Fixed Points of Contractive Mapping in Ultra Metric Space

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2017

Fixed point theory

Metric space

Contractive mapping

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DECLARATION

I, hereby declare that this thesis neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work presented in this thesis has been submitted in support of an application for any degree or qualification of this or any other Institute of learning.

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***MASTER OF SCIENCE (MS) IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS & STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY ISLAMABAD.

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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2017

DEDICATION

This work is dedicated

To

My Parents, Family, Friends,

Valued Teacher Prof. Dr. Muhammad Arshad Zia

for Supporting and Encouraging me.

ACKNOWLEDGEMENTS

First and foremost, all praise to **'ALLAH'**, Lord of the world, the Almighty, and I find no words to thank **'ALLAH'**, who created me and all the universe around us. Who is the sole creator of all the things tinier than an electron to huge galaxies. I am thankful to the Prophet Muhammad (S.A.W.) whose teachings are a blessing for the whole mankind. May Allah guide us and the whole humanity to the right path.

Words of gratitude and appreciation do not always convey the depth of one's feelings, yet I wish to record my thanks to my most respected supervisor and honorable teacher **Prof. Dr. Muhammad Arshad Zia**, who encouraged me very much in time, made me realize and enjoy the hard work. I am obliged to him for his able guidance, help, immense encouragement and limitless patience, which made massive task accomplished.

I pay my thanks to the whole faculty of Department of Mathematics and Statistics International Islamic University Islamabad. I also feel much pleasure in acknowledging nice company of my friends Latif-ur-Rahman, Awais Ahmad, Waseem Ahmad, Asad Khan, Muhammad Jehanger Khan, Ubaid-ur-Rahman and many other who support and encourage me directly or indirectly in my research work.

In the end I want to pay my attribute to my parents whose love and guidance always gave me a ray of hope in the darkness of desperation. I am thankful to all of my family members specially my father for their support and well wishing, enormous love, support, encouragements and constant patience. I cannot forget their prayers for me throughout my life, without them this effort would have been nothing.

Muhammad Junaid

CONTENTS

Preface

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the "Banach Contraction Principle" (BCP) which is one of the most important results of analysis and considered to be the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting. The BCP has been generalized in many different directions. In fact, there is vast amount of literature dealing with extensions / generalizations of this remarkable theorem. Fixed point theory is an essential subject which works as a bridge between pure and applied mathematics, if in pure mathematics it can solve non linear and trancendental equations then in applied mathematics it is quite helpful to work out the differential equations (ODE and PDE). Fixed point theorem deals with the assurance that a mapping T on a set X has one or more fixed points, i.e. the functional equation $Tx = x$ has one or more soloutions. A large variety of the problems of analysis in applied mathematics relates to finding soloutions of nonlinear functional equations which can be formulated in terms of finding the fixed point of a nonlinear mappings. In fact, fixed point theorems are extremely substantial tools for proving the existence and uniqueness of the solutions to various mathematical models.

Van Roovij, A. C. M.[20] for the first time introduced the concept of the ultrametric space and non-Archimedean functional analysis in 1978. In a metric, if the triangular inequality is replaced by the stronger triangular inequality ($d(x, z) \leq$

$max\{d(x,y),d(y,z)\})$ then it is called ultrametric and the corresponding space becomes ultrametric space or non-Archimedean space. Due to the strong triangular inequality the ultrametric space exhibits very strange properties that every point inside the ball is its centre ,the two intersecting balls are contained in each other ,every ball is both open and closed and geometrically every triangle is either isosceles or equilatral. The modus opereandi to obtain a fixed point in ultrametric space is totaly different from usaul complete metric space,wherein we take a sequence to be cauchy through the picard's iteration while in ultrametric space we take open balls and establish and order between them and through the Zorn's lemma we obtain a fixed point. The ultrametric space should be spherically complete as compared to just complete metric space, because spherical completness is a stronger condition whereas completness i.e every sphericall complete metric space is a complete metric space but the converse is not true. Let (X,d) be a metric space and $T : X \to X$ be a self mapping. The mapping T is called contractive if $d(Tx,Ty) < d(x,y)$ for all $x, y \in X$ with $x \neq y$ and nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x, y \in X$. It is well known that contractive and nonexpensive mappings are necessarily continuous on the whole domain and not have a fixed point in complete metric space.

Example[13] Let $X = (-\infty,\infty)$ endowed with the usual metric and $T : X \to X$ defined by

$$Tx = x + \frac{1}{1+e^x}$$

for all $x \in X$. Notice that X is complete and T is contractive mapping but T does not have a fixed point.

Example[12] Let $X = [0, \infty)$ endowed with the usual metric and $T : X \to X$ defined by

$$Tx = 1 + x$$

for all $x \in X$. Notice that X is complete and T is a nonexpensive mapping but T does not have a fixed point. To overcome this difficulty we use spherically complete ultrametric space.

C. Petalas, F. Vidalis [13] and Lj. Gajic [4] studied fixed point theorems of contractive type maps on a spherically complete ultrametric spaces which are generalizations of the Banach fixed point theorems. Rao etal.,[16] obtained coincidence point theorems for three and four self maps in ultrametric. Kubiaczyk etal.,[10] extend the fixed point theorems from the single-valued maps to the set- valued contractive maps. Then Gajic [5] gave the result for the multivalue mapping. Again, Rao [17] proved some common fixed point theorems for a pair of maps of Jungck type on a spherically complete ultrametricspace. Zhang et al.,[?] introduced generalized weak-contraction, which is a generalization of banach contraction principle. Recently, R. Pant [12] obtained some new fixed point theorems for set-valued contractive and nonexpansive mappings in the setting of ultrametric spaces.

Chapter 1, is essentially an introduction, where we fix notations and terminology to be used. It is a survey aimed at recalling some basic definitions and facts. While some of the classical and recent results about fixed point existence in ultrametric space are also presented in this chapter.

Chapter 2, concerned with the study of unique fixed point results in ultrametric space on B.E Rhoades [18], listed contractive mapping and extended the M. Edelstein[3] and pant[12] results for Junck type mappings which are a generalization of Banach contraction principle.

Chapter 3, is allocated to find the fixed point in ultrametric space, using F-contraction and extended the resluts to Juncks type mappings.

Chapter 1

Preliminaries

The aim of this chapter is to present basic definitions and to explain the terminology nsed through out this dissertation ,this chapter consists of two sections ,the first section is about the definitions and ultrametric space while the second is about some theorems without proof.

1.1 Basic Concepts

Definition 1.1.1 [9] Let (X, d) be a metric space. (i) A point $x \in X$ is said to be a fixed point of mapping $f : X \to X$ if $x = fx$.

(ii) f is called contraction if there exists a fixed constant $h < 1$ such that

$$d(f(x), f(y)) \leq hd(x, y) \tag{1.1}$$

for all $x, y \in X$.

A contraction mapping is also known as Banach contraction. If we replace the inequality with strict inequality and $h = 1$, then f is called contractive (or strict contractive).

If (1.1) holds for $h = 1$, then f is called nonexpansive and if (1.1) holds for fixed $h < \infty$, then f is called Lipschitz continuous. Clearly, for the mapping f, the following obvious implications hold:contraction ⇒contractive⇒ nonexpansive⇒Lipschitz continuous.

Definition 1.1.2. [20] Let (X, d) be a metric space. If the metric d satifies

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;

(ii) $d(x, y) = d(y, x)$ (symetric);

(iii) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$

for all $x, y, z \in X$ is called strong triangle inequality;

It is called ultrametric on X. Pair (X, d) is an ultrametric space.

Definition 1.1.3. [2] A self mapping $T : X \to X$ on the metric space (X, d) is said to be quasi-contraction if $d(Tx, Ty) \leq k. \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ where $0 \leq k < 1$.

Definition1.1.4[18] A self mapping $T : X \to X$ on the metric space (X, d) is said to be contractive mapping if $d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}$ for all $x, y \in X, x \neq y$.

Definition 1.1.5 [20] An ultrametric space is said to spherically complete if the intersection of nested balls in X is non-empty.

Definition 1.1.6. An element $x \in X$ is said to be a coincidence point of $f : X \to X$ and $T : X \to C(X)$ if $fx \in Tx$. We denote $C(f, T) = \{x \in X \text{ such that } fx \in Tx\}$ the set of coincidence points of f and T.

Definition 1.1.7. Let (X, d) be an ultrametric space, and $f : X \to X$ and $T : X \to C(X)$. f and T are said to be coincidentally commuting at $z \in X$ if $fz \in Tz$ implies $fTz \subseteq Tfz$.

Definition 1.1.8. Let $C(X)$ denote the class of all non empty compact subsets of X.

For $A, B \in C(X)$ the Hausdorff metric is defined as $H(A,B) = \max\{\sup_{x\in B} d(x,A), \sup_{y\in A} d(x,b)$

where $d(x,A) = \inf d(x,a)$, $a \in A$.

Definition 1.1.9 [18] A self mapping $T : X \to X$ on the metric space (X,d) is said to be contractive mapping if $d(Tx,Ty) < \max\{d(y,Tx), d(x,Ty)\}$; for all $x, y \in X, x \neq y$.

Definition 1.1.10 Edelstein[3] for all $x, y \in X$ and $x \neq y$ then $d(Tx,Ty) < d(x,y)$.

Definition 1.1.11 A self mapping $T : X \to X$ on the metric space (X,d) is said to be contractive mapping if $d(Tx,Ty) < \max\{d(x,Ty), d(y,Tx), d(x,y)\}$; for all $x, y \in X, x \neq y$.

Definition 1.1.12 Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be F-contraction if there exist $\tau > 0$ such that for all $x, y \in X$,

$$[d(Tx,Ty)] > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))$$

Let R be the set of real number and $\mathcal{F}$ be the set of all function $F : (0,\infty) \to R$ satisfying the following condition.

(i) F is strictly increasing.i.e for $x, y \in (0,\infty)$ such that $x < y$, $F(x) < F(y)$.

(ii) For each seuence $\{a_n\}_{n=1}^{\infty}$ of positive number $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(iii) There exist $K \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

1.2 SOME PREVIOUS THEOREM ABOUT THE FIXED POINT IN ULTRAMETRIC SPACE

Theorem 1.2.1 [4] Let (X, d) be a spherically complete ultrametricspace. If $T : X \to X$ is a mapping such that $d(Tx, Ty) < max\{d(x, y), d(x, Tx), d(y, Ty)\}$;for all $x, y \in X, x \neq y$.

Then T has a unique fixed point in X.

Theorem 1.2.2 *(Zorn's lemma).* Let S be a partially ordered set. If every totally ordered subset of S has an upper bound, then S contains a maximal element.

Theorem 1.2.3 Let (X, d) be a compact metric space and $T : X \to X$ mapping. Assume that

$\frac{1}{2}d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$; for $x, y \in X$;. Then T has a unique fixed point.

Theorem 1.2.4 Let (X, d) be a spherically complete ultrametric space and $T : X \to C(X)$ a set-valued mapping. Assume that

$\frac{1}{2}d(x, Tx) < d(x, y)$ implies $H(Tx, Ty) < d(x, y)$ for $x, y \in X$. Then T has a fixed point.

Theorem 1.2.5[5] Let (X, d) be the spherically complete ultrametric space if $T : X \to 2_C^X$ is such that $H(Tx, Ty) < \{d(x, y), d(x, Tx), d(y, Ty)\}$ for any $x, y \in X$, $x \neq y$. Then T has a fixed point (i.e there exist $x \in X$, such that $x \in Tx$).

Theorem 1.2.6 Let X be a metric space and $T : X \to X$ a generalized nonexpansive mapping. Then for all $x; y \in X$,

(a) $d(Tx, T^2x) \leq d(x, Tx)$;

(b) Either $\frac{1}{2}d(x, Tx) \leq d(x, y)$ or $\frac{1}{2}d(T^2x, Tx) \leq d(Tx, y)$;

(c) Either $d(Tx, Ty) \leq d(x, y)$ or $d(T^2x, Ty) \leq d(Tx, y)$.

Chapter 2

Some Fixed Point Results on Ultrametric space

In metric fixed point theory the contractive conditions on underlying functions play an important role for finding solution of fixed point problems. Banach contraction principle is a fundamental result in metric fixed point theory. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle. B.E Rhoades [18], listed contractive mapping which is a generalization of Banach contraction principle. Van Roovij.A.C.M [20] introduced a new metric called Ultrametric and C.Petalas and F.Vidalis[13] and later on Gajic Lj proved a fixed point results in Ultrametric space as a generalization of the Banach contraction principle. Over the couple of years, it has been generalized in different directions by several mathematician .

Main Results

Theorem 1 *Let (X,d) be a spherically complete ultrametricspace. If $T : X \longrightarrow X$ is a mapping such that for every $x, y \in X, x \neq y$*

$$d(Tx,Ty) < \max\{d(x,Ty), d(y,Tx), d(x,y)\} \tag{2.1}$$

Then T has a unique fixed point.

Proof. *Let $S_a = S(a; d(a,Ta))$ denote the closed sphere centered at a with radii $d(a,Ta)$ and let F be the collection of these sphere for all $a \in X$. The relation $S_a \preceq S_b$ iff $S_b \subseteq S_a$ is a partial order on F. Now consider a totally order subfamily F_1 of F. Since (X,d) is spherically complete. We have that*

$$\bigcap_{S_a \in F_1} S_a = S \neq \emptyset.$$

Let $b \in S$ and $S_a \in F_1$. Also let $x \in S_b$ then

$$d(x,b) \leq d(b,Tb) \ \leq \max\{d(b,a), d(a,Ta), d(Ta,Tb)\}.$$

Using equation (2.1) we have

$$d(x,b) \leq d(b,Tb) \ < \max\{d(b,a), d(a,Ta), \max\{d(a,Tb), d(b,Ta), d(a,b)\}\}.$$

As

$$d(a,Tb) \leq \max\{d(a,b), d(b,Tb)\},$$

and

$$d(b,Ta) \leq \max\{d(b,a), d(a,Ta)\}.$$

Therefore

$$\begin{aligned} d(x,b) &< \max\{d(b,a), d(a,Ta), \max\{d(a,b), d(b,Tb)\}, \max\{d(b,a), d(a,Ta)\}, d(a,b)\} \\ &< \max\{d(a,b), d(a,Ta)\} = d(a,Ta) \\ &\Rightarrow d(x,b) \leq d(a,Ta). \end{aligned}$$

Now

$$d(x,a) \leq \max\{d(x,b), d(b,a)\} \leq d(a,Ta).$$

So $x \in S_a$ implies that $S_b \subseteq S_a$ for all $S_a \in F_1$. Hence S_b is an upper bound of F for the family F_1. So by Zorn's lemma F has a maximal element say $S_z, z \in X$. Now we are going to prove that $Tz = z$. Suppose $Tz \neq z$, i.e. $d(z,Tz) > 0$. Now

$$d(Tz,TTz) < \max\{d(z,TTz), d(Tz,Tz), d(z,Tz)\}.$$

As

$$d(z,TTz) \leq \max d(z,Tz), d(Tz,TTz).$$

Therefore

$$d(Tz,TTz) < \max\{d(z,Tz), d(Tz,TTz), d(z,Tz)\},$$

implies that

$$d(Tz,TTz) < \max\{d(z,Tz), d(Tz,TTz)\} = d(z,Tz)$$

which implies that

$$d(Tz,TTz) < d(z,Tz).$$

let $y \in B_{Tz}$ implies

$$d(y,Tz) \leq d(Tz,TTz) < d(z,Tz)$$

implies that

$$d(y,Tz) < d(z,Tz)$$

as

$$d(y,z) \leq \max\{d(y,Tz), d(Tz,z)\} = d(z,Tz)$$

implie

$$d(y,z) \leq d(z,Tz)$$

this means that $y \in S_z$ *and that* $S_{Tz} \subseteq S_z$. *On the other hand* $z \notin S_{Tz}$.*Since*

$$d(z,Tz) > d\ (Tz,TTz)$$

So $S_{Tz} \subsetneq S_z$, *which is a contradiction to the maximality of* S_z. *Hence* $z = Tz$.

For the uniquness let $Tu = u$ *be another fixed point, for* $u \neq z$ *we have that*

$$d(z,u) = d\ (Tz,Tu) < \max\{d(z,Tu), d(u,Tz), d(z,u)\} = d\ (z,u)$$

implies

$$d\ (z,u) < d\ (z,u)$$

which is a contradiction, so $z = u$

Example 2 *Let* $(X = R, d)$ *is a discrete metric space which is an ultrametric space.*

$$d(a,b) = \begin{cases} 0 & if\ a = b \\ 1 & if\ a \neq b \end{cases}$$

Let $Ta = c$ *then* $Tc = c$ *is the fixed point where c is any real constant.*

∎

Theorem 3 *Let* (X,d) *a spherically complete ultrametric space.if* $T : X \longrightarrow 2^X_C$ *is such that for any* $a, b \in X$, $a \neq b$ *satisfying condition*

$$H(Ta,Tb) < \max\{d(a,Tb), d(b,Ta), d(a,b)\}.$$

Then T *has a fixed point on* X.

Proof. Let $S_a = S(a; d(a, Ta))$ denote a closed sphere centered at a of radii $d\ (a, Ta) = \inf_{z \in Ta} d\ (a, z)$ for all $a \in X$ and let F be the collection of these sphere. Then the relatio$S_a \preceq S_b\ iff\ \ S_b \subseteq S_a$ is a partial ordered on F. Let F_1 be a totally ordered subfamily of F. Since (X, d) is spherically complete. So

$$\bigcap_{S_a \in F_1} S_a = S \neq \emptyset.$$

Let $b \in S_a \in F_1$ obviously $b \in S_a$ as $S_a \in F_1$,so

$$d(b, a) \leq d\ (a, Ta).$$

Take $u \in Ta$ such that

$$d\ (a, u) = d\ (a, Ta)$$

(it is possible becaus Ta is non empty compact set). If $a = b$ then $S_a = S_b$.Assume that $a \neq b$. ∎

let $x \in S_b$ which implies that

$$d\ (x, b) \leq d\ (b, Tb) \leq \inf_{v \in Tb} d(b, v)$$

$$\leq \max\left\{d(b, a), d(a, u), \inf_{v \in Tb} d(u, v)\right\}$$

$$\leq \max\left\{d(a, Ta), H(Ta, Tb)\right\}$$

Using equation (1)

$$d(x, b) \leq d\ (b, Tb)\ < \max\left\{d(a, Ta), \max\left\{d(a, Tb), d(b, a)\right\}\right\}$$

As

$$d(a, Tb) \leq \max\left\{d(a, b), d(b, Tb)\right\}$$

and

$$d(b, Ta) \le \max\{d(b,a), d(a,Ta)\}.$$

Therefore

$$d(x,b) \le \max\{d(a,Ta), \max\{d(a,b), d(b,Tb)\}, \max\{d(b,a), d(a,Ta)\}, d(a,b)\}$$

$$= \max\{d(a,Ta), d(b,a), d(b,Tb)\}$$

$$= d(a,Ta)$$

which implies that

$$d(x,b) \le d(a,Ta)$$

Now

$$d(x,a) \le \max\{d(x,b), d(b,a)\} \le d(a,Ta)$$

implies that

$$d(x,a) \le d(a,Ta).$$

So $x \in S_a$ and $S_b \subseteq S_a$ for all $S_a \in F_1$. Hence S_b is the upper bound of F for the family F_1. So by Zorn's lemma F has a maximal element say S_z . We are going to prove that $z \in Tz$. Suppose $z \notin Tz$ then there exist $\overline{z} \in Tz$ such that

$$d(z,\overline{z}) = d(z,Tz).$$

Let us prove that $S_{\overline{z}} \subseteq S_z$

$$d(\overline{z},T\overline{z}) \le H(Tz,T\overline{z})$$

$$< \max\{d(z,T\overline{z}), d(\overline{z},Tz), d(z,\overline{z})\}$$

$$\leq \max\{\max\{d(z,\overline{z}), d(\overline{z}, T\overline{z})\}, \max\{d(\overline{z}, z), d(z, Tz)\}, d(z, \overline{z})\}$$

$$\leq \max\{d(\overline{z}, T\overline{z}), d(z, Tz), d(z, \overline{z})\}$$

$$= \max\{d(\overline{z}, T\overline{z}), d(z, Tz)\}$$

which is possible only for

$$d(\overline{z}, T\overline{z}) < d(z, Tz)$$

which implies that

$$d(\overline{z}, T\overline{z}) \ < d(z, Tz).$$

Let $y \in S_{\overline{z}}$ implies that

$$d(y, \overline{z}) \leq d(\overline{z}, T\overline{z}) < d(z, Tz)$$

implies that

$$d(y, \overline{z}) < d(z, Tz)$$

As

$$d((y, z)) \leq \max\{d(y, \overline{z}), d(\overline{z}, z)\}$$

$$= d(z, Tz)$$

$y \in S_z$ implies that $S_{\overline{z}} \subsetneq S_z$ as $z \notin S_{\overline{z}}$ which is a contradiction to the maximality of S_z. Hence $z \in Tz$.

Theorem 4 *Let (X, d) a spherically complete ultrametric space. If f and T are self mapping satisfying*

$$TX \subseteq fX \tag{2.2}$$

$$d(Tx, Ty) < \max\{d(fx, fy), d(fx, Ty), d(fy, Tx)\}, x \neq y \tag{2.3}$$

Then there exist $z \in X$ such that $z = Tz$. Further if f and T are coincidentally commuting at z. Then z is the unique common fixed point of f and T.

Proof. *Let $S_a = (fa; d(fa, Ta))$ denote the closed sphere centered at fa with radii $d(fa, Ta)$. Let F be the collection of these sphere for all $a \in X$. Then the relation $S_a \preceq S_b$ iff $S_b \subseteq S_a$ is a partial order on F. Let F_1 be a totally ordered subfamily of F. Since (X, d) is spherically complete, we have*

$$\bigcap_{S_a \in F_1} Sa = S \neq \emptyset$$

Let $fb \in S$ and $S_a \in F_1$ then $fb \in S_a$ hence

$$d(fb, fa) \leq d(fa, Ta) \tag{2.4}$$

if $a = b$ then $S_a = S_b$. Assume that $a \neq b$.Let $x \in S_b$ then

$$d(x, fb) \leq d(fb, Tb)$$

$$\leq \max\{d(fb, fa), d(fa, Ta), d(Ta, Tb)\}$$

$$= \max\{d(fa, Ta), d(Ta, Tb)\}$$

$$< \max\{d(fa, Ta), \max\{d(fa, fb), d(fa, Tb), d(fb, Ta)\}\}$$

As

$$d(fa, Tb) \leq \max\{d(fa, fb), d(fb, Tb)\}$$

and

$$d(fb, Ta) \leq \max\{d(fb, fa), d(fa, Ta)\}$$

Therefore

$$d(x, fb) \leq d(fb, Tb) < \max\{d(fa, Ta), \max\{d(fa, fb), \max\{d(fa, fb), d(fb, Tb)\}, \max\{d(fb, f$$

$$= d(fa, Ta) \qquad (2.5)$$

$$\Rightarrow d(x, fb) \leq d(fa, Ta)$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\}$$

$$\leq d(fa, Ta) \text{ from } (2.4) \text{ and } (2.5)$$

Thus $x \in S_a$. *Hence* $S_b \subseteq S_a$ *for any* $S_a \in F_1$. *Thus* S_b *is an upper bound in* F *for the familyF*$_1$ *and hence by the Zorn's lemma* F *has a maximal element say* S_z, $z \in X$. *We are going to prove that* $fz = Tz$. *Suppose that* $fz \neq Tz$,*since* $Tz \in TX \subseteq fX$,*there exist* $w \in X$ *such that* $Tz = fw$.*Clearly* $z \neq w$.

Now from (2.3) *we have*

$$\begin{aligned} d(fw, Tw) &= d(Tz, Tw) \\ &< \max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\} \\ &= d(fz, fw) \\ &\Rightarrow d(fw, Tw) < d(fz, fw) \end{aligned}$$

Thus $fz \notin S_w$, *hence* $S_z \subsetneq S_w$ *which is a contradiction to the maximality of* S_z, *hence* $fz = Tz$.

Further assume that f *and* T *are coincidently commuting at* z *then*

$$f^2 z = f(fz) = f(Tz) = Tfz = TTz = T^2 z$$

Suppose $fz \neq z$ *from* (2.3) *we have*

$$d\ (Tfz, Tz) < \max\left\{d\left(f^2z, fz\right), d\left(f^2z, Tz\right), d\left(fz, Tfz\right)\right\}$$

$$= d\left(Tfz, Tz\right)$$

$$\Longrightarrow d\left(Tfz, Tz\right) < d\left(Tfz, Tz\right).$$

which is a contradiction so $fz = z$ *thus*

$$fz = Tz = z.$$

For uniqueness let u *be another fixed point such that*

$$u = fu = Tu$$

and $u \neq z$*.Now*

$$d\left(Tz, Tu\right) < \max\left\{d\left(fz, fu\right), d\left(fu, Tz\right), d\left(fz, Tu\right)\right\}$$

$$= \max\left\{d\left(fz, Tz\right), d\left(fz, Tu\right)\right\}$$

$$= \max\left\{d\left(Tz, fz\right), d\left(fz, Tu\right)\right\}$$

$$= d\left(Tz, Tu\right)$$

$$\Longrightarrow d\left(Tz, Tu\right) < d\left(Tz, Tu\right)$$

Which is a contradiction so $u = z$*.* ■

Example 5 *Let* (X, d) *is discrete metric space*

$$d(x,y) = \begin{cases} 0 & if\, x = y \\ \\ 1 & if\, x = y \end{cases}$$

Let $Tx = 2$ *and* $fx = \frac{x+2}{2}$ *has a common fixed point* $x = 2$.

Theorem 6 *Let* (X, d) *a spherically complete ultrametricspace. Let* $f : X \to X$ *and* $T :$ $X \to C(X)$ *be satisfying*

$$Tx \subseteq fX \text{ for all } x \in X \tag{2.6}$$

$$H(Tx, Ty) < \max\{d(fx, fy), d(fx, Ty), d(fy, Tx)\}, \text{for all } x, y \in X, x \neq y \tag{2.7}$$

Then there exist $z \in X$ *such that* $fz \in Tz$.

Further assume that

$$d(fx, fu)\, a \leq H(Tfy, Tu), \text{for all } x, y, u \in X \text{ with } fx \in Ty \tag{2.8}$$

and

$$f \text{ and } T \text{ are coincidentally commuting at } z. \tag{2.9}$$

Then fz *is the unique common fixed point of* f *and* T.

Proof. *Let* $S_a = (fa; d(fa, Ta))$ *denote the closed sphere centered at* fa *with radius* $d(fa, Ta)$ *and let* F *be the collection of these sphere for all* $a \in X$. *Then the* *relation* $S_a \preceq S_b$ *iff* $S_b \subseteq S_a$ *is a partial order on* F. *Let* F_1 *be a totally order subfamily of* F. *Since* (X,d) *is spherically complete, we have*

$$\bigcap_{S_a \in F_1} S_a = S \neq \emptyset.$$

Let $fb \in S$ *and* $S_a \in F_1$, *then* $fb \in S_a$. *Hence*

$$d(fb, fa) \leq d(fa, Ta) \tag{2.10}$$

if $a = b$ *then* $S_a = S_b$. *Assume that* $a \neq b$. *Let* $x \in S_b$ *then*

$$d(x, fb) \leq d(fb, Tb).$$

Since Ta *is compact there exist* $u \in Ta$ *such that*

$$d(fa, u) = d(fa, Ta) \tag{2.11}$$

consider

$$d(fb, Tb) = \inf_{c \in Tb} d(fb, c) \leq \max\left\{d(fb, fa), d(fa, u), \inf_{c \in Tb} d(u, c)\right\}$$

$$\leq \max\{d(fa, Ta), d(Ta, Tb)\} \text{ from } (2.10) \text{ and } (2.11)$$

$$< \max\{d(fa, Ta), d(fb, Tb)\} \text{ from } (2.10) \text{ and } (2.7)$$

thus

$$d(fb, Tb) < d(fa, Ta) \tag{2.12}$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\} \leq d(fa, Ta), \text{from } (2.10) \text{ and } (2.12).$$

Thus $x \in S_a$ *and* $S_b \subseteq S_a$ *for any* $S_a \in F$. *Thus* S_b *is an upper bound in* F *for the family* F_1*and hence Zorn's lemma* F *has a maximal element say* $S_z,\ z \in X$.

We are going to prove that $fz \in Tz$. *Suppose* $fz \notin Tz$,*since* Tz *is compact,there exist* $k \in Tz$ *such that*

$$d(fz, Tz) = d(fz, k)$$

From (2.6) *there exist* $w \in X$ *such that* $k = fw$,*thus*

$$d(fw, Tz) = d(fw, fw) \tag{2.13}$$

Now

$$d(fw, Tw) \leq H(Tz, Tw) < \max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\} = d(fz, fw)$$

Hence $fz \notin S_w$ *thus* $S_z \nsubseteq S_w$ *which is a contradiction to the maximality of* S_z. *Hence* $fz \in Tz$. *Further assume that* (2.8) *and* (2.9) *write* $fz = p$,*then* $p \in Tz$. *from*(2.8) *we have*

$$d(p, fp) = d(fz, fp)$$

$$\leq H(Tfz, Tp) = H(Tp, Tp) = 0$$

implies that $fp = p$.*From*(2.9) *we have*

$$p = fp \in fTz \subseteq Tfz = Tp$$

Thus $fz = p$ *is a common fixed point of* f *and* T. *Suppose* $q \in X, q \neq p$ *is such that* $q = pq \in Tq$ *from* (2.7) *and* (2.8) *we have*

$$d(p, q) = d(fp, fq) \leq H(Tfp, Tq)$$

$$= H(Tp, Tq) < \max\{d(fp, fq), d(fp, Tp), d(fq, Tq)\} = d(p, q)$$

implies $p = q$ thus $p = fz$ is the unique common fixed point of f and T. ∎

Theorem 7 *Let (X, d) be a spherically complete ultrametric space. If $S, T : X \to X$ ara mapping such that*

$$d(Tx, Ty) < \{d(Sx, Sy), d(Sx, TSy), d(Sy, TSx)\} \forall x, y \in X, x \neq y \tag{2.14}$$

$$d(Sx, Sy) < d(x, y) \tag{2.15}$$

$$TSx = STx \; \forall x \in X \tag{2.16}$$

Then S and T have a unique common fixed point in X.

Proof. *Using condition (2.15) and (2.16) in (2.14) we have*

$$d(Tx, Ty) < \max\{d(Sx, Sy), d(Sx, TSy), d(Sy, TSx)\}$$

which implies that

$$\begin{aligned} d(Tx, Ty) &< \max\{d(x, y), d(TSy, Sx), d(TSx, Sy)\} \\ &= \max\{d(x, y), d(STy, Sx), d(STx, Sy)\} \\ &< \max\{d(x, y), d(Ty, x), d(Tx, y)\} \\ &= \max\{d(x, y), d(x, Ty), d(y, Tx)\} \end{aligned}$$

Using theorem(1) T has a unique fixed point .i.e there exist $z \in X$ such that $z = Tz$.

Now we are to prove that $z = Sz$.*Suppose that* $z \neq Sz$

$$\begin{aligned} d(z, Sz) &= d(Tz, STz) = d(Tz, TSz) \\ &< \max\left\{d\left(Sz, S^2z\right), d\left(Sz, TS^2z\right), d\left(S^2z, TSz\right)\right\} \\ &= \max\left\{d\left(Sz, S^2z\right), d\left(Sz, S^2Tz\right), d\left(S^2z, TSz\right)\right\} \\ &= \max\left\{d(z, Sz), d(Sz, SSTz), d(SSz, STz)\right\} \\ &< \max\left\{d(z, Sz), d(z, STz), d(Sz, Tz)\right\} \\ &= \max\left\{d(z, Sz), d(z, Sz), d(Sz, z)\right\} = d(z, Sz) \end{aligned}$$

which implies that

$$d(z, Sz) < d(z, Sz)$$

Which is a contradiction and hence $z = Sz$.

Uniquness: If possible let z *and* w *be two distinct fixed point of* S *and* T *then*

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \\ &< \max\left\{d(Sz, Sw), d(Sz, TSw), d(Sw, TSz)\right\} \\ &= \max\left\{d(z, w), d(z, w), d(w, z)\right\} \\ d(z, w) &< d(z, w) \end{aligned}$$

which is not possible and hence $z = w$. *So* z *is the unique fixed point of* S *and* T.

NOTE: If we put $S = I$ *(identity map) then the theorem*(7) *reduce to theorem* (1). ∎

Theorem 8 *Let* (X, d) *be a spherically complete ultrametric space. If* T, f *and* g *are self maps on* X *satisfying*

$$g(x) \subseteq f(x) \tag{2.17}$$

$$d(g(x), g(y)) < \max\{d(f(Tx), f(Ty)), d(f(Tx), g(Ty)), d(f(Ty), g(Tx))\}, \text{for all } x, y \in X,\ x \neq y \tag{2.18}$$

$$d(Tx, Ty) < d(x, y) \tag{2.19}$$

$$T(f(x)) = f(T(x)) \, and T(g(x)) = g(T(x)), \text{for all } x \in X. \tag{2.20}$$

Then $Tz = fz = gz$. Further if f and g are commutative then there exist a unique common fixed point of T, f and g.

Proof. *Using equation(2.19) and (2.20) equation(2.18) becomes*

$$d(g(x), g(y)) < \max\{d(T(fx), T(fy)), d(T(fx), T(gy)), d(T(fy), T(gx))\}$$

$$\Rightarrow d(g(x), g(y)) < \max\{d(fx, fy), d(fx, gy), d(fy, gx)\}$$

by theorem (4) z is the unique common fixed point of f and g.i.e $z = fz = gz$. Now we are going to prove that $z = Tz$. Suppose $z \neq Tz$,then

$$d(z, Tz) = d(gz, Tgz) = d(gz, gTz) < \max\left\{d\left(fTz, fT^2z\right), d\left(fTz, gT^2z\right), d\left(fT^2z, gTz\right)\right\}$$

$$= \max\left\{d\left(Tfz, T^2fz\right), d\left(Tfz, T^2gz\right), d\left(T^2fz, Tgz\right)\right\}$$

$$= \max\left\{d\left(Tz, T^2z\right), d\left(Tz, T^2z\right), d\left(T^2z, Tz\right)\right\}$$

$$= d\left(Tz, T^2z\right) < d(z, Tz) \text{ from equation (2.19)}$$

$\Rightarrow d(z, Tz) < d(z, Tz)$ which is a contradiction hence $z = Tz$.Thus $z = fz = gz = Tz$.

Uniqueness: Let w be a different fixed point of f, g and T such that $w \neq z$. Then

$$d(z, w) = d(gz, gw) < \max\{d(fTz, fTw), d(fTz, gTw), d(fTw, gTz)\}$$

$$= \max\{d(z,w), d(z,w), d(z,w)\}$$

$$\Rightarrow d(z,w) < d(z,w)$$

which is not possible. Hence $z = w$*.So* z *is the unique common fixed point of* f, g *and* T*.*

Remark:If we put $T = I$ *(identity map) then the theorem*(8) *reduce to theorem* (4) *.i.e* $d(gx, gy) < \max\{d(fx, fy), d(fx, gy), d(fy, gx)\}$. ■

Chapter 3

Fixed point in ultrametric space using F-contraction

In this chapter, we establish some results on coincidence and common fixed points for a single-valued map, a pair of single-valued maps and of single-valued map with a multi-valued map in an Ultrametric space which satisfy F-contraction. Our theorems generalize and extend the theorems of Mishra and Pant[Generalization of some fixed point theorems in ultrametric spaces, Adv. Fixed Point Theory, 4(1)(2014), 41- 47], there by generalizes some known results in the existing literature.

Theorem 9 *Let (X,d) be a spherically complete ultrametric space and let $T : X \to X$ be a single value map such that*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(\max\{d(x,y), d(x,Ty), d(y,Tx)\}), \textit{for all } x,y \in X, F \in \tag{3.1}$$

Then T has a unique fixed point.

Proof. *For $a \in X$ let $S_a = S(a; d(a, Ta))$ denote the closed sphere with center at a and radius $d(a, Ta)$. Let A be the collection of these spheres for all $a \in X$. Then the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally ordered subfamily A_1 of A. Since (X, d) is spherically complete, we have*

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset.$$

Let $b \in S$ and $S_a \in A_1$. If $x \in S_b$ then

$$d(x, b) \leq d(b, Tb) \leq \max\{d(b, a), d(a, Ta), d(Ta, Tb)\}$$

Using equation (3.1) *we have*

$$d(x, b) \leq d(b, Tb) < \max\{d(b, a), d(a, Ta), \max\{d(a, Tb), d(b, Ta), d(a, b)\}\}.$$

As

$$d(a, Tb) \leq \max\{d(a, b), d(b, Tb)\}$$

and

$$d(b, Ta) \leq \max\{d(b, a), d(a, Ta)\}$$

Therefore

$$d(x, b) \leq \max\{d(b, a), d(a, Ta), \max\{d(a, b), d(b, Tb)\}, \max\{d(b, a), d(a, Ta), d(a, b)\}\}$$

$$\leq \max\{d(b, a), d(a, Ta), d(b, Tb)\}$$

$$\leq \max\{d(a, b), d(a, Ta)\}$$

$$= d(a, Ta)$$

which implies

$$d(x,b) \leq d(a,Ta).$$

Now

$$d(x,a) \leq \max\{d(x,b), d(b(,a))\} \leq d(a,Ta)$$

So $x \in S_a \Rightarrow S_b \subseteq S_a$ *for all* $S_a \in A_1$. *Hence* S_b *is an upper bound of* A *for the family* A_1. *So by Zorn's lemma* A *has a maximal element say* S_z *for some* $z \in X$. *We are to prove that* $z = Tz$. *Suppose* $z \neq Tz$. *Using equation*(3.1)

$$\tau + F(d(Tz, TTz)) \leq F(\max\{d(z,Tz), d(z,TTz), d(Tz,Tz)\}) = F\left(\max\left\{d(z,Tz), d\left(z,T^2z\right)\right\}\right)$$

As

$$d(z,TTz) \leq \max\{d(z,Tz), d(Tz,TTz)\}$$

therefore

$$\tau + F(d(Tz,TTz)) \leq F(\max\{d(z,Tz), d(Tz,TTz), d(z,Tz)\})$$

$$\Rightarrow \tau + F(d(Tz,TTz)) \leq F\left(\max\left\{d(z,Tz), d\left(Tz,T^2z\right)\right\}\right)$$

which implies that

$$F\left(d\left(Tz,T^2z\right)\right) < F(d(z,Tz))$$

and since F *is increasing function so*

$$d\left(Tz,T^2z\right) < d(z,Tz).$$

Now if $y \in S_{Tz}$ *then*

$$d(y,Tz) \leq d\left(Tz,T^2z\right) < d(z,Tz)$$

and

$$d(y,z) \leq \max\{d(y,Tz), d(Tz,z)\} = d(z,Tz)$$

which implies that $y \in S_z$.*Hence* $S_{Tz} \subseteq S_z$. *Since* $d(z,Tz) > d(Tz,T^2z)$ *implies* $z \notin S_{Tz}$. *Therefore* $S_{Tz} \subsetneq S_z$ *which is a contradiction to the maximality of* S_z. *Hence* $z = Tz$. *So* z *is the common fixed point of* T.

Uniquness: Let w *be a different fixed point. For* $w \neq z$ *we have from equation* (3.1)

$$\tau + F(d(z,w)) = \tau + F(d(Tz,Tw)) \leq F(\max\{d(z,w), d(z,Tw), d(w,Tz)\}) = F(d(z,w))$$

which implies that

$$F(d(z,w)) < F(d(z,w))$$

$$\Rightarrow d(z,w) < d(z,w),$$

a contradiction therefore $z = w$. *Hence* z *is the unique common fixed point of* T. ∎

Example 10 *Let* $(X = R, d)$ *is a discrete metric space*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Let $Tx = c$ *then* $Tc = c$ *is the fixed point where* c *is any real constant.*

Theorem 11 *Let* (X,d) *be a spherically complete ultrametric space. If* T *and* f *are single value maps on* X *satisfying*

$$T(X) \subseteq f(X) \tag{3.2}$$

$$d(Tx,Ty)>0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(\max\{d(fx,fy), d(fx,Ty), d(fy,Tx)\}) \tag{3.3}$$

for all $x, y \in X, x \neq y$ *where* $F \in \mathcal{F}, \tau > 0$. *Then there exist* $z \in X$ *such that* $fz = Tz$. *Further if* T *and* f *are coincidentally commuting at* z *then* z *is the unique common fixed point of* T *and* f.

Proof. *For* $a \in X$ *let* $S_a = S(fa; d(fa, Ta))$ *denote the closed sphere with center at* fa *and radius* $d(fa, Ta)$. *Let* A *be the collection of these sphere for all* $a \in X$ *then the relation* $S_a \preceq S_b$ *if and only if* $S_b \subseteq S_a$ *is a partial order on* A. *Now consider a totally order subfamily* A_1 *of* A. *Since* (X, d) *is spherically complete, we have*

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset.$$

Let $fb \in S$ *and* $S_a \in A_1$ *then* $fb \in S_a$. *Hence*

$$d(fb, fa) \le d(fa, Ta) \tag{3.4}$$

Using equation (3.3)

$$\tau + F(d(Ta, Tb)) \le F(\max\{d(fa, fb), d(fa, Tb), d(fb, Ta)\})$$

Which implies

$$F(d(Ta, Tb)) < F(\max\{d(fa, fb), d(fa, Tb), d(fb, Ta)\}).$$

Hence

$$d(Ta, Tb) < \max\{d(fa, fb), d(fa, Tb), d(fb, Ta)\} \tag{3.5}$$

If $a = b$ *then* $S_a = S_b$. *Let* $a \neq b$ *and let* $x \in S_b$ *then*

$$d(x, fb) \leq d(fb, Tb) \leq \max\{d(fb, fa), d(fa, Ta), d(Ta, Tb)\}$$

$$= \max\{d(fa, Ta), d(Ta, Tb)\} \text{ from } (3.4)$$

$$< \max\{d(fa, Ta), d(fa, fb), d(fa, Tb), d(fb, Ta)\} = d(fa, Ta)$$

which implies that

$$d(x, fb) \leq d(fa, Ta) \tag{3.6}$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\}$$

$$\leq d(fa, Ta) \text{ from } (3.4) \text{ and } (3.6).$$

Thus $x \in S_a$. *Hence* $S_b \subseteq S_a$ *for any* $S_a \in A_1$. *Thus* S_b *is an upper bound in* A *for the family* A_1 *and hence by the Zorn's lemma* A *has a maximal element say* S_z *for some* $z \in X$. *We are going to prove that* $fz = Tz$. *Suppose* $fz \neq Tz$. *Since* $Tz \in TX \subseteq fX$ *there exist* $w \in X$ *such that* $Tz = fw$, *clearly* $w \neq z$. *Consider*

$$\tau + F(d(fw, Tw)) = \tau + F(d(Tz, Tw))$$

$$\leq F(\max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\})$$

which implies that

$$F(d(fw, Tw)) < F(\max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\}) = F(d(fz, fw))$$

which implies that

$$F(d(fw, Tw)) < F(d(fz, fw))$$

Thus

$$d(fw, Tw) < d(fz, fw)$$

Hence $fz \not\subseteq S_w$. *Therefore* $S_z \not\subseteq S_w$ *which is a contradiction to the maximality of* S_z. *Hence* $fz = Tz$.

Since f *and* T *are coincidentally commuting at* z *then*

$$f^2z = f(fz) = f(Tz) = T(fz) = T^2z$$

Now to show that $fz = z$. *Suppose* $fz \neq z$, *then we have*

$$\tau + F(d(Tfz, Tz)) \leq F\left(\max\left\{d\left(f^2z, fz\right), d\left(f^2z, Tz\right), d(fz, Tfz)\right\}\right)$$

$$= F(d(Tfz, Tz))\,\tau + F(d(Tfz, Tz))$$

$$\leq F(d(Tfz, Tz))$$

implies that

$$F(d(Tfz, Tz)) < F(d(Tfz, Tz))$$

which gives

$$d(Tfz, Tz) < d(Tfz, Tz)$$

a contradiction. Hence $fz = z$.*Thus* $fz = Tz = z$. *Therefore* z *is the common fixed point of* f *and* T.

Uniqueness: Let w *be a different fixed point. For* $w \neq z$ *we have*

$$\tau + F(d(z, w)) = \tau + F(d(Tz, Tw)) \leq F(\max\{d(fz, fw), d(fw, Tz), d(fz, Tw)\})$$

Hence we have

$$F(d(z, w)) < F(\max\{d(fz, fw), d(fw, Tz), d(fz, Tw)\})$$

$$= F(d(z,w))$$

$$\Rightarrow F(d(z,w)) < F(d(z,w))$$

which implies that

$$d(z,w) < d(z,w)$$

which is a contradiction. Therefore $z = w$. Hence z is the unique common fixed point of f and T. ■

Example 12 *Let (X,d) is discrete metric space*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Let $Tx = 3$ and $fx = \frac{x+3}{3}$ has a common fixed point $x = 3$.

Theorem 13 *Let (X,d) be a spherically complete ultrametric space. If $T : X \to 2_C^X$ is a multivalue mapping such that*

$$H(Tx,Ty) > 0 \Rightarrow \tau + F(H(Tx,Ty)) \leq F(\max\{d(x,y), d(x,Ty), d(y,Tx)\}) \tag{3.7}$$

for all $x, y \in X, F \in \mathcal{F}$ $\tau > 0$. Then T has a fixed point on X.

Proof. For $a \in X$ let $S_a = S(a; d(a,Ta))$ denote the closed sphere with center at a and radius $d(a,Ta) = \inf_{z \in Ta} d(a,z) > 0$. Let A he the collection of these sphere,the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally order subfamily A_1 of A. Since (X,d) is spherically complete, we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset$$

Let $b \in S$ and $S_a \in A_1$ 0bviously $b \in S_a$ as $S_a \in A_1$.So

$$d(b,a) \leq d(a,Ta).$$

Take $u \in Ta$ such that

$$d(a,u) = d(a,Ta).$$

(it is possible because Ta is non empty compact set). If $a = b$ then $S_a = S_b$. Assume that $a \neq b$ let $x \in S_b$ which implies that

$$d(x,b) \leq d(b,Tb) \leq \inf_{v \in Tb} d(b,v) \leq \max\left\{d(b,a), d(a,u), \inf_{v \in Tb} d(u,v)\right\}$$

$$\leq \max\{d(a,Ta), H(Ta,Tb)\}$$

Using equation (3.7)

$$d(x,b) \leq d(b,Tb) < \max\{d(a,Ta), \max\{d(a,Tb), d(b,Ta), d(a,b)\}\}$$

As

$$d(a,Tb) \leq \max\{d(a,b), d(b,Tb)\}$$

and

$$d(b,Ta) \leq \max\{d(b,a), d(a,Ta)\}.$$

Therefore

$$d(x,b) \leq \max\{d(a,Ta), \max\{d(a,b), d(b,Tb)\}, \max\{d(b,a), d(a,Ta)\}, d(a,b)\}$$

$$= \max\{d(a,Ta), d(b,a), d(b,Tb)\} = d(a,Ta)$$

$$\Rightarrow d(x,b) \leq d(a,Ta)$$

Now

$$d(x,a) \leq \max\{d(x,b), d(b,a)\} \leq d(a,Ta)$$

$$\Rightarrow d(x,a) \leq d(a,Ta)$$

So $x \in S_a$ and $S_b \subseteq S_a$ for all $S_a \in A_1$. Hence S_b is the upper bound of A for the family A_1. So by Zorn's lemma A has a maximal element say S_z. We are going to prove that $z \in Tz$. Suppose $z \notin Tz$ then there exist $\overline{z} \in Tz$ such that

$$d(z,\overline{z}) = d(z,Tz).$$

Let us prove that $S_{\overline{z}} \subseteq S_z$.

$$d(\overline{z},T\overline{z}) \leq \tau + F(H(Tz,T\overline{z})) \leq F(\max\{d(z,T\overline{z}), d(\overline{z},Tz), d(z,\overline{z})\})$$

$$\leq F(\max\{\max\{d(z,\overline{z}), d(\overline{z},T\overline{z})\}, \max\{d(\overline{z},z), d(z,Tz)\}, d(z,\overline{z})\})$$

$$\leq F(\max\{d(\overline{z},T\overline{z}), d(z,Tz), d(z,\overline{z})\}) = F(\max\{d(\overline{z},T\overline{z}), d(z,Tz)\})$$

$$\Rightarrow d(\overline{z},T\overline{z}) \leq \tau + F(H(Tz,T\overline{z})) \leq F(\max\{d(\overline{z},T\overline{z}), d(z,Tz)\})$$

$$\Rightarrow d(\overline{z},T\overline{z}) \leq F(H(Tz,T\overline{z})) < F(\max\{d(\overline{z},T\overline{z}), d(z,Tz)\})$$

$$\Rightarrow d(\overline{z},T\overline{z}) \leq H(Tz,T\overline{z}) < \max\{d(\overline{z},T\overline{z}), d(z,Tz)\}$$

which is possible only for

$$d(\overline{z},T\overline{z}) < d(z,Tz)$$

this implies that

$$d(\overline{z},T\overline{z}) < d(z,Tz)$$

Let $y \in S_{\overline{z}} \Rightarrow d(y,\overline{z}) \leq d(\overline{z},T\overline{z}) < d(z,Tz) \Rightarrow d(y,\overline{z}) < d(z,Tz)$. As $d(y,z) \leq \max\{d(y,\overline{z}), d(\overline{z},z)\} = d(z,Tz)$, $y \in S_z$ implies that $S_{\overline{z}} \subsetneq S_z$ as $z \notin S_{\overline{z}}$ which is a contradiction to the maximality of S_z. Hence $z \in Tz$. ■

Theorem 14 *Let (X, d) be a spherically complete ultrametric space. If $f : X \to X$ is a single value map and $T : X \rightsquigarrow C(X)$ is a mutivalue map satisfying*

$$Tx \subseteq fX \text{ for all } x \in X \tag{3.8}$$

and

$$H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(\max\{d(fx, fy), d(fx, Ty), d(fy, Tx)\}) \tag{3.9}$$

For all $x, y \in X$, where $F \in \mathcal{F}$, $\tau > 0$.Then there exist $z \in X$ such that $fz \in Tz$.Further if

$$d(fx, fu) \leq H(Tfy, Tu),$$

for all $x, y, u \in X$ with $fx \in Ty$ and f and T are coincidentally commuting at z then Tz is the unique common fixed point of f and T.

Proof. For $a \in X$ let $S_a = S(fa; d(fa, Ta))$ denote the closed sphere with center at fa and radius $d(fa, Ta)$. Let A be the collection of these sphere for all $a \in X$ the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally order subfamily A_1 of A. Since (X, d) is spherically complete, we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset$$

let $fb \in S$ and $S_a \in A_1$ then $fb \in S_a$. Hence

$$d(fb, fa) \leq d(fa, Ta) \tag{3.10}$$

If $a = b$ then $S_a = S_b$. Assume that $a \neq b$ let $x \in S_b$ then $d(x, fb) \leq d(fb, Tb)$. Since Ta is compact there exist $u \in Ta$ such that

$$d(fa, u) = d(fa, Ta) \tag{3.11}$$

Using (3.9)

$$\tau + F(H(Ta,Tb)) \leq F(\max\{d(fa,fb), d(fa,Tb), d(fb,Ta)\})$$

$$\Rightarrow F(H(Ta,Tb)) < F(\max\{d(fa,fb), d(fa,Tb), d(fb,Ta)\})$$

$$\Rightarrow H(Ta,Tb) < \max\{d(fa,fb), d(fa,Tb), d(fb,Ta)\}$$

$$\Rightarrow H(Ta,Tb) < \max\{d(fa,fb), d(fa,fb), d(fb,Tb), d(fb,fa), d(fa,Ta)\}$$

$$= \max\{d(fa,fb), d(fa,Ta), d(fb,Tb)\} = \max\{d(fa,Ta), d(fb,Tb)\} \text{ from (3.8)}$$

$$\Rightarrow H(Ta,Tb) < \max\{d(fa,Ta), d(fb,Tb)\}$$

Now consider

$$d(fb,Tb) = \inf_{c \in Tb} d(fb,c) \leq \max\left\{d(fb,fa), d(fa,u), \inf_{c \in Tb} d(u,c)\right\}$$

$$\leq \max\{d(fa,Ta), d(Ta,Tb)\} \text{ from (3.10) and (3.11)}$$

$$< \max\{d(fa,Ta), d(fb,Tb)\} \text{ from (3.10) and (3.9)}$$

Thus

$$d(fb,Tb) < d(fa,Ta) \tag{3.12}$$

Now

$$d(x,fa) \leq \max\{d(x,fb), d(fb,fa)\} \leq d(fa,Ta) \text{ from (3.10) and (3.12)}.$$

Thus $x \in S_a$. Hence $S_b \subseteq S_a$ for any $S_a \in A_1$. Thus S_b is an upper bound in A for the family A_1 and hence by Zorn's lemma A has a maximal element say S_z, $z \in X$. We are going to prove that $fz \in Tz$. Suppose that $fz \notin Tz$. Since Tz is compact, there exist $k \in Tz$ such that

$$d(fz,Tz) = d(fz,k).$$

Since $Tx \subseteq fX$ so there exist $w \in X$ such that $k = fw$.Thus

$$d(fz, Tz) = d(fz, fw)$$

clearly $w \neq z$ using equation (3.9)

$$\tau + F(H(Tz, Tw)) \leq F(\max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\})$$

$$\Rightarrow F(H(Tz, Tw)) < F(\max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\})$$

which implies that

$$H(Tz, Tw) < \max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\}$$

Now

$$d(fw, Tw) \leq H(Tz, Tw) < \max\{d(fz, fw), d(fz, Tw), d(fw, Tz)\} = d(fz, Tw)$$

$$\Rightarrow d(fw, Tw) < d(fz, Tw).$$

Hence $fz \notin S_w$. Thus $S_z \subsetneqq S_w$ which is a contradiction to the maximality of S_z. Hence $fz \in Tz$.

Now consider

$$d(fz, f^2z) = d(fz, ffz) \leq H(Tfz, Tfz) = 0$$

$$\Rightarrow ffz = fz.$$

Thus

$$fz = ffz \in fTz \subseteq Tfz.$$

Hence fz is the common fixed point of f and T.

Uniqueness: Let Tw be another fixed point such that $fz \neq fw$.Using equation (3.9)

$$\tau + F(H(Tfz, Tw)) \leq F(\max\{d(ffz, fw), d(ffz, Tw), d(fw, Tfz)\})$$

$$\Rightarrow F\left(H\left(Tfz,Tw\right)\right) < F\left(\max\left\{d\left(ffz,fw\right),d\left(ffz,Tw\right),d\left(fw,Tfz\right)\right\}\right)$$

$$\Rightarrow H\left(Tfz,Tw\right) < \max\left\{d\left(ffz,fw\right),d\left(ffz,Tw\right),d\left(fw,Tfz\right)\right\}.$$

Now consider

$$d\left(fz,fw\right) \leq H\left(Tfz,Tw\right) < \max\left\{d\left(ffz,fw\right),d\left(ffz,Tw\right),d\left(fw,Tfz\right)\right\} < d\left(fz,fw\right)$$

$$\Rightarrow d\left(fz,fw\right) < d\left(fz,fw\right)$$

a contradiction so $fz = fw$. Hence fz is the unique common fixed point of f and T.

■

Theorem 15 *Let (X,d) be a spherically complete ultrametric space and let $T : X \rightarrow X$ be a single value map such that*

$$d\left(Tx,Ty\right) > 0 \Rightarrow \tau + F\left(d\left(Tx,Ty\right)\right) \leq F\left(\max\left\{d\left(x,Tx\right),d\left(y,Ty\right)\right\}\right), \text{for all } x,y \in X, \tau > 0 \tag{3.13}$$

Then T has a unique fixed point.

Proof. For $a \in X$ let $S_a = S\left(a; d\left(a,Ta\right)\right)$ denote the closed sphere with center at a and radius $d\left(a,Ta\right)$. Let A be the collection of these spheres for all $a \in X$. Then tbe relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally ordered subfamily A_1 of A. Since (X,d) is spherically complete ,we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset.$$

Let $b \in S \Rightarrow b \in S_a$ as $S_a \in A_1$. Hence

$$d\left(a,b\right) \leq d\left(a,Ta\right).$$

If $a = b$ then $S_a = S_b$. Assume that $a \neq b$.Let $x \in S_b$ then

$$d(x,b) \leq d(b,Tb) \leq \max\{d(b,a), d(a,Ta), d(Ta,Tb)\}$$

$$= \max\{d(a,Ta), d(Ta,Tb)\}.$$

Case1:if

$$d(Ta,Tb) \leq d(a,Ta)$$

then

$$d(x,b) \leq d(a,Ta).$$

Case2:if

$$d(a,Ta) \leq d(Ta,Tb)$$

then

$$d(x,b) \leq d(b,Tb) \leq d(Ta,Tb)$$

$$< \max\{d(a,Ta), d(b,Tb)\} \text{ from } (3.13)$$

If

$$d(b,Tb) < d(a,Ta)$$

then

$$d(x,b) \leq d(a,Ta).$$

And if

$$d(a,Ta) < d(b,Tb),$$

then

$$d(b,Tb) < d(b,Tb),$$

a contradiction. Therefore,

$$d(x,b) \leq d(a,Ta).$$

So $x \in S_a$ hence $S_b \subseteq S_a$ for any S_a in A_1.Thus S_b is the upper bound for the family A_1 in A, and hence by Zorn's lemma A has a maximal element say S_z for some $z \in X$. We are to prove that $z = Tz$. Suppose that $z \neq Tz$. Using equation(3.13)

$$\tau + F\left(d\left(Tz, T^2z\right)\right) \leq F(\max\left\{d(z,Tz), d\left(Tz, T^2z\right)\right\})$$

$$\Rightarrow F\left(d\left(Tz, T^2z\right)\right) < F(\max\left\{d(z,Tz), d\left(Tz, T^2z\right)\right\})$$

$$\Rightarrow F\left(d\left(Tz, T^2z\right)\right) < F\left(d(z,Tz)\right).$$

Since F is increasing function so

$$d\left(Tz, T^2z\right) < d(z,Tz).$$

Now if $y \in S_{Tz}$ then

$$d(y,Tz) \leq d\left(Tz, T^2z\right) < d(z,Tz).$$

And

$$d(y,z) \leq \max\{d(y,Tz), d(Tz,z)\} = d(z,Tz)$$

$$\Rightarrow d(y,z) \leq d(z,Tz)$$

which implies $y \in S_z$. Hence $S_{Tz} \subseteq S_z$.Since $d(z,Tz) > d(Tz, T^2z)$ implies $z \notin S_{Tz}$. Therefore $S_{Tz} \nsubseteq S_z$ which is a contradiction to the maximality of S_z. Hence $z = Tz$ is the common fixed point of T.

Uniqueness: Let w be a different fixed point by equation(3.13) for $w \neq z$ we have

$$\tau + F\left(d(z,w)\right) = \tau + F\left(d(Tz,Tw)\right) \leq F\left(\max\left\{d(z,Tz), d(w,Tw)\right\}\right)$$

$$\Rightarrow F(d(z,w)) < F(\max\{d(z,Tz), d(w,Tw)\})$$

$$\Rightarrow d(z,w) < \max\{d(z,Tz), d(w,Tw)\}$$

$$\Rightarrow d(z,w) < 0$$

Which is a contradiction so $w = z$. Hence z is the unique common fixed point of T.

■

Theorem 16 *Let (X,d) be a spherically complete ultrametric space. If $T : X \to 2^X_C$ is a multivalue mapping such that*

$$H(Tx,Ty) > 0 \Rightarrow \tau + F(H(Tx,Ty)) \leq F(\max\{d(x,Tx), d(y,Ty)\}) \text{ for all } x,y \in X,\ F \in \mathcal{F}, \tau > \tag{3.14}$$

Then T has a fixed point on X.

Proof. For $a \in X$ let $S_a = S(a; d(a,Ta))$ denote the closed sphere with center at a and radius $d(a,Ta)$. Let A be the collection of these spheres for all $a \in X$. Then the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally ordered subfamily A_1 of A. Since (X,d) is spherically complete ,we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset.$$

Let $b \in S \Rightarrow b \in S_a$ as $S_a \in A_1$. Hence

$$d(a,b) \leq d(a,Ta).$$

Let $b \in S \Rightarrow b \in S_a$ as $S_a \in A_1$, hence $d(a,b) \leq d(a,Ta)$. Take $u \in Ta$ such that $d(a,u) = d(a,Ta)$.(it is possible because Ta is non empty compact set). If $a = b$

then $S_a = S_b$. Assume that $a \neq b$, let $x \in S_b$, so

$$d(x,b) \leq d(b,Tb) \leq \inf_{v \in Tb} d(b,v) \leq \max\left\{d(b,a), d(a,u), \inf_{v \in Tb} d(u,v)\right\}$$

$$\leq \max\{d(a,Ta), H(Ta,Tb)\} \leq \max\{d(a,Ta), \max\{d(a,Ta), d(b,Tb)\}\} \text{ from } (3.14)$$

$$= d(a,Ta)$$

$\Rightarrow d(x,b) \leq d(a,Ta)$. So $x \in S_a$ hence $S_b \subseteq S_a$ for any S_a in A_1. Thus S_b is the upper bound for the family A_1 in A, and hence by Zorn's lemma A has a maximal element say S_z for some $z \in X$. We are to prove that $z \in Tz$. Suppose that $z \notin Tz$. Then there exist $\overline{z} \in Tz$ such that

$$d(z,\overline{z}) = d(z,Tz).$$

Now

$$d(\overline{z},T\overline{z}) \leq \tau + F(H(Tz,T\overline{z})) \leq F(\max\{d(z,Tz), d(\overline{z},T\overline{z})\})$$

$$= F(d(z,Tz))$$

$$\Rightarrow d(\overline{z},T\overline{z}) \leq F(H(Tz,T\overline{z})) < F(d(z,Tz))$$

$$\Rightarrow d(\overline{z},T\overline{z}) \leq H(Tz,T\overline{z}) < d(z,Tz)$$

$$\Rightarrow d(\overline{z},T\overline{z}) < d(z,Tz).$$

Let $y \in S_{\overline{z}}$ implies that

$$d(y,\overline{z}) \leq d(\overline{z},T\overline{z}) < d(z,Tz)$$

$$\Rightarrow d(y,\overline{z}) < d(z,Tz).$$

As

$$d(y,z) \leq \max\{d(y,\overline{z}), d(\overline{z},z)\} = d(z,Tz)$$

$$\Rightarrow d(y,z) \leq d(z,Tz)$$

$y \in S_z \Rightarrow S_{\overline{z}} \subsetneq S_z$, as $z \notin S_{\overline{z}}$ which is a contradiction to the maximality of S_z. Hence $z \in Tz$.

Now we extend the idea to pair of junck type mapping. ■

Theorem 17 *Let (X,d) be a spherically complete ultrametric space. If $T$and f are single value mapps on X satisfying*

$$TX \subseteq fX \tag{3.15}$$

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(\max\{d(fa,Ta), d(fb,Tb)\}) \textit{ for all } x,y \in X, x \neq y \tag{3.16}$$

where $F \in \mathcal{F}, \tau > 0$. Then there exist $z \in X$ such that $fz = Tz$. Further if T and f are coincidently commuting at z. Then z is the unique common fixed point of T and f.

Proof. For $a \in X$ let $S_a = S(fa; d(fa,Ta))$ denote the closed sphere with center at fa and radius $d(fa,Ta)$. Let A be the collection of these sphere for all $a \in X$ then the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A.Now consider a totally order subfamily A_1 of A. Since (X,d) is spherically complete, we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset.$$

Let $fb \in S$ and $S_a \in A_1$ then $fb \in S_a$.Hence

$$d(fb, fa) \leq d(fa, Ta) \tag{3.17}$$

Using equation (3.16)

$$\tau + F(d(Ta, Tb)) \leq F(\max\{d(fa, Ta), d(fb, Tb)\})$$

$$\Rightarrow F(d(Ta, Tb)) < F(\max\{d(fa, Ta), d(fb, Tb)\})$$

$$\Rightarrow d(Ta, Tb) < \max\{d(fa, Ta), d(fb, Tb)\} \tag{3.18}$$

If $a = b$ then $S_a = S_b$. Suppose $a \neq b$ let $x \in S_b$

$$\Rightarrow d(x, fb) \leq d(fb, Tb) \leq \max\{d(fb, fa), d(fa, Ta), d(Ta, Tb)\}$$

$$< \max\{d(fb, fa), d(fa, Ta), \max\{d(fa, Ta), d(fb, Tb)\}\} \text{ using } (3.18)$$

$$= d(fa, Ta) \text{ using equation } (3.17) \tag{3.19}$$

$$\Rightarrow d(x, fb) \leq d(fa, Ta).$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\} \leq d(fa, Ta) \text{ from } (3.17) \text{ and } (3.19).$$

Thus $x \in S_a \Rightarrow S_b \subseteq S_a$ for any $S_a \in A_1$. Thus S_a is an upper bound in A for the family A_1 and hence by Zorn's lemma A has a maximal element say S_z for some $z \in X$. We are going to prove that $fz = Tz$.Suppose $fz \neq Tz$.Since $Tz \in TX \subseteq fX$ there exist $w \in X$ such that $Tz = fw$ and $z \neq w$. Now consider

$$\tau + F(d(fw, Tw)) = \tau + F(d(Tz, Tw)) \leq F(\max\{d(fz, Tz), d(fw, Tw)\})$$

$$= F(d(fz, fw))$$

$$\Rightarrow \tau + F(d(fw, Tw)) \leq F(d(fz, fw))$$

$$\Rightarrow F(d(fw, Tw)) < F(d(fz, fw))$$

$$\Rightarrow d(fw, Tw) < d(fz, fw).$$

Let $y \in S_w$ implies that

$$d(y, fw) \leq d(fw, Tw) < d(fz, fw) = d(fz, Tz)$$

$$\Rightarrow d(y, fw) < d(fz, Tz).$$

As

$$d(y, fz) \leq \max\{d(y, fw), d(fw, fz)\} = d(fz, Tz), \text{ as } Tz = fw.$$

So

$$d(y, fz) \leq d(fz, Tz).$$

As

$$d(y, fz) \leq \max\{d(y, fw), d(fw, fz)\} = d(fz, Tz), as\ Tz = fw$$

$$\Rightarrow d(y, fz) \leq d(fz, Tz).$$

As $y \in S_z \Rightarrow S_w \subsetneqq S_z$ as $fz \notin S_w$ which is a contradiction to the maximality of S_z. Hence $Tz = fz$.

Suppose f and T are coincidentally commuting at z then

$$f^2 z = f(fz) = f(Tz) = T(fz) = T^2 z.$$

To show that $fz = z$. Suppose that $fz \neq z$. Now

$$\tau + F(d(Tfz, Tz)) \leq F\left(\max\left\{d\left(f^2 z, Tfz\right), d(fz, Tz)\right\}\right)$$

$$\Rightarrow F\left(d\left(Tfz,Tz\right)\right) < F\left(\max\left\{d\left(f^2z,Tfz\right),d\left(fz,Tz\right)\right\}\right)$$

$$\Rightarrow d\left(Tfz,Tz\right) < \max\left\{d\left(f^2z,Tfz\right),d\left(fz,Tz\right)\right\}$$

$$\Rightarrow d\left(Tfz,Tz\right) < 0$$

Which is a contradiction, hence $fz = z$. Therefore z is the common fixed point of f and T.

Uniqueness: Let w be another fixed point for $w \neq z$, we have

$$\tau + F\left(d\left(z,w\right)\right) = \tau + F\left(d\left(Tz,Tw\right)\right) \leq F\left(\max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}\right)$$

$$\Rightarrow F\left(d\left(Tz,Tw\right)\right) < F\left(\max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}\right)$$

$$\Rightarrow d\left(Tz,Tw\right) < \max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}$$

$$\Rightarrow d\left(Tz,Tw\right) < \max\left\{d\left(Tz,Tz\right),d\left(Tw,Tw\right)\right\} = 0$$

$$\Rightarrow d\left(Tz,Tw\right) < 0$$

Which is a contradiction.which is a contradiction. Therefore $z = w$.Hence z is the unique common fixed point of f and T. ■

Example 18 *Let $(X = R, d)$ is discrete metric space*

$$d\left(x,y\right) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

Let $Tx = 6$ and $fx = \frac{x+6}{6}$ has a common fixed point $x = 6$.

Theorem 19 *Let (X, d) be a spherically complete ultrametric space. If $f : X \to X$ is a single value map and $T : X \to C\left(X\right)$ is a mutivalue map satisfying*

$$Tx \subseteq fX \text{ for all } x \in X \tag{3.20}$$

and

$$H(Tx,Ty) > 0 \Rightarrow \tau + F(H(Tx,Ty)) \leq F(\max\{d(fx,Tx), d(fy,Ty)\}) \quad (3.21)$$

For all $x, y \in X$, where $F \in \mathcal{F}$, $\tau > 0$. Then there exist $z \in X$ such that $fz \in Tz$. Further if

$$d(fx, fu) \leq H(Tfy, Tu),$$

for all $x, y, u \in X$ with $fx \in Ty$ and f and T are coincidentally commuting at z then Tz is the unique common fixed point of f and T.

Proof. For $a \in X$ let $S_a = S(fa; d(fa, Ta))$ denote the closed sphere with center at fa and radius $d(fa, Ta)$. Let A be the collection of these sphere for all $a \in X$ the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A.Now consider a totally order subfamily A_1 of A. Since (X, d) is spherically complete, we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset$$

let $fb \in S$ and $S_a \in A_1$ then $fb \in S_a$.Hence

$$d(fb, fa) \leq d(fa, Ta) \quad (3.22)$$

if $a = b$ then $S_a = S_b$.Assume that $a \neq b$ let $x \in S_b$ then $d(x, fb) \leq d(fb, Tb)$. Since Ta is compact there exist $u \in Ta$ such that

$$d(fa, u) = d(fa, Ta) \quad (3.23)$$

Using (3.21)

$$\tau + F(H(Ta, Tb)) \leq F(\max\{d(fa, Ta), d(fb, Tb)\})$$

$$\Rightarrow F\left(H\left(Ta,Tb\right)\right) < F\left(\max\left\{d\left(fa,Ta\right),d\left(fb,Tb\right)\right\}\right)$$

$$\Rightarrow H\left(Ta,Tb\right) < \max\left\{d\left(fa,Ta\right),d\left(fb,Tb\right)\right\}$$

Now consider

$$d\left(fb,Tb\right) = \inf_{c\in Tb} d\left(fb,c\right) \leq \max\left\{d\left(fb,fa\right),d\left(fa,u\right),\inf_{c\in Tb} d\left(u,c\right)\right\}$$

$$\leq \max\left\{d\left(fa,Ta\right),H\left(Ta,Tb\right)\right\} \text{ from } (3.22) \text{ and } (3.23)$$

$$< \max\left\{d\left(fa,Ta\right),d\left(fb,Tb\right)\right\} \text{ from } (3.22) \text{ and } (3.21)$$

Thus

$$d\left(fb,Tb\right) < d\left(fa,Ta\right) \tag{3.24}$$

Now

$$d\left(x,fa\right) \leq \max\left\{d\left(x,fb\right),d\left(fb,fa\right)\right\} \leq d\left(fa,Ta\right) \text{ from } (3.22) \text{ and } (3.24).$$

Thus $x \in S_a$ hence $S_b \subseteq S_a$ for any $S_a \in A_1$. Thus S_b is an upper bound in A for the family A_1 and hence by Zorn's lemma A has a maximal element say $S_z,\ z \in X$. We are going to prove that $fz \in Tz$.Suppose that $fz \notin Tz$. Since Tz is compact, there exist $k \in Tz$ such that

$$d\left(fz,Tz\right) = d\left(fz,k\right).$$

Since $Tx \subseteq fX$ so there exist $w \in X$ such that $k = fw$.Thus

$$d\left(fz,Tz\right) = d\left(fz,fw\right)$$

clearly $w \neq z$ using equation (3.21)

$$\tau + F\left(H\left(Tz,Tw\right)\right) \leq F\left(\max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}\right)$$

$$\Rightarrow F\left(H\left(Tz,Tw\right)\right) < F\left(\max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}\right)$$

which implies that

$$H\left(Tz,Tw\right) < \max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}$$

$$= \max\left\{d\left(fz,fw\right),d\left(fw,Tw\right)\right\} = d\left(fz,Tw\right)$$

$$\Rightarrow H\left(Tz,Tw\right) < d\left(fz,Tw\right)$$

Now

$$d\left(fw,Tw\right) \leq H\left(Tz,Tw\right) < d\left(fz,Tw\right)$$

$$\Rightarrow d\left(fw,Tw\right) < d\left(fz,Tw\right).$$

Hence $fz \notin S_w$.Thus $S_z \subsetneq S_w$ which is a contradiction to the maximality of S_z. Hence $fz \in Tz$.

Now consider

$$d\left(fz,f^2z\right) = d\left(fz,ffz\right) \leq H\left(Tfz,Tfz\right) = 0$$

$$\Rightarrow ffz = fz.$$

Thus

$$fz = ffz \in fTz \subseteq Tfz.$$

Hence fz is the common fixed point of f and T.

Uniqueness: Let fw be another fixed point such that $fz \neq fw$. Using equation (3.21)

$$\tau + F\left(H\left(Tfz,Tw\right)\right) = \tau + F\left(H\left(Tz,Tw\right)\right) \leq F\left(\max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}\right)$$

$$\Rightarrow F\left(H\left(Tfz,Tw\right)\right) < F\left(\max\left\{d\left(fz,Tz\right),d\left(fw,Tw\right)\right\}\right)$$

$$\Rightarrow H(Tfz, Tw) < \max\{d(fz, Tz), d(fw, Tw)\}.$$

Now consider

$$d(z, w) = d(fz, fw) \leq H(Tfz, Tw) < \max\{d(fz, Tz), d(fw, Tw)\} = 0$$

$$\Rightarrow d(z, w) < 0$$

a contradiction so $fz = fw$.Hence fz is the unique common fixed point of $f$and T.

■

Example 20 *Let $X = (-\infty, \infty)$ endowed with the usual metric and $T : X \to X$ defined by $Tx = x + \frac{1}{1+e^x}$ for all $x \in X$.Note that X is complete and T is contractive mapping but T does not have a fixed point.*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Let $Tx = 3$ and $fx = \frac{x+3}{3}$ has a common fixed point $x = 3$.

Theorem 21 *Let (X, d) be a spherically complete ultrametric space and let $T : X \to X$ be a single value map such that*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\max\{d(x, Ty), d(y, Tx)\}), \text{for all } x, y \in X, \tau > 0 \tag{3.25}$$

Then T has a unique fixed point.

Proof. For $a \in X$ let $S_a = S(a; d(a, Ta))$ denote the closed sphere with center at a and radius $d(a, Ta)$. Let A be the collection of these spheres for all $a \in X$. Then the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally ordered subfamily A_1 of A.Since (X,d) is spherically complete ,we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset.$$

Let $b \in S \Rightarrow b \in S_a$ as $S_a \in A_1$.Hence

$$d(a,b) \leq d(a,Ta).$$

If $a = b$ then $S_a = S_b$. Assume that $a \neq b$.Let $x \in S_b$ then

$$d(x,b) \leq d(b,Tb) \leq \max\{d(b,a), d(a,Ta), d(Ta,Tb)\}$$

$$= \max\{d(a,Ta), d(Ta,Tb)\}.$$

Case1:if

$$d(Ta,Tb) \leq d(a,Ta)$$

then

$$d(x,b) \leq d(a,Ta).$$

Case2:if

$$d(a,Ta) \leq d(Ta,Tb)$$

then

$$d(x,b) \leq d(b,Tb) \leq d(Ta,Tb)$$

$$< \max\{d(a,Tb), d(b,Ta)\} \text{ from } (3.25)$$

As

$$d(a,Tb) \leq \max\{d(a,b), d(b,Tb)\}$$

and

$$d(b,Ta) \le \max\{d(b,a), d(a,Ta)\}.$$

so

$$d(x,b) \le d(b,Tb) \le \max\{\max\{d(a,b), d(b,Tb)\}, \max\{d(b,a), d(a,Ta)\}\}$$

$$= \max\{d(b,a), d(a,Ta), d(b,Tb)\} \le \max\{d(a,Ta), d(b,Tb)\}$$

$$\Rightarrow d(x,b) \le d(b,Tb) \le \max\{d(a,Ta), d(b,Tb)\}$$

if

$$d(b,Tb) \le d(a,Ta)$$

then

$$d(x,b) \le d(a,Ta).$$

if

$$d(a,Ta) < d(b,Tb)$$

then

$$d(x,b) \le d(b,Tb) < d(b,Tb)$$

$$\Rightarrow d(b,Tb) < d(b,Tb)$$

Which is a contradiction, therefore

$$d(x,b) \le d(a,Ta)$$

so $x \in S_b$.Now

$$d(x,a) \le \max\{d(x,b), d(b,a)\} \le \max\{d(x,b), d(a,Ta)\}$$

$$= d(a, Ta)$$

$$\Rightarrow d(x, a) \leq d(a, Ta)$$

So $x \in S_a$ hence $S_b \subseteq S_a$ for any S_a in A_1.Thus S_b is the upper bound for the family A_1 in A, and hence by Zorn's lemma A has a maximal element say S_z for some $z \in X$. We are to prove that $z = Tz$. Suppose that $z \neq Tz$. Using equation(3.25)

$$\tau + F(d(Tz, TTz)) \leq F(\max\{d(z, TTz), d(Tz, Tz)\})$$

As

$$d(z, TTz) \leq \max\{d(z, Tz), d(Tz, TTz)\}$$

Therefore

$$\tau + F(d(Tz, TTz)) \leq F \max\{d(z, Tz), d(Tz, TTz)\}$$

$$= F(d(z, Tz))$$

$$\Rightarrow \tau + F(d(Tz, TTz)) \leq F(d(z, Tz))$$

$$\Rightarrow F(d(Tz, TTz)) < F(d(z, Tz))$$

Since F is increasing function

$$d(Tz, TTz) < d(z, Tz).$$

Now if $y \in S_{Tz}$ then

$$d(y, Tz) \leq d(Tz, TTz) < d(z, Tz).$$

And

$$d(y, z) \leq \max\{d(y, Tz), d(Tz, z)\} = d(z, Tz)$$

$$\Rightarrow d(y,z) \leq d(z,Tz)$$

Which implies $y \in S_z$. Hence $S_{Tz} \subseteq S_z$.Since $d(z,Tz) > d(Tz,T^2z)$ implies $z \notin S_{Tz}$. Therefore $S_{Tz} \subsetneq S_z$ which is a contradiction to the maximality of S_z. Hence $z = Tz$ is the common fixed point of T.

Uniqueness: Let w be a different fixed point by equation(3.25) for $w \neq z$ we have

$$\tau + F(d(z,w)) = \tau + F(d(Tz,Tw)) \leq F(\max\{d(z,Tw), d(w,Tz)\})$$

$$\leq F(\max\{\max\{d(z,w), d(w,Tw)\}, \max\{d(w,z), d(z,Tz)\}\}) = F(d(z,w))$$

$$\Rightarrow \tau + F(d(z,w)) \leq F(d(z,w))$$

$$\Rightarrow F(d(z,w)) < F(d(z,w))$$

therefore

$$d(z,w) < d(z,w)$$

Which is a contradiction. ■

Example 22 *Let (X,d) is discrete metric space*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Let $Tx = c$.Then $Tc = c$ is the common fixed point where c is any real constant.

Theorem 23 *Let (X,d) be a spherically complete ultrametric space. If $T : X \to 2^X_C$ is a multivalue mapping such that*

$$H(Tx,Ty) > 0 \Rightarrow \tau + F(H(Tx,Ty)) \leq F(\max\{d(x,Ty), d(y,Tx)\}) \text{ for all } x,y \in X,\ F \in \mathcal{F}, \tau > 0 \tag{3.26}$$

Then T has a fixed point on X.

Proof. For $a \in X$ let $S_a = S(a; d(a, Ta))$ denote the closed sphere with center at a and radius $d(a, Ta)$. Let A be the collection of these spheres for all $a \in X$. Then the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally ordered subfamily A_1 of A. Since (X, d) is spherically complete ,we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset.$$

Let $b \in S \Rightarrow b \in S_a$ as $S_a \in A_1$. Hence

$$d(a, b) \leq d(a, Ta).$$

Take $u \in Ta$ such that $d(a, u) = d(a, Ta)$.(it is possible because Ta is non empty compact set). If $a = b$ then $S_a = S_b$. Assume that $a \neq b$, let $x \in S_b$, so

$$d(x, b) \leq d(b, Tb) \leq \inf_{v \in Tb} d(b, v) \leq \max \left\{ d(b, a), d(a, u), \inf_{v \in Tb} d(u, v) \right\}$$

$\leq \max\{d(a, Ta), H(Ta, Tb)\} \leq \max\{d(a, Ta), \max\{d(a, Tb), d(b, Ta)\}\}$ from (3.26)

As

$$d(a, Tb) \leq \max\{d(a, b), d(b, Tb)\}$$

and

$$d(b, Ta) \leq \max\{d(b, a), d(a, Ta)\}$$

therefore

$$d(x, b) \leq \max\{d(a, Ta), \max\{d(a, b), d(b, Tb)\}, \max\{d(b, a), d(a, Ta)\}\}$$

$$= \max\{d(b,a), d(a,Ta), d(b,Tb)\} = d(a,Ta)$$

$$\Rightarrow d(x,b) \leq d(a,Ta).$$

Now

$$d(x,a) \leq \max\{d(x,b), d(b,a)\} \leq d(a,Ta)$$

$$\Rightarrow d(x,a) \leq d(a,Ta)$$

So $x \in S_a$ hence $S_b \subseteq S_a$ for any S_a in A_1. Thus S_b is the upper bound for the family A_1 in A, and hence by Zorn's lemma A has a maximal element say S_z for some $z \in X$. We are to prove that $z \in Tz$. Suppose that $z \notin Tz$. Then there exist $\overline{z} \in Tz$ such that

$$d(z,\overline{z}) = d(z,Tz).$$

Now

$$d(\overline{z},T\overline{z}) \leq \tau + F(H(Tz,T\overline{z})) \leq F(\max\{d(z,T\overline{z}), d(\overline{z},Tz)\})$$

$$\leq F(\max\{\max\{d(z,\overline{z}), d(\overline{z},T\overline{z})\}, \max\{d(\overline{z},z), d(z,Tz)\}\})$$

$$\leq F(\max\{d(z,Tz), d(\overline{z},T\overline{z})\}) = F(d(z,Tz))$$

This implies

$$d(\overline{z},T\overline{z}) < F(H(Tz,T\overline{z})) < F(d(z,Tz))$$

$$\Rightarrow d(\overline{z},T\overline{z}) < H(Tz,T\overline{z}) < d(z,Tz)$$

$$\Rightarrow d(\overline{z},T\overline{z}) < d(z,Tz).$$

Let $y \in B_{\overline{z}}$ implies that

$$d(y,\overline{z}) \leq d(\overline{z},T\overline{z}) < d(z,Tz)$$

$$\Rightarrow d(y,\overline{z}) < d(z,Tz).$$

As

$$d(y,z) \le \max\{d(y,\overline{z}), d(\overline{z},z)\} = d(z,Tz)$$

$$\Rightarrow d(y,z) \le d(z,Tz)$$

$y \in S_z \Rightarrow S_{\overline{z}} \subsetneqq S_z$, as $z \notin S_{\overline{z}}$ which is a contradiction to the maximality of S_z. Hence $z \in Tz$.

Now we extrnd the idea to pair of junck type mapping. ■

Theorem 24 *Let (X,d) be a spherically complete ultrametric space. If T and f are single value mapps on X satisfying*

$$TX \subseteq fX \tag{3.27}$$

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(\max\{d(fa,Tb), d(fb,Ta)\}) \text{ for all } x,y \in X, x \ne y \tag{3.28}$$

where $F \in \mathcal{F}, \tau > 0$. Then there exist $z \in X$ such that $fz = Tz$. Further if T and f are coincidently commuting at z. Then z is the unique common fixed point of T and f.

Proof. For $a \in X$ let $S_a = S(fa; d(fa,Ta))$ denote the closed sphere with center at fa and radius $d(fa,Ta)$. Let A be the collection of these sphere for all $a \in X$ then the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally order subfamily A_1 of A.Since (X,d) is spherically complete, we have

$$\bigcap_{S_a \in A_1} S_a = B \ne \emptyset.$$

Let $fb \in S$ and $S_a \in A_1$ then $fb \in S_a$.Hence

$$d(fb, fa) \leq d(fa, Ta) \tag{3.29}$$

Using equation (3.28)

$$\tau + F(d(Ta, Tb)) \leq F(\max\{d(fa, Tb), d(fb, Ta)\})$$

$$\Rightarrow F(d(Ta, Tb)) < F(\max\{d(fa, Tb), d(fb, Ta)\})$$

$$\Rightarrow d(Ta, Tb) < \max\{d(fa, Ta), d(fb, Tb)\} \tag{3.30}$$

If $fa = fb$ then $S_a = S_b$.Suppose$fa \neq fb$ let $x \in S_b$

$$\Rightarrow d(x, fb) \leq d(fb, Tb) \leq \max\{d(fb, fa), d(fa, Ta), d(Ta, Tb)\}$$

$$< \max\{d(fb, fa), d(fa, Ta), \max\{d(fa, Tb), d(fb, Ta)\}\} \text{ using } (3.30)$$

As

$$d(fa, Tb) \leq \max\{d(fa, fb), d(fb, Tb)\}$$

and

$$d(fb, Ta) \leq \max\{d(fb, fa), d(fa, Ta)\}.$$

Therefore

$$d(x, fb) \leq d(fb, Tb) \leq \max\{d(fb, fa), d(fa, Ta), \max\{\max\{d(fa, fb), d(fb, Tb)\}, \max\{d(fb, fa), d(\ldots$$

$$= d(fa, Ta)$$

$$\Rightarrow d(x, fb) \leq d(fa, Ta)$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\} \leq d(fa, Ta)$$

Thus $x \in S_a \Rightarrow S_b \subseteq S_a$ for any $S_a \in A_1$. Thus S_a is an upper bound in A for the family A_1 and hence by Zorn's lemma A has a maximal element say S_z for some $z \in X$. We are going to prove that $fz = Tz$.Suppose $fz \neq Tz$. Since $Tz \in TX \subseteq fX$ there exist $w \in X$ such that $Tz = fw$ and $z \neq w$.Now consider

$$\tau + F(d(fw,Tw)) = \tau + F(d(Tz,Tw)) \leq F(\max\{d(fz,Tw), d(fw,Tz)\})$$

As

$$d(fz,Tw) \leq \max\{d(fz,fw), d(fw,Tw)\}$$

So

$$\tau + F(d(fw,Tw)) \leq F(\max\{\max\{d(fz,fw), d(fw,Tw)\}, d(Tz,Tz)\})$$

$$\Rightarrow F(d(fw,Tw)) < F(\max\{d(fz,fw), d(fw,Tw)\})$$

$$\Rightarrow d(fw,Tw) < \max\{d(fz,fw), d(fw,Tw)\} = d(fz,fw)$$

$$\Rightarrow d(fw,Tw) < d(fz,fw).$$

Let $y \in S_w$ implies that

$$d(y,fw) \leq d(fw,Tw) < d(fz,fw) = d(fz,Tz)$$

$$\Rightarrow d(y,fw) < d(fz,Tz).$$

As

$$d(y,fz) \leq \max\{d(y,fw), d(fw,fz)\} = d(fz,Tz), \text{ as } Tz = fw.$$

So

$$d(y,fz) \leq d(fz,Tz).$$

As

$$d(y, fz) \leq \max\{d(y, fw), d(fw, fz)\} = d(fz, Tz), as\ Tz = fw$$

$$\Rightarrow d(y, fz) \leq d(fz, Tz).$$

As $y \in S_z \Rightarrow S_w \subsetneqq S_z$ as $fz \notin S_w$ which is a contradiction to the maximality of S_z. Hence $Tz = fz$. Suppose f and T are coincidentally commuting at z then

$$f^2 z = f(fz) = f(Tz) = T(fz) = T^2 z.$$

To show that $fz = z$. Suppose that $fz \neq z$.Now

$$\tau + F(d(Tfz, Tz)) \leq F\left(\max\left\{d\left(f^2 z, Tz\right), d(fz, Tfz)\right\}\right)$$

$$\Rightarrow F(d(Tfz, Tz)) < F\left(\max\left\{d\left(f^2 z, Tz\right), d(fz, Tfz)\right\}\right)$$

$$\Rightarrow d(Tfz, Tz) < \max\left\{d\left(f^2 z, fz\right), d\left(fz, f^2 z\right)\right\}$$

$$= d\left(f^2 z, f^2 z\right) = 0$$

$$\Rightarrow d(Tfz, Tz) < 0$$

Which is a contradiction, hence $fz = z$. Therefore z is the common fixed point of f and T. Uniqueness:Let w he another fixed point for $w \neq z$, we have

$$\tau + F(d(z, w)) = \tau + F(d(Tz, Tw)) \leq F(\max\{d(fz, Tw), d(fw, Tz)\})$$

$$\Rightarrow F(d(z, w)) < F(\max\{d(fz, Tw), d(fw, Tz)\})$$

$$\Rightarrow d(z, w) < \max\{d(fz, Tw), d(fw, Tz)\}$$

$$= \max\{d(z, w), d(w, z)\} = d(z, z) = 0$$

$$\Rightarrow d(z, w) < 0$$

Which is a contradiction.So z is the unique common fixed point of f and T. ∎

Example 25 *Let $(X = R, d)$ is discrete metric space*

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

Let $Tx = 3$ and $Sx = \frac{x+3}{3}$ has a common fixed point $x = 3$.

Theorem 26 *Let (X, d) be a spherically complete ultrametric space.If $f : X \to X$ is a single value map and $T : X \to C(X)$ is a mutivalue map satisfying*

$$Tx \subseteq fX \text{ for all } x \in X \tag{3.31}$$

and

$$H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(\max\{d(fx, Ty), d(fy, Tx)\}) \tag{3.32}$$

for all $x, y \in X$, where $F \in \mathcal{F}$, $\tau > 0$.Then there exist $z \in X$ such that $fz \in Tz$.Further if

$$d(fx, fu) \leq H(Tfy, Tu),$$

for all $x, y, u \in X$ with $fx \in Ty$ and f and T are coincidentally commuting at z then Tz is the unique common fixed point of f and T.

Proof. For $a \in X$ let $S_a = S(fa; d(fa, Ta))$ denote the closed sphere with center at fa and radius $d(fa, Ta)$. Let A be the collection of these sphere for all $a \in X$ the relation $S_a \preceq S_b$ if and only if $S_b \subseteq S_a$ is a partial order on A. Now consider a totally order subfamily A_1 of A. Since (X, d) is spherically complete, we have

$$\bigcap_{S_a \in A_1} S_a = S \neq \emptyset$$

let $fb \in S$ and $S_a \in A_1$ then $fb \in S_a$..Hence

$$d(fb, fa) \leq d(fa, Ta) \tag{3.33}$$

if $fa = fb$ then $S_a = S_b$.Assume that $fa \neq fb$ let $x \in S_b$ then $d(x, fb) \leq d(fb, Tb)$ .Since Ta is compact there exist $u \in Ta$ such that

$$d(fa, u) = d(fa, Ta) \tag{3.34}$$

using (3.32)

$$\tau + F(H(Ta, Tb)) \leq F(\max\{d(fa, Tb), d(fb, Ta)\})$$

$$\Rightarrow F(H(Ta, Tb)) < F(\max\{d(fa, Tb), d(fb, Ta)\})$$

$$\Rightarrow H(Ta, Tb) < \max\{d(fa, Tb), d(fb, Ta)\}$$

Now consider

$$d(fb, Tb) = \inf_{c \in Tb} d(fb, c) \leq \max\left\{d(fb, fa), d(fa, u), \inf_{c \in Tb} d(u, c)\right\}$$

$$\leq \max\{d(fa, Ta), H(Ta, Tb)\} \text{ from (3.33) and (3.34)}$$

$$< \max\{d(fa, Ta), d(fa, Tb), d(fb, Ta)\}$$

$$\Rightarrow d(x, fb) \leq d(fb, Tb) < \max\{d(fa, Ta), d(fa, Tb), d(fb, Ta)\}$$

As

$$d(fa, Tb) \leq \max\{d(fa, fb), d(fb, Tb)\}$$

and

$$d(fb, Ta) \leq \max\{d(fb, fa), d(fa, Ta)\}$$

Therefore

$$d(x, fb) \leq \max\{d(fa, Ta), \max\{d(fa, fb), d(fb, Tb)\}, \max\{d(fb, fa), d(fa, Ta)\}\}$$

$$= d(fa, Ta)$$

$$\Rightarrow d(x, fb) \leq d(fa, Ta)$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\} \leq d(fa, Ta)$$

$$\Rightarrow d(x, fa) \leq d(fa, Ta)$$

Thus $x \in S_a$ hence $S_b \subseteq S_a$ for any $S_a \in A_1$. Thus S_b is an upper bound in A for the family A_1 and hence by Zorn's lemma A has a maximal element say $S_z,\ z \in X$. We are going to prove that $fz \in Tz$. Suppose that $fz \notin Tz$. Since Tz is compact, there exist $k \in Tz$ such that

$$d(fz, Tz) = d(fz, k).$$

Since $Tx \subseteq fX$ so there exist $w \in X$ such that $k = fw$. Thus

$$d(fz, Tz) = d(fz, fw)$$

clearly $w \neq z$ using equation (3.32)

$$\tau + F(H(Tz, Tw)) \leq F(\max\{d(fz, Tw), d(fw, Tz)\})$$

$$\Rightarrow F(H(Tz, Tw)) < F(\max\{d(fz, Tw), d(fw, Tz)\})$$

which implies that

$$H(Tz, Tw) < \max\{d(fz, Tw), d(fw, Tz)\}$$

$$\leq \max\{\max\{d(fz,fw),d(fw,Tw)\},\max\{d(fw,fz),d(fz,Tz)\}\}$$

$$= \max\{d(fz,fw),d(fz,Tz)\} = d(fz,Tz)$$

$$H(Tz,Tw) < d(fz,Tz)$$

Now

$$d(fw,Tw) \leq H(Tz,Tw) < d(fz,Tz).$$

$$\Rightarrow d(fw,Tw) < d(fz,Tz)$$

Let $y \in S_w$ then

$$d(y,fw) \leq d(fw,Tw) < d(fz,Tz)$$

As

$$d(y,fz) \leq \max\{d(y,fw),d(fw,fz)\} = d(fz,Tz)$$

$$\Rightarrow d(y,fz) \leq d(fz,Tz)$$

Where $y \in S_z$ implies that $S_z \subsetneqq S_w$ as $fz \notin S_w$ which is a contradiction to the maximality of S_z. Hence $fz \in Tz$.

Now consider

$$d(fz,f^2z) = d(fz,ffz) \leq H(Tfz,Tfz) = 0$$

$$\Rightarrow ffz = fz.$$

Thus

$$fz = ffz \in fTz \subseteq Tfz.$$

Hence fz is the common fixed point of f and T.

Uniqueness: Let fw be another fixed point such that $fz \neq fw$.Using equation (3.32)

$$\tau + F(H(Tfz,Tw)) = \tau + F(H(Tz,Tw)) \leq F(\max\{d(fz,Tw),d(fw,Tz)\})$$

$$\Rightarrow F\left(H\left(Tfz, Tw\right)\right) < F\left(\max\left\{d\left(fz, Tw\right), d\left(fw, Tz\right)\right\}\right)$$

$$\Rightarrow H\left(Tfz, Tw\right) < \max\left\{d\left(fz, Tw\right), d\left(fw, Tz\right)\right\}.$$

Now consider

$$d\left(z, w\right) = d\left(fz, fw\right) \leq H\left(Tfz, Tw\right) < \max\left\{d\left(fz, Tw\right), d\left(fw, Tz\right)\right\} = d\left(z, w\right)$$

$$\Rightarrow d\left(z, w\right) < d\left(z, w\right)$$

a contradiction so $fz = fw$. Hence fz is the unique common fixed point of f and T.

■

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