.3.19 Definition [15]

Let U be an initial universal , $P(U)$ be the power set of U, E be the set of all parameters and $A, B \subseteq E$

Let (F, A) and (G, B) be the two soft sets over a common universe U.

Then the basic Union of the two soft sets (F, A) and (G, B) is defined as the soft sets

$(H, C) = (F, A) \vee (G, B)$

where $C = A \times B$ such that

$H(e_1, e_2) = F(e_1)\ U\ G(e_2)\ \forall\ (e_1, e_2) \in A \times B$.

1.3.20 Theorem [2]

If (F, A), (G, B) and (H, C) are three soft sets over U, then

1. $((F, A) \wedge (G, B)) \wedge (H, C) = (F, A) \wedge ((G, B) \wedge (H, C))$

2. $((F, A) \vee (G, B)) \vee (H, C) = (F, A) \vee ((G, B) \vee (H, C))$

3. $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

4. $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

The following remark shows that the parameter sets on both sides of the above assertions 3 and 4 are inconsistent in general.

1.3.21 Remark [2]

Let (F, A), (G, B) and (H, C) be soft sets over a common universe U. The soft set $(F, A) \wedge ((G, B) \vee (H, C))$ on left side of 3 has the parameter set $A \times (B \times C)$ and the soft set $((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$ on right side of 3 has a set of parameters as $(A \times B) \times (A \times C)$. But in [15] we can not find any notion which ensure

$A \times (B \times C) = (A \times B) \times (A \times C)$. Hence in Proposition 2.6 [15], two statements

1. $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

2. $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

are not true.

Chapter 2

Soft Matrix Theory and its Decision Making

In this chapter we review the paper of Naim Cagman and Serdar Enginoglu [5].

2.1 Soft Matrices

2.1.1 Definition [5]

Let U be an initial universal, $P(U)$ be the power set of U, E be the set of all parameter and $A \subseteq E$.

A soft set (f_A, E) over U is defined by the set of order pairs.

$$(f_A,\ E) = \{(f_A(e),\ e) : f_A(e) \in P(U),\ e \in E\}$$

where $f_A : E \to P(U)$ such that $f_A(e) = \phi$ if e$\notin A$.

Here f_A is called approximation function of the soft set $(f_A,\ E)$. The set $(f_A,\ E)$ is called e-approximate soft set. The element $f(e)$ is called the e-approximate value, which consists of related object of the perameter $e \in E$

2.1.2 Definition [5]

Let $(f_A,\ E)$ be an approximate soft set over U. Then a unique subset of $U \times E$ is defined by

$$R_A = \{(u,\ e) : u \in f_A(e),\ e \in E\}$$

is called approximate relation.

2.1.3 Definition [5]

Let us define a mapping $\chi_{R_A} : U \times E \to \{0, 1\}$ such that

$$\chi_{R_A}(u,e) = \begin{cases} 1, & \text{if} \quad (u,e) \in R_A \\ 0, & \text{if} \quad (u,e) \notin R_A. \end{cases}$$

If $U = \{u_1, u_2, u_3, \ldots, u_m\}$ and $E = \{e_1, e_2, e_3, \ldots, e_n\}$ and $A \subseteq E$, then R_A can be presented by a table as in the following form

R_A	e_1	e_2	$\ldots$	e_n
u_1	$\chi_{R_A}(u_1, e_1)$	$\chi_{R_A}(u_1, e_2)$	$\ldots$	$\chi_{R_A}(u_1, e_n)$
u_2	$\chi_{R_A}(u_2, e_1)$	$\chi_{R_A}(u_2, e_2)$	$\ldots$	$\chi_{R_A}(u_2, e_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\chi_{R_A}(u_m, e_1)$	$\chi_{R_A}(u_m, e_2)$	$\ldots$	$\chi_{R_A}(u_m, e_n)$

if $a_{ij} = \chi_{R_A}(u_i, e_j)$ we define a matrix

$$[a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Which is called an $m \times n$ soft matrix of the soft set (f_A, E) on a universe U. Various types of products for the elements of $SM_{m \times n}$ are defined in the following we reconsider these products.

According to the definition, soft set $(f_A, E)$is uniquely characterized by the matrix $[a_{ij}]$. It means that a soft set (f_A, E) is formally equal to its soft matrix $[a_{ij}]_{m \times n}$. Therefore we shell identify any soft set with its soft matrix and is use these two concept as interchangeable

The set of all $m \times n$ soft matrices over U will be denoted by $SM_{m \times n}$. From now on we shell delete the subscripts $m \times n$ of $[a_{ij}]_{m \times n}$ we use $[a_{ij}]$ instead of $[a_{ij}]_{m \times n}$

2.1.4 Example [5]

Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of parameters. If $A = \{e_2, e_3, e_4\}$ and $f_A(e_2) = \{u_2, u_4\}$, $f_A(e_3) = \phi.$, $f_A(e_4) = U$, then we write a soft set $(f_A, E) = \{(\{u_2, u_4\}, e_2), (U, e_4)\}$ and then the relation form of (f_A, E) is written by $R_A = \{(u_2, e_2), (u_4, e_2), (u_1, e_4), (u_2, e_4), (u_3, e_4), (u_4, e_4)\}$ hence the soft matrix $[a_{ij}]$ is written by

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.1.5 Definition [5]

Let $[a_{ij}] \in SM_{m\times n}$. Then $[a_{ij}]$ is called

1. A zero matrix is denoted by $[0]$, if $a_{ij} = 0$ for all i and j
2. An A-universal soft matrix $[\tilde{a}_{ij}]$, if $a_{ij} = 1$ for $j \in I_A = \{j : e_j \in A\}$ and $i = 1$, 2, 3, ..., m
3. A universal soft set matrix denoted by $[1]$, if $a_{ij} = 1$ for all i and j

2.1.6 Example [5]

Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E = \{e_1, e_2, e_3, e_4\}$ is a set of parameters and $[a_{ij}], [c_{ij}], [d_{ij}] \in SM_{5\times 4}$.

If $A = \{e_1, e_3\}$ and $f_A(e_1) = \phi$, $f_A(e_3) = \phi$ then $[a_{ij}] = [0]$ is a zero soft matrix written by

$$[a_{ij}] = [0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $C = \{e_1, e_2\}$ and $f_C(e_1) = U$, $f_C(e_2) = U$. Then $[c_{ij}] = [\check{c}_{ij}]$ is a C-Universal soft matrix written by

$$[c_{ij}] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

If $D = E$ and $f_D(e_i) = U$, for all $e_i \in D$. then $[d_{ij}] = [1]$ is a Universal soft matrix written by

$$[d_{ij}] = [1] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

2.1.7 Definition [5]

Let $[a_{ij}] \in SM_{m\times n}$. Then

1. $[a_{ij}]$ is the soft submatrix of $[b_{ij}]$, denoted by $[a_{ij}] \tilde{\subseteq} [b_{ij}]$, if $a_{ij} \leq b_{ij}$ for all i and j.

2. $[a_{ij}]$ is the proper soft submatrix of $[b_{ij}]$, denoted by $[a_{ij}] \tilde{\subset} [b_{ij}]$, if $a_{ij} \leq b_{ij}$ for at least one item $a_{ij} < b_{ij}$ all i and j.

3. $[a_{ij}]$ is the soft equal matrix of $[b_{ij}]$, denoted by $[a_{ij}] = [b_{ij}]$, if $a_{ij} = b_{ij}$ for all i and j.

2.1.8 Definition [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then the soft matrix $[c_{ij}]$ is called

1. Union of $[a_{ij}]$ and $[b_{ij}]$, denoted by $[a_{ij}] \tilde{\cup} [b_{ij}] = [c_{ij}]$, if $[c_{ij}] = \max\{a_{ij}, b_{ij})$ for all i and j.

2. Intersection of $[a_{ij}]$ and $[b_{ij}]$, denoted by $[a_{ij}]\tilde{\cap}[b_{ij}] = [c_{ij}]$, if $[c_{ij}] = \min\{a_{ij}, b_{ij})$ for all i and j.

3. Complement of $[a_{ij}]$, denoted by $[a_{ij}]^o = [c_{ij}]$, if $c_{ij} = 1 - a_{ij}$ for all i and j.

2.1.9 Definition [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then $[a_{ij}]$ and $[b_{ij}]$ are disjoint, if $[a_{ij}] \tilde{\cap} [b_{ij}] = [0]$ for all i and j.

2.1.10 Example [5]

Assume that $[a_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\qquad [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Then

$$[a_{ij}]\tilde{\cup}[b_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad [a_{ij}]\tilde{\cap}[b_{ij}] = [0], \qquad [a_{ij}]^o = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

2.1.11 Proposition [5]

Let $[a_{ij}] \in SM_{m\times n}$. Then

1. $[[a_{ij}]^\circ]^\circ = [a_{ij}]$
2. $[0]^\circ = [1]$.

2.1.12 Proposition [5]

Let $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}] \in SM_{m\times n}$. Then

1. $[a_{ij}] \tilde{\subseteq} [1]$
2. $[0] \tilde{\subseteq} [a_{ij}]$
3. $[a_{ij}] \tilde{\subseteq} [a_{ij}]$
4. $[a_{ij}] \tilde{\subseteq} [b_{ij}]$ and $[b_{ij}] \tilde{\subseteq} [c_{ij}] \Longrightarrow [a_{ij}] \tilde{\subseteq} [c_{ij}]$.

2.1.13 Proposition [5]

Let $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}] \in SM_{m\times n}$.Then

1. $[a_{ij}] = [b_{ij}]$ and $[b_{ij}] = [c_{ij}] \Leftrightarrow [a_{ij}] = [c_{ij}]$
2. $[a_{ij}] \tilde{\subseteq} [b_{ij}]$ and $[b_{ij}] \tilde{\subseteq} [a_{ij}] \Leftrightarrow [a_{ij}] = [b_{ij}]$.

2.1.14 Proposition [5]

Let $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}] \in SM_{m\times n}$.Then

1. $[a_{ij}] \tilde{\cup} [a_{ij}] = [a_{ij}]$
2. $[a_{ij}] \tilde{\cup} [0] = [a_{ij}]$
3. $[a_{ij}] \tilde{\cup} [1] = [1]$
4. $[a_{ij}] \tilde{\cup} [a_{ij}]^o = [1]$
5. $[a_{ij}] \tilde{\cup} [b_{ij}] = [b_{ij}] \tilde{\cup} [a_{ij}]$
6. $([a_{ij}] \tilde{\cup} [b_{ij}]) \tilde{\cup} [c_{ij}] = [a_{ij}] \tilde{\cup} ([b_{ij}] \tilde{\cup} [c_{ij}])$.

2.1.15 Proposition [5]

Let $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}] \in SM_{m\times n}$.Then

1. $[a_{ij}] \tilde{\cap} [a_{ij}] = [a_{ij}]$
2. $[a_{ij}] \tilde{\cap} [0] = [0]$
3. $[a_{ij}] \tilde{\cap} [1] = [a_{ij}]$
4. $[a_{ij}] \tilde{\cap} [a_{ij}]^o = [0]$
5. $[a_{ij}] \tilde{\cap} [b_{ij}] = [b_{ij}] \cap [a_{ij}]$
6. $([a_{ij}] \tilde{\cap} [b_{ij}]) \tilde{\cap} [c_{ij}] = [a_{ij}] \tilde{\cap} ([b_{ij}] \tilde{\cap} [c_{ij}])$.

2.1.16 Proposition [5]

Let $[a_{ij}]$, $[b_{ij}]$ and $[c_{ij}] \in SM_{m\times n}$. Then De Morgan's laws are valid

1. $([a_{ij}] \tilde{\cap} [b_{ij}])^0 = [a_{ij}]^0 \tilde{\cup} [b_{ij}]^0$
2. $([a_{ij}] \tilde{\cup} [b_{ij}])^0 = [a_{ij}]^0 \tilde{\cap} [b_{ij}]^0$.

Proof. For all i and j

i.

$$\begin{aligned}([a_{ij}] \cap [b_{ij}])^0 &= [\max\{a_{ij}, b_{ij}\}]^o \\ &= [1 - \max\{a_{ij}, b_{ij}\}] \\ &= [\min\{1 - a_{ij}, 1 - b_{ij}\}] \\ &= [a_{ij}]^o \tilde{\cap} [b_{ij}]^o\end{aligned}$$

ii.

It can be proved similarly ■

2.1.17 Example [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{5\times 4}$ as in Example 2.1.10. Then

$$([a_{ij}] \tilde{\cup} [b_{ij}])^0 = [a_{ij}]^0 \tilde{\cap} [b_{ij}]^0 .= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

and

$([a_{ij}] \tilde{\cap} [b_{ij}])^0 = [a_{ij}]^0 \tilde{\cup} [b_{ij}]^0 = [1]$

2.1.18 Proposition [5]

Let $[a_{ij}]$, $[b_{ij}]$ and $[c_{ij}] \in SM_{m\times n}$. Then

1. $[a_{ij}] \tilde{\cup} ([b_{ij}]) \tilde{\cap} [c_{ij}]) = ([a_{ij}] \tilde{\cup} [b_{ij}]) \tilde{\cap} ([a_{ij}] \tilde{\cup} [c_{ij}])$
2. $[a_{ij}] \tilde{\cap} ([b_{ij}]) \tilde{\cup} [c_{ij}]) = ([a_{ij}] \tilde{\cap} [b_{ij}]) \tilde{\cap} ([a_{ij}] \tilde{\cap} [c_{ij}])$.

2.2 Product of Soft Matrices

In this section we define four special products of soft matrices to construct soft decision making methods.

2.2.1 Definition [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then And product of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$\wedge : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $\quad [a_{ij}] \wedge [b_{ik}] = [c_{ip}]$

Where $c_{ip} = \min(a_{ij}, b_{ik})$ such that $p = n(j-1)+k$.

2.2.2 Definition [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then Or- product of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$\vee : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $\quad [a_{ij}] \vee [b_{ik}] = [c_{ip}]$

Where $c_{ip} = \max(a_{ij}, b_{ik})$ such that $p = n(j-1)+k$.

2.2.3 Definition [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$.Then And-Not-product of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$\overline{\wedge} : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $\quad [a_{ij}] \overline{\wedge} [b_{ik}] = [c_{ip}]$

Where $c_{ip} = \min(a_{ij}, 1-b_{ik})$ such that $p = n(j-1)+k$.

2.2.4 Definition [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then Or-Not- product of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$\underline{\vee} : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}$, $\quad [a_{ij}] \underline{\vee} [b_{ik}] = [c_{ip}]$

Where $c_{ip} = \max(a_{ij}, 1-b_{ik})$ such that $p = n(j-1)+k$.

2.2.5 Example [5]

Assume that $[a_{ij}]$, $[b_{ij}] \in SM_{5\times 4}$

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad [b_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Then

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&1 \\ 0&0&0&0&1&0&0&1&1&0&0&1&1&0&0&1 \\ 0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \end{bmatrix}$$

similarly we can find the other product $[a_{ij}] \vee [b_{ik}]$, $[a_{ij}] \bar{\wedge} [b_{ik}]$, $[a_{ij}] \veebar [b_{ik}]$ note that the commutativity is not valid for the soft matrices.

2.2.6 Proposition [5]

Let $[a_{ij}]$, $[b_{ij}] \in SM_{(m \times n)}$. Then the following De Morgan's types of result are true

1. $([a_{ij}] \wedge [b_{ij}])^0 = [a_{ij}]^0 \vee [b_{ij}]^0$
2. $([a_{ij}] \vee [b_{ij}])^0 = [a_{ij}]^0 \wedge [b_{ij}]^0$.
3. $([a_{ij}] \veebar [b_{ij}])^0 = [a_{ij}]^0 \bar{\wedge} [b_{ij}]^0$.
4. $([a_{ij}] \bar{\wedge} [b_{ij}])^0 = [a_{ij}]^0 \veebar [b_{ij}]^0$.

2.3 Soft min-max Decision Making

In this section we construct a soft max-min decision making(SMmDM) method by using soft max-min decision function which is also defined here. The method selects optimum alternative from the set of all alternatives

2.3.1 Definition [5]

Let $[c_{ij}] \in SM_{m \times n^2}$, $I_K = \{p : \exists\, i,\ c_{ip} \neq 0,\ (k-1)n < p \leq kn\}$ for all $k \in I = \{1, 2, 3, \ldots, n\}$. Then the soft max-min decision function, denoted by Mm, is defined as follows

$$Mm : SM_{m \times n^2} \to SM_{m \times 1}, \qquad Mm\,[c_{ip}] = \left[\max_{k \in I}\{t_k\}\right]$$

where

$$t_k = \begin{cases} \min\limits_{p \in I_k}\{c_{ip}\}, & \text{if } I_K \neq \varphi \\ 0, & \text{if } I_K = \varphi \end{cases}$$

the one column soft matrix $Mm\,[c_{ip}]$ is called Max-min decision soft matrix.

2.3.2 Definition [5]

Let $U = \{u_1, u_2, \ldots, u_n\}$ be initial universe and $Mm\,[c_{ip}] = [d_{i1}]$. Then a subset of U can be obtained by using $[d_{i1}]$ as in he following way

$opt_{[d_{i1}]}(U) = \{u_i : u_i \in U, d_{i1} = 1\}$

which is called the optimum solution.

Now, by using the definitions we can construct a SMmDM method by the following algorithm.

Step 1: Choose feasible subsets of the set of parameters,

Step 2: construct the soft matrix for each set of parameters,

Step 3: find a convenient product of the soft matrices,

Step 4: find a max min decision soft matrix,

Step 5: find an optimum set of U.

Note that, by the similar way, we can define soft min max, soft min min and soft max max decision making methods

which may be denoted by SmMDM, SmmDM, SMMDM respectively. One of them may be more useful than others according to the type of the problems.

2.4 Applications

Assume that a real estate agent has a set of different types of houses $U = \{u_1, u_2, u_3, u_4, u_5\}$ which may be characterized by a set of parameters $E = \{e_1, e_2, e_3, e_4\}$. For $j = 1, 2, 3, 4$ the parameters e_j stand for "in good location", "cheap", "modern", "large", respectively. Then we can give the following examples.

2.4.1 Example [5]

Suppose that a married couple, Mr. X and Mrs. X, come to the real estate agent to buy a house. If each partner has to consider their own set of parameters, then we select a house on the basis of the sets of partners' parameters by using the SMmDM as follows.

Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E = \{e_1, e_2, e_3, e_4\}$ is a set of all parameters.

Step 1: First, Mr. X and Mrs. X have to choose the sets of their parameters, $A = \{e_2, e_3, e_4\}$ and $B = \{e_1, e_3, e_4\}$, respectively.

Step 2: Then we can write the following soft matrices which are constructed according to their parameters.

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad [b_{ik}] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Now, we can find a product of the soft matrices $[a_{ij}]$ and $[b_{ik}]$ by using And-product as follows

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, we use And-product since both Mr. X and Mrs. X's choices have to be considered.

Step 4: We can find a max min decision soft matrix as

$$\mathrm{Mm}([a_{ij}] \wedge [b_{ik}]) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 5: Finally, we can find an optimum set of U according to Mm.$[a_{ij}] \wedge [b_{ik}]$

$optMm._{[a_{ij}] \wedge [b_{ik}]}.(U) = \{u_1\}$

where u_1 is an optimum house to buy for Mr. X and Mrs. X.

Note that the optimal set of U may contain more than one element.

Similarly, we can also use the other products $([a_{ij}] \vee [b_{ik}])$, $([a_{ij}] \overline{\wedge} [b_{ik}])$and $([a_{ij}] \underline{\vee} [b_{ik}])$ for the other convenient problems.

Chapter 3

Soft Matrix Theory and its Decision Making: A New Approach

In this chapter we are going to define new type of products which are binary and satisfies associative laws and distributive laws.

3.1 Binary-Product Of The Soft Matrices

3.1.1 Definition [5]

Let $[a_{ij}], [b_{ij}] \in SM_{m\times n}$. Then,

i. And-product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$$\wedge : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2},\ [a_{ij}] \wedge [b_{ik}] = [c_{ip}]$$

Where $c_{ip} = \min(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$.

ii. Or-product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$$\vee : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2},\ [a_{ij}] \vee [b_{ik}] = [c_{ip}]$$

Where $c_{ip} = \max(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$.

iii. And Not-product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$$\overline{\wedge} : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2},\ [a_{ij}] \overline{\wedge} [b_{ik}] = [c_{ip}]$$

Where $c_{ip} = \min(a_{ij}, 1 - b_{ik})$ such that $p = n(j-1) + k$.

iv. Or Not-product $[a_{ij}]$ and $[b_{ij}]$ is defined by

$$\veebar : SM_{m\times n} \times SM_{m\times n} \rightarrow SM_{m\times n^2},\ [a_{ij}] \veebar [b_{ik}] = [c_{ip}]$$

Where $c_{ip} = \max(a_{ij},\ 1 - b_{ik})$ such that $p = n(j-1) + k$.

From above definition it is clear that all these products are not binary operations. Even if we consider two soft square matrices from $SM_{m\times m}$, any of above mentioned product will not give us a soft square matrix from $SM_{m\times m}$. As above mentioned products are not binary operations, therefore there is no question of associativity in these soft matrix product.

In the following products, for soft matrices are redefine so that they happen to be associate binary operations for the elements of $SM_{m\times n}$. These products will be called Binary-Product or simply we can write it as B-Products.

3.1.2 Definition

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then And-B-Product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$$\wedge \ : \ SM_{m\times n} \times SM_{m\times n} \rightarrow SM_{m\times n},\ [a_{ij}] \wedge [b_{ik}] = [d_{iq}] = \left[\bigvee_{p=(q-1)n+1}^{qn} (c_{ip})\right]$$

for all $i = 1,\ 2,\ ...,\ m$ and $q = 1,\ 2,\ ...,\ n$.

Where $c_{ip} = \min(a_{ij},\ b_{ik})$ such that $p = n(j-1) + k$.

3.1.3 Definition

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then Or-B- product of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$$\vee \ : \ SM_{m\times n} \times SM_{m\times n} \rightarrow SM_{m\times n},\ [a_{ij}] \vee [b_{ik}] = [d_{iq}] = \left[\bigwedge_{p=(q-1)n+1}^{qn} (c_{ip})\right]$$

for all $i = 1,\ 2,\ ...,\ m$ and $q = 1,\ 2,\ ...,\ n$.

Where $c_{ip} = \max(a_{ij},\ b_{ik})$ such that $p = n(j-1) + k$

3.1.4 Definition

Let $[a_{ij}]$, $[b_{ij}] \in SM_{m\times n}$. Then And -Not-B-product of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$$\bar{\wedge} \ : SM_{m\times n} \times SM_{m\times n} \rightarrow SM_{m\times n},\ [a_{ij}]\, \bar{\wedge}\, [b_{ik}] = [d_{iq}] = \left[\bigvee_{p=(q-1)n+1}^{qn} (c_{ip})\right]$$

for all $i = 1,\ 2,\ ...,\ m$ and $q = 1,\ 2,\ ...,\ n$.

Where $c_{ip} = \min(a_{ij},\ 1 - b_{ik})$ such that $p = n(j-1) + k$

3.1.5 Definition

Let $[a_{ij}], [b_{ij}] \in SM_{m\times n}$. Then Or-Not-B-product of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$$\veebar \ : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n},\ [a_{ij}] \veebar [b_{ik}] = [d_{iq}] = \left[\bigwedge_{p=(q-1)n+1}^{qn} (c_{ip}) \right]$$

for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$.

Where $c_{ip} = \max(a_{ij}, 1 - b_{ik})$ such that $p = n(j-1) + k$.

3.1.6 Theorem

The And -B-Product is a binary product.

Proof. Let $[a_{ij}]$ and $[b_{ij}] \in SM_{m\times n}$. Then And-B-Product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$\wedge : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n}$

$$[a_{ij}] \wedge [b_{ik}] = [d_{iq}]$$

Where $d_{iq} = \left(\bigvee_{p=(q-1)n+1}^{qn} (e_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

where $e_{ip} = \min(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$, then $[a_{ij}] \wedge [b_{ik}] = [d_{iq}]$ ■

3.1.7 Theorem

The Or-B-Product is a binary product.

Proof. Let $[a_{ij}]$ and $[b_{ij}] \in SM_{m\times n}$. Then Or-B-Product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$\vee : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n}$

$[a_{ij}] \vee [b_{ik}] = [g_{iq}]$

Where $g_{iq} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (f_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $f_{ip} = \max(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$ ■

3.1.8 Theorem

The And-Not-B-Product is a binary product.

Proof. Let $[a_{ij}]$ and $[b_{ij}] \in SM_{m\times n}$. Then And-Not-B-Product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$\bar{\wedge} : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n}$

$$[a_{ij}]\ \bar{\wedge}\ [b_{ik}] = [d_{iq}]$$

Where $d_{iq} = \left(\bigvee_{p=(q-1)n+1}^{qn} (e_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $e_{ip} = \min(a_{ij}, 1 - b_{ik})$ such that $p = n(j-1) + k$ ■

3.1.9 Theorem

The Or-Not-B-Product is a binary product.

Proof. Let $[a_{ij}]$ and $[b_{ij}] \in SM_{m\times n}$. Then Or-Not-B-Product of $[a_{ij}]$ and $[b_{ij}]$ is defined by

$\underline{\vee} : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n}$

$$[a_{ij}]\ \underline{\vee}\ [b_{ik}] = [g_{iq}]$$

where $g_{iq} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (f_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $f_{ip} = \max(a_{ij}, 1 - b_{ik})$ such that $p = n(j-1) + k$ ■

3.1.10 Example

(And-B-Product)

Let $U = \{u_1, u_2, u_3\}$ be the universal set, $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_3, e_4\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that

$f_A(e_1) = \{u_1, u_2\}$

$f_A(e_2) = \{u_2, u_3\}$

$f_A(e_3) = f_A(e_4) = \phi$

$R_A = \{(u_1, e_1), (u_2, e_2); (u_2, e_2), (u_3, e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that
$f_B(e_3) = U$
$f_B(e_4) = \{u_1, u_3\}$
$f_B(e_1) = f_B(e_2) = \phi$
$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[d_{ip}] = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } d_{ip} = \min(a_{ij}, b_{ik}) \text{ such that } p = n(j-1) + k$$

$$\text{Where } y_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (d_{ip}) \right) \qquad \text{for all } i = 1, 2, 3 \text{ and } q = 1, 2, 3, 4$$

Then
$[Y] = [y_{iq}]_{3\times 4} = [a_{ij}] \wedge [b_{ik}]$

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

3.1.11 Example

(Or-B-Product)

Let $U = \{u_1, u_2, u_3\}$ be the universal set, $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_3, e_4\}$ be the subsets of E.

Let $f_A : E \to P(U)$, be such that

$f_A(e_1) = \{u_1, u_2\}$

$f_A(e_2) = \{u_2, u_3\}$

$f_A(e_3) = f_A(e_4) = \phi$

$R_A = \{(u_1, e_1), (u_2, e_2), (u_2, e_2), (u_3, e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that

$f_B(e_3) = U$

$f_B(e_4) = \{u_1, u_3\}$

$f_B(e_1) = f_B(e_2) = \phi$

$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[a_{ij}] \vee [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[f_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{and } f_{ip} = \max(a_{ij},\ b_{ik}) \text{ such that } p = n(j-1)+k$$

$$\text{if } x_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{4} (f_{ip}) \right) \qquad \text{for all } i = 1,2,3 \text{and } q = 1,2,3,4$$

so

$[X] = [x_{iq}]_{3\times 4} = [a_{ij}] \vee [b_{ik}]$

$$[a_{ij}] \vee [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

3.1.12 Example

(And-Not-B-Product)

Let $U = \{u_1,\ u_2,\ u_3\}$ be the universal set, $E = \{e_1,\ e_2,\ e_3,\ e_4\}$ be the set of parameters and $A = \{e_1,\ e_2\}$, $B = \{e_3,\ e_4\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that

$f_A(e_1) = \{u_1,\ u_2\}$

$f_A(e_2) = \{u_2,\ u_3\}$

$f_A(e_3) = f_A(e_4) = \phi$

$R_A = \{(u_1,\ e_1),\ (u_2,\ e_2),\ (u_2,\ e_2),\ (u_3,\ e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that

$f_B(e_3) = U$

$f_B(e_4) = \{u_1,\ u_3\}$

$f_B(e_1) = f_B(e_2) = \phi$
$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[a_{ij}] \,\overline{\wedge}\, [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \overline{\wedge} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[d_{ip}] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $d_{ip} = \min(a_{ij},\ 1 - b_{ik})$ such that $p = n(j-1) + k$

where $y_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (d_{ip}) \right)$ for all $i = 1, 2, 3$ and $q = 1, 2, 3, 4$

Then
$[Y] = [y_{iq}]_{3\times 4} = [a_{ij}] \,\overline{\wedge}\, [b_{ik}]$

$$[a_{ij}] \,\overline{\wedge}\, [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

3.1.13 Example

(Or-Not-B-Product)

Let $U = \{u_1, u_2, u_3\}$ be the universal set, $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_3, e_4\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that

$f_A(e_1) = \{u_1, u_2\}$
$f_A(e_2) = \{u_2, u_3\}$
$f_A(e_3) = f_A(e_4) = \phi$
$R_A = \{(u_1, e_1), (u_2, e_2), (u_2, e_2), (u_3, e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that
$f_B(e_3) = U$
$f_B(e_4) = \{u_1, u_3\}$
$f_B(e_1) = f_B(e_2) = \phi$
$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[a_{ij}] \veebar [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \veebar \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[f_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

and $f_{ip} = \max(a_{ij},\ 1 - b_{ik})$ such that $p = n(j-1) + k$

$$\text{Where } x_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{4} (f_{ip}) \right) \qquad \text{for all } i = 1,\ 2,\ 3 \text{ and } q = 1,\ 2,\ 3,\ 4$$

so

$[X] = [x_{iq}]_{3\times 4} = [a_{ij}] \underline{\vee} [b_{ik}]$

$$[a_{ij}] \underline{\vee} [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

so the above example shows that the defined product is a binary operation

3.1.14 Theorem

The associative law holds with respect to And-B-Product.

Proof. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in SM_{m\times n}$.

Then And-B-Product of $[a_{ij}]$ and $[b_{ij}]$ is define by

$\wedge : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n}$

$$[a_{ij}] \wedge [b_{ik}] = [d_{iq}]$$

$$\text{where } d_{iq} = \left(\bigvee_{p=(q-1)n+1}^{qn} (e_{ip}) \right) \qquad \text{for all } i = 1,\ 2,\ \ldots,\ m \text{ and } q = 1,\ 2,\ \ldots,\ n$$

and $e_{ip} = \min(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$

now

$([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}] = [d_{iq}] \wedge [c_{ij}]$

$$([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}] = [h_{iq}]$$

$$\text{where } h_{iq} = \left(\bigvee_{p=(q-1)n+1}^{qn} (s_{ip}) \right) \qquad \text{for all } i = 1,\ 2,\ \ldots,\ m \text{ and } q = 1,\ 2,\ \ldots,\ n$$

and $s_{ip} = \min(d_{ij}, c_{ik})$ such that $p = n(j-1) + k$

now R.H.S

$$[b_{ij}] \wedge [c_{ik}] = [g_{iq}]$$

$$\text{Where } g_{iq} = \left(\bigvee_{p=(q-1)n+1}^{qn} (f_{ip}) \right) \qquad \text{for all } i = 1,\ 2,\ \ldots,\ m \text{ and } q = 1,\ 2,\ \ldots,\ n$$

and $f_{ip} = \min(b_{ij}, c_{ik})$ such that $p = n(j-1) + k$

$[a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}]) = [a_{ij}] \wedge ([g_{iq}])$

Now

$$[a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}]) = [h_{iq}]$$

Where $h_{iq} = \left(\bigvee_{p=(q-1)n+1}^{qn} (t_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $t_{ip} = \min(a_{ij}, g_{ik})$ such that $p = n(j-1) + k$

than

$([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}] = [a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}])$. ■

3.1.15 Theorem

The associative law holds with respect to Or-B-Product.

Proof. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in SM_{m\times n}$.

Then Or-Product of $[a_{ij}]$ and $[b_{ij}]$ is define by

$\vee : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n}$

$$[a_{ij}] \vee [b_{ik}] = [d_{iq}]$$

Where $d_{iq} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (e_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $e_{ip} = \max(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$

$([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}] = [d_{iq}] \vee [c_{ij}]$

$$([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}] = [h_{iq}]$$

Where $h_{iq} = \left[\bigwedge_{p=(q-1)n+1}^{qn} (s_{ip}) \right]$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $s_{ip} = \max(d_{ij}, c_{ik})$ such that $p = n(j-1) + k([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}] = [h_{iq}]$

now R.H.S

$$[b_{ij}] \vee [c_{ik}] = [g_{iq}]$$

Where $g_{iq} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (f_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $f_{ip} = \max(b_{ij}, c_{ik})$ such that $p = n(j-1) + k$

Now

$[a_{ij}] \vee ([b_{ik}] \vee [c_{ij}]) = [a_{ij}] \vee ([g_{iq}])$

$$[a_{ij}] \vee ([b_{ik}] \vee [c_{ij}]) = [h_{iq}]$$

Where $h_{iq} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (t_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $t_{ip} = \max(a_{ij}, g_{ik})$ such that $p = n(j-1) + k$

$([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}] = [a_{ij}] \vee ([b_{ik}] \vee [c_{ij}])$. ■

3.1.16 Theorem

The associative law holds with respect to And-Not-B-Product.

$([a_{ij}] \bar{\wedge} [b_{ik}]) \bar{\wedge} [c_{ij}] = [a_{ij}] \bar{\wedge} ([b_{ik}] \bar{\wedge} [c_{ij}])$

Proof. Strightforword. ■

3.1.17 Theorem

The associative law holds with respect to Or-Not-B-Product.

$([a_{ij}] \underline{\vee} [b_{ik}]) \underline{\vee} [c_{ij}] = [a_{ij}] \underline{\vee} ([b_{ik}] \underline{\vee} [c_{ij}])$

Proof. Strightforword. ■

3.1.18 Example

(Associative law with respect to And-B-Product)

Let $U = \{u_1, u_2, u_3\}$ be the universal set, $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_3, e_4\}$, $C = \{e_2, e_3\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that

$f_A(e_1) = \{u_1, u_2\}$

$f_A(e_2) = \{u_2, u_3\}$

$f_A(e_3) = f_A(e_4) = \phi$

$R_A = \{(u_1, e_1), (u_2, e_2), (u_2, e_2), (u_3, e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that

$f_B(e_3) = U$

$f_B(e_4) = \{u_1, u_3\}$

$f_B(e_1) = f_B(e_2) = \phi$
$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and $f_C : E \to P(U)$ be such that
$f_C(e_2) = \{u_2\}$
$f_C(e_3) = \{u_2, u_3\}$
$f_C(e_1) = f_C(e_4) = \phi$
$R_C = \{(u_2, e_2), (u_2, e_3), (u_3, e_3)\}$

R_C	e_1	e_2	e_3	e_4
u_1	0	0	0	0
u_2	0	1	1	0
u_3	0	0	1	0

$$C = [c_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now to prove $([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}] = [a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}])$
Firstly we Find that $([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}]$

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[d_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $d_{ip} = \min(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$

Where $y_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (d_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

so

$[Y] = [y_{iq}]_{3\times4} = [a_{ij}] \wedge [b_{ik}]$

$$[y_{iq}]_{3\times4} = [a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now

$([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}] = [y_{ij}] \wedge [c_{ik}]$

$$[y_{ij}] \wedge [c_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[e_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } e_{ip} = \min(y_{ij},\ c_{ik}) \text{ such that } p = n(j-1) + k$$

Where $w_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (e_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

so

$$[W] = [w_{iq}]_{3\times4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$[w_{iq}] = [y_{ij}] \wedge [c_{ik}]$

$[w_{iq}] = ([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}]$

$$([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now we find $[a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}])$

$$[b_{ik}] \wedge [c_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let

$$[f_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{and } f_{ip} = \min(b_{ij},\ c_{ik}) \text{ such that } p = n(j-1)+k$$

Where $v_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (f_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

so

$[V] = [v_{iq}]_{3\times 4} = [b_{ik}] \wedge [c_{ij}]$

$$[v_{iq}] = [b_{ik}] \wedge [c_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$[a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}]) = [a_{ij}] \wedge [v_{iq}]$

$$[a_{ij}] \wedge [v_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[g_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } g_{ip} = \min(b_{ij},\ c_{ik}) \text{ such that } p = n(j-1)+k$$

Where $s_{iq} = \left[\bigvee_{p=(q-1)4+1}^{q4} (g_{ip}) \right]$ for all$i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

so

$$[S] = [s_{iq}]_{3\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$[s_{iq}] = [a_{ij}] \wedge [v_{iq}]$
$[s_{iq}] = [a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}])$
so

$$[a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}]) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

there fore
$([a_{ij}] \wedge [b_{ik}]) \wedge [c_{ij}] = [a_{ij}] \wedge ([b_{ik}] \wedge [c_{ij}])$
now it can satisfy the associative property

3.1.19 Example

(Associative law over Or-B-Product)

Let $U = \{u_1, u_2, u_3\}$ be the universal set, $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_3, e_4\}$, $C = \{e_2, e_3\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that
$f_A(e_1) = \{u_1, u_2\}$
$f_A(e_2) = \{u_2, u_3\}$
$f_A(e_3) = f_A(e_4) = \phi$
$R_A = \{(u_1, e_1), (u_2, e_2), (u_2, e_2), (u_3, e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and $f_B : E \to P(U)$ be such that
$f_B(e_3) = U$
$f_B(e_4) = \{u_1, u_3\}$
$f_B(e_1) = f_B(e_2) = \phi$
$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$[Y] = [y_{iq}] = [a_{ij}] \vee [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now

$([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}] = [y_{iq}] \vee [c_{ij}]$

$$[y_{ij}] \vee [c_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[e_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{and } e_{ip} = \max(y_{ij},\ c_{ik}) \text{ such that } p = n(j-1)+k$$

Where $w_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{q4} (e_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

so

$$[W] = [w_{iq}]_{3\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$[w_{iq}] = [y_{ij}] \vee [c_{ik}]$

$[w_{iq}] = ([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}]$

$$([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now we find $[a_{ij}] \vee ([b_{ij}] \vee [c_{ij}])$

$$[b_{ij}] \vee [c_{ik}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let

$$[f_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{and } f_{ip} = \max(b_{ij},\ c_{ik}) \text{ such that } p = n(j-1)+k$$

Where $v_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{q4} (f_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

so

$$[v_{iq}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$[V] = [v_{iq}]_{3\times 4} = [b_{ij}] \vee [c_{ik}]$

$$[b_{ij}] \vee [c_{ik}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

now

$[a_{ij}] \vee ([b_{ij}] \vee [c_{ij}]) = [a_{ij}] \vee [v_{iq}]$

$$[a_{ij}] \vee [v_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[g_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{and } g_{ip} = \max(b_{ij},\ c_{ik}) \text{ such that } p = n(j-1)+k$$

Where $s_{iq} = \left[\bigwedge_{p=(q-1)4+1}^{q4} (g_{ip}) \right]$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

so

$$[S] = [s_{iq}]_{3\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

as $[s_{iq}] = [a_{ij}] \vee [v_{ij}]$

$[s_{iq}] = [a_{ij}] \vee ([b_{ij}] \vee [c_{ij}])$

so

$$[a_{ij}] \vee ([b_{ij}] \vee [c_{ij}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

there fore

$([a_{ij}] \vee [b_{ik}]) \vee [c_{ij}] = [a_{ij}] \vee ([b_{ik}] \vee [c_{ij}])$

Similarly we can prove

$([a_{ij}] \barwedge [b_{ik}]) \barwedge [c_{ij}] = [a_{ij}] \barwedge ([b_{ik}] \barwedge [c_{ij}])$

$([a_{ij}] \veebar [b_{ik}]) \veebar [c_{ij}] = [a_{ij}] \veebar ([b_{ik}] \veebar [c_{ij}])$

3.1.20 Theorem

Or-B-Product is distributive over And-B-Product.

Proof. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in SM_{m\times n}$.

Then

$$[b_{ij}] \wedge [c_{ik}] = [d_{iq}]$$

Where $d_{iq} = \left(\bigvee_{p=(q-1)n+1}^{qn} (e_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $e_{ip} = \min(a_{ij}, b_{ik})$ such that $p = n(j-1)+k$

now

$[a_{ij}] \vee ([b_{ij}] \wedge [c_{ij}]) = [a_{ij}] \vee [d_{ik}]$

$$[a_{ij}] \vee [d_{ik}] = [g_{iq}]$$

if $g_{iq} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (f_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

Where $f_{ip} = \max(a_{ij}, b_{ik})$ such that $p = n(j-1)+k$

$[a_{ij}] \vee ([b_{ij}] \wedge [c_{ij}]) = [a_{ij}] \vee [d_{ik}] = [g_{ij}]$

Now R.H.S

$$[a_{ij}] \wedge [b_{ik}] = [h_{iq}]$$

Where $h_{iq} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (t_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

and $t_{ip} = \max(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$

Now

$$[a_{ij}] \vee [c_{ik}] = [s_{iq}]$$

Where $s_{ij} = \left(\bigwedge_{p=(q-1)n+1}^{qn} (v_{ip}) \right)$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

And $v_{ip} = \max(a_{ij}, c_{ik})$ such that $p = n(j-1) + k$

$$[h_{ij}] \wedge [s_{ip}] = [x_{iq}]$$

Where $x_{iq} = \left[\bigvee_{p=(q-1)n+1}^{qn} (y_{ip}) \right]$ for all $i = 1, 2, ..., m$ and $q = 1, 2, ..., n$

And $y_{ip} = \min(h_{ij}, s_{ik})$ such that $p = n(j-1) + k$

so

$[a_{ij}] \vee ([b_{ij}] \wedge [c_{ij}]) = ([a_{ij}] \wedge [b_{ij}]) \wedge ([a_{ij}] \vee [c_{ij}])$ ■

3.1.21 Example

(Or-B-product is distributive over And-B-Product)

Let $U = \{u_1, u_2, u_3\}$ be the universal set, $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_3, e_4\}$, $C = \{e_2, e_3\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that

$f_A(e_1) = \{u_1, u_2\}$

$f_A(e_2) = \{u_2, u_3\}$

$f_A(e_3) = f_A(e_4) = \phi$

$R_A = \{(u_1, e_1), (u_2, e_2), (u_2, e_2), (u_3, e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that

$f_B(e_3) = U$

$f_B(e_4) = \{u_1, u_3\}$

$f_B(e_1) = f_B(e_2) = \phi$

$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and $f_C : E \to P(U)$ be such that

$f_C(e_2) = \{u_2\}$

$f_C(e_3) = \{u_2, u_3\}$

$f_C(e_1) = f_C(e_4) = \phi$

$R_C = \{(u_2, e_2), (u_2, e_3), (u_3, e_3)\}$

R_C	e_1	e_2	e_3	e_4
u_1	0	0	0	0
u_2	0	1	1	0
u_3	0	0	1	0

$$C = [c_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To prove $[a_{ij}] \vee ([b_{ij}] \wedge [c_{ij}]) = ([a_{ij}] \vee [b_{ij}]) \wedge ([a_{ij}] \vee [c_{ij}])$

L.H.S

firstly we find

$$[b_{ij}] \wedge [c_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[g_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{and } g_{ip} = \min(b_{ij},\ c_{ik}) \text{ such that } p = n(j-1)+k$$

$$\text{Where } s_{iq} = \left(\bigvee_{p=(q-1)n+1}^{q4} (g_{ip}) \right) \qquad \text{for all } i = 1,\ 2,\ 3 \text{ and } q = 1,\ 2,\ 3,\ 4$$

then

$$[s_{iq}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

so

$$[s_{iq}] = [b_{ij}] \wedge [c_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now

$[a_{ij}] \vee ([b_{ij}] \wedge [c_{ij}]) = [a_{ij}] \vee [s_{iq}]$

$$[a_{ij}] \vee [s_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[f_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{and } f_{ip} = \max(a_{ij},\ s_{ik}) \text{ such that } p = n(j-1)+k$$

Where $e_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{q4} (f_{ip}) \right)$ for all $i = 1, 2, 3$ and $q = 1, 2, 3, 4$

$$[a_{ij}] \vee [s_{iq}] = [e_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now R.H.S

$$([a_{ij}] \vee [b_{ij}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[d_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and $d_{ip} = \max(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$

where $x_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{q4} (d_{ip}) \right)$ for all $i = 1, 2, 3$ and $q = 1, 2, 3, 4$

$$([a_{ij}] \vee [b_{ij}]) = [x_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now

$$([a_{ij}] \vee [c_{ij}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[y_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $y_{ip} = \max(a_{ij}, c_{ik})$ such that $p = n(j-1) + k$

Where $z_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{q4} (y_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

$$[z_{iq}] = ([a_{ij}] \vee [c_{ij}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

now

$$([x_{ij}] \wedge [z_{ik}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[g_{ip}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $g_{ip} = \min(x_{ij},\ z_{ik})$ such that $p = n(j-1) + k$

Where $h_{iq} = \left(\bigvee_{p=(q-1)n+1}^{q4} (u_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

then

$$[h_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$([x_{ij}] \wedge [z_{ik}]) = [h_{iq}]$

so $([a_{ij}] \vee [b_{ij}]) \wedge ([a_{ij}] \vee [c_{ij}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

so

$[a_{ij}] \vee ([b_{ij}] \wedge [c_{ij}]) = ([a_{ij}] \vee [b_{ij}]) \wedge ([a_{ij}] \vee [c_{ij}])$.

3.1.22 Theorem

And-B-Product is distributive over Or-B-Product.

$[a_{ij}] \wedge ([b_{ij}] \vee [c_{ij}]) = ([a_{ij}] \wedge [b_{ij}]) \vee ([a_{ij}] \wedge [c_{ij}])$

Proof. Strightforword ■

3.1.23 Remark

And-Not-B-Product is not distributive over Or-Not-B-Product.

$[a_{ij}] \bar{\wedge} ([b_{ij}] \veebar [c_{ij}]) \neq ([a_{ij}] \bar{\wedge} [b_{ij}]) \veebar ([a_{ij}] \bar{\wedge} [c_{ij}])$

3.1.24 Example

Let $U = \{u_1, u_2, u_3\}$ be the universal set, $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_3, e_4\}$, $C = \{e_2, e_3\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that

$f_A(e_1) = \{u_1, u_2\}$

$f_A(e_2) = \{u_2, u_3\}$

$f_A(e_3) = f_A(e_4) = \phi$

$R_A = \{(u_1, e_1), (u_2, e_2), (u_2, e_2), (u_3, e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that

$f_B(e_3) = U$

$f_B(e_4) = \{u_1, u_3\}$

$f_B(e_1) = f_B(e_2) = \phi$

$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and $f_C : E \to P(U)$ be such that

$f_C(e_2) = \{u_2\}$

$f_C(e_3) = \{u_2, u_3\}$
$f_C(e_1) = f_C(e_4) = \phi$
$R_C = \{(u_2, e_2), (u_2, e_3), (u_3, e_3)\}$

R_C	e_1	e_2	e_3	e_4
u_1	0	0	0	0
u_2	0	1	1	0
u_3	0	0	1	0

$$C = [c_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[b_{ij}] \veebar [c_{ik}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \veebar \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[f_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and $f_{ip} = \max(b_{ij},\ 1 - c_{ik})$ such that $p = n(j-1) + k$

Where $e_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{q4} (f_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

$$[e_{iq}] = [b_{ij}] \veebar [c_{ik}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now
$[a_{ij}] \barwedge ([b_{ij}] \veebar [c_{ij}]) = [a_{ij}] \barwedge [e_{iq}]$

$$[a_{ij}] \barwedge [e_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \barwedge \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[g_{ip}] = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\text{and } g_{ip} = \min(a_{ij},\ 1-e_{ik}) \text{ such that } p = n(j-1)+k$$

$$\text{Where } s_{iq} = \left(\bigvee_{p=(q-1)n+1}^{q4} (g_{ip})\right) \qquad \text{for all } i = 1,\ 2,\ 3 \text{ and } q = 1,\ 2,\ 3,\ 4$$

then

$$[s_{iq}] = \begin{bmatrix} 0&0&0&0 \\ 1&1&0&0 \\ 0&1&0&0 \end{bmatrix}$$

so

$$[a_{ij}] \bar{\wedge} [e_{ik}] = [s_{iq}] = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&1&1 \end{bmatrix}$$

$$[a_{ij}] \bar{\wedge} ([b_{ij}] \underline{\vee} [c_{ij}]) = [s_{iq}] = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&1&1 \end{bmatrix}$$

Now R.H.S

$([a_{ij}] \bar{\wedge} [b_{ij}]) \underline{\vee} ([a_{ij}] \bar{\wedge} [c_{ij}])$

$$[a_{ij}] \bar{\wedge} [b_{ik}] = \begin{bmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 0&1&0&0 \end{bmatrix} \bar{\wedge} \begin{bmatrix} 0&0&1&1 \\ 0&0&1&0 \\ 0&0&1&1 \end{bmatrix}$$

$$[f_{ip}] = \begin{bmatrix} 1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&1&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\text{and } f_{ip} = \min(a_{ij},\ 1-b_{ik}) \text{ such that } p = n(j-1)+k$$

$$\text{Where } x_{iq} = \left(\bigvee_{p=(q-1)n+1}^{q4} (f_{ip})\right) \qquad \text{for all } i = 1,\ 2,\ 3 \text{ and } q = 1,\ 2,\ 3,\ 4$$

then

$$[x_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

so

$$[a_{ij}] \overline{\wedge} [b_{ik}] = [x_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$([a_{ij}] \overline{\wedge} [c_{ik}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \overline{\wedge} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[g_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } g_{ip} = \min(a_{ij},\ 1 - c_{ik}) \text{ such that } p = n(j-1) + k$$

Where $y_{iq} = \left(\bigvee_{p=(q-1)n+1}^{q4} (g_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

then

$$[y_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

so

$$([a_{ij}] \overline{\wedge} [c_{ik}]) = [y_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$([x_{ij}] \underline{\vee} [y_{ik}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \underline{\vee} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[z_{ip}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

where $z_{ip} = \max(x_{ij},\ 1 - y_{ik})$ such that $p = n(j-1) + k$

if $h_{iq} = \left(\bigwedge_{p=(q-1)4+1}^{q4} (z_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

$$[h_{iq}] = ([x_{ij}] \veebar [y_{ik}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$([a_{ij}] \barwedge [b_{ij}]) \veebar ([a_{ij}] \barwedge [c_{ij}]) = ([x_{ij}] \veebar [y_{ik}]) = [h_{iq}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

so$[a_{ij}] \barwedge ([b_{ij}] \veebar [c_{ij}]) \neq ([a_{ij}] \barwedge [b_{ij}]) \veebar ([a_{ij}] \barwedge [c_{ij}])$

3.1.25 Remark

Or-Not-B-Product is not distributive over And-Not-B-Product.

$[a_{ij}] \veebar ([b_{ij}] \barwedge [c_{ij}]) \neq ([a_{ij}] \veebar [b_{ij}]) \barwedge ([a_{ij}] \veebar [c_{ij}])$

3.1.26 Remark

Let $[a_{ij}],\ [b_{ij}],\ \in SM_{m\times n}$ and $\circledast \in (\wedge,\ \vee,\ \barwedge,\ \veebar)$ be the binary operation. Then $[a_{ij}] \circledast [b_{ij}] \neq [b_{ij}] \circledast [a_{ij}]$.

3.1.27 Example

Let $U = \{u_1,\ u_2,\ u_3\}$ be the universal set, $E = \{e_1,\ e_2,\ e_3,\ e_4\}$ be the set of parameters and $A = \{e_1,\ e_2\}$, $B = \{e_3,\ e_4\}$ be the subsets of E.

Let $f_A : E \to P(U)$ be such that

$f_A(e_1) = \{u_1,\ u_2\}$

$f_A(e_2) = \{u_2,\ u_3\}$

$f_A(e_3) = f_A(e_4) = \phi$

$R_A = \{(u_1,\ e_1),\ (u_2,\ e_2),\ (u_2,\ e_2),\ (u_3,\ e_2)\}$

R_A	e_1	e_2	e_3	e_4
u_1	1	0	0	0
u_2	1	1	0	0
u_3	0	1	0	0

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f_B : E \to P(U)$ be such that
$f_B(e_3) = U$
$f_B(e_4) = \{u_1, u_3\}$
$f_B(e_1) = f_B(e_2) = \phi$
$R_B = \{(u_1, e_3), (u_2, e_3), (u_3, e_3), (u_1, e_4), (u_3, e_4)\}$

R_B	e_1	e_2	e_3	e_4
u_1	0	0	1	1
u_2	0	0	1	0
u_3	0	0	1	1

$$B = [b_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[d_{ip}] = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $d_{ip} = \min(a_{ij}, b_{ik})$ such that $p = n(j-1) + k$

if $y_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (d_{ip}) \right)$ for all $i = 1, 2, 3$ and $q = 1, 2, 3, 4$

Then

$[Y] = [y_{iq}]_{3\times 4} = [a_{ij}] \wedge [b_{ik}]$

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now

$$[b_{ij}] \wedge [a_{ik}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[f_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

where $f_{ip} = \min(a_{ij},\ b_{ik})$ such that $p = n(j-1) + k$

if $e_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (f_{ip}) \right)$ for all $i = 1,\ 2,\ 3$ and $q = 1,\ 2,\ 3,\ 4$

Then

$[E] = [e_{iq}]_{3\times 4} = [b_{ij}] \wedge [a_{ik}]$

$$[b_{ij}] \wedge [a_{ik}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

hence

$[a_{ik}] \wedge [b_{ij}] \neq [b_{ij}] \wedge [a_{ik}]$.

Therefore commutative law does not hold with respect to And-B-Product

similarly

$[a_{ik}] \vee [b_{ij}] \neq [b_{ij}] \vee [a_{ik}]$

$[a_{ik}]\, \overline{\wedge}\, [b_{ij}] \neq [b_{ij}]\, \overline{\wedge}\, [a_{ik}]$

$[a_{ik}]\, \underline{\vee}\, [b_{ij}] \neq [b_{ij}]\, \underline{\vee}\, [a_{ik}]$

3.1.28 Theorem

Let $SM_{m\times n}$ be the collection of all the soft matrices and $\circledast \in \{\wedge,\ \vee,\ \overline{\wedge},\ \underline{\vee}\}$ be the binary operations, then $(SM_{m\times n},\ \circledast)$ is a semigroup.

Proof. Straightforward. ■

3.1.29 Theorem

Let $SM_{m\times n}$ be the collection of all the soft matrices and $\circledast$, $o \in \{\wedge, \vee\}$ be the binary operations, Then $(SM_{m\times n}, \circledast, o)$ is a semiring.

Proof. Straightforward. ■

3.2 Soft Matrix Decision Making

In this section we construct a soft matrix decision making with the help of soft matrix decision function and then select an optimum solution from the decision soft matrix.

3.2.1 Definition

let $[a_{ij}], [b_{ij}] \in SM_{m\times n}$, and let $[c_{ij}]$ be the product of $[a_{ij}]$ and $[b_{ij}]$. Then the soft matrix decision function, denoted SMDF is define as follows

$$SMDF : SM_{m\times n} \to SM_{m\times 1}$$

$$SMDF[c_{ij}] = \begin{bmatrix} \dfrac{\sum_{j=1}^{n}\{c_{ij}\}}{n} \end{bmatrix} \text{ where } i = 1,\ 2,\ ...,\ m$$

the one column soft matrix $SMDF[c_{ij}]$ is called decision soft matrix.

3.2.2 Definition

let $U = \{u_1, u_2, \ldots u_n\}$ be initial universe and $SMDF[c_{ij}] = [d_{i1}]$. Then a subset of U can be obtained by using $[d_{i1}]$ as in he following way

$optm_{[d_{i1}]}(U) = \{u_i : u_i \in U, \max(d_{i1})\}$

3.2.3 Applications

Assume that a person wants to seek admission in Ph.D. program and the universal set contain different universities $U = \{u_1,\ u_2,\ u_3,\ u_4,\ u_5\}$, which may be characterized by a set of parameters $E = \{e_1,\ e_2,\ e_3,\ e_4\}$. For $j = 1,\ 2,\ 3,\ 4$ the parameters e_j stand for "Part time studies", "less Fee ", "Full time studies" and "Located near Islamabad" respectively. Then we can give the following examples.

3.2.4 Example

Suppose that two Students, Mr. A and Mr. B, come to the contact with each other and want to get admission. If each of them has to consider their own set of parameters, then we select a University on the basis of the sets of partners' parameters by using the Soft Matrix Decision as follows.

Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$is a universal set and $E = \{e_1, e_2, e_3, e_4\}$ is a set of all parameters.

Mr. A and Mr. B have to choose the sets of their parameters, $A = \{e_2, e_3, e_4\}$ and $B = \{e_1, e_3, e_4\}$ respectively.

Then we can write the following soft matrices which are constructed according to their parameters.

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad [b_{ik}] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we can find a product of the soft matrices $[a_{ij}]$ and $[b_{ik}]$ by using And-B-product as follows

Now we apply And-B-product since both Mr. A and Mr. B choices have to be considered.

$$[d_{ip}] = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&1 \\ 0&0&0&0&1&0&0&1&1&0&0&1&1&0&0&1 \\ 0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1 \end{bmatrix}$$

$$\text{and } d_{ip} = \min(a_{ij}, b_{ik}) \text{ such that } p = n(j-1)+k$$

Where $y_{iq} = \left(\bigvee_{p=(q-1)4+1}^{q4} (d_{ip}) \right)$ for all $i = 1, 2, 3, 4, 5$and $q = 1, 2, 3, 4$

Then

$[Y] = [y_{iq}]_{5\times 4} = [a_{ij}] \wedge [b_{ik}]$

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

We can find a decision soft matrix as

$$MDF\left([a_{ij}] \wedge [b_{ik}]\right) = \begin{bmatrix} 0.5 \\ 0.75 \\ 0.5 \\ 0.25 \\ 0.5 \end{bmatrix}$$

we can find an optimum set of U according to $MDF\left([a_{ij}] \wedge [b_{ik}]\right)$

$optm_{MDF([a_{ij}] \wedge [b_{ik}])}(U) = \{u_2\}$, where u_2 is an optimum University for Mr. A and Mr. B.

Note that the optimal set of U may contain more than one element.

Similarly, we can also use the other products $([a_{ij}] \vee [b_{ik}])$, $([a_{ij}] \overline{\wedge} [b_{ik}])$and $([a_{ij}] \underline{\vee} [b_{ik}])$ for the other convenient problems.

Conclusion

The soft set theory has been used in different fields. The results of this thesis show that the B-products are binary. Further it is shown that associative laws as well as distributive laws holds. At the end of this thesis we highlighted that soft matrix decision making on the basis of soft set theory is useful. The example of a student who is looking for some university for Ph.D. is also given in this thesis. These type of products can also be defined in fuzzy soft matrices. and we can also take the products of the soft sets and then convert it into soft matrices and can compair the result in both the cases. This Converse can be applied in both soft matrices and fuzzy soft matrices.

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Conclusion

The soft set theory has been used in different fields. The results of this thesis show that the B-products are binary. Further it is shown that associative laws as well as distributive laws holds. At the end of this thesis we highlighted that soft matrix decision making on the basis of soft set theory is useful. The example of a student who is looking for some university for Ph.D. is also given in this thesis. These type of products can also be defined in fuzzy soft matrices. and we can also take the products of the soft sets and then convert it into soft matrices and can compair the result in both the cases. This Converse can be applied in both soft matrices and fuzzy soft matrices.

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