A New Odd Generalized X-Family of Distributions: Statistical Aspects and Applications

By

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A Thesis
Submitted in the Partial Fulfillment of the
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MASTER OF SCIENCE
In
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CERTIFICATE

A New Odd Generalized X-Family of Distribution:

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A dissertation submitted in the partial fulfillment of the requirements for the degree of Master of Science in STATISTICS We accept this dissertation as conforming to the required standard

1.――――――――――	2.――――――――――
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Declaration

I, Sumyyah Khaliq, hereby sincerely certify that the work contained in this dissertation, *"A New Odd Generalized X-Family of Distributions: Statistical Aspects and Applications"* is unique and otherwise credited. No other university in Pakistan or abroad has submitted this work in full or in part for another degree.

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Thesis Completion Certificate From Supervisor

The thesis entitled "A New Odd Generalized X-Family of Distribution: Statistical Aspects and Applications" the research work of MS degree in Statistics submitted by Sumyyah Khaliq, 197-FBAS/MSST/F22 has been completed under my direction and supervision. I allow her to submit this thesis for further process in order to get a Master of Science degree from the department of Mathematics and Statistics in accordance with IIUI rules and regulations because I am satisfied with the quality of her work.

Dr. Maryam Siddiqa
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Acknowledgement

With the grace of Allah Almighty, I have experienced the blessings of Allah Almighty to explore the abilities and opportunities in the way of learning. i am thankful to Allah Almighty for changing the impossible to possible. I always seek His constant guidance in every moment of my life.

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I am greatful from my heart to my siblings for their constant motivation and encouragement, and support, to bring this research in the light of the day.

SUMYYAH KHALIQ

Dedicated

To

My Mother

Kaneez Fatima

(A strong and gentle soul who is the center of my happiness, who taught me to trust in Allah, to believe in hard work, and that a little can be a lot.)

My Father

Abdul Khaliq(Deceased)

(In hazy memories of my father who died when i was a child. This thesis is a tribute to your constant presence in my life.♡)

Abstract

Statistical distributions systematically analyze variability in real-world problems by providing suitable models. To address limitations in existing distributions, numerous new distributions have been developed and are continuously studied. Researchers are dedicating significant efforts to modifying these well-known distributions to more accurately capture the complex nature of data and enhance their efficacy. In the similar vein, Odd Generated distributions have their own importance in modeling the real life data in these days due to limited resources, time and expenses. In view of that, the goal of the current research proposal is to derive more flexible and valid form of current classical Dagum distribution by combining with a new generator Sin Exponentiated Odd Generalized Pareto.

In this research work, A new more flexible distribution known as"Sin Exponentiated Odd Generalized Pareto X family of distribution (SEOGPX)" has been proposed. A novel and more versatile generator is introduced in the initial approach. In the subsequent approach, this new generator is applied to the baseline distribution to develop an extension of the Dagum distribution. By considering both empirical data sets and simulated data, this research have extensively studied various statistical characteristics, including the quantile function, moments, incomplete r^{th} moment, hazard function, survival function, order statistics, and Rényi entropy. The SEOGPD model exhibited significant flexibility in modeling the dataset compared to other distributions evaluated in the study.

Nomenclature

- AIC Akaike information criterion
- BIC Bayesian information criterion
- CAIC Consistent Akaike information criterion
- CDF Cumulative density function
- D Dagum distribution
- ED Exponentiated Dagum
- lD Lomax Dagum
- McD McDonald Dagum
- MSE Mean square error
- MLE Maximum likelihood Estimation
- MGF Moment generating function
- OLD Odd Lomax Dagum
- PDF Probability density function
- SEOGPD Sin Exponentiated Odd Generalized Pareto Dagum
- SEOGPG Sin Exponentiated Odd Generalized Pareto Gumbel

- SEOGPL Sin Exponentiated Odd Generalized Pareto Logistic
- TLD Topp-Leone Dagum
- WD Weighted Dagum

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Historical Background

Continuous probability distributions play a crucial role in statistical theory and applications, providing a framework for modeling and analyzing random variables that can take on an infinite number of values within a specified range. In the realm of continuous probability distributions, the concept of a probability density function (PDF) takes center stage. The PDF is function that describes the likelihood of a continuous random variable falling within a particular interval.

Probability distributions in survival analysis are fundamental for modeling the time until the occurrence of an event of interest, such as death, disease onset, or equipment failure. Commonly used distributions include the exponential distribution, which assumes a constant hazard rate and is characterized by its simplicity and memoryless property. The Weibull distribution is more flexible, accommodating increasing or decreasing hazard rates over time, depending on its shape parameter. The log-normal distribution, which models situations where the logarithm of survival time follows a normal distribution, is suitable for data with a long right tail. Another important distribution is the Gompertz distribution, often used in demography for modeling human mortality. These distributions provide different ways to estimate and interpret the survival function, hazard function, and other related metrics, enabling more accurate predictions and insights into the survival data. By selecting an appropriate distribution, researchers can better understand the underlying mechanisms influencing the time-to-event and make more informed decisions in fields such as medicine, engineering, and social sciences.

Generators in the context of statistical distributions are powerful tools used to modify existing probability distributions, enhancing their flexibility to better fit complex real-world data. These generators transform baseline distributions can capture a wider range of behaviors and patterns in the data. By employing the generators, statisticians and researchers can develop more nuanced and accurate statistical models and distributions, enabling a deeper understanding of the underlying processes and improving predictive power across diverse applications in medicine, engineering, and social sciences. Eugene et al.,(2002) [4] proposed beta generated (G) family of distributions, Zografos and Balakrishnan(2009) [5] proposed gamma-G family, Cordeiro and de Castro(2011) [6] defined Kumaraswamy-G family, Bourguignon et al.,(2014) [8] introduced weibull-G family, Al-Shomrani et al.,(2016) [15] proposed TL family of distributions and Cordeiro et al.(2019) [18] introduced the OD-G family of distributions.

The use of trigonometric functions to enhance the flexibility of existing distributions is a relatively recent and innovative approach in statistical modeling and survival analysis. Trigonometric functions, such as sine and cosine, offer a high degree of mathematical adaptability, allowing for smooth and continuous transformations of baseline distributions. By integrating these functions, researchers can introduce periodic components and oscillatory behavior into the models, which can be particularly useful for capturing cyclic patterns and seasonality in the data. This method can significantly

improve the properties of existing distributions, such as their shape and tail behavior, leading to better fitting models that more accurately reflect the complexities of real-world phenomena.
Modern statistical developments are in, e.g. Kumar et al.,(2015) [13]; Jamal and Chesneau(2019) [19] and Souza et al.,(2019) [21]. In specifically, Souza et al.,(2019) [21] established a new Sin-G distribution with the CDF ;

$$D_l(x) = \sin\left\{\frac{\pi}{2}L(x)\right\}, \quad x \epsilon R$$

Where $L(x)$ is baseline distribution function of any distribution and Souza et al.,(2019) [21] also proposed the new Cos-G class. They both are very simple trigonometric classes , these both classes have a remarkable degree of flexibility in statistical modeling due to the smoothly occurring periods of the trigonometric functions. This feature is illustrated by the development of the Cos-G and Sin-G class of trigonometric distribution, with favorable outcomes in comparison to useful model competitors.

1.2 Preliminaries

1.2.1 Odd Exponential Generalized Distribution

Tahir et al.,(2015) [14] proposed the new class of distribution called the odd generalized exponential family. Moreover, in recent years, some new generators of distributions based on the exponential distribution such as the OGE distributions and OEG distributions were proposed and analyzed by Tahir et al.,(2015) [14].
The Cumulative Distribution Function (CDF) of OGE family is given as:

$$F(x) = \left(1 - e^{-\eta\frac{G(x,\kappa)}{\bar{G}(x,\kappa)}}\right)^{\tau} \quad {}_{\eta,\kappa>0}^{x>0,} \tag{1.2.1}$$

Where $\bar{G}(x,\kappa) = 1 - G(x,\kappa)$ is the survival function and $G(x)$ is the CDF of a continuous type distribution.
Then the corresponding probability density function is:

$$f(x) \quad = \quad \frac{\eta\tau g(x,\kappa)}{\bar{G}(x,\kappa)}\left(1 - e^{-\eta\frac{G(x,\kappa)}{\bar{G}(x,\kappa)}}\right)^{\tau-1} e^{-\eta\frac{G(x,\kappa)}{\bar{G}(x,\kappa)}} \quad {}_{\substack{x>0,\\ \eta,\kappa>0}} \tag{1.2.2}$$

Where $f(x)$ is the Probability Density Function (PDF) of a continuous type distribution and η and τ are two additional parameters.

1.2.2 Generalized Pareto Distribution

The Generalized Pareto Distribution (GPD) was introduced by Maurice Fréchet, a French mathematician, in the early $20th$ century. Fréchet made significant contributions to the field of probability and statistics, and he introduced the GPD as a way to model extreme values in a distribution.
The CDF of GPD is;

$$Q_G(z;\hbar) \quad = \quad 1 - \{1 + D(z;\hbar)\}^{-\frac{1}{\hbar}}$$

where $D(z;\hbar)$ is the CDF of any contiunous distribution and $'\hbar'$ is the shape parameter.
While the corresponding Probability Density Function (PDF) is given by:

$$q_G(z;\hbar) = \frac{1}{\hbar}\{1 + D(z;\hbar)\}^{-\frac{1}{\hbar}-1} d(z;\hbar)$$
$$z = \frac{x-\mu}{\sigma} \quad {}_{\substack{x>\mu,\\ \mu(-\infty,\infty)}}$$

where $d(z;\hbar)$ is the PDF of any contiunous distribution and $'\mu'$ and $'\sigma'$ are the location and scale parameters.respectively.

The Generalized Pareto Distribution (GPD) is a probability distribution used in extreme value theory to model the tails of a distribution. It is often employed to assess and model extreme events, such as unusually large values in a dataset. The GPD is commonly applied in fields such as finance, meteorology, and hydrology to estimate the probability of extreme events, such as floods, high wind speeds, or financial market crashes. The choice of the GPD for modeling extreme values is often motivated by its ability to capture the tail behavior of the distribution, allowing for a more accurate estimation of extreme quantiles.

1.2.3 Dagum Distribution

The Dagum distribution is a continuous probability distribution. The distribution was first introduced in the early 1970s Dagum,Camilo (1977) [3], it is named after Camilo Dagum, who proposed it in a series of papers in the 1970s.

- the shape parametr(p)
- the scale parametr(a)
- the location parameter(b)

The CDF of introduced distribution is:

$$D(z;\hbar) = \left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}$$

While the corresponding PDF is given by:

$$d(z;\hbar) = \frac{ap}{z}\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^{a}+1\right\}^{p+1}}\right]_{\substack{a,b,p>0,\\ x>0}}$$

where a, b and p are the scale ,location and shape parameters respectively.
The Dagum distribution is employed in various fields, including economics, finance, and reliability engineering, to model data with heavy tails and positive skewness. It is particularly useful for analyzing income distribution, lifetime data, and other phenomena where extreme values play a significant role. The Dagum distribution can be used to model the distribution of lifetimes or durations of products, making it relevant in certain contexts related to reliability and survival analysis. In applications such as reliability engineering or actuarial science, where the focus is on the lifespan of components, products, or individuals, the Dagum distribution may be employed to describe the distribution of time until failure or other relevant events.

1.2.4 Validity

The validity of a probability density function (PDF) is vitally important to ensure that the function meets the necessary benchmark for a valid probability distribution. A valid PDF must satisfy two main conditions:

- Probability Density Fuction of distribution must be positive
- Integral of PDF must be equal to 1

the validity is mathematically defined as:

$$\int_{-\infty}^{\infty} f(w)dw = 1$$

where $f(w)$ is the PDF of distribution.Estimate the validity of PDF verify that the function accurately represents a probability function.

1.2.5 Survival Function

Survival function is the branch of statistics and its find extensive application in the analysis and modeling of time-to-event data, encompassing the lifespan of products, the duration until a system failure, or the time until a medical event occurs.Survival analysis deals with the study of time until an event of interest occurs, such as the failure of a machine, the occurrence of a disease, or death.The survival function is denoted as $S(w)$. Survival functions are commonly employed in a range of areas, such as medical research, engineering, and social sciences, to study and simulate time-to-event data.
The survival function $S(w)$ is defined as;

$$S(w) = P(W > w) = 1 - F(w)$$

where $'P'$ is the probability $,'w'$ is the some time and and $'W'$ is the random variable denoting the time of death.As time increases the survival function can be increasing or decreasing and approaches to zero.

1.2.6 Hazard Function

The hazard function denotes the rate of failure occurring at a specific moment, taking into account the survival until that moment.The hazard function is mathematically defined as the ration of the PDF$f(w)$ of an event occurring in a small time interval around time t to the length of that time interval S(w).
The hazard function is defined as:

$$h(w) = \frac{f(w)}{s(w)} \quad where \quad S(w) = 1 - F(w)$$

where $h(w)$denotes the hazard function ,$f(w)$ is thr probability density function and $S(w)$ is the survival function.The hazard rate is a measure of tendency to fail, if the value of the hazard function is large, there will be greater probability of failure.

1.2.7 Quantile Function

A distribution can be effectively described by employing the quantile function, which serves as a significant measure.The summarization of the spread and central tendency of a data-set is an invaluable tool, offering significant benefits in numerous scenarios.The Quantile function is also known as the inverse cumulative distribution function(CDF).
The quantile function of a distribution is defined as,

$$W = F^{-1}(x)$$

Where $F^{-1}(x)$ is the Inverse Cumulative Distribution Function (ICDF) of any distribution and W is quantile function.

1.2.8 Moment Generating Function

A moment generating function (MGF) is a mathematical concept used in probability theory and statistics to uniquely characterize a probability distribution. It is a way of encoding all the moments of a random variable into a single function.The moment generating function of a random variable W is denoted by $M_W(t)$ and is defined as:

$$M_W(t) = E[e^{tW}]$$

where:

- t is a real-valued parameter.
- E$[e^{tW}]$ denotes the expected value operator.
- W is a random variable.

The moment generating function is particularly useful in deriving moments of distributions, finding distributions of linear combinations of random variables, and proving theorems related to convergence of random variables.

1.2.9 Rényi Entropy

Rényi entropy is a generalization of Shannon entropy, named after Alfréd Rényi, a Hungarian mathematician. It provides an alternative way to measure the uncertainty or randomness in a probability distribution.For a discrete random variable W with probability mass function $p(w)$.The Rényi entropy of order ℓ is defined as:

$$H_{W;R}(\ell) = \frac{1}{1-\ell} \log[H_X(\ell)]$$

where;

$$H_X(\ell) = \int_R f_W^{\ell}(w)d(w), for \quad \ell > 0 \qquad \ell \neq 1$$

Rényi entropy has applications in various fields such as information theory, statistical physics, and machine learning.

1.2.10 Order Statistics

In the context of a set of data points, the order statistics provide valuable insights by presenting the values of these data points in ascending order. This

allows researchers and analysts to better understand the distribution and characteristics of the data, enabling them to make informed decisions and draw meaningful conclusions.Order statistics are a fundamental concept in the fields of statistics and probability theory, focusing on the arrangement and ranking of observations within a given sample.
For example, consider a sample of n observations:$Z_1, Z_2, Z_3, ...Z_n$. The order statistics are denoted as follows:

- The minimum value:$Z_(1)$
- The second smallest value:$Z_(2)$
- The third smallest value:$Z_(3)$
- And so on, until:
- The maximum value:$Z_(n)$

These order statistics can be useful in various statistical analyses, such as determining percentiles, constructing confidence intervals, and performing hypothesis tests.
Let $Z_\imath(\imath \leq n)$ be random variable in order that are independently and identically distributed.Then the corresponding CDF is given by

$$F_{Z_{(v:n)}}(z) = \sum_{\imath=v}^{n} \left(\begin{array}{c} n \\ \imath \end{array} \right) [F_Z(z)]^\imath \, [1 - F_Z(z)]^{n-1}$$

1.2.11 Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) is a statistical technique utilized to determine the optimal values of parameters within a model by maximizing the likelihood of observing the provided data.This process involves the formulation of a likelihood function based on both the model and data,followed by the utilization of optimization techniques to identify the parameter values that result in the observed data being most probable.

Let $W_1, W_2, W_3, ...W_n$ a random sample from a distribution with probability density function (pdf) or probability mass function (PMF) denoted by $f(w; \emptyset)$ where $\emptyset$ represents the parameter to be estimated.The likelihood function $L(\emptyset)$ is defined as the joint probability density function of the sample given the parameter's $\emptyset$,which is defined as:

$$L(w; \emptyset) = \prod_{i=1}^{n} f(W_i; \emptyset)$$

1.2.12 Goodness of Fit Criteria

Goodness of fit criteria are employed to evaluate the degree to which a statistical model accurately represents the observed data. These criteria offer quantitative measures that enable the assessment of how well a model describes the data. Several commonly used goodness of fit criteria exist, which aid in determining the adequacy of a model in describing the observed data.

- Akaike Information Criterion (AIC)
- Bayesian Information Criterion (BIC)
- Consistent Akaike Information Criterion (CAIC)

Akaike Information Criterion (AIC)

The Akaike Information Criterion (AIC) serves as an indicator of the comparative excellence of a statistical model with respect to a specific dataset. It imposes a penalty on models with a higher number of parameters, with the goal of striking a balance between the model's goodness of fit and its complexity. Smaller AIC values are indicative of models that fit the data more effectively. It can be defined as:

$$AIC = 2q - 2\log(L)$$

where,

- $\log(L)$represents the Log-Likelihood function calculated using the M-LEs.
- q is number of parameters in the model.

Bayesian Information Criterion (BIC)

BIC is similar to AIC but places a stronger penalty on models with more parameters. It is calculated as;

$$BIC = q\log(m) - 2\log(L)$$

where,

- $\log(L)$ represents the Log-Likelihood function calculated using the M-LEs.
- q is number of parameters in the model.
- m is sample size.

Like AIC, lower BIC values indicate better-fitting models.

Consistent Akaike Information Criterion (CAIC)

The Consistent Akaike Information Criterion (CAIC) is a modified version of the Akaike Information Criterion (AIC) that specifically addresses the issue of bias in finite sample sizes. It is particularly beneficial in cases where the sample size is small or when the number of parameters in the model is relatively large compared to the sample size.
The CAIC is defined as:

$$CAIC = -2\log(L) + 2d\frac{n}{n-d-1}$$

where,

- $\log(L)$ represents the Log-Likelihood function calculated using the M-LEs.
- d is number of parametersin the model.
- n is sample size.

1.2.13 Simulation

A simulation study might make use of a straightforward model or one that is highly complex,depending on the sort of research for which the model is produced.The major purpose of simulation research is to understand the real system.Simulation study may have sophisticated model or simple one depending on research type.The main goal of the researchers is to create a simulation model that takes into account every imaginable factor.A simulation study is carried out in statistics to evaluate the effectiveness of the new statistical strategy that the researcher has suggested.

1.3 Research Objectives

- To propose a new Sin Exponentiated Odd Generalized Pareto type generator.
- To generate a new more flexible extension of Dagum distribution by using new trigonometric generator.
- To explore the proposed distribution's statistical properties.
- To utilize the MLE technique for obtaining parameter estimates.
- To determine the effeciency of the distribution under study by using both real and simulated datasets.

1.4 Study Outline

This thesis is categorized into five chapters.

- In Chapter 1, is introductory in nature and provides genesis of the probability distributions. And a new methodology is proposed for generating a new family of Sin Exponentiated Odd Generalized Pareto Dagum and its general mathematical properties are derived.
- In Chapter 2, A brief survey of the literature on existing Odd Generated distributions and their applications are given.
- In chapter 3, Sin Exponentiated Odd Generalized Pareto X-Family of generator is introduced by using the proposed methodology. The structural and statistical properties including moments, moment generating function, entropy, survival function, hazard function and order statistics are derived.
- In chapter 4, Dagum distribution is developed by using the proposed methodology. The structural and statistical properties including moments, moment generating function, entropy, survival function, hazard function and order statistics are derived.
- In Chapter 5, The parameters are estimated by using maximum likelihood estimation technique. The simulation study is carried out to examine the performance of the proposed distribution. The proposed SEOGPD model is fitted on real-life data sets to highlight its compatibility. The thesis discussion conclusion have been reported.

Chapter 2

Literature Review

Numerous studies have suggested the utilization of diverse generated classes to expand the parameter space within distributions. These enhanced distributions have demonstrated their applicability in modeling data across a wide range of disciplines. In this chapter literature survey relating to the current study is presented.
Abonongo, A. I. L., (2024) [30] introduces the transformed sin Dagum distribution, a modification of the Dagum distribution based on the transformed sin-G family. It effectively models positively skewed, approximately symmetric, and decreasing datasets, and can handle both monotonic and non-monotonic hazard rates. They derived various statistical properties, including the quantile function, moments, and order statistics, and use maximum likelihood estimation for parameter estimation. Monte Carlo simulations confirm the consistency of the estimators. Applied to real-life datasets, the transformed sine Dagum distribution demonstrates a superior parametric fit compared to competing models.

El-Khabeary et al., (2024) [28] study that Multivariate distributions, such as the bivariate Dagum distribution, are essential in economic, social, and business fields but are often challenging for non-specialists due to their complex mathematical forms. This paper provides a statistical table with exact values for the bivariate Dagum cumulative distribution in cases of dependence and independence of random variables, as introduced by K. M. El-Khabeary et al., (2023) [27]. They also present an approximate distribution using numerical double integration, along with the corresponding statistical table and some numerical examples. Finally, They conduct a comparison study between the exact and approximate values.

Sherwani et al., (2023) [25] propose the MOED, featuring four shape parameters and one scale parameter, is introduced. To streamline calculations, alternative expressions for the MOED are derived. The study explores various properties of the MOED, including dispersion, central tendency, hazard rate, and survival rate. Additionally, network properties such as moments, quantile function, median, and MD random number generator are derived. Parameters are estimated using the MLE method, with performance assessed via Monte Carlo Simulation. The MOED's application and performance are evaluated using three real-life data sets, comparing it to the MOD distribution, Exponentiated Dagum (ED), Dagum (D), Burr III, K-Burr III, KLL, and LL distributions. Four information criteria are utilized to compare these distributions, showing the MOED's superior performance.

Peter O Koleoso., (2023) [26] explores an OLD distribution has been constructed. The ODG family of distributions served as the model for the distribution. With the help of pertinent supporting plots of the PDF, hazard function, survival function, etc., the structural properties of the distribution have been determined. They investigate the OLD distribution's performance by fitting it to two skewed datasets in addition to the Dagum, Lomax, and WD distributions (WeiDaD, DaD, and LomD). Compared to other distributions taken into consideration in the study, the distribution showed a great deal of flexibility in modeling the two datasets.

Tanis et al., (2019) [22] introduce the approximation Bayes estimation issue with three parameters for the log-Dagum distribution. First, the MLEs for unknown parameters of the LD distribution are produced, together with asymptotic confidence intervals based on these estimators. Moreover, the Tierney and Kadane approximations are used to produce estimated Bayes estimators under squared error loss function for the unknown parameters of this distribution. A Monte-Carlo simulation research is conducted to evaluate the mean square errors and biases of the approximation Bayes estimators and maximum likelihood estimators. The study of actual data for this distribution is finally completed.

Dey et al., (2017) [17] discuss From multiple perspectives, the characteristics and approaches for estimating the unknown parameters of three-parameter Dagum distribution are discussed. Various mathematical and statistical properties of the Dagum distribution, including quantiles, moments, moment generating function, hazard rate, mean residual lifetime, mean past lifetime, MD about the mean and median, various entropies, Bonferroni and Lorenz curves, and order statistics, are explored. Although the primary focus is on estimation from a frequentist perspective, several estimation methods are briefly discussed, including MLE, moments, L-moment, percentile-based, least squares, maximum product of spacings, minimum distances, Cramér-von-Mises, Anderson-Darling, and right-tail Anderson-Darling estimators. Monte Carlo simulations are conducted to compare the estimation performances of the proposed approaches for both small and large samples.

Tahir et al., (2016) [16] introduce the WD distribution, a new lifetime model based on the Weibull G class. This model is characterized by a flexible density function that can be symmetrical, left-skewed, right-skewed, or reversed-J shaped. It exhibits various hazard rate shapes, including constant, increasing, decreasing, upside-down bathtub, bathtub, and reversed-J shaped. They derived several structural properties, including quantile functions, ordinary and incomplete moments, and probability weighted moments. Additionally, They provide explicit expressions for Rényi and q-entropies and derive the density function of order statistics as a mixture of Dagum densities. Using maximum likelihood estimation, They demonstrated the model's effectiveness through a simulation study and applications to real data, where it outperforms the beta-Dagum, McDonaldDagum, and Dagum models.

Silva et al., (2015) [12] explore a novel model with five parameters known as the extended Dagum distribution. Among other distributions, the proposed model includes the log-logistic and Burr III distributions as special examples. The moments, mean deviations, generating and quantile functions, Bonferroni, Lorenz, and Zenga curves are all derived. We acquire the order statistics' density function. The maximum likelihood approach is utilized to estimate the parameters. The information matrix observed is established. The significance of the new model is demonstrated through an application to actual data.

Oluyede, B. O.,and Ye, Y., (2014) [11] proposes a new distributions connected to WD. The Dagum distribution's probability-weighted moments and the ensuing weighted distributions are shown. Many WD distributions, including length-biased Dagum, proportional hazard moment Dagum, proportional reverse hazard moment Dagum, and Dagum distributions as special instances, are included in this family of distributions. The WD distribution's entropy and Fisher information are obtained. They used the MLE process to estimate the model parameters. They offer examples and a comparison of this model with the generalized Lindley, generalized gamma, and WGG distributions.

Chapter 3

Sin Exponentiated Odd Generalized Pareto X-Family of Distribution

3.1 Introduction

Generators are tools used to modify the existing distributions by adjusting specific characteristics. The main focus of this chapter is to introduce a new generator by using the trignometric function and flexibility of existing distributions is a relatively recent and innovative approach in statistical probability modeling.
This approach is especially popular among researchers,Eugene et al.,(2002) [4] proposed beta generated (G) family of distributions, Zografos and Balakrishnan(2009) [5] proposed gamma-G family, Cordeiro and de Castro(2011) [6] defined Kumaraswamy-G family, Bourguignon et al.,(2014) [8] introduced weibull-G family, Al-Shomrani et al.,(2016) [15] proposed TL family of distributions and Cordeiro et al.(2019) [18] introduced the OD-G family of distributions.

3.2 Sin Exponentiated Odd Generalized X-Family of Distribution

3.2.1 Sin Exponentiated Odd Distribution

In introduction it is discussed that Tahir et al.,(2015) [14] proposed the new class of distribution called the odd generalized exponential family. Then the PDF and CDF of Sin Odd-Generalized Exponential X-Family of distribution is;

$$S_{OE}(y;\alpha;\beta) = \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{Q(y)}{\bar{Q}(y)}}\right)^{\beta}\right\} \quad {\scriptstyle y>0, \atop \alpha,\beta>0} \tag{3.2.1}$$

$$\begin{aligned} s_{OE}(y;\alpha;\beta) &= \frac{\alpha\beta\pi q(y)}{2\bar{Q}(y)}\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{Q(y)}{\bar{Q}(y)}}\right)^{\beta}\right\} \\ &\times\left(1-e^{-\alpha\frac{Q(y)}{\bar{Q}(y)}}\right)^{\beta-1} e^{-\alpha\frac{Q(y)}{\bar{Q}(y)}} \quad {\scriptstyle y>0, \atop \alpha,\beta>0} \end{aligned} \tag{3.2.2}$$

Where $\bar{Q}(y) = 1 - Q(y)$

3.2.2 The Generalized Pareto X-Family of Distribution

The PDf and CDf of Generalized Pareto X-Family of distribution was introduced by Maurice Fréchet, in the early $20th$ century.

$$Q_G(z;\hbar) = 1 - \{1 + D(z)\}^{\frac{1}{\hbar}} \tag{3.2.3}$$

where $\quad z = \frac{x-\mu}{\sigma} \quad x \geq \mu$

$$q_G(z;\hbar) = \frac{1}{\hbar}\{1 + D(z)\}^{-\frac{1}{\hbar}-1} D(z) \tag{3.2.4}$$

Where $\hbar$ is the shape parameter.

3.3 Sin Exponentiated Odd Generalized Pareto X-Family of Distribution(SEOGPX)

By using (3.2.3), (3.2.4) into (3.2.1) and (3.2.2) respectively,
The CDF of SEOGPX family of distribution is,

$$S_{SEOGPX}(z) = \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \tag{3.3.1}$$

Given by is the relevant probability density function(PDF) is,

$$\begin{aligned} s_{SEOGPX}(z) &= \frac{\alpha\beta\pi}{2\hbar}\frac{\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]} \\ &\times e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \\ &\times\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1} \\ &\times\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \end{aligned} \tag{3.3.2}$$

3.3.1 Expansion of CDF and PDF

The CDF of expended form of Sin-G family of distribution Souza et al.,(2019) [21] is given as

$$\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{Q(y)}{\bar{Q}(y)}}\right)^{\beta}\right\} = \sum_{n=0}^{\infty}\frac{(-1)^{i}}{(2n+1)!}x^{2n+1}$$

So, The CDF of the SEOGPX family of distribution is

$$\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)!}$$
$$\times\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}^{2i+1}$$

$$S_{SEOGPX}(z)=\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)!}\left(\frac{\pi}{2}\right)^{2i+1}$$
$$\times\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)} \tag{3.3.3}$$

Now expanded form of PDF: By taking derivative of (3.3.3)

$$s_{SEOGPX}(z;\hbar)=\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)!}\left(\frac{\pi}{2}\right)^{2i+1}\beta(2i+1)$$
$$\times\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)-1}$$
$$\times\frac{d}{dz}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)$$

$$s_{SEOGPX}(z;\hbar) = \frac{\alpha\beta}{\hbar}\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)!}\left(\frac{\pi}{2}\right)^{2i+1}(2i+1)$$
$$\times\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)-1}$$
$$\times e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}$$
$$\times\frac{\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^2} \tag{3.3.4}$$

Consider

$$e^{-\alpha \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \left(1-e^{-\alpha \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)-1}$$

As the binomial series is:

$$(1-z)^{a-1} = \sum_{b=0}^{\infty} (-1)^b \begin{pmatrix} a-1 \\ b \end{pmatrix} z^b \qquad (i)$$

Applying on above considered term

$$= \sum_{j=0}^{\infty} (-1)^j \begin{pmatrix} \beta(2i+1)-1 \\ j \end{pmatrix} e^{-\alpha j \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \times e^{-\alpha \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}$$

$$= \sum_{j=0}^{\infty} (-1)^j \begin{pmatrix} \beta(2i+1)-1 \\ j \end{pmatrix} e^{-\alpha(j+1) \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}$$

put all in equation (3.3.4)

$$\begin{aligned} s_{SEOGPX}(z) = & \frac{\alpha\beta}{\hbar} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{(2i+1)!} \left(\frac{\pi}{2}\right)^{2i+1} \\ & \times (2i+1) \begin{pmatrix} \beta(2i+1)-1 \\ j \end{pmatrix} \\ & \times e^{-\alpha(j+1) \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \\ & \times \frac{\{1+D(z)\}^{-\frac{1}{\hbar}-1} D(z)}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^2} \end{aligned} \qquad (3.3.5)$$

Now consider

$$e^{-\alpha(j+1)\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}$$

As the exponential expansion is;

$$e^{-\lambda(\varsigma+1)\frac{G(x)}{\bar{G}(x)}} = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k (\varsigma+1)^k}{k!} \frac{G(x)^k}{\bar{G}(x)^k}$$

Applying on the above considered term

$$= \sum_{k=0}^{\infty} \frac{\alpha^k (j+1)^k}{k!} \frac{\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^k}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^k}$$

So

$$e^{-\alpha(j+1)\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k (j+1)^k}{k!} \times \frac{\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^k}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^k}$$

so the equation (3.3.5) becomes;

$$s_{SEOGPX}(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{k+1}\beta}{\hbar k!} \frac{(-1)^{i+j+k}(j+1)^k}{(2i+1)!} \times \left(\frac{\pi}{2}\right)^{2i+1} (2i+1) \begin{pmatrix} \beta(2i+1)-1 \\ j \end{pmatrix} \times \frac{\{1+D(z)\}^{-\frac{1}{\hbar}-1} D(z)}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^2} \times \frac{\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^k}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^k} \tag{3.3.6}$$

Now consider

$$\frac{\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^{k}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{2}[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{k}}$$

$$=[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{-(k+2)}\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^{k}$$

As the generalized binomial expansion is;

$$(1-x)^{-n}=\sum_{h=0}^{\infty}(-1)^{h}\begin{pmatrix} n \\ h \end{pmatrix} x^{h} \qquad (ii)$$

Applying on the above considered term

$$=\sum_{l=0}^{\infty}(-1)^{l}\begin{pmatrix} -(k+2) \\ l \end{pmatrix}\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^{l} \times\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^{k}$$

$$=\sum_{l=0}^{\infty}(-1)^{l}\begin{pmatrix} -(k+2) \\ l \end{pmatrix}\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^{k+l}$$

the equation (3.3.6) becomes

$$\begin{aligned} s_{SEOGPX}(z)=&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^{k+1}\beta}{\hbar k!}\frac{(-1)^{i+j+k+l}(j+1)^{k}}{(2i+1)!} \\ &\times\left(\frac{\pi}{2}\right)^{2i+1}(2i+1)\begin{pmatrix} \beta(2i+1)-1 \\ j \end{pmatrix} \\ &\times\begin{pmatrix} -(k+2) \\ l \end{pmatrix}\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z) \\ &\times\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^{k+l} \end{aligned} \qquad (3.3.7)$$

Now consider

$$\{1-(1+D(z))^{-\frac{1}{\hbar}}\}^{k+l}\{1+D(z)\}^{-\frac{1}{\hbar}-1}$$

As the binomial series is;

$$(1-x)^n=\sum_{h=0}^{\infty}(-1)^h\begin{pmatrix} n \\ h \end{pmatrix}x^h \qquad (iii)$$

Applying on the above considered term

$$=\sum_{e=0}^{\infty}(-1)^e\begin{pmatrix} k+l \\ e \end{pmatrix}\{(1+D(z))^{-\frac{1}{\hbar}}\}^e \times\{1+D(z)\}^{-\frac{1}{\hbar}-1}$$

$$=\sum_{e=0}^{\infty}(-1)^e\begin{pmatrix} k+l \\ e \end{pmatrix}\{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)}$$

Now the equation (3.3.7) becomes;

$$s_{SEOGPX}(z)=\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\frac{\alpha^{k+1}\beta}{\hbar k!}\frac{(-1)^{i+j+k+l+e}(j+1)^k}{(2i+1)!} \times\left(\frac{\pi}{2}\right)^{2i+1}(2i+1)\begin{pmatrix} \beta(2i+1)-1 \\ j \end{pmatrix} \times\begin{pmatrix} -(k+2) \\ l \end{pmatrix}\begin{pmatrix} k+l \\ e \end{pmatrix} \times D(z)\{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)}$$

$$s_{SEOGPX}(z)=\frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}A_{ijkle}D(z)\{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)} \qquad (3.3.8)$$

Where,

$$A_{ijkle} = \frac{\alpha^{k+1}}{k!} \frac{(-1)^{i+j+k+l+e}(j+1)^k}{(2i+1)!} \left(\frac{\pi}{2}\right)^{2i+1} (2i+1) \times \binom{\beta(2i+1)-1}{j} \binom{-(k+2)}{l} \binom{k+l}{e}$$

3.3.2 Statistical properties of SEOGPX

3.3.3 Survival Function

The survival function of SEOGPX family of distribution is obtained as,

$$S(z) = 1 - T(z) \tag{3.3.9}$$

By substituting (3.3.1) into (3.3.9)

$$S_{SOE-PX}(z) = 1 - \sin\left\{\frac{\pi}{2}\left(1 - e^{-\alpha \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \tag{3.3.10}$$

3.3.4 Hazard Function

The hazard function, in survival analysis, represents the instantaneous risk of an event occurring at time t, given that the subject has survived up to that time.This function helps in assessing the probability of failure or an event occurring within a very short time interval,

$$h_{SEOGPX}(z) = \frac{t(z)}{1 - T(z)} \tag{3.3.11}$$

By putting (3.3.1) and (3.3.2) into (3.3.11)

$$
\begin{aligned}
h_{SEOGPX}(z) =& \frac{\alpha\beta\pi}{2\hbar}\frac{\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]} \\
&\times e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \\
&\times \left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1} \\
&\times \frac{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}{1-\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}
\end{aligned}
\tag{3.3.12}
$$

3.3.5 Quantile Function

The Quantile function of SEOGPX family of distribution is,

$$S_{SEOGPX}(z) = P$$

By adding the results of equation (3.3.1) in above expression we get;

$$\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} = P \tag{3.3.13}$$

where P is an usual uniform random number.We must first solve (3.3.13) for z in order to determine the inverse CDF. so taking the $\sin^{-1}$ on both sides of

(3.3.13)

$$\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}=\sin^{-1}P$$

$$\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}=\frac{2}{\pi}\sin^{-1}P$$

$$1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}=e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}}$$

Taking log on both sides,

$$\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}=-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}$$

$$\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]}=-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}$$

$$1-\{1+D(z)\}^{-\frac{1}{\hbar}}=-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}\times[1-\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}]$$

$$1-\{1+D(z)\}^{-\frac{1}{\hbar}}=-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}\times\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}$$

$$\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\}\left[1-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}\right]=-\frac{\log}{\alpha}\times\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}$$

$$\begin{aligned}\{1-\{1+D(z)\}^{-\frac{1}{\hbar}}\} &= -\frac{\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}}{\left[1-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}\right]}\\ \{1+D(z)\}^{-\frac{1}{\hbar}} &= 1+\frac{\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}}{\left[1-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}\right]}\end{aligned}$$

$$z = D^{-1}\left[1+\frac{\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}}{\left[1-\frac{\log}{\alpha}\left\{1-\left(\frac{2}{\pi}\sin^{-1}P\right)^{\frac{1}{\beta}}\right\}\right]}\right]^{-\hbar}-1 \tag{3.3.14}$$

3.3.6 Moment Generating Function

The moment generating function (MGF) of a random variable X is defined as $M_X(t) = E[e^{tX}]$, where E denotes the expectation.

$$M_z(t) = E[exp^{tz}] = \int_R exp^{tx} f(z)dz$$

$$\Longrightarrow exp^{tz} = \sum_{r=0}^{\infty}\frac{(tz)^r}{r!}$$

$$M_z(t) = \sum_{r=0}^{\infty}\frac{t^r u'}{r!}$$

using equation (3.3.8), the SEOGPX family of distributions moment generating function is,

$$M_z(t) = \frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\sum_{r=0}^{\infty} A_{ijkle}\frac{t^r}{r!}\int_R z^r D(z) \times\{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)}dz \tag{3.3.15}$$

3.3.7 Moments

$$u'_r = E(z^r) = \int_R z^r t(z)dz$$

The SEOGPX familys rth ordinary moment by using equation (3.3.8) is given as,

$$u'_r = \frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty} A_{ijkle}\int_R z^r D(z)\{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)}dz \tag{3.3.16}$$

By putting r=1,2,3,...... we can obtain $1^{st}, 2^{nd}, 3^{rd}$...
ordinary moments of distribution.

3.3.8 Incomplete rth Moment

The rth incomplete moment by definition is:

$$\phi_r(t) = \int_{-\infty}^{t} z^r f(z)dz$$

The rth incomplete moments of SEOGPX family of distribution
by using (3.3.8) is:

$$\phi_r(t) = \frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\sum_{r=0}^{\infty} A_{ijkle}\frac{t^r}{r!}\int_{-\infty}^{t} z^r D(z) \times\{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)}dz \tag{3.3.17}$$

3.3.9 Mean Residual Life

The estimated remaining life duration of a unit that survives at time t. It is defined as,

$$m(t) = \frac{1}{1-F(t)}\left[u - \int_{-\infty}^{t} zf(z)dz\right] - t$$

Mean residual life of SEOGPX family of distribution by using (3.3.8 is,

$$\begin{aligned} m(t) =& \frac{1}{1-T(t)}\Bigg[u - \frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e,r=0}^{\infty} A_{ijkle}\frac{t^r}{r!}\int_{-\infty}^{t} zD(z) \\ & \times \{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)}dz\Bigg] - t \end{aligned} \tag{3.3.18}$$

3.3.10 Rényi Entropy

The R'enyi entropy Renyi (1961) [2] is defined as;

$$I_R = \frac{1}{1-\delta}\log\left[\int_R t^{\delta}(z)dz\right] \quad \delta \neq 1 \quad \delta > 0 \tag{3.3.19}$$

By using equation (3.3.2)

$$\begin{aligned} t^{\delta}(z) &= \left(\frac{\alpha\beta\pi}{2\hbar}\right)^{\delta}[\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)]^{\delta} \\ &\times \frac{1}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{2\delta}} \\ &\times e^{-\alpha\delta\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \\ &\times \left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\delta(\beta-1)} \\ &\times \left[\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}\right]^{\delta} \end{aligned} \tag{3.3.20}$$

consider

$$\left[\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}\right]^{\delta}$$

By using the series expansions for the $\cos$ function;

$$\left\{cos\frac{\pi}{2}G(x)\right\}^{i}=\sum_{j=0}^{\infty}b_j(i)\left(\frac{\pi}{2}G(x)\right)$$

$$\left[\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}\right]^{\delta}$$
$$=\sum_{p=0}^{\infty}b_p(\delta)\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}^{2p}$$

$$\begin{aligned}t^{\delta}(z)=&\sum_{p=0}^{\infty}b_p(\delta)\left(\frac{\pi}{2}\right)^{2p}\left(\frac{\alpha\beta\pi}{2\hbar}\right)^{\delta}\\&\times[\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)]^{\delta}\\&\times\frac{1}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{2\delta}}\\&\times e^{-\alpha\delta\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\\&\times\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2p+\delta)-\delta}\end{aligned}\tag{3.3.21}$$

Now consider

$$\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2p+\delta)-\delta}$$

again using binomial series (iii)

$$= \sum_{q=0}^{\infty}(-1)^q \begin{pmatrix} \beta(2p+\delta)-\delta \\ q \end{pmatrix} e^{-\alpha q \frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}$$

so the equation (3.3.21) becomes;

$$\begin{aligned} t^{\delta}(z) &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q b_p(\delta)\left(\frac{\pi}{2}\right)^{2p}\left(\frac{\alpha\beta\pi}{2\hbar}\right)^{\delta} \\ &\times[\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)]^{\delta} \\ &\times\frac{1}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{2\delta}} \\ &\times\begin{pmatrix} \beta(2p+\delta)-\delta \\ q \end{pmatrix} e^{-\alpha(q+\delta)\delta\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \end{aligned} \quad (3.3.22)$$

Substituting (3.3.22) in equation (3.3.19)

$$\begin{aligned} I_R &= \frac{1}{1-\delta}\log\left[\left(\frac{\alpha\beta\pi}{2\hbar}\right)^{\delta}\delta\int_R\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}W_{pq}\right. \\ &\times\frac{[\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)]^{\delta}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{2\delta}} \\ &\left.\times e^{-\alpha(q+\delta)\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}dz\right] \end{aligned} \quad (3.3.23)$$

where

$$W_{pq} = (-1)^q b_p\left(\frac{\pi}{2}\right)^{2p}\begin{pmatrix} \beta(2p+\delta)-\delta \\ q \end{pmatrix}$$

3.3.11 Order Statistics

Let $\zeta_1, \zeta_2, \zeta_3, , , \zeta_n$ be followed by the PDF of the ordered random t^{th} order statistics is defined as

$$t_{h:n} = \frac{t(\zeta)}{\beta(h, n-h+1)}\sum_{m=0}^{n-h}(-1)^m\begin{pmatrix} n-h \\ m \end{pmatrix}T(\zeta)^{m+h-1}$$

Using expanded form of CDF and PDF of SEOGPX distribution from (3.3.3) and (3.3.8)

$$T(\zeta)^{m+h-1} = S^{m+h-1}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)(m+h-1)}$$

where;

$$S^{m+h-1} = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)!}\left(\frac{\pi}{2}\right)^{2i+1}$$

$$\begin{aligned} t_{h:n} &= \frac{\beta}{\hbar\beta(h,n-h+1)}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\sum_{m=0}^{n-h}A_{ijkle}(-1)^m \\ &\times\begin{pmatrix} n-h \\ m \end{pmatrix} S^{m+h-1}D(z)\{1+D(z)\}^{-(\frac{e+1}{\hbar}+1)} \\ &\times\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)(m+h-1)} \end{aligned} \qquad (3.3.24)$$

Now consider

$$\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)(m+h-1)}$$

let

$$u = (2i+1)(m+h-1)$$

$$\begin{aligned} \Rightarrow \left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta(2i+1)(m+h-1)} \\ = \left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta u} \end{aligned}$$

again using binomial series (iii)

$$\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta u} = \sum_{s=0}^{\infty}(-1)^{s}\binom{\beta u}{s} \times e^{-\alpha s\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}$$

consider

$$e^{-\alpha s\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}$$

using the power series expansion

$$e^{-x}=\sum_{i=0}^{\infty}\frac{(-1)^{i}x^{i}}{i!}$$

so the above considered term becomes;

$$e^{-\alpha s\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} = \sum_{s=0}^{\infty}\sum_{v=0}^{\infty}(-1)^{s+v}\binom{\beta u}{s} \times\frac{\left\{\alpha s\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}\right\}^{v}}{v!}$$

Substituting back in equation (3.3.24)

$$\begin{aligned} t_{h:n} &= \frac{\beta}{\hbar\beta(h,n-h+1)}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\sum_{m=0}^{n-h}\sum_{s,v=0}^{\infty}A_{ijkle} \\ &\times\frac{(-1)^{m+s+v}}{v!}S^{m+h-1}\binom{n-h}{m}\binom{\beta u}{s} \\ &\times\left\{\alpha s\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}\right\}^{v}D(z) \\ &\times\{1+D(z)\}^{-(\frac{e+1}{\hbar}+1)} \end{aligned}$$

$$
\begin{aligned}
t_{h:n} \quad = \quad & \frac{\beta}{\hbar\beta(h,n-h+1)} \sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\sum_{m=0}^{n-h}\sum_{s,v=0}^{\infty} Z_c \\
& \times\left\{\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}\right\}^{v} \\
& \times D(z)\{1+D(z)\}^{-(\frac{e+1}{\hbar}+1)} \qquad (3.3.25)
\end{aligned}
$$

Where;

$$
Z_c = A_{ijkle}\frac{(-1)^{m+s+v}(\alpha s)^{v}}{v!}S^{m+h-1}\begin{pmatrix} n-h \\ m \end{pmatrix}\begin{pmatrix} \beta u \\ s \end{pmatrix}
$$

3.3.12 Parameter Estimation

Since the parameters of the probability model are unknown, they need to be estimated using sample data. In this section, the maximum likelihood estimation method is applied to estimate these unknown parameters.
The likelihood-function(L) is defined as,

$$
L = \prod_{w=1}^{n} t(z_w;\partial) \quad where \quad \partial > 0
$$

As the PDF of SEOGPD by (3.3.2) is,

$$
\begin{aligned}
s_{SEOGPX}(z) = & \frac{\alpha\beta\pi}{2\hbar}\frac{\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]} \\
& \times e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}} \\
& \times \left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1} \\
& \times \cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}
\end{aligned}
$$

$$
\begin{aligned}
L &= \prod_{w=1}^{n}\left[\frac{\alpha\beta\pi}{2\hbar}\frac{\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}\right.\\
&\quad\times e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}\\
&\quad\left.\times\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}\right]
\end{aligned}
$$

The log(L) is;

$$
\begin{aligned}
Log(L) &= n\log\alpha+n\log\beta+n\log\frac{\pi}{2}+\left(-\frac{1}{\hbar}-1\right)\\
&\quad\times\log\sum_{w=1}^{n}(1+D(z_w;\hbar))+\log\sum_{w=1}^{n}d(z_w;\hbar)\\
&\quad-2\log\sum_{w=1}^{n}[1-\{1-(1+D(z_w;\hbar))^{-\frac{1}{\hbar}}\}]\\
&\quad-\alpha\sum_{w=1}^{n}\frac{\{1-(1+D(z_w;\hbar))^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+D(z_w;\hbar))^{-\frac{1}{\hbar}}\}]}\\
&\quad+(\beta-1)\log\sum_{w=1}^{n}\left(1-e^{-\alpha\frac{\{1-(1+D(z_w;\hbar))^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+D(z_w;\hbar))^{-\frac{1}{\hbar}}\}]}}\right)\\
&\quad+\log\sum_{w=1}^{n}\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \quad (3.3.26)
\end{aligned}
$$

3.3.13 Some Special Cases

Gumbel Type-2 Distribution

Ogunde et al.,(2020) [23] proposed Gumbel Type-2 Distribution, it's CDF and PDF is;

$$
\begin{aligned}
F(x) &= e^{-bx^{-a}}\\
f(x) &= abx^{-a-1}e^{-bx^{-a}}
\end{aligned}
$$

Where b is shape parameter.

Submitting it into (3.3.1) and (3.3.2)

$$G_{SEOGPT2G}(z) = \sin\left\{\frac{\pi}{2}\left(1 - e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}}\right)^{\beta}\right\}$$

$$g_{SEOGPT2G}(z) = \frac{\alpha\beta\pi}{2\hbar}\frac{(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}-1}abx^{-a-1}e^{-bx^{-a}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]} \times e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}} \times \left(1 - e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}}\right)^{\beta-1} \times \cos\left\{\frac{\pi}{2}\left(1 - e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}}\right)^{\beta}\right\}$$

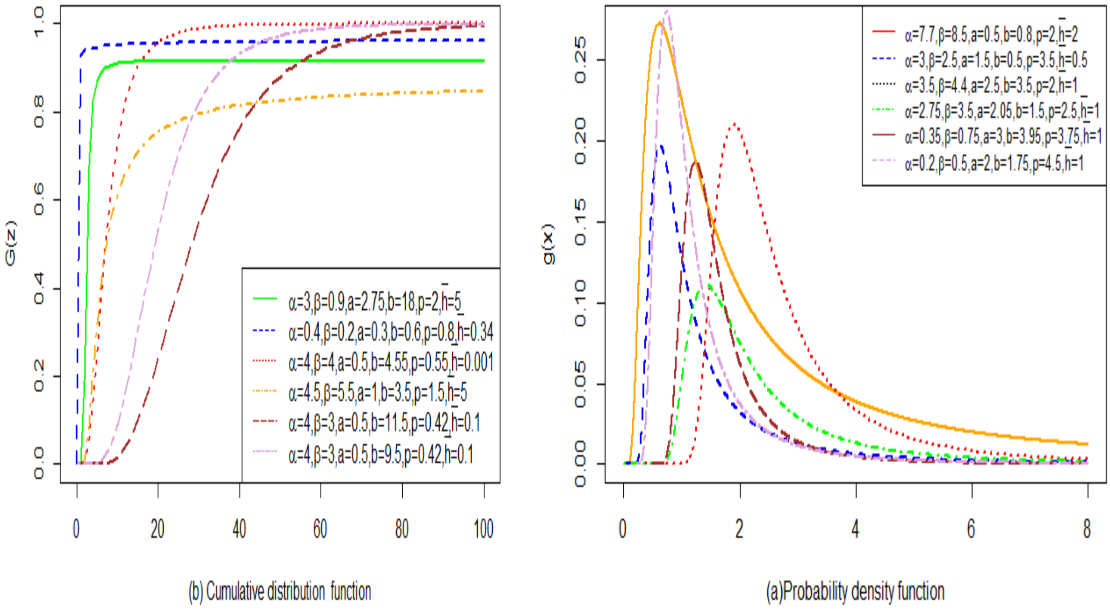

Figure 3.1: Plot of the CDF and PDF of SEOGPG

The survival and Hazard rate
function are

$$S_{SEOGPT2G}(z) = 1 - \sin\left\{\frac{\pi}{2}\left(1 - e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}}\right)^{\beta}\right\}$$

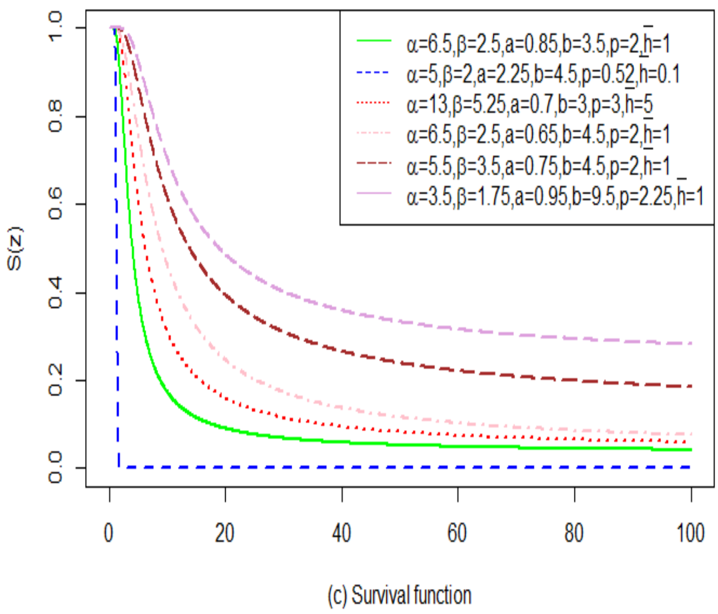

Figure 3.2: Plot of Survival Function of SEOGPG

$$H_{SEOGT2G}(z) = \frac{\alpha\beta\pi}{2\hbar}\frac{(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}-1}abx^{-a-1}e^{-bx^{-a}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]} \times e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}} \times \left(1-e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}}\right)^{\beta-1} \frac{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}}\right)^{\beta}\right\}}{1-\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+e^{-bx^{-a}})^{-\frac{1}{\hbar}}\}\right]}}\right)^{\beta}\right\}}$$

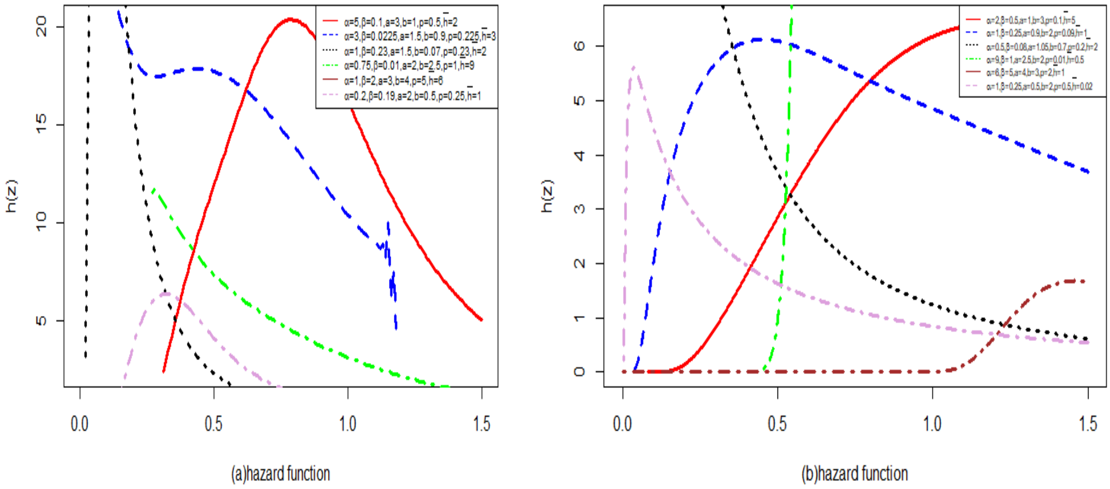

Figure 3.3: Plot of Hazard Function of SEOGPG

Logistic Distribution

Verhulst, P. F., (1838) [1] introduced the Logistic Distribution, it's CDF and PDF is;

$$
\begin{aligned}
F(x) &= \frac{1}{1+e^{-x}} \\
f(x) &= \frac{e^{-x}}{(1+e^{-x})^2}
\end{aligned}
$$

Submitting it into (3.3.1) and (3.3.2)

$$
L_{SEOGPL}(z) = \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{h}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{h}}\right]}}\right)^{\beta}\right\}
$$

$$l_{SEOGPL}(z) = \frac{\alpha\beta\pi}{2\hbar} \frac{\left(1+\frac{1}{1+e^{-x}}\right)^{-\frac{1}{\hbar}-1}\left\{\frac{e^{-x}}{(1+e^{-x})^2}\right\}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]^2}$$

$$\times e^{-\alpha \frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}} \left(1-e^{-\alpha \frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}}\right)^{\beta-1}$$

$$\times \cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha \frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}}\right)^{\beta}\right\}$$

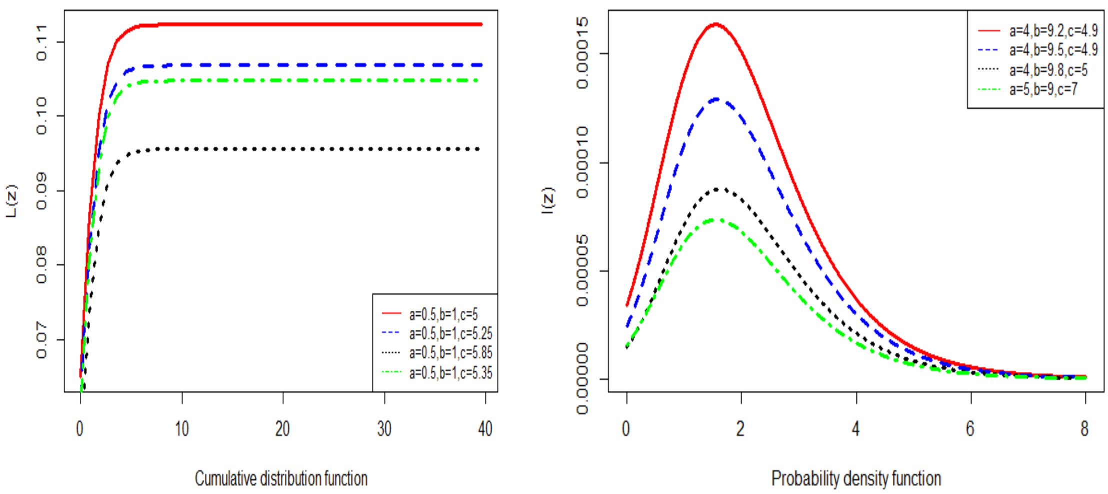

Figure 3.4: Plot of CDF and PDF of SEOGPL

The survival and Hazard rate function are:

$$S_{SEOGPL}(z) = 1 - \sin\left\{\frac{\pi}{2}\left(1 - e^{-\alpha \frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}}\right)^{\beta}\right\}$$

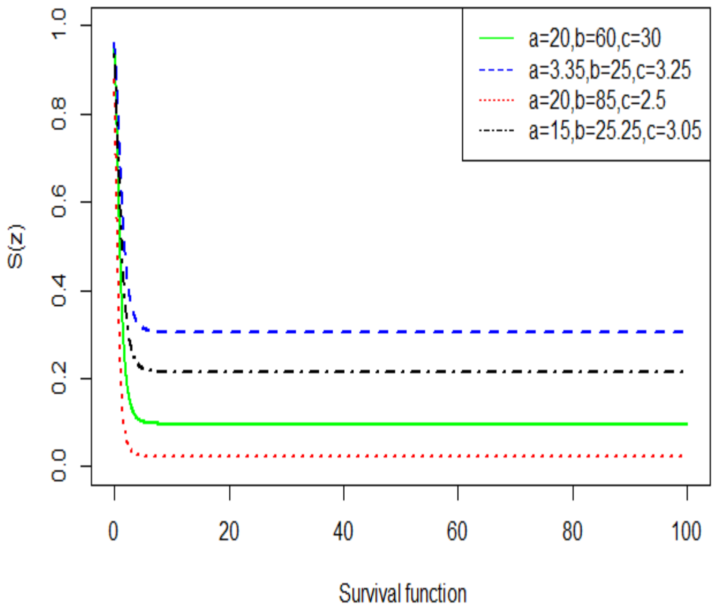

Figure 3.5: Plot of Survival Function of SEOGPL

$$H_{SEOGPL}(z) = \frac{\alpha\beta\pi}{2\hbar}\frac{\left(1+\frac{1}{1+e^{-x}}\right)^{-\frac{1}{\hbar}-1}\left\{\frac{e^{-x}}{(1+e^{-x})^{2}}\right\}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]^{2}}$$

$$\times e^{-\alpha\frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}}\left(1-e^{-\alpha\frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}}\right)^{\beta-1}$$

$$\frac{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}}\right)^{\beta}\right\}}{1-\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}}{\left[1-1-\left(1+\frac{1}{(1+e^{-x})}\right)^{-\frac{1}{\hbar}}\right]}}\right)^{\beta}\right\}}$$

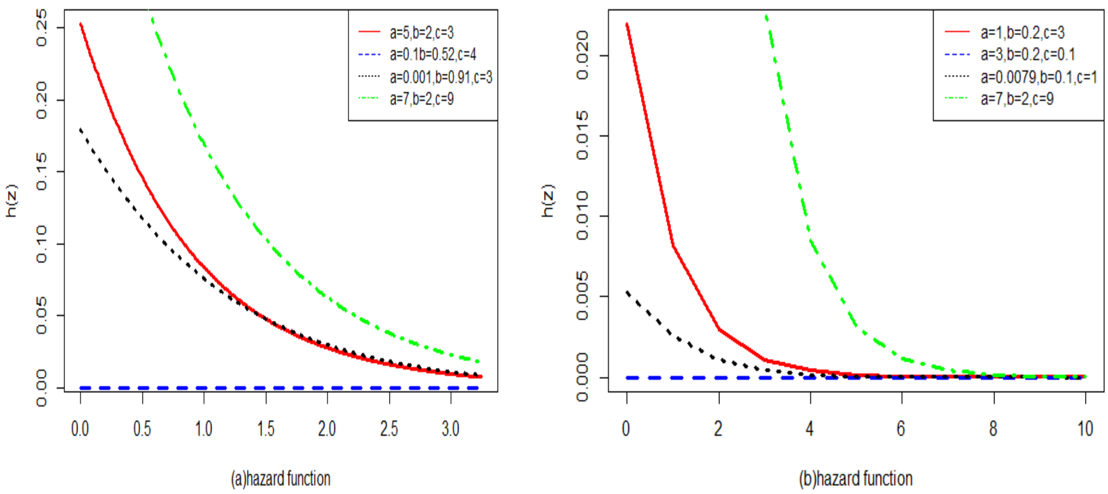

Figure 3.6: Plot of Hazard Function of SEOGPL

Chapter 4

Sin Exponentiated Odd Generalized Pareto Dagum Distribution

The Dagum distribution, named after Camilo Dagum, is a continuous probability distribution primarily used in economics to model income distribution. It is a flexible distribution that can capture the heavy tails and skewness often observed in real-world income data. The distribution is also utilized in other fields such as finance, hydrology, and reliability engineering. The Dagum distribution is known for its ability to model both the lower and upper tails of income distribution, making it suitable for analyzing economic inequality and wealth distribution. Its parameters can be estimated using various statistical methods, such as maximum likelihood estimation.

4.1 SEOGP Dagum Distribution

The distribution was first introduced in the early 1970s Dagum,Camilo (1977) [3],it is named after Camilo Dagum, who proposed it in a series of papers in the 1970s. The PDF and CDF of Dagum distribution is;

$$D(z) = \frac{ap}{z}\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^{a}+1\right\}^{p+1}}\right] \quad (4.1.1)$$

$$D(z) = \left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p} \quad (4.1.2)$$

Where p , a and b are the 'Shape' , 'Scale' and 'Location' parameters respectively.

By using $(4.1.1)$ and $(4.1.2)$ into $(3.3.1)$,$(3.3.2)$

The SEOGP Dagum distribution's CDF and PDF is,

$$S(z)_{SEOGPD}(z) = \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \quad (4.1.3)$$

$$\begin{aligned} s(z)_{SEOGPD}(z) = & \frac{\alpha\beta ap\pi}{2\hbar x}\left[1+\{1+(\frac{x}{b})^{-a}\}\right]^{-\frac{1}{\hbar}-1} \\ & \times\left\{\frac{(\frac{x}{b})^{ap}}{((\frac{x}{b})^{a}+1)^{p+1}}\right\} \\ & \times e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}} \\ & \times\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1} \\ & \times\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \\ & \times\left[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}\right]^{-2} \end{aligned} \quad (4.1.4)$$

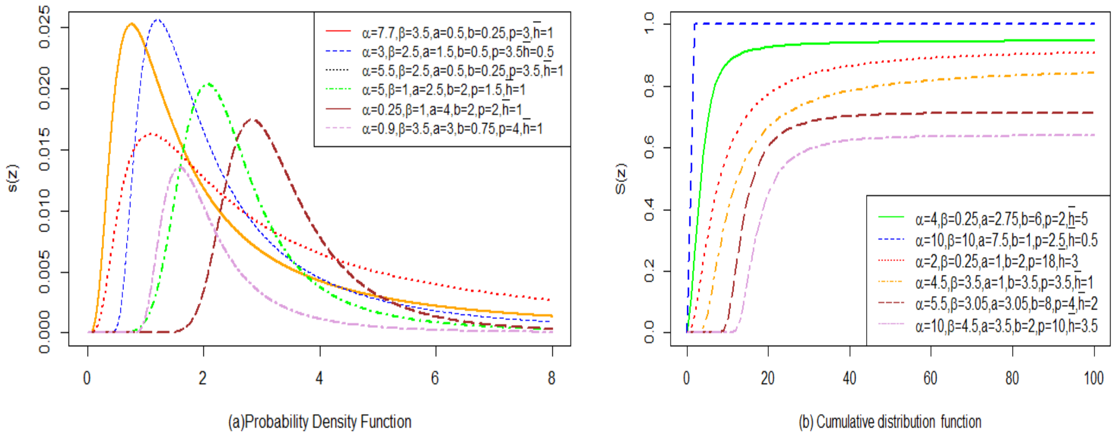

Figure 4.1: Plot of CDF and PDF of SEOGPD

Figure 4.1(a) presents the probability density function of the SEOGPD distribution. It is evident that the SEOGPD distribution exhibits positive skewness, characterized by a right-tailed curve. Additionally, the curve appears flatter when different parameter values are varied.

Figure 4.1(b) illustrates the cumulative distribution function (CDF) of the SEOGPD distribution. The curves exhibit a rising behavior at their peaks, and a monotonically increasing trend is observed throughout.

4.1.1 Validity of SEOGP Dagum Distribution

The entire area under the curve is one.

$$\int_{-\infty}^{\infty} f(z)dz = 1$$

By using the PDF of dagum distribution from

equation (4.1.2) and (4.1.4)

$$\int_0^\infty \frac{\alpha\beta ap\pi}{2\hbar x}\left[1+\{1+(\frac{x}{b})^{-a}\}\right]^{-\frac{1}{\hbar}-1}\left\{\frac{(\frac{x}{b})^{ap}}{((\frac{x}{b})^a+1)^{p+1}}\right\}$$
$$\times e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}$$
$$\times\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}$$
$$\times\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}$$
$$\times\left[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}\right]^{-2}dz$$

Let

$$w=\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}$$

when

$$z\longrightarrow 0 \qquad then \qquad w\longrightarrow 0$$
$$z\longrightarrow \infty \qquad then \qquad w\longrightarrow \frac{\pi}{2}$$

$$
\begin{aligned}
dw &= \frac{\alpha\beta ap\pi}{2\hbar x}\left[1+\{1+(\frac{x}{b})^{-a}\}\right]^{-\frac{1}{\hbar}-1}\left\{\frac{(\frac{x}{b})^{ap}}{((\frac{x}{b})^{a}+1)^{p+1}}\right\} \\
&\times e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}} \\
&\times\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1} \\
&\times\left[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}\right]^{-2}dz
\end{aligned}
$$

So,

$$
\begin{aligned}
\int_0^{\infty} t(z)dz &= \int_0^{\frac{\pi}{2}} \cos w dw \\
\int_0^{\infty} \cos w dw &= \sin w \\
\int_0^{\frac{\pi}{2}} \cos w dw &= \sin w\Big|_0^{\frac{\pi}{2}} \\
&= \sin\frac{\pi}{2} - \sin 0 \\
\sin\frac{\pi}{2} &= 1
\end{aligned} \tag{4.1.5}
$$

4.2 Statistical Properties

4.2.1 Survival Function

By using CDF from (4.1.3)

$$
S_{SEOGPD}(z) = 1-\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \tag{4.2.1}
$$

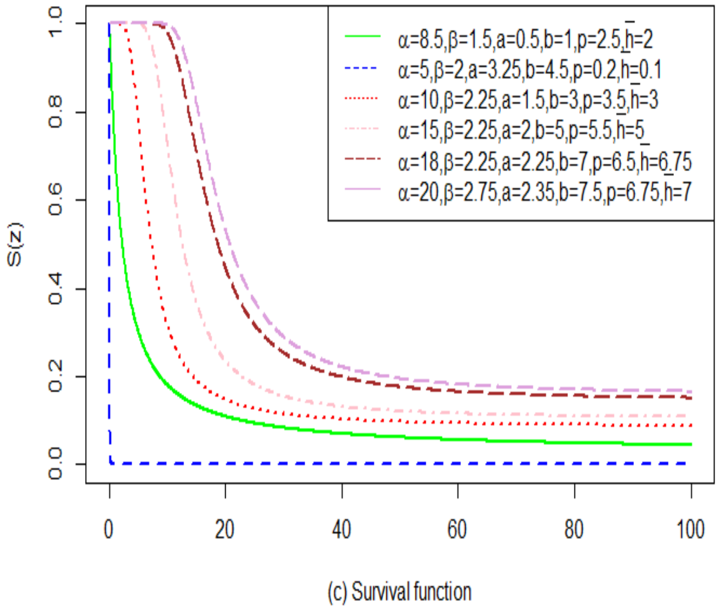

Figure 4.2: Plot of Survival Function of SEOGPD

Figure 4.2 depicts the survival function of the SEOGPD distribution. The figure demonstrates that the curves trend downward, indicating a decreasing pattern. Variation in parameters results in a further decreasing trend, leading to a gradual decline in the survival function.

4.2.2 Hazard Function

By using (4.1.3) and (4.1.4)

$$
\begin{aligned}
h_{SEOGPD}(Z) =& \frac{\alpha\beta ap\pi}{2\hbar x}\left[1+\{1+(\frac{x}{b})^{-a}\}\right]^{-\frac{1}{\hbar}-1}\left\{\frac{(\frac{x}{b})^{ap}}{((\frac{x}{b})^{a}+1)^{p+1}}\right\} \\
&\times e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}} \\
&\times\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1} \\
&\times\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \\
&\times\frac{\left[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}\right]^{-2}}{1-\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}
\end{aligned}
\tag{4.2.2}
$$

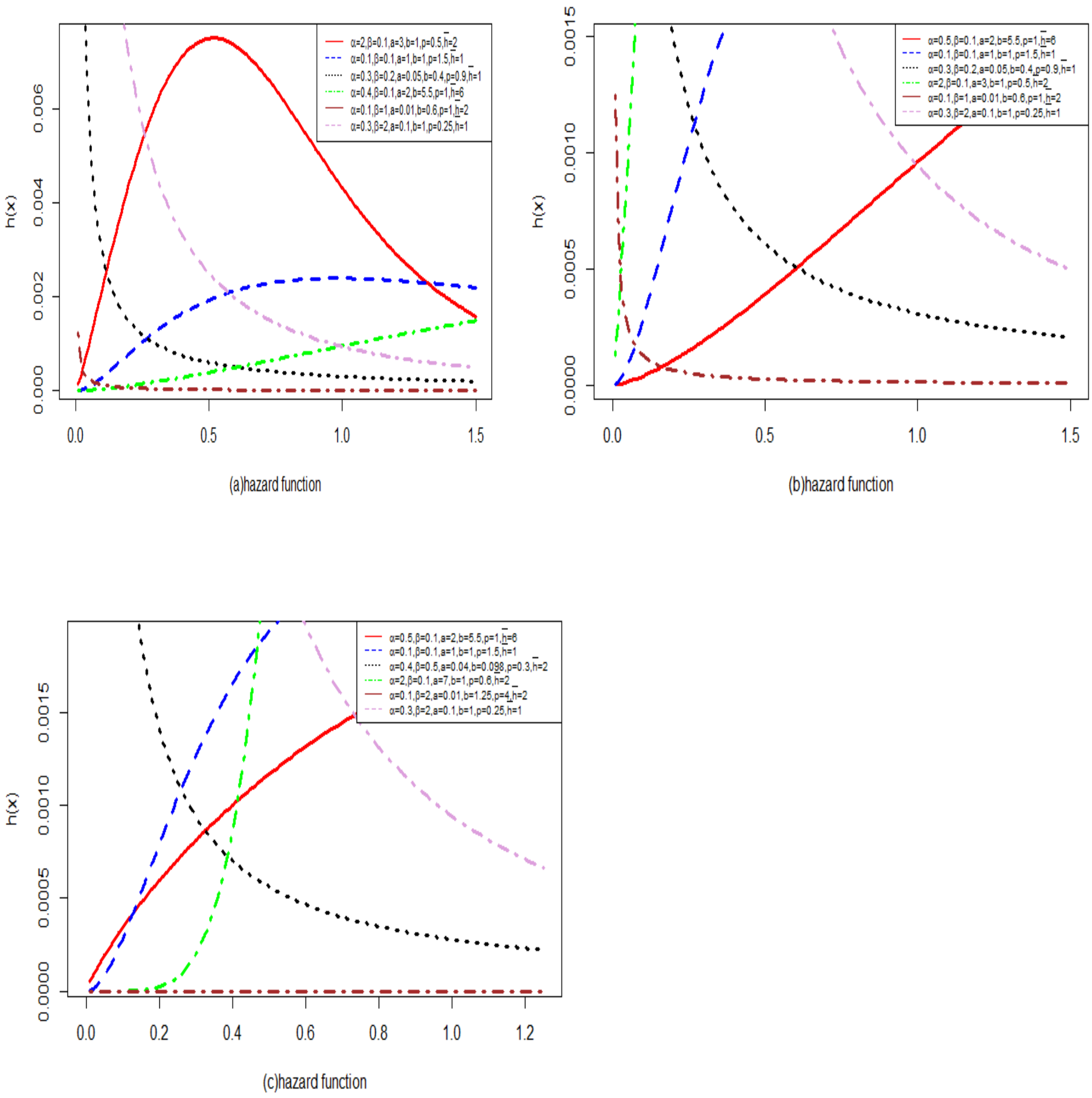

Figure 4.3: Plot of Hazard Function of SEOGPD

A comprehensive hazard function graph of SEOGPD distribution is presented in Figure (4.3(a)), (4.3(b)) and (4.3(c)) that exhibits all trends-initially decreasing, constant, and finally increasing-can be best represented by a bathtub-shaped hazard function. It can be seen that the curve illustrates an initial high hazard rate due to early failures, a subsequent period of low and constant hazard rate indicating a stable operational phase.

4.2.3 Quantile Function

As the CDF of SEOGPD from (4.1.3)

$$S_{SEOGPD}(z) = \sin\left\{\frac{\pi}{2}\left(1 - e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \tag{4.2.3}$$

$$S_{SEOGPD}(z) = U$$

$$\sin\left\{\frac{\pi}{2}\left(1 - e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} = U$$

$$\frac{\pi}{2}\left(1 - e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta} = \sin^{-1} U$$

$$\left(1 - e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right)^{\beta} = \frac{2}{\pi}\sin^{-1} U$$

$$\left(1 - e^{-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]}}\right) = \left(\frac{2}{\pi}\sin^{-1} U\right)^{\beta^{-1}}$$

$$e^{\frac{-\alpha(1-\{1+(1+(xb^{-1})^{-a})^{-p}\}^{-\hbar^{-1}})}{[1-\{1-(1+(1+(xb^{-1})^{-a})^{-p})^{-\hbar^{-1}}\}]}} = 1 - \left(\frac{2}{\pi}\sin^{-1} U\right)^{\beta^{-1}}$$

Taking log on b/s

$$-\alpha\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]} = \log\left\{1 - \left(\frac{2}{\pi}\sin^{-1} U\right)^{\frac{1}{\beta}}\right\}$$

$$\frac{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}]} = -\frac{1}{\alpha}\log\left\{1 - \left(\frac{2}{\pi}\sin^{-1} U\right)^{\beta^{-1}}\right\}$$

$$1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}} = \begin{aligned}[t] &[1-\{1-(1+(1+(\tfrac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\}] \\ &\times\left[-\tfrac{1}{\alpha}\log\left\{1-\left(\tfrac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right]\end{aligned}$$

$$\begin{aligned} 1-&\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}} \\ &=-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}+\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\} \\ &\quad\times\{1-(1+(1+(\frac{x}{b})^{-a})^{-p})^{-\frac{1}{\hbar}}\} \end{aligned}$$

$$\begin{aligned} \{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}\}&\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\} \\ &=-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\} \end{aligned}$$

$$\{1-\{1+(1+(\frac{x}{b})^{-a})^{-p}\}^{-\frac{1}{\hbar}}\}=-\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta-1}\right\}\right\}}$$

$$\left\{1+\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right\}^{-\frac{1}{\hbar}}=1+\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\}}$$

$$1+\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}=\left[1+\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\}}\right]^{-\hbar}$$

$$\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}=\left[1+\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta-1}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\}}\right]^{-\hbar}-1$$

$$1+\left(\frac{x}{b}\right)^{-a}=\left[\left[1+\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\}}\right]^{-\hbar}-1\right]^{-\frac{1}{p}}$$

$$\left(\frac{x}{b}\right)^{-a}=\left[\left[1+\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\}}\right]^{-\hbar}-1\right]^{-\frac{1}{p}}-1$$

$$\frac{x}{b}=\left[\left[\left[1+\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\}}\right]^{-\hbar}-1\right]^{-\frac{1}{p}}-1\right]^{-\frac{1}{a}}$$

$$x=b\left[\left[\left[1+\frac{\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}}{\left\{1-\frac{1}{\alpha}\log\left\{1-\left(\frac{2}{\pi}\sin^{-1}U\right)^{\beta^{-1}}\right\}\right\}}\right]^{-\hbar}-1\right]^{-\frac{1}{p}}-1\right]^{-\frac{1}{a}} \tag{4.2.4}$$

4.2.4 Moments

As r^{th} moment of SEOGPX family of distribution from (3.3.15)

$$u'_r=\frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}A_{ijkle}\int_R z^r D(z)\{1+D(z)\}^{-\left(\frac{e+1}{\hbar}+1\right)}dz$$

By using PDF and CDF of dagum distribution
from (4.1.1) and (4.1.2)

$$u'_r=\frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}A_{ijkle}\int_0^{\infty} z^r\frac{ap}{z}\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^{a}+1\right\}^{p+1}}\right] \times\left\{1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\right\}^{-\left(\frac{e+1}{\hbar}+1\right)}dz \tag{4.2.5}$$

$$u'_r = K\int_0^\infty z^r \frac{ap}{z}\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^a+1\right\}^{p+1}}\right] \times \left\{1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\right\}^{-\left(\frac{e+1}{\hbar}+1\right)} dz \tag{4.2.6}$$

Let,

$$K = \frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty} A_{ijkle}$$

Consider

$$\int_0^\infty z^r \frac{ap}{z}\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^a+1\right\}^{p+1}}\right]\left\{1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\right\}^{-\left(\frac{e+1}{\hbar}+1\right)} dz$$

Put

$$\begin{aligned}
\left\{1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\right\} &= \hbar \\
\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p} &= \hbar - 1 \\
1+\left(\frac{z}{b}\right)^{-a} &= (\hbar-1)^{-\frac{1}{p}} \\
\left(\frac{z}{b}\right)^{-a} &= (\hbar-1)^{-\frac{1}{p}} - 1 \\
\frac{z}{b} &= \{(\hbar-1)^{-\frac{1}{p}} - 1\}^{-\frac{1}{a}} \\
\Rightarrow z &= b\{(\hbar-1)^{-\frac{1}{p}} - 1\}^{-\frac{1}{a}}
\end{aligned}$$

$$\frac{ap}{z}\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^a+1\right\}^{p+1}}\right] dz = d\hbar$$

when

$$z \longrightarrow 0 \qquad then \qquad \hbar \longrightarrow 0$$

$$z \longrightarrow \infty \qquad then \qquad \hbar \longrightarrow 1$$

$$= \int_0^1 b^r \{(\hbar - 1)^{-\frac{1}{p}} - 1\}^{-\frac{r}{a}} \hbar^{-\left(\frac{e+1}{\hbar}+1\right)} d\hbar$$

Consider;

$$\{(\hbar - 1)^{-\frac{1}{p}} - 1\}^{-\frac{r}{a}}$$

As binomial serirs for negative exponents,

$$(x-1)^{-n} = \sum_{q=0}^{\infty} (-1)^q \begin{pmatrix} -n \\ q \end{pmatrix} x^q$$

After applying the series on above considered expression, we get,

$$=b^r \int_0^1 \hbar^{-\left(\frac{e+1}{\hbar}+1\right)} \sum_{t=0}^{\infty} (-1)^t \begin{pmatrix} -\frac{r}{a} \\ t \end{pmatrix} (\hbar - 1)^{-\frac{t}{p}} d\hbar$$

$$=b^r (-1)^t \begin{pmatrix} -\frac{r}{a} \\ t \end{pmatrix} \sum_{t=0}^{\infty} \int_0^1 \hbar^{-\left(\frac{e+1}{\hbar}+1\right)} (\hbar - 1)^{-\frac{t}{p}} d\hbar$$

$$=b^r (-1)^t \begin{pmatrix} -\frac{r}{a} \\ t \end{pmatrix} \sum_{t=0}^{\infty} \int_0^1 \hbar^{-\left(\frac{e+1}{\hbar}+1\right)} (\hbar - 1)^{\left(-\frac{t}{p}+1\right)-1} d\hbar$$

As the beta function of 1^{st}kind is

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \beta(\alpha, \beta)$$

So,

$$b^r (-1)^t \begin{pmatrix} -\frac{r}{a} \\ t \end{pmatrix} \sum_{t=0}^{\infty} \int_0^1 \hbar^{-\left(\frac{e+1}{\hbar}+1\right)} (\hbar - 1)^{\left(-\frac{t}{p}+1\right)-1} d\hbar$$

$$=b^r(-1)^t\sum_{t=0}^{\infty}\begin{pmatrix}-\frac{r}{a}\\ t\end{pmatrix}\beta\left\{\left(-\frac{e+1}{\hbar}\right),\left(\frac{-t}{p}+1\right)\right\}$$

Substituting it into (4.2.6)

$$u_r' =Kb^r(-1)^t\sum_{0}^{\infty}\begin{pmatrix}-\frac{r}{a}\\ t\end{pmatrix}\beta\left\{\left(-\frac{e+1}{\hbar}\right),\left(\frac{-t}{p}+1\right)\right\}$$

$$u_r' =\frac{\beta}{\hbar}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e,t=0}^{\infty}A_{ijkle}b^r(-1)^t\sum_{0}^{\infty}\begin{pmatrix}-\frac{r}{a}\\ t\end{pmatrix}$$

$$\beta\left\{\left(-\frac{e+1}{\hbar}\right),\left(\frac{-t}{p}+1\right)\right\}$$

4.2.5 Renyi Entropy

By using general form of Renyi Entropy of SEOGPX family of distribution from (4.1)

$$I_R = \frac{1}{1-\delta}\log\left[\left(\frac{\alpha\beta\pi}{2\hbar}\right)^{\delta}\delta\int_R\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}W_{pq}\right.$$

$$\left.\times\frac{[\{1+D(z)\}^{-\frac{1}{\hbar}-1}D(z)]^{\delta}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]^{2\delta}}e^{-\alpha(q+\delta)\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}}\right]dz$$

By using PDF and CDF of Dagum distribution

$$I_R = \frac{1}{1-\delta}\log\left[\left(\frac{\alpha\beta\pi}{2\hbar}\right)^{\delta}\delta\int_R\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}W_{pq}\right.$$

$$\times\frac{\left[\left\{1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\right\}^{-\frac{1}{\hbar}-1}\frac{ap}{z}\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^{a}+1\right\}^{p+1}}\right]\right]^{\delta}}{\left[1-\left\{1-\left(1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\right)^{-\frac{1}{\hbar}}\right\}\right]^{2\delta}}$$

$$\left.\times e^{-\alpha(q+\delta)\frac{1-\left\{1+\left\{1+\left(zb^{-1}\right)^{-a}\right\}^{-p}\right\}^{-\frac{1}{\hbar}}}{\left[1-\left\{1-\left(1+\left\{1+(zb^{-1})^{-a}\right\}^{-p}\right)^{-\frac{1}{\hbar}}\right\}\right]}}\right]dz$$

4.2.6 Order Statistics

The general form of Order Statistics of SEOGPX family of distribution from (4.2)

$$\begin{aligned} t_{h:n} &= \frac{\beta}{\hbar\beta(h,n-h+1)}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\sum_{m=0}^{n-h}\sum_{s,v=0}^{\infty} Z_c \\ &\quad \times\left\{\frac{1-\{1+D(z)\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+D(z))^{-\frac{1}{\hbar}}\}]}\right\}^{v} \\ &\quad \times D(z)\{1+D(z)\}^{-(\frac{e+1}{\hbar}+1)} \end{aligned}$$

by using PDF and CDF of Dagum distribution from (4.1.1) and (4.1.2),

$$\begin{aligned} t_{h:n} &= \frac{ap\beta}{z\hbar\beta(h,n-h+1)}\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{e=0}^{\infty}\sum_{m=0}^{n-h}\sum_{s,v=0}^{\infty} Z_c \\ &\quad \times\left[\frac{1-\{1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\}^{-\frac{1}{\hbar}}}{\left[1-\{1-(1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p})^{-\frac{1}{\hbar}}\}\right]}\right]^{v} \\ &\quad \times\left[\frac{\left(\frac{z}{b}\right)^{ap}}{\left\{\left(\frac{z}{b}\right)^{a}+1\right\}^{p+1}}\right]\left\{1+\left\{1+\left(\frac{z}{b}\right)^{-a}\right\}^{-p}\right\}^{-(\frac{e+1}{\hbar}+1)} \end{aligned} \tag{4.2.7}$$

4.2.7 Parameter Estimation

By using PDF and CDF of baseline distribution from (4.1.1) and (4.1.2) in (4.3),

$$
\begin{aligned}
Log(L) \quad = \quad & n\log\alpha + n\log\beta + n\log\frac{\pi}{2} + \left(-\frac{1}{\hbar} - 1\right) \\
& \times \log\sum_{w=1}^{n}(1 + \left[\left\{1 + \left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]) + \log\sum_{w=1}^{n}\left[\frac{ap}{z_w}\left[\frac{\left(\frac{z_w}{b}\right)^{ap}}{\left\{\left(\frac{z_w}{b}\right)^{a} + 1\right\}^{p+1}}\right]\right] \\
& -2\log\sum_{w=1}^{n}[1 - \{1 - (1 + \left[\left\{1 + \left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}] \\
& -\alpha\sum_{w=1}^{n}\frac{\{1 - (1 + \left[\left\{1 + \left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1 - \{1 - (1 + \left[\left\{1 + \left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]} \\
& +(\beta - 1)\log\sum_{w=1}^{n}\left(1 - e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right) \\
& +\log\sum_{w=1}^{n}\cos\left\{\frac{\pi}{2}\left(1 - e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \qquad (4.2.8)
\end{aligned}
$$

Differentiating the log-likelihood function of SEOGPD with respect to $\alpha, \beta, a, b, p, \hbar$ respectively and then putting results;

$$\frac{\partial LogL(z_w)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{w=1}^{n} \frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}$$

$$+ (\beta-1)\log\sum_{w=1}^{n} \frac{1}{\left(1-e^{-\alpha\frac{\left\{1-(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p})^{-\frac{1}{\hbar}}\right\}}{[1-\{1-\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{-\frac{1}{\hbar}}\}]}}\right)}$$

$$\times e^{-\alpha\frac{\left\{1-(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p})^{-\frac{1}{\hbar}}\right\}}{[1-\{1-\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{-\frac{1}{\hbar}}\}]}}$$

$$\times \frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}$$

$$- \sum_{w=1}^{n} \frac{1}{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}$$

$$\times \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}$$

$$\times \frac{\pi\beta}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}$$

$$\times e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}} \tag{4.2.9}$$

$$
\begin{aligned}
\frac{\partial LogL(z_w)}{\partial \beta} &= \frac{n}{\beta} + \log \sum_{w=1}^{n} \left(1 - e^{-\alpha \frac{\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}} \right) \\
&\quad - \Sigma_{w=1}^{n} \frac{1}{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha \frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}} \\
&\quad \times \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha \frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\} \\
&\quad \times \frac{\pi}{2}\left(1-e^{-\alpha \frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta} \\
&\quad \times log\left(1-e^{-\alpha \frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right) \qquad (4.2.10)
\end{aligned}
$$

$$\frac{\partial LogL(z_w)}{\partial p} = -\Big(-\frac{1}{\hbar}-1\Big)\sum_{w=1}^{n}\frac{1}{(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big])}$$

$$\times\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}^{-p}\log\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}$$

$$+\Sigma_{w=1}^{n}\frac{1}{\Big[\frac{ap}{z_w}\Big[\frac{(\frac{z_w}{b})^{ap}}{\{(\frac{z_w}{b})^{a}+1\}^{p+1}}\Big]\Big]}\Big[\Big[\frac{a}{z_w}\Big[\frac{(\frac{z_w}{b})^{ap}}{\{(\frac{z_w}{b})^{a}+1\}^{p+1}}\Big]\Big]$$

$$+\frac{1}{z_w\Big\{(\frac{z_w}{b})^{a}+1\Big\}^{p+2}}\Big(\frac{z_w}{b}\Big)^{ap}[\{\Big(\frac{z_w}{b}\Big)+1\}+a\log\Big(\frac{z_w}{b}\Big)-(p+1)\log\{\Big(\frac{z_w}{b}\Big)^{a}+1\}]\Big]$$

$$-\frac{2}{\hbar}\sum_{w=1}^{n}\frac{1}{[1-\{1-(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big])^{-\frac{1}{\hbar}}\}]}\Big(1+\Big[\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}^{-p}\Big]\Big)^{-\frac{1}{\hbar}-1}$$

$$\times\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}^{-p}\log\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}+\frac{\alpha}{\hbar}\log\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}^{-p-1}$$

$$\times\Big(1+\Big[\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}^{-p}\Big]\Big)^{-\frac{1}{\hbar}}-(\beta-1)\sum_{w=1}^{n}\frac{1}{\left(1-e^{-\alpha\frac{\{1-(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big])^{-\frac{1}{\hbar}}\}]}}\right)}$$

$$\times e^{-\alpha\frac{\{1-(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big])^{-\frac{1}{\hbar}}\}]}}\frac{\alpha}{h\Big(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big]\Big)^{-\frac{1}{\hbar}}}\log\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}$$

$$\times\Big\{1+\Big(\frac{z_w}{b}\Big)^{-a}\Big\}^{-p-1}-\sum_{w=1}^{n}\frac{1}{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\Big[\Big\{1+(\frac{z_w}{b})^{-a}\Big\}^{-p}\Big])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}$$

$$\times \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}$$

$$\times\frac{\pi\beta}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}$$

$$\times e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\frac{\alpha}{h\left(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\right)^{-\hbar}}$$

$$\times \log\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p-1} \tag{4.2.11}$$

$$\frac{\partial LogL(z_w)}{\partial a} = -p\left(-\frac{1}{\hbar}-1\right)\sum_{w=1}^{n}\frac{1}{(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])}$$

$$\times\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p-1}\log\left(\frac{z_w}{b}\right)^{-a}+\sum_{w=1}^{n}\frac{1}{\left[\frac{ap}{z_w}\left[\frac{\left(\frac{z_w}{b}\right)^{ap}}{\left\{\left(\frac{z_w}{b}\right)^{a}+1\right\}^{p+1}}\right]\right]}$$

$$\times\left[\frac{p\left(\frac{z_w}{b}\right)^{ap}}{z_w\left\{\left(\frac{z}{b}\right)^{a}+1\right\}^{p+1}}+\frac{ap\left(\frac{z_w}{b}\right)^{ap}\log\left(\frac{z_w}{b}\right)\left[ap\left\{\left(\frac{z_w}{b}\right)^{a}+1\right\}-(p+1)\left(\frac{z_w}{b}\right)^{a}\right]}{\left\{\left(\frac{z_w}{b}\right)^{a}+1\right\}^{p+2}}\right]$$

$$+\sum_{w=1}^{n}\frac{2}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}$$

$$\times\left[\frac{p\left(\frac{z_w}{b}\right)^{ap}\log\left(\frac{z_w}{b}\right)}{\hbar\left[1+\left\{1+\left(\frac{z_w}{b}\right)^{-ap}\right\}\right]^{-\frac{1}{\hbar}+1}\left\{1+\left(\frac{z_w}{b}\right)^{-ap}\right\}^{2}}\right]$$

$$+\alpha\sum_{w=1}^{n}\frac{\frac{p}{\hbar}\log\left(\frac{z_w}{b}\right)\left[1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]^{-\frac{1}{\hbar}-1}\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p-1}\left(\frac{z_w}{b}\right)^{-a}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]^{2}}$$

$$+\frac{\alpha p}{\hbar}(\beta-1)\sum_{w=1}^{n}\frac{e^{\frac{\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]}{\left[1-\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]\right]}}}{\left(1-e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)}\frac{\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]}{\left[1-\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]\right]}$$

$$\times\left(\frac{z_w}{b}\right)^{-a}\log\left(\frac{z_w}{b}\right)\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-1}$$

$$
-\sum_{w=1}^{n}\frac{1}{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}
$$

$$
\times\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}
$$

$$
\times\frac{\pi\beta}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}
$$

$$
\times e^{\frac{\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]}{\left[1-\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]\right]}}\frac{\alpha p}{\hbar}\frac{\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]}{\left[1-\left[1-\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-\frac{p}{\hbar}}\right]\right]}
$$

$$
\left(\frac{z_w}{b}\right)^{-a}\log\left(\frac{z_w}{b}\right)\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-1} \qquad (4.2.12)
$$

$$\frac{\partial LogL(z_w)}{\partial b} = -\left(-\frac{1}{\hbar}-1\right)\sum_{w=1}^{n} \frac{p\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p-1} a z_w^{-a} b^{a-1}}{\left[1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]}$$

$$+ n\sum_{w=1}^{n} \frac{\frac{ap}{z_w} z_w^{ap} b^{-ap-1}\left\{a\left(\frac{z_w}{b}\right)^{a} + a - ap\right\}}{\left[\frac{ap}{z_w}\left[\frac{\left(\frac{z_w}{b}\right)^{ap}}{\left\{\left(\frac{z_w}{b}\right)^{a}+1\right\}^{p+1}}\right]\right]\left\{1+\left(\frac{z_w}{b}\right)^{a}\right\}^{p+1}}$$

$$-\frac{2np}{\hbar}\sum_{w=1}^{n} \frac{(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}$$

$$\times \left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p-1}\left(\frac{z_w}{b}\right)^{-a}\log\left(\frac{z_w}{b}\right)$$

$$+\alpha\sum_{w=1}^{n} \frac{paz_w^{1-a}}{\hbar b^{a+2}}\left[1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]^{\frac{1}{\hbar}-1}\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p-1}$$

$$-(\beta-1)\sum_{w=1}^{n} \frac{1}{\left(1-e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)}$$

$$\times e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\frac{\alpha paz_w^{1-a}}{\hbar b^{a+2}}\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{\frac{1}{\hbar}+p+1}$$

$$\times\left[1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]^{2\left(\frac{1}{\hbar}+p+1\right)}$$

$$
-\sum_{w=1}^{n}\frac{1}{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}
$$

$$
\times\sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}
$$

$$
\times\frac{\pi\beta}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}
$$

$$
\times e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\frac{az_w\left(\frac{z_w}{b}\right)^{-a}}{b^2}
\tag{4.2.13}
$$

$$
\begin{aligned}
\frac{\partial LogL(z_w)}{\partial \hbar} =& -\frac{2\log\sum_{w=1}^{n}\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)}{\hbar^2} + \alpha\frac{\log\left[1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]}{\hbar^2} \\
& \frac{\alpha}{\hbar^2}\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{\frac{1}{\hbar}} \\
& \times 1-\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{-\frac{1}{\hbar}}\log\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right) \\
& -(\beta-1)\sum_{w=1}^{n}\frac{e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}}{\left(1-e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)} \\
& \times \alpha\frac{\log\left[1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]}{\hbar^2} \\
& -(\beta-1)\sum_{w=1}^{n}\frac{e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}}{\left(1-e^{-\alpha\frac{\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)} \\
& \times \frac{\alpha}{\hbar^2}\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{\frac{1}{\hbar}} \\
& \times 1-\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{-\frac{1}{\hbar}}\log\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)
\end{aligned}
$$

$$
-\sum_{w=1}^{n} \frac{1}{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}
$$

$$
\times \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}
$$

$$
\times \left\{\frac{\pi\beta}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}\right\}
$$

$$
\times e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}
$$

$$
\times \alpha\frac{\log\left[1+\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]}{\hbar^{2}}
$$

$$
-\sum_{w=1}^{n} \frac{1}{\cos\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{zw}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}}
$$

$$\times \sin\left\{\frac{\pi}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta}\right\}$$

$$\times\left\{\frac{\pi\beta}{2}\left(1-e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}\right)^{\beta-1}\right\}$$

$$\times e^{-\alpha\frac{1-\{1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right]\}^{-\frac{1}{\hbar}}}{[1-\{1-(1+\left[\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right])^{-\frac{1}{\hbar}}\}]}}$$

$$\times\frac{\alpha}{\hbar^2}\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{\frac{1}{\hbar}}$$

$$\times 1-\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right)^{-\frac{1}{\hbar}}\log\left(1+\left\{1+\left(\frac{z_w}{b}\right)^{-a}\right\}^{-p}\right) \quad (4.2.14)$$

The form of equations is not closed. Therefore it seems difficult to simply calculate the values of parameter. To obtain MLEs,we utilize an iteration process similar to the Newton Raphson technique, which is employed in mathematics.

Chapter 5

Applications and Simulations:

This chapter applies the Sin Exponentiated Odd Generalized Pareto Dagum (SEOGPD) distribution to the real dataset to demonstrate the practical benefits and applicability of the proposed model. Maximum likelihood estimators (MLEs) for the model parameters are calculated, and several goodness-of-fit statistics are computed. In order to assess the performance of the Maximum likelhood Estimates (MLE's), some Monte Carlo simulation study for various sample sizess has been conducted. The study utilized R-language software for data generation, ensuring robustness across iterations to evaluate the suitability of the fitted models. .

5.1 Application

The estimated CDFs of the data sets and the PDFs of the fitted distributions are presented to visually assess the goodness of fit. Additionally, Probability-Probability (P-P) plots are discussed to further evaluate the fit of the distributions. We compare the Sin Exponentiated Odd Generalized Pareto Dagum (SEOGPD) Distribution with Weighted Dagum (WD), Topp Leone Dagum (TLD) distribution and Dagum (D) distribution. For SEOGPD distribution

we estimate the unknown parameters by MLE to obtain the numerical values and then we used these estimated values to obtain the the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). In general the smaller the value of AIC, CAIC, BIC, the better the fit to the data.
These information criterions are defined as:

$$AIC = 2p - 2(l)$$
$$CAIC = \frac{2pn}{n-p-1} - 2(l)$$
$$BIC = p\log(n) - 2(l)$$

Where l denotes the Log-Likelihood function evaluated at the MLEs, n is the sample size and p is the number of parameters.

Data set:Failure of ball bearing Broderick O. Oluyede and Yuan Ye (2014) [9]: The dataset includes 22 values, represents the number of million revolutions before failure of each of 22 ball bearing in a life testing experiment.The data are: 17.88, 28.92, 33.00,41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12,93.12, 98.64, 105.12, 127.92, 128.04, 173.40. Summary of Data set is given below:

Table 5.1: Summary of the data set.

Min.	Max.	Q_1	Median	Mean	Q_3
17.88	173.40	46.40	62.68	70.71	90.87

Table 5.2: MLEs of the data sets' considered distributional parameters.

	α	β	a	b	p	$\hbar$
SEOGPD	**22.2675**	**3.1174**	**0.0260**	**23.2743**	**3.7373**	**0.1059**
WD	156.36	2.8208	97.1548	0.5	1	-
WD	113.10	2.5908	101.60	0	0.5	-
D	138.47	2.5890	124.19	-	-	-
TLD	2.2143	9.8580	15.2610	1.2920	-	-

The SEOGPD distribution's maximum likelihood estimates are shown in Table 5.2. Figure 4.4 shows the Sin Exponentiated Odd Generalized Pareto Dagum distribution for failure of ball bearing data set in both its theory base and empirical density and distribution function forms. The Q-Q and P-P plot of the data is shown in Fig. 5.2.The dataset's summary is shown in Table 5.1

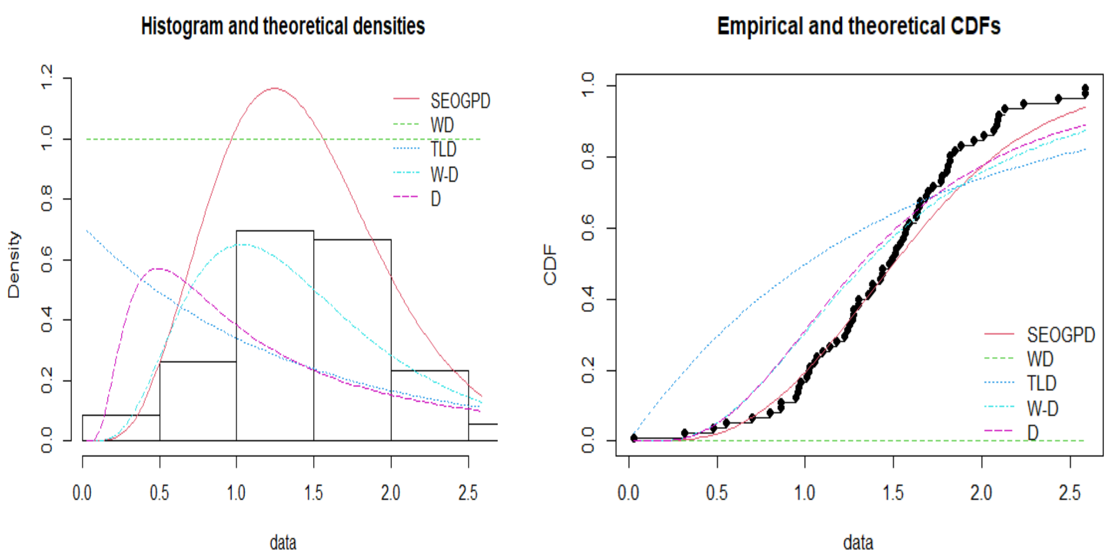

Figure 5.1: SEOGPD's empirical and theoretical CDF and PDF

??

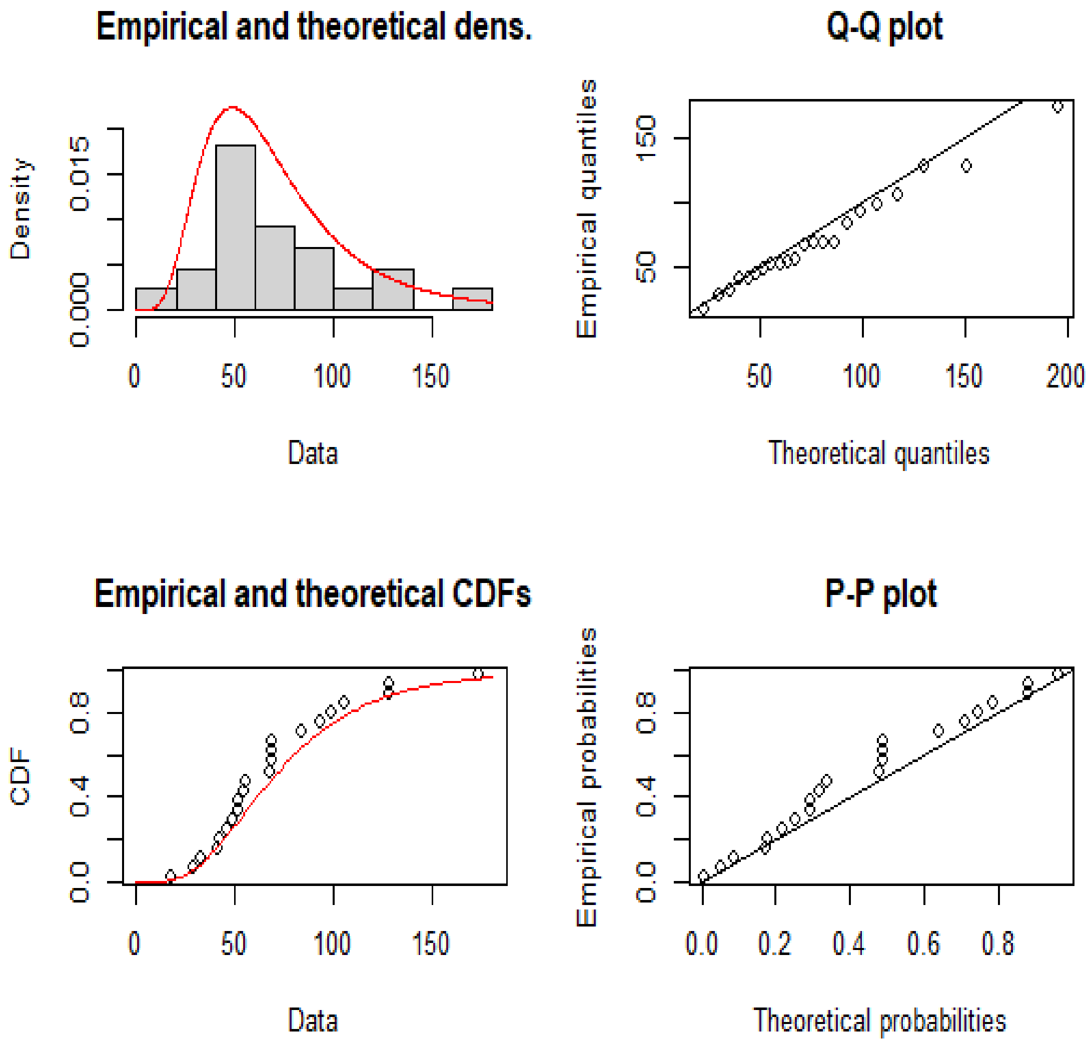

Figure 5.2: Empirical and theory-based CDF and PDF of SEOGPD with Q-Q and P-Plots

Table 5.3: Goodness of fit tests.

	AIC	CAIC	BIC
SEOGPD	**139.8679**	**145.4679**	**146.4142**
WD	189.2	190.6	192.5
WD	192.7	194.0	195.9
D	192.6	193.9	195.9
TLD	225.6006	227.9535	229.9648

The distribution that best fits the data is the one with the lowest AIC, CAIC and BIC value. Table 5.3 makes it very clear that the Sin Exponentiated Odd Generalized Pareto Dagum distribution has lower AIC, CAIC, BIC values than other distributions.As a result, the SEOGPD distribution gives a better fit for the failure of ball bearing dataset.

5.2 Simulation

Equation (4.2.4) facilitates data generation from the SEOGPD distribution for simulation analysis. The experiment was conducted 5000 times using sample sizes n = 5, 10, 15, 20, 30, 50, 100, 200, 300, 400, and 500, with fixed parameter values (α=22.26,β=12.75,a=0.22,b=23.27,p=3.73,$\hbar$=0.10). Mean square error (MSE) and average bias were estimated for each sample size. The results indicate that increasing sample size leads to decreased MSE, reflecting improved parameter estimation precision. Table 5.4 presents the Maximum Likelihood Estimates of parameters for the SEOGPD distribution, highlighting better estimation performance with larger sample sizes.

Table 5.4: SEOGPD distribution's estimated mean, MSE, and bias, together with the actual value of α, β, a , b , p and $\hbar$.

	TrueValues:	$\alpha = 22.26$	$\beta = 12.75$	$a = 0.22$	$b = 23.27$	$p = 3.73$	$\hbar = 0.09$
n		$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{p}$	$\hat{\hbar}$
	MLE	21.422196	16.522723	1.662633	13.212436	0.118125	1.731983
5	MSE	584.1792	79.51356	5.222816	41.68017	32.58051	9.540323
	Bias	23.73434	1.886987	1.853664	2.865974	4.036395	2.273738
	MLE	26.59837108	15.49031330	2.08404102	21.61127325	0.07967888	2.06706141
10	MSE	574.8169	70.61935	5.178344	32.1801	31.99716	6.605873
	Bias	23.1546	6.493652	1.984118	2.442783	3.94276	2.130953
	MLE	25.24526667	10.51031352	1.933614	21.08029	0.086033	2.326776
15	MSE	574.0595	61.1189	5.395929	27.79096	30.4017	6.510134
	Bias	23.3302	5.950414	1.987326	2.472379	3.903733	2.099888
	MLE	18.311391	15.1787984	3.05787577	19.4193613	0.04727273	1.5427904
20	MSE	568.1449	56.38444	5.382622	23.04276	29.80969	6.36907
	Bias	22.25758	4.204921	2.035229	2.465319	3.88241	2.076168
	MLE	28.5458095	10.3217341	10.3217341	15.9218487	0.1212167	2.7451825
30	MSE	559.8328	55.53155	5.381536	20.66568	28.79468	6.049398
	Bias	23.72167	0.6397051	2.092392	2.509564	4.048388	2.120159
	MLE	25.91077923	11.087508	2.02125982	18.35803611	0.0819831	2.343994
50	MSE	554.9358	45.98361	5.357567	19.82587	28.49601	6.010648
	Bias	23.21626	2.733883	2.135666	2.605664	3.852379	2.03456
	MLE	27.083567	18.565236	1.654387	23.038749	0.1005674	2.034479
100	MSE	535.1887	13.8246	5.167891	16.05231	24.65018	5.80416
	Bias	23.67285	7.220444	2.154102	2.611894	3.94421	2.057745
	MLE	22.702967	15.190682	2.639692	16.506047	0.061300	1.874427
200	MSE	535.1437	12.92415	4.955364	15.30843	23.78674	5.482224
	Bias	23.0051	0.472614	2.176154	2.909089	3.897607	2.072236
	MLE	25.806402	15.743974	3.239066	18.170094	0.0524978	2.010955
300	MSE	534.2956	8.975011	4.912563	14.10367	22.62131	5.106631
	Bias	22.3795	1.972983	2.080646	2.806317	4.019328	2.097709
	MLE	22.231129	13.786307	2.6964054	19.4368647	0.0584625	1.8902943
400	MSE	509.1411	6.630032	4.782713	14.28973	19.01862	4.528489
	Bias	22.98282	3.264705	2.130274	2.976935	3.975925	2.067014
	MLE	25.202923	18.102866	2.7419861	23.535192	0.055537	1.946666
500	MSE	507.31	3.22392	4.759553	13.1892	17.02352	4.32846
	Bias	22.99664	7.29364	2.130087	2.911927	4.011864	2.107747

5.3 Conclusion

In this research work, A new more flexible distribution known as"Sin Exponentiated Odd Generalized Pareto-X family of distribution (SEOGPX)" has been proposed. A novel and more versatile generator is introduced in the initial approach. In the subsequent approach, this new generator is applied to the baseline distribution to develop an extension of the Dagum distribution. Several statistical properties of the new distribution are analyzed, including moments, incomplete r^{th} moment, hazard function, survival function, order statistics, and Rényi entropy.

The SEOGPD model exhibited significant flexibility in modeling the dataset compared to other distributions evaluated in the study, highlighting its superior efficacy over several well-established distributions.Given that the SEOG-PD distribution has hazard ratio functions that can take the form of an upside-down bathtub, a bathtub shape, as well as increasing, decreasing, constant, and increasing-decreasing-increasing patterns depending on different parameter values, it is considered a flexible distribution for modeling. The simulation findings show that as sample sizes rise, the mean squared errors (MSEs), gradually decline.By comparing the values of criteria AIC, CAIC, and BIC statistics, it is evident that the values for the SEOGPD distribution are significantly lower than those of other existing distributions.

5.4 Recommendations

Future research could explore using this model on different datasets and compare its performance with newer distributions. The estimation of parameters for the Sin Exponentiated Odd Generalized Pareto Dagum (SEOG-PD) distribution can be studied using various methods, including the ordinary least squares method, weighted least squares method, Bayesian estimation, and the method of parametric bootstrap. This could lead to advancements in statistical analysis and modeling in various scientific areas.

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