The influence of magnetic field on peristaltic flow of Oldroyd-8 constant fluid in a curved channel

By

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- Magnetic fields
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- Curved Channel.

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as confirming to the required standard.

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Dedicated to

I dedicate this dissertation to my family, especially to my father and mother for instilling the importance of hard work and higher education.

Acknowledgements

In the name of Allah, the Beneficent, the merciful, all praises to Allah for providing me opportunity to come to one of the best universities of my homeland and granting me the capability to do work efficiently.

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Preface

Peristalsis is an important mechanism generated by the propagation of waves along the walls of a channel or a tube. The peristaltic flows are very significant in physiology and industry. It involves in the chyme movement in the gastrointestinal tract, the vasomotion of small blood vessels such as venules, capillaries, arterioles and so forth etc. In industrial applications, it is involved in artificial heart and ortho-pumps, transport of toxic material and others. In view of such physiological and industrial applications, the peristaltic flow has been studied with great interest by various researchers, for viscous and non-Newtonian fluids [*1-8*]. Since most of the fluids, for instance, polymer solutions, soaps, blood, ketchup, shampoo and greases are of non-Newtonian nature. Because of their wide applications in technology viscoelastic fluids have gained significant importance in the last few years [*9-11*]. In fact, the physiological duct and glandular ducts are curved in shape. It is observed that curved channels in industrial and physiological processes are more practical than the straight channel. Some important studies related to the Peristaltic flow in curved channel are cited in [*12-16*]. Magneto-hydrodynamic is the science which deals with motion of a highly conducting fluid across the magnetic field. Some researchers have considered the effect of magnetic field on peristaltic flow of Newtonian and non-Newtonian fluids [*17-18*].

This thesis is organized in the following way.

Chapter 1 contains some basic definitions and equations.\newline

Chapter 2 is related with the peristaltic transport of a Newtonian fluid in a curved channel [*16*]. Analytic solution is carried out under long wavelength approximation. The effect of a pertinent parameter on the axial velocity, pressure gradient, pressure rise and streamlines is also examined graphically.\newline

In chapter 3, we studied the influence of magnetic field on peristaltic flow of Oldroyd –8 constant fluid in a curved channel. The reduced equations are solved by using perturbation method and finite difference method. The various impacts of emerging parameters are studied in detail through graphs.

Contents

Chapter 1

Preliminaries

1.1 Introduction

The aim of this chapter is to provide the relevant definitions and laws.

1.2 Fluid

A substance that continuously deform under an applied shear stress is known as fluid.

1.3 Classification of fluids

1.3.1 Ideal fluids

A fluid with zero viscosity is called ideal fluid i.e, it offers no resistance.

1.3.2 Real fluids

It is a fluid in which viscosity is not zero.

1.3.3 Newtonian fluids

In a fluid if the viscous stresses that arises from its flow, at every point, are directly proportional to the strain rate then the fluid is said to be Newtonian.

1.3.4 Non-Newtonian fluids

A non-Newtonian fluid is a fluid whose flow properties differ in any way from those of a newtonian fluids. Most commonly the viscosity of non-Newtonian fluids is dependent on shear rate.

1.4 Flow

A material goes under deformation when different forces act upon it. If the deformation is continuously increases without limit, then the phenomena is known as flow.

1.5 Types of flows

1.5.1 Uniform flow

A flow in which velocities of liquid particles at all sections of the pipe or channel are same. This term is generally applied to flow in channel.

1.5.2 Non-uniform flow

A flow in which the velocities of the liquid particles are not same at all sections of the pipe.

1.5.3 Laminar flow

It is a flow in which each liquid particle has a definite path and the path of individual particles do not cross each other.

1.5.4 Turbulent flow

It is a flow in which each liquid particle does not have a definite path and the path of individual particles also cross each other.

1.5.5 Steady and unsteady flows

A flow in which the quantity of liquid flowing per second is constant is called steady flow. A steady flow may b uniform or non-uniform. whereas a flow in which the quantity of liquid flowing per second is not constant is called unsteady flow.

1.5.6 Incompressible and compressible flows

A flow in which the volume and the density of the flowing fluid does not change during the flow. All liquids are generally considered to have incompressible fluid.

A compressible flow is one in which the volume and thus the density of the flowing fluid changes during the flow. All the gases are generally considered to have compressible fluid.

1.6 Basic definitions

1.6.1 Density

It is defined as mass per unit volume. Mathematically, it is denoted by ρ and defined as

$$\rho = \frac{m}{V}, \tag{1.1}$$

where ρ is the density, m is the mass, and V is the volume.

1.6.2 Viscosity

In fluids the measure of its resistance to gradual deformation by shear stress is called viscosity. Mathematically, viscosity is the ratio of shear stress to shear strain i.e.

$$\text{viscosity} = \mu = \frac{\text{shear stress}}{\text{shear strain}}, \tag{1.2}$$

where μ is called the coefficient of viscosity.

1.6.3 Viscoelasticity

The property of material exhibiting both viscous and elastic characteristics when undergoing deformation is called viscoelasticity. Kinds of fluids which describe such behaviour are known as viscoelastic.

1.6.4 Shear thinning effect

It is an effect in which viscosity decreases with increasing rate of shear stress. Fluids describing such effects are termed as Pseudoclassic.

1.6.5 Shear thickening effect

Shear thickening effect is one in which viscosity of a fluid increases with the rate of shear stress. Fluids which posses such effect are called dilatant.

1.6.6 Body force

Body forces are those forces which depend upon the mass of the fluid in control volume e.g weight, electromagnetic force and gravity.

1.6.7 Magnetic field

The magnetic influence of electric currents and magnetic materials is said to be magnetic field. It is a vector field, because at any given point, it is specified by both direction and magnitude.

1.6.8 Magnetohydrodynamics

Magnetohydrodynamics is the science that deals with the study of dynamics of electrically conducting fluids.

1.6.9 Curvilinear coordinate system

A coordinate system for Euclidean space in which the coordinate lines may be curved is known as curvilinear coordinate system.

1.6.10 Stream function

A stream function is a function which describes the form of pattern of flow, or in other words it is the discharge per unit thickness. Mathematically, for steady state two dimensional flow field, we may have

$$\psi = f(x, y), \tag{1.3}$$

where ψ is the stream function, f is the coefficient of stream function and x, y are coordinates of the the point where the stream function is required to be found out.

A stream function a mathematical expression that describes flow field in terms of either mass flow rate, for compressible fluid, or volume flow rate, for incompressible fluid.

1.7 Dimensionless number

1.7.1 Reynolds number

Ratio of the inertial forces to the viscous forces is said to be the Reynolds number. It is denoted by the symbol Re. Mathematically,

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}}, \tag{1.4}$$

1.7.2 Wave number

It is defined as

$$\delta = \frac{a_1}{\lambda}, \tag{1.5}$$

where $a_1$is the half width of the channel and λ is the wavelength of the wave.

1.7.3 Hartmann number

The Hartman number is the ratio of lorentz force to the viscous force. It is denoted by

$$Ha = \frac{\text{Lorentz force}}{\text{viscous force}}, \tag{1.6}$$

in which Ha is the Hartman number.

1.7.4 Curvature parameter

It is defined as

$$k = \frac{R^*}{a_1}, \tag{1.7}$$

where R^* is the radius and a_1 is the half thickness of the channel.

1.8 Maxwell's equation

The set of four equation which relate the electric and magnetic field to their sources, charge density and current density are Maxwell's equations. These equations are given by

$$\nabla.\mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{1.8}$$

$$\nabla.\mathbf{B} = 0, \tag{1.9}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.10}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{1.11}$$

where ρ is the total charge density, ϵ_0 is the permittivity of the free space, μ is the permeability of free space and $\mathbf{J}$ is the total current density.

1.9 Equation of continuity

Matter cannot be made or destroyed, and so the total mass of a fluid element must remain the same. Thus if the density of a fluid element decreases, its volume must expand accordingly. This expansion causes a divergence of the velocity field, giving the conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0, \tag{1.12}$$

Above, ρ is the fluid density. Eq. (1.12) is called continuity equation. It is also named as law of conservation of mass. For incompressible fluid ρ is constant. Thus Eq. (1.12) reduces into the following

$$\nabla.\mathbf{V} = 0, \tag{1.13}$$

where

$$\nabla.\mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{1.14}$$

1.10 Equation of Momentum

In an isolated system, the total momentum of the system remains same when some bodies act upon one another. In inertial frame of reference, the general form of equations of fluid motion or the law of conservation of momentum is

$$\rho\frac{d\mathbf{V}}{dt} = \nabla.\mathbf{T} + \mathbf{b}, \tag{1.15}$$

where $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{b}$ is the body force.

1.11 Method of solution

Most of the problems encountered in fluid mechanics are nonlinear. To find the exact solution of these nonlinear problems is very difficult and sometimes impossible. Therefore various methods have been developed to solve nonlinear differential equations. Among these finite difference and perturbation are widely used to solve non linear differential equation.

1.11.1 Finite difference method

In this method, the derivative in the differential equation (and in the boundary conditions as well) are replaced by appropriate finite difference and differential equation is therefor reduced to a system of algebraic equations. The solution of algebraic equations then gives the dependent variables at discrete values of independent variables.

1.11.2 Perturbation solution

Perturbation theory consist of mathematical methods for finding an approximate solution to a problem, by starting from the exact solution of a related problem. This methods rely on there being a dimensionless parameter in the problem that may b small or large. The solution is given by few terms of expansion. We consider an equation

$$\psi = 1 + \epsilon\psi^2, \tag{1.16}$$

for $\epsilon = 0$

$$\psi = 1, \tag{1.17}$$

for small $\epsilon(\neq 0)$

$$\psi = 1 + \epsilon\psi_1 + \epsilon^2\psi_2 + ... \tag{1.18}$$

and Eq. (1.16) becomes

$$(\epsilon\psi_1 + \epsilon^2\psi_2 + ...) = \epsilon(1 + \epsilon\psi_1 + \epsilon^2\psi_2 + ...)^2 \tag{1.19}$$

Expanding for small ϵ we write the above equation

$$(\epsilon\psi_1 + \epsilon^2\psi_2 + ...) = \epsilon(1 + \epsilon^2\psi_1^2 + 2\epsilon\psi_1 + 2\epsilon^2\psi_2 + ...). \tag{1.20}$$

Chapter 2

Peristaltic transport of Newtonian fluid in a curved channel

2.1 Introduction

This chapter is detailed review of a paper by Ali et al. [16]. This chapter focuses on peristaltic transport of viscous fluid in a curved channel. The analysis has been carried out under the assumption of long wavelength and low Reynolds number. The closed form solutions are obtained for stream function, axial velocity, and pressure gradient. The influence of curvature is analyzed on various flow quantities of interest.

2.2 Mathematical formulation

We consider a curved channel of half width a coiled in a circle with center O and radius R^*. The velocity components in radial $(\overline{R})$ and axial $(\overline{X}\ )$ directions are $(\overline{V})$ and $(\overline{U})$, respectively. Sinusoidal waves of small amplitude b are imposed on the flexible walls of the channel. The peristaltic flow is induced because of the transverse deflections of the channel walls. The inertial effects are taken small. The geometries of channel walls are defined as follows:

$$\overline{H}(\overline{X},\bar{t}) = a + b\sin[\frac{2\pi}{\lambda}(\overline{X} - c\bar{t})], \quad \text{upper wall}$$

$$-\overline{H}(\overline{X},\bar{t}) = -a - b\sin[\frac{2\pi}{\lambda}(\overline{X} - c\bar{t})], \quad \text{lower wall} \tag{2.1}$$

in which c is the speed and λ is the wavelength. For the geometry under consideration, the dimensional equations of motion are

$$\frac{\partial}{\partial \overline{R}}(\overline{R} + R^*)\overline{V} + R^*\frac{\partial \overline{U}}{\partial \overline{X}} = 0, \tag{2.2}$$

$$\frac{\partial \overline{V}}{\partial \bar{t}} + \overline{V}\frac{\partial \overline{V}}{\partial \overline{R}} + \frac{R^*\overline{U}}{R^* + \overline{R}}\frac{\partial \overline{V}}{\partial \overline{X}} - \frac{\overline{U}^2}{R^* + \overline{R}} = -\frac{\partial \bar{p}}{\partial \overline{R}}$$
$$+ \nu\left[\frac{1}{R^* + \overline{R}}\frac{\partial}{\partial \overline{R}}\left\{(\overline{R} + R^*)\frac{\partial \overline{V}}{\partial \overline{R}}\right\} + \left(\frac{R^*}{R^* + \overline{R}}\right)^2\frac{\partial^2 \overline{V}}{\partial \overline{X}^2} - \frac{\overline{V}}{(R^* + \overline{R})^2}\right.$$
$$\left. - \frac{2R^*}{(R^* + \overline{R})^2}\frac{\partial \overline{U}}{\partial \overline{X}}\right], \tag{2.3}$$

$$\frac{\partial \overline{U}}{\partial \bar{t}} + \overline{V}\frac{\partial \overline{U}}{\partial \overline{R}} + \frac{R^*\overline{U}}{R^* + \overline{R}}\frac{\partial \overline{U}}{\partial \overline{X}} + \frac{\overline{U}\,\overline{V}}{R^* + \overline{R}} = -\frac{R^*}{\overline{R} + R^*}\frac{\partial \bar{p}}{\partial \overline{X}}$$
$$+ \nu\left[\frac{1}{R^* + \overline{R}}\frac{\partial}{\partial \overline{R}}\left\{(\overline{R} + R^*)\frac{\partial \overline{U}}{\partial \overline{R}}\right\} + \left(\frac{R^*}{R^* + \overline{R}}\right)^2\frac{\partial^2 \overline{U}}{\partial \overline{X}^2} - \frac{\overline{U}}{(R^* + \overline{R})^2}\right.$$
$$\left. + \frac{2R^*}{(R^* + \overline{R})^2}\frac{\partial \overline{V}}{\partial \overline{X}}\right]. \tag{2.4}$$

In above equations $\bar{p}$ is the pressure, ν is the kinematic viscosity, and $\bar{t}$ is the time. In the laboratory frame $(\overline{R}, \overline{X})$, the flow is unsteady. However, it can be treated as steady in a coordinate system $(\bar{r}, \bar{x})$ moving with the wave speed c (wave frame). The two frames can be related in the following from:

$$\bar{x} = \overline{X} - c\bar{t},\ \bar{r} = \overline{R},\ \bar{u} = \overline{U} - c,\ \bar{v} = \overline{V}, \tag{2.5}$$

where $\overline{v}$ and $\overline{u}$ are the velocity components along $\overline{r}$ and $\overline{x}$-directions in the wave frame. Using Eq. (2.5) into Eqs. (2.2)–(2.4) reduced to

$$\frac{\partial}{\partial \overline{r}}(\overline{r}+R^*)\overline{v}+R^*\frac{\partial \overline{u}}{\partial \overline{x}}=0, \tag{2.6}$$

$$-c\frac{\partial \overline{v}}{\partial \overline{x}}+\overline{v}\frac{\partial \overline{v}}{\partial \overline{r}}+\frac{R^*(\overline{u}+c)}{R^*+\overline{r}}\frac{\partial \overline{v}}{\partial \overline{x}}-\frac{(\overline{u}+c)^2}{R^*+\overline{r}}=-\frac{\partial \overline{p}}{\partial \overline{r}}$$
$$+\nu\left[\frac{1}{R^*+\overline{r}}\frac{\partial}{\partial \overline{r}}\left\{(\overline{r}+R^*)\frac{\partial \overline{v}}{\partial \overline{r}}\right\}+\left(\frac{R^*}{R^*+\overline{r}}\right)^2\frac{\partial^2 \overline{v}}{\partial \overline{x}^2}-\frac{\overline{v}}{(R^*+\overline{r})^2}\right.$$
$$\left.-\frac{2R^*}{(R^*+\overline{r})^2}\frac{\partial \overline{u}}{\partial \overline{x}}\right], \tag{2.7}$$

$$-c\frac{\partial \overline{u}}{\partial \overline{x}}+\overline{v}\frac{\partial \overline{u}}{\partial \overline{r}}+\frac{R^*(\overline{u}+c)}{R^*+\overline{r}}\frac{\partial \overline{u}}{\partial \overline{x}}+\frac{(\overline{u}+c)\overline{v}}{R^*+\overline{r}}=-\frac{R^*}{\overline{r}+R^*}\frac{\partial \overline{p}}{\partial \overline{x}}$$
$$+\nu\left[\frac{1}{R^*+\overline{r}}\frac{\partial}{\partial \overline{r}}\left\{(\overline{r}+R^*)\frac{\partial \overline{u}}{\partial \overline{r}}\right\}+\left(\frac{R^*}{R^*+\overline{r}}\right)^2\frac{\partial^2 \overline{u}}{\partial \overline{x}^2}-\frac{\overline{u}}{(R^*+\overline{r})^2}\right.$$
$$\left.+\frac{2R^*}{(R^*+\overline{r})^2}\frac{\partial \overline{v}}{\partial \overline{x}}\right]. \tag{2.8}$$

Introducing the following dimensionless quantities:

$$x=\frac{2\pi\overline{x}}{\lambda},\ \eta=\frac{\overline{r}}{a},\ u=\frac{\overline{u}}{c},\ v=\frac{\overline{v}}{c},\ Re=\frac{\rho c a}{\mu},\ P=\frac{2\pi a^2}{\lambda\mu c}\overline{p},$$
$$h=\frac{\overline{H}}{a},\ \delta=\frac{2\pi a}{\lambda},\ k=\frac{R^*}{a}, \tag{2.9}$$

and defining the stream function ψ by

$$u=-\frac{\partial\psi}{\partial\eta},\quad v=\delta\frac{k}{\eta+k}\frac{\partial\psi}{\partial x}, \tag{2.10}$$

then Eq. (2.6) is identically satisfied and Eqs. (2.7) and (2.8), under long wavelength and low Reynolds number approximations, reduced to the following dimensionless form:

$$\frac{\partial P}{\partial \eta} = 0, \tag{2.11}$$

$$-\frac{\partial P}{\partial x} - \frac{1}{k}\frac{\partial}{\partial \eta}\left\{(\eta + k)\frac{\partial^2 \psi}{\partial \eta^2}\right\} - \frac{1}{k(\eta + k)}\left(1 - \frac{\partial \psi}{\partial \eta}\right) = 0, \tag{2.12}$$

From above equations, one obtains

$$\frac{\partial^2}{\partial \eta^2}\left\{(\eta + k)\frac{\partial^2 \psi}{\partial \eta^2}\right\} + \frac{\partial}{\partial \eta}\left\{\frac{1}{(\eta + k)}\left(1 - \frac{\partial \psi}{\partial \eta}\right)\right\} = 0. \tag{2.13}$$

2.3 Flow rate and boundary conditions

The dimensional volume flow rate in laboratory frame is given by

$$Q = \int_{-\bar{H}}^{\bar{H}} \bar{U}\, d\overline{R} \tag{2.14}$$

in which $\overline{H}$ is a function of $\overline{X}$ and $\bar{t}$. The above expression in wave frame becomes

$$F = \int_{-\bar{h}}^{\bar{h}} \overline{u}\, d\overline{r}, \tag{2.15}$$

where $\bar{h}$ is function of $\overline{x}$ alone. From (2.9), (2.14) and (2.15), we can write

$$Q = F + 2c\bar{H}. \tag{2.16}$$

The time-averaged flow over a period T at a fixed position $\bar{X}$ is

$$\bar{Q} = \frac{1}{T}\int_0^T Q\, dt. \tag{2.17}$$

Substituting Eq. (2.16) into Eq. (2.17) and then integrating, one can write

$$\bar{Q} = q + 2ac, \tag{2.18}$$

by defining the dimensionless mean flows θ, in the laboratory frame and q in the wave frame, according to

$$\theta = \frac{\bar{Q}}{ac}, \quad q = \frac{F}{ac}, \tag{2.19}$$

Eq. (2.18) reduces to

$$\theta = q + 2, \tag{2.20}$$

where

$$q = -\int_{-h}^{h} \frac{\partial \psi}{\partial \eta}\, d\eta = -(\psi(h) - \psi(-h)). \tag{2.21}$$

We note that $h(x)$ and $(-h(x))$ represent the dimensionless form of the surface of the peristaltic walls

$$h(x) = 1 + \phi \sin x,$$

$$-h(x) = -1 - \phi \sin x. \tag{2.22}$$

Where $\phi = \frac{b}{a}$

By the choice of $\psi(h) = -\frac{q}{2}$ and $\psi(-h) = \frac{q}{2}$ and thus the boundary conditions in terms of the stream function ψ are

$$\psi = -\frac{q}{2}, \quad \frac{\partial \psi}{\partial \eta} = 1 \quad \text{at} \quad h,$$

$$\psi = \frac{q}{2}, \quad \frac{\partial \psi}{\partial \eta} = 1 \quad \text{at} \quad -h. \tag{2.23}$$

The dimensionless pressure rise over one wavelength is defined by

$$\Delta p = \int_{0}^{2\pi} \frac{dp}{dx}\, dx. \tag{2.24}$$

2.4 Solution of the problem

The solution of Eq. (2.13) with the boundary conditions Eq.(2.23) is in the form

$$\psi = (\eta + k) + \frac{C1}{2}\left\{(\ln(\eta + k) - 1)\frac{(\eta + k)^2}{2}\right\}$$

$$+ C_2 \frac{(\eta + k)^2}{4} + C_3 \ln(\eta + k) + C_4. \tag{2.25}$$

The axial velocity and the pressure gradient is

$$u = 1 + \frac{C1}{2}\left[(\frac{\eta+k}{2} + (\eta+k)\ln(\eta+k) - 1\right]$$

$$+ C_2\frac{\eta+k}{2} + \frac{C_3}{\eta+k}, \tag{2.26}$$

$$\frac{dp}{dx} = 8h(2h+q)\Bigg[-4h^2k^2 + (h^2-k^2)^2((\ln(k-h))^2 + (\ln(k+h))^2)$$

$$-2(h^2-k^2)^2\ln(k-h)\ln(k+h)\Bigg]^{-1}, \tag{2.27}$$

where

$$C_1 = -8hk(2h+q)\Bigg[-4h^2k^2 + (h^2-k^2)^2((\ln(k-h))^2 + (\ln(k+h))^2)$$

$$-2(h^2-k^2)^2\ln(k-h)\ln(k+h)\Bigg]^{-1},$$

$$C_2 = -2(2h+q)(2hk + (k-h)^2\ln(k-h) - (k+h)^2\ln(k+h))\Bigg[-4h^2k^2$$

$$+(h^2-k^2)^2((\ln(k-h))^2 + (\ln(k+h))^2) - 2(h^2-k^2)^2\ln(k-h)\ln(k+h)\Bigg]^{-1},$$

$$C_3 = (h^2-k^2)^2\ln\left(\frac{k-h}{k+h}\right)(2h+q)\Bigg[-4h^2k^2 + (h^2-k^2)^2((\ln(k-h))^2$$

$$+(\ln(k+h))^2) - 2(h^2-k^2)^2\ln(k-h)\ln(k+h)\Bigg]^{-1},$$

$$C_4 = 2hk(2h^3 - 2hk^2 + h^2q + k^2q) + (h^2 - k^2)^2 \ln\left(\frac{k-h}{k+h}\right)(-2h - q + (2(h+k)$$

$$+ q)\ln(k-h) + (2h - 2k + q)\ln(k+h))\Bigg[-4h^2k^2 + (h^2 - k^2)^2((\ln(k-h))^2$$

$$+ (\ln(k+h))^2) - 2(h^2 - k^2)^2 \ln(k-h)\ln(k+h)\Bigg]^{-1}.$$

2.5 Graphs and Discussion

In order to analyze the effect of pertinent parameter namely i.e curvature of the channel (k) on the axial velocity, pressure gradient, pressure rise and the stream function ψ, a set of Figs. 2.1 to 2.4 were prepared. Fig. 2.1 represents the variation of u for various values of k. It is observed that for large values of k the velocity profile is symmetric about the axis of the channel and the maxima occur at $\eta = 0$. However for small values of k the profiles are not symmetric about $\eta = 0$ and maxima shifts towards the negative values of η. Fig. 2.2 shows the variation of $\frac{dp}{dx}$ per wavelength for different values of k. It is found that the magnitude of $\frac{dp}{dx}$ decreases in going from curved to straight channel. Fig. 2.3 depicts the variation of pressure rise per wavelength $\triangle P$ with mean flow rate θ for different values of k. In co-pumping and free pumping the pumping rate for straight channel is greater in magnitude as compared to curved channel. Fig. 2.4 is plotted to discuss the trapping phenomenon for various values of k. We observed that for small values of k the bolus is not symmetric about $\eta = 0$ and is pushed towards the lower wall. However, as k increases the results of straight channel can be recovered.

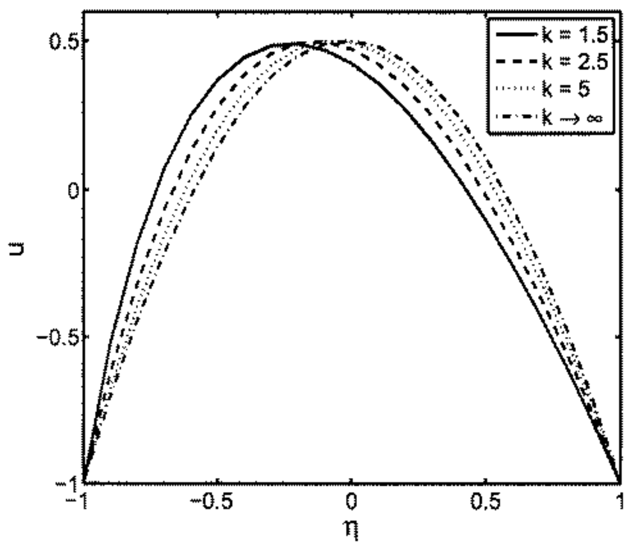

Figure 2.1: Variation of $u(\eta)$ for different values of k with $\phi = 0.8$ and $\theta = 2$.

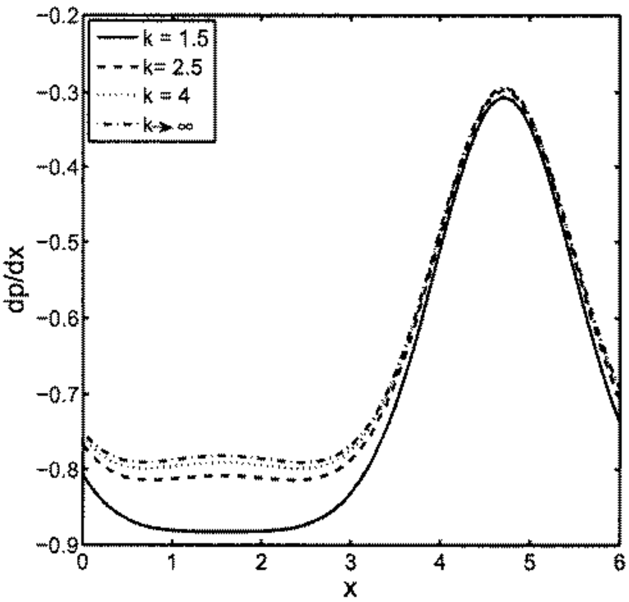

Figure 2.2: Variation of $\frac{dp}{dx}$ for different values of k with $\phi = 0.2$ and $\theta = 0.5$.

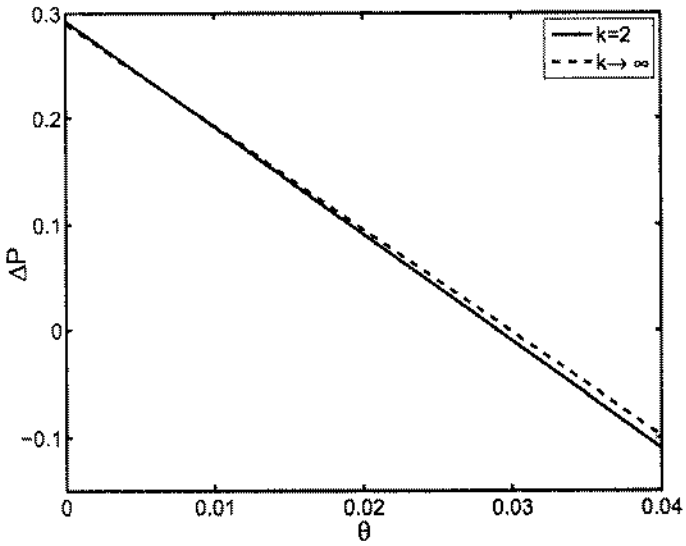

Figure 2.3: Variation of $\triangle P$ for different values of k with $\phi = 0.1$

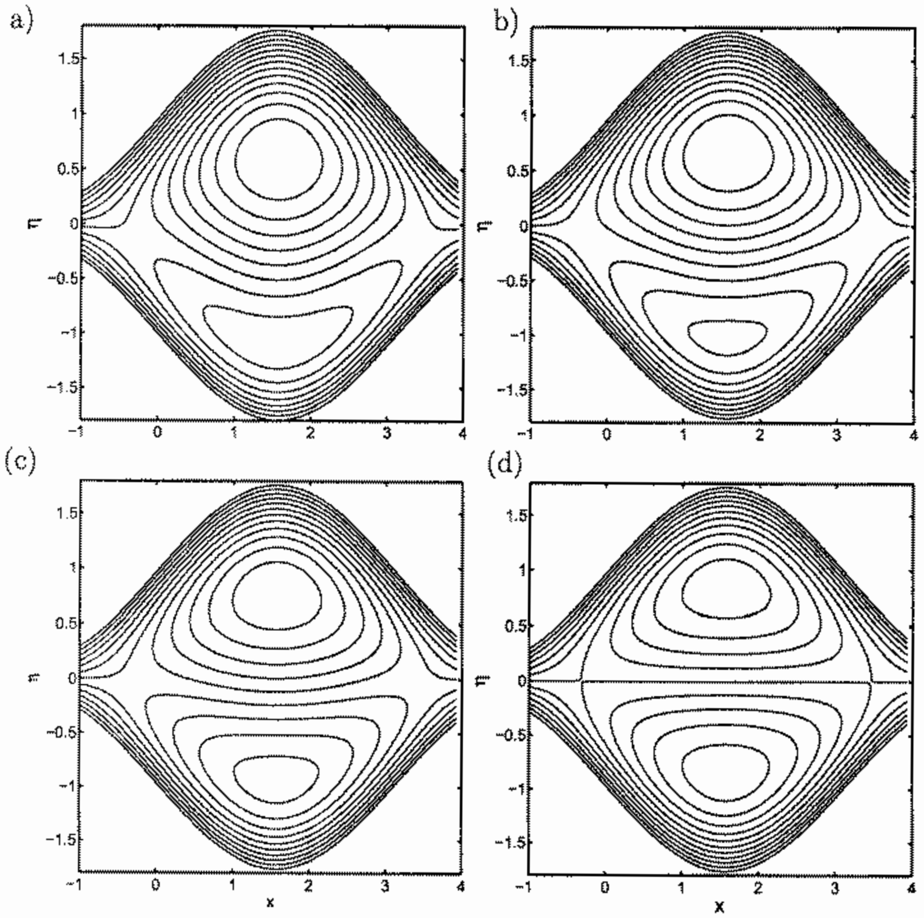

Figure 2.4: Streamlines for $k = 3.5$ (panel a), $k = 5$ (panel b), $k = 10$ (panel c) and $k \to \infty$ (panel d). The other parameters are θ=1.5 and $\phi = 0.8$.

Chapter 3

The influence of magnetic field on peristaltic flow of Oldroyd 8-constant fluid in a curved channel

3.1 Introduction

The aim of present chapter is to investigate the the effect of magnetic field on peristaltic motion of Oldroyd 8 - constant fluid in a curved channel. The governing equations for Oldroyd 8-constant model are derived in a curved channel. The long wavelength and low Reynolds number assumptions have been used to simplify the nonlinear differential equations. Solutions are obtained by two techniques: analytically and numerically with the help of perturbation and finite difference scheme respectively. The effects of emerging parameters have been analyzed by plotting the graph of pressure, stream function and velocity profile.

3.2 Mathematical formulation

Let us consider the MHD Oldroyd 8-constant fluid in a curved channel of radius R^* and uniform thickness $2a$ coiled in a circle with centre O. We denote the axial and radial direction by $\overline{X}$ and $\overline{R}$ respectively. The components of the velocity field in the radial and axial directions are $(\overline{V})$ and $(\overline{U})$ respectively. A uniform magnetic field B_0 is applied in the radial direction. The induced magnetic field is neglected by considering the small magnetic Reynolds number. The shape of the wall is same as in the previous chapter. The relevant equations governing the flow can be expressed as

$$\overline{\nabla}\cdot\overline{\mathbf{V}}=0, \tag{3.1}$$

$$\rho\frac{d\overline{\mathbf{V}}}{d\overline{t}}=\overline{\nabla}\cdot\overline{\mathbf{T}}+\overline{\mathbf{J}}\times\overline{\mathbf{B}}, \tag{3.2}$$

where

$$\overline{\mathbf{J}}\times\overline{\mathbf{B}}=-\frac{\sigma B_0^2\overline{U}}{(\overline{R}+R^*)^2}e_X, \tag{3.3}$$

in which $\frac{d}{dt}$ is the material derivative, $\overline{\mathbf{J}}$ indicates the electric current density and σ is the electric conductivity of the fluid.

The constitutive equations for Oldroyd 8 - constant fluid are

$$\overline{\mathbf{T}}=-P\overline{\mathbf{I}}+\overline{\mathbf{S}}, \tag{3.4}$$

$$\overline{\mathbf{S}}+\lambda_1\frac{D\overline{\mathbf{S}}}{D\overline{\mathbf{t}}}+\frac{\lambda_3}{2}(\overline{\mathbf{S}}\,\overline{\mathbf{A}}_1+\overline{\mathbf{A}}_1\overline{\mathbf{S}})+\frac{\lambda_5}{2}tr(\overline{\mathbf{S}})\overline{\mathbf{A}}_1+\frac{\lambda_6}{2}[tr(\overline{\mathbf{S}}\,\overline{\mathbf{A}}_1)]\overline{\mathbf{I}}$$

$$=\mu\left[\overline{\mathbf{A}}_1+\lambda_2\frac{D\overline{\mathbf{A}}_1}{D\overline{\mathbf{t}}}+\lambda_4(\overline{\mathbf{A}}_1)^2+\frac{\lambda_7}{2}[tr(\overline{\mathbf{A}}_1)^2]\overline{\mathbf{I}}\right], \tag{3.5}$$

$$\overline{\mathbf{A}}_1=L+L^T,\quad L=\text{grad}\mathbf{V}. \tag{3.6}$$

Where $\overline{\mathbf{T}}$ is the cauchy stress tensor, P is the pressure, $\overline{\mathbf{I}}$ is the identity tensor, $\overline{\mathbf{S}}$ is the extra stress tensor, $\overline{\mathbf{A}}_1$ is the first Rivlin-Ericksen tensor, μ is the dynamic viscosity, $\lambda_i (i = 1,\ 2,\ ...,\ 7)$ are material constants of the fluid. To make the problem steady, we define the following transformations:

$$\overline{p}(\overline{r}, \overline{x}) = \overline{p}(\overline{R}, \overline{X}, \overline{t}),\ \overline{x} = \overline{X} - c\overline{t},\ \overline{r} = \overline{R},\ \overline{u} = \overline{U} - c,\ \overline{v} = \overline{V}, \tag{3.7}$$

in the above expression, $\overline{v}, \overline{u}$ are the velocity components and $\overline{p}(\overline{r}, \overline{x})$ is the pressure in the wave frame respectively. Using the transformation (3.7), the Eqs. (3.1), (3.2) and (3.4) to (3.6) reduced to

$$\frac{\partial}{\partial \overline{r}}(\overline{r} + R^*)\overline{v} + R^* \frac{\partial \overline{u}}{\partial \overline{x}} = 0, \tag{3.8}$$

$$\rho\left[-c\frac{\partial \overline{v}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{v}}{\partial \overline{r}} + \frac{R^*(\overline{u}+c)}{R^*+\overline{r}}\frac{\partial \overline{v}}{\partial \overline{x}} - \frac{(\overline{u}+c)^2}{R^*+\overline{r}}\right] = -\frac{\partial \overline{p}}{\partial \overline{r}} +$$

$$\frac{1}{R^*+\overline{r}}\frac{\partial}{\partial \overline{r}}\left\{(\overline{r}+R^*)\overline{S}_{\overline{r}\overline{r}}\right\} + \frac{R^*}{R^*+\overline{r}}\frac{\partial \overline{S}_{\overline{r}\overline{x}}}{\partial \overline{x}} - \frac{\overline{S}_{\overline{x}\overline{x}}}{R^*+\overline{r}}, \tag{3.9}$$

$$\rho\left[-c\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{r}} + \frac{R^*(\overline{u}+c)}{R^*+\overline{r}}\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{(\overline{u}+c)\overline{v}}{R^*+\overline{r}}\right] = -\frac{R^*}{\overline{r}+R^*}\frac{\partial \overline{p}}{\partial \overline{x}}$$

$$+\frac{1}{(R^*+\overline{r})^2}\frac{\partial}{\partial \overline{r}}\left\{(\overline{r}+R^*)^2\overline{S}_{\overline{x}\overline{r}}\right\} + \frac{R^*}{R^*+\overline{r}}\frac{\partial \overline{S}_{\overline{x}\overline{x}}}{\partial \overline{x}} - \frac{\sigma(\overline{u}+c)B_0^2}{(\overline{r}+R^*)^2}, \tag{3.10}$$

$$S_{\overline{r}\overline{r}} + \lambda_1\Bigg\{\left(-c\frac{\partial}{\partial \overline{x}} + \overline{v}\frac{\partial}{\partial \overline{r}} + \frac{R^*}{\overline{r}+R^*}(\overline{u}+c)\frac{\partial}{\partial \overline{x}} - 2\frac{\partial \overline{v}}{\partial \overline{r}}\right)S_{\overline{r}\overline{r}} - \frac{2R^*}{\overline{r}+R^*}$$

$$\frac{\partial \overline{v}}{\partial \overline{x}}S_{\overline{r}\overline{x}}\Bigg\} + \left\{(2\lambda_3+\lambda_6+\lambda_5)S_{\overline{r}\overline{r}} + (\lambda_5-\lambda_6)S_{\overline{x}\overline{x}}\right\}\frac{\partial \overline{v}}{\partial \overline{r}} + (\lambda_3+\lambda_6)$$

$$\left\{\frac{\partial \overline{u}}{\partial \overline{r}} + \frac{R^*}{\overline{r}+R^*}\frac{\partial \overline{v}}{\partial \overline{x}} - \frac{\overline{u}+c}{\overline{r}+R^*}\right\}S_{\overline{r}\overline{x}} = \mu\Bigg[2\frac{\partial \overline{v}}{\partial \overline{r}} + 2\lambda_2\Bigg\{-c\frac{\partial^2 \overline{v}}{\partial \overline{x}\partial \overline{r}} + \overline{v}\frac{\partial^2 \overline{v}}{\partial \overline{r}^2}$$

$$-2\left(\frac{\partial \overline{v}}{\partial \overline{r}}\right)^2 + \frac{R^*}{\overline{r}+R^*}\left((\overline{u}+c)\frac{\partial^2 \overline{v}}{\partial \overline{x}\partial \overline{r}} - \frac{\partial \overline{v}}{\partial \overline{x}}\frac{\partial \overline{u}}{\partial \overline{r}}\right) - \left(\frac{R^*}{\overline{r}+R^*}\right)^2\left(\frac{\partial \overline{v}}{\partial \overline{x}}\right)^2$$

$$+\frac{R^*}{(\overline{r}+R^*)^2}(\overline{u}+c)\frac{\partial \overline{v}}{\partial \overline{x}}\Bigg\} + (\lambda_4+\lambda_7)\Bigg\{4\left(\frac{\partial \overline{v}}{\partial \overline{r}}\right)^2 + \Bigg(\frac{\partial \overline{u}}{\partial \overline{r}} + \frac{R^*}{\overline{r}+R^*}\frac{\partial \overline{v}}{\partial \overline{x}}$$

$$-\frac{\overline{u}+c}{\overline{r}+R^*}\Bigg)^2\Bigg\}\Bigg], \tag{3.11}$$

$$S_{\bar{r}\bar{x}} + \lambda_1\bigg\{\bigg(-c\frac{\partial}{\partial\bar{x}} + \bar{v}\frac{\partial}{\partial\bar{r}} + \frac{R^*}{\bar{r}+R^*}(\bar{u}+c)\frac{\partial}{\partial\bar{x}}\bigg)S_{\bar{r}\bar{x}} + (\frac{\bar{u}+c}{\bar{r}+R^*} - \frac{\partial\bar{u}}{\partial\bar{r}})S_{\bar{r}\bar{r}}$$

$$- \frac{R^*}{\bar{r}+R^*}\frac{\partial\bar{v}}{\partial\bar{x}}S_{\bar{x}\bar{x}}\bigg\} + (\frac{\lambda_3}{2} + \frac{\lambda_5}{2})(S_{\bar{r}\bar{r}} + S_{\bar{x}\bar{x}})\bigg(\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{R^*}{\bar{r}+R^*}\frac{\partial\bar{v}}{\partial\bar{x}} - \frac{\bar{u}+c}{\bar{r}+R^*}\bigg)$$

$$= \mu\bigg[\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{R^*}{\bar{r}+R^*}\frac{\partial\bar{v}}{\partial\bar{x}} - \frac{\bar{u}+c}{\bar{r}+R^*} + \lambda_2\bigg\{-2c\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{r}} + \frac{R^*}{\bar{r}+R^*}\bigg(-c\frac{\partial^2\bar{v}}{\partial\bar{x}^2}$$

$$+ \bar{v}\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{r}} + (\bar{u}+c)\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{r}} + 2\frac{\partial\bar{v}}{\partial\bar{r}}\frac{\partial\bar{v}}{\partial\bar{x}}\bigg) + \frac{\bar{v}(\bar{u}+c)}{(\bar{r}+R^*)^2} - \frac{R^*}{(\bar{r}+R^*)^2}\bigg(\bar{v}\frac{\partial\bar{v}}{\partial\bar{x}}$$

$$+ (\bar{u}+c)\frac{\partial\bar{u}}{\partial\bar{x}}\bigg) + \frac{1}{\bar{r}+R^*}\bigg(-c\frac{\partial\bar{u}}{\partial\bar{x}} - \bar{v}\frac{\partial\bar{u}}{\partial\bar{r}} + 2(\bar{u}+c)\frac{\partial\bar{v}}{\partial\bar{r}}\bigg) - 2\frac{\partial\bar{u}}{\partial\bar{r}}\frac{\partial\bar{v}}{\partial\bar{r}}$$

$$+ (\frac{R^*}{\bar{r}+R^*})^2(\bar{u}+c)\frac{\partial^2\bar{v}}{\partial\bar{x}^2} + \bar{v}\frac{\partial^2\bar{u}}{\partial\bar{r}^2}\bigg\}\bigg], \tag{3.12}$$

$$S_{\bar{x}\bar{x}} + \lambda_1\bigg\{\bigg(-c\frac{\partial}{\partial\bar{x}} + \bar{v}\frac{\partial}{\partial\bar{r}} + \frac{R^*}{\bar{r}+R^*}(\bar{u}+c)\frac{\partial}{\partial\bar{x}} + 2\frac{\partial\bar{v}}{\partial\bar{r}}\bigg)S_{\bar{x}\bar{x}} + 2\bigg(\frac{\bar{u}+c}{\bar{r}+R^*}$$

$$- \frac{\partial\bar{u}}{\partial\bar{r}}\bigg)S_{\bar{r}\bar{x}}\bigg\} - \bigg\{(2\lambda_3 + \lambda_5 + \lambda_6)S_{\bar{x}\bar{x}} + (\lambda_5 - \lambda_6)S_{\bar{r}\bar{r}}\bigg\}\frac{\partial\bar{v}}{\partial\bar{r}} + (\lambda_3 + \lambda_6)$$

$$\bigg(\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{R^*}{\bar{r}+R^*}\frac{\partial\bar{v}}{\partial\bar{x}} - \frac{\bar{u}+c}{\bar{r}+R^*}\bigg)S_{\bar{r}\bar{x}} = \mu\bigg[-2\frac{\partial\bar{v}}{\partial\bar{r}} + 2\lambda_2\bigg\{-c\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{r}} - \bar{v}\frac{\partial^2\bar{v}}{\partial\bar{r}^2}$$

$$- \frac{R^*}{\bar{r}+R^*}\bigg((\bar{u}+c)\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{r}} + \frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{r}}\bigg) - 2\bigg(\frac{\partial\bar{v}}{\partial\bar{r}}\bigg)^2 - \bigg(\frac{\partial\bar{u}}{\partial\bar{r}}\bigg)^2 - \bigg(\frac{\bar{u}+c}{\bar{r}+R^*}\bigg)^2$$

$$+ \frac{\bar{u}+c}{\bar{r}+R^*}\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{(\bar{u}+c)R^*}{(\bar{r}+R^*)^2}\frac{\partial\bar{v}}{\partial\bar{x}}\bigg\} + (\lambda_4 + \lambda_7)\bigg\{4\bigg(\frac{\partial\bar{v}}{\partial\bar{r}}\bigg)^2$$

$$+ \bigg(\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{R^*}{\bar{r}+R^*}\frac{\partial\bar{v}}{\partial\bar{x}} - \frac{\bar{u}+c}{\bar{r}+R^*}\bigg)^2\bigg\}\bigg]. \tag{3.13}$$

Introducing the following dimensionless parameters

$$x = \frac{\bar{x}}{\lambda},\ r = \frac{\bar{r}}{a},\ u = \frac{\bar{u}}{c},\ v = \frac{\bar{v}}{\delta c},\ p = \frac{a^2}{\lambda\mu c}\bar{p},\ S_{ij} = \frac{a\overline{S}_{\bar{i}\bar{j}}}{\mu c},$$

$$\delta = \frac{a}{\lambda},\ Ha = B_0\sqrt{\frac{\sigma}{\mu}},\ \lambda_i = \frac{c\lambda_i}{a},\ i = 1, 2, ..., 7, \tag{3.14}$$

where δ is the wave number and Ha the Hartman number. The dimensionless form of governing Eqs. (3.8) – (3.13) become

$$\frac{\partial}{\partial r}(\delta(r+k)v) + \delta k\frac{\partial u}{\partial x} = 0, \tag{3.15}$$

$$Re\delta\left[-\delta\frac{\partial v}{\partial x} + \delta^2 v\frac{\partial v}{\partial r} + \frac{k\delta^2(u+1)}{k+r}\frac{\partial v}{\partial x} - \frac{(u+1)^2}{k+r}\right] = -\frac{\partial p}{\partial r}$$

$$+\frac{\delta}{k+r}\frac{\partial}{\partial r}\{(k+r)S_{rr}\} + \frac{k\delta^2}{k+r}\frac{\partial S_{rx}}{\partial x} - \frac{\delta S_{xx}}{k+r}, \tag{3.16}$$

$$Re\left[-\delta\frac{\partial u}{\partial x} + \delta v\frac{\partial u}{\partial r} + \frac{k\delta(u+1)}{k+r}\frac{\partial u}{\partial x} + \frac{\delta(u+1)v}{k+r}\right] = -\frac{k}{k+r}\frac{\partial p}{\partial x}$$

$$+\frac{1}{k+r)^2}\frac{\partial}{\partial r}\{(k+r)^2 S_{xr}\} + \frac{k\delta}{k+r}\frac{\partial S_{xx}}{\partial x} - \frac{Ha^2(u+1)}{(k+r)^2}, \tag{3.17}$$

$$S_{rr} + \lambda_1\delta\left\{-\frac{\partial}{\partial x} + v\frac{\partial}{\partial r} + \frac{k}{k+r}(u+1)\frac{\partial}{\partial x}\right\}S_{rr} - 2\lambda_1\delta\frac{\partial v}{\partial r}S_{rr}$$

$$-2\lambda_1\frac{\delta^2 k}{k+r}\frac{\partial v}{\partial x}S_{rx} + 2\lambda_3\delta\frac{\partial v}{\partial r}S_{rr} + \lambda_5\delta\big(S_{rr}+S_{xx}\big)\frac{\partial v}{\partial r} + \lambda_6\delta\big(S_{rr}$$

$$-S_{xx}\big)\frac{\partial v}{\partial r} + (\lambda_3+\lambda_6)\left(\frac{\partial u}{\partial r} + \frac{k\delta^2}{k+r}\frac{\partial v}{\partial x} - \frac{u+1}{k+r}S_{rx}\right) = 2\delta\frac{\partial v}{\partial r}$$

$$+2\lambda_2\delta^2\bigg\{-\frac{\partial^2 v}{\partial x\partial r} - 4\left(\frac{\partial v}{\partial r}\right)^2 + v\frac{\partial^2 v}{\partial r^2} + \frac{k}{k+r}\bigg((u+1)\frac{\partial^2 v}{\partial x\partial r}$$

$$-\frac{\partial v}{\partial x}\frac{\partial u}{\partial r}\bigg) - 2\left(\frac{k\delta}{k+r}\right)^2\left(\frac{\partial v}{\partial x}\right)^2 + \frac{k}{(k+r)^2}(u+1)\frac{\partial v}{\partial x}\bigg\} + (\lambda_4+\lambda_7)$$

$$\left\{4\delta^2\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r} + \frac{k\delta^2}{k+r}\frac{\partial v}{\partial x} - \frac{u+1}{k+r}\right)^2\right\}, \tag{3.18}$$

$$S_{rx}+\lambda_1\delta\left\{-\frac{\partial}{\partial x}+v\frac{\partial}{\partial r}+\frac{k}{k+r}(u+1)\frac{\partial}{\partial x}\right\}S_{rx}+\lambda_1\left(\frac{u+1}{k+r}-\frac{\partial u}{\partial r}\right)S_{rr}$$

$$-\lambda_1\delta^2\frac{k}{k+r}\frac{\partial v}{\partial x}S_{xx}+\frac{(\lambda_3+\lambda_5)}{2}(S_{rr}+S_{xx})\left(\frac{\partial u}{\partial r}+\frac{k\delta^2}{k+r}\frac{\partial v}{\partial x}-\frac{u+1}{k+r}\right)$$

$$=\frac{\partial u}{\partial r}+\frac{k\delta^2}{k+r}\frac{\partial v}{\partial x}-\frac{u+1}{k+r}+\lambda_2\delta\Bigg\{-2\frac{\partial^2 u}{\partial x\partial r}+\frac{k}{k+r}\bigg(-\delta^2\frac{\partial^2 v}{\partial x^2}$$

$$+\delta^2 v\frac{\partial^2 v}{\partial x\partial r}+(u+1)\frac{\partial^2 u}{\partial x\partial r}+2\delta^2\frac{\partial v}{\partial r}\frac{\partial v}{\partial x}\bigg)+\frac{v(u+1)}{(k+r)^2}-\frac{k}{(k+r)^2}\bigg(v\frac{\partial v}{\partial x}$$

$$+(u+1)\frac{\partial u}{\partial x}\bigg)+\frac{1}{k+r}\left(-\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial r}+2(u+1)\frac{\partial v}{\partial r}\right)-2\frac{\partial u}{\partial r}\frac{\partial v}{\partial \overline{r}}$$

$$+\left(\frac{k\delta}{k+r}\right)^2(u+1)\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 u}{\partial r^2}\Bigg\},$$

$$(3.19)$$

$$S_{xx}+\lambda_1\delta\left\{-\frac{\partial}{\partial x}+v\frac{\partial}{\partial r}+\frac{k}{k+r}(u+1)\frac{\partial}{\partial x}\right\}S_{xx}+2\lambda_1\delta\frac{\partial v}{\partial r}S_{xx}$$

$$+2\lambda_1\left(\frac{u+1}{k+r}-\frac{\partial u}{\partial r}\right)S_{rx}-2\lambda_3\delta\frac{\partial v}{\partial r}S_{xx}+\delta\bigg(\lambda_6(S_{rr}-S_{xx})-\lambda_5(S_{rr}$$

$$+S_{xx})\bigg)\frac{\partial v}{\partial r}+(\lambda_3+\lambda_6)\left(\frac{\partial u}{\partial r}+\frac{k\delta^2}{k+r}\frac{\partial v}{\partial x}-\frac{u+1}{k+r}S_{rx}\right)=-2\delta\frac{\partial v}{\partial r}$$

$$+2\lambda_2\Bigg\{\delta^2\bigg(-\frac{\partial^2 v}{\partial x\partial r}-v\frac{\partial^2 v}{\partial r^2}-\frac{k}{k+r}\left((u+1)\frac{\partial^2 v}{\partial x\partial r}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial r}\right)$$

$$-2\left(\frac{\partial v}{\partial r}\right)^2+\frac{(u+1)k}{(k+r)^2}\frac{\partial v}{\partial x}\bigg)-\left(\frac{\partial u}{\partial r}\right)^2+\frac{2(u+1)}{k+r}\frac{\partial u}{\partial r}-\left(\frac{u+1}{k+r}\right)^2\Bigg\}$$

$$+(\lambda_4+\lambda_7)\left\{4\delta^2\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{\partial u}{\partial r}+\frac{k\delta^2}{k+r}\frac{\partial v}{\partial x}-\frac{u+1}{k+r}\right)^2\right\}. \qquad (3.20)$$

The components of velocity in the form of stream function ψ are written as

$$u=-\frac{\partial\psi}{\partial r},\quad v=\frac{k}{r+k}\frac{\partial\psi}{\partial x}, \qquad (3.21)$$

Using Eq. (3.21), Eq. 3.15 is identically satisfied. Under Long wavelength and low Reynolds number approximations, one can find from Eqs. (3.16) – (3.20) that

$$\frac{\partial p}{\partial r} = 0, \tag{3.22}$$

$$\frac{\partial p}{\partial x} = \frac{1}{k(k+r)}\frac{\partial}{\partial r}\left\{(k+r)^2 S_{rx}\right\} - \frac{Ha^2}{k(k+r)}(1-\frac{\partial \psi}{\partial r}), \tag{3.23}$$

$$S_{rr} - (\lambda_3+\lambda_6)\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r}\right)S_{rx} = (\lambda_4+\lambda_7)\left(\frac{\partial^2\psi}{\partial r^2}\right.$$

$$\left. + \frac{1-\frac{\partial\psi}{\partial r}}{k+r}\right)^2, \tag{3.24}$$

$$S_{rx} + \left(\lambda_1 S_{rr} - (\frac{\lambda_3+\lambda_5}{2})(S_{rr}+S_{xx})\right)(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r}) =$$

$$-\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r}\right). \tag{3.25}$$

$$S_{xx} - (\lambda_3+\lambda_6-2\lambda_1)\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r}\right)S_{rx} = (\lambda_4 - 2\lambda_2$$

$$+\lambda_7)\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r}\right)^2, \tag{3.26}$$

By eliminating the pressure from Eq. (3.22) and Eq. (3.23), yields

$$\frac{\partial}{\partial r}\left[\frac{1}{k(k+r)}\frac{\partial}{\partial r}\left\{(k+r)^2 S_{rx}\right\} - \frac{Ha^2}{k(k+r)}(1-\frac{\partial\psi}{\partial r})\right] = 0. \tag{3.27}$$

By solving Eqs.(3.24 – 3.26) simultaneously, we get

$$S_{rx} = -\left\{\frac{1+\alpha_1(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r})^2}{1+\alpha_2(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r})^2}\right\}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1-\frac{\partial\psi}{\partial r}}{k+r}\right). \tag{3.28}$$

Where

$$\alpha_1 = \lambda_1(\lambda_4+\lambda_7) - (\lambda_3+\lambda_5)(\lambda_4+\lambda_7-\lambda_2 \text{ and}$$

$$\alpha_2 = \lambda_1(\lambda_3+\lambda_6) - (\lambda_3+\lambda_5)(\lambda_3+\lambda_6-\lambda_1).$$

Substituting Eq. (3.28) into Eqs. (3.23) and (3.27), we finally obtained

$$\frac{\partial}{\partial r}\left[\frac{1}{k(k+r)}\frac{\partial}{\partial r}\left\{(k+r)^2\{\frac{1+\alpha_1(\frac{\partial^2\psi}{\partial r^2}+\frac{1-\frac{\partial\psi}{\partial r}}{k+r})^2}{1+\alpha_2(\frac{\partial^2\psi}{\partial r^2}+\frac{1-\frac{\partial\psi}{\partial r}}{k+r})^2}\}(-\frac{\partial^2\psi}{\partial r^2}-\frac{1-\frac{\partial\psi}{\partial r}}{k+r})\right\}\right.$$

$$\left.-\frac{Ha^2(1-\frac{\partial\psi}{\partial r})}{k(k+r)}\right]=0. \tag{3.29}$$

$$\frac{\partial p}{\partial x}=\frac{1}{k(k+r)}\frac{\partial}{\partial r}\left\{(k+r)^2\{\frac{1+\alpha_1(\frac{\partial^2\psi}{\partial r^2}+\frac{1-\frac{\partial\psi}{\partial r}}{k+r})^2}{1+\alpha_2(\frac{\partial^2\psi}{\partial r^2}+\frac{1-\frac{\partial\psi}{\partial r}}{k+r})^2}\}(-\frac{\partial^2\psi}{\partial r^2}-\frac{1-\frac{\partial\psi}{\partial r}}{k+r})\right\}$$

$$-\frac{Ha^2(1-\frac{\partial\psi}{\partial r})}{k(k+r)}, \tag{3.30}$$

3.3 Boundary conditions

The boundary conditions are same as in the previous chapter.

3.4 Perturbation solution

To obtain the solution of nonlinear Eqs. (3.29) and (3.30), we proceed under the assumption that α_2 and α_1 are small and we write

$$\psi=\psi_0+\alpha_2\psi_1+O(\alpha_2^2),$$

$$q=q_0+\alpha_2 q_1+O(\alpha_2^2),$$

$$\frac{dp}{dx}=\frac{dp_0}{dx}+\alpha_2\frac{dp_1}{dx}+O(\alpha_2^2), \tag{3.31}$$

where

$$\psi_0 = \psi_{00} + \alpha_1\psi_{01} + O(\alpha_1^2),$$

$$\psi_1 = \psi_{10} + \alpha_1\psi_{11} + O(\alpha_1^2),$$

$$q_0 = q_{00} + \alpha_1 q_{01} + O(\alpha_1^2),$$

$$q_1 = q_{10} + \alpha_1 q_{11} + O(\alpha_1^2),$$

$$\frac{dp_0}{dx} = \frac{dp_{00}}{dx} + \alpha_1\frac{dp_{01}}{dx} + O(\alpha_1^2),$$

$$\frac{dp_1}{dx} = \frac{dp_{10}}{dx} + \alpha_1\frac{dp_{11}}{dx} + O(\alpha_1^2), \tag{3.32}$$

substituting Eqs. (3.31) and (3.32) into Eqs. (3.29) – (3.30) and equating the coefficients of like powers of α_2 and α_1, we obtain a system of equations of different orders:

3.4.1 Zeroth order system

$$(k+r)^4\frac{\partial^4\psi_{00}}{\partial r^4} + 2(k+r)^3\frac{\partial^3\psi_{00}}{\partial r^3} - (1+Ha^2)(k+r)^2\frac{\partial^2\psi_{00}}{\partial r^2}$$

$$+ (1+Ha^2)(k+r)\frac{\partial\psi_{00}}{\partial r} = (1+Ha^2)(k+r), \tag{3.33}$$

$$\frac{\partial p_{00}}{\partial x} = -\frac{1}{k}\left[(k+r)\frac{\partial^3\psi_{00}}{\partial r^3} + \frac{\partial^2\psi_{00}}{\partial r^2} + (1+Ha^2)\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{k+r}\right], \tag{3.34}$$

subject to the boundary conditions:

$$\psi_{00} = -\frac{q_{00}}{2},\ \frac{\partial\psi_{00}}{\partial r} = 1 \text{ at } r = h,$$

$$\psi_{00} = \frac{q_{00}}{2},\ \frac{\partial\psi_{00}}{\partial r} = 1 \text{ at } r = -h. \tag{3.35}$$

3.4.2 First order system

$$
\begin{aligned}
&(k+r)^4\frac{\partial^4\psi_{01}}{\partial r^4}+2(k+r)^3\frac{\partial^3\psi_{01}}{\partial r^3}-(1+Ha^2)(k+r)^2\frac{\partial^2\psi_{01}}{\partial r^2}+(1+Ha^2)\\
&(k+r)\frac{\partial\psi_{01}}{\partial r}=-3\left(\frac{\partial^2\psi_{00}}{\partial r^2}+\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{k+r}\right)\Bigg[+2(k+r)^4\left(\frac{\partial^3\psi_{00}}{\partial r^3}-\frac{\frac{\partial^2\psi_{00}}{\partial r^2}}{k+r}\right.\\
&\left.-\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{(k+r)^2}\right)^2+(k+r)^4\left(\frac{\partial^4\psi_{00}}{\partial r^4}-\frac{\frac{\partial^3\psi_{00}}{\partial r^3}}{k+r}+2\frac{\frac{\partial^2\psi_{00}}{\partial r^2}}{(k+r)^2}+2\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{(k+r)^3}\right)\\
&+3(k+r)^3\left(\frac{\partial^2\psi_{00}}{\partial r^2}+\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{k+r}\right)\left(\frac{\partial^3\psi_{00}}{\partial r^3}-\frac{\frac{\partial^2\psi_{00}}{\partial r^2}}{k+r}-\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{(k+r)^2}\right)\Bigg],
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
\frac{\partial p_{01}}{\partial x}&=-\frac{1}{k}\Bigg[(k+r)\frac{\partial^3\psi_{01}}{\partial r^3}+\frac{\partial^2\psi_{01}}{\partial r^2}-(1+Ha^2)\frac{\frac{\partial\psi_{01}}{\partial r}}{k+r}+\left(\frac{\partial^2\psi_{00}}{\partial r^2}\right.\\
&\left.+\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{k+r}\right)^2\left(3(k+r)\frac{\partial^3\psi_{00}}{\partial r^3}-\frac{\partial^2\psi_{00}}{\partial r^2}-\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{k+r}\right)\Bigg],
\end{aligned}
\tag{3.37}
$$

along with boundary conditions:

$$
\begin{aligned}
\psi_{01}&=-\frac{q_{01}}{2},\ \frac{\partial\psi_{01}}{\partial r}=0 \text{ at } r=h,\\
\psi_{01}&=\frac{q_{01}}{2},\ \frac{\partial\psi_{01}}{\partial r}=0 \text{ at } r=-h.
\end{aligned}
\tag{3.38}
$$

3.4.3 Second order system

$$
\begin{aligned}
&(k+r)^4\frac{\partial^4\psi_{10}}{\partial r^4}+2(k+r)^3\frac{\partial^3\psi_{10}}{\partial r^3}-(1+Ha^2)(k+r)^2\frac{\partial^2\psi_{10}}{\partial r^2}\\
&+(1+Ha^2)(k+r)\frac{\partial\psi_{10}}{\partial r}=3\left(\frac{\partial^2\psi_{00}}{\partial r^2}+\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{k+r}\right)\Bigg[Ha^2\\
&\left((k+r)^2\frac{\partial^2\psi_{00}}{\partial r^2}+(k+r)(1-\frac{\partial\psi_{00}}{\partial r})\right)\left(\frac{\partial^2\psi_{00}}{\partial r^2}+\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{k+r}\right)\\
&+2(k+r)^4\left(\frac{\partial^3\psi_{00}}{\partial r^3}-\frac{\frac{\partial^2\psi_{00}}{\partial r^2}}{k+r}-\frac{(1-\frac{\partial\psi_{00}}{\partial r})}{(k+r)^2}\right)^2\Bigg],
\end{aligned}
\tag{3.39}
$$

$$\frac{\partial p_{10}}{\partial x} = -\frac{1}{k}\Bigg[(k+r)\frac{\partial^3 \psi_{10}}{\partial r^3} + \frac{\partial^2 \psi_{10}}{\partial r^2} - (1+Ha^2)\frac{\frac{\partial \psi_{10}}{\partial r}}{k+r} - \Bigg(\frac{\partial^2 \psi_{00}}{\partial r^2}$$

$$+ \frac{(1-\frac{\partial \psi_{00}}{\partial r})}{k+r}\Bigg)^2 \Bigg(3(k+r)\frac{\partial^3 \psi_{00}}{\partial r^3} - \frac{\partial^2 \psi_{00}}{\partial r^2} - \frac{(1-\frac{\partial \psi_{00}}{\partial r})}{k+r}\Bigg)\Bigg], \tag{3.40}$$

with corresponding boundary conditions:

$$\psi_{10} = -\frac{q_{10}}{2}, \ \frac{\partial \psi_{10}}{\partial r} = 0 \text{ at } r = h,$$

$$\psi_{10} = \frac{q_{10}}{2}, \ \frac{\partial \psi_{10}}{\partial r} = 0 \text{ at } r = -h. \tag{3.41}$$

3.4.4 Solution of zeroth order system

Solving zeroth order system, we obtain the stream function

$$\psi_{00} = c_1 + (r+k)^2 c_2 + (r+k)^{1-\sqrt{1+Ha^2}} c_3$$

$$+ (r+k)^{1+\sqrt{1+Ha^2}} c_4 + (r+k), \tag{3.42}$$

the axial velocity is

$$u_{00} = -2(r+k)c_2 - (1-\sqrt{1+Ha^2})(r+k)^{-\sqrt{1+Ha^2}} c_3$$

$$-(1+\sqrt{1+Ha^2})(r+k)^{\sqrt{1+Ha^2}} c_4 - 1, \tag{3.43}$$

and pressure gradient becomes

$$\frac{\partial p_{00}}{\partial x} = \frac{1}{k}(2Ha^2 c_2). \tag{3.44}$$

3.4.5 Solution of first order system

The solution of the first order system gives

$$\psi_{01} = c_5 + (r+k)^2 c_6 + (r+k)^{1-\sqrt{1+Ha^2}} c_7$$

$$+ (r+k)^{1+\sqrt{1+Ha^2}} c_8 + (r+k)^{-1-3\sqrt{1+Ha^2}} L_1$$

$$+ (r+k)^{-1+3\sqrt{1+Ha^2}} L_2 + (r+k)^{-1-\sqrt{1+Ha^2}} L_3$$

$$+ (r+k)^{-1+\sqrt{1+Ha^2}} L_4, \tag{3.45}$$

$$u_{01} = -2(r+k)c_6 - (1-\sqrt{1+Ha^2})(r+k)^{-\sqrt{1+Ha^2}}c_7$$

$$-(1+\sqrt{1+Ha^2})(r+k)^{\sqrt{1+Ha^2}}c_8 + (1+3\sqrt{1+Ha^2})$$

$$(r+k)^{-2-3\sqrt{1+Ha^2}}L_1 - (-1+3\sqrt{1+Ha^2})(r+k)^{-2+3\sqrt{1+Ha^2}}L_2$$

$$+(1+\sqrt{1+Ha^2})(r+k)^{-2-\sqrt{1+Ha^2}}L_3 - (-1+\sqrt{1+Ha^2})$$

$$(r+k)^{-2+\sqrt{1+Ha^2}}L_4, \tag{3.46}$$

$$\frac{\partial p_{01}}{\partial x} = \frac{1}{k}(2Ha^2c_6). \tag{3.47}$$

3.4.6 Solution of second order system

$$\psi_{10} = c_9 + (r+k)^2c_{10} + (r+k)^{1-\sqrt{1+Ha^2}}c_{11}$$

$$+(r+k)^{1+\sqrt{1+Ha^2}}c_{12} + (r+k)^{-1-3\sqrt{1+Ha^2}}L_5$$

$$+(r+k)^{-1+3\sqrt{1+Ha^2}}L_6 + (r+k)^{-1-\sqrt{1+Ha^2}}L_7$$

$$+(r+k)^{-1+\sqrt{1+Ha^2}}L_8, \tag{3.48}$$

$$u_{10} = -2(r+k)c_{10} - (1-\sqrt{1+Ha^2})(r+k)^{-\sqrt{1+Ha^2}}c_{11}$$
$$-(1+\sqrt{1+Ha^2})(r+k)^{\sqrt{1+Ha^2}}c_{12} + (1+3\sqrt{1+Ha^2})$$

$$(r+k)^{-2-3\sqrt{1+Ha^2}}L_5 - (-1+3\sqrt{1+Ha^2})(r+k)^{-2+3\sqrt{1+Ha^2}}L_6$$

$$+(1+\sqrt{1+Ha^2})(r+k)^{-2-\sqrt{1+Ha^2}}L_7 - (-1+\sqrt{1+Ha^2})$$

$$(r+k)^{-2+\sqrt{1+Ha^2}}L_8, \tag{3.49}$$

$$\frac{\partial p_{10}}{\partial x} = \frac{1}{k}(2Ha^2c_{10}). \tag{3.50}$$

where

$$L_1 = -\frac{Ha^6c_3^3}{12 + 8Ha^2 + 12\sqrt{1+Ha^2}},\ L_2 = -\frac{Ha^6c_4^3}{12 + 8Ha^2 - 12\sqrt{1+Ha^2}},$$

$$L_3 = -\frac{3Ha^6c_3^2c_4}{4 + 4\sqrt{1+Ha^2}},\ L_4 = -\frac{3Ha^6c_4^2c_3}{4 - 4\sqrt{1+Ha^2}},$$

$$L_5 = -L_1,\ L_6 = -L_2,\ L_7 = -L_3,\ L_8 = -L_4,$$

$$L_9 = 2\sqrt{1+Ha^2}\{k^2((k+h)^{\sqrt{1+Ha^2}} - (k-h)^{\sqrt{1+Ha^2}})^2 + h^2((k+h)^{\sqrt{1+Ha^2}}$$

$$+ (k-h)^{\sqrt{1+Ha^2}})^2\} + 2kh(Ha^2+2)L_{18},$$

$$L_{10} = 2kh(1+3\sqrt{1+Ha^2})(1-\sqrt{1+Ha^2})L_{14} - 2(1+\sqrt{1+Ha^2})L_{26}$$

$$+ (1+3\sqrt{1+Ha^2})L_{27} + (1-\sqrt{1+Ha^2})L_{28},$$

$$L_{11} = 2kh(1-3\sqrt{1+Ha^2})(1-\sqrt{1+Ha^2})L_{15} - 2(1-2\sqrt{1+Ha^2})L_{29}$$

$$+ (1-3\sqrt{1+Ha^2})L_{30} + (1-\sqrt{1+Ha^2})L_{31},$$

$$L_{12} = 2Ha^2khL_{16} - 2L_{32} + (1+\sqrt{1+Ha^2})(2+L_{33}),$$

$$L_{13} = 2kh(1-\sqrt{1+Ha^2})^2L_{17} + (1-\sqrt{1+Ha^2})L_{35},$$

$$L_{14} = (k+h)^{-2-2\sqrt{1+Ha^2}} - (k-h)^{-2-2\sqrt{1+Ha^2}},$$

$$L_{15} = (k+h)^{-2+4\sqrt{1+Ha^2}} - (k-h)^{-2+4\sqrt{1+Ha^2}},$$

$$L_{16} = (k+h)^{-2} - (k-h)^{-2},$$

$$L_{17} = (k+h)^{-2+2\sqrt{1+Ha^2}} - (k-h)^{-2+2\sqrt{1+Ha^2}},$$

$$L_{18} = (k-h)^{2\sqrt{1+Ha^2}} - (k+h)^{2\sqrt{1+Ha^2}},$$

$$L_{19} = (k-h)^{1+\sqrt{1+Ha^2}}(k+h)^{2\sqrt{1+Ha^2}} - (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{2\sqrt{1+Ha^2}}$$

$$L_{20} = (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-2-2\sqrt{1+Ha^2}} - (k-h)^{1+\sqrt{1+Ha^2}}(k+h)^{-2-2\sqrt{1+Ha^2}},$$

$$L_{21} = (k-h)^{1+\sqrt{1+Ha^2}}(k-h)^{-2+4\sqrt{1+Ha^2}} - (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-2+4\sqrt{1+Ha^2}},$$

$$L_{22} = (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-2} - (k+h)^{-2}(k-h)^{1+\sqrt{1+Ha^2}}$$

$$L_{23} = (k-h)^{1+\sqrt{1+Ha^2}}(k+h)^{-2+2\sqrt{1+Ha^2}} - (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-2+2\sqrt{1+Ha^2}},$$

$$L_{24} = (k-h)^{1+\sqrt{1+Ha^2}}(k+h)^{2\sqrt{1+Ha^2}} - (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{2\sqrt{1+Ha^2}},$$

$$L_{25} = (k+h)^{1+\sqrt{1+Ha^2}} - (k-h)^{1+\sqrt{1+Ha^2}},$$

$$L_{26} = (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-1-3\sqrt{1+Ha^2}} + (k-h)^{1+\sqrt{1+Ha^2}}(k+h)^{-1-3\sqrt{1+Ha^2}},$$

$$L_{27} = (k+h)^{-2}(k-h)^{-2-2\sqrt{1+Ha^2}} + (k-h)^{-2}(k+h)^{-2-2\sqrt{1+Ha^2}},$$

$$L_{28} = (k-h)^{-2\sqrt{1+Ha^2}} + (k+h)^{-2\sqrt{1+Ha^2}},$$

$$L_{29} = (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-1+3\sqrt{1+Ha^2}} + (k-h)^{1+\sqrt{1+Ha^2}}(k+h)^{-1+3\sqrt{1+Ha^2}},$$

$$L_{30} = (k-h)^2(k+h)^{-2+4\sqrt{1+Ha^2}} + (k-h)^2(k+h)^{-2+4\sqrt{1+Ha^2}},$$

$$L_{31} = (k-h)^{4\sqrt{1+Ha^2}} + (k+h)^{4\sqrt{1+Ha^2}},$$

$$L_{32} = (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-1-\sqrt{1+Ha^2}} + (k-h)^{1+\sqrt{1+Ha^2}}(k-h)^{-1-\sqrt{1+Ha^2}},$$

$$L_{33} = (k-h)^{-2}(k+h)^2 + (k+h)^{-2}(k-h)^2,$$

$$L_{34} = (k+h)^{1+\sqrt{1+Ha^2}}(k-h)^{-1+\sqrt{1+Ha^2}} + (k-h)^{1+\sqrt{1+Ha^2}}(k+h)^{-1+\sqrt{1+Ha^2}},$$

$$L_{35} = (k+h)^2(k-h)^{-2+2\sqrt{1+Ha^2}} + (k-h)^2(k+h)^{-2+2\sqrt{1+Ha^2}} + (k-h)^{2\sqrt{1+Ha^2}} + (k+h)^{2\sqrt{1+Ha^2}} - 2L_{34},$$

$$L_{36} = 2\sqrt{1+Ha^2}\{k^2((k+h)^{\sqrt{1+Ha^2}} - (k-h)^{\sqrt{1+Ha^2}})^2 + h^2((k+h)^{\sqrt{1+Ha^2}} + (k-h)^{\sqrt{1+Ha^2}})^2 + 2kh((k-h)^{2\sqrt{1+Ha^2}} + (k+h)^{2\sqrt{1+Ha^2}})\},$$

$$c_1 = -\frac{q_{00}}{2} - (k+h) + \frac{(F_{00}+2h)}{2L_9}\{(k+h)^2(Ha^2+2)L_{18} + L_{36},$$

$$c_2 = -\frac{1}{2L_9}Ha^2(q_{00}+2h)L_{18},$$

$$c_3 = \frac{1}{L_9}(q_{00}+2h)(1+\sqrt{1+Ha^2})L_{19},$$

$$c_4 = \frac{1}{L_9}(q_{00}+2h)(1-\sqrt{1+Ha^2})L_{25},$$

$$c_5 = -\frac{q_{01}}{2} - (k+h)^2 c_6 - (k+h)^{1-\sqrt{1+Ha^2}}c_7 - (k+h)^{1+\sqrt{1+Ha^2}}c_8$$

$$- (k+h)^{-1-3\sqrt{1+Ha^2}}L_1 - (k+h)^{-1+3\sqrt{1+Ha^2}}L_2 - (k+h)^{-1-\sqrt{1+Ha^2}}L_3$$

$$- (k+h)^{-1+\sqrt{1+Ha^2}}L_4,$$

$$c_6 = -\frac{q_{01}}{2L_9}L_{18} + \frac{1}{2L_9(1-\sqrt{1+Ha^2})L_{25}}[(1-\sqrt{1+Ha^2})$$

$$\{(1+3\sqrt{1+Ha^2})L_1L_{14} + (1-3\sqrt{1+Ha^2})L_2L_{15} + (1+\sqrt{1+Ha^2})^2L_3L_{16}$$

$$+ (1-\sqrt{1+Ha^2})^2L_4L_{17}\}L_9 - Ha^2\{(L_1L_{10}+L_2L_{11}+L_3L_{12}+L_4L_{13})L_{18}\}],$$

$$c_7 = \frac{1}{L_9}q_{01}(1+\sqrt{1+Ha^2})L_{19} + \frac{1}{(1-\sqrt{1+Ha^2})L_9L_{25}}$$

$$[(1+3\sqrt{1+Ha^2})L_1L_9L_{20} + (-1+3\sqrt{1+Ha^2})L_2L_9L_{21}$$

$$+ (1+\sqrt{1+Ha^2})L_3L_9L_{22} + (-1+\sqrt{1+Ha^2})^2L_4L_9L_{23}$$

$$+ (1+\sqrt{1+Ha^2})\{L_1L_{10}+L_2L_{11}+L_3L_{12}+L_4L_{13}\}L_{24}],$$

$$c_8 = \frac{1}{L_9}[(1-\sqrt{1+Ha^2})L_{25}F_{01} + L_1L_{10} + L_2L_{11} + L_3L_{12} + L_4L_{13}],$$

$$c_9 = -\frac{q_{10}}{2} - (k+h)^2c_{10} - (k+h)^{1-\sqrt{1+Ha^2}}c_{11} - (k+h)^{1+\sqrt{1+Ha^2}}c_{12}$$

$$- (k+h)^{-1-3\sqrt{1+Ha^2}}L_5 - (k+h)^{-1+3\sqrt{1+Ha^2}}L_6 - (k+h)^{-1-\sqrt{1+Ha^2}}L_7$$

$$- (k+h)^{-1+\sqrt{1+Ha^2}}L_8,$$

$$c_{10} = -\frac{q_{10}}{2L_9}L_{18} + \frac{1}{2L_9(1-\sqrt{1+Ha^2})L_{25}}[(1-\sqrt{1+Ha^2})$$

$$\{(1+3\sqrt{1+Ha^2})L_5L_{14} + (1-3\sqrt{1+Ha^2})L_6L_{15} + (1+\sqrt{1+Ha^2})^2L_7L_{16}$$

$$+(1-\sqrt{1+Ha^2})^2L_8L_{17}\}L_9 - Ha^2\{(L_5L_{10}+L_6L_{11}+L_7L_{12}+L_8L_{13})L_{18}\}],$$

$$c_{11} = \frac{1}{L_9}q_{10}(1+\sqrt{1+Ha^2})L_{19} + \frac{1}{(1-\sqrt{1+Ha^2})L_9L_{25}}$$

$$[(1+3\sqrt{1+Ha^2})L_5L_9L_{20} + (-1+3\sqrt{1+Ha^2})L_6L_9L_{21}$$

$$+(1+\sqrt{1+Ha^2})L_7L_9L_{22} + (-1+\sqrt{1+Ha^2})^2L_8L_9L_{23}$$

$$+(1+\sqrt{1+Ha^2})\{L_5L_{10}+L_6L_{11}+L_7L_{12}+L_8L_{13}\}L_{24}],$$

$$c_{12} = \frac{1}{L_9}[(1-\sqrt{1+Ha^2})L_{25}q_{10} + L_5L_{10} + L_6L_{11} + L_7L_{12} + L_8L_{13}].$$

3.5 Graphs and Discussion

In this section, we have presented the solution of Oldroyd-8-constant fluid graphically. Fig. 3.1 shows the comparison between numerical solution and perturbation for pressure rise and velocity profile. It is seen from 3.1a, 3.1b and 3.1c that for different values of α_1 and α_2 the perturbation and numerical solution show a very good agreement. It is observed from Fig. 3.1d that there is a difference between the numerical solution and perturbation towards the positive values of r. To observe the behaviour of the emerging parameters involved in the expression of pressure rise, axial velocity and stream function ψ Figures. 3.2-3.7 have been displayed. The effect of α_1, α_2, k and Ha on the pressure rise are shown in Fig. 3.2. It is observed from Fig. 3.2a that pressure rise increases with increasing α_1 in peristaltic pumping region and free pumping region while it decreases in the augmented pumping region.

It is depicted that Fig. 3.2b has an opposite behaviour as compared to the Fig. 3.2a. Fig. 3.2c prepared to see the behaviour of k. We observed the pressure rise increase with an increase in k. Fig. 3.2d shows that the pressure rise decreases with an increase in Ha in augmented pumping and free pumping region. Fig. 3.3 is plotted to illustrate the effects of α_1, α_2, k, and, Ha on velocity profile. In Fig. 3.3a the effects of α_1 are studied. It is observed that velocity increases with the increase in α_1 near the centre of the curved channel. The influence of α_2 on the velocity field is illustrated in Fig. 3.3b. It is noted that α_2 yields an effect opposite to that of α_1. Fig. 3.3c depicts that with the increase in values of k the velocity profile is symmetric about the axis of the curved channel however, for small values of k the profiles are not symmetric about $r = 0$. Fig. 3.3d indicates that with the increase in the value of Ha causes decrease in velocity field near the center of the curved channel. It is due to the fact magnetic field provides resistance to the flow. The velocity increases with an increase in the upper half of the curved channel, whereas it has opposite behaviour in the lower half of the curved channel. The stream lines pattern for different values of α_1, α_2, k, and, Ha is studied in Figures. 3.4-3.7. The stream lines for different values of α_1 are shown in Fig. 3.4. It is evident that the trapped bolus which is as whole increases in size with the increase in α_1. The effect of α_2 can be seen through Fig. 3.5. It is observed that with the increase in α_2 the size of trapped bolus decreases. In Fig. 3.6 stream lines are plotted for different values of curvature parameter k. We observed that with an increase in curvedness of the channel the size of the trapped bolus increases in the lower half of the curved channel and decrease in the upper half of the curved channel. The stream lines for different values of Ha are shown in Fig. 3.7. we noticed that it is similar to that presented in Fig. 3.6.

3.6 Conclusion

In this study, we have developed the governing equations for peristaltic flow of an incompressible conducting Oldroyd 8-constant fluid under long wavelength and low Reynolds number assumptions. The major findings have been listed below.

(1) The behaviour of α_1 and α_2 on the velocity u and pressure rise p are opposite.

(2) The effect of increasing in magnetic parameter Ha decrease in velocity.

(3) The bolus size decreases in the upper half of the curved channel while it increases in the lower half of the curved channel by increasing magnetic parameter Ha.

(4) The curved channel for large curvature parameter is reduced into the straight channel.

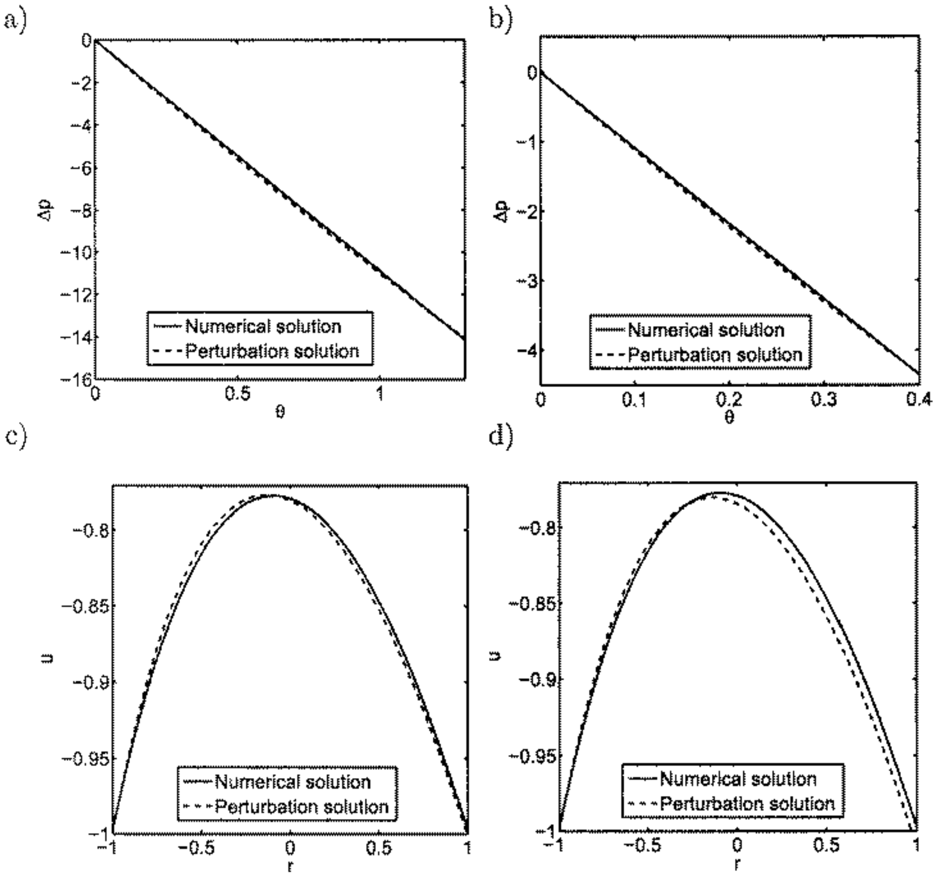

Figure 3.1: (a) Comparison of Δp for fixed $\phi = 0.001$, $k = 1.5$, $Ha = .75$, $\alpha_1 = .002$ and $\alpha_2 = .001$. (b) Comparison of Δp for fixed $\phi = 0.001$, $k = 1.5$, $Ha = .75$, $\alpha_1 = .02$ and $\alpha_2 = .01$. (c) Comparison of u for fixed $k = 2.5$, $\phi = 0.003$, $\alpha_1 = .01$, $\alpha_2 = .02$ and $Ha = 1$. (d) Comparison of u for fixed $k = 2.5$, $\phi = 0.003$, $\alpha_1 = .1$, $\alpha_2 = .2$ and $Ha = 1$.

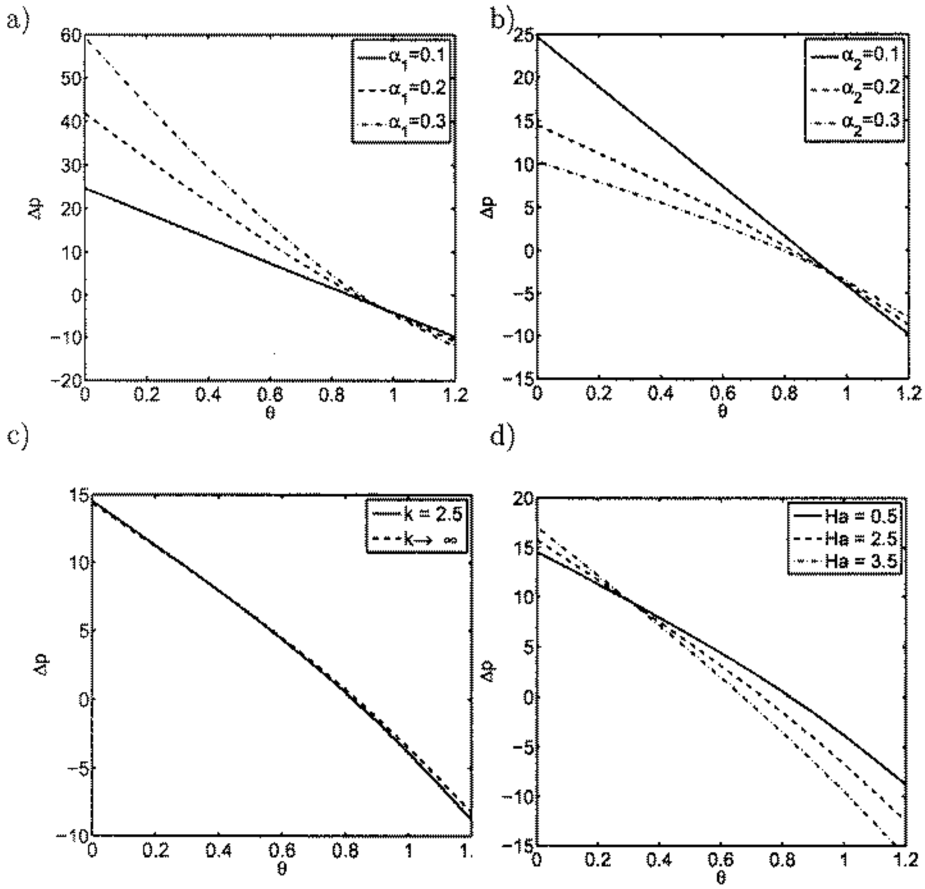

Figure 3.2: (a) Variation of Δp for different values of α_1 with $\phi = 0.6$, $k = 2.5$, $Ha = 0.5$ and $\alpha_2 = 0.1$. (b) Variation of Δp for different values of α_2 with $\phi = 0.9$, $k = 2.5$, $Ha = 0.5$ and $\alpha_1 = 0.1$. (c) Variation of Δp for different values of k with $\phi = 0.6$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$ and $Ha = 0.5$. (d) Variation of Δp for different values of Ha with $\phi = 0.6$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$ and $k = 2.5$.

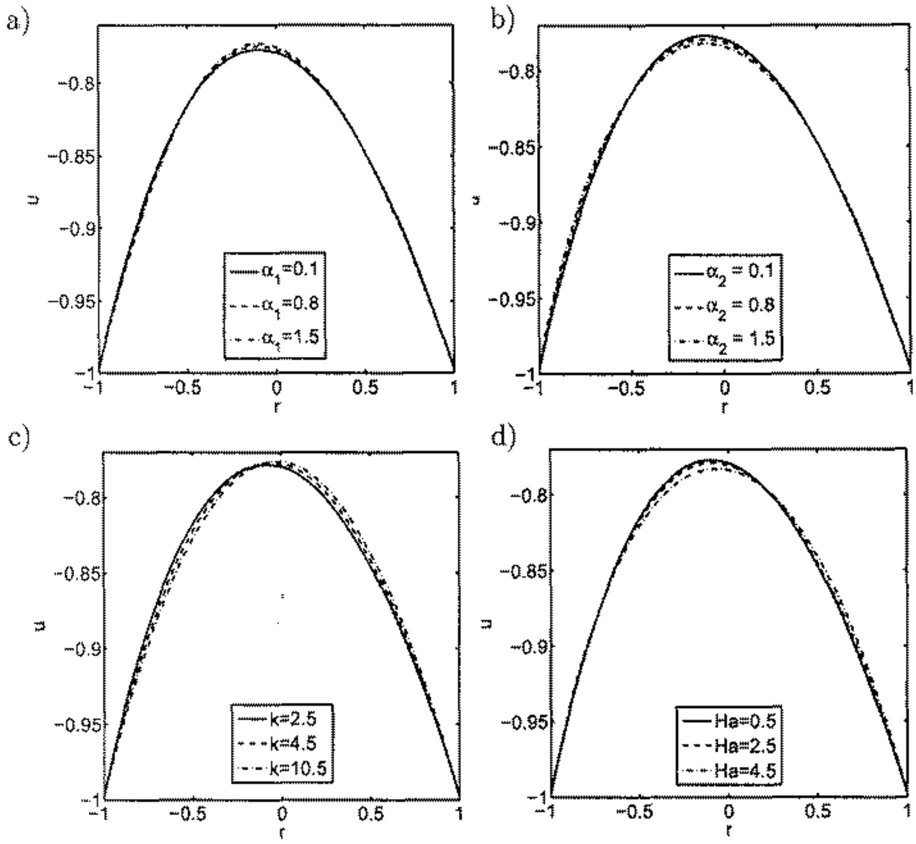

Figure 3.3: (a) Variation of $u(r)$ for different values of α_1 with $\phi = 0.6$, $k = 2.5$, $\alpha_2 = 0.1$ and $Ha = 0.5$. (b) Variation of $u(r)$ for different values of α_2 with $\phi = 0.6$, $k = 2.5$, $\alpha_1 = 0.1$ and $Ha = 0.5$. (c) Variation of $u(r)$ for different values of k with $\phi = 0.6$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$ and $Ha = 0.5$. (d) Variation of $u(r)$ for different values of Ha with $\phi = 0.6$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$ and $k = 2.5$.

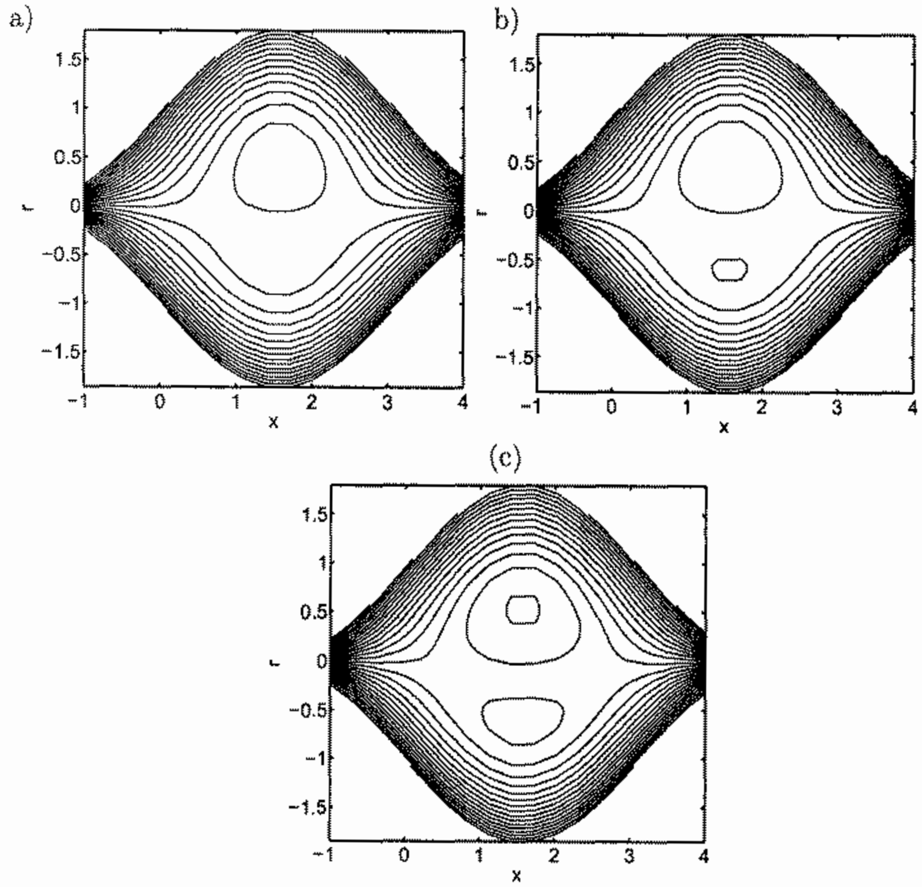

Figure 3.4: Streamlines for $\alpha_1 = 0.2$ (panel *a*), $\alpha_1 = 0.4$ (panel *b*), and $\alpha_1 = 0.6$ (panel *c*). The other parameters are $\phi = 0.9$, $\alpha_2 = 0.4$, $k = 8$ and $Ha = 3.5$.

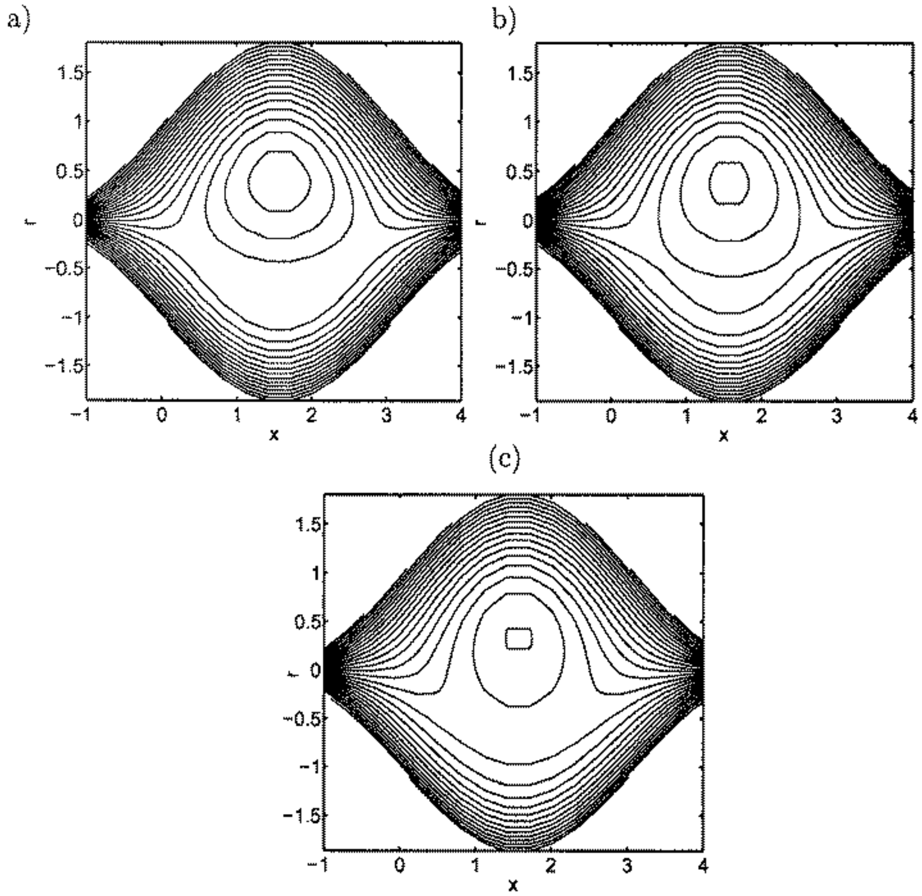

Figure 3.5: Streamlines for $\alpha_2 = 0.4$ (panel a), $\alpha_2 = 0.6$ (panel b) and $\alpha_2 = 0.8$) (panel c). The other parameters are ϕ=0.9, $\alpha_1 = 1.4$, $k = 4$ and $Ha = 0.4$.

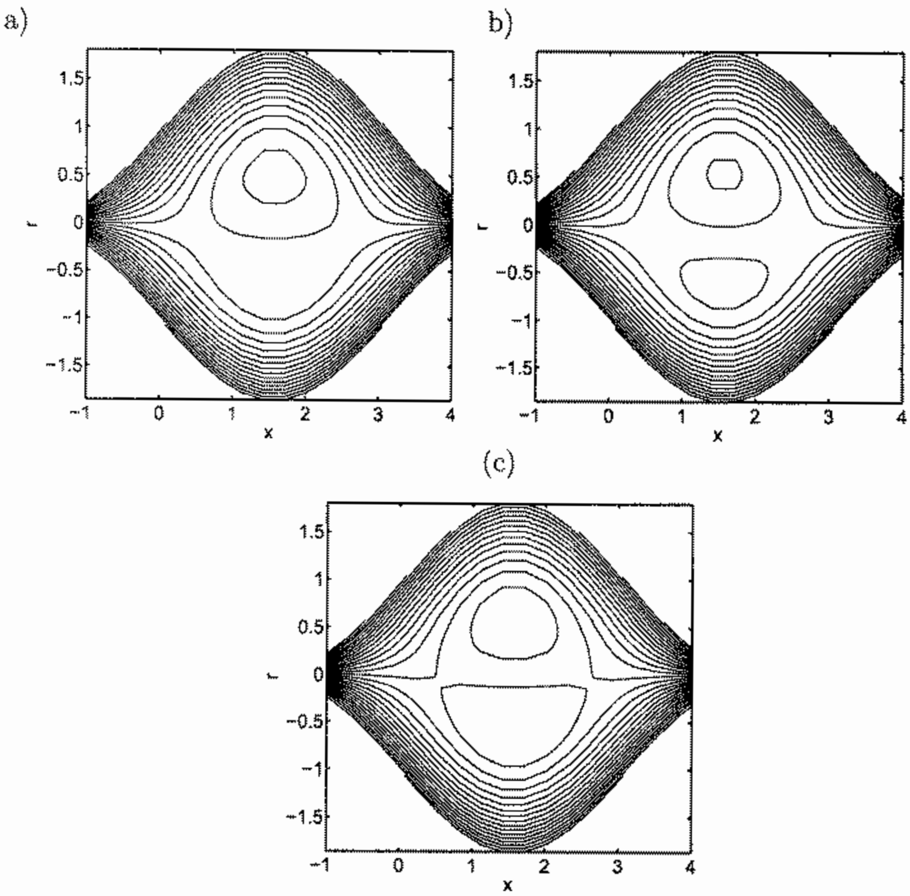

Figure 3.6: Streamlines for $k = 6$ (panel a), $k = 8$ (panel b) and $k = 12$(panel c). The other parameters are ϕ=0.9, $\alpha_1 = 0.4$, $\alpha_2 = 0.2$ and $Ha = 3.5$.

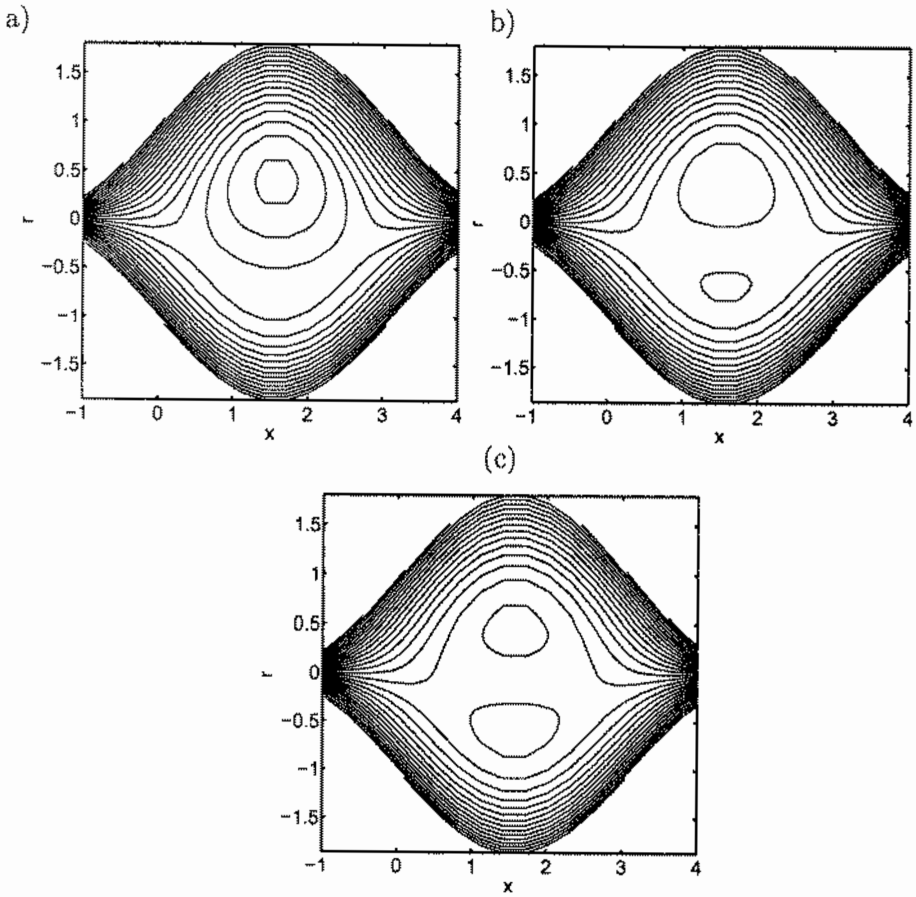

Figure 3.7: Streamlines for $Ha = 0$ (panel a), $Ha = 3$ (panel b) and $Ha = 3.5$ (panel c). The other parameters are ϕ=0.9, $\alpha_1 = 0.4$, $\alpha_2 = 0.2$ and $k = 4$.

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