PROPERTIES OF SOME SPECIAL K-FUNCTIONS AND THEIR APPLICATIONS

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ISLAMABAD-PAKISTAN

2017

Matrix K- Functions

Hypergeometric

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS DOCTOR OF PHILOSOPHY IN MATHEMATICS AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD.

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2017

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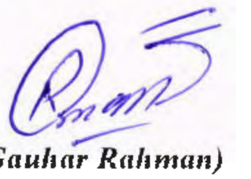

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TO

MY PARENTS

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Acknowledgements

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List of Publications

1. G. Rahman, A. Ghaffar, S. D. Purohit, S. Mubeen, M. Arshad, *On the hypergeometric matrix k-functions*, Bulletin of Mathematical Analysis and Applications, Vol. 8, 4(2016), Pages 98-111.

2. S. Mubeen, S. D. Purohit, M. Arshad, G. Rahman, *Extension of k-gamma, k-beta functions and k-beta distribution*, Journal of Mathematical Analysis, Vol. 7, 5(2016), Pages 118-131.

3. G. Rahman, M. Arshad, S. Mubeen, *Some results on a generalized hypergeometic k-functions*, Bulletin of Mathematical Analysis and Applications, Vol. 8, 3(2016), Pages 66-77.

4. S. Mubeen, C. G. Kokologiannaki, G. Rahman, M. Arshad, *Properties of generalized hypergeometric k-functions via k-fractional calculus*, Far East Journal of Applied Mathematics, (2017) (Accepted).

5. K. S. Nisar, G. Rahman, J. Choi, S. Mubeen, M. Arshad, *Generalized hypergeometric k-functions via (k, s)-fractional calculus*, J. Nonlinear Sci. Appl., 10 (2017), 1791-1800.

6. K. S. Nisar, G. Rahman, D. Baleanu, S. Mubeen, M. Arshad, *The (k, s)-fractional calculus of k-Mittag-Leffler function*, Advances in Difference Equations (2017) 2017:118.

7. G. Rahman, K. S. Nisar, S. Mubeen, M. Arshad, Generalized fractional integration of k-Bessel function, Advances in Studies and Contemporary Mathematics, (2017), (Accepted).

Notations

The following symbols are used to represent the text available in the research work.

The set of integers:	$\mathbb{Z}$
The set of natural numbers:	$\mathbb{N}$
The set of complex numbers:	$\mathbb{C}$
The set of positive complex numbers:	$\mathbb{C}^+$
The set of real numbers:	$\mathbb{R}$
The set of positive real numbers:	$\mathbb{R}^+$
Any real numbers:	$a, b, c, d, p, q, u, v, w, r, s$
Any complex numbers:	$\alpha, \beta, \gamma, \delta, \rho, \omega, \mu, \nu$
Any integral numbers:	i, j, m, n
Pochhammer's symbol:	$(\alpha)_n$
Pochhammer k-symbol:	$(\alpha)_{n,k}$
Gamma function:	$\Gamma(x)$
Gamma k-function :	$\Gamma_k(x)$
Extended gamma function:	$\Gamma_b(x)$
Extended gamma k-function:	$\Gamma_{b,k}(x)$
Beta function:	$\mathcal{B}(x, y)$
Beta k-function :	$\mathcal{B}_k(x, y)$
Extended beta function:	$\mathcal{B}_b(x, y)$
Extended beta k-function:	$\mathcal{B}_{b,k}(x, y)$
Hypergeometric function:	${}_2F_1\Big[\alpha, \beta; \gamma; z\Big]$
Generalized hypergeometric function:	${}_pF_q\Big[\alpha_1, \cdots, \alpha_p; \beta_1, cdots, \beta_q; z\Big]$
Wright-hypergeometric function:	${}_p\Psi_q\Big[\alpha_1, \cdots, \alpha_p; \beta_1, cdots, \beta_q; z\Big]$
Hypergeometric k-function:	${}_2F_{1,k}\Big[\alpha, \beta; \gamma; z\Big]$
Confluent hypergeometric function:	${}_1F_1\Big[\beta; \gamma; z\Big]$

Confluent hypergeometric k-function:	${}_1F_{1,k}\Big[\beta;\gamma;z\Big]$
Generalized hypergeometric function:	${}_2R_1\Big[\alpha,\beta;\gamma;\tau;z\Big]$
Generalized hypergeometric k-function:	${}_2R_{1,k}\Big[\alpha,\beta;\gamma;\tau;z\Big]$
Mittag-Leffler function:	$E^{\gamma}_{\alpha,\beta}(z)$
Mittag-Leffler k-function:	$E^{\gamma}_{k,\alpha,\beta}(z)$
Bessel functions:	$J_v(z), I_v(z)$
Bessel k-function:	$W^{k}_{v,c}$
Expected value of a random variable X:	$E(X)$
Variance of a random variable X:	$\sigma^2(X)$
Fractional integral operator:	I^{μ}_{0}
k-Fractional integral operator:	$I^{\mu}_{0,k}$
Fractional Differential operator:	D^{μ}_{0}
k-Fractional Differential operator:	$D^{\mu}_{0,k}$
(k,s)-Fractional integral operator:	${}^{s}_{k}I^{\mu}_{0}$
(k,s)-Fractional Differential operator:	${}^{s}_{k}D^{\mu}_{0}$
Erdélyi-Kober fractional operators;	$K^{+}_{\eta,\alpha}f, K^{-}_{\eta,\alpha}f$

Preface

The theory of special functions constitutes an important part of mathematics. In the last three centuries, the necessity of solving the problems arising in the fields of hydrodynamics, control theory, classical mechanics stimulated the development of the theory of special functions of one and several variables. Special functions have also extensive applications in pure mathematics as well as in applied mathematics such as electrical current, fluid dynamics, heat conduction, solutions of wave equations, moments of inertia and quantum mechanics, etc.

Mathematical models of physical phenomena contain, as a rule, ordinary differential equations, partial differential equations or systems of such equations. However, only very few of the equations which arise from physically interesting problems can be solved in the class of elementary functions. Thus there exists a necessity of extending the class of studied functions. New functions were usually defined as solutions to differential equations or systems of such equations and were called special functions. This was the way in which the gamma and hypergeometric functions came into existence. The general name of these functions are called special functions. The gamma function has an impressive number of different representations, including series, limit and integral forms, each offering their own particular advantage in different applications. Apart from its central role in pure mathematics, the gamma function has an important role in the study of the analytic solutions of many problems in area of applied sciences, astrophysics, diffraction and plasma wave theory, fluid flow, nuclear and molecular physics, probability and engineering. The gamma function was first established by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial function to non-integer numbers (real and complex numbers). Latter, this function was studied by other mathematicians like Adrien Marie Legender (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901).

The basic element in the theory of special functions is the Pochhammer's symbol because this element plays a vital role in the structures or relations of most of the special functions. It was introduced by L. A. Pochhammer and is defined for $\alpha \neq 0$ and $n \in \mathbb{N}$ as

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1).$$

It is also known as the rising factorial function because the term $(\alpha)_n$ is expressed in the product of n factors, starting with α and with each factor large by unity than the preceding factor.

Special functions is very wide and there is an unlimited number of special functions which were investigated and studied by many researchers. It is quite impossible to discuss the history of all the members of this field. That is why, we discuss the brief introduction of few special functions of our interest. Our major concern here is with the gamma, beta and hypergeometric functions and their application in theory of fractional calculus.

This thesis consist of six chapters. In first chapter, we deal with the study of the hypergeometric k-functions i.e., $F_k(P, Q; R; z)$ having the matrix arguments P, Q and R and these matrix arguments satisfy the matrix differential equation in terms of the new parameter $k > 0$ which is the improved version of generalization of classical hypergeometric matrix functions. Further, we obtain an integral representation of $F_k(P, Q; R; z)$ for the cases where Q, R and $R - Q$ are positive stable matrices with the property that $QR = RQ$ by using the definitions of gamma and beta matrix k-functions recently defined by the researchers.

In second chapter, we introduce extended gamma, beta k-functions and extended beta k-distribution. These newly defined functions are the generalized forms of the extended gamma, beta functions and extended beta distribution. Also, we further generalize the said functions and prove some of their properties.

In chapter 3, the authors introduce τ-Gauss hypergeometric k-functions. Some of the differential properties, integral representation, contiguous relations and differential formulas of the generalized hypergeometric k-functions ${}_2R_{1,k}(a, b; c; \tau; z)$ are established.

In chapter 4, we prove the various properties of generalized hypergeometric k-functions which includes the generalized Riemann-Liouville k-fractional integrals, differential operators and the Lebesgue measurable real valued or complex valued functions. Also, we obtain the (k, s) fractional integral and differential formulas of generalized hypergeometric k-functions ie., τ-Gauss hypergeometric k-function.

In chapter five, we deal with the Mittag-Leffler k-function and investigate the (k, s)-fractional integrals and differentials of such function. Further, we considered a number of certain consequences of the main results.

In chapter six, we study the two integral transforms which involving the Gauss hypergeometric function as its kernels. We prove some compositions formulas for such a generalized fractional integrals with Bessel k-functions. The results are established in terms of generalized Wright type hypergeometric function and generalized hypergeometric series. Also, Also, we present some corresponding assertions for Riemann-Liouville and Erdélyi-Kober fractional integral transforms.

I am grateful to my Supervisors **Prof. Dr. Muhammad Arshad**, Dean, Faculty of Basic and Applied Sciences, International Islamic University Islamabad, for providing research atmosphere and facilities in the Department. I would like to express my sincere and heartiest gratitude to my Co-supervisor **Dr. Shahid Mubeen**, Associate Professor, University of Sargodha, for his guidance, affection, deep consideration and active co-operation that made possible this work to meet its end successfully well in time.

Gauhar Rahman **2017**

Chapter 1

On the Hypergeometric Matrix k-Functions

In this chapter [96], we deal with gamma and beta matrix k-functions. We introduce the Gauss hypergeometric matrix k-functions $F_k(P,Q;R;z)$ and its various properties. The necessary convergence conditions of hypergeometric matrix k-functions $F_k(P,Q;R;z)$ on the boundary of unit disc are discussed. Also, we define some results of an integral representations of hypergeometric matrix k-functions.

1.1 Introduction

Most of the special functions appeared in mathematical physics, engineering, analytic functions and mathematical statistical are special cases of hypergeometric functions ([59,60], [87], [100], [106], [110]). A function of matrix argument is a real or complex valued function of the elements of a matrix. These matrix function arises in the literature of Statistical distributions [13], Lie groups theory [41], and in connection with Laguerre matrix polynomials and system of second order differential equations for matrix arguments, orthogonal matrix polynomial and second order differential equations, Hermite and Legendre second order differential equations and the resultant polynomial families ([43], [44], [45], [46]). Also, many researchers ([35], [37], [38], [42, 47]) have defined matrix computation, Bessel function of matrix arguments, ordinary differential equation of matrix arguments, properties of gamma and beta matrices and hypergeometric matrix arguments. The operational calculus of emerging theory of orthogonal matrix polynomials ([21], [22], [23]) propose the study of hypergeometric matrix k-functions

In this chapter, we consider the well known gamma and beta matrix k-functions. For this continuation of our purpose, we recall the following matrix analog formula

$$\beta_k(P,Q) = \frac{\Gamma_k(P)\Gamma_k(Q)}{\Gamma_k(P+Q)} \tag{1.1.1}$$

see [77]. Recently the researchers have worked on special k-functions ([72], [74], [76]). Mubeen *et al.* [75] defined the solution of hypergeometric k-differential equations. We also prove that if matrices Q and R commutes and are positive stables where positive stable means if every eigenvalue of the matrix has positive real part, then $F_k(P,Q;R;z)$ is a solution of the second order differential equation

$$kz(1-kz)\omega'' - kzP\omega' + (R-(Q+kI)kz)\omega' - PQ\omega = 0.$$

For any arbitrary matrix P in $\mathbb{C}^{r\times r}$ and for invertible matrix R such that whose eigenvalues are positive integers, then we can prove that the matrix polynomial

$$kz(1-kz)\omega'' - kzP\omega' + (R+(n-k)kIz)\omega' - nP\omega = 0$$

has n degree of solutions where $n \in \mathbb{Z}^+$. Throughout in this chapter, for a matrix $P \in \mathbb{C}^{r\times r}$ and $\delta(P)$ denotes the spectrum of a matrix p, which contains the set of all the eigenvalues of P. The 2-norm of matrix P is denoted and defined by

$$\| P \| = \sup_{x\neq 0} \frac{\| Px \|_2}{\| x \|_2}, \tag{1.1.2}$$

where for a z in $C^{r\times r}$, the Euclidean norm of z is defined by $\| z \|_2 = (zTz)^{\frac{1}{2}}$. Let us define the real numbers $\alpha(P)$ and $\beta(P)$ by

$$\alpha(P) = \max\{\Re(z) : z \in \delta(P)\}, \qquad \beta(P) = \min\{\Re(z) : z \in \delta(P)\}. \quad (1.1.3)$$

Let the holomorphic functions of the complex plane $f(z)$ and $g(z)$ be defined in an open set Ω for the complex variable z and the matrix P is in $\mathbb{C}^{r\times r}$ with $\delta(P) \subset \Omega$, then from the properties of matrix functional calculus ([20]), we have

$$f(P)g(P) = g(P)f(P). \quad (1.1.4)$$

The reciprocal of gamma k-function denoted by $\Gamma_k^{-1} = \frac{1}{\Gamma_k}$ is an entire function of the complex variable. Like wise the image of the inverse gamma matrix k-function acting on the matrix P, denoted by $\Gamma_k^{-1}(P)$ is a well defined matrix for $k > 0$. Now, if the matrix $P + nkI$ is invertible for every integer $n \geq 0$ and $k > 0$,
then the gamma matrix k-function Γ_k is invertible and its inverse coincides with $\Gamma_k^{-1}(P)$, and recently Mubeen *et al.* [77] defined

$$P(P + kI)(P + 2kI) \cdots (P + (n-1)kI)\Gamma_k^{-1}(P + nkI) = \Gamma_k^{-1}(P), n \geq 1, k > 0 \quad (1.1.5)$$

In the same paper, they introduced by using the condition that $P + nkI$ is invertible matrix, then equation (1.1.5) can be written as

$$P(P + kI)(P + 2kI) \cdots (P + (n-1)kI) = \Gamma_k(P + nkI)\Gamma_k^{-1}(P), n \geq 1, k > 0, \quad (1.1.6)$$

and like the Pochhammer k-symbol for any matrix P in $\mathbb{C}^{r\times r}$ by application of the matrix functional calculus, they defined

$$(P)_{n,k} = P(P + kI)(P + 2kI) \cdots (P + (n-1)kI), \quad n > 0, \quad (P)_0 = I. \quad (1.1.7)$$

The Schur deposition of a matrix P is given by ([35])

$$\|e^{tP}\| \leq e^{t\alpha(P)} \sum_{i=0}^{r-1} \frac{(\|P\| r^{\frac{1}{2}} t)}{i!}, \quad t \geq 0. \tag{1.1.8}$$

1.2 On Gamma, Beta Matrix k-Functions

In this section, we used the property of commutativity of matrices and extend the matrix framework of gamma and beta k-functions. We recall the following results recently defined by Mubeen *et al.* [77].

Definition 1.2.1. For a positive stable matrix P in $\mathbb{C}^{r\times r}$, we define the gamma matrix k-function as

$$\Gamma_k(P) = \lim_{n\to\infty} n! k^n (P)_{n,k}^{-1} (nk)^{\frac{P}{k}-I}, \tag{1.2.1}$$

where $n \geq 1$ is an integer and $k > 0$.

Definition 1.2.2. For a positive stable matrices P and Q in $\mathbb{C}^{r\times r}$, we define beta matrix k-function as

$$\beta_k(P,Q) = \frac{1}{k} \int_0^1 t^{\frac{P}{k}-I} (1-t)^{\frac{Q}{k}-I} dt. \tag{1.2.2}$$

Hence, we defined that if the positive stable matrices P and Q are commuting ie., $PQ = QP$, then $\beta_k(P,Q) = \beta_k(Q,P)$, and for symmetry of beta matrix k-functions comutativity is one of the necessary condition see [77].

Lemma 1.2.1. *Let $P, Q \in \mathbb{C}^{r\times r}$ be two positive matrices such that $PQ = QP$ and the matrix $P + Q + mkI$ is invertible for all integer $m \geq 0$ and $k > 0$.*
If $n \geq 0$, then the following relations hold true:
(i)

$$\beta_k(P, Q + nkI) = (P+Q)_{n,k}^{-1} (Q)_{n,k} \beta_k(P,Q),$$

(ii)

$$\beta_k(P + nkI, Q + nkI) = (P+Q)_{2n,k}^{-1} (Q)_{n,k} \beta_k(P,Q).$$

Proof. (i) If we put $n = 0$, then the proof is obvious. Let us assume that $0 < m \leq n$ and using the condition that $PQ = QP$, we have

$$\begin{aligned}
\beta_k(P, Q + mkI) &= \frac{1}{k}\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q}{k}+(m-1)I}dt \\
&= \frac{1}{k}\lim_{\delta\to 0}\int_\delta^{1-\delta} t^{\frac{P}{k}-I}(1-t)^{\frac{Q}{k}+(m-1)I}dt \\
&= \frac{1}{k}\lim_{\delta\to 0}\int_\delta^{1-\delta} t^{\frac{P+Q}{k}+(m-2)I}(1-t)^{\frac{Q}{k}+(m-1)I}t^{-(\frac{Q}{k}+(m-1)I)}dt \\
&= \frac{1}{k}\lim_{\delta\to 0}\int_\delta^{1-\delta} u(t)v(t)dt, \qquad (1.2.3)
\end{aligned}$$

where

$$u(t) = (1-t)^{\frac{Q}{k}+(m-1)I}t^{-(\frac{Q}{k}+(m-1)I)}t^{\frac{P}{k}}, \quad v(t) = t^{\frac{P+Q}{k}+(m-2)I}.$$

Integrating equation (1.2.3) by parts, we get

$$\begin{aligned}
\beta_k(P, Q + mkI) &= \lim_{\delta\to 0}[k(P+Q+(m-1)kI)^{-1}(1-t)^{\frac{Q}{k}+(m-1)I}t^{\frac{P}{k}}]_{t=\delta}^{t=1-\delta} \\
&+ \lim_{\delta\to 0} k(P+Q+(m-1)kI)^{-1} \\
&\times \int_\delta^{1-\delta}\{\frac{1}{k}(Q+(m-1)kI)(1-t)^{\frac{Q}{k}+(m-2)I}t^{\frac{P}{k}} \\
&+ \frac{1}{k}(Q+(m-1)kI)(1-t)^{\frac{Q}{k}+(m-1)}t^{\frac{P}{k}}\}dt \\
&= k(P+Q+(m-1)kI)^{-1}(Q+(m-1)kI) \\
&\times \frac{1}{k}\int_0^1 (1-t)^{\frac{Q}{k}+(m-1)}t^{\frac{P}{k}}dt \\
&= (P+Q+(m-1)kI)^{-1}(Q+(m-1)kI) \\
&\times \beta_k(P, Q+(m-1)kI).
\end{aligned}$$

Hence by using an induction, we obtain

$$\beta_k(P, Q + nkI) = (P+Q)^{-1}_{n,k}(Q)_{n,k}\beta_k(P,Q).$$

(ii). Taking $\hat{P} = P + nkI$ where $n \geq 1$. Then by (i) it becomes

$$\beta_k(\hat{P}, Q + nkI) = (\hat{P} + Q)^{-1}_{n,k}(Q)_{n,k}\beta_k(\hat{P}, Q). \tag{1.2.4}$$

As $PQ = QP$, therefore we have, $\hat{P}Q = Q\hat{P}$ and $\beta_k(\hat{P}, Q) = \beta_k(Q, \hat{P})$. By (1.2.4) it becomes

$$\beta_k(\hat{P}, Q + nkI) = (\hat{P} + Q)^{-1}_{n,k}(Q)_{n,k}\beta_k(Q, \hat{P}). \tag{1.2.5}$$

Also by (i), we have

$$\beta_k(Q, P + nkI) = (Q + P)^{-1}_{n,k}(P)_{n,k}\beta_k(Q, P) = (Q + P)^{-1}_{n,k}(P)_{n,k}\beta_k(P, Q). \tag{1.2.6}$$

By equations (1.2.4) and (1.2.5), we get

$$\begin{aligned} \beta_k(P + nkI, Q + nkI) = \beta_k(\hat{P}, Q + nkI) &= (P + Q + nkI)^{-1}_{n,k}(Q)_{n,k}(P)_{n,k} \quad (1.2.7)\\ &\times (Q + P)^{-1}_{n,k}\beta_k(P, Q). \quad (1.2.8) \end{aligned}$$

Now by definition, we have $(P + Q + nkI)_{n,k}(Q + P)_{n,k} = (P + Q)_{2n,k}$. Hence by substituting in equation (1.2.7), we get the required result as

$$\beta_k(P + nkI, Q + nkI) = (P + Q)^{-1}_{2n,k}(P)_{n,k}(Q)_{n,k}\beta_k(P, Q).$$

□

Lemma 1.2.2. *Let P and Q be commuting matrices in $\mathbb{C}^{r\times r}$ such that P, Q and $P + Q$ are positive stable matrices, then*

$$\beta_k(P, Q) = \Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P + Q).$$

Proof. Since the matrices P and Q are stable and also $PQ = QP$, we can write it as

$$\Gamma_k(P)\Gamma_k(Q) = (\int_0^\infty u^{P-I}e^{-\frac{u^k}{k}}du)(\int_0^\infty v^{Q-I}e^{-\frac{v^k}{k}}dv). \tag{1.2.9}$$

By changing of variables $x = \frac{u^k}{u^k+v^k}$ and $y = u^k + v^k$, then equation (1.2.9) becomes

$$\begin{aligned} \Gamma_k(P)\Gamma_k(Q) &= \int_0^\infty\int_0^1 (xy)^{\frac{1}{k}(P-I)}e^{-\frac{1}{k}(xy)}\frac{1}{k}x^{\frac{1}{k}-I}y^{\frac{1}{k}}(y(1-x))^{\frac{1}{k}(Q-I)}e^{-\frac{1}{k}(y(1-x))}\\ &\times \frac{1}{k}y^{\frac{1}{k}-I}(1-x)^{\frac{1}{k}}dxdy \end{aligned}$$

$$= \left(\frac{1}{k}\int_0^\infty (y)^{\frac{1}{k}(P+Q)-I} e^{-\frac{y}{k}} dy\right)\left(\frac{1}{k}\int_0^1 x^{\frac{P}{k}-I}(1-x)^{\frac{Q}{k}-I}dx\right). \qquad (1.2.10)$$

Now by replacing $y = t^k$ in the first integral of (1.2.10), we get

$$\begin{aligned} \Gamma_k(P)\Gamma_k(Q) &= \left(\int_0^\infty t^{P+Q-I} e^{-\frac{t^k}{k}} dt\right)\left(\frac{1}{k}\int_0^1 x^{\frac{P}{k}-I}(1-x)^{\frac{Q}{k}-I}dx\right) \\ &= \Gamma_k(P+Q)\beta_k(P,Q). \end{aligned}$$

□

Definition 1.2.3. Let us consider P and Q be two commuting matrices in $\mathbb{C}^{r\times r}$ such that for all integer $n \geq 0$ and satisfy the condition

$$P + nkI, \qquad Q + nkI, \qquad P + Q + nkI \quad \forall \quad k > 0, \qquad (1.2.11)$$

are invertible matrices.
Let $\alpha(P,Q) = \min\{\alpha(P), \alpha(Q), \alpha(P+Q)\}$ and let $n_0 = n_0(P,Q) = [|\alpha(P,Q)|] + 1$, where $[|\alpha(P,Q)|]$ denotes the entire part function. Then beta k-function $\beta_k(P,Q)$ is defined by

$$\beta_k(P,Q) = (P)^{-1}_{n_0,k}(Q)^{-1}_{n_0,k}(P+Q)_{2n_0,k}\beta_k(P+n_0kI, Q+n_0kI). \qquad (1.2.12)$$

Theorem 1.2.3. *Let P and Q be two commuting matrices in $\mathbb{C}^{r\times r}$ satisfying the condition (1.2.11) for all integer $n \geq 0$. then*

$$\beta_k(P,Q) = \Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P+Q).$$

Proof. Suppose that $n_0 = n_0(P,Q)$ be defined in definition 1.2.3, then we can write

$$\beta_k(P,Q) = (P)^{-1}_{n_0,k}(Q)^{-1}_{n_0,k}(P+Q)_{2n_0,k}\beta_k(P+n_0kI, Q+n_0kI),$$

where $P + nkI$ and $Q + nkI$ are positive stable matrices. By (1.2.4) we can write

$$\begin{aligned} \Gamma_k(P) &= \Gamma_k(P+n_0kI)(P+(n_0-1)kI)^{-1}\cdots(P+kI)^{-1}P^{-1} \\ &= \Gamma_k(P+n_0kI)(P)^{-1}_{n_0,k}, \end{aligned}$$

$$\Gamma_k(Q) = \Gamma_k(Q+n_0kI)(Q)^{-1}_{n_0,k}$$

and

$$\Gamma_k(P+Q) = \Gamma_k(P+Q+2n_0kI)(P+Q)^{-1}_{2n_0,k}.$$

Since $PQ = QP$, we can write

$$\Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P+Q)$$

$$= \Gamma_k(P + n_0kI)\Gamma_k(Q + n_0kI)\Gamma_k^{-1}(P + Q + 2n_0kI)(P)_{n_0,k}^{-1}(Q)_{n_0,k}^{-1}(P + Q)_{2n_0,k}. \tag{1.2.13}$$

Since we know that the matrices $P + n_0kI$, $Q + n_0kI$ and $P + Q + 2n_0kI$ are positive stable, so by Lemma 1.2.2, we get

$$\Gamma_k(P + n_0kI)\Gamma_k(Q + n_0kI)\Gamma_k^{-1}(P + Q + 2n_0kI) = \beta_k(P + n_0kI, Q + n_0kI), \tag{1.2.14}$$

and by Lemma 1.2.1 (ii), we have

$$\beta_k(P + n_0kI, Q + n_0kI) = (P)_{n_0,k}(Q)_{n_0,k}(P + Q)_{2n_0,k}^{-1}\beta_k(P, Q). \tag{1.2.15}$$

Hence by (1.2.13)-(1.2.15), it follows that

$$\beta_k(P, Q) = \Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P + Q).$$

□

1.3 On the Hypergeometric Matrix k-Functions

In this section, we define the hypergeometric matrix k-function which is denoted by $F_k(P, Q; R; z)$ where $k > 0$ and defined as

$$F_k(P, Q; R; z) = \sum_{n=0}^{\infty} \frac{(P)_{n,k}(Q)_{n,k}(R)_{n,k}^{-1}}{n!} z^n, \tag{1.3.1}$$

where the matrices P, Q and R are in $\mathbb{C}^{r\times r}$ such that $R + nkI$ is invertible matrix for all $n \geq 0$. Now we prove that the hypergeometric matrix k-function converges for $|z| = 1$ and $k > 0$.

Theorem 1.3.1. *Let P, Q and R be positive stable matrices in $\mathbb{C}^{r\times r}$ such that*

$$\beta(R) > \alpha(P) + \alpha(Q). \tag{1.3.2}$$

Then the series (1.3.1) is absolutely convergent for $|z| = 1$.

Proof. Assume that there exist a positive number δ, then by hypothesis (1.3.2) we have

$$\beta(R) - \alpha(P) - \alpha(Q) = 2\delta. \tag{1.3.3}$$

Now let us write

$$(nk)^{1+\frac{\delta}{k}}[\frac{1}{n!}(P)_{n,k}(Q)_{n,k}(R)^{-1}_{n,k}]$$

$$\begin{aligned}
&= \frac{(nk)^{1+\frac{\delta}{k}}}{n!}\frac{(n-1)!k^{n-1}(nk)^{\frac{P}{k}}(nk)^{-\frac{P}{k}}(P)_{n,k}}{(n-1)!k^{n-1}} \\
&\times \frac{(n-1)!k^{n-1}(nk)^{\frac{Q}{k}}(nk)^{-\frac{Q}{k}}(Q)_{n,k}}{(n-1)!k^{n-1}}(R)^{-1}_{n,k}(nk)^{\frac{R}{k}}(nk)^{-\frac{R}{k}} \\
&= \frac{(nk)^{1+\frac{\delta}{k}}}{n}(\frac{(nk)^{-\frac{P}{k}}(P)_{n,k}}{(n-1)!k^{n-1}})(nk)^{\frac{P}{k}}k^{n-1}(\frac{(nk)^{-\frac{Q}{k}}(Q)_{n,k}}{(n-1)!k^{n-1}})(nk)^{\frac{Q}{k}} \\
&\times k^{n-1}(n-1)!(R)^{-1}_{n,k}(nk)^{\frac{R}{k}}(nk)^{-\frac{R}{k}}
\end{aligned}$$

or

$$(nk)^{1+\frac{\delta}{k}}[\frac{1}{n!}(P)_{n,k}(Q)_{n,k}(R)^{-1}_{n,k}]$$

$$\begin{aligned}
&= k^{n}(nk)^{\frac{\delta}{k}}(\frac{(nk)^{-\frac{P}{k}}(P)_{n,k}}{(n-1)!k^{n-1}})(nk)^{\frac{P}{k}}(\frac{(nk)^{-\frac{Q}{k}}(Q)_{n,k}}{(n-1)!k^{n-1}}) \\
&\qquad \times (nk)^{\frac{Q}{k}}((n-1)!k^{n-1}(R)^{-1}_{n,k}(nk)^{\frac{R}{k}})(nk)^{-\frac{R}{k}}.
\end{aligned} \tag{1.3.4}$$

By (1.1.8), we are taking into account that $\alpha(-R) = -\beta(R)$ thus we can write

$$\begin{aligned}
\|(nk)^{\frac{P}{k}}\| \quad \|(nk)^{\frac{Q}{k}}\| \quad \|(nk)^{-\frac{R}{k}}\| &\le (nk)^{\frac{1}{k}(\alpha(P)+\alpha(Q)-\beta(R))}\{\sum_{j=0}^{r-1}\frac{(\|P\|r^{\frac{1}{2}}\ln nk)^{j}}{kj!}\} \\
&\times \{\sum_{j=0}^{r-1}\frac{(\|Q\|r^{\frac{1}{2}}\ln n)^{j}}{kj!}\}\{\sum_{j=0}^{r-1}\frac{(\|R\|r^{\frac{1}{2}}\ln n)^{j}}{kj!}\}.
\end{aligned}$$

By (1.3.3), we obtain

$$\|(nk)^{\frac{P}{k}}\| \quad \|(nk)^{\frac{Q}{k}}\| \quad \|(nk)^{-\frac{R}{k}}\| \le (nk)^{-\frac{2\delta}{k}}\{\sum_{j=0}^{r-1}\frac{[\max\{\|P\|,\|Q\|,\|R\|\}r^{\frac{1}{2}}]^{j}}{kj!}(\ln nk)^{j}\}^{3}. \tag{1.3.5}$$

Thus with the aid of (1.3.3)-(1.3.5) and for $|z| = 1$, we get

$$\begin{aligned}
&\lim_{n\to\infty}(nk)^{1+\frac{\delta}{k}}\|\frac{(P)_{n,k}(Q)_{n,k}(R)^{-1}_{n,k}z^n}{n!}\| \\
\leq\ & \lim_{n\to\infty} k^n(nk)^{-\frac{\delta}{k}}\|\frac{(nk)^{-\frac{P}{k}}(P)_{n,k}}{(n-1)!k^{n-1}}\| \quad \|(nk)^{\frac{P}{k}}\| \\
\times\ & \|\frac{(nk)^{-\frac{Q}{k}}(Q)_{n,k}}{(n-1)!k^{n-1}}\| \quad \|(nk)^{\frac{Q}{k}}\| \\
\times\ & \quad \|(n-1)!k^{n-1}(R)^{-1}_{n,k}(nk)^{\frac{R}{k}}\| \quad \|(nk)^{-\frac{R}{k}}\| \\
\leq\ & \|\Gamma_k^{-1}(P)\| \quad \|\Gamma_k^{-1}(Q)\| \quad \|\Gamma_k(R)\| \\
\times\ & \lim_{n\to\infty} k^{2n-2}(nk)^{-\frac{\delta}{k}}\{\sum_{j=0}^{r-1}\frac{[\max\{\|P\|,\|Q\|,\|R\|\}r^{\frac{1}{2}}]^j}{kj!} \\
\times\ & (\ln nk)^j\}^3 \\
=\ & 0,
\end{aligned}$$

because

$$\lim_{n\to\infty} n^{-\frac{\delta}{k}}(\ln nk)^j = 0, \quad \forall \quad j \geq 0, \quad k > 0.$$

Thus

$$\lim_{n\to\infty}(nk)^{1+\frac{\delta}{k}}\|\frac{(P)_{n,k}(Q)_{n,k}(R)^{-1}_{n,k}z^n}{n!}\| = 0; \quad |z| = 1,$$

therefore the series (1.3.1) is absolutely convergent for $|z| = 1$. Now we show that under certain condition the hypergeometric matrix k-function $F_k(P,Q;R;z)$ is a solution of matrix differential equation of bilateral type. □

Theorem 1.3.2. *Let R is matrix in $\mathbb{C}^{r\times r}$ satisfying $R+nkI$ is invertible matrix and $QR = RQ$. Then $F_k(P,Q;R;z)$ is the solution of*

$$kz(1-kz)W'' - kzPW' + W'(R - kz(Q+kI)) - PQW = 0, \quad 0 \leq |z| < 1 \quad (1.3.6)$$

satisfying $F_k(P,Q;R;0) = I$.

Proof. By the given hypothesis $QR = RQ$, so we can write

$$F_{n,k} = \frac{(P)_{n,k}(Q)_{n,k}(R)^{-1}_{n,k}}{n!} = \frac{(P)_{n,k}(R)^{-1}_{n,k}(Q)_{n,k}}{n!}.$$

Let us denote

$$w(z) = F_k(P,Q;R;z) = \sum_{n=0}^{\infty} F_{n,k} z^n, \quad |z| < 1. \tag{1.3.7}$$

Since $W(z)$ is a power series convergent for $|z| < 1$, so it is termwise differentiable in the given domain and

$$w'(z) = \sum_{n=1}^{\infty} nF_{n,k} z^{n-1}, \quad W''(z) = \sum_{n=2}^{\infty} n(n-1)F_{n,k} z^{n-2}, \quad |z| < 1.$$

Hence

$$\begin{aligned}
& kz(1-kz)W'' - kzPW' + W'(R - kz(Q+kI)) - PQW \\
= & \sum_{n=2}^{\infty} nk(n-1)F_{n,k} z^{n-1} - \sum_{n=2}^{\infty} nk^2(n-1)F_{n,k} z^n - P\sum_{n=1}^{\infty} nkF_{n,k} z^n \\
+ & \sum_{n=1}^{\infty} nF_{n,k} R z^{n-1} - \sum_{n=1}^{\infty} nkF_{n,k}(Q+kI)z^n - \sum_{n=0}^{\infty} PF_{n,k} Q z^n,
\end{aligned}$$

replacing $n = n+1$ in the first and fourth summation, we obtain

$$\begin{aligned}
& kz(1-kz)W'' - kzPW' + W'(R - kz(Q+kI)) - PQW \\
= & \sum_{n=1}^{\infty} nk(n+1)F_{n+1,k} z^n - \sum_{n=2}^{\infty} nk^2(n-1)F_{n,k} z^n - P\sum_{n=1}^{\infty} nkF_{n,k} z^n \\
+ & \sum_{n=1}^{\infty} nF_{n,k} R z^n - \sum_{n=1}^{\infty} nkF_{n,k}(Q+kI)z^n - \sum_{n=0}^{\infty} PF_{n,k} Q z^n \\
= & \sum_{n=1}^{\infty} \{nk(n+1)F_{n,k} - nk^2(n-1)F_{n,k} - nkPF_{n,k} + (n+1)F_{n+1,k}R \\
- & nkF_{n,k}(Q+kI) - PF_{n,k}Q\}z^n + 2kF_{2,k}z - kPF_{1,k}z + F_{1,k}R + 2F_{2,k}Rz \\
- & F_{1,k}(Q+kI)kz - PF_{0,k}Q - PF_{1,k}Qz = 0.
\end{aligned}$$

By equating the coefficients of each power z^n and noting that $F_{0,k} = I$, we get

$$\begin{aligned}
z^0 \quad &: \quad F_{1,k}R - PIQ = 0, \\
z^1 \quad &: \quad 2kF_{2,k} - kPF_{1,k} + 2F_{2,k}R - F_{1,k}(Q+kI)kz - PF_{1,k}Q \\
&= \quad 2F_{2,k}(kI+R) - PF_{1,k}(kI+Q) - F_{1,k}(Q+kI)k = 0 \\
\vdots \quad &\vdots
\end{aligned}$$

$$\Rightarrow \quad F_{n+1,k} = \frac{(P+nkI)F_{n,k}(Q+nkI)(R+nkI)^{-1}}{n+1}.$$

Hence $W(z) = F_k(P, Q; R; z)$ is the solution of (1.3.6) satisfying $W(0) = I$.

□

Corollary 1.3.3. *Let R be a matrix in $\mathbb{C}^{r\times r}$ satisfying that $R + nkI$ is invertible matrix for $n \geq 0$ and let P be an arbitrary matrix in $\mathbb{C}^{r\times r}$ and n be a positive integer. Then equation*

$$kz(1-kz)W'' - kzPW' + W'(R + z(n-k)kI) + nPW = 0 \tag{1.3.8}$$

has matrix polynomial solutions of degree n.

Proof. Let $Q = -nI$, then by theorem 1.3.2 the function $W(z) = F_k(P, -nI; R; z)$ satisfies (1.3.6) for $Q = -nI$. Hence

$$W(z) = F_k(P, Q; R; z) = \sum_{l=0}^{n} \frac{(P)_{l,k}(-nI)_{l,k}(R)_{l,k}}{l!} z^l$$

is a matrix polynomial of degree n of (1.3.8). □

1.4 An Integral Representation of Hypergeometric matrix k-function

In this section, we define the integral representation of hypergeometric matrix k-function. If y and b are complex numbers with $|y| < 1$, then the Taylor series expansion of $(1-ky)^{-\frac{b}{k}}$ about $y = 0$ is given by [15]

$$(1-ky)^{-\frac{b}{k}} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} y^n, \quad |y| < 1, \quad a \in \mathbb{C}. \tag{1.4.1}$$

Let $f_{n,k}(a)$ be a function defined by

$$f_{n,k}(a) = \frac{(a)_{n,k}}{n!} y^n = \frac{a(a+k)(a+2k)\cdots(a+(n-1)k)}{n!} y^n, \quad a \in \mathbb{C} \quad k > 0, \tag{1.4.2}$$

for a fixed complex numher y with $|y| < 1$. Clearly the function $f_{n,k}$ is an holomorphic function of variable a defined in the complex plane for $k > 0$. For a given closed disc

$D_\alpha = \{a \in \mathbb{C} : |a| \leq \alpha\}$, we have

$$|f_{n,k}(a)| \leq \frac{(|a|)_{n,k}|y|^n}{n!} \leq \frac{(\alpha)_{n,k}|y|^n}{n!}, \quad n \geq 0, \quad |a| \leq \alpha, \quad k > 0.$$

Since

$$\sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}|y|^n}{n!} \leq +\infty,$$

so by the Weierstrass theorem for the convergence of holomorphic functions [15, 100] it follows that

$$g(a) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} y^n = (1-ky)^{-\frac{a}{k}}$$

is holomorphic in R for $k > 0$. Thus by holomorphic functional calculus [20], for any matrix $P \in \mathbb{C}^{r\times r}$, the image of g acting on P gives

$$(1-ky)^{-\frac{P}{k}} = g(P) = \sum_{n=0}^{\infty} \frac{(P)_{n,k}}{n!} y^n, \quad |y| < 1, \tag{1.4.3}$$

where

$$(P)_{n,k} = P(P+kI)\cdots(P+(n-1)kI), \quad k > 0.$$

Assume that the matrices Q and R in $\mathbb{C}^{r\times r}$ with the conditions $QR = RQ$ and Q, R and $R-Q$ are positive stable matrices. Thus by (1.1.5), (1.1.7) and with the aid of the condition that Q, R and $R-Q$ are positive stable matrices, we obtain

$$\begin{aligned}(Q)_{n,k}(R)_{n,k}^{-1} &= \Gamma_k^{-1}(Q)\Gamma_k(Q+nkI)\Gamma_k(R)\Gamma_k^{-1}(R+nkI),\\ &= \Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)\Gamma_k(R-Q)\Gamma_k(Q+nkI)\Gamma_k^{-1}(R+nkI)\Gamma_k(R). \quad (1.4.4)\end{aligned}$$

By positive stability condition of the matrices and by Lemma 1.2.2 this implies that

$$\frac{1}{k}\int_0^1 t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}dt$$

$$= \beta_k(Q+nkI, R-Q) = \Gamma_k(R-Q)\Gamma_k(Q+nkI)\Gamma_k^{-1}(R+nkI), \qquad (1.4.5)$$

by (1.4.4) and (1.4.5), we get

$$(Q)_{n,k}(R)_{n,k}^{-1} = \Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)[\frac{1}{k}\int_0^1 t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}dt]\Gamma_k(R). \quad (1.4.6)$$

Hence, for $|z|<1$ we can write

$$\begin{aligned} F_k(P,Q;R;z) &= \sum_{n=0}^{\infty}\frac{(P)_{n,k}(Q)_{n,k}(R)_{n,k}^{-1}}{n!}z^n \\ &= \sum_{n=0}^{\infty}\frac{(P)_{n,k}\Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)\Gamma_k(R)z^n}{n!} \\ &\times [\frac{1}{k}\int_0^1 t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}dt] \end{aligned}$$

$$= \sum_{n=0}^{\infty}[\frac{1}{k}\int_0^1 \frac{(P)_{n,k}\Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}\Gamma_k(R)z^n}{n!}dt]. \quad (1.4.7)$$

Now let us consider

$$S_{n,k}(t) = \frac{(P)_{n,k}\Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}\Gamma_k(R)z^n}{n!}, \quad 0\le t\le 1,$$

and for $0<t<1$ and $n\ge 0$, we have

$$\|S_{n,k}(t)\|$$

$$\le \frac{(\|P\|)_{n,k}\|\Gamma_k^{-1}(Q)\|\|\Gamma_k^{-1}(R-Q)\|\|\Gamma_k(R)\|\|t^{\frac{Q}{k}-I}\|\|(1-t)^{\frac{R-Q}{k}-I}\|\|z\|^n}{n!}, k>0. \quad (1.4.8)$$

By using (1.1.8), the above equation becomes

$$\|t^{\frac{Q}{k}-I}\| \quad \|(1-t)^{\frac{R-Q}{k}-I}\| \leq t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}[\sum_{j=0}^{r-1}\frac{(\|Q-kI\|r^{\frac{1}{2}}\ln t)^j}{kj!}] \times [\sum_{j=0}^{r-1}\frac{(\|R-Q-kI\|r^{\frac{1}{2}}\ln t)^j}{kj!}]$$

and noting that for $0<t<1$, we have $\ln t < t < 1$ and $\ln(1-t) < 1-t < 1$, hence from above relation, we obtain

$$\|t^{\frac{Q}{k}-I}\| \quad \|(1-t)^{\frac{R-Q}{k}-I}| \leq \Lambda t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}, \quad 0<t<1, \quad k>0 \quad (1.4.9)$$

where

$$\Lambda = [\sum_{j=0}^{r-1}\frac{(\|Q-kI\|r^{\frac{1}{2}})^j}{kj!}][\sum_{j=0}^{r-1}\frac{(\|R-Q-kI\|r^{\frac{1}{2}}\ln t)^j}{kj!}]. \quad (1.4.10)$$

Now, consider the sum of the convergent series be

$$S = \sum_{n=0}^{r-1}\frac{(\|P\|)_{n,k}|z|^n}{n!}, \quad |z|<1 \quad k>0, \quad (1.4.11)$$

then by (1.4.7)-(1.4.10), we obtain

$$\sum_{n=0}^{\infty}\|S_{n,k}(t)\| \leq \phi(t) = \frac{1}{k}[L\Lambda S t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}], 0<t<1, k>0, \quad (1.4.12)$$

where

$$L = \|\Gamma_k^{-1}(Q)\| \quad \|\Gamma_k^{-1}(R-Q)\| \quad \|\Gamma_k(R)\|.$$

Since $\alpha(Q)>0$, $\alpha(R-Q)>0$ and $k>0$, then the function $\phi(t) = \frac{1}{k}[L\Lambda t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}]$ is integrable and

$$\int_0^1 \phi(t)dt = L\Lambda S B_k(\alpha(Q), \alpha(R-Q)).$$

Thus by dominated convergence theorem ([26]), the series and the integral can be computed in (1.4.7) and using $QR = RQ$, we can write

$$F_k(P,Q;R;z)$$
$$= \frac{1}{k}\int_0^1 \{\sum_{n=0}^{\infty}(\frac{(P)_{n,k}(tz)^n}{n!})t^{\frac{Q}{k}-I}(1-t)^{\frac{R-Q}{k}-I}\}dt\Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)\Gamma_k(R). \quad (1.4.13)$$

Now by (1.4.3), we obtain

$$\sum_{n=0}^{\infty}\frac{(P)_{n,k}(tz)^n}{n!} = (1-ktz)^{-\frac{P}{k}}, \quad |z|<1, \quad 0<t<1, \quad (1.4.14)$$

and (1.4.13) becomes

$$F_k(P,Q;R;z)$$
$$= \Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)\Gamma_k(R)\frac{1}{k}\int_0^1 t^{\frac{Q}{k}-I}(1-t)^{\frac{R-Q}{k}-I}(1-ktz)^{-\frac{P}{k}}dt. \quad (1.4.15)$$

By summarizing the above result, we established the following theorem:

Theorem 1.4.1. *Let P, Q and R be the matrices in $\mathbb{C}^{r\times r}$ with the conditions $QR = RQ$ and Q, R, $R-Q$ are positive stable matrices. Then for $|z|<1$, we have*

$$F_k(P,Q;R;z) = \Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)\Gamma_k(R)[\frac{1}{k}\int_0^1 t^{\frac{Q}{k}-I}(1-t)^{\frac{R-Q}{k}-I}(1-ktz)^{-\frac{P}{k}}dt]. \quad (1.4.16)$$

Corollary 1.4.2. *Let P, Q and R be matrices in $\mathbb{C}^{r\times r}$ and let $\hat{\alpha}(Q,R) = \min\{\alpha(Q), \alpha(R), \alpha(R-Q)\}$ and $n_1 = n_1(Q,R) = [|\hat{\alpha}(Q,R)|]+1$, where $[|\hat{\alpha}(Q,R)|]$ denotes the entire part functions. Suppose that $QR = RQ$, and*

$$\begin{aligned} \sigma(Q) &\subset R\sim\{-n; n\geq n_1, n\in Z\} \\ \sigma(R-Q) &\subset R\sim\{-n; n\geq n_1, n\in Z\} \\ \sigma(R) &\subset R\sim\{-2n; n\geq n_1, n\in Z\}. \end{aligned}$$

Then for $|z| < 1$, *we have*

$$\begin{aligned} F_k(P, Q + n_1kI; R + 2n_1kI; z) &= \Gamma_k^{-1}(Q + n_1kI)\Gamma_k(R - Q + n_1kI) \\ &\times \Gamma_k(R + 2n_1kI) \\ &\times \frac{1}{k}\int_0^1 t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}+(n-1)I} \\ &\times (1 - ktz)^{-\frac{P}{k}}dt. \end{aligned}$$

Proof. Consider the matrices P, $\hat{Q} = Q + nkI$, $\hat{R} = R + nkI$ and $\hat{R}$, $\hat{Q}$, $\hat{R} - \hat{Q} = R - Q + n_1kI$ are positive stable matrices. It is now a consequence of Theorem 1.4.1. □

Chapter 2

Extension of Gamma, Beta k-Functions and Beta k-Distribution

In this chapter [97], we derive the extended form of gamma and beta k-functions some of their properties. Also, we derive extended k-beta distribution which is the extended form of k-beta distribution. We establish further generalization of extended gamma, beta k-functions and beta k-distribution and their properties.

2.1 Introduction

In this section, we present some fundamental relations of gamma, beta, extended gamma, extended beta functions and extended beta distribution introduced in ([6], [7], [82], [87]). The gamma function is defined by

$$\Gamma(\sigma_1) = \int_0^\infty t^{\sigma_1 - 1} e^{-t} dt, \quad \Re(\sigma_1) > 0.$$

In another way, it is defined as

$$\Gamma(\sigma_1) = \lim_{n \to \infty} \frac{n! n^{\sigma_1 - 1}}{(\sigma_1)_n}$$

where $(\sigma_1)_n$ denotes the Pochhammer symbol which is defined by

$$(\sigma_1)_n = \begin{cases} \sigma_1(\sigma_1+1)(\sigma_1+2)\cdots(\sigma_1+n-1); & for \quad n \geq 1, \sigma_1 \neq 0 \\ 1 \quad if \quad n=0 \end{cases}$$

and

$$\Gamma(\sigma_1+1) = \sigma_1\Gamma(\sigma_1).$$

The relation between Pochhammer symbol and gamma function is given below

$$(\sigma_1)_n = \frac{\Gamma(\sigma_1+n)}{\Gamma(\sigma_1)}.$$

The well-known beta function is given by

$$\begin{aligned} B(\sigma_1,\sigma_2) &= \frac{\Gamma(\sigma_1)\Gamma(\sigma_2)}{\Gamma(\sigma_1+\sigma_2)} \\ &= \int_0^\infty t^{\sigma_1}(1-t)^{\sigma_2}dt, \Re(\sigma_1)>0, \Re(\sigma_2)>0. \end{aligned}$$

Chaudhry and Zubair [6] defined the following extended form of gamma function

$$\Gamma_b(\sigma_1) = \int_0^\infty t^{\sigma_1-1}e^{-t-bt^{-1}}dt, \quad \Re(\sigma_1)>0, b \geq 0. \tag{2.1.1}$$

When $b=0$, then Γ_b tends to the classical gamma function Γ. Also, Chaudhry *et al.* [7] defined the following extended form of Eulers beta function

$$B(\sigma_1,\sigma_2;b) = \int_0^\infty t^{\sigma_1-1}(1-t)^{\sigma_2-1}e^{-\frac{b}{t(1-t)}}dt, \tag{2.1.2}$$

where $\Re(b)>0, \Re(\sigma_1)>0, \Re(\sigma_1)>0$. When $b=0$, then $B_0(\sigma_1,\sigma_2) = B(\sigma_1,\sigma_2)$. They also defined the following extended beta distribution

$$f(z) = \begin{cases} \frac{1}{B(\sigma_1,\sigma_2;b)}z^{\sigma_1-1}(1-z)^{\sigma_2-1}e^{-\frac{b}{t(1-t)}}; & 0 \leqq z \leqq 1; \sigma_1,\sigma_2,b>0 \\ 0, \quad elsewhere. \end{cases} \tag{2.1.3}$$

Recently, the researchers ([4], [9]- [12], [70]) have been considered various extension of the well-known special functions. Diaz *et al.* ([15], [14], [16]) have investigated gamma and beta k-functions and proved their various properties. They also defined zeta k-functions and hypergeometric k-functions by using the Pochhammer k-symbols. For $k > 0$ and $z \in \mathbb{C}$, the gamma k-function is defined by

$$\Gamma_k(\sigma_1) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{\sigma_1}{k}-1}}{(\sigma_1)_{n,k}}. \tag{2.1.4}$$

Its integral representation is also given by,

$$\Gamma_k(\sigma_1) = \int_0^\infty t^{\sigma_1-1}e^{\frac{-t^k}{k}}dt \tag{2.1.5}$$

and

$$\Gamma_k(\sigma_1 + k) = \sigma_1\Gamma_k(\sigma_1). \tag{2.1.6}$$

$$\Gamma_k(\sigma_1) = k^{\frac{\sigma_1}{k}-1}\Gamma(\frac{\sigma_1}{k}) \tag{2.1.7}$$

The relation between Pochhammer k-symbol and gamma k-function is given as

$$(\sigma_1)_{n,k} = \frac{\Gamma_k(\sigma_1 + nk)}{\Gamma_k(\sigma_1)}.$$

The well-known beta k-function is defined as

$$B_k(\sigma_1, \sigma_2) = \frac{1}{k}\int_0^\infty t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}-1}dt. \tag{2.1.8}$$

The relation between gamma k-function and beta k-function is

$$B_k(\sigma_1, \sigma_2) = \frac{\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}{\Gamma_k(\sigma_1+\sigma_2)}, \Re(\sigma_1) > 0, \Re(\sigma_2) > 0. \tag{2.1.9}$$

These studies were then followed by the works of Mansour [64], Kokologiannaki [56], Krasniqi [57,58] and Merovci [66] elaborating and strengthening the scope of gamma and beta k-functions. Very recently, Mubeen and Habibullah [78] defined k-fractional integration and gave its application. Mubeen and Habibullah [72] established the integral representations of hypergeometric k-functions by using the properties of Pochhammer k-symbols, gamma and beta k-functions. In 2011, Özergin et al. [82] have further generalized the extended gamma and beta function, which are respectively defined as

$$\Gamma_p^{\lambda,\rho}(\sigma_1) = \int_0^\infty t^{\sigma_1 - 1} \quad {}_1F_1(\lambda; \rho; -t - \frac{p}{t})dt \tag{2.1.10}$$

where $p \geq 0$, $|\arg(1-z)| < \pi < p$ and $\Re(\rho) > \Re(\lambda) > 0$ and

$$B_p^{\lambda,\rho}(\sigma_1, \sigma_1) = \int_0^1 t^{\sigma_1 - 1}(1-t)^{\sigma_2 - 1} \quad {}_1F_1(\lambda; \rho; -\frac{p}{t(1-t)})dt \tag{2.1.11}$$

where $\Re(\lambda) > 0$, $\Re(\rho) > 0$, $\Re(p) > 0, \Re(\sigma_1) > 0, \Re(\sigma_2) > 0$. These functions are also called the extended gamma and beta functions

2.2 Extended Gamma and Beta k-Functions

In this section, we introduce the following extended form of gamma, beta k-functions and some other properties related to these functions

$$\Gamma_{b,k}(\sigma_1) = \int_0^\infty t^{\sigma_1 - 1} e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}} dt, \quad \Re(\sigma_1) > 0, b \geq 0, k > 0. \tag{2.2.1}$$

When $b = 0$, then $\Gamma_{b,k}$ tends to the gamma k-function Γ_k defined in section 2.1, if $k = 1$, then $\Gamma_{b,k}$ tends to Γ_b defined in [6] and if both $b = 0$ and $k = 1$, then $\Gamma_{b,k}$ tends to classical gamma function Γ. The extended Eulers beta k-function is defined as

$$B_k(\sigma_1, \sigma_1; b) = \frac{1}{k} \int_0^1 t^{\frac{\sigma_1}{k} - 1}(1-t)^{\frac{\sigma_2}{k} - 1} e^{-\frac{b^k}{kt(1-t)}} dt, \tag{2.2.2}$$

where $k > 0$, $\Re(b) > 0, \Re(\sigma_1) > 0, \Re(\sigma_2) > 0$. When $b = 0$, then $B_0(\sigma_1, \sigma_1) = B(\sigma_1, \sigma_1)$.

It is obvious that, if $k = 1$, then $B_{b,k}$ tends to $B_b(\sigma_1, \sigma_1)$ defined in section 2.1, if $b = 0$, then $B_{b,k}(\sigma_1, \sigma_1)$ tends to beta k-function $B_k(\sigma_1, \sigma_1)$ defined in section 2.1 and if both $b = 0$ and $k = 1$, then $B_{b,k}(\sigma_1, \sigma_1)$ tends to the Eular's beta function $B(\sigma_1, \sigma_1)$.

These extensions will be seen to extremely useful, in that most properties of the gamma and beta k-functions.

2.3 Properties of Extended Gamma k-function

This section is devoted to various properties of extended gamma k-function.

Theorem 2.3.1. *Prove that the following difference formula holds for $k > 0$*

$$\Gamma_{b,k}(\sigma_1 + k) = \sigma_1 \Gamma_{b,k}(\sigma_1) + b^k \Gamma_{b,k}(\sigma_1 - k), b \geq 0. \tag{2.3.1}$$

Proof. For $k > 0$, consider $\mathfrak{M}$ be the operator of Mellin integral transform defined by

$$\mathfrak{M}\{f(t); \sigma_1\} =< t_+^{\sigma_1 - 1}, f(t) >= \int_0^\infty t^{\sigma_1 - 1} f(t) dt. \tag{2.3.2}$$

Then, $\Gamma_{k,b}$ is a Mellin transform of the function $f(t) = e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}$ in σ_1, i.e.,

$$\Gamma_{b,k}(\sigma_1) = \mathfrak{M}\{e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}; \sigma_1\}.$$

By using the relationship

$$\mathfrak{M}\{f'(t); \sigma_1\} = -(\sigma_1 - 1)\mathfrak{M}\{f(t); \sigma_1 - 1\},$$

between the Mellin transform of a function and its derivative, we obtain

$$-(\sigma_1 - 1)\Gamma_{b,k}(\sigma_1 - 1) = \mathfrak{M}\{(-t^{k-1} + b^k t^{-k-1})e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}; \sigma_1\},$$

which implies to give

$$-(\sigma_1 - 1)\Gamma_{b,k}(\sigma_1 - 1) = -\Gamma_{b,k}(\sigma_1 - 1 + k) + b^k \Gamma_{b,k}(\sigma_1 - 1 - k).$$

Replacing σ_1 by $\sigma_1 + 1$, we get the proof of (2.3.1) □

Remark 2.3.1. If letting $b = 0$, then we get the result of gamma k-function as

$$\Gamma_k(\sigma_1 + k) = \sigma_1 \Gamma_k(\sigma_1).$$

Similarly if $k = 1$, then we get the result of extended gamma function see [8] and if both $b = 0$ and $k = 1$, then we have a result of classical gamma function.

Theorem 2.3.2. *Assume that* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\Gamma_{b,k}(\frac{\sigma_1}{p} + \frac{\sigma_2}{q}) \leq (\Gamma_{b,k}(\sigma_1))^{\frac{1}{p}} (\Gamma_{b,k}(\sigma_2))^{\frac{1}{q}}. \tag{2.3.3}$$

Proof. For $k > 0$, we have

$$\begin{aligned} \Gamma_{b,k}(\frac{\sigma_1}{p} + \frac{\sigma_2}{q}) &= \int_0^\infty t^{\frac{\sigma_1}{p} + \frac{\sigma_2}{q} - 1} e^{-\frac{t^k}{k} - \frac{b^k}{kt^k}} dt \\ &= \int_0^\infty \left(t^{\sigma_1 - 1} e^{-\frac{t^k}{k} - \frac{b^k}{kt^k}} \right)^{\frac{1}{p}} \left(t^{\sigma_2 - 1} e^{-\frac{t^k}{k} - \frac{b^k}{kt^k}} \right)^{\frac{1}{q}} dt. \end{aligned}$$

Now, using Hölder inequality [94], we get

$$\Gamma_{b,k}(\frac{\sigma_1}{p} + \frac{\sigma_2}{q}) \leq (\Gamma_{b,k}(\sigma_1))^{\frac{1}{p}} (\Gamma_{b,k}(\sigma_2))^{\frac{1}{q}}$$

which is exactly (2.3.3). □

Corollary 2.3.3. *Prove that the following inequality holds for* $\sigma_1 > 0, \sigma_2 > 0, b \geq 0$

$$\Gamma_{b,k}(\frac{\sigma_1 + \sigma_2}{2}) \leq \sqrt{(\Gamma_{b,k}(\sigma_1))(\Gamma_{b,k}(\sigma_2))}. \tag{2.3.4}$$

Proof. For $k > 0$, setting $p = 2 = q$ in (2.3.3), we obtain the required result (2.3.4).

$$\Gamma_{b,k}(\frac{\sigma_1 + \sigma_2}{2}) \leq \sqrt{(\Gamma_{b,k}(\sigma_1))(\Gamma_{b,k}(\sigma_2))} \leq \frac{1}{2} \left(\Gamma_{b,k}(\sigma_1) + \Gamma_{b,k}(\sigma_1), \right) \tag{2.3.5}$$

where $\sigma_1 > 0, \sigma_1 > 0, b \geq 0$. □

Theorem 2.3.4. *Prove that the following reflection formula holds true*

$$b^{\sigma_1} \Gamma_{b,k}(-\sigma_1) = \Gamma_{b,k}(\sigma_1), \Re(b) > 0. \tag{2.3.6}$$

Proof. For $k > 0$, substituting $t = b\tau^{-1}$ in (2.2.1), we have

$$\Gamma_{b,k}(x) = b^{\sigma_1} \int_0^\infty \tau^{-\sigma_1 - 1} e^{-\frac{\tau^k}{k} - \frac{b^k}{k\tau^k}} d\tau$$

which is exactly (2.3.6). □

Corollary 2.3.5. *Prove that the following result holds true for $k > 0$,*

$$\Gamma_{b,k}(k - \sigma_1) = b^{\sigma_1} \left[\; \Gamma_{b,k}(\sigma_1 + k) - \sigma_1 \Gamma_{b,k}(\sigma_1) \; \right]. \tag{2.3.7}$$

Proof. For $k > 0$, replacing σ_1 by $-\sigma_1$ in (2.3.1) and (2.3.6), we have

$$\Gamma_{b,k}(k - \sigma_1) = b^k \Gamma_{b,k}(-(\sigma_1 + k)) - \sigma_1 \Gamma_{b,k}(-\sigma_1) \tag{2.3.8}$$

and

$$b^{-\sigma_1} \Gamma_{b,k}(\sigma_1) = \Gamma_{b,k}(-\sigma_1), \Re(b) > 0. \tag{2.3.9}$$

Now, using (2.3.9) in (2.3.8), we get the required result of (2.3.7). □

2.4 Integral Representation of Extended Beta k-Function

In this section, we prove some various properties of extended beta k-functions such as integral representations, Mellin transforms, relations with extended k-gamma functions.

Theorem 2.4.1. *Prove that the following integral representation holds true*

$$\int_0^\infty b^{s-1} B_k(\sigma_1, \sigma_2; b) db = \Gamma_k(s) B_k(\sigma_1 + s, \sigma_1 + s), \quad \Re(s) > 0, \Re(\sigma_1 + s) > 0, \Re(\sigma_2 + s) > 0. \tag{2.4.1}$$

Proof. For $k > 0$, multiplying (2.2.2) by b^{s-1} and integrating both sides with respect to b from $b = 0$ to $b = \infty$, we have

$$\int_0^\infty b^{s-1} B_k(\sigma_1, \sigma_2; b) db = \int_0^\infty b^{s-1} \left(\; \frac{1}{k} \int_0^1 t^{\frac{\sigma_1}{k} - 1} (1 - t)^{\frac{\sigma_2}{k} - 1} e^{-\frac{b^k}{kt(1-t)}} dt \; \right) db. \tag{2.4.2}$$

Interchanging the order of integration in (2.4.2), we have

$$\int_0^\infty b^{s-1} B_k(\sigma_1, \sigma_2; b)db = \frac{1}{k}\int_0^1 t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}-1}\left(\int_0^\infty b^{s-1}e^{-\frac{b^k}{kt(1-t)}}db\right)dt. \qquad (2.4.3)$$

However, the above integral can be written in term of gamma k-function by taking $u = \frac{b}{t^{\frac{1}{k}}(1-t)^{\frac{1}{k}}}$, we have

$$\begin{aligned}\int_0^\infty b^{s-1}e^{-\frac{b^k}{kt(1-t)}}db &= t^{\frac{s}{k}}(1-t)^{\frac{s}{k}}\int_0^\infty u^{s-1}e^{-\frac{u^k}{k}}du \\ &= t^{\frac{s}{k}}(1-t)^{\frac{s}{k}}\Gamma_k(s), \Re(s) > 0, 0 < t < 1.\end{aligned}$$

Now, using this result in (2.4.3), we obtain our desired result.
By settining $s = k$ in (2.4.1), we obtain the following relation

$$\int_0^\infty b^{k-1}B_k(\sigma_1, \sigma_2; b)db = B_k(\sigma_1 + k, \sigma_2 + k).$$

□

Remark 2.4.1. The usual integral representation of extended beta function can be recover from them by taking $k = 1$. Similarly, we can recover the integral representation of beta k-function by taking $b = 0$ and that of integral representation of Euler's beta function can be recover by taking $b = 0$ and $k = 1$.

Theorem 2.4.2. *Prove that the following integral representations for extended beta k-function hold true*

$$B_k(\sigma_1, \sigma_2; b) = \frac{2}{k}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{2\sigma_1}{k}-1}(\sin\theta)^{\frac{2\sigma_2}{k}-1}e^{\frac{-b^k}{k}\sec^2\theta\csc^2\theta}d\theta, \qquad (2.4.4)$$

$$B_k(\sigma_1, \sigma_2; b) = \frac{1}{k}e^{-\frac{2b^k}{k}}\int_0^\infty \frac{u^{\frac{\sigma_1}{k}-1}}{(1+u)^{\frac{\sigma_1+\sigma_2}{k}}}e^{\frac{-b^k}{k}(u+u^{-1})}du, \qquad (2.4.5)$$

$$B_k(\sigma_1, \sigma_2; b) = \frac{1}{k}2^{1-\frac{\sigma_1+\sigma_2}{k}}\int_{-1}^1(1+t)^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_1+\sigma_2}{k}}e^{\frac{-4b^k}{k(1-t^2)}}dt, \qquad (2.4.6)$$

and

$$B_k(\sigma_1,\sigma_2;b)=\frac{1}{k}(c-a)^{1-\frac{\sigma_1+\sigma_2}{k}}\int_a^c(u-a)^{\frac{\sigma_1}{k}-1}(c-u)^{\frac{\sigma_2}{k}-1}e^{-\frac{b^k}{k}\frac{(c-a)^2}{(u-a)(c-u)}}du. \quad (2.4.7)$$

Proof. The proofs of (2.4.4)-(2.4.7) are straight forward. The relation (2.4.4) follows by using the transformation $t=\cos^2\theta$ in (2.2.2) and similarly (2.4.5),(2.4.6) and (2.4.7) follow from (2.2.2) by using the transformations $t=\frac{u}{1+u}$, $t=\frac{1+t}{2}$ and $t=\frac{u-a}{c-a}$ respectively. □

Theorem 2.4.3. *Prove the following functional relation of extended gamma k-function*

$$B_k(\sigma_1,\sigma_2+k;b)+B_k(\sigma_1+k,y;b)=B_k(\sigma_1,\sigma_2;b). \quad (2.4.8)$$

Proof. Consider the left hand side of (2.4.8), we have

$$B_k(\sigma_1,\sigma_2+k;b)+B_k(\sigma_1+k,y;b)=\frac{1}{k}\int_0^1\left\{\ t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}}+t^{\frac{\sigma_1}{k}}(1-t)^{\frac{\sigma_2}{k}-1}\ \right\}e^{-\frac{b^k}{kt(1-t)}}dt,$$

after a simple algebraic manipulation, we get

$$B_k(\sigma_1,\sigma_2+k;b)+B_k(\sigma_1+k,y;b)=\frac{1}{k}\int_0^1 t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}-1}e^{-\frac{b^k}{kt(1-t)}}dt,$$

which completes the desired proof. □

Theorem 2.4.4. *The following integral representations for gamma k-function hold true:*

$$\Gamma_{b,k}(\sigma_1)\Gamma_{b,k}(\sigma_2)=\frac{2}{k}\int_0^\infty r^{2\frac{\sigma_1+\sigma_2}{k}-1}e^{\frac{-r^2}{k}}B_k(\sigma_1,\sigma_2;\frac{b}{r^2})dr, Re(b)>0, Re(x)>0, Re(\sigma_2)>0. \quad (2.4.9)$$

Proof. Substituting $t=\eta^2$ in (2.2.1), we get

$$\Gamma_{b,k}(\sigma_1)=2\int_0^\infty\eta^{2\sigma_1-1}e^{\frac{-\eta^{2k}}{k}-\frac{b^k}{k\eta^{2k}}}d\eta. \quad (2.4.10)$$

Therefore,

$$\Gamma_{b,k}(\sigma_1)\Gamma_{b,k}(\sigma_2)=4\int_0^\infty\int_0^\infty\eta^{2\sigma_1-1}\zeta^{2\sigma_2-1}e^{\frac{-\eta^{2k}}{k}-\frac{\zeta^{2k}}{k}}e^{-\frac{b^k}{k\eta^{2k}}-\frac{b^k}{k\zeta^{2k}}}d\eta d\zeta. \quad (2.4.11)$$

The substitution $\eta = (r\cos\theta)^{\frac{1}{k}}$ and $\zeta = (r\sin\theta)^{\frac{1}{k}}$ in (2.4.11) yields:

$$\Gamma_{b,k}(\sigma_1)\Gamma_{b,k}(\sigma_2) = \frac{4}{k^2}\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2\frac{\sigma_1+\sigma_2}{k}-1}e^{-\frac{r^2}{k}}(\cos\theta)^{\frac{2\sigma_1}{k}-1}(\sin\theta)^{\frac{2\sigma_2}{k}-1}\exp[-\frac{b^k}{kr^2\sin^2\theta\cos^2\theta}]dr d\theta. \tag{2.4.12}$$

Interchanging the order of integration on the left hand side in (2.4.12), we get

$$\Gamma_{b,k}(\sigma_1)\Gamma_{b,k}(\sigma_2) = \frac{2}{k}\int_0^{\infty} r^{2\frac{\sigma_1+\sigma_2}{k}-1}e^{-\frac{r^2}{k}}\left(\frac{2}{k}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{2\sigma_1}{k}-1}(\sin\theta)^{\frac{2\sigma_2}{k}-1}\exp[-\frac{\frac{b^k}{r^2}}{k\sin^2\theta\cos^2\theta}]d\theta\right)dr. \tag{2.4.13}$$

From (2.4.4) and (2.4.13), the proof of the theorem is complete. □

Remark 2.4.2. If letting $k = 1$, then we have a result of the product of two extended gamma functions and if letting $b = 0$, then we have the product of two gamma k-functions and if both $b = 0$ and $k = 1$, then we get the result of classical gamma functions.

Theorem 2.4.5. *Prove the following Mellin transform representation for extended beta k-function*

$$B_k(\sigma_1, \sigma_2; b) = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} \frac{\Gamma_k(s)\Gamma_k(\sigma_1+s)\Gamma_k(\sigma_2+s)}{\Gamma_k(\sigma_1+\sigma_2+2s)}b^{-s}ds, Re(b) > 0. \tag{2.4.14}$$

Proof. Applying the Mellin transform on both sides of (2.4.1), we have

$$\mathfrak{M}\{B(\sigma_1, \sigma_2; b \to s)\} = \Gamma_k(s)B_k(\sigma_1 + s, \sigma_2 + s). \tag{2.4.15}$$

Now, taking the inverse Mellin transform of both sides of (2.4.15), we obtain

$$B_k(\sigma_1, \sigma_2; b) = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} \Gamma_k(s)B_k(\sigma_1 + s, \sigma_2 + s)b^{-s}ds, \Re(b) > 0. \tag{2.4.16}$$

The substitution for $B_k(\sigma_1 + s, \sigma_2 + s) = \frac{\Gamma_k(\sigma_1+s)\Gamma_k(\sigma_2+s)}{\Gamma_k(\sigma_1+\sigma_2+2s)}$ in (2.4.16) completes the proof of (2.4.14).

□

2.5 Application of Extended Beta k-Function

In this section, we introduced extended beta k-distribution by using the definition of extended beta k-function. Also, we introduced mean, variance and moment generating functions of extended beta k-distribution.

$$f_k(z) = \begin{cases} \frac{1}{kB_k(\sigma_1,\sigma_2;b)} z^{\frac{\sigma_1}{k}-1}(1-z)^{\frac{\sigma_2}{k}-1} e^{-\frac{b^k}{kt(1-t)}}; 0 \leqq z \leqq 1; \sigma_1, \sigma_2, k > 0, \\ 0, \quad elsewhere. \end{cases} \tag{2.5.1}$$

Its distribution function $F_k(z)$ is given by

$$F_k(z) = \begin{cases} 0, \quad z < 0 \\ \int\limits_0^z \frac{1}{kB_k(\sigma_1,\sigma_2;b)} z^{\frac{\sigma_1}{k}-1}(1-z)^{\frac{\sigma_2}{k}-1} e^{-\frac{b^k}{kt(1-t)}} dz; 0 \leqq z \leqq 1; \sigma_1, \sigma_2 > 0, \\ 0, \quad z > 1. \end{cases} \tag{2.5.2}$$

This can also be written as

$$F_k(z) = \frac{B_{z,k}(\sigma_1, \sigma_2; b)}{B_k(\sigma_1, \sigma_2; b)}, \tag{2.5.3}$$

where

$$B_{z,k}(\sigma_1, \sigma_2; b) = \frac{1}{k} \int\limits_0^z z^{\frac{\sigma_1}{k}-1}(1-z)^{\frac{\sigma_2}{k}-1} e^{-\frac{b^k}{kt(1-t)}} dz; \qquad 0 \leqq z \leqq 1; \sigma_1, \sigma_2 > 0 \tag{2.5.4}$$

is the extended incomplete beta k-function.

The mean of the extended beta k-distribution is given by

$$\mu_k = E(Z) = \frac{B_k(\sigma_1 + k, \sigma_2; b)}{B_k(\sigma_1, \sigma_2)} \tag{2.5.5}$$

and the variance of extended beta k-distribution is defined by

$$\sigma_k^2 = E(Z^2) - (E(Z))^2 = \frac{B_k(\sigma_1, \sigma_2; b) B_k(\sigma_1 + 2k, \sigma_2; b) - B_k^2(\sigma_1 + k, \sigma_2; b)}{B_k^2(\sigma_1, \sigma_2; b)}. \tag{2.5.6}$$

The moment generating function of the beta k-distribution is

$$M_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(Z^n) = \frac{1}{B_k(\sigma_1, \sigma_2; b)} \sum_{n=0}^{\infty} B_k(\sigma_1 + nk, \sigma_2; b) \frac{t^n}{n!}. \tag{2.5.7}$$

Remark 2.5.1. If we letting $k = 1$, then we obtain all the results of extended beta distribution see [7], similarly if we take $b = 0$, then we get all the results of k-beta distribution [88] and if both $b = 0$ and $k = 1$, then we get the results of beta distribution.

2.6 Further Application of Extended Gamma and Beta k-Functions

Here, we investigate some further application of extended gamma and beta k-functions, which we introduced in section 2.2.

$$\Gamma_{b,k}^{\lambda,\rho}(\sigma_1) = \int_0^{\infty} t^{\sigma_1 - 1} \cdot {}_1F_{1,k}\left(\lambda; \rho; -\tfrac{t^k}{k} - \tfrac{b^k}{kt^k} \right) dt, \tag{2.6.1}$$

where $\Re(\lambda) > 0, \Re(\rho) > 0, \Re(b) > 0, \Re(\sigma_1) > 0, k > 0,$

and

$$B_{b,k}^{\lambda,\rho}(\sigma_1, \sigma_2) = \frac{1}{k} \int_0^1 t^{\frac{\sigma_1}{k} - 1} (1-t)^{\frac{\sigma_2}{k} - 1} \cdot {}_1F_{1,k}\left(\lambda; \rho; -\tfrac{b^k}{kt(1-t)} \right) dt, \tag{2.6.2}$$

where $\Re(\lambda) > 0, \Re(\rho) > 0, \Re(b) > 0, \Re(\sigma_1) > 0, \Re(\sigma_2) > 0, k > 0$. It is obvious that, if we letting $k \to 1$ and, then we get the generalize extended gamma and beta functions see [82]. Similarly, if $\lambda = \rho$ then we get the results which we have introduced in section 2.2 and if both $b = 0$, $k = 1$ and $\lambda = \rho$, then we bave the results of classical gamma and beta functions.

2.7 Properties of Generalized Extended Gamma and Beta k-Functions

Here, we discuss different integral representation and some properties of new generalized gamma and beta k-functions.

Theorem 2.7.1. *The following integral representation for generalized gamma k-function holds:*

$$\Gamma_{b,k}^{\lambda,\rho}(s) = \frac{\Gamma_k(\rho)}{k\Gamma_k(\lambda)\Gamma_k(\rho-\lambda)} \int_0^1 \Gamma_{b\mu^2,k} \mu^{\frac{\lambda-s}{k}-1}(1-\mu)^{\frac{\rho-\lambda}{k}-1} d\mu. \tag{2.7.1}$$

Proof. Using the integral representation of confluent hypergeometric k-function, we have

$$\Gamma_{b,k}^{\lambda,\rho}(s) = \frac{\Gamma_k(\rho)}{k\Gamma_k(\lambda)\Gamma_k(\rho-\lambda)} \int_0^\infty \int_0^1 u^{s-1} \exp[-\frac{u^k t}{k} - \frac{b^k t}{k u^k}] t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\rho-\lambda}{k}-1} dt du.$$

Now, using one to one transformation i.e. except possibly at the boundaries and maps the region onto itself. $v = ut^{\frac{1}{k}}$, $\mu = t$ in the above equation, we get

$$\Gamma_{b,k}^{\lambda,\rho}(s) = \frac{\Gamma_k(\rho)}{k\Gamma_k(\lambda)\Gamma_k(\rho-\lambda)} \int_0^\infty \int_0^1 v^{s-1} \exp[-\frac{v^k}{k} - \frac{b^k \mu^2}{k v^k}] dv \mu^{\frac{\lambda-s}{k}-1}(1-\mu)^{\frac{\rho-\lambda}{k}-1} d\mu.$$

From the uniform convergence of the integrals, interchanging the order of integration, we have

$$\begin{aligned}\Gamma_{b,k}^{\lambda,\rho}(s) &= \frac{\Gamma_k(\rho)}{k\Gamma_k(\lambda)\Gamma_k(\rho-\lambda)} \int_0^1 [\int_0^\infty v^{s-1} \exp[-\frac{v^k}{k} - \frac{b^k \mu^2}{k v^k}] dv] \mu^{\frac{\lambda-s}{k}-1}(1-\mu)^{\frac{\rho-\lambda}{k}-1} d\mu \\ &= \frac{\Gamma_k(\rho)}{k\Gamma_k(\lambda)\Gamma_k(\rho-\lambda)} \int_0^1 \Gamma_{b\mu^2,k} \mu^{\frac{\lambda-s}{k}-1}(1-\mu)^{\frac{\rho-\lambda}{k}-1} d\mu.\end{aligned}$$

Hence the result follows. □

Remark 2.7.1. The case when $k=1$ in above theorem, we get [82]

$$\Gamma_b^{\lambda,\rho}(s)=\frac{\Gamma(\rho)}{\Gamma(\lambda)\Gamma(\rho-\lambda)}\int_0^1 \Gamma_{b\mu^2}\mu^{\lambda-s-1}(1-\mu)^{\rho-\lambda-1}d\mu.$$

Similarly, if $b=0$ then

$$\begin{aligned}\Gamma_k^{\lambda,\rho}(s) &= \frac{\Gamma_k(\rho)}{k\Gamma_k(\lambda)\Gamma_k(\rho-\lambda)}\int_0^1 \Gamma_k(s)\mu^{\frac{\lambda-s}{k}-1}(1-\mu)^{\frac{\rho-\lambda}{k}-1}d\mu \\ &= \frac{\Gamma_k(\rho)\Gamma_k(\lambda-s)\Gamma_k(s)}{\Gamma_k(\lambda)\Gamma_k(\rho-s)}.\end{aligned} \tag{2.7.2}$$

Theorem 2.7.2. *For the generalized extended beta k-function, we have the following integral representation*

$$\int_0^\infty b^{s-1}B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2)db=B_k(\sigma_1+s,\sigma_2+s)\Gamma_k^{\lambda,\rho}(s) \tag{2.7.3}$$

where $\Re(s)>0, \Re(\sigma_1+s)>0, \Re(\sigma_2+s)>0, \Re(b)>0, \Re(\lambda)>0, \Re(\rho)>0.$

Proof. Multiplying (2.6.2) by b^{s-1} and integrating with respect to b from $b=0$ to $b=\infty$, we get

$$\int_0^\infty b^{s-1}B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2)db=\frac{1}{k}\int_0^\infty b^{s-1}\int_0^1 t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}-1}{}_1F_{1,k}\left(\lambda;\rho;-\tfrac{b^k}{kt(1-t)}\right)dtdb. \tag{2.7.4}$$

From the uniform convergence of the integral, the order of integration in (2.7.4) can be interchanged. Therefore, we have

$$\int_0^\infty b^{s-1}B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2)db=\frac{1}{k}\int_0^1 t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}-1}\int_0^\infty b^{s-1}{}_1F_{1,k}\left(\lambda;\rho;-\tfrac{b^k}{kt(1-t)}\right)dbdt. \tag{2.7.5}$$

Now, using the one to one transformation (except possibly at the boundaries and maps the region onto itself) $v=\frac{b}{t^{\frac{1}{k}}(1-t)^{\frac{1}{k}}}$, $\mu=t$ in (2.7.5), we get

$$\int_0^\infty b^{s-1}B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2)db=\frac{1}{k}\int_0^1 \mu^{\frac{\sigma_1+s}{k}-1}(1-\mu)^{\frac{\sigma_2+s}{k}-1}d\mu\int_0^\infty v^{s-1}{}_1F_{1,k}\left(\lambda;\rho;-\tfrac{v^k}{k}\right)dv.$$

Therefore, using (2.6.1), we have

$$\int_0^\infty b^{s-1} B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2)db = B_k(\sigma_1+s,\sigma_2+s)\Gamma_k^{\lambda,\rho}(s).$$

□

Theorem 2.7.3. *The following integral representations holds for generalized beta k-function:*

$$B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2) = \frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2\sigma_1}{k}-1}(\sin\theta)^{\frac{2\sigma_2}{k}-1} \quad {}_1F_{1,k}\left(\ \lambda;\rho;-\tfrac{b^k}{k}\sec^2\theta\csc^2\theta\ \right)d\theta \tag{2.7.6}$$

and

$$B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2) = \frac{1}{k}\int_0^\infty \frac{u^{\frac{\sigma_1}{k}-1}}{(1+u)^{\frac{\sigma_1+\sigma_2}{k}}} \quad {}_1F_{1,k}\left(\ \lambda;\rho;-2\tfrac{b^k}{k}-\tfrac{b^k}{k}(u+\tfrac{1}{u})\ \right)du. \tag{2.7.7}$$

Proof. Consider

$$B_{b,k}^{\lambda,\rho} = \frac{1}{k}\int_0^1 t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}-1} \quad {}_1F_{1,k}\left(\ \lambda;\rho;-\tfrac{b^k}{kt(1-t)}\ \right)dt.$$

Letting $t=\cos^2\theta$, we get

$$B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2) = \frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2\sigma_1}{k}-1}(\sin\theta)^{\frac{2\sigma_2}{k}-1} \quad {}_1F_{1,k}\left(\ \lambda;\rho;-\tfrac{b^k}{k}\sec^2\theta\csc^2\theta\ \right)d\theta.$$

On the other hand to prove (2.7.7), letting $t=\frac{u}{1+u}$ in (2.6.2), we get the required result of (2.7.7). □

Theorem 2.7.4. *The following functional relation holds true:*

$$B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2+k) + B_{b,k}^{\lambda,\rho}(\sigma_1+k,y) = B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2) \tag{2.7.8}$$

Proof.

$$B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2+k) + B_{b,k}^{\lambda,\rho}(\sigma_1+k,y)$$

$$
\begin{aligned}
&= \frac{1}{k}\int_0^1 t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}} \quad {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{b^k}{kt(1-t)} \right) dt \\
&+ \frac{1}{k}\int_0^1 t^{\frac{\sigma_1}{k}}(1-t)^{\frac{\sigma_2}{k}-1} \quad {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{b^k}{kt(1-t)} \right) dt \\
&= \frac{1}{k}\int_0^1 [t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}} + t^{\frac{\sigma_1}{k}}(1-t)^{\frac{\sigma_2}{k}-1}] \quad {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{b^k}{kt(1-t)} \right) dt \\
&= \frac{1}{k}\int_0^1 t^{\frac{\sigma_1}{k}-1}(1-t)^{\frac{\sigma_2}{k}-1} \quad {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{b^k}{kt(1-t)} \right) dt \\
&= B_{b,k}^{\lambda,\rho}(\sigma_1,\sigma_2).
\end{aligned}
$$

□

Theorem 2.7.5. *The following relation between generalized extended gamma and beta k-functions holds:*

$$
\Gamma_{b,k}^{\lambda,\rho}(\sigma_1)\Gamma_{b,k}^{\lambda,\rho}(\sigma_2) = \frac{4}{k^2}\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2\frac{\sigma_1+\sigma_2}{k}-1}(\cos\theta)^{\frac{2\sigma_1}{k}-1}(\sin\theta)^{\frac{2\sigma_2}{k}-1}
$$
$$
\times {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{1}{k}r^2\cos^2\theta-\tfrac{b^k}{kr^2\cos^2\theta} \right) \cdot {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{1}{k}r^2\sin^2\theta-\tfrac{b^k}{kr^2\sin^2\theta} \right) \qquad (2.7.9)
$$

Proof. Substituting $t=\eta^2$ in (2.6.1), we get

$$
\Gamma_{b,k}^{\lambda,\rho}(\sigma_1) = 2\int_0^{\infty} \eta^{2\sigma_1-1} \quad {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{1}{k}\eta^{2k}-\tfrac{b^k}{k\eta^{2k}} \right) d\eta. \qquad (2.7.10)
$$

Therefore

$$
\begin{aligned}
\Gamma_{b,k}^{\lambda,\rho}(\sigma_1)\Gamma_{b,k}^{\lambda,\rho}(\sigma_2) &= 4\int_0^{\infty}\int_0^{\infty} \eta^{2\sigma_1-1}\zeta^{2\sigma_2-1} \quad {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{1}{k}\eta^{2k}-\tfrac{b^k}{k\eta^{2k}} \right) \\
&\times \quad {}_1F_{1,k}\left(\lambda;\rho;-\tfrac{1}{k}\zeta^{2k}-\tfrac{b^k}{k\zeta^{2k}} \right) d\eta d\zeta.
\end{aligned}
$$

Letting $\eta=(r\cos\theta)^{\frac{1}{k}}$ and $\zeta=(r\sin\theta)^{\frac{1}{k}}$ in the above equality, we get the required result in (2.7.9). □

Remark 2.7.2. By putting $b = 0$ and $\lambda = \rho$ in (2.7.9), we obtain the well-known relation

$$B_k(\sigma_1, \sigma_2) = \frac{\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}{\Gamma_k(\sigma_1 + \sigma_2)}.$$

Chapter 3

Some Results on a Generalized Hypergeometric k-Functions

In this chapter [95], we introduce the generalized hypergeometric k-functions ie., τ-Gauss hypergeometric k-functions. Some recurrence relations, integral representation and differential properties of the generalized hypergeometric k-functions have been investigated.

3.1 Introduction

The Gauss hypergeometric function ${}_2F_1(\alpha_1, \alpha_2; \alpha_3; z)$ plays an important role in mathematical analysis and its applications. Most of the special functions appeared in physics, engineering and probability theory are particular cases of gamma, beta and hypergeometric functions ([54], [55], [59], [60], [87], [85], [99]). Wright [119] introduced the Wright type hypergeometric function in the following form

$$ {}_p\Psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + An) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\rho_1 + B_1 n) \cdots \Gamma(\rho_q + B_q n)}, \tag{3.1.1} $$

where A_r and B_s are positive real numbers such that

$$1+\sum_{s=1}^{q} B_s - \sum_{r=1}^{p} A_r > 0.$$

The Wright type hypergeometric function (3.1.1) is slight different from the generalized hypergeometric function ${}_pF_q(z)$. This generalized function has been further considered by Malovichko [63]. Dotsenko [19] introduced the following hypergeometric function as:

$${}_2R_1^{\omega,\mu}(z) = {}_2R_1(\alpha_1,\alpha_2;\alpha_3;\omega;\mu;z) = \frac{\Gamma(\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_3-\alpha_2)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+n)\Gamma(\alpha_2+\frac{\omega}{\mu}n)}{\Gamma(\alpha_3+\frac{\omega}{\mu}n)}\frac{z^n}{n!} \tag{3.1.2}$$

and its integral representation is expressed in the form

$${}_2R_1^{\omega,\mu}(z) = \frac{\mu\Gamma(\alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_3-\alpha_2)}\int_0^1 t^{\mu\alpha_2-1}(1-t^{\mu})^{\alpha_3-\alpha_2-1}(1-zt^{\omega})^{-\alpha_1}dt \tag{3.1.3}$$

where $\Re(\alpha_3) > \Re(\alpha_2) > 0$. In 2001, Virchenko *et al.* [112] have investigated that the function ${}_2R_1^{\omega,\mu}(z)$ is not symmetric with respect to α_1 and α_2. In the same paper, they defined τ-Gauss hypergeometric function ${}_2R_1^{\tau}(z)$ as

$${}_2R_1^{\tau}(z) = {}_2R_1(\alpha_1,\alpha_2;\alpha_3;\tau;z) = \frac{\Gamma(\alpha_3)}{\Gamma(\alpha_3-\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\Gamma(\alpha_2+\tau n)}{\Gamma(\alpha_3+\tau n)}\frac{z^n}{n!};\tau>0,|z|<1 \tag{3.1.4}$$

and its integral representation is defined as

$${}_2R_1^{\tau}(z) = {}_2R_1(\alpha_1,\alpha_2;\alpha_3;\tau;z) = \frac{\Gamma(\alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_3-\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-t)^{\alpha_3-\alpha_2-1}(1-zt^{\tau})^{-\alpha_1}dt \tag{3.1.5}$$

or

$${}_2R_1^{\tau}(z) = {}_2R_1(\alpha_1,\alpha_2;\alpha_3;\tau;z) = \frac{\Gamma(\alpha_3)}{\tau\Gamma(\alpha_2)\Gamma(\alpha_3-\alpha_2)}\int_0^1 t^{\frac{\alpha_2}{\tau}-1}(1-t^{\frac{1}{\tau}})^{\alpha_3-\alpha_2-1}(1-zt)^{-\alpha_1}dt. \tag{3.1.6}$$

The same authors have also defined the following contiguous function relations for ${}_2R_1^{\tau}(z)$ as

$$(\alpha_2 - \alpha_1\tau)R = \alpha_2 R(\alpha_2 + 1) - \alpha_1\tau R(\alpha_1 + 1), \tag{3.1.7}$$

$$(\alpha_3 - \alpha_1\tau - 1)R = (\alpha_3 - 1)R(\alpha_3 - 1) - \alpha_1\tau R(\alpha_1 + 1), \tag{3.1.8}$$

$$(\alpha_3 - \alpha_2 - 1)R = (\alpha_3 - 1)R(\alpha_3 - 1) - \alpha_2 R(\alpha_2 + 1), \tag{3.1.9}$$

and

$$\alpha_3 R = (\alpha_3 - \alpha_2)R(\alpha_3 + 1) - \alpha_2 R(\alpha_2 + 1). \tag{3.1.10}$$

where for simplicity $R =_2 R_1^{\tau}(z) = R(\alpha_1, \alpha_2; \alpha_3; \tau; z)$ and $R(\alpha_1 + 1) = R(\alpha_1 + 1, \alpha_2; \alpha_3; \tau; z)$ etc., have been used. For more details about the theory of Wright type hypergeometric series and for its properties, ([49], [89], [90], [92], [99], [114]). Diaz and Pariguan [15] have defined gamma and beta k-functions which are defined in Chapter 2 (see (2.1.4)-(2.1.8)).

Also, they proved tbe following relation

$$\sum_{n=0}^{\infty}(\alpha_1)_{n,k}\frac{z^n}{n!} = (1 - kz)^{\frac{-\alpha_1}{k}}. \tag{3.1.11}$$

The Researchers ([3], [4], [5], [24], [36], [58], [64]) have established various properties of gamma and beta k-functions and Kokologiannaki [56] introduced zeta k-function as

$$\zeta(z, s) = \sum_{n=0}^{\infty}\frac{1}{(z + nk)^s}, \quad k, z > 0, s > 1, \tag{3.1.12}$$

$$m^{mj}(\frac{z}{m})_{j,k}(\frac{z + k}{m})_{j,k}\cdots(\frac{z + (m-1)k}{m})_{j,k} = (z)_{mj,k}, \tag{3.1.13}$$

The same authors have also defined the following contiguous function relations for ${}_2R_1^{\tau}(z)$ as

$$(\alpha_2 - \alpha_1\tau)R = \alpha_2 R(\alpha_2 + 1) - \alpha_1\tau R(\alpha_1 + 1), \tag{3.1.7}$$

$$(\alpha_3 - \alpha_1\tau - 1)R = (\alpha_3 - 1)R(\alpha_3 - 1) - \alpha_1\tau R(\alpha_1 + 1), \tag{3.1.8}$$

$$(\alpha_3 - \alpha_2 - 1)R = (\alpha_3 - 1)R(\alpha_3 - 1) - \alpha_2 R(\alpha_2 + 1), \tag{3.1.9}$$

and

$$\alpha_3 R = (\alpha_3 - \alpha_2)R(\alpha_3 + 1) - \alpha_2 R(\alpha_2 + 1). \tag{3.1.10}$$

where for simplicity $R =_2 R_1^{\tau}(z) = R(\alpha_1, \alpha_2; \alpha_3; \tau; z)$ and $R(\alpha_1 + 1) = R(\alpha_1 + 1, \alpha_2; \alpha_3; \tau; z)$ etc., have been used. For more details about the theory of Wright type hypergeometric series and for its properties, ([49], [89], [90], [92], [99], [114]). Diaz and Pariguan [15] have defined gamma and beta k-functions which are defined in Chapter 2 (see (2.1.4)-(2.1.8)).

Also, they proved the following relation

$$\sum_{n=0}^{\infty}(\alpha_1)_{n,k}\frac{z^n}{n!} = (1 - kz)^{\frac{-\alpha_1}{k}}. \tag{3.1.11}$$

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$$m^{mj}(\frac{z}{m})_{j,k}(\frac{z + k}{m})_{j,k}\cdots(\frac{z + (m-1)k}{m})_{j,k} = (z)_{mj,k}, \tag{3.1.13}$$

$$(z)_{mj,k} = \frac{\Gamma_k(z+mjk)}{\Gamma_k(z)},$$

and

$$\sum_{j=0}^{\infty} \frac{z^j}{j!} = e^z. \tag{3.1.14}$$

For more details about the theory of special k-functions like gamma k-function, beta k-function, hypergeometric k-function, solutions of hypergeometric k-differential equations, contiguous k-function relations, inequalities with applications and integral representations involving gamma and beta k-functions, recurrence relations and integral representation for Appell k-series and so forth ([78], [73], [74], [75], [76], [93]). In 2012, Mubeen and Habibullah [72] have defined an iutegral representation of some hypergeometric k-functions as

$$F_k[(\alpha_1,k),(\alpha_2,k);(\alpha_3,k);z] = \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)} \int_0^1 t^{\frac{\alpha_2}{k}-1}(1-t)^{\frac{\alpha_3-\alpha_2}{k}-1}(1-ktz)^{\frac{-\alpha_1}{k}}dt.$$

3.2 The Generalized Hypergeometric k-Function.

The Wright type hypergeometric k-function is defined in the following form

$${}_2R_{1,k}^{\omega,\mu}(z) = {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\omega;\mu;z) = \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\frac{\omega}{\mu}nk)}{\Gamma_k(\alpha_3+\frac{\omega}{\mu}nk)} \frac{z^n}{n!}, \quad k>0. \tag{3.2.1}$$

Theorem 3.2.1. *If $\Re(\alpha_3) > \Re(\alpha_2) > 0$, then the function ${}_2R_{1,k}^{\omega,\mu}(z)$ can be expressed in the following form*

$${}_2R_{1,k}^{\omega,\mu}(z) = \frac{\mu\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)} \int_0^1 t^{\mu\frac{\alpha_2}{k}-1}(1-t^{\mu})^{\frac{\alpha_3-\alpha_2}{k}-1}(1-zt^{\omega})^{\frac{-\alpha_1}{k}}dt, \quad k>0. \tag{3.2.2}$$

Proof. Let us consider

$$
\begin{aligned}
{}_2R^{\omega,\mu}_{1,k}(z) &= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\frac{\omega}{\mu}nk)}{\Gamma_k(\alpha_3+\frac{\omega}{\mu}nk)}\frac{z^n}{n!}\\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\frac{\omega}{\mu}nk)\Gamma_k(\alpha_3-\alpha_2)}{\Gamma_k(\alpha_3+\frac{\omega}{\mu}nk)}\frac{z^n}{n!}\\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\sum_{n=0}^{\infty}\Gamma_k(\alpha_1+nk)B_k(b+\frac{\omega}{\mu}nk,\alpha_3-\alpha_2)\frac{z^n}{n!}\\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\sum_{n=0}^{\infty}\Gamma_k(\alpha_1+nk)[\frac{1}{k}\int_0^1 t^{\frac{\alpha_2}{k}+\frac{\omega}{\mu}n-1}(1-t)^{\frac{\alpha_3-\alpha_2}{k}-1}dt]\frac{z^n}{n!}
\end{aligned}
$$

$$
= \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}[\sum_{n=0}^{\infty}\Gamma_k(\alpha_1+nk)\frac{z^n t^{\frac{\omega}{\mu}n}}{n!}]\int_0^1 t^{\frac{\alpha_2}{k}-1}(1-t)^{\frac{\alpha_3-\alpha_2}{k}-1}dt. \tag{3.2.3}
$$

Now since

$$
(1-kzt)^{-\frac{\alpha_1}{k}} = \frac{1}{\Gamma_k(\alpha_1)}\sum_{n=0}^{\infty}\Gamma_k(\alpha_1+nk)\frac{(tz^n)}{n!} \tag{3.2.4}
$$

and taking into account

$$
(1-kzt^{\frac{\omega}{\mu}})^{-\frac{\alpha_1}{k}} = \frac{1}{\Gamma_k(\alpha_1)}\sum_{n=0}^{\infty}\Gamma_k(\alpha_1+nk)\frac{z^n t^{\frac{\omega}{\mu}n}}{n!}. \tag{3.2.5}
$$

Hence by substituting (3.2.5) in (3.2.3), we obtain

$$
{}_2R^{\omega,\mu}_{1,k}(z) = \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\int_0^1 t^{\frac{\alpha_2}{k}-1}(1-t)^{\frac{\alpha_3-\alpha_2}{k}-1}(1-kzt^{\frac{\omega}{\mu}})^{-\frac{\alpha_1}{k}}dt. \tag{3.2.6}
$$

Thus after a simplification, we get the required result as:

$$
{}_2R^{\omega,\mu}_{1,k}(z) = \frac{\mu\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\int_0^1 t^{\mu\frac{\alpha_2}{k}-1}(1-t^{\mu})^{\frac{\alpha_3-\alpha_2}{k}-1}(1-kzt^{\omega})^{-\frac{\alpha_1}{k}}dt.
$$

□

3.3 The τ-Gauss Hypergeometric k-Function ${}_2R^{\tau}_{1,k}(z)$

In this section, we defined the τ-Gauss hypergeometric k-function and some of its properties. As ${}_2R^{\omega,\mu}_{1,k}(z)$ is not symmetric with respect to α_1 and α_2. So by substituting $\frac{\omega}{\mu} = \tau > 0$ in (3.2.1), then we have

$$\begin{aligned} {}_2R^{\tau}_{1,k}(z) &= {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z) \\ &= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1 + nk)\Gamma_k(\alpha_2 + \tau nk)}{\Gamma_k(\alpha_3 + \tau nk)} \frac{z^n}{n!}, k > 0, \tau > 0. \qquad (3.3.1) \end{aligned}$$

Its integral representation is expressed in the following form:

$${}_2R^{\tau}_{1,k}(z) = \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3 - \alpha_2)} \int_0^1 t^{\frac{\alpha_2}{k}-1}(1-t)^{\frac{\alpha_3-\alpha_2}{k}-1}(1-kzt^{\tau})^{-\frac{\alpha_1}{k}} dt \qquad (3.3.2)$$

and by change of variable, we obtain

$${}_2R^{\tau}_{1,k}(z) = \frac{\Gamma_k(\alpha_3)}{\tau k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3 - \alpha_2)} \int_0^1 t^{\frac{\alpha_2}{\tau k}-1}(1-t^{\frac{1}{\tau}})^{\frac{\alpha_3-\alpha_2}{k}-1}(1-kzt)^{-\frac{\alpha_1}{k}} dt. \qquad (3.3.3)$$

Definition 3.3.1. We define the contiguous function to ${}_2R^{\tau}_{1,k}(z)$ as a function which is obtained by increasing or decreasing one of the parameters by $\pm k$ where $k > 0$. For simplicity, we use the following notations ${}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z) = R_k$, ${}_2R_{1,k}(\alpha_1 + k, \alpha_2; \alpha_3; \tau; z) = R_k(\alpha_1 + k)$, ${}_2R_{1,k}(\alpha_1, \alpha_2 + k; \alpha_3; \tau; z) = R_k(\alpha_2 + k)$, etc.

Lemma 3.3.1. *For ${}_2R^{\tau}_{1,k}(z)$ and its contiguous functions, the following relations satisfy*

$$(\alpha_2 - \alpha_1\tau)R_k = \alpha_2 R_k(\alpha_2 + k) - \alpha_1\tau R_k(\alpha_1 + k), \qquad (3.3.4)$$

$$(\alpha_3 - \alpha_1\tau - k)R_k = (\alpha_3 - k)R_k(\alpha_3 - k) - \alpha_1\tau R_k(\alpha_1 + k), \qquad (3.3.5)$$

$$(\alpha_3 - \alpha_2 - k)R_k = \alpha_3 R_k(\alpha_3 - k) - \alpha_2 R_k(\alpha_1 + k), \qquad (3.3.6)$$

$$\alpha_3 R_k = (\alpha_3 - \alpha_2)R_k(\alpha_3 + k) - \alpha_2 R_k(\alpha_2 + k; \alpha_3 + k), \qquad (3.3.7)$$

and

$$\Gamma_k(\alpha_2)\Gamma_k(\alpha_3+\tau k)R_k = \Gamma_k(\alpha_2)\Gamma_k(\alpha_3+\tau k)R_k(\alpha_1+k) - kz\Gamma_k(\alpha_3)\Gamma_k(\alpha_2+\tau k)R_k(\alpha_1+k;\alpha_2+k;\alpha_3+k). \quad (3.3.8)$$

Proof. To prove the first relation (3.3.4), we have

$$\alpha_2 R_k(\alpha_2+k) = \frac{\alpha_2\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2+k)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+k+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^n}{n!}$$

$$= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^n}{n!}(\alpha_2+\tau nk) \quad (3.3.9)$$

and

$$\alpha_1\tau R_k(\alpha_1+k) = \frac{\alpha_1\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_1+k)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+k+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^n}{n!}$$

$$= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_1)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+nk)}\frac{\tau z^n}{n!}(\alpha_1+nk). \quad (3.3.10)$$

Subtracting (3.3.10) from (3.3.9), we get the required relation (3.3.4). Now to prove relation (3.3.5), we have

$$(\alpha_3-k)R_k(\alpha_3-k) = \frac{(\alpha_3-k)\Gamma_k(\alpha_3-k)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+k+\tau nk)}{\Gamma_k(\alpha_3-k+\tau nk)}\frac{z^n}{n!}$$

$$= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+k+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^n}{n!}(\alpha_3-k+\tau nk) \quad (3.3.11)$$

and

$$\alpha_1\tau R_k(\alpha_1+k) = \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_1)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{\tau z^n}{n!}(\alpha_1+nk). \quad (3.3.12)$$

Thus subtracting (3.3.12) from (3.3.11), we get the desired relation. In the same manner, we can prove (3.3.6)-(3.3.8). □

Lemma 3.3.2. *If $\tau \in \mathbb{N}$ $(\tau = n)$, then the following relation holds*

$$
{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;n;z) = A
\times {}_{n+1}F_{n,k}[(\alpha_1,k),(\frac{\alpha_2}{n},k),\cdots,(\frac{\alpha_2+(n-1)k}{n},k);(\frac{\alpha_3}{n},k),(\frac{\alpha_3+k}{n},k),\cdots,(\frac{\alpha_3+(n-1)k}{n},k);z], \tag{3.3.13}
$$

where

$$
A = n^{-\frac{\delta}{k}} \frac{\Gamma_k(\alpha_3)\Gamma_k(\frac{\alpha_2}{n})\Gamma_k(\frac{\alpha_2+k}{n})\cdots\Gamma_k(\frac{\alpha_2+(n-1)k}{n}))}{\Gamma_k(\alpha_2)\Gamma_k(\frac{\alpha_3}{n})\Gamma_k(\frac{\alpha_3+k}{n})\cdots\Gamma_k(\frac{\alpha_3+(n-1)k}{n})}, \quad \delta = \alpha_3 - \alpha_2.
$$

Proof. Let us consider

$$
{}_{n+1}F_{n,k}[(\alpha_1,k),(\frac{\alpha_2}{n},k),\cdots,(\frac{\alpha_2+(n-1)k}{n},k);(\frac{\alpha_3}{n},k),(\frac{\alpha_3+k}{n},k),\cdots,(\frac{\alpha_3+(n-1)k}{n},k);z]
$$

$$
= \frac{\Gamma_k(\frac{\alpha_3}{n})\cdots\Gamma_k(\frac{\alpha_3+(n-1)k}{n})}{\Gamma_k(\alpha_1)\Gamma_k(\frac{\alpha_2}{n})\cdots\Gamma_k(\frac{\alpha_2+(n-1)k}{n})}
\times \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\frac{\alpha_2}{n}+nk)\Gamma_k(\frac{\alpha_2+k}{n}+nk)\cdots\Gamma_k(\frac{\alpha_2+(n-1)k}{n}+nk)}{\Gamma_k(\frac{\alpha_3}{n}+nk)\Gamma_k(\frac{\alpha_3+k}{n}+nk)\cdots\Gamma_k(\frac{\alpha_3+(n-1)k}{n}+nk)} \frac{z^n}{n!}
$$

$$
= \frac{n^{\frac{\alpha_3}{k}}\Gamma_k(\frac{\alpha_3}{n})\cdots\Gamma_k(\frac{\alpha_3+(n-1)k}{n})}{n^{\frac{\alpha_3}{k}}\Gamma_k(\alpha_1)\Gamma_k(\frac{\alpha_2}{n})\cdots\Gamma_k(\frac{\alpha_2+(n-1)k}{n})} \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+n^2k)}{\Gamma_k(\alpha_3+n^2k)} \frac{z^n}{n!}. \tag{3.3.14}
$$

By substituting (3.3.14) in right hand side of (3.3.13), we get

$$
\frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+n^2k)}{\Gamma_k(\alpha_3+n^2k)} \frac{z^n}{n!} = {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;n;z).
$$

□

3.4 Differentiation Formulas

In this section, we derive some basic differentiation formulas by the help of following lemmas.

Lemma 3.4.1. *If $k > 0$, then*

$$\frac{d}{dz}[{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z)] = \alpha_1\frac{\Gamma_k(\alpha_3)\Gamma_k(\alpha_2+\tau k)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_3+\tau k)} \quad {}_2R_{1,k}(\alpha_1+k,\alpha_2+\tau k;\alpha_3+\tau k;\tau;z). \tag{3.4.1}$$

Proof. Consider

$$\frac{d}{dz}[{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z)] = \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_1)}\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^n}{n!}.$$

Thus, we can write

$$\frac{d}{dz}[{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z)] = \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_1)}\sum_{n=1}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^{n-1}}{(n-1)!}. \tag{3.4.2}$$

Now replace $n-1$ by n in (3.4.2), we obtain

$$\begin{aligned}\frac{d}{dz}[{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z)] &= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_1)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+k+nk)\Gamma_k(\alpha_2+\tau k+\tau nk)}{\Gamma_k(\alpha_3+\tau k+\tau nk)}\frac{z^n}{n!}\\ &= \alpha_1\frac{\Gamma_k(\alpha_3)\Gamma_k(\alpha_3+\tau k)\Gamma_k(\alpha_2+\tau k)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_2+\tau k)\Gamma_k(\alpha_1+k)\Gamma_k(\alpha_3+\tau k)}\\ &\times \sum_{n=0}^{\infty}\frac{\Gamma_k(a+k+nk)\Gamma_k(\alpha_2+\tau k+\tau nk)}{\Gamma_k(\alpha_3+\tau k+\tau nk)}\frac{z^n}{n!}\\ &= \alpha_1\frac{\Gamma_k(\alpha_3)\Gamma_k(\alpha_2+\tau k)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_3+\tau k)} \quad {}_2R_{1,k}(\alpha_1+k,\alpha_2+\tau k;\alpha_3+\tau k;\tau;z).\end{aligned}$$

□

Lemma 3.4.2. *If $k > 0$, then*

$$\frac{d}{dz}[z^{\frac{\alpha_1}{k}} \quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z) = \frac{1}{k}[\alpha_1 z^{\frac{\alpha_1}{k}-1} \quad {}_2R_{1,k}(\alpha_1+k,\alpha_2;\alpha_3;\tau;z)]. \tag{3.4.3}$$

Proof. Let us consider

$$\frac{d}{dz}[z^{\frac{\alpha_1}{k}} \quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z)]$$

$$\begin{aligned}&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)}\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^{n+\frac{\alpha_1}{k}}}{n!}\\ &= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\alpha_1+nk)\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{z^{n+\frac{\alpha_1}{k}-1}}{n!}(\frac{\alpha_1}{k}+n)\\ &= \frac{1}{k}[\alpha_1 z^{\frac{\alpha_1}{k}-1} \quad {}_2R_{1,k}(\alpha_1+k,\alpha_2;\alpha_3;\tau;z)].\end{aligned}$$

□

Similarly the following differentiation formulas holds for $k > 0$

$$\frac{d^n}{dz^n}[{}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z)] = \frac{\Gamma_k(\alpha_1 + nk)\Gamma_k(\alpha_2 + \tau nk)\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\Gamma_k(\alpha_3 + \tau nk)} \times {}_2R_{1,k}(\alpha_1 + nk, \alpha_2 + \tau nk; \alpha_3 + \tau nk; \tau; z), \quad (3.4.4)$$

$$\frac{d^n}{dz^n}[z^{\frac{\alpha_1}{k}+n-1} \quad {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z)] = \frac{\Gamma_k(\alpha_1 + nk)}{k\Gamma_k(\alpha_1)} z^{\frac{\alpha_1}{k}-1} \quad {}_2R_{1,k}(\alpha_1 + nk, \alpha_2; \alpha_3; \tau; z), \quad (3.4.5)$$

and

$$(\alpha_1)\, {}_2R_{1,k}(\alpha_1 + k, \alpha_2; \alpha_3; \tau; z) = (kz\frac{d}{dz} + \alpha_1) \quad {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z). \quad (3.4.6)$$

To prove the result (3.4.6), we have

$$\begin{aligned}
&\alpha_1[{}_2R_{1,k}(\alpha_1 + k, \alpha_2; \alpha_3; \tau; z) - {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z)] \\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} [\frac{\alpha_1\Gamma_k(\alpha_1 + k + nk)\Gamma_k(\alpha_2 + \tau nk)}{\Gamma_k(\alpha_1 + k)\Gamma_k(\alpha_3 + \tau nk)} \\
&- \frac{\alpha_1\Gamma_k(\alpha_1 + nk)\Gamma_k(\alpha_2 + \tau nk)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_3 + \tau nk)}]\frac{z^n}{n!} \\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1 + nk)\Gamma_k(\alpha_2 + \tau nk)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_3 + \tau nk)}[\alpha_1 + nk - \alpha_1]\frac{z^n}{n!} \\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)} \sum_{n=1}^{\infty} \frac{\Gamma_k(\alpha_2 + \tau nk)}{\Gamma_k(\alpha_1)\Gamma_k(\alpha_3 + \tau nk)} k \frac{z^n}{(n-1)!} \\
&= kz\frac{d}{dz} \quad {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z).
\end{aligned}$$

This implies that

$$\alpha_1\, {}_2R_{1,k}(\alpha_1 + k, \alpha_2; \alpha_3; \tau; z) = (kz\frac{d}{dz} + \alpha_1){}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z).$$

3.5 Integral Formulas of ${}_2R^{\tau}_{1,k}(z)$

In this section, we derive some integral formulas in term of k, where $k > 0$.

Theorem 3.5.1. *If $\Re(\alpha_3 - \alpha_2) > 1 - \frac{1}{\tau k}$, $\Re(\alpha_3 - \alpha_2) > 0$, then ${}_2R^{\tau}_{1,k}(z)$ can be expressed in the following integral forms:*

$$
{}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \frac{1}{\tau}; z) = \frac{2\Gamma_k(\alpha_3)}{\tau k \Gamma_k(\alpha_2)\Gamma_k(\alpha_3 - \alpha_2)} \int_0^{\infty} \frac{(\sinh\phi)^{2\frac{\alpha_2}{\tau k}-1}(\cosh\phi + 1)^{\frac{1}{\tau}+\frac{\alpha_1}{k}-\frac{(\alpha_2+\alpha_3)}{\tau k}}}{[1 + kz + (1 - kz)\cosh\phi]^{\frac{\alpha_1}{k}}}
$$

$$
\times \left[(\cosh\phi + 1)^{\frac{1}{\tau}} - (\cosh\phi - 1)^{\frac{1}{\tau}}\right]^{\frac{\alpha_3-\alpha_2}{k}-1} d\phi, \quad (3.5.1)
$$

and

$$
{}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \frac{1}{\tau}; z) = \frac{4\Gamma_k(\alpha_3)}{\tau k \Gamma_k(\alpha_2)\Gamma_k(\alpha_3 - \alpha_2)} \int_0^{\infty} \frac{(\cosh\phi)^{\frac{2}{\tau}-\frac{2\alpha_3}{\tau k}+\frac{2\alpha_1}{k}-1}(\cosh\phi - 1)^{\frac{(\alpha_2+\alpha_3)}{\tau k}-\frac{\alpha_1}{k}-\frac{1}{\tau}}}{[1 + kz + (1 - kz)\cosh\phi]^{\frac{\alpha_1}{k}}}
$$

$$
\times \left[(\cosh\phi + 1)^{\frac{1}{\tau}} - (\cosh\phi - 1)^{\frac{1}{\tau}}\right]^{\frac{\alpha_3-\alpha_2}{k}-1} d\phi. \quad (3.5.2)
$$

Proof. To prove (3.5.1), using the substitution $t^{\tau} = \tanh^2 \frac{\phi}{2}$ in (3.3.2) then

$$
\begin{aligned}
{}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \frac{1}{\tau}; z) &= \frac{2\Gamma_k(\alpha_3)}{\tau k \Gamma_k(\alpha_2)\Gamma_k(\alpha_3 - \alpha_2)} \int_0^{\infty} \frac{(\tanh^2 \frac{\phi}{2})^{\frac{1}{\tau}(\frac{\alpha_2}{k}-1)}(1 - \tanh^2 \frac{\phi}{2})^{\frac{1}{\tau}(\frac{\alpha_3-\alpha_2}{k}-1)}}{(1 - kz(\tanh^2 \frac{\phi}{2}))^{-\frac{\alpha_1}{k}}} \\
&\times \ (\tanh^2 \frac{\phi}{2})^{\frac{1}{\tau}-1} \tanh \frac{\phi}{2} \frac{1}{\cosh^2 \frac{\phi}{2}} d\phi.
\end{aligned}
$$

Now taking into account that

$$
\cosh\phi - 1 = \frac{\sinh^2\phi}{\cosh\phi + 1},
$$

and after simplification, we get

$$
\begin{aligned}
{}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \frac{1}{\tau}; z) &= \frac{2\Gamma_k(\alpha_3)}{\tau k \Gamma_k(\alpha_2)\Gamma_k(\alpha_3 - \alpha_2)} \int_0^{\infty} \frac{(\sinh\phi)^{2\frac{\alpha_2}{\tau k}-1}(\cosh\phi + 1)^{\frac{1}{\tau}+\frac{\alpha_1}{k}-\frac{(\alpha_2+\alpha_3)}{\tau k}}}{[1 + kz + (1 - kz)\cosh\phi]^{\frac{\alpha_1}{k}}} \\
&\times \ \left[(\cosh\phi + 1)^{\frac{1}{\tau}} - (\cosh\phi - 1)^{\frac{1}{\tau}}\right]^{\frac{\alpha_3-\alpha_2}{k}-1} d\phi.
\end{aligned}
$$

Similarly, using the substitution $t^\tau = \tanh^2 \frac{\phi}{2}$ in (3.3.2) and then taking the following into account, we will get the required integral (3.5.2)

$$\cosh\phi + 1 = \frac{\sinh^2\phi}{\cosh\phi - 1}.$$

□

Theorem 3.5.2. *If $\Re(\alpha_3) > \Re(\alpha_2) > 0$, $s \neq -1$, then the following relation holds:*

$${}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z) = \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\int_0^\infty s^{\frac{\alpha_2}{k}}(1+s)^{-\frac{\alpha_3}{k}}[1-kz(\frac{s}{s+1})^\tau]^{-\frac{\alpha_1}{k}}ds. \tag{3.5.3}$$

Proof. Let us consider (3.3.2)

$${}_2R^\tau_{1,k}(z) = \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\int_0^1 t^{\frac{\alpha_2}{k}-1}(1-t)^{\frac{\alpha_3-\alpha_2}{k}-1}(1-kzt^\tau)^{-\frac{\alpha_1}{k}}dt.$$

Now replacing t by $\frac{s}{s+1}$, then $dt = \frac{1}{(s+1)^2}ds$. Thus, we can write it as

$$\begin{aligned}{}_2R^\tau_{1,k}(z) &= \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\int_0^\infty (\frac{s}{s+1})^{\frac{\alpha_2}{k}-1}(1-(\frac{s}{s+1}))^{\frac{\alpha_3-\alpha_2}{k}-1}\\ &\times [1-kz(\frac{s}{s+1})^\tau]^{-\frac{\alpha_1}{k}}\frac{1}{(s+1)^2}ds\\ &= \frac{\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\int_0^\infty s^{\frac{\alpha_2}{k}-1}(1+s)^{-\frac{\alpha_3}{k}}[1-kz(\frac{s}{s+1})^\tau]^{-\frac{\alpha_1}{k}}ds.\end{aligned}$$

□

Corollary 3.5.3. *The substitution $s = \sinh^2\phi$ in (3.5.3) leads to the following integral representation as*

$${}_2R^\tau_{1,k}(z) = \frac{2\Gamma_k(\alpha_3)}{k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)}\int_0^\infty(\sinh\phi)^{\frac{2\alpha_2}{k}-1}(\cosh\phi)^{-\frac{2\alpha_3}{k}+1}[1-kz(\tanh\phi)^{2\tau}]^{-\frac{\alpha_1}{k}}d\phi. \tag{3.5.4}$$

The following integral representation can be easily derived from theorem 3.5.2.

$$ {}_2R^{\tau}_{1,k}(z) = \frac{2\Gamma_k(\alpha_3)}{\tau k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)} \int_0^{\frac{\pi}{2}} \frac{(\sin\lambda)^{\frac{2\alpha_2}{\tau k}-1}(1-\sin^{\frac{2}{\tau}}\lambda)^{\frac{\alpha_3-\alpha_2}{k}-1}}{(1-kz\sin^2\lambda)^{\frac{\alpha_1}{k}}\cos\lambda} d\lambda, \quad (3.5.5) $$

$$ = \frac{2\Gamma_k(\alpha_3)}{\tau k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)} \int_0^{\pi} \frac{(\sinh\frac{\lambda}{2})^{\frac{2\alpha_2}{\tau k}-1}(1-\sin^{\frac{2}{\tau}}\frac{\lambda}{2})^{\frac{\alpha_3-\alpha_2}{k}-1}}{(1-k\frac{z}{2}+k\frac{z}{2}\cos\lambda)^{\frac{\alpha_1}{k}}\cos\frac{\lambda}{2}} d\lambda \quad (3.5.6) $$

and

$$ = \frac{2\Gamma_k(\alpha_3)}{\tau k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)} \int_0^{\infty} \frac{(\tanh\lambda)^{\frac{2\alpha_2}{\tau k}-1}(1-\tanh^{\frac{2}{\tau}}\lambda)^{\frac{\alpha_3-\alpha_2}{k}-1}}{(1-kz\tanh^2\lambda)^{\frac{\alpha_1}{k}}\cosh^2\lambda} d\lambda. \quad (3.5.7) $$

To prove (3.5.5), we may write theorem 3.5.2 as

$$ {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;z) = \frac{\Gamma_k(\alpha_3)}{\tau k\Gamma_k(\alpha_2)\Gamma_k(\alpha_3-\alpha_2)} \int_0^{\infty} s^{\frac{\alpha_2}{\tau k}}(1+s)^{-\frac{\alpha_3}{\tau k}}[1-kz(\frac{s}{s+1})]^{-\frac{\alpha_1}{k}} ds. $$

Now by replacing $s = \tan^2\lambda$, then after simplification we get the required integral representation. Similarly we can prove (3.5.6) and (3.5.7).

Chapter 4

Properties of Generalized Hypergeometric k-Functions via Generalized k-Fractional Calculus

In this chapter ([80]), we introduce the generalized k-fractional integral and differential operators and obtained generalized k-fractional integration and differentiation formulas of generalized hypergeometric k-function. Also, we investigate investigate a certain number of their consequences containing the said function in their kernels.

4.1 Introduction

Special functions mainly the hypergeometric functions are helpful in solving the problems in fields of mathematical physics, engineering and other mathematical sciences [53], [61], [101].

We begin with the Gauss hypergeometric function ${}_2F_1(\alpha_1, \alpha_2; \alpha_3; z)$ is defined [87] as

$$ {}_2F_1(\alpha_1, \alpha_2; \alpha_3; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\alpha_3)_n} \frac{z^n}{n!}, (|z| < 1), \tag{4.1.1} $$

where the denominator parameters are neither zero nor negative integer.

The generalized hypergeometric function has been defined [24] as

$$ {}_pF_q\left[\begin{array}{c}\alpha_1,\cdots,\alpha_p;z\\ \beta_1,\cdots,\beta_q\end{array}\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},(p=q+1,|z|<1), \tag{4.1.2} $$

where the denominator parameters are neither zero nor negative integer.

Many researchers ([50], [112], [113], [114], [89], [90], [84], [91], [27], [28], [29], [30]). have been made several generalization of hypergeometric function. Virchenko *et al.* [112] defined τ-generalization of Gauss hypergeometric function ${}_2R_1(\alpha_1,\alpha_2;\alpha_3;\tau,z)$ and its integral representation which are defined in (3.1.4) and (3.1.6) (see Chapter 3). Rao *et al.* [92] obtained many properties for the generalized hypergeometric function ${}_2R_1(\alpha_1,\alpha_2;\alpha_3;\tau;z)$. Recently many researchers Prajapapati *et al.* [84], Prajapapati and Shukla [107] and Srivastava *et al.* [109] used fractional calculus approach in the study of integral operator and generalized Mittag-Leffler function. In 2007, Diaz and Pariguan [15] have introduced an improved generalized version of the classical gamma and beta functions called them gamma and beta k-functions and proved some relations of gamma k-function and Pochhammer k-symbol as defined in Chapter 2. They [15] used Pochhammer k-symbols and defined hypergeometric k-function as:

$$ {}_2F_{1,k}(\alpha_1,\alpha_2;\alpha_3;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}\frac{z^n}{n!},(|z|<1,k>0), \tag{4.1.3} $$

where $\alpha_1,\alpha_2,\alpha_3\in\mathbb{C}$ and $\alpha_3\neq 0,-1,2,\cdots$.

Mubeen and Habibullah [72, 78] defined the following k-fractional integral and its related result by

$$ I_k^{\alpha_1}(f(z))=\frac{1}{k\Gamma_k(\alpha_1)}\int\limits_o^z(x-t)^{\frac{\alpha_1}{k}-1}f(t)dt, \tag{4.1.4} $$

$$I_k^{\alpha_1}(z^{\frac{\alpha_2}{k}-1}) = \frac{\Gamma_k(\alpha_2)}{\Gamma_k(\alpha_1+\alpha_2)} z^{\frac{\alpha_1}{k}+\frac{\alpha_2}{k}-1} \tag{4.1.5}$$

and

$$I_k^{\alpha_1}((x-u)^{\frac{\alpha_2}{k}-1}) = \frac{\Gamma_k(\alpha_2)}{\Gamma_k(\alpha_1+\alpha_2)}(x-u)^{\frac{\alpha_1}{k}+\frac{\alpha_2}{k}-1}. \tag{4.1.6}$$

The generalized k-fractional integral and its related results are defined in [104] as:

$$ {}_k^s I_a^{\mu} f(z) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_a^z (z^{s+1} - \eta^{s+1})^{\frac{\mu}{k}-1} \eta^s f(t)dt, \tag{4.1.7}$$

where $x \in [a,b]$, $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$.

$$ {}_k^s I_a^{\mu}[(\eta^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}] = \frac{\Gamma_k(\lambda)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\lambda+\mu)}(z^{s+1} - a^{s+1})^{\frac{\lambda+\mu}{k}-1}. \tag{4.1.8}$$

Very recently, Rahman *et al.* [95] defined generalized hypergeometric k-function ${}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z)$ as:

$$ {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau, z) = \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2 + n\tau k)}{\Gamma_k(\alpha_3 + n\tau k)} \frac{z^n}{n!}, \tau > 0, |z| < 1, \tag{4.1.9}$$

and they also defined an integral representation and recurrence relations of generalized hypergeometric k-function ${}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; z)$.

Definition 4.1.1. Let α_1, α_2, α_3, μ, $\nu \in \mathbb{C}$ such that the following summation can be defined.

$$\begin{aligned} &{}_3R_{2,k}(\alpha_1, \alpha_2, \alpha_3; \mu, \nu; \tau, \rho; z) \\ &:= \frac{\Gamma_k(\mu)\Gamma_k(\nu)}{\Gamma_k(\alpha_2)\Gamma_k(\alpha_3)} \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2 + n\tau k)\,\Gamma_k(\alpha_3 + n\rho k)}{\Gamma_k(\mu + n\tau k)\,\Gamma_k(\nu + n\rho k)} \frac{z^n}{n!} \end{aligned} \tag{4.1.10}$$

$$\left(k,\ \tau,\ \rho \in \mathbb{R}^+;\ |z| < 1\right).$$

4.2 The Generalized k-Fractional Integrals and Differentials of Generalized Hypergeometric k-Functions

In this section, we define the generalized k-fractional and differential formulas of generalized hypergeometric k-functions. For this purpose, we first define the following generalized operator as:

Definition 4.2.1. If $k > 0$ and $\alpha_1, \alpha_2, \alpha_3, \omega \in \mathbb{C}$, $\Re(\alpha_1) > 0$, $\Re(\alpha_2) > 0$, $\Re(\alpha_3) > 0$, then

$$({}_k^s\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2} f)(z) = \frac{1}{k}\int_a^z (z^{s+1} - \zeta^{s+1})^{\frac{\alpha_3}{k}-1} \quad {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; \omega(z^{s+1} - \zeta^{s+1})^\rho)\zeta^s f(\tau)d\tau, \tag{4.2.1}$$

where $x > a$.

When setting $s = 0$ in(4.2.1), then we get the following result

$$({}_k\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2} f)(z) = \int_a^z (x - \zeta)^{\frac{\alpha_3}{k}-1} \quad {}_2R_{1,k}(\alpha_1, \alpha_2; \alpha_3; \tau; \omega(x - \zeta)^\rho)f(\tau)d\tau, \tag{4.2.2}$$

Similarly, if $s = 0$ and $k = 1$, then (4.2.1) becomes

$$(\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2} f)(z) = \int_a^z (x - \zeta)^{\alpha_3-1} \quad {}_2R_1(\alpha_1, \alpha_2; \alpha_3; \tau; \omega(x - \zeta)^\rho)f(\tau)d\tau, \tag{4.2.3}$$

see [99].

To define the following generalized k-fractional integral and differential operators, first we define the well-known Lebesgue measurable real or complex valued function $L[a, b]$ such that

$$L(\alpha_1, \alpha_2) = \left\{ f : ||f||_1 = \int_a^b |f(t)|dt < \infty \right\}. \tag{4.2.4}$$

Definition 4.2.2. Assume that $f(z) \in L(a,b)$; $\mu \in \mathbb{C}$; $\mathbb{R}(\mu) > 0$ and $k > 0$, then the left and right generalized sided k-fractional integral operators of order μ are respectively defined as:

$$
\begin{aligned}
{}^{s}_{a,k}D^{-\mu}_{x,}f(z) &= {}^{s}_{a,k}I^{\mu}_{z}f(z) = {}^{s}_{k}I^{\mu}_{a+}f(z) \\
&= ({}^{s}_{k}I^{\mu}_{a+}f)(z) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int\limits_{a}^{z} \frac{f(t)}{(z^{s+1}-\eta^{s+1})^{1-\frac{\mu}{k}}}dt, (x > a) \quad (4.2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{s}_{\rho,k}D^{-\mu}_{\alpha_2}f(z) &= {}^{s}_{\rho,k}I^{\mu}_{\alpha_2}f(z) = {}^{s}_{k}I^{\mu}_{a-}f(z) = ({}^{s}_{k}I^{\mu}_{a-}f)(z) \\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int\limits_{z}^{\alpha_2} \frac{f(t)}{(z^{s+1}-\eta^{s+1})^{1-\frac{\mu}{k}}}dt, (x < \alpha_2) \quad (4.2.6)
\end{aligned}
$$

Definition 4.2.3. Suppose $k > 0$, $s \in \Re\backslash\{-1\}$, $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ and $n = [\Re(\mu)] + 1$, then the left and right sided generalized k-fractional differential operators are respectively defined as:

$$
({}^{s}_{k}D^{\mu}_{a+}f)(z) = \Big(\frac{1}{z^s}\frac{d}{dz}\Big)^n \Big(k^n \quad {}^{s}_{k}I^{nk-\mu}_{a+}f\Big)(z), \quad (4.2.7)
$$

$$
({}^{s}_{k}D^{\mu}_{a-}f)(z) = \Big(-\frac{1}{z^s}\frac{d}{dz}\Big)^n \Big(k^n \quad {}^{s}_{k}I^{nk-\mu}_{a-}f\Big)(z). \quad (4.2.8)
$$

Setting $s = 0$, then we get the result defined in [?]. Also, when $k = 1$ and $s = 0$, then the generalized left and right sided k-fractional integrals and derivatives will lead to the well known fractional integrals and derivatives (see [99]).

Theorem 4.2.1. *Let $k, \tau, \rho \in \mathbb{R}^+$, $s \in \mathbb{R}\setminus\{-1\}$, and $m \in \mathbb{N}$. Also, let $x, a \in \mathbb{R}$ with $x > a$ and $x \neq 0$. Further, let $\alpha_1, \alpha_2, \tau, \omega, c \in \mathbb{C}$ be such that the involved summations can be defined. Then*

$$
\begin{aligned}
&\Big(\frac{1}{z^s}\frac{d}{dz}\Big)^m \Big\{(z^{s+1}-a^{s+1})^{\frac{c}{k}-1}\, {}_2R_{1,k}\left(\alpha_1,\alpha_2;\alpha_3;\tau,\omega(z^{s+1}-a^{s+1})^{\rho}\right)\Big\} \\
&= \frac{\Gamma_k(c)}{\Gamma_k(c-mk)} \frac{(s+1)^m}{k^m} \left(z^{s+1}-a^{s+1}\right)^{\frac{c}{k}-m-1} \quad (4.2.9) \\
&\times {}_3R_{2,k}\Big(\alpha_1,\alpha_2,c;\alpha_3,c-mk;\tau,\rho;\omega(z^{s+1}-a^{s+1})^{\rho}\Big).
\end{aligned}
$$

Proof. Let $\mathcal{L}$ be the left-hand side of (4.2.9). Using (4.1.9) and interchanging the order of summation and differentiation, we have

$$\begin{aligned}\mathcal{L} = & \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+n\tau k)}{\Gamma_k(\alpha_3+n\tau k)} \frac{\omega^n}{n!} \\ & \times \left\{ \left(\frac{1}{z^s}\frac{d}{dz}\right)^m \left(z^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-1} \right\}.\end{aligned} \tag{4.2.10}$$

We find

$$\begin{aligned}& \left(\frac{1}{z^s}\frac{d}{dz}\right)^m \left(z^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-1} \\ & = (s+1)^m \left(\rho n+\frac{c}{k}-1\right)\cdots\left(\rho n+\frac{c}{k}-m\right) \left(z^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-m-1} \\ & = (s+1)^m \frac{\Gamma\left(\rho n+\frac{c}{k}\right)}{\Gamma\left(\rho n+\frac{c}{k}-m\right)} \left(z^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-m-1}.\end{aligned} \tag{4.2.11}$$

Using (2.1.7), we get

$$\frac{\Gamma\left(\rho n+\frac{c}{k}\right)}{\Gamma\left(\rho n+\frac{c}{k}-m\right)} = \frac{\Gamma_k\left(c+n\rho k\right)}{k^m\,\Gamma_k\left(c-mk+n\rho k\right)}. \tag{4.2.12}$$

Combining (4.2.11) with (4.2.12) into (4.2.10), we obtain

$$\begin{aligned}\mathcal{L} = & \frac{(s+1)^m}{k^m} \left(z^{s+1}-a^{s+1}\right)^{\frac{c}{k}-m-1} \\ & \times \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+n\tau k)\,\Gamma_k(c+n\rho k)}{\Gamma_k(\alpha_3+n\tau k)\,\Gamma_k(c-mk+n\rho k)} \frac{\left\{\omega\left(z^{s+1}-a^{s+1}\right)^{\rho}\right\}^n}{n!},\end{aligned} \tag{4.2.13}$$

which, upon expressing in terms of (4.1.10), leads to the right-hand side of (4.2.9). □

Theorem 4.2.2. *Suppose $k>0$, $s\neq -1$, then the following result holds true:*

$$\begin{aligned}& {}^s_kI^{\mu}_{a+}[(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}\;{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})] \\ & \quad = \frac{(x-\alpha_1)^{\frac{\mu+\alpha_3}{k}-1}\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+\mu)}\;{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3+\mu;\tau;\omega(z^{s+1}-a^{s+1})^{\tau}),\end{aligned} \tag{4.2.14}$$

$$\begin{aligned}& {}^s_kD^{\mu}_{a+}[(\eta^{s+1}-a^{s+1})^{\frac{c}{k}-1}\;{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})] \\ & = \Gamma_k(c)\frac{(z^{s+1}-a^{s+1})^{\frac{c-\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}\Gamma_k(c-\mu)}\;{}_3R_{2,k}(\alpha_1,\alpha_2;\alpha_3;c-\mu;\tau;\omega(z^{s+1}-a^{s+1})^{\tau})].\end{aligned} \tag{4.2.15}$$

Proof.

$$ {}^{s}_{k}I^{\mu}_{a+}[(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})] $$

$$
\begin{aligned}
&= \frac{1}{k\Gamma_k(\mu)}\int_a^z \frac{(t-\alpha_1)^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(z^{s+1}-a^{s+1})^{\tau})]}{(x-t)^{1-\frac{\mu}{k}}}dt \\
&= \frac{1}{k\Gamma_k(\mu)}\frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{\omega^n}{n!}\int_a^z \frac{(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}}{(z^{s+1}-\eta^{s+1})^{1-\frac{\mu}{k}}}(\eta^{s+1}-a^{s+1})^{\tau n}dt \\
&= \frac{1}{k\Gamma_k(\mu)}\frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{\omega^n}{n!}\int_a^z \frac{(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}+\tau n-1}}{(z^{s+1}-\eta^{s+1})^{1-\frac{\mu}{k}}}dt \\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{\omega^n}{n!}\left(\ {}^{s}_{k}I^{\mu}_{\alpha_1+}[(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}+\tau n-1}].\ \right)
\end{aligned}
$$

The use of (4.1.7) gives

$$ {}^{s}_{k}I^{\mu}_{a+}[(z^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})] $$

$$
\begin{aligned}
&= \frac{\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{\omega^n}{n!}\frac{\Gamma_k(\alpha_3+\tau nk)}{\Gamma_k(\alpha_3+\mu+\tau nk)}(z^{s+1}-a^{s+1})^{\frac{\mu+\alpha_3}{k}+\tau n-1} \\
&= \frac{(z^{s+1}-a^{s+1})^{\frac{\mu+\alpha_3}{k}-1}\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\mu+\tau nk)}\frac{(\omega(x-a)^{\tau})^n}{n!} \\
&= \frac{(z^{s+1}-a^{s+1})^{\frac{\mu+\alpha_3}{k}-1}\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+\mu)}\left\{\frac{\Gamma_k(\alpha_3+\mu)}{\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\mu+\tau nk)}\frac{(\omega(x-\alpha_1)^{\tau})^n}{n!}\right\} \\
&= \frac{(z^{s+1}-a^{s+1})^{\frac{\mu+\alpha_3}{k}-1}\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+\mu)}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3+\mu;\tau;\omega(z^{s+1}-a^{s+1})^{\tau}).
\end{aligned}
$$

This completes the proof of (4.2.14).
Now, we have to prove

$$ {}^{s}_{k}D^{\mu}_{a+}[(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})] $$

$$ = \left(\frac{1}{z^s}\frac{d}{dz}\right)^n\left\{\ k^n\cdot{}^{s}_{k}I^{nk-\mu}_{a+}[(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})]\ \right\} $$

and using (4.2.14) this takes the following form

$$ {}_{k}^{s}D_{a+}^{\mu}[(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})] $$

$$ = \quad (\frac{1}{z^s}\frac{d}{dz})^n\left\{\; k^n\frac{(z^{s+1}-a^{s+1})^{\frac{\alpha_3-\mu}{k}+n-1}\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+nk-\mu)}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3+nk-\mu;\tau;\omega(z^{s+1}-a^{s+1})^{\tau})]\;\right\}. $$

Applying (4.2.9), we have

$$ {}_{k}^{s}D_{a+}^{\mu}[(\eta^{s+1}-a^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-a^{s+1})^{\tau})] $$

$$ \Gamma_k(\alpha_3)\left\{\;\frac{(z^{s+1}-a^{s+1})^{\frac{\alpha_3-\mu}{k}-1}}{\Gamma_k(\alpha_3-\mu)}\quad {}_3R_{2,k}(\alpha_1,\alpha_2,c;\alpha_3,c-\mu k;\tau;\omega(z^{s+1}-a^{s+1})^{\tau})]\;\right\}. $$

This completes the desired proof. □

Corollary 4.2.3. *If $s = 0$ and $k = 1$, then (4.2.14) and (4.2.15) reduce to the following results of [99] as:*

$$ I_{\alpha_1+}^{\mu}[(x-a)^{\alpha_3-1}\quad {}_2R_1(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(x-a)^{\tau})] = \frac{(x-a)^{\mu+\alpha_3-1}\Gamma(\alpha_3)}{\Gamma(\alpha_3+\mu)}\quad {}_2R_1(\alpha_1,\alpha_2;\alpha_3+\mu;\tau;\omega(x-a)^{\tau}), \quad (4.2.16) $$

$$ D_{\alpha_1+}^{\mu}[(x-a)^{\alpha_3-1}\quad {}_2R_1(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(x-a)^{\tau})] = \Gamma(\alpha_3)\frac{(x-a)^{\alpha_3-\mu-1}}{\Gamma(\alpha_3-\mu)}\quad {}_2R_1(\alpha_1,\alpha_2;\alpha_3-\mu;\tau;\omega(x-a)^{\tau})]. \quad (4.2.17) $$

4.3 Some Properties of the Operator $({}_{k}^{s}\Re_{\alpha_1+;\tau,c}^{\omega;a,b}f)(z)$

Theorem 4.3.1. *Let $k > 0$, $s \neq 1$ and $\tau > 0$, then the following result holds:*

$$ ({}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}(\eta^{s+1}-a^{s+1})^{\frac{\mu}{k}-1})(z) $$

$$ = (z^{s+1}-a^{s+1})^{\frac{\mu+\alpha_3}{k}-1}\Gamma_k(\mu)\frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_3+\mu)}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3+\mu;\tau;\omega(z^{s+1}-a^{s+1})^{\tau}). \quad (4.3.1) $$

Proof. From (4.2.1)

$$({}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}f)(z)=\frac{1}{k}\int_a^z(z^{s+1}-\eta^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(z^{s+1}-\eta^{s+1})^{\tau})\eta^s f(t)dt.$$

Therefore, we have

$$({}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}(\eta^{s+1}-a^{s+1})^{\frac{\mu}{k}-1})(z)$$

$$\begin{aligned}
&= \frac{1}{k}\int_a^z(z^{s+1}-\eta^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(z^{s+1}-\eta^{s+1})^{\tau})(\eta^{s+1}-a^{s+1})^{\frac{\mu}{k}-1}\eta^s dt\\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{\omega^n}{n!}\left(\frac{1}{k}\int_a^z(\eta^{s+1}-a^{s+1})^{\frac{\mu}{k}-1}(z^{s+1}-\eta^{s+1})^{\frac{\alpha_3}{k}+\tau n-1}\eta^s dt\right)\\
&= \frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\tau nk)}\frac{\omega^n(z^{s+1}-a^{s+1})^{\frac{\alpha_3+\mu}{k}+\tau n-1}}{n!}\beta_k(\alpha_3+\tau nk,\mu)\\
&= (z^{s+1}-a^{s+1})^{\frac{\alpha_3+\mu}{k}-1}\Gamma_k(\mu)\frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_3+\mu)}\left\{\frac{\Gamma_k(\alpha_3+\mu)}{\Gamma_k(\alpha_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}\Gamma_k(\alpha_2+\tau nk)}{\Gamma_k(\alpha_3+\mu+\tau nk)}\frac{(\omega(z^{s+1}-a^{s+1})^{\tau})^n}{n!}\right\}\\
&= (z^{s+1}-a^{s+1})^{\frac{\alpha_3+\mu}{k}-1}\Gamma_k(\mu)\frac{\Gamma_k(\alpha_3)}{\Gamma_k(\alpha_3+\mu)}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3+\mu;\tau;\omega(z^{s+1}-a^{s+1})^{\tau}).
\end{aligned}$$

this completes the desired proof. □

Theorem 4.3.2. *Let $k>0$, $\tau>0$ and $b>a$, then the following result holds true:*

$$({}_{k}^{s}I_{a+}^{\mu}[{}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}f](z)=\frac{\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+\mu)}({}_{k}^{s}\Re_{a+;\tau,\alpha_3+\mu}^{\omega;\alpha_1,\alpha_2}f)(z)=({}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}[{}_{k}^{s}I_{a+}^{\mu}f])(z) \tag{4.3.2}$$

holds for any function $f\in L(\alpha_1,\alpha_2)$.

Proof. From (4.1.7) and (4.2.5), we have

$$({}_{k}^{s}I_{a+}^{\mu}[{}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}f])(z)$$

$$\begin{aligned}
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^z\frac{[{}_{k}^{s}R_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}f)(t)]}{(z^{s+1}-\eta^{s+1})^{1-\frac{\mu}{k}}}dt\\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k^2\Gamma_k(\mu)}\int_a^z(z^{s+1}-\eta^{s+1})^{\frac{\mu}{k}-1}\\
&\times\left[\int_a^t(\eta^{s+1}-u^{s+1})^{\frac{\alpha_3}{k}-1}\quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-u^{s+1})^{\tau})u^s f(u)du\right]dt.
\end{aligned}$$

By interchanging the order of integration, we get

$$({}_k^sI^{\mu}_{a+}[{}_k^s\Re^{\omega;\alpha_1,\alpha_2}_{a+;\tau,\alpha_3}f])(z)$$

$$= \frac{1}{k}\int_a^z \left[\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_u^z (z^{s+1}-\eta^{s+1})^{\frac{\mu}{k}-1}(\eta^{s+1}-u^{s+1})^{\frac{\alpha_3}{k}-1}. {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\eta^{s+1}-u^{s+1})^{\tau})dt \right]$$

$$\times \quad u^s f(u)du.$$

Substituting $(\eta^{s+1}-u^{s+1}) = \lambda^{s+1}$, we obtain

$$({}_k^sI^{\mu}_{\alpha_1+}[{}_k^s\Re^{\omega;\alpha_1,\alpha_2}_{a+;\tau,\alpha_3}f])(z)$$

$$= \frac{1}{k}\int_a^z \left[\frac{1}{k\Gamma_k(\mu)} \int_0^{z^{s+1}-u^{s+1}} (z^{s+1}-u^{s+1}-\lambda^{s+1})^{\frac{\mu}{k}-1}(\lambda^{s+1})^{\frac{\alpha_3}{k}-1}. {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\lambda^{s+1})^{\tau})\lambda^s d\lambda \right]$$

$$\times \quad u^s f(u)du.$$

$$= \frac{1}{k}\int_a^z \left[\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_0^{z^{s+1}-u^{s+1}} \frac{(\lambda^{s+1})^{\frac{\alpha_3}{k}-1} \quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\lambda^{s+1})^{\tau})}{(z^{s+1}-u^{s+1}-\lambda^{s+1})^{1-\frac{\mu}{k}}}\lambda^s d\lambda \right] u^s f(u)du. \tag{4.3.3}$$

By using (4.2.5) and applying (4.2.14), we get

$$({}_k^sI^{\mu}_{\alpha_1+}[{}_k^s\Re^{\omega;\alpha_1,\alpha_2}_{a+;\tau,\alpha_3}f])(z)$$

$$= \frac{\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}k\Gamma_k(\alpha_3+\mu)}$$

$$\times \left[\int_a^z (z^{s+1}-u^{s+1})^{\frac{\mu+\alpha_3}{k}-1}. {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3+\mu;\tau;\omega(z^{s+1}-u^{s+1})^{\tau}) \right] u^s f(u)du$$

$$= \frac{\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+\mu)}$$

$$\times \left[\frac{1}{k}\int_a^z (z^{s+1}-u^{s+1})^{\frac{\mu+\alpha_3}{k}-1}. {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3+\mu;\tau;\omega(z^{s+1}-u^{s+1})^{\tau})u^s f(u)du \right]$$

thus, we get

$$({}_k^sI^{\mu}_{a+}[{}_k^s\Re^{\omega;\alpha_1,\alpha_2}_{a+;\tau,\alpha_3}f])(z) = \frac{\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+\mu)}{}_k^s\Re^{\omega;a,b}_{a+;\tau,\alpha_3+\mu}f(z) \tag{4.3.4}$$

this is the required proof of (4.3.2).
To prove the second part, we begin from the right hand side of (4.3.2) and using (4.2.1), we have

$$({}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}[{}_{k}^{s}I_{a+}^{\mu}f])(z)$$

$$= \frac{1}{k}\int_a^z (z^{s+1}-\eta^{s+1})^{\frac{\alpha_3}{k}-1} \quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(z^{s+1}-\eta^{s+1})^{\tau})\eta^{s}[{}_{k}^{s}I_{a+}^{\mu}f](t)dt$$

$$= \frac{1}{k}\int_a^z (z^{s+1}-\eta^{s+1})^{\frac{\alpha_3}{k}-1} \quad {}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(z^{s+1}-\eta^{s+1})^{\tau})\eta^{s}$$

$$\times \left(\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_a^t \frac{f(u)}{(\eta^{s+1}-u^{s+1})^{1-\frac{\mu}{k}}}du \right) dt.$$

By interchanging the order of integration, we get

$$({}_{k}^{s}\Re_{a+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}[{}_{k}^{s}I_{a+}^{\mu}f])(z)$$

$$= \frac{1}{k}\int_a^z \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}$$

$$\times \left[\int_u^z (z^{s+1}-\eta^{s+1})^{\frac{\alpha_3}{k}-1}(\eta^{s+1}-u^{s+1})^{\frac{\mu}{k}-1}.{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(z^{s+1}-\eta^{s+1})^{\tau})\eta^{s}dt \right] f(u)du.$$

Substituting $(z^{s+1}-\eta^{s+1})=\lambda^{s+1}$

$$({}_{k}^{s}\Re_{\alpha_1+;\tau,\alpha_3}^{\omega;\alpha_1,\alpha_2}[{}_{k}^{s}I_{a+}^{\mu}f])(z)$$

$$= \frac{1}{k}\int_a^z \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}$$

$$\times \left[\int_{z^{s+1}-u^{s+1}}^{0} (\lambda^{s+1})^{\frac{\alpha_3}{k}-1}(z^{s+1}-\lambda^{s+1}-u^{s+1})^{\frac{\mu}{k}-1}.{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\lambda^{s+1})^{\tau})\lambda^{s}(-d\lambda) \right]$$

$$\times \; u^{s}f(u)du$$

$$= \frac{1}{k}\int_a^z \frac{1}{k\Gamma_k(\mu)}$$

$$\times \left[\int_0^{z^{s+1}-u^{s+1}} (\lambda^{s+1})^{\frac{\alpha_3}{k}-1}(z^{s+1}-\lambda^{s+1}-u^{s+1})^{\frac{\mu}{k}-1}.{}_2R_{1,k}(\alpha_1,\alpha_2;\alpha_3;\tau;\omega(\lambda^{s+1})^{\tau})\lambda^{s}d\lambda \right]$$

$$\times \; u^{s}f(u)du.$$

Again by making the use of (4.2.5) and applying (4.2.14), we get

$$({}_{k}^{s}\Re^{\omega;\alpha_1,\alpha_2}_{a+;\tau,\alpha_3}[{}_{k}^{s}I^{\mu}_{a+}f])(z) = \frac{\Gamma_k(\alpha_3)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\alpha_3+\mu)} \cdot {}_{k}^{s}\Re^{\omega;\alpha_1,\alpha_2}_{a+;\tau,\alpha_3+\mu}f(z) \qquad (4.3.5)$$

Thus (4.3.4) and (4.3.5) completes the desired proof of (4.3.3). □

Chapter 5

The Generalized k-Fractional Calculus of Mittag-Leffler k-Function

In this chapter [81], we consider the generalized k-fractional calculus defined in Chapter 4 and define generalized Riemann-Liouville k-fractional integral and differential formulas of Mittag-Leffler k-function.

5.1 Introduction

Fractional calculus and its applications have recently paid more attentions. In Mathematics, it is a extremely more helpful to find out differentials and integrals with the real or complex numbers order of fractional calculus. The researchers Miller and Ross [67] and Kiryakova [53] introduced a brief description of fractional calculus operators, some of their properties and applications. Atangana and Baleanu [1] have further extent this study by considering the derivative based upon Mittag-Leffler function. For further study of fractional calculus the reader may study the work of researchers ([50], [86], [109]). The Integral inequalities are more important as these are helpful in the study of various courses differential and integral equations [68].

Recently many researchers have introduced integral inequalities by using fractional integral operators. In recent few years, the theory of k-fractional integral has paid more attention. Díaz and Pariguan [15] have investigated the Pochhammer k-symbol which is defined as:

$$(\sigma)_{n,k} = \begin{cases} 1, (n = 0, \sigma \in \mathbb{C}) \\ \sigma(\sigma + k) \cdots (\sigma + (n-1)k), (n \in \mathbb{N}, \sigma \in \mathbb{C}, k > 0) \end{cases} \tag{5.1.1}$$

Mubeen and Habibullah [78] introduced k-fractional integral and its various properties defined in Chapter 4 (see 4.1.4). The k-fractional integral defined as:

Recently Sarikaya *et al.* [104] have introduced the generalized Riemann-Liouville k-fractional integral of order $\mu > 0$ is defined in 4.1.7 (see Chapter 4).

Recently the researchers ([2], [79] [111]) used the idea of generalized k-fractional integrals and estalished fractional integral inequalities. The Mittag-Leffler function is defined in [69] by

$$E_{\vartheta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\vartheta n + 1)}, z \in \mathbb{C}; \Re(\vartheta) > 0. \tag{5.1.2}$$

The generalized form of (5.1.2) is defined in [115] by

$$E_{\vartheta,\lambda}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\vartheta n + \lambda)}, z, \lambda \in \mathbb{C}; \Re(\vartheta) > 0. \tag{5.1.3}$$

The readers may follow the work of ([33], [50], [48], [34] and [62]) and the work of Saigo and Kilbas [103] for generalizations and applications Mittag-Leffler functions. In recent years, the Mittag-Leffler function (5.1.2) and some of its different generalizations and applications have been considered numerically in the complex plane $\mathbb{C}$ [39, 105]. Prabhakar [83] have have a generalized Mittag-Leffler function $E^{\sigma}_{\vartheta,\lambda}(z)$.

Srivastava and Tomovski [109] have established further the generalization of Mittag-Leffler function in the form $E^{\sigma,q}_{\vartheta,\lambda}(z)$.

Dorrego [17] defined the Mittag-Leffler k-function $E^{\sigma}_{k,\vartheta,\lambda}(z)$ (where $k > 0$) in the following form:

$$E^{\sigma}_{k,\vartheta,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n + \lambda)} \frac{z^n}{n!}, \tag{5.1.4}$$

where $\vartheta, \lambda, \sigma \in \mathbb{C}$, $\Re(\vartheta) > 0$, $\Re(\lambda) > 0$, $\Re(\sigma) > 0$, $k > 0$ and $(\sigma)_{n,k}$ is the Pochhammer k-symbol defined in (5.1.1).

5.2 The Generalized k-Fractional Integrals and Differentials of Mittag-Leffler k-Functions

In continuation of the study of generalized k-fractional calculus, we define integral the following integral operators in term of (k, s) as follow:

Definition 5.2.1. If $k > 0$ and $\vartheta, \sigma, \omega \in \mathbb{C}$, $\Re(\vartheta) > 0$, $\Re(\lambda) > 0$, $\Re(\sigma) > 0$, then

$$({}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda} f)(z) = \frac{1}{k} \int_a^z (z^{s+1} - \tau^{s+1})^{\frac{\lambda}{k}-1} E^{\sigma}_{k,\vartheta,\lambda}(\omega(z^{s+1} - \tau^{s+1})^{\frac{\vartheta}{k}}) \tau^s f(\tau) d\tau, \tag{5.2.1}$$

where $x > \vartheta$. When $s = 0$, then (5.2.1) reduces to the operator

$$({}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda} f)(z) = \int_a^z (x - \tau)^{\frac{\lambda}{k}-1} E^{\sigma}_{\vartheta,\lambda}(\omega(x - \tau)^{\frac{\vartheta}{k}}) f(\tau) d\tau, \tag{5.2.2}$$

see [18]. It is clear that, if $\omega = 0$ and $k = 1$ then (5.2.2) reduces to the well-known fractional integral operator defined as:

$$(I^{\mu}_{a+} f)(z) = \frac{1}{\Gamma(\mu)} \int_a^z \frac{f(\tau)}{(x - \tau)^{1-\lambda}} d\tau, (\mathbb{R}(\mu) > 0). \tag{5.2.3}$$

Here, we recall the generalized k-fractional order integrations and differentiations which are defined by the operators ${}^{s}_{k}I^{\mu}_{a+}$, ${}^{s}_{k}I^{\mu}_{\lambda-}$, $D^{\mu}_{\vartheta+,k}$ and $D^{\mu}_{\vartheta-,k}$ in Chapter 4 (see (4.2.5)-(4.2.8)).

Definition 5.2.2. The generalized form of ${}^{s}_{k}D^{\mu}_{a+}$ defined in (4.2.7) is denoted by the operator ${}^{s}_{k}D^{\mu,\nu}_{a+}$ where μ is the order such that $0 < \mu < 1$ and ν is the type of this generalized k-fractional derivative operator such that $0 < \nu < 1$, which is defined as:

$$\left({}^{s}_{k}D^{\mu,\nu}_{a+}f \right)(z) = [{}^{s}_{k}I^{\nu(k-\mu)}_{\vartheta+}(\frac{1}{z^s}\frac{d}{dz})(k \quad {}^{s}_{k}I^{(1-\nu)(k-\mu)}_{a+}f)](z). \tag{5.2.4}$$

It is clear that, setting $\nu = 0$ then (5.2.4) reduces to the generalized (4.2.5).

Lemma 5.2.1. *For $k > 0$, $s \neq 1$, the following result for k-fractional derivative operator ${}^{s}_{k}D^{\mu,\nu}_{a+}$ defined in(5.2.4) holds true:*

$$\left({}^{s}_{k}D^{\mu,\nu}_{a+}[(\tau^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}]\right)(z) = \frac{\Gamma_k(\lambda)}{(s+1)^{-\frac{\mu}{k}}\Gamma_k(\lambda-\mu)}(z^{s+1} - a^{s+1})^{\frac{\lambda-\mu}{k}-1}, \tag{5.2.5}$$

with $x > \vartheta$, $0 < \mu < 1$, $0 < \nu < 1$ and $\Re(\lambda) > 0$.

Proof. We obtain from equation (??) that

$$\begin{aligned}&\left({}^{s}_{k}I^{(1-\nu)(k-\mu)}_{a+}[(\tau^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}]\right)(z)\\ &= \frac{\Gamma_k(\lambda)}{(s+1)^{\frac{(1-\nu)(k-\mu)}{k}}\Gamma_k((1-\nu)(k-\mu)+\lambda)}(z^{s+1} - a^{s+1})^{\frac{(1-\nu)(k-\mu)+\lambda}{k}-1}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{z^s}\frac{d}{dz}\left({}^{s}_{k}I^{(1-\nu)(k-\mu)}_{a+}[(t^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}]\right)(z)\\ &= \frac{[(1-\nu)(k-\mu)+\lambda-k]\Gamma_k(\lambda)}{k(s+1)^{\frac{(1-\nu)(k-\mu)}{k}-1}\Gamma_k((1-\nu)(k-\mu)+\lambda)}(z^{s+1} - a^{s+1})^{\frac{(1-\nu)(k-\mu)+\lambda}{k}-2},\end{aligned}$$

which by applying the relation given in (2.1.6), yields

$$\begin{aligned}\left({}^{s}_{k}D^{\mu,\nu}_{a+}[(\tau^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}] \right)(z) &= \frac{\Gamma_k(\lambda)}{\Gamma_k((1-\nu)(k-\mu)+\lambda-k)}\\ &\times \quad [{}^{s}_{k}I^{\nu(k-\mu)}_{a+}(z^{s+1} - a^{s+1})^{\frac{(1-\nu)(k-\mu)+\lambda}{k}-2}](z)\\ &= \frac{\Gamma_k(\lambda)}{(s+1)^{\frac{\nu(k-\mu)+(1-\nu)(k-\mu)}{k}-1}\Gamma_k((1-\nu)(k-\mu)+\lambda-k)}\\ &\times \quad \frac{\Gamma_k((1-\nu)(k-\mu)+\lambda-k)}{\Gamma_k(\nu(k-\mu)+(1-\nu)(k-\mu)+\lambda-k)}(z^{s+1} - a^{s+1})^{\frac{\lambda-\mu}{k}-1}\\ &= \frac{\Gamma_k(\lambda)}{(s+1)^{-\frac{\mu}{k}}\Gamma_k(\lambda-\mu)}(z^{s+1} - a^{s+1})^{\frac{\lambda-\mu}{k}-1},\end{aligned}$$

which is the desired proof. □

Theorem 5.2.2. *For $k > 0$, $s \neq -1$, the following result always holds true:*

$$\left(\frac{1}{z^s}\frac{d}{dz}\right)^m\left[(z^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})\right]$$

$$= \frac{(s+1)^m(z^{s+1}-a^{s+1})^{\frac{\lambda}{k}-m-1}}{k^m}E^{\sigma}_{k,\vartheta,\lambda-mk}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}}). \tag{5.2.6}$$

Proof. Let $\mathcal{L}$ be the left-hand side of (5.2.6). Using (5.1.4) and interchanging the order of summation and differentiation, we have

$$\begin{aligned}\mathcal{L} = &\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n+\lambda)}\frac{\omega^n}{n!}\\ &\times\left\{\left(\frac{1}{z^s}\frac{d}{dz}\right)^m\left(z^{s+1}-a^{s+1}\right)^{\frac{\vartheta}{k}n+\frac{\lambda}{k}-1}\right\}.\end{aligned} \tag{5.2.7}$$

We find

$$\begin{aligned}&\left(\frac{1}{z^s}\frac{d}{dz}\right)^m\left(z^{s+1}-a^{s+1}\right)^{\frac{\vartheta}{k}n+\frac{\lambda}{k}-1}\\ &=(s+1)^m\left(\frac{\vartheta}{k}n+\frac{\lambda}{k}-1\right)\cdots\left(\frac{\vartheta}{k}n+\frac{\lambda}{k}-m\right)\left(z^{s+1}-a^{s+1}\right)^{\frac{\vartheta}{k}n+\frac{\lambda}{k}-m-1}\\ &=(s+1)^m\frac{\Gamma\left(\frac{\vartheta}{k}n+\frac{\lambda}{k}\right)}{\Gamma\left(\frac{\vartheta}{k}n+\frac{\lambda}{k}-m\right)}\left(z^{s+1}-a^{s+1}\right)^{\frac{\vartheta}{k}n+\frac{\lambda}{k}-m-1}.\end{aligned} \tag{5.2.8}$$

Using (2.1.7), we get

$$\frac{\Gamma\left(\frac{\vartheta}{k}n+\frac{\lambda}{k}\right)}{\Gamma\left(\frac{\vartheta}{k}n+\frac{\lambda}{k}-m\right)}=\frac{\Gamma_k\left(\lambda+n\vartheta\right)}{k^m\,\Gamma_k\left(\lambda-mk+n\vartheta\right)}. \tag{5.2.9}$$

Combining (5.2.8) with (5.2.9) into (5.2.7), we obtain

$$\begin{aligned}\mathcal{L}=&\frac{(s+1)^m}{k^m}\left(z^{s+1}-a^{s+1}\right)^{\frac{\lambda}{k}-m-1}\\ &\times\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n+\lambda)}\frac{\left\{\omega\left(z^{s+1}-a^{s+1}\right)^{\frac{\vartheta}{k}}\right\}^n}{n!},\end{aligned} \tag{5.2.10}$$

which completes the desired proof. □

Theorem 5.2.3. *Suppose $k > 0$, $s \neq 1$, then the following results hold true:*

$$
{}^{s}_{k}I^{\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})](z)
= \frac{(z^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}}E^{\sigma}_{k,\vartheta,\lambda+\mu}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}}), \quad (5.2.11)
$$

$$
{}^{s}_{k}D^{\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})](z)
= \frac{(z^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}}E^{\sigma}_{k,\vartheta,\lambda-\mu}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})], \quad (5.2.12)
$$

and

$$
{}^{s}_{k}D^{\mu,\nu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})](z)
= \frac{(z^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}}E^{\sigma}_{k,\vartheta,\lambda-\mu}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})]. \quad (5.2.13)
$$

Proof.

$$
\begin{aligned}
&{}^{s}_{k}I^{\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})]\\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^z \frac{(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})\tau^s}{(z^{s+1}-\tau^{s+1})^{1-\frac{\mu}{k}}}d\tau\\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}\omega^n}{\Gamma_k(\vartheta n+\lambda)n!}\\
&\times \int_a^z (\tau^{s+1}-a^{s+1})^{\frac{\lambda+\sigma n}{k}-1}(z^{s+1}-\tau^{s+1})^{\frac{\mu}{k}-1}\tau^s d\tau
\end{aligned}
$$

Substituting $\tau^{s+1}=a^{s+1}+y(z^{s+1}-a^{s+1})$, this implies $\tau^s d\tau = (\frac{z^{s+1}-a^{s+1}}{s+1})dy$, we have

$$
{}^{s}_{k}I^{\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})]
$$

$$= \sum_{n=0}^{\infty} \frac{(\sigma)_{n,k}\omega^n}{\Gamma_k(\vartheta n+\lambda)n!}(z^{s+1}-a^{s+1})^{\frac{\lambda+\mu+\vartheta n}{k}-1}\frac{(s+1)^{\frac{-\mu}{k}}}{k\Gamma_k(\mu)}\int_0^1 (1-y)^{\frac{\lambda+\vartheta n}{k}-1}y^{\frac{\mu}{k}-1}dy$$

$$= \sum_{n=0}^{\infty} \frac{(\sigma)_{n,k}\omega^n}{\Gamma_k(\vartheta n+\lambda)n!}(z^{s+1}-a^{s+1})^{\frac{\lambda+\mu+\vartheta n}{k}-1}.\frac{\Gamma_k(\vartheta n+\lambda)\Gamma_k(\mu)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\mu)\Gamma_k(\vartheta n+\lambda+\mu)}$$

$$= \frac{(z^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}}\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}\omega^n(z^{s+1}-a^{s+1})^{\frac{\vartheta n}{k}-1}}{\Gamma_k(\vartheta n+\lambda+\mu)n!}$$

$$= \frac{(z^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}}E^{\sigma}_{k,\vartheta,\lambda+\mu}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}}).$$

This completes the proof of (5.2.11).
Now, we have

$$\begin{aligned}&{}^{s}_{k}D^{\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})]\\ &=\Big(\frac{1}{z^s}\frac{d}{dz}\Big)^n\Big\{k^n\ \ {}^{s}_{k}I^{nk-\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})\Big\}\end{aligned}$$

and using (5.2.11) this takes the following form:

$$\begin{aligned}&{}^{s}_{k}D^{\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})]\\ &=k^n\Big(\frac{1}{z^s}\frac{d}{dz}\Big)^n\Big\{(s+1)^{\frac{\mu}{k}-n}(z^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}+n-1}E^{\sigma}_{k,\vartheta,\lambda-\mu+nk}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})\Big\}.\end{aligned}$$

Applying (5.2.6), we have

$$\begin{aligned}&{}^{s}_{k}D^{\mu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})](z)\\ &=\Big\{(s+1)^{\frac{\mu}{k}}(z^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}E^{\sigma}_{k,\vartheta,\lambda-\mu}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})\Big\}.\end{aligned}$$

This completes the desired proof.
Now to prove (5.2.13), we have

$$\begin{aligned}&\Big({}^{s}_{k}D^{\mu,\nu}_{a+,k}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})]\Big)(z)\\ &=\left({}^{s}_{k}D^{\mu,\nu}_{a+}\Big[\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n+\lambda)}\frac{\omega^n}{n!}(\tau^{s+1}-a^{s+1})^{\frac{\vartheta n+\lambda}{k}-1}\Big]\right)(z).\end{aligned}$$

This can be written as:

$$= \sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n+\lambda)}\frac{\omega^n}{n!}\Big(\ {}^{s}_{k}D^{\mu,\nu}_{a+}[(\tau^{s+1}-a^{s+1})^{\frac{\vartheta n+\lambda}{k}-1}]\ \Big)(z).$$

By applying (5.2.5), we get

$$\begin{aligned}
&\left({}_k^sD_{a+}^{\mu,\nu}[(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}E_{k,\vartheta,\lambda}^{\sigma}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})]\right)(z)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\vartheta n+\lambda)}\frac{\omega^n}{n!}.\frac{\Gamma_k(\vartheta n+\lambda)}{(s+1)^{-\frac{\mu}{k}}\Gamma_k(\vartheta n+\lambda-\mu)}(z^{s+1}-a^{s+1})^{\frac{\vartheta n+\lambda-\mu}{k}-1}\\
&=\frac{(z^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}}\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n+\lambda-\mu)}\frac{[\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}}]^n}{n!}\\
&=\frac{(z^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}}E_{k,\vartheta,\lambda-\mu}^{\sigma}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})
\end{aligned}$$

which completes the desired proof. □

Remark 5.2.1. Setting $s=0$ in (5.2.11), (5.2.12) and (5.2.13), then we have the well known results (see [18]). Similarly setting $s=0$ and $k=1$, then we get the results derived in [109].

5.3 Some Properties of the Operator $({}_k^s\varepsilon_{a+;\vartheta,\lambda}^{\omega;\sigma}f)(z)$

Theorem 5.3.1. *For $k>0$, the following result holds true:*

$$\begin{aligned}
&\left({}_k^s\varepsilon_{a+;\vartheta,\lambda}^{\omega;\sigma}[(\tau^{s+1}-a^{s+1})^{\frac{\mu}{k}-1}]\right)(z)\\
&=\frac{(z^{s+1}-a^{s+1})^{\frac{\mu+\lambda}{k}-1}\Gamma_k(\mu)}{(s+1)}E_{k,\vartheta,\lambda+\mu}^{\sigma}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})f(t)dt. \qquad (5.3.1)
\end{aligned}$$

Proof. From (5.2.1), we have

$$\begin{aligned}
&\left({}_{k}^{s}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}[(\tau^{s+1}-a^{s+1})^{\frac{\mu}{k}-1}]\right)(z)\\
&=\frac{1}{k}\int_a^z (z^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}-1}(\tau^{s+1}-a^{s+1})^{\frac{\mu}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(z^{s+1}-\tau^{s+1})^{\frac{\vartheta}{k}})\tau^{s}d\tau\\
&=\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n+\lambda))}\frac{\omega^n}{n!}\left(\frac{1}{k}\int_a^z(\tau^{s+1}-a^{s+1})^{\frac{\mu}{k}-1}(z^{s+1}-\tau^{s+1})^{\frac{\lambda+\vartheta n}{k}-1}\tau^{s}d\tau\right)\\
&=\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{(s+1)\Gamma_k(\vartheta n+\lambda))}\frac{\omega^n(z^{s+1}-a^{s+1})^{\frac{\lambda+\vartheta n+\mu}{k}-1}}{n!}B_k(\lambda+\vartheta n,\mu)\\
&=\frac{(z^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}\Gamma_k(\mu)}{(s+1)}\left\{\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}}{\Gamma_k(\vartheta n+\lambda))}\frac{\omega^n(z^{s+1}-a^{s+1})^{\frac{\vartheta n}{k}}}{.}\frac{\Gamma_k(\mu)\Gamma_k(\vartheta n+\lambda)}{\Gamma_k(\vartheta n+\lambda+\mu)}\right\}\\
&=\frac{(z^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}\Gamma_k(\mu)}{(s+1)}E^{\sigma}_{k,\vartheta,\lambda+\mu}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}}),
\end{aligned}$$

which completes the desired proof. □

Theorem 5.3.2. *The following result is holds true for $x\in[a,b]$;*

Proof. Assume that $\Sigma=[a,b]\times[a,b]$ and $P:\Sigma\to\mathbb{R}$ such that $P(z,\tau)=[(z^{s+1}-\tau^{s+1})\tau^{s}]$ for all $x\in[a,b]$. It is obvious that $P=P_{+}+P_{-}$ where

$$P_{+}(z,\tau)=\begin{cases}(z^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}-1}\tau^{s}; & a\le\tau\le z\le b\\ 0; & a\le z\le t\le b,\end{cases}$$

and

$$P_{-}(z,\tau)=\begin{cases}(\tau^{s+1}-z^{s+1})^{\frac{\lambda}{k}-1}z^{s}; & a\le\tau\le z\le b\\ 0; & a\le z\le\tau\le b.\end{cases}$$

As P is measurable on Σ, therefore we can write

$$\begin{aligned}
\int_a^b P(z,\tau)d\tau &= \int_a^z P(z,\tau)d\tau\\
&=\int_a^z (z^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}-1}\tau^{s}d\tau\\
&=\frac{k}{\lambda}(z^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}}.
\end{aligned}$$

Hence, we obtain

$$\int_a^b P(z,\tau)E^{\sigma}_{k,\vartheta,\lambda}(\omega(z^{s+1}-\tau^{s+1})^{\frac{\vartheta}{k}})d\tau$$
$$=\int_a^z P(z,\tau)E^{\sigma}_{k,\vartheta,\lambda}(\omega(z^{s+1}-\tau^{s+1})^{\frac{\vartheta}{k}})d\tau$$
$$=\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}\omega^n}{\Gamma_k(\vartheta n+\lambda)n!}\int_a^z (z^{s+1}-\tau^{s+1})^{\frac{\lambda+\vartheta}{k}-1}\tau^s d\tau$$
$$=\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}(\omega(z^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})^n}{\Gamma_k(\vartheta n+\lambda)n!}\frac{k}{\lambda+\vartheta n}(z^{s+1}-a^{s+1})^{\frac{\lambda}{k}}.$$

By using repeated integral, we have

$$\int_a^b\left(\int_a^b P(z,\tau)E^{\sigma}_{k,\vartheta,\lambda}(\omega(x-\tau)^{\frac{\vartheta}{k}})|f(z)|d\tau\right)dz$$
$$=\int_a^b |f(z)|\left(\int_a^b P(z,\tau)E^{\sigma}_{k,\vartheta,\lambda}(\omega(x-\tau)^{\frac{\vartheta}{k}})d\tau\right)dz$$
$$=\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}(\omega)^n}{\Gamma_k(\vartheta n+\lambda)n!}\frac{k}{\lambda+\vartheta n}$$
$$\times\int_a^b (z^{s+1}-a^{s+1})^{\frac{\lambda+\vartheta n}{k}}|f(z)|dz$$
$$\leq\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}(\omega(b^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})^n}{\Gamma_k(\vartheta n+\lambda)n!}\frac{k^2}{(\lambda+\vartheta n)(\lambda+\vartheta n+k)}$$
$$\times(b^{s+1}-a^{s+1})^{\frac{\lambda}{k}+1}\int_a^b |f(z)|dz$$
$$\leq(b^{s+1}-a^{s+1})^{\frac{\lambda}{k}+1}\sum_{n=0}^{\infty}\frac{(\sigma)_{n,k}(\omega(b^{s+1}-a^{s+1})^{\frac{\vartheta}{k}})^n}{\Gamma_k(\vartheta n+\lambda)n!}$$
$$\times\frac{k^2}{(\lambda+\vartheta n)(\lambda+\vartheta n+k)}\|f\|_1\leq\infty.$$

Therefor the function $Q:\Sigma\to\mathbb{R}$ such that $Q(z,t)=P(z,\tau)f(z)$ is integrable on Σ by Tonelli's theorem. Thus, by Fubini's theorem $\int_a^b P(z,\tau)E^{\sigma}_{k,\vartheta,\lambda}(\omega(z^{s+1}-\tau^{s+1})^{\frac{\vartheta}{k}})^n f(z)dz$ is an integrable function on $[a,b]$, as a function of $t\in[a,b]$. Thus, ${}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}f(z)$ exists. □

Theorem 5.3.3. *Suppose $k>0$, $s\neq 1$, then following result holds true:*

$$({}^{s}_{k}I^{\mu}_{a+}[{}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}f](z)=\frac{1}{(s+1)^{\frac{\mu}{k}}}({}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda+\mu}f)(z)=({}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}[{}^{s}_{k}I^{\mu}_{\vartheta+}f])(z) \tag{5.3.2}$$

holds for any $f\in L(\alpha,\lambda)$.

Proof. From equations (5.2.1) and (4.2.5), we observe

$$\begin{aligned}
&({}^{s}_{k}I^{\mu}_{a+}[{}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}f])(z)\\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^z \frac{[{}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}f)(\tau)]}{(z^{s+1}-\tau^{s+1})^{1-\frac{\mu}{k}}}\tau^s d\tau\\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k^2\Gamma_k(\mu)}\int_a^z (z^{s+1}-\tau^{s+1})^{\frac{\mu}{k}-1}\\
&\times \left[\int_a^\tau (\tau^{s+1}-u^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-u^{s+1})^{\frac{\vartheta}{k}})f(u)u^s du\right]\tau^s d\tau.
\end{aligned}$$

By interchanging the order of integration, we obtain

$$\begin{aligned}
&({}^{s}_{k}I^{\mu}_{a+}[{}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}f])(z)\\
&=\frac{1}{k}\int_a^z\left[\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_u^z (z^{s+1}-\tau^{s+1})^{\frac{\mu}{k}-1}(\tau^{s+1}-u^{s+1})^{\frac{\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda}(\omega(\tau^{s+1}-u^{s+1})^{\frac{\vartheta}{k}})\tau^s d\tau\right]\\
&\times u^s f(u)du.
\end{aligned}$$

By applying (5.2.11), we have

$$\begin{aligned}
&({}^{s}_{k}I^{\mu}_{a+}[{}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}f])(z)\\
&=\left[\frac{1}{k(s+1)^{\frac{\mu}{k}}}\int_a^z (z^{s+1}-u^{s+1})^{\frac{\mu+\lambda}{k}-1}E^{\sigma}_{k,\vartheta,\lambda+\mu}(\omega(z^{s+1}-u^{s+1})^{\frac{\vartheta}{k}})u^s f(u)du\right]
\end{aligned}$$

thus, we get

$$({}^{s}_{k}I^{\mu}_{a+}[{}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda}f])(z) = \frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}^{s}_{k}\varepsilon^{\omega;\sigma}_{a+;\vartheta,\lambda+\mu}f\right)(z) \tag{5.3.3}$$

To prove the second part, consider the rhs of (5.3.2) then by applying (5.2.1), we get

$$({}_k^s\varepsilon_{a+;\vartheta,\lambda}^{\omega;\sigma}[{}_k^sI_{a+}^{\mu}f])(z)$$

$$\begin{aligned}
&= \frac{1}{k}\int_a^z (z^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}-1}E_{k,\vartheta,\lambda}^{\sigma}(\omega(z^{s+1}-t^{s+1})^{\frac{\vartheta}{k}})[{}_k^sI_{a+}^{\mu}f](\tau)\tau^s d\tau \\
&= \frac{1}{k}\int_a^z (z^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}-1}E_{k,\vartheta,\lambda}^{\sigma}(\omega(z^{s+1}-\tau^{s+1})^{\frac{\vartheta}{k}})\left(\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^{\tau}\frac{f(u)}{(\tau^{s+1}-u^{s+1})^{1-\frac{\mu}{k}}}u^s du\right)d\tau.
\end{aligned}$$

By interchanging the order of integration, we have

$$({}_k^s\varepsilon_{a+;\vartheta,\lambda}^{\omega;\sigma,q}[{}_k^sI_{a+}^{\mu}f])(z)$$

$$\begin{aligned}
&= \frac{1}{k}\int_a^z \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\left[\int_u^z (z^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}-1}(\tau^{s+1}-u^{s+1})^{\frac{\mu}{k}-1}E_{k,\vartheta,\lambda}^{\sigma}(\omega(z^{s+1}-\tau^{s+1})^{\frac{\vartheta}{k}})\tau^s d\tau\right] \\
&\times \; u^s f(u)du.
\end{aligned}$$

Again by making the use of (4.2.5) and applying (5.2.11), we obtain

$$({}_k^s\varepsilon_{a+;\vartheta,\lambda}^{\omega;\sigma}[{}_k^sI_{\vartheta+}^{\mu}f])(z) = \frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}_k^s\varepsilon_{a+;\vartheta,\lambda+\mu}^{\omega;\sigma}\right)f(z). \tag{5.3.4}$$

Thus (5.3.3) and (5.3.4) complete the desired proof of (5.3.2). □

Chapter 6

Generalized Fractional Integration of Bessel k-Function

In this Chapter [98], we deal with two integral transforms which involving the Gauss hypergeometric function as its kernels. We prove some compositions formulas for such a generalized fractional integrals with Bessel k-function. The results are established in terms of generalized Wright type hypergeometric function and generalized hypergeometric series. Also, some corresponding assertions for RiemannLiouville and ErdélyiKober fractional integral transforms are established.

6.1 Introduction and Preliminaries

The Gauss hypergeometric function is defined as:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{6.1.1}$$

where $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \cdots$ and $(\lambda)_n$ is the Pochhammer symbol defined for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ as:

$$(\lambda)_0 = 1, \qquad (\lambda)_n = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1); n \in \mathbb{N}. \tag{6.1.2}$$

The series defined in (6.1.1) is absolutely convergent for $|z| < 1$ and $|z| = 1$ [24]. Saigo [102] introduced the following left and right sided generalized integral transforms defined for $x > 0$ respectively as:

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta} f\right)(x) = \frac{x^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)} \times \int_0^x (x-t)^{\alpha_1-1} \; {}_2F_1\left(\alpha_1+\alpha_2, -\eta; \alpha_1; 1-\tfrac{t}{x} \right) f(t)dx, \tag{6.1.3}$$

and

$$\left(I_{-}^{\alpha_1,\alpha_2,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha_1)} \times \int_x^\infty (x-t)^{\alpha_1-1} t^{-\alpha_1-\alpha_2} \; {}_2F_1\left(\alpha_1+\alpha_2, -\eta; \alpha_1; 1-\tfrac{x}{t} \right) f(t)dx, \tag{6.1.4}$$

where $\alpha_1, \alpha_2, \eta \in \mathbb{C}$ and $\Re(\alpha_1) > 0$ and ${}_2F_1(a, b; c; z)$ is Gauss hypergeometric function defined in (6.1.1). When $\alpha_2 = -\alpha_1$, then (6.1.2) and (6.1.4) will lead to the classical Riemann-Liouville left and right-sided fractional integrals of order $\alpha_1 \in \mathbb{C}$, $\Re(\alpha_1) > 0$, (see [101]):

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta} f \right)(x) = \frac{x^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)} \int_0^x (x-t)^{\alpha_1-1} f(t)dx (x > 0), \tag{6.1.5}$$

and

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta} f \right)(x) = \frac{1}{\Gamma(\alpha_1)} \int_x^\infty (x-t)^{\alpha_1-1} t^{-\alpha_1-\alpha_2} f(t)dx (x > 0). \tag{6.1.6}$$

If $\alpha_2 = 0$, then equations (6.1.3) and (6.1.4) will reduce to the well known Erdélyi-Kober fractional defined as:

$$\left(I_{0+}^{\alpha_1,0,\eta} f \right)(x) = \left(K_{\eta,\alpha_1}^{+} f \right)(x) = \frac{x^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)} \int_0^x (x-t)^{\alpha_1-1} t^{\eta} f(t) dx \qquad (6.1.7)$$

and

$$\left(I_{0+}^{\alpha_1,0,\eta} f \right)(x) = \left(K_{\eta,\alpha_1}^{-} f \right)(x) = \frac{x^{\eta}}{\Gamma(\alpha_1)} \int_x^\infty (x-t)^{\alpha_1-1} t^{-\alpha_1-\eta} f(t) dx, \qquad (6.1.8)$$

where $\alpha_1, \eta \in \mathbb{C}$, $\Re(\alpha_1) > 0$ see [101].

The generalized Bessel k-function defined in [71] as:

$$W_{v,c}^{k}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+v+k)n!} \left(\frac{z}{2}\right)^{2n+\frac{v}{k}}, \qquad (6.1.9)$$

where $k > 0$, $v > -1$, and $c \in \mathbb{R}$ and $\Gamma_k(z)$ is the gamma k-function defined in Chapter 2.

If $k \to 1$ and $c = 1$, then the generalized Bessel k-function defined in (6.1.9) reduces to the well known classical Bessel function J_v defined in [25]. For further detail about Bessel k-function and its properties ([31], [32]).

The generalized hypergeometric function ${}_pF_q(z)$ is defined in [24] as:

$${}_pF_q(z) = \quad {}_pF_q. \left[\begin{matrix} (\alpha_1), (\alpha_1). \cdots (\alpha_p) \\ \\ (\beta_1), (\beta_2), \cdots (\beta_q) \end{matrix} \quad ; z \right]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \qquad (6.1.10)$$

where $\alpha_i, \beta_j \in \mathbb{C}$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$ and $\beta_j \neq 0, -1, -2, \cdots$ and $(z)_n$ is the Pochhammer symbols. Also, the following identity of Gauss hypergeometric function holds:

$$_2F_1(\alpha_1, \alpha_2; \alpha_3; 1) = \frac{\Gamma(\alpha_3)\Gamma(\alpha_3 - \alpha_1 - \alpha_2)}{\Gamma(\alpha_3 - \alpha_1)\Gamma(\alpha_3 - \alpha_2)}; \Re(\alpha_3 - \alpha_1 - \alpha_2) > 0, \qquad (6.1.11)$$

(see [24], [101]).

The Wright type hypergeometric function is defined (see [116]- [118]) by the following series as:

$$_p\Psi_q(z) = \ _p\Psi_q \left[\begin{array}{ll} (\alpha_i, A_i)_{1,p} & \\ & ; z \\ (\beta_j, B_j)_{1,q} & \end{array} \right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!}, \qquad (6.1.12)$$

where α_r and μ_s are real positive numbers such that

$$1 + \sum_{s=1}^{q} \beta_s - \sum_{r=1}^{p} \alpha_r > 0. \qquad (6.1.13)$$

The equation (6.1.12) differs from the generalized hypergeometric function $_pF_q(z)$ defined (6.1.10) only by a constant multiplier. The generalized hypergeometric function $_pF_q(z)$ is a special case of $_p\Psi_q(z)$ for $A_i = B_j = 1$, where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$:

$$\frac{1}{\prod_{j=1}^{q} \Gamma(\beta_j)} \ _pF_q \left[\begin{array}{ll} (\alpha_1), \cdots (\alpha_p) & \\ & ; z \\ (\beta_1), \cdots (\beta_q) & \end{array} \right] = \frac{1}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \ _p\Psi_q \left[\begin{array}{ll} (\alpha_i, 1)_{1,p} & \\ & ; z \\ (\beta_j, 1)_{1,q} & \end{array} \right]. \qquad (6.1.14)$$

For various properties of this functions see [51].

Lemma 6.1.1. *([52]) Let $\alpha_1, \alpha_2, \eta \in \mathbb{C}$, $\Re(\alpha_1) > 0$ and $\lambda > \max[0, \alpha_2 - \mathfrak{n}]$, then the following relation holds:*

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta} t^{\lambda-1} \right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda+\eta-\alpha_2)}{\Gamma(\lambda-\alpha_2)\Gamma(\lambda+\alpha_1+\eta)} x^{\lambda-\alpha_2-1}. \tag{6.1.15}$$

Lemma 6.1.2. *([52]) Let $\alpha_1, \alpha_2, \eta \in \mathbb{C}$, $\Re(\alpha_1) > 0$ and $\lambda > \max[0, \alpha_2 - \mathfrak{n}]$, then the following relation holds:*

$$\left(I_{-}^{\alpha_1,\alpha_2,\eta} t^{\lambda-1} \right)(x) = \frac{\Gamma(\eta-\lambda+1)\Gamma(\alpha_2-\lambda+1)}{\Gamma(1-\lambda)\Gamma(\alpha_1+\alpha_2+\eta-\lambda+1)} x^{\lambda-\alpha_2-1}. \tag{6.1.16}$$

In the same paper, they define the following left and right sided Erdélyi-Kober fractional integral as:

$$\left(K_{\eta,\alpha_1}^{+} t^{\lambda-1} \right)(x) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\alpha_1+\eta)} x^{\lambda-1}, \tag{6.1.17}$$

where $\Re(\alpha_1) > 0$, $\Re(\lambda) > -\Re(\eta)$, and

$$\left(K_{\eta,\alpha_1}^{-} t^{\lambda-1} \right)(x) = \frac{\Gamma(\eta-\lambda+1)}{\Gamma(\alpha_1+\eta-\lambda+1)} x^{\lambda-1}, \tag{6.1.18}$$

where $\Re(\lambda) < 1 + \Re(\eta)$.

6.2 Representation of Generalized Fractional Integrals in Term of Wright Functions

In this section, we introduce the generalized left-sided fractional integration (6.1.3) of the Bessel k-functions (6.1.9). It is given by the following result.

Theorem 6.2.1. *Assume that α_1, α_2, η, λ, $v \in \mathbb{C}$ be such that*

$$\Re(\frac{v}{k}) > -1, \Re(\alpha_1) > 0, \Re(\frac{\lambda}{k} + \frac{v}{k}) > \max[0, \Re(\alpha_2 - \eta)], \tag{6.2.1}$$

then the following result holds:

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(t)\right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}}$$

$$\times\, {}_2\Psi_3\left[\begin{matrix}(\frac{\lambda}{k}+\frac{v}{k},2),(\frac{\lambda}{k}+\frac{v}{k}+\eta-\alpha_2,2k)\\ \\ (\frac{\lambda}{k}+\frac{v}{k}-\alpha_2,2),(\frac{\lambda}{k}+\frac{v}{k}+\alpha_1+\eta,2),(\frac{v}{k}+1,1)\end{matrix}\;\middle|\; -\frac{cx^2}{4k}\right]. \tag{6.2.2}$$

Proof. Note that the condition (6.1.13) is satisfied so therefore ${}_2\Psi_3(z)$ is defined. Now, from (6.1.3) and (6.1.9), we have

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(t)\right)(x) = \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{\frac{v}{k}+2n}}{\Gamma_k(v+k+nk)n!}\left(I_{0+,k}^{\alpha_1,\alpha_2,\eta}t^{\frac{\lambda+v}{k}+2n-1}\right)(x)$$

By (6.2.1) and for any $n=0,1,2,\cdots$, $\Re(\frac{\lambda}{k}+\frac{v}{k}+2n)\geq\Re(\frac{\lambda}{k}+\frac{v}{k})>\max[0,\Re(\alpha_2-\eta)]$. Applying equation (6.1.16), we obtain

$$\left(I_{0+,k}^{\alpha_1,\alpha_2,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(t)\right)(x) = \frac{x^{\frac{\lambda+v}{k}-\alpha_2-1}}{2^{\frac{v}{k}}}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\frac{v}{k}+\frac{\lambda}{k}+2n)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\eta-\alpha_2+2n)}{\Gamma(\frac{v}{k}+\frac{\lambda}{k}-\alpha_2+2n)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\alpha_1+\eta+2n)\Gamma_k(\frac{v}{k}+1+n)k^{\frac{v}{k}}}$$
$$\times\frac{(-cx^2)^n}{(4k)^n n!}. \tag{6.2.3}$$

By (6.1.12), we obtain

$$\left(I_{0+,k}^{\alpha_1,\alpha_2,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(t)\right)(x)$$
$$=\frac{x^{\frac{v}{k}+\frac{\lambda}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}}\,{}_2\Psi_3\left[\begin{matrix}(\frac{v}{k}+\frac{\lambda}{k},2),(\frac{v}{k}+\frac{\lambda}{k}+\eta-\alpha_2,2)\\ \\ (\frac{v}{k}+\frac{\lambda}{k}-\alpha_2,2),(\frac{v}{k}+\frac{\lambda}{k}+\alpha_1+\eta,2),(\frac{v}{k}+1,1)\end{matrix}\;\middle|\; -\frac{cx^2}{4k}\right].$$

This is the required proof of (6.2.2). □

Corollary 6.2.2. *Assume that* α_1, λ, $v \in \mathbb{C}$ *be such that* $\Re(\frac{v}{k}) > -1$, $\Re(\alpha_1) > 0$, $\Re(\frac{\lambda}{k} + \frac{v}{k}) > 0$, *then the following result holds:*

$$\left(I_{0+}^{\alpha_1} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t) \right)(x) = \frac{x^{\frac{v}{k}+\frac{\lambda}{k}+\alpha_1-1}}{(2k)^{\frac{v}{k}}}$$

$$\times \quad {}_1\Psi_2 \left[\begin{matrix} (v+\lambda, 2k) \\ \\ (\frac{v}{k}+\frac{\lambda}{k}+\alpha_1, 2), (\frac{v}{k}+1, k) \end{matrix} \middle| -\frac{cx^2}{4k} \right]. \qquad (6.2.4)$$

Proof. By substituting $\alpha_2 = -\alpha_1$ in (6.2.2), we obtain the required result. □

Corollary 6.2.3. *Assume that* α_1, η, λ, $v \in \mathbb{C}$ *be such that* $\Re(\frac{v}{k}) > -1$, $\Re(\alpha_1) > 0$, $\Re(\frac{\lambda}{k} + \frac{v}{k}) > 0$, *then the following formula holds:*

$$\left(K_{\alpha_1,\eta}^{+} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t) \right)(x) = \frac{x^{\frac{v}{k}+\frac{\lambda}{k}-1}}{(2k)^{\frac{v}{k}}}$$

$$\times \quad {}_1\Psi_2 \left[\begin{matrix} (\frac{v}{k}+\frac{\lambda}{k}+\eta, 2) \\ \\ (\frac{v}{k}+\frac{\lambda}{k}+\alpha_1+\eta, 2), (\frac{v}{k}+1, 1) \end{matrix} \middle| -\frac{cx^2}{4k} \right]. \qquad (6.2.5)$$

Proof. By setting $\alpha_2 = 0$ in (6.2.2), we get the desired result. □

Theorem 6.2.4. *Assume that* α_1, α_2, η, λ, $v \in \mathbb{C}$ *and* $k > 0$ *be such that*

$$\Re(\frac{v}{k}) > -1, \Re(\alpha_1) > 0, \Re(\frac{\lambda}{k} - \frac{v}{k}) < 1 + \min[\Re(\alpha_2), \Re(\eta)], \qquad (6.2.6)$$

then the following result holds:

$$\left(I_{0-}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(\tfrac{1}{t}) \right)(x) = \frac{x^{\frac{\lambda-v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}}$$

$$\times \quad {}_2\Psi_3\left[\begin{array}{c} (1+\alpha_2-\frac{\lambda}{k}+\frac{v}{k},2),(1-\frac{\lambda}{k}+\frac{v}{k}+\eta,2) \\ \\ (1-\frac{\lambda}{k}+\frac{v}{k},2),(1+\alpha_2+\alpha_1+\eta-\frac{\lambda}{k}+\frac{v}{k},2),(\frac{v}{k}+1,k) \end{array} \Big| -\frac{c}{4x^2}\right]. \tag{6.2.7}$$

Proof. Note that the condition (6.1.13) is satisfied so therefore ${}_2\Psi_3(z)$ is defined. Now, from (6.1.4) and (6.1.9), we have

$$\left(I_{0-}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(\tfrac{1}{t}) \right)(x) = \sum_{n=0}^{\infty} \frac{(-c)^n (\frac{1}{2})^{\frac{v}{k}+2n}}{\Gamma_k(v+k+nk)n!} \left(I_{0-}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}+\frac{v}{k}-2n-1} \right)(x)$$

By (6.2.6) and for any $k>0$ and $n=0,1,2,\cdots$, $\Re(\frac{\lambda}{k}-\frac{v}{k}-2n-1) \leq 1+\Re(\frac{\lambda}{k}-\frac{v}{k}-1) < 1+\min[\alpha_2,\Re(\eta)]$. Applying equation (6.1.16), we obtain

$$\begin{aligned} &\left(I_{0-}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(\frac{1}{t}) \right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}} \\ &\times \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_2-\frac{\lambda}{k}+\frac{v}{k}+1+2n)\Gamma(\eta-\frac{\lambda}{k}+\frac{v}{k}+1+2n)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k}+2n)\Gamma(\alpha_1+\alpha_2+\eta-\frac{\lambda}{k}+\frac{v}{k}+1+2n)\Gamma(\frac{v}{k}+1+n)} \\ &\times \frac{(-c)^n}{(4kx^2)^n n!}. \end{aligned} \tag{6.2.8}$$

By (6.1.12), we obtain

$$\left(I_{0-}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(\frac{1}{t}) \right)(x) = \frac{x^{\frac{\frac{\lambda}{k}-\frac{v}{k}-\alpha_2}{k}-1}}{(2k)^{\frac{v}{k}}}$$

$$\times_2 \Psi_3\left[\begin{array}{c} (\alpha_2-\frac{\lambda}{k}+\frac{v}{k}+1,2),(\eta-\frac{\lambda}{k}+\frac{v}{k}+1,2) \\ \\ (1-\frac{\lambda}{k}+\frac{v}{k}.2),(\alpha_1+\alpha_2+\eta-\frac{\lambda}{k}+\frac{v}{k}+1,2),(\frac{v}{k}+1,1) \end{array} \Big| -\frac{c}{4kx^2}\right].$$

This is the required proof of (6.2.7). □

Corollary 6.2.5. *Assume that* α_1, η λ, $v \in \mathbb{C}$ *and* $k>0$ *be such that* $\Re(\frac{v}{k}) > -1$, $0 < \Re(\alpha_1) < 1-\Re(\frac{\lambda}{k}-\frac{v}{k})$, *then the following result holds:*

$$\left(I_{0+}^{\alpha_1} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(\tfrac{1}{t}) \right)(x) \quad = \quad \frac{x^{\frac{\lambda}{k}-\frac{v}{k}+\alpha_1-1}}{(2k)^{\frac{v}{k}}}$$

$$\times \quad {}_1\Psi_2 \left[\begin{matrix} (1-\alpha_1-\frac{\lambda}{k}+\frac{v}{k},2) \\ \\ (1-\frac{\lambda}{k}+\frac{v}{k},2),(\frac{v}{k}+1,1) \end{matrix} \middle| -\frac{c}{4kx^2} \right]. \tag{6.2.9}$$

Corollary 6.2.6. *Assume that* α_1, η, λ, $v \in \mathbb{C}$ *and* $k > 0$ *be such that* $\Re(\frac{v}{k}) > -1$, $\Re(\alpha_1) > 0$, $\Re(\frac{\lambda}{k}+\frac{v}{k}) < 1+\max[0,\Re(\eta)]$, *then the following formula holds:*

$$\left(K^{-}_{\alpha_1,\eta} t^{\frac{\lambda}{k}-1} W^{k}_{v,c}(\tfrac{1}{t}) \right)(x) = \frac{x^{\frac{\lambda}{k}-\frac{v}{k}-1}}{(2k)^{\frac{v}{k}}}$$

$$\times \quad {}_1\Psi_2 \left[\begin{matrix} (1+-\frac{\lambda}{k}+\frac{v}{k}+\eta,2) \\ \\ (1-\frac{\lambda}{k}+\frac{v}{k}+\alpha_1+\eta,2),(\frac{v}{k}+1,1) \end{matrix} \middle| -\frac{c}{4kx^2} \right]. \tag{6.2.10}$$

6.3 Representation in Terms of Generalized Hypergeometric Functions

In this section, we introduce the generalized fractional integrals of Bessel k-function in term of generalized hypergeometric function. First we consider the following well known results:

$$\Gamma(2\mu) = \frac{2^{2\mu-1}}{\sqrt{\pi}}\Gamma(\mu)\Gamma(\mu+\frac{1}{2}); \mu \in \mathbb{C} \tag{6.3.1}$$

and

$$(\mu)_{2n} = 2^{2n}(\frac{\mu}{2})_n(\frac{\mu+1}{2})_n, \mu \in \mathbb{C}, n \in \mathbb{N}. \tag{6.3.2}$$

We represent the following theorems containing the generalized hypergeometric function.

Theorem 6.3.1. *Assume that α_1, α_2, η, λ, $v \in \mathbb{C}$ be such that*

$$\Re(\frac{v}{k}) > -1, \Re(\alpha_1) > 0, \Re(\frac{\lambda}{k} + \frac{v}{k}) > \max[0, \Re(\alpha_2 - \eta)], \tag{6.3.3}$$

and let $\frac{\lambda}{k} + \frac{v}{k}$, $\frac{\lambda}{k} + \frac{v}{k} + \eta - \alpha_2 \neq 0, -1, \cdots$, then the following result holds:

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^k(t)\right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}} \frac{\Gamma(\frac{\lambda}{k}+\frac{v}{k})\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\eta-\alpha_2)}{\Gamma(\frac{\lambda}{k}+\frac{v}{k}-\alpha_2)\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\alpha_1+\eta)\Gamma(\frac{v}{k}+1)}$$
$$\times {}_4F_5 \left[\begin{array}{c} \frac{\lambda}{2k}+\frac{v}{2k}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{1}{2}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta-\alpha_2}{2}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta-\alpha_2+1}{2} \\ \\ \frac{v}{k}+1, \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\alpha_2}{2}, \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\alpha_2+1}{2}, \frac{\lambda}{k}+\frac{v}{k}+\frac{\alpha_1+\eta}{2}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\alpha_1+\eta+1}{2} \end{array} \middle| -\frac{cx^2}{4k} \right]. \tag{6.3.4}$$

Proof. Note that ${}_4F_5$ defined in (6.3.4) exit as the series is absolutely convergent. Now, using (??) with $z = \frac{v}{k} + 1$ and (6.2.3) and applying (6.3.2) with z being replaced by $\frac{\lambda}{k} + \frac{v}{k}$, $\frac{\lambda}{k} + \frac{v}{k} + \eta - \alpha_2$ and $\frac{\lambda}{k} + \frac{v}{k} + \alpha_1 + \eta$, we have

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^k(t)\right)(x) = \frac{x^{\frac{\lambda+v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\frac{v}{k}+\frac{\lambda}{k})\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\eta-\alpha_2)}{\Gamma(\frac{v}{k}+\frac{\lambda}{k}-\alpha_2)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\alpha_1+\eta)\Gamma(\frac{v}{k}+1)}$$
$$\times \frac{(\frac{v}{k}+\frac{\lambda}{k})_{2n}(\frac{v}{k}+\frac{\lambda}{k}+\eta-\alpha_2)_{2n}}{(\frac{v}{k}+\frac{\lambda}{k}-\alpha_2)_{2n}(\frac{v}{k}+\frac{\lambda}{k}+\alpha_1+\alpha_2)_{2n}} \frac{(-cx^2)^n}{(4k)^n n!}$$
$$= \frac{x^{\frac{\lambda+v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}} \frac{\Gamma(\frac{v}{k}+\frac{\lambda}{k})\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\eta-\alpha_2)}{\Gamma(\frac{v}{k}+\frac{\lambda}{k}-\alpha_2)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\alpha_1+\eta)\Gamma(\frac{v}{k}+1)}$$
$$\times \sum_{n=0}^{\infty} \frac{(\frac{v}{2k}+\frac{\lambda}{2k})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{1}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\eta-\alpha_2}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\eta-\alpha_2}{2})_n}{(\frac{v}{k}+1)(\frac{v}{2k}+\frac{\lambda}{2k}-\frac{\alpha_2}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}-\frac{\alpha_2}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\alpha_1+\eta}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\alpha_1+\eta+1}{2})_n}$$
$$\times \frac{(-cx^2)^n}{(4k)^n n!}.$$

Thus, in accordance with equation (6.1.10), we get the required result (6.3.4). □

Corollary 6.3.2. *Assume that α_1, λ, $v \in \mathbb{C}$ be such that $\Re(\frac{v}{k}) > -1$, $\Re(\alpha_1) > 0$, $\Re(\frac{\lambda}{k} + \frac{v}{k}) > 0$ and $\frac{\lambda}{k} + \frac{v}{k} = 0, -1, \cdots$, then the following result holds:*

$$\left(I_{0+}^{\alpha_1,\alpha_2,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t)\right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}+\alpha_1-1}}{(2k)^{\frac{v}{k}}} \frac{\Gamma(\frac{\lambda}{k}+\frac{v}{k})}{\Gamma(\frac{\lambda}{k}+\frac{v}{k}-\alpha_2)\Gamma(\frac{v}{k}+1)}$$

$$\times {}_2F_3 \left[\begin{array}{c} \frac{\lambda}{2k}+\frac{v}{2k}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{1}{2}, \\ \qquad\qquad\qquad\qquad | -\frac{cx^2}{4k} \\ \frac{v}{k}+1, \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\alpha_2}{2}, \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\alpha_2+1}{2} \end{array} \right]. \tag{6.3.5}$$

Proof. By substituting $\alpha_2 = -\alpha_1$ in (6.3.4), we obtain the required result. □

Corollary 6.3.3. *Assume that α_1, η, λ, $v \in \mathbb{C}$ be such that $\Re(\frac{v}{k}) > -1$, $\Re(\alpha_1) > 0$, $\Re(\frac{\lambda}{k} + \frac{v}{k}) > 0$ and let $\frac{\lambda}{k} + \frac{v}{k} + \eta - \alpha_2 \neq 0, -1, \cdots$, then the following result holds:*

$$\left(K_{\alpha_1,\eta}^{+} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t)\right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}-1}}{(2k)^{\frac{v}{k}}} \frac{\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\eta)}{\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\alpha_1+\eta)\Gamma(\frac{v}{k}+1)}$$

$$\times {}_2F_3 \left[\begin{array}{c} \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta}{2}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta+1}{2} \\ \qquad\qquad\qquad\qquad | -\frac{cx^2}{4k} \\ \frac{v}{k}+1, \frac{\lambda}{k}+\frac{v}{k}+\frac{\alpha_1+\eta}{2}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\alpha_1+\eta+1}{2} \end{array} \right]. \tag{6.3.6}$$

Proof. By setting $\alpha_2 = 0$ in (6.3.4), we get the desired result. □

Theorem 6.3.4. *Assume that α_1, α_2, η, λ, $v \in \mathbb{C}$ and $k > 0$ be such that*

$$\Re(\frac{v}{k}) > -1, \Re(\alpha_1) > 0, \Re(\frac{\lambda}{k} - \frac{v}{k}) < 1 + \min[\Re(\alpha_2), \Re(\eta)], \tag{6.3.7}$$

and let $\frac{\alpha_2-\lambda}{k}+\frac{v}{k}+1,\ \eta-\frac{\lambda}{k}+\frac{v}{k}+1\neq 0,-1,\cdots$, *then the following result holds:*

$$\left(I_{0-}^{\alpha_1,\alpha_2,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\frac{1}{t})\right)(x)=\frac{x^{\frac{\lambda}{k}-\frac{v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}}$$
$$\times\frac{\Gamma(\alpha_2-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\eta-\frac{\lambda}{k}+\frac{v}{k}+1)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k})\Gamma(\alpha_1+\alpha_2+\eta-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\frac{v}{k}+1)}$$
$$\times_4F_5\left[\begin{matrix}\frac{\alpha_2+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k},\frac{\alpha_2+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k},\frac{\eta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k},\frac{\eta+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k}\\ \\ \frac{v}{k}+1,\frac{1}{2}-\frac{\lambda}{2k}+\frac{v}{2k},1-\frac{\lambda}{2k}+\frac{v}{2k},\frac{\alpha_1+\alpha_2+\eta+1}{2}-\frac{\lambda}{k}+\frac{v}{k},\frac{\alpha_1+\eta+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k}\end{matrix}\ \Big|-\frac{c}{4kx^2}\right].\tag{6.3.8}$$

Proof. Using (??) with $z=\frac{v}{k}+1$ and (6.2.8) and applying (6.3.2) with z being replaced by $\alpha_2-\frac{\lambda}{k}+\frac{v}{k}+1$, $1-\frac{\lambda}{k}+\frac{v}{k}$ and $\alpha_2-\frac{\lambda}{k}+\frac{v}{k}+\alpha_1+\eta+1$, we have

$$\left(I_{0-}^{\alpha_1,\alpha_2,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\frac{1}{t})\right)(x)=\frac{x^{\frac{\lambda}{k}-\frac{v}{k}-\alpha_2-1}}{(2k)^{\frac{v}{k}}}$$
$$\frac{\Gamma(\alpha_2-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\eta-\frac{\lambda}{k}+\frac{v}{k}+1)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k})\Gamma(\alpha_1+\alpha_2+\eta-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\frac{v}{k}+1)}$$
$$\times\sum_{n=0}^{\infty}\frac{(\frac{\alpha_2+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\alpha_2}{2}-\frac{\lambda}{2k}+\frac{v}{2k}+1)_n(\frac{\eta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\eta}{2}-\frac{\lambda}{2k}+\frac{v}{2k}+1)_n}{(\frac{v}{k}+1)_n((\frac{1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n)(1-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\alpha_1+\alpha_2+\eta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\alpha_1+\alpha_2+\eta}{2}-\frac{\lambda}{2k}+\frac{v}{2k}+1)_n}$$
$$\times\frac{(-c)^n}{(4kx^2)^n n!}.$$

By (6.1.10), we obtain the required given in (6.3.8). □

Corollary 6.3.5. *Assume that* α_1, η λ, $v\in\mathbb{C}$ *and* $k>0$ *be such that* $\Re(\frac{v}{k})>-1$, $0<\Re(\alpha_1)<1-\Re(\frac{\lambda}{k}-\frac{v}{k})$, *and let* $\frac{\lambda}{k}-\frac{v}{k}+\alpha_1\neq 1,2,\cdots$ *then the following result holds:*

$$\left(\ I_{0+}^{\alpha_1}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\tfrac{1}{t})\ \right)(x)\ =\ \frac{x^{\frac{\lambda}{k}-\frac{v}{k}+\alpha_1-1}}{(2k)^{\frac{v}{k}}}\frac{\Gamma(-\alpha_1-\frac{\lambda}{k}+\frac{v}{k}+1)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k})\Gamma(\frac{v}{k}+1)}$$

$$\times \quad {}_2F_3\left[\begin{matrix} \frac{-\alpha_2+1}{2} - \frac{\lambda}{2k} + \frac{v}{2k}, \frac{-\alpha_1+2}{2} - \frac{\lambda}{2k} + \frac{v}{2k}, \\ \frac{v}{k}+1, \frac{1}{2} - \frac{\lambda}{2k} + \frac{v}{2k}, 1 - \frac{\lambda}{2k} + \frac{v}{2k}. \end{matrix} \middle| -\frac{c}{4kx^2}\right]. \quad (6.3.9)$$

Corollary 6.3.6. *Assume that α_1, η, λ, $v \in \mathbb{C}$ and $k > 0$ be such that $\Re(\frac{v}{k}) > -1$, $\Re(\alpha_1) > 0$, $\Re(\frac{\lambda}{k}+\frac{v}{k}) < 1+\max[0, \Re(\eta)]$ and let $\frac{\lambda}{k}-\frac{v}{k}-\eta \neq 1, 2, \cdots$, then the following formula holds:*

$$\left(K^{-}_{\eta,\alpha_1} t^{\frac{\lambda}{k}-1} W^{k}_{v,c}(\tfrac{1}{t}) \right)(x) = \frac{x^{\frac{\lambda}{k}-\frac{v}{k}-1}}{(2k)^{\frac{v}{k}}} \frac{\Gamma(\eta - \frac{\lambda}{k} + \frac{v}{k} + 1)}{\Gamma(1 - \frac{\lambda}{k} + \frac{v}{k})\Gamma(\frac{v}{k}+1)}$$

$$\times \quad {}_2F_3\left[\begin{matrix} \frac{\eta+1}{2} - \frac{\lambda}{2k} + \frac{v}{2k}, \frac{\eta+2}{2} - \frac{\lambda}{2k} + \frac{v}{2k} \\ \frac{v}{k}+1, \frac{\alpha_1+\eta+1}{2} - \frac{\lambda}{k} + \frac{v}{k}, \frac{\alpha_1+\eta+2}{2} - \frac{\lambda}{2k} + \frac{v}{2k} \end{matrix} \middle| -\frac{c}{4kx^2}\right]. \quad (6.3.10)$$

Corollary 6.3.5 and 6.3.6 follow from theorem 6.3.4 in respective cases $\alpha_2 = -\alpha_1$ and $\alpha_2 = 0$.

Conclusion

In this research work, We have derived several properties of some special k-functions. We have worked on gamma, beta and hypergeometric k-functions and derived gamma, beta and hypergeometric k-functions with matrix arguments. We have proved their various properties such as integral representation, relation between gamma and beta matrix k-function and differential equations of hypergeometric matrix k-function. Also, we have worked on extended gamma and beta function and derived extended gamma, beta k-functions and their various properties. We have used the idea of gamma and beta k-function and established (τ, k)-hypergeometric function. Working in the theory of fractional calculus, we have proved a number fractional integral and differential formulas of (τ, k)-hypergeometric function. Also, we have studied the generalized fractional integrals containing hypergeometric function in their kernels and established generalized fractional integration formulas of k-Bessel function. Further, we have established (k, s)-fractional calculus of (τ, k)-hypergeometric function and k-Mittag-Leffler function. Some various generalized fractional integral and differential formulas of (τ, k)-hypergeometric function and k-Mittag-Leffler function have been obtained. Note that If $k = 1$, we get the classical results throughout the research work.

Future research: The theory of special k-functions is presented formally in the beginning of this century. It is a very fruitful field having vast applications in different fields of mathematics, physics, statistics and other natural sciences. There is a scope of this theory to find the k-analogue of several special functions with properties and applications. We can established certain inequalities involving the extended gamma and beta k-functions. We can apply this theory in special k-function and introduced extension of some special k-functions. This theory can also be used in fractional calculus in order to introduce extended k-fractional derivatives and integral operators.

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Appendix

Contribution of Research During PhD

1 G. Rahman, D. Báleanu, M. Al-Qurashi, S. D. Purohit, S. Mubeen, M. Arshad, *Extended Mittag-Leffler function via fractional calculus*, J. Nonlinear Sci. Appl. (Accepted) (2017).

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15 G. Rahman, K. S. Nisar, S. Mubeen, M. Arshad, *Opial-Type Inequalities Of Extended Mittag-Leffler Function*, (to be appeared).

16 G. Rahman, K. S. Nisar, S. Mubeen, M. Arshad, *The Extended k-Mittag-Leffler Function And Its Properties*, (to be appeared).

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