Decision Making Based on Cubic Sets

By

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A Dissertation
Submitted in the Partial Fulfillment of the
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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF SCIENCE in MATHEMATICS***

We accept this dissertation as conforming to the required standard.

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2015

DECLARATION

I hereby, declare, that this thesis neither as a whole nor as apart thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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Dedicated
To
My
Loving Parents
And respectful teachers.

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Preface

The idea of fuzzy sets (FSs) was first proposed by Zadeh[31] and has achieved a huge success in many areas. The concept of fuzzy sets was generalized as intuitionistic fuzzy sets (IFSs) by Atanassov. In Xu[25] proposed some geometric aggregation operators, like the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and applied IFGH operator to multi-criteria decision-making problems with intuitionistic fuzzy knowledge. Some of the arithmetic aggregation operators like intuitionistic fuzzy weighted averaging (IFWA) etc. were introduced by Xu [25]. Tursken (1986) and Gorzaleczany (1987) gave the idea of so-called interval-valued fuzzy sets (IVFSs) which was considered to be further general form of a fuzzy set, but really there is solid bond between IFSs and IVFSs. Both the IFSs and IVFSs were further generalized by Gargov (1989), named as interval-valued intuitionistic fuzzy sets (IVIFSs). For IVIFSs some aggregation operators, labelled as the interval-valued intuitionistic fuzzy weighted geometric aggregation (IIFWGA) operator and the interval-valued intuitionistic fuzzy weighted arithmetic aggregation (IIFWAA) operator were introduced, and utilized these operators to decision making problems involving multi-criteria with the help of the score function of interval-valued intuitionistic fuzzy information.

Multi-criteria decision making (MCDM) is a procedure which enables a decision maker to point out the best alternative among the provided alternatives. The fast development in diverse fields like management, economics and engineering etc. has made the situation more complex for a single decision maker to deal with all dimensions of a problem. In general, many decision makers are involved in the decision making. In the whole process of decision making, the information to make the decision about alternatives is usually fuzzy or uncertain due to increase in the complexity of politico -socio -economic situation of the world. PROMETHEE, AHP, TOPSIS etc. are representative methods which were introduced to solve MCDM problems in various fields like business, engineering and industries. Such approaches in decision making has a drawback that they generally consider the decision making with certain information of decision values and weights, this thing makes these methods less effective when dealing with fuzzy or uncertain information.

Technique for order preference by similarity to ideal solution (TOPSIS) is one of the familiar classical MCDM method was introduced by Hwang and Yoon [11]. The base of the TOPSIS is a concept that chosen alternative must have the shortest distance from the positive ideal solution and the longest distance from the negative ideal solution. During the application of TOPSIS mainly exact values of performance ratings and weight of criterion are considered. However in real world situation normally data (attributes etc.) is often not so exact and precise, therefore these situations usually fall under huge fuzziness. Due to this reason many researchers have extended the TOPSIS method to fuzzy environment. This extended approach has great use in fuzzy multi-criteria decision making problems. Triantaphyllou and Lin [6] introduced the fuzzy TOPSIS approach depending upon fuzzy arithmetic operations. Then Chen [6] extended the TOPSIS technique for the fuzzy group decision making problems by defining a crisp Euclidean distance between any two fuzzy values. Later Tsaur using the TOPSIS approach converted the fuzzy multi-criteria decision making problem to non-fuzzy multi-criteria problem.

The procedure in which one finds the right suppliers who can provide the customer with the right product, at suitable price, at suitable time and in the right quantity is known as supplier selection. Supplier selection is considered back bone for establishing an efficient and effective supply chain. Selection of an inappropriate supplier could harm the whole supply chains at the economic and management levels badly. Now a days without good suppliers it is almost impossible to produce low cast, high quality products in a successful manner. The main objective of the supplier selection procedure is to minimize purchase risk, enhance over all worth to the buyer and construct the confident building measures between suppliers and buyers on long term basis. In simple words, the major goals of supply chain management is to minimize supply chain risk, curtail down production costs, increase revenue, improve customer services, optimization of time cycle, customer satisfaction and revenue. There is no doubt that supplier selection is a multi-criteria decision making (MCDM) problem which has many conflicting factors in its account like price, quality and delivery etc. Multi-criteria decision making is being heavily affected by these factors. There are also many factors which affect the performance of a supplier.

In this thesis, we will solve different multi-criteria decision making problems having information in cubic set values. We solve the multi-criteria group decision making problem with the help of well-known SIR method and TOPSIS technique. We introduce different aggregating operators for cubic sets. A score function has been proposed to solve different multi-criteria decision making problems in this thesis.

Structure of the Thesis

The thesis is organized chapter wise as follows:

Chapter 1:

This chapter is introductory and sets up the background for the problems taken in the thesis. It overviews fuzzy sets, interval-valued fuzzy set, cubic set, decision making, alternatives, criteria, rate or weight, TOPSIS method, SIR method etc.

Chapter 2:

In this chapter, we will solve different multi-criteria decision making problems having information in cubic set values. In the first section of this chapter we deal with the multi-criteria decision making problem using a score function. In the later section we solve the multi-criteria group decision making problem with the help of well-known SIR method.

Chapter 3:

In this chapter, we solve the multi-criteria decision making problems using a new approach, named as TOPSIS. The first section is comprising of a procedure to solve the multi-criteria decision making problem through TOPSIS approach while using score function. We have observed that the aid of score function makes the use of TOPSIS technique very simple. The second section relates to the solution of the multi-criteria group decision making problem through TOPSIS technique, without any aid of score function.

Contents

Chapter 1

Preliminaries

In this chapter, we present a concise summary of basic definitions and preliminary results, which will be helpful in the subsequent chapters of this dissertation. For undefined terms and notions, we reffer to [1, 2, 3, 4, 8, 12, 13, 21, 22, 23, 25, 26, 27, 33].

1.1 Definitions and Examples

In this section, we review some basic defnitions and notions.

1.1.1 Definition[10]

Decision maker is any individual, group of individuals which can play a role, directly or indirectly in the decision process for whom the decision-aid tools are developed and implemented.

1.1.2 Definition[10]

Alternatives are the possibilities, one has to choose from. Alternatives can be identified or even developed. We will follow $B = \{B_i : i = 1, 2, 3, \ \dots .m\}$ to denote the alternatives.

1.1.3 Definition[10]

Criteria are the characteristics or requirements that each alternative must possess to a greater or lesser extent. Usually are rated on how they possess each criterion. We will follow $C = \{C_i : i = 1, 2, 3, \ \dots .n\}$ to denote the criteria.

1.1.4 Definition[10]

Rate or weight of the criteria is the value that indicates the relative importance of one criterion in a particular decision process. It depends upon decision maker preference.

1.1.5 Definition[10]

Decision making can be regarded as the mental processes resulting in the selection of a course of action among several alternatives. Every decision making process produces a final choice. The output can be an action or an opinion of choice.

1.1.6 Definition[10]

A function which measures the accuracy of possible alternatives is known as score function. It is the measure of the calibration of a set of possible options. We will denote score function by M.

1.1.7 Definition[24]

The procedure in which one finds the right suppliers who can provide the customer with the right product, at suitable price, at suitable time and in the right quantity is known as supplier selection.

1.1.8 Definition[32]

A linguistic variable means a variable whose values are words or sentences in a natural or artificial language.

1.1.9 Example

Age is a linguistic variable if its values are linguistic rather than numerical, i.e. , young, not young, very young, quite young, old, not very old and not very young, etc. , rather than 19, 24, 27, 29, ... , .

1.1.10 Definition[10]

The method of tackling the process of decision making in the presence of multiple criteria or objectives is known as multi-criteria decision making (MCDM).

1.2 Fuzzy sets

1.2.1 Definition[31]

A fuzzy set is a pair $(X,\ \lambda)$ where X is a set and λ is a function from X to the unit closed interval $I = [0,\ 1]$, i.e. $\lambda : X \longrightarrow [0,\ 1]$. For each $x \in X$, $\lambda(x)$ is called the grade of membership of x in $(X,\ \lambda)$ and λ is called membership function of $(X,\ \lambda)$.

A fuzzy subset $\lambda : X \longrightarrow [0,\ 1]$ is non-empty if λ is not a zero map. Note that $\lambda(x) \in [0,\ 1]$ for all $x \in X$.

1.2.2 Definition[31]

Let X be a non-empty subset. Then for any $A \subseteq X$ the characteristic function of A is denoted by C_A defined by $C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

for $x \in X$.

1.2.3 Definition[31]

A fuzzy subset of X of the form

$$\lambda(z) = \begin{cases} t & \text{if } z = x \\ 0 & \text{if } z \neq x \end{cases},$$

is called the fuzzy point with support x and value t, where $t \in (0,\ 1]$. It is usually denoted by x_t.

1.2.4 Definition[31]

The complement of the fuzzy subset of X of the form $\lambda(x)$ is denoted and defined as

$$\lambda^c(x) = 1 - \lambda(x).$$

1.2.5 Remarks

1. Two fuzzy subsets λ and μ of a set X are said to be disjoint if there is no $x \in X$ such that $\lambda(x) = \mu(x)$. If $\lambda(x) = \mu(x)$ for each $x \in X$, then we say that λ and μ are equal and write $\lambda = \mu$.

2. Let λ and μ be two fuzzy subsets of non-empty set X. Then λ is said to be included in μ *i.e.*, $\lambda \subseteq \mu$ if and only if $\lambda(x) \leq \mu(x)$ for all $x \in X$.

3. Let λ and μ be two fuzzy subsets of non-empty set X. Then λ is said to be properly included in μ *i.e.*, $\lambda \subset \mu$ if and only if $\lambda(x) < \mu(x)$ for all $x \in X$.

4. The union of any family $\{\lambda_i : i \in \Omega\}$ of fuzzy subsets λ_i of a non-empty set X is denoted by $(\underset{i\in\Omega}{\cup} \lambda_i)$ and defined by $(\underset{i\in\Omega}{\cup} \lambda_i)(x) = \underset{i\in\Omega}{\sup} \lambda_i(x) = \underset{i\in\Omega}{\vee} \lambda_i(x)$, for all $x \in X$. Moreover $(\underset{i\in\Omega}{\cup} \lambda_i)$ is smallest fuzzy subset which containing λ_i.

5. The intersection of any family $\{\lambda_i : i \in \Omega\}$ of fuzzy subsets λ_i of a non-empty set X is denoted by $(\underset{i\in\Omega}{\cap} \lambda_i)$ and defined by $(\underset{i\in\Omega}{\cap} \lambda_i)(x) = \underset{i\in\Omega}{\inf} \lambda_i(x) = \underset{i\in\Omega}{\wedge} \lambda_i(x)$, for all $x \in X$. Moreover $(\underset{i\in\Omega}{\cap} \lambda_i)$ is largest fuzzy subset which is contained in λ_i.

1.3 Interval-valued fuzzy sets (IVFs)

1.3.1 Definition[30]

An interval number means a closed subinterval of $I = [0,\ 1]$. We denote the set of all interval numbers by $[I]$.

1.3.2 Definition[30]

Let X be a non empty set. A function $\beta : X \longrightarrow [I]$ is called an interval-valued fuzzy set (briefly, an IVF set). Let $[I]^x$ denote the set of all IVF sets in X. For every $\beta \in [I]^x$ and $x \in X$, $\beta(X) = [\beta^-(x),\ \beta^+(x)]$, is called the degree of membership function of an element x to β, where $\beta^- : X \longrightarrow I$ and $\beta^+ : X \longrightarrow I$ are fuzzy sets in X which are called lower fuzzy set and upper fuzzy set in X, resp.

The complement β^c of $\beta \in [I]^x$ is defined as follows $\beta^c(x) = \beta(x)^c$ for all $x \in X$, i.e. $\beta^c(x) = [1 - \beta^+(x),\ 1 - \beta^-(x)]$ for all $x \in X$.

1.4 Cubic sets (CSs)

1.4.1 Definition[30]

Let X be a non-empty set. By cubic set in X we mean a structure $A = \{\langle x,\ \beta(x),\ \lambda(x)\rangle \mid x \in X\}$ in which β is an IVF set in X and λ is fuzzy set in X. A cubic set $A = \{\langle x,\ \beta(x),\ \lambda(x)\rangle \mid x \in X\}$ is simply denoted by $A = \langle \beta,\ \lambda\rangle$.

1.4.2 Example

Let X be a non-empty set. Let β be an IVF set in X. Then $A = \{\langle x,\ \beta(x),\ 1(x)\rangle \mid x \in X\}$, $B = \{\langle x,\ \beta(x),\ 0(x)\rangle \mid x \in X\}$ and $C = \{\langle x,\ \beta(x),\ \lambda(x)\rangle \mid x \in X\}$, where $\lambda(x) = \frac{\beta^-(x) + \beta^+(x)}{2}$ are cubic sets in X.

1.4.3 Definition[30]

Let X be a non-empty set. A cubic set $A = \langle \beta,\ \lambda \rangle$ in X is called an internal cubic set (briefly ICS) if $\beta^-(x) \leq \lambda(x) \leq \beta^+(x)$ for all $x \in X$.

1.4.4 Example

Let $A = \{\langle x,\ \beta(x),\ \lambda(x)\rangle \mid x \in I\}$ be a cubic set in I. If $\beta(x) = [0.3,\ 0.7]$ and $\lambda(x) = 0.4$ for all $x \in I$, then A is an ICS.

1.4.5 Definition[30]

Let X be a non-empty set. A cubic set $A = \langle \beta,\ \lambda \rangle$ in X is called an external cubic set (briefly ECS) if $\lambda(x) \notin (\beta^-(x),\ \beta^+(x))$ for all $x \in X$.

1.4.6 Example

Let $A = \{\langle x,\ \beta(x),\ \lambda(x)\rangle \mid x \in I\}$ be a cubic set in I. If $\beta(x) = [0.3,\ 0.7]$ and $\lambda(x) = 0.8$ for all $x \in I$, then A is an ECS.

1.4.7 Example

If $\beta(x) = [0.3,\ 0.7]$ and $\lambda(x) = x$ for all $x \in I$, then A is neither an ICS nor an ECS.

1.4.8 Theorem

Let $A = \langle \beta,\ \lambda \rangle$ be a cubic set in X which is not an ECS. Then there exist $x \in X$ such that $\lambda(x) \in \left(\beta^-(x),\ \beta^+(x)\right)$.

1.4.9 Theorem

Let $A = \langle \beta,\ \lambda \rangle$ be a cubic set in X. If A is both an ICS and an ECS, then $(\forall\ x \in X)$ $(\lambda(x) \in U(A) \cup L(A))$ *where* $U(A) = \{\beta^+(x) \mid x \in X\}$ and $L(A) = \{\beta^-(x) \mid x \in X\}$.

1.4.10 Definition[30]

Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be cubic sets in X. Then we define

(a) (Equality) $A = B \Leftrightarrow \beta = \gamma \text{ and } \lambda = \mu$.

(b) (P-order) $A \subseteq_P B \Leftrightarrow \beta \subseteq \gamma \text{ and } \lambda \leq \mu$.

(c) (R-order) $A \subseteq_R B \Leftrightarrow \beta \subseteq \gamma \text{ and } \lambda \geq \mu$.

1.4.11 Definition[30]

For any $A_i = \{\langle\ x,\ \beta_i(x),\ \lambda_i(x)\ \rangle \mid x \in X\}$ where $i \in \Lambda$, we define

$$(a)\ \bigcup_{P\, i \in \Lambda} A_i = \left\{\langle\ x,\ \left(\bigcup_{i \in \Lambda} \beta_i\right)(x),\ \left(\bigvee_{i \in \Lambda} \lambda_i\right)(x)\ \rangle \mid x \in X\right\} \quad \text{(P-union)}$$

$$(b)\ \bigcap_{P\, i \in \Lambda} A_i = \left\{\langle\ x,\ \left(\bigcap_{i \in \Lambda} \beta_i\right)(x),\ \left(\bigwedge_{i \in \Lambda} \lambda_i\right)(x)\ \rangle \mid x \in X\right\} \quad \text{(P-intersection)}$$

$$(c)\ \bigcup_{R\, i \in \Lambda} A_i = \left\{\langle\ x,\ \left(\bigcup_{i \in \Lambda} \beta_i\right)(x),\ \left(\bigwedge_{i \in \Lambda} \lambda_i\right)(x)\ \rangle \mid x \in X\right\} \quad \text{(R-union)}$$

(d) $\cap_{R \, i \in \Lambda} A_i = \left\{ \langle \, x, \, \left(\underset{i \in \Lambda}{\cap} \beta_i \right) \, (x), \, \left(\underset{i \in \Lambda}{\vee} \lambda_i \right) \, (x) \, \rangle \mid x \in X \right\}$ (R-intersection)

1.4.12 Defintion[30]

The complement of $A = \langle \beta, \lambda \rangle$ is defined to be the cubic set $A^c = \{ \langle x, \, \beta^c(x), \, 1 - \lambda(x) \rangle \mid x \in$

Obviously $(A^c)^c = A$. Also for any $A_i = \{ \langle \, x, \, \beta_i(x), \, \lambda_i(x) \, \rangle \mid x \in X \}$ where $i \in \Lambda$, we have $\left(\underset{i \in \Lambda}{\cup_P} A_i \right)^c = \underset{i \in \Lambda}{\cup_P} (A)^c$ and $\left(\underset{i \in \Lambda}{\cap_P} A_i \right)^c = \underset{i \in \Lambda}{\cap_P} (A_i)^c$. Also we have $\left(\underset{i \in \Lambda}{\cup_R} A_i \right)^c = \underset{i \in \Lambda}{\cap_R} (A_i)^c$ and $\left(\underset{i \in \Lambda}{\cap_R} A_i \right)^c = \underset{i \in \Lambda}{\cup_R} (A_i)^c$.

1.4.13 Proposition

For any cubic sets $A = \langle \beta, \lambda \rangle$, $B = \langle \gamma, \mu \rangle$, $C = \langle \delta, \nu \rangle$ and $D = \langle \epsilon, \xi \rangle$, we have

(1) if $A \subseteq_P B$ and $B \subseteq_P C$ then $A \subseteq_P C$.

(2) if $A \subseteq_P B$ then $B^c \subseteq_P A^c$.

(3) if $A \subseteq_P B$ and $A \subseteq_P C$ then $A \subseteq_P B \cap_P C$.

(4) if $A \subseteq_P B$ and $C \subseteq_P B$ then $A \cup_P C \subseteq_P B$.

(5) if $A \subseteq_P B$ and $C \subseteq_P D$ then $A \cup_P C \subseteq_P B \cup_P D$ and $A \cap_P C \subseteq_P B \cap_P D$.

(6) if $A \subseteq_R B$ and $B \subseteq_R C$ then $A \subseteq_R C$.

(7) if $A \subseteq_R B$ then $B^c \subseteq_R A^c$.

(8) if $A \subseteq_R B$ and $A \subseteq_R C$ then $A \subseteq_R B \cap_R C$.

(9) if $A \subseteq_R B$ and $C \subseteq_R B$ then $A \cup_R C \subseteq_R B$.

(10) if $A \subseteq_R B$ and $C \subseteq_R D$ then $A \cup_R C \subseteq_R B \cup_R D$ and $A \cap_R C \subseteq_R B \cap_R D$.

1.4.14 Theorem

Let $A = \langle \beta, \lambda \rangle$ be a cubic set in X. If A is an ECS (resp. ICS), then A^c is an ICS (resp. ICS).

1.4.15 Theorem

Let $\{A_i = \langle \beta_i, \lambda_i \rangle \mid i \in \Lambda\}$ be a family of ICSs in X. Then the P-union and P-intersection of $\{A_i = \langle \beta_i, \lambda_i \rangle \mid i \in \Lambda\}$ are ICSs in X.

1.4.16 Remark

P-union and P-intersection of ECSs need not to be an ECS. The following example illustrates this.

1.4.17 Example

Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in $I = [0, 1]$ in which $\beta(x) = [0.3, 0.5]$, $\lambda(x) = 0.8$, $\gamma(x) = [0.7, 0.9]$, $\mu(x) = 0.4$ for all $x \in I$.

(1) We know that $A \cup_P B = \{\langle x, \gamma(x), \lambda(x) \rangle \mid x \in I\}$ and $\lambda(x) \in (\gamma^-(x), \gamma^+(x))$ for all $x \in I$. Hence $A \cup_P B$ is not an ECS in I.

(2) We know that $A \cap_P B = \{\langle x, \beta(x), \mu(x) \rangle \mid x \in I\}$ and $\mu(x) \in (\beta^-(x), \beta^+(x))$ for all $x \in I$. Hence $A \cap_P B$ is not an ECS in I.

1.4.18 Remark

R-union and R-intersection of ICSs need not to be an ICS. The following example illustrates this.

1.4.19 Example

Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be ICSs in $I = [0,\ 1]$ in which $\beta(x) = [0.3,\ 0.5]$, $\lambda(x) = 0.4$, $\gamma(x) = [0.7,\ 0.9]$, $\mu(x) = 0.8$ for all $x \in I$.

(1) We know that $A \cup_R B = \{\langle x,\ \gamma(x),\ \lambda(x)\ \rangle \mid x \in I\}$ and $\lambda(x) \notin (\gamma^-(x),\ \gamma^+(x))$ for all $x \in I$. Hence $A \cup_R B$ is not an ICS in I.

(2) We know that $A \cap_R B = \{\langle x,\ \beta(x),\ \mu(x)\ \rangle \mid x \in I\}$ and $\mu(x) \notin (\beta^-(x),\ \beta^+(x))$ for all $x \in I$. Hence $A \cap_R B$ is not an ICS in I.

1.4.20 Remark

R-union and R-intersection of ECSs need not to be an ECS. The following example illustrates this.

1.4.21 Example

Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be ECSs in $I = [0,\ 1]$ in which $\beta(x) = [0.2,\ 0.4]$, $\lambda(x) = 0.7$, $\gamma(x) = [0.6,\ 0.8]$, $\mu(x) = 0.9$ for all $x \in I$.

(1) We know that $A \cup_R B = \{\langle x,\ \gamma(x),\ \lambda(x)\ \rangle \mid x \in I\}$ and $\lambda(x) \in (\gamma^-(x),\ \gamma^+(x))$ for all $x \in I$. Hence $A \cup_R B$ is not an ECS in I.

(2) Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be ECSs in $I = [0,\ 1]$ in which $\beta(x) = [0.2,$

0.4], $\lambda(x) = 0.1$, $\gamma(x) = [0.6, 0.8]$, $\mu(x) = 0.3$ for all $x \in I$. Then $A \cap_R B = \{\langle x, \beta(x), \mu(x) \rangle \mid x \in I\}$ and $\mu(x) \in (\beta^-(x), \beta^+(x))$ for all $x \in I$. Thus $A \cap_R B$ is not an ECS in I.

1.4.22 Theorem

Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ICSs in X such that

$$\max \{\beta^-(x), \gamma^-(x)\} \leq (\lambda, \mu)(x)$$

for all $x \in X$. Then R-union of A and B is an ICS in X.

1.4.23 Theorem

Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ICSs in X satisfying the following inequality

$$\min \{\beta^+(x), \gamma^+(x)\} \geq (\lambda, \mu)(x)$$

for all $x \in X$. Then R-intersection of A and B is an ICS in X.

1.4.24 Remark

Suppose that $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ in X, if we exchange μ and λ, we denote the cubic sets by $A^* = \langle \beta, \mu \rangle$ and $B^* = \langle \gamma, \lambda \rangle$ resp. For two ECSs $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ in X, two cubic sets A^* and B^* may not be ICSs in X as observed in the following example.

1.4.25 Example

(1) Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in $I = [0, 1]$ in which $\beta(x) = [0.6, 0.7]$, $\lambda(x) = 0.8$, $\gamma(x) = [0.3, 0.4]$, $\mu(x) = 0.2$ for all $x \in I$. Then we know that $A^* = \langle \beta, \mu \rangle$ and $B^* = \langle \gamma, \lambda \rangle$ are not an ICSs in X because $\mu(0.5) = 0.2 \notin [0.6, 0.7] = A(0.5)$ and $\lambda(0.5) = 0.8 \notin [0.3, 0.4] = B(0.5)$.

(2) Let $X = \{a, b\}$ be a set. Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in X defined by the following table.

X	$\beta(x)$	$\lambda(x)$
a	$[0.2, 0.3]$	0.1
b	$[0.5, 0.6]$	0.7

X	$\gamma(x)$	$\mu(x)$
a	$[0.4, 0.5]$	0.9
b	$[0.7, 0.9]$	0.4

Then we come to know that $A^* = \langle \beta, \mu \rangle$ and $B^* = \langle \gamma, \lambda \rangle$ are not ICSs in X because $\mu(a) = 0.9 \notin [0.2, 0.3] = A(a)$ and $\lambda(a) = 0.1 \notin [0.4, 0.5] = B(a)$.

1.4.26 Remark

P-union of two ECSs in X need not be an ICS in X. We observe this in the following example.

1.4.27 Example

Let $X = \{a, b, c\}$ be a set. Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in X defined by the following table.

X	$\beta(x)$	$\lambda(x)$
a	$[0.3,\ 0.5]$	0.7
b	$[0.2,\ 0.4]$	0.65
c	$[0.35,\ 0.45]$	0.75

X	$\gamma(x)$	$\mu(x)$
a	$[0.6,\ 0.8]$	0.35
b	$[0.25,\ 0.55]$	0.1
c	$[0.7,\ 0.85]$	0.4

Then we come to know that $A \cup_P B = \langle \beta \cup \gamma,\ \lambda \vee \mu \rangle$ is not an ICS in X because $\langle \lambda \vee \mu \rangle (b) = 0.65 \notin [0.25,\ 0.55] = (\beta \cup \gamma)(b)$.

1.4.28 Theorem

For two ECSs $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ in X, if $A^* = \langle \beta,\ \mu \rangle$ and $B^* = \langle \gamma,\ \lambda \rangle$ are ICSs in X , then P-union $A \cup_P B$ of $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ is an ICS in X.

1.4.29 Theorem

Let A and B be ECSs in X such that A^* and B^* are ICS in X. Then the P-intersection of A and B is an ICS in X.

1.4.30 Remark

If A and B are ECSs in X, two cubic sets A^* and B^* may not be ECSs in X as illustrated below.

1.4.31 Example

Let $X = \{a,\ b\}$ be a set. Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be ECSs in X defined by the following table.

X	$\beta(x)$	$\lambda(x)$
a	$[0.2,\ 0.3]$	0.1
b	$[0.5,\ 0.6]$	0.7

X	$\gamma(x)$	$\mu(x)$
a	$[0.4,\ 0.5]$	0.9
b	$[0.7,\ 0.9]$	0.4

Then we come to know that $A^* = \langle \beta,\ \mu \rangle$ and $B^* = \langle \gamma,\ \lambda \rangle$ are not ECSs in X because $\mu(b) = 0.4 \notin (0.3,\ 0.6) = A(a)$ and $\lambda(a) = 0.5 \in (0.4,\ 0.5) = B(a)$.

1.4.32 Theorem

Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be ECSs in X such that $A^* = \langle \beta,\ \mu \rangle$ and $B^* = \langle \gamma,\ \lambda \rangle$ are ECS in X. Then the P-union of A and B is an ECS in X.

1.4.33 Theorem

Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be ECSs in X such that

$$\min\left\{\max\ \left\{\beta^+(x),\ \gamma^-(x)\ \right\},\ \max\ \left\{\beta^-(x),\ \gamma^+(x)\ \right\}\right\} \geq (\lambda \wedge \mu)(x)$$

$$> \max\left\{\min\ \left\{\beta^+(x),\ \gamma^-(x)\ \right\},\ \min\ \left\{\beta^-(x),\ \gamma^+(x)\ \right\}\right\}$$

for all $x \in X$. Then the P-intersection of A and B is an ECS in X.

1.4.34 Remark

For two ECSs $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ which satisfy the condition

$$\min\left\{\max\ \left\{\beta^+(x),\ \gamma^-(x)\ \right\},\ \max\ \left\{\beta^-(x),\ \gamma^+(x)\ \right\}\right\} > (\lambda \wedge \mu)(x)$$

$$= \max\left\{\min\ \left\{\beta^+(x),\ \gamma^-(x)\ \right\},\ \min\ \left\{\beta^-(x),\ \gamma^+(x)\ \right\}\right\}$$

for all $x \in X$, the P-intersection of A and B may not be an ECS in X. We justify it with following example.

1.4.35 Example

Let $X = \{a,\ b,\ c\}$ be a set. Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be ECSs in X defined by the following table.

X	$\beta(x)$	$\lambda(x)$
a	$[0.2,\ 0.6]$	0.7
b	$[0.3,\ 0.7]$	0.3
c	$[0.2,\ 0.6]$	0.9

X	$\gamma(x)$	$\mu(x)$
a	$[0.3,\ 0.7]$	0.3
b	$[0.2,\ 0.6]$	0.7
c	$[0.4,\ 0.7]$	0.4

Then we know that $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ satisfy the following condition

$$\min\left\{\max\ \left\{\beta^{+}(x),\ \gamma^{-}(x)\ \right\},\ \max\ \left\{\beta^{-}(x),\ \gamma^{+}(x)\ \right\}\right\} > (\lambda \wedge \mu)(x)$$

$$= \max\left\{\min\ \left\{\beta^{+}(x),\ \gamma^{-}(x)\ \right\},\ \min\ \left\{\beta^{-}(x),\ \gamma^{+}(x)\ \right\}\right\}$$

But $A \cap_P B = \langle \beta \cap \gamma, (\lambda \wedge \mu)\ \rangle$ is not an ECS in X because $(\lambda \wedge \mu)(a) = 0.3 \in [0.2,\ 0.6] = \left((\beta \cap \gamma)^{-}(a),\ (\beta \cap \gamma)^{+}(a)\right)$.

1.4.36 Theorem

Let $A = \langle \beta,\ \lambda \rangle$ and $B = \langle \gamma,\ \mu \rangle$ be cubic sets in X such that

$$\min\left\{\max\ \left\{\beta^{+}(x),\ \gamma^{-}(x)\ \right\},\ \max\ \left\{\beta^{-}(x),\ \gamma^{+}(x)\ \right\}\right\} = (\lambda \wedge \mu)(x)$$

$$= \max\left\{\min\ \left\{\beta^{+}(x),\ \gamma^{-}(x)\ \right\},\ \min\ \left\{\beta^{-}(x),\ \gamma^{+}(x)\ \right\}\right\}$$

for all $x \in X$. Then the P-intersection of A and B is both an ECS and an ICS in X.

1.4.37 Remark

P-union of two ECSs A and B may not be an ECS. As shown in the following example.

1.4.38 Example

Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in I defined by

$$\beta(x) = \begin{cases} [0.15,\ 0.25] \text{ if } 0 \leq x < 0.5 \\ [0.6,\ 0.7] \text{ if } 0.5 \leq x \leq 1 \end{cases},\ \lambda(x) = \begin{cases} 0.5x +\ 0.5 \text{ if } 0 \leq x < 0.5 \\ 0.3 \text{ if } 0.5 \leq x \leq 1 \end{cases}$$

$$\gamma(x) = \begin{cases} [0.8,\ 0.9] \text{ if } 0 \leq x < 0.5 \\ [0.1,\ 0.2] \text{ if } 0.5 \leq x \leq 1 \end{cases},\ \mu(x) = \begin{cases} 0.4 \text{ if } 0 \leq x < 0.5 \\ x \text{ if } 0.5 \leq x \leq 1 \end{cases}$$

Then

$$\beta \cup \gamma(x) = \begin{cases} [0.8,\ 0.9] \text{ if } 0 \leq x < 0.5 \\ [0.6,\ 0.7] \text{ if } 0.5 \leq x \leq 1 \end{cases}$$

$$(\lambda \vee \mu)(x) = \begin{cases} 0.5x + 0.5 \text{ if } 0 \leq x < 0.5 \\ x \text{ if } 0.5 \leq x \leq 1 \end{cases}$$

But $A \cup_P B$ is not an ECS because

$(\lambda \vee \mu)(0.65) = 0.65 \in (0.65,\ 0.7) = \left((\beta \cup \gamma)^-(0.65),\ (\beta \cup \gamma)^+(0.65)\right).$

1.4.39 Theorem

Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in X such that

$\min\left\{\max\ \{\beta^+(x),\ \gamma^-(x)\ \},\ \max\ \{\beta^-(x),\ \gamma^+(x)\ \}\right\} > (\lambda \vee \mu)(x)$

$\geq \max\left\{\min\ \{\beta^+(x),\ \gamma^-(x)\ \},\ \min\ \{\beta^-(x),\ \gamma^+(x)\ \}\right\}$

for all $x \in X$. Then the P-union of A and B is an ECS in X.

1.4.40 Theorem

Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in X . If for each $x \in X$

$$\min \left\{ \max \left\{ \beta^{+}(x), \gamma^{-}(x) \right\}, \max \left\{ \beta^{-}(x), \gamma^{+}(x) \right\} \right\} > (\lambda \wedge \mu)(x)$$

$$\geq \max \left\{ \min \left\{ \beta^{+}(x), \gamma^{-}(x) \right\}, \min \left\{ \beta^{-}(x), \gamma^{+}(x) \right\} \right\}$$

Then the R-union of A and B is an ECS in X.

1.4.41 Remark

If $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ are ECSs in X satisfying the following condition

$$\min \left\{ \max \left\{ \beta^{+}(x), \gamma^{-}(x) \right\}, \max \left\{ \beta^{-}(x), \gamma^{+}(x) \right\} \right\} = (\lambda \wedge \mu)(x)$$

$$> \max \left\{ \min \left\{ \beta^{+}(x), \gamma^{-}(x) \right\}, \min \left\{ \beta^{-}(x), \gamma^{+}(x) \right\} \right\}$$

for all $x \in X$. Then the R-union of A and B may not be an ECS in X. Next example is an evident of this.

1.4.42 Example

Let $X = \{a, b, c\}$ be a set. Let $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ be ECSs in X defined by the following table.

X	$\beta(x)$	$\lambda(x)$
a	[0.1, 0.8]	0.9
b	[0.3, 0.6]	0.6
c	[0.4, 0.5]	0.5

X	$\gamma(x)$	$\mu(x)$
a	[0.2, 0.7]	0.7
b	[0.1, 0.7]	0.8
c	[0.3, 0.8]	0.9

Then we know that $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ satisfy the following condition

$$\min \left\{ \max \left\{ \beta^{+}(x), \gamma^{-}(x) \right\}, \max \left\{ \beta^{-}(x), \gamma^{+}(x) \right\} \right\} = (\lambda \wedge \mu)(x)$$

$$> \max\left\{\min \ \left\{\beta^{+}(x), \ \gamma^{-}(x) \ \right\}, \ \min \ \left\{\beta^{-}(x), \ \gamma^{+}(x) \ \right\}\right\}$$

But $A \cup_R B = \langle \beta \cup \gamma, (\lambda \wedge \mu) \ \rangle$ is not an ECS in X because $(\lambda \wedge \mu)(c) = 0.5 \in (0.4, \ 0.8) = \left((\beta \cup \gamma)^{-}(c), \ (\beta \cup \gamma)^{+}(c)\right)$.

1.4.43 Theorem

Let $A = \langle \beta, \ \lambda \rangle$ and $B = \langle \gamma, \ \mu \rangle$ be ECSs in X such that

$$\min\left\{\max \ \left\{\beta^{+}(x), \ \gamma^{-}(x) \ \right\}, \ \max \ \left\{\beta^{-}(x), \ \gamma^{+}(x) \ \right\}\right\} \geq (\lambda \vee \mu)(x)$$

$$> \max\left\{\min \ \left\{\beta^{+}(x), \ \gamma^{-}(x) \ \right\}, \ \min \ \left\{\beta^{-}(x), \ \gamma^{+}(x) \ \right\}\right\}$$

for all $x \in X$. Then the R-intersection of A and B is an ECS in X.

1.4.44 Remark

If $A = \langle \beta, \ \lambda \rangle$ and $B = \langle \gamma, \ \mu \rangle$ are ECSs in X satisfying the following condition

$$\min\left\{\max \ \left\{\beta^{+}(x), \ \gamma^{-}(x) \ \right\}, \ \max \ \left\{\beta^{-}(x), \ \gamma^{+}(x) \ \right\}\right\} > (\lambda \vee \mu)(x)$$

$$= \max\left\{\min \ \left\{\beta^{+}(x), \ \gamma^{-}(x) \ \right\}, \ \min \ \left\{\beta^{-}(x), \ \gamma^{+}(x) \ \right\}\right\}$$

for all $x \in X$. Then the R-intersection of A and B may not be an ECS in X. Next example is an evident of this.

1.4.45 Example

Let $X = \{a, \ b, \ c\}$ be a set. Let $A = \langle \beta, \ \lambda \rangle$ and $B = \langle \gamma, \ \mu \rangle$ be ECSs in X defined by the following table.

X	$\beta(x)$	$\lambda(x)$
a	[0.2, 0.4]	0.1
b	[0.5, 0.8]	0.5
c	[0.6, 0.8]	0.4

X	$\gamma(x)$	$\mu(x)$
a	[0.3, 0.6]	0.3
b	[0.4, 0.7]	0.2
c	[0.7, 0.9]	0.7

Then we know that $A = \langle \beta, \lambda \rangle$ and $B = \langle \gamma, \mu \rangle$ satisfy the following condition

$\min \left\{ \max \ \left\{ \beta^{+}(x), \gamma^{-}(x) \right\}, \ \max \ \left\{ \beta^{-}(x), \gamma^{+}(x) \right\} \right\} > (\lambda \vee \mu)(x)$

$= \max \left\{ \min \ \left\{ \beta^{+}(x), \gamma^{-}(x) \right\}, \ \min \ \left\{ \beta^{-}(x), \gamma^{+}(x) \right\} \right\}$

But $A \cap_R B = \langle \beta \cap \gamma, (\lambda \vee \mu) \rangle$ is not an ECS in X because $(\lambda \vee \mu)(b) = 0.5 \in (0.4, 0.7) = \left((\beta \cap \gamma)^{-}(b), \ (\beta \cap \gamma)^{+}(b) \right)$.

1.5 General TOPSIS technique

We can carry out the TOPSIS technique as follows

Step 1: Form an evaluation matrix comprising of m alternatives and n criteria, with the intersection of each alternative and criteria given as u_{ij} , thus we have a matrix $(u_{ij})_{m \times n}$.

Step 2: Then the matrix $(u_{ij})_{m \times n}$ is normalised to form the matrix $R = (r_{ij})_{m \times n}$ with the following normalisation method

$$r_{ij} = \frac{u_{ij}}{\sqrt{\sum_{i=1}^{m} (u_{ij})^2}} \quad \forall j.$$

Step 3: Calculation of the weighted normalised decision matrix as $(V_{ij})_{m \times n} = (w_{ij} \times r_{ij}), 1 \le i \le m$. Here $w_{ij} = \frac{W_J}{\sum_{j=1}^{n} W_J}$ so that $\sum_{j=1}^{n} w_j = 1$ and W_j is the original weight given to each criteria.

Step 4: Determination of the positive and negative ideal solutions: Both the positive ideal solution V_j^+ and the negative ideal solution V_j^- are given as

$$\begin{aligned} V_j^+ &= \{(\max V_{ij},\ j \in J_+), (\min V_{ij},\ j \in J_-), 1 \le i \le m\} \\ V_j^- &= \{(\min V_{ij},\ j \in J_+), (\max V_{ij},\ j \in J_-), 1 \le i \le m\} \end{aligned}$$

where J_+ indicates the benefit criteria and J_- indicates the non - beneficial criteria.

Step 5: Calculations of the separation measures i.e. the distances d_i^+ and d_i^- from the positive ideal and negative ideal solution resp. as:

$$\begin{aligned} d_i^+ &= \sqrt{\sum_{j=i}^{n} (\ V_{ij} - V_j^+)^2} \\ d_i^- &= \sqrt{\sum_{j=i}^{n} (\ V_{ij} - V_j^-)^2} \end{aligned}$$

Step 6: Determination of the relative nearness of alternatives to the ideal solution:

$$S_i^+ = \frac{d_i^-}{d_i^- + d_i^+}$$

where $0 \le S_i^+ \le 1$. Alternatives with the greater magnitudes of nearness are preferred.

1.6 An overview of SIR method

Here we discuss the main theme of superiority and inferiority ranking method. In order to execute the SIR method we firstly, form the superiority and inferiority matrices. These matrices are obtained with the help of superiority and inferiority indexes of alternatives. The superiority and inferiority indexes are obtained through preference intensity of the given alternatives. After the formation of superiority matrix and inferiority matrix, we use standard MCDM aggregation procedures to aggregate superiority and inferiority indexes into two type of global preference indexes: the superiority flow (S-flow) and the inferiority flow (I-flow), which represent the global intensity of superiority and inferiority of each alternative. Finally, the alternative with the higher S-flow and the lower I-flow is considered to be best choice.

Chapter 2

Multicriteria decision making using a score function and SIR method

In this chapter, we will solve different multicriteria decision making problems having information in cubic set values. In the first section of this chapter we deal with the multicriteria decision making problem using a score function. In the later section we solve the multicriteria group decision making problem with the help of well known SIR method.

2.1 Multicriteria decision making based on cubic set

In the this section we have suggested the application of cubic set for decision making problems having multicriteria. Our proposed an accuracy function or the score function does not lead to the paradox of the difficult decision to the alternatives. We briefly introduce some aggregation operators for cubic sets. We suggest a score function, and then we provide two examples to justify that the suggested function is more suitable in the process of decision making. We have established a algorithm to recognize the best alternative. We make the use of cubic set weighted geometric aggregation (CSWGA) and cubic set weighted arithmetic aggregation (CSWAA) operators to aggregate cubic set information related to each alternative, and then give ranking to the alternatives and choose the appropriate one in view of the accuracy measures of the aggregated cubic set information corresponding to score function. We show the worth of the adopted method by presenting illustrative examples.

2.1.1 Score function

Let $B= ([b, c], d)$ be a cubic set value (CSV), a score function M of cubic set value is suggested by the formula given below

$$M(B) = \frac{b + c - 1 + d}{2} \tag{1}$$

where $M(B) \in [-1, +1]$.

2.1.2 Definition

Let $B_j (1 \leq j \leq n) \in \text{CS}(X)$, where $\text{CS}(X)$ is the collection of all cubic sets in X. The weighted arithematic average operator is defined by $\mathbf{F}_w(B_1, B_2, ..., B_n) =$

$$\sum_{j=1}^{n} w_j B_j = \left(\left[1 - \underset{j=1}{\overset{n}{\pi}} \left(1 - \beta_j^-(x)\right)^{w_j}, 1 - \underset{j=1}{\overset{n}{\pi}} \left(\left(1 - \beta_j^+(x)\right)^{w_j} \right) \right], \left[\underset{j=1}{\overset{n}{\pi}} \mu_j^{wj}(x) \right] \right) \quad (2)$$

where w_j is the weight of $B_j (1 \leq j \leq n)$, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Especially assume $w_j = \frac{1}{n}$ ($j = 1, 2, ..., n$) then, $\mathbf{F}_w$ is known as an arithematic average operator for CSs.

2.1.3 Definition

Let $B_j(1 \leq j \leq n) \in \mathrm{CS}(X)$. The weighted geometric average operator is defined by $\mathbf{G}_w(B_1, B_2, ..., B_n) =$

$$\underset{j=1}{\overset{n}{\pi}} B_j^{w_j} = \left(\left[\underset{j=1}{\overset{n}{\pi}} \beta_j^{-w_j}(x), \underset{j=1}{\overset{n}{\pi}} \beta_j^{+w_j}(x) \right], \left[1 - \underset{j=1}{\overset{n}{\pi}} \left(1 - \mu_j(x)\right)^{wj} \right] \right) \quad (3)$$

where w_j is the weight of $B_j(1 \leq j \leq n)$, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Especially assume $w_j = \frac{1}{n}$ ($j = 1, 2, ..., n$) then, $\mathbf{G}_w$ is known as geometric average operator for CSs.

The aggregation results $\mathbf{F}_w$ & $\mathbf{G}_w$ are still $\mathrm{CS}(X)$.

2.1.4 Example

If internal cubic set values for different alternatives are $B_1 = ([0.3,\ 0.5],\ 0.4)$ and $B_2 = ([0.5,\ 0.7],\ 0.6)$, the wanted option is selected in view of score function. After using equation (1), we have

$$\begin{aligned} M(B_1) &= \frac{0.3+0.5-1+0.4}{2} = 0.1 \\ M(B_1) &= \frac{0.5+0.7-1+0.6}{2} = 0.4 \end{aligned}$$

Obviously the alternative B_2 has prefrence over B_1.

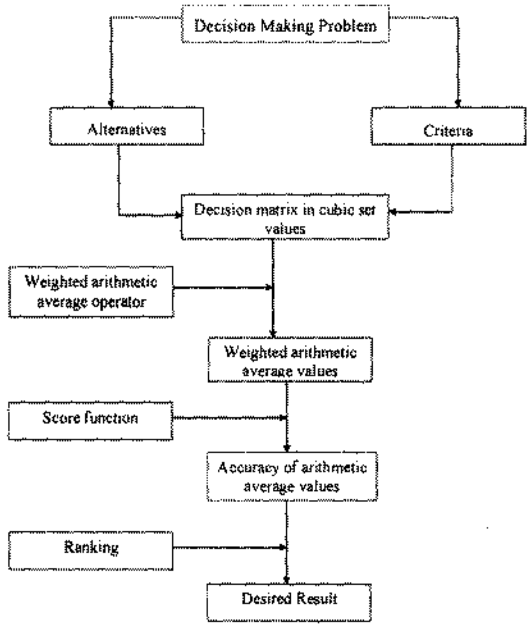
2.1.5 Example

If external cubic set values for two diffrent alternatives are $B_1 = ([0.3,\ 0.4],\ 0.5)$ and $B_2 = ([0.4,\ 0.5],\ 0.6)$, the desired alternative is chosen with the help of score function. By using equation (1) we get

$$\begin{aligned} M(B_1) &= \frac{0.3+0.4-1+0.5}{2} = 0.10 \\ M(B_2) &= \frac{0.4+0.5-1+0.6}{2} = 0.25 \end{aligned}$$

clearly the alternative B_2 has advantage over B_1.

2.1.6 Multicriteria cubic set decision making method based on the score function



Flow chart of the proposed method.

Here we are going to present a method for tackling of multicriteria cubic set decision making problems along with weights. Suppose that $B=\{B_1, B_2, ..., B_m\}$ is a collection of alternatives and also suppose that $C=\{C_1, C_2, ..., C_n\}$ is a set of criteria. Consider the criterion C_j $(1 \leq j \leq n)$, recommended by the decision maker has weight $w_j, w_j \in [0,1]$ and $\sum\limits_{j=1}^{n} w_j = 1$. In this situation, the characterstic of the alternative B_i is represented by a cubic set as

$$B_i = \left\{ \left\langle C_j, \left[B^-(C_j), B^+(C_j)\right], \left[\mu(C_j)\right] \right\rangle \mid C_j \in C \right\}.$$

The cubic set value which is the pair of IVFS and fuzzy number, i.e.

$$(B_i(C_j) = [b_{ij}, c_{ij}], \mu(C_j) = d_{ij}) \text{ is denoted by } \alpha_{ij} = ([b_{ij},\ c_{ij}],\ d_{ij})$$

Since $[b_{ij},\ c_{ij}] \subseteq [0,1]$ & $d_{ij} \in [0,1]$. Therefore a decision matrix of the form $D = (\alpha_{ij})$ can be formulated. The aggregating cubic set value α_i for $B_i\,(1 \leq i \leq m)$ is $\alpha_i = ([b_i,\ c_i], d_i) = \mathbf{F}_{iw}(\alpha_{i1}, \alpha_{i2,\ldots,} \alpha_{in})$ or $\alpha_i = ([b_i,\ c_i], d_i) = \mathbf{G}_{iw}(\alpha_{i1}, \alpha_{i2,\ldots,} \alpha_{in})$ which is obtained by using equation (2) or Eq. (3), in accordance with every row of the decision matrix. We will use Eq. (3) to calculate the accuracy $M(\alpha_i)$ of aggregating cubic set value $\alpha_i\,(1 \leq i \leq m)$ to rank the alternatives $B_i\,(1 \leq i \leq m)$ and then to choose the suitable one(s). Simply, the of process decision making for the suggested technique can be described by the following steps.

Step(a). Obtain the CSWAA values by applying Eq. (2) if we prefer the influence of group, on th other hand get the CSWGA values with the help of Eq. (3).

Step(b). Obtain the accuracy $M(\alpha_i)$ of cubic set value $\alpha_i\,(1 \leq i \leq m)$ by the application of Eq. (1).

Step (c). Rank the alternatives $B_i\,(1 \leq i \leq m)$ and choose the best one(s) in comparison with $M(\alpha_i)\,(1 \leq i \leq m)$.

2.1.7 Illustrative Examples

This section is consisting of two examples. First example adapted from [18] to solve a decision making problem along with multicriteria to potray the suggested cubic decision making method in the spectrum of reallity, as well as the validity of the effectiveness of the suggested algorithm.

Here is a set of people provided with four options for the investation of their money: (1) B_1 is a company of car; (2) B_2 is a company of food; (3) B_3 is a company of computer; (4) B_4 is a company of arms. The investor must have to decide by keeping in mind these three criteria: (1) C_1 is the analysis of risk; (2) C_2 is the analysis of growth; (3) C_3 is the analysis of enviromental impact. Now decider will evaluate the four possible alternatives under the above mentioned criteria, as provided in the following matrices. First we consider the matrix D_1 consisting of internal cubic set values.

$$D_1 = \begin{bmatrix} ([0.1,0.3],0.2) & ([0.2,0.4],0.3) & ([0.3,0.6],0.4) \\ ([0.5,0.7],0.5) & ([0.3,0.4],0.3) & ([0.7,0.8],0.7) \\ ([0.3,0.5],0.4) & ([0.7,0.9],0.8) & ([0.6,0.8],0.7) \\ ([0.4,0.6],0.4) & ([0.1,0.2],0.2) & ([0.6,0.8],0.7) \end{bmatrix}$$

Now assume that the weights of C_1, C_2 & C_3 are $0.35, 0.25$ and 0.40 resp. Then we use the following algorithm.

Step 1. Eq. (2) provides us the CSWAA value α_i for $B_i\,(i = 1, 2, ..., 4)$.

$\alpha_1 = ([0.2097, 0.4615], 0.2921)$

$\alpha_2 = ([0.5566, 0.6967], 0.5035)$

$\alpha_3 = ([0.5472, 0.7682], 0.5950)$

$\alpha_4 = ([0.3827, 0.5243], 0.3678)$

Step 2. By applying Eq. (1) we can compute $M(\alpha_i)$ where $i = 1, 2, 3, 4$ as $M(\alpha_1) = 0.4817$, $M(\alpha_2) = 0.3784$, $M(\alpha_3) = 0.4552$, $M(\alpha_4) = 0.1374$.

Step 3. Awarding ranks to all alternatives in view of the accuracy degree of $M(\alpha_i)$ $(i = 1, 2, 3, 4) : B_1 > B_3 > B_2 > B_4$, and thus the best alternative is B_1.

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Now we consider the matrix D_2 consisting of external cubic set values.

$$D_2 = \begin{bmatrix} ([0.4, 0.5], 0.3) & ([0.4, 0.6], 0.2) & ([0.1, 0.3], 0.5) \\ ([0.6, 0.7], 0.2) & ([0.5, 0.7], 0.2) & ([0.4, 0.7], 0.1) \\ ([0.3, 0.6], 0.1) & ([0.5, 0.6], 0.4) & ([0.5, 0.6], 0.3) \\ ([0.7, 0.8], 0.1) & ([0.6, 0.7], 0.3) & ([0.3, 0.4], 0.2) \end{bmatrix}$$

Consider the same weights for C_1, C_2 & C_3 as mentioned above and use the following algorithm.

Step 1. Applying Eq. (2) we get the CSWAA value α_i for $B_i\,(i = 1, 2, ..., 4)$.

$\alpha_1 = ([0.2944, 0.4590], 0.3325)$

$\alpha_2 = ([0.5026, 0.7000], 0.1516)$

$\alpha_3 = ([0.4375, 0.6000], 0.2195)$

$\alpha_4 = ([0.5476, 0.6565], 0.1737)$

Step 2. Using Eq. (1) we can compute $M(\alpha_i)\,(i = 1, 2, 3, 4.)$ as $M(\alpha_1) = 0.0430$, $M(\alpha_2) = 0.1771$, $M(\alpha_3) = 0.1285$, $M(\alpha_4) = 0.1889$.

Step 3. By ranking all alternatives in view of the accuracy degree of $M(\alpha_i)$ $(i = 1, 2, 3, 4) : B_4 > B_2 > B_3 > B_1$, and thus the alternative B_4 is the best one.

Finally we consider the matrix D_3 consisting of cubic set values which are neither internal cubic set values nor external cubic set values.

$$D_3 = \begin{bmatrix} ([0.3, 0.7], 0.1) & ([0.3, 0.7], 0.2) & ([0.3, 0.7], 0.4) \\ ([0.3, 0.7], 0.4) & ([0.3, 0.7], 0.5) & ([0.4, 0.7], 0.1) \\ ([0.3, 0.7], 0.7) & ([0.3, 0.7], 0.8) & ([0.3, 0.7], 0.6) \\ ([0.2, 0.5], 1) & ([0.2, 0.5], 0.3) & ([0.2, 0.5], 0.6) \end{bmatrix}$$

Again using the similar procedure as stated above with similar weights we have

$M(\alpha_1) = 0.1036$, $M(\alpha_2) = 0.1215$, $M(\alpha_3) = 0.3403$, $M(\alpha_4) = 0.3017$ so $B_3 > B_4 > B_2 > B_1$ and thus the alternative B_3 is the most wishful one.

Now we present another example in this section in which we want to investigate the suitability of an S-box to image encryption applications. We have been provided with nine different alternatives of S-boxes: (1) B_1 is Plain Image; (2) B_2 is Advanced Encryption Standard; (3) B_3 is Affine Power Affine; (4) B_4 is Gray; (5) B_5 is S_8; (6) B_6 is Liu; (7) B_7 is Prime; (8) B_8 is Xyi; (9) B_9 is Skipjack. We have to decide in accordance with the following criteria: (1) C_1 is the entropy analysis; (2) C_2 is the contrast analysis; (3) C_3 is the average correlation analysis; (4) C_4 is the energy analysis; (5) C_5 is the homogeneity analysis; (6) C_6 is the mean of absolute deviation analysis. The nine possible alternatives are to be sorted out using the cubic set information by the decider from the given criteria as presented in the form of following matrix.

$$D = \begin{bmatrix} ([0.1,0.2],0.3) & ([0.1,0.3],0.2) & ([0.3,0.4],0.1) & ([0.4,0.5],0.6) & ([0.3,0.6],0.5) & ([0.5,0.6],0.4) \\ ([0.5,0.7],0.4) & ([0.3,0.4],0.2) & ([0.7,0.8],0.6) & ([0.4,0.5],0.3) & ([0.6,0.7],0.2) & ([0.4,0.7],0.1) \\ ([0.3,0.5],0.4) & ([0.7,0.9],0.8) & ([0.6,0.8],0.7) & ([0.5,0.6],0.3) & ([0.7,0.8],0.1) & ([0.1,0.3],0.5) \\ ([0.4,0.6],0.4) & ([0.1,0.2],0.2) & ([0.3,0.6],0.4) & ([0.3,0.4],0.1) & ([0.3,0.4],0.2) & ([0.6,0.7],0.3) \\ ([0.1,0.3],0.3) & ([0.5,0.6],0.7) & ([0.2,0.4],0.3) & ([0.6,0.8],0.7) & ([0.1,0.2],0.2) & ([0.3,0.5],0.1) \\ ([0.5,0.6],0.2) & ([0.4,0.7],0.6) & ([0.5,0.7],0.9) & ([0.8,0.9],0.8) & ([0.4,0.6],0.3) & ([0.7,0.8],0.2) \\ ([0.7,0.8],0.9) & ([0.4,0.7],0.5) & ([0.4,0.6],0.2) & ([0.7,0.9],0.2) & ([0.8,0.9],0.7) & ([0.2,0.5],0.4) \\ ([0.8,0.9],0.7) & ([0.7,0.9],0.8) & ([0.1,0.2],0.1) & ([0.3,0.2],0.1) & ([0.5,0.6],0.1) & ([0.4,0.8],0.6) \\ ([0.8,0.9],0.6) & ([0.6,0.9],0.7) & ([0.3,0.5],0.6) & ([0.4,0.7],0.3) & ([0.4,0.6],0.5) & ([0.1,0.2],0.3) \end{bmatrix}$$

Now we assume the same weight for each of C_1, C_2 ,..., C_6, that is 0.167 and use the following algorithm.

Step 1. We calculate the CSWAA value α_i for $B_i\,(i = 1, 2, ..., 9)$ with the aid of

Eq. (2).

$\alpha_1 = ([0.3035, 0.4592], 0.2922)$

$\alpha_2 = ([0.5096, 0.6646], 0.2501)$

$\alpha_3 = ([0.5330, 0.7200], 0.3797)$

$\alpha_4 = ([0.3575, 0.5170], 0.2334)$

$\alpha_5 = ([0.3350, 0.5194], 0.3025)$

$\alpha_6 = ([0.5884, 0.7499], 0.4088)$

$\alpha_7 = ([0.5912, 0.7845], 0.4068)$

$\alpha_8 = ([0.5330, 0.7242], 0.2567)$

$\alpha_9 = ([0.4942, 0.7272], 0.4670)$

Step 2. By applying Eq. (3) we can compute $M(\alpha_i)$ where $i = 1, 2, ..., 9$ as $M(\alpha_1) = 0.0275$, $M(\alpha_2) = 0.2122$, $M(\alpha_3) = 0.3164$, $M(\alpha_4) = 0.0540$, $M(\alpha_5) = 0.0785$, $M(\alpha_6) = 0.3736$, $M(\alpha_7) = 0.8913$, $M(\alpha_8) = 0.7570$, $M(\alpha_9) = 0.3342$.

Step 3. After awarding ranks to all alternatives in view of the accuracy degree of $M(\alpha_i)$ $(i = 1, 2\ , ..., 9.)$: $B_7 > B_8 > B_6 > B_9 > B_3 > B_2 > B_5 > B_4 > B_1$ and thus the alternative B_7 is the most desired one.

2.2 A multicriteria group decision making approach with cubic SIR method

In this section we propose a multicriteria group decision making method based on cubic sets. Cubic sets basically consist of interval valued fuzzy set and a fuzzy set.

In group decision making process several experts are involved to get a suitable result. The weigths of the experts are usually predetermined, however some experts assign same weight to each alternative. We suggest cubic set Superiority and Inferiority Ranking (CS-SIR) approach for application in group decision making problems. In view of cubic set we define operators to integrate the individual piont of views into group piont of views. We get Superiority (Inferiority) for each alternative by setting a threshold function. We order alternatives according to our suggested CS-SIR ranking rules and shape them into a final decision. We will illustrate this procedure with an example related to mamagement field. A detail description of cubic set SIR method has been provided. An Illustration of the multicriteria group decision making (MCGDM) problem with reference to the field of management has been included in this section.

2.2.1 Algorithm for cubic set SIR approach

In group multi-criteria decision making problems we have multiple individuals who assess alternatives based on mutiple criteria. Here we present different decision factors for group MCDM problem and their representative symbols as.

(1) Alternative sets by $B_i(1 \leq i \leq n)$.

(2) Decision makers set by $e_k(1 \leq k \leq l)$.

(3) Criterion sets as $C_j(1 \leq j \leq m)$.

(4) Decision maker weights by $\omega_k(1 \leq k \leq l)$ $\omega_k = ((\beta_k^-(x), \beta_k^+(x), \mu_k(x))$.

(5) Criterion weights by $w_j(1 \leq j \leq m)$ $w_j^{(k)} = (\beta_j^{-(k)}(x), \beta_j^{+(k)}(x), \mu_j^{(k)}(x))$.

(6) Individual decision matrix as $d_{ij}^{(k)} = (\beta_{ij}^{-(k)}(x),\, \beta_{ij}^{+(k)}(x),\, \mu_{ij}^{(k)}(x))$.

(7) Group-integrated decision matrix is represented as $\bar{\bar{d}}_{ij} = (\beta_{ij}^{-}(x), \beta_{ij}^{+}(x), \bar{\mu}_{ij}(x))$ and weight as $w_j = (\beta_j^{-}(x), \beta_j^{+}(x), \bar{\mu}_j(x))$.

Score function

In decision making a score function measures the accuracy of possible alternatives. It is the measure of the calibration of a set of possible options. For a cubic set $B = ([b, c], d)$ we propose a score function which is defined as

$$M(B) = \frac{b+c-1+d}{2}, \ where\ M(B) \in [-\mathbf{1}, +\mathbf{1}]. \tag{1}$$

Some of the properties of this score function are as follows:

$$\begin{aligned} \text{(i) } M(B) &= 0 \text{ implies } b+c+d=1 \\ \text{(ii) } M(B) &= 1 \text{ implies } b+c+d=3 \\ \text{(iii) } M(B) &= -1 \text{ implies } b+c+d=-1 \\ \text{(iv) if } M(B_1) &< M(B_2) \text{ then } B_1 < B_2 \end{aligned} \tag{2}$$

The cubic set SIR method comprises of the following steps.

Step1: Determination of the individual measure degree ξ_k.

The weights which are given to the decision makers are assigned in fuzzy literature which are defined using cubic sets. Table 1 gives an example of the term measure on " Importance" and "Quality" on different levels.

Level	"Importance" Measure	"Quality" Measure	Cubic Set value (CSV)
1	Extremely Important (EI)	Extremely Positive (EP)	$([1,1],0)$
2	Great Important (GI)	Absolutely Positive (AP)	$([0.90,1],0.2)$
3	Very Important (VI)	Very Very Positive (VVP)	$([0.80,0.90],0.2)$
4	Important (I)	Very Positive (VP)	$([0.70,0.80],0.1)$
5	Medium (M)	Positive (P)	$([0.60,0.90],0.3)$
6	Less Important (LI)	Medium (M)	$([0.50,0.70],0.4)$
7	Unimportant (UI)	Negative (N)	$([0.40,0.60],0.5)$
8	Not Important (NI)	Very Negative (VN)	$([0.1,0.3],0.9)$
9	Unconsidered (UC)	Extremely Negative (EN)	$([0,0],1)$

Table 1. "Imoprtance" and "Quality" ranked in cubic set values.

Let $\omega_k = (\beta_k^-, \beta_k^+, \mu_k)$ be cubic set values confirmed based on table 1.

The normalized Euclidean Distance is given by

$$D_k(\omega_k, \omega^+) = \sqrt{\frac{1}{2}((\beta_k^- - \beta^-)^2 + (\beta_k^+ - \beta^+)^2 + (\mu_k - \mu^+)^2)} \tag{3}$$

where $\omega^+ = (\beta^-, \beta^+, \mu) = (1,1,0)$.

Now similarity measure is obtained as

$$\xi_k(\omega_k, \omega^+) = \frac{D_k(\omega_k, \omega^+)}{D_k(\omega_k, \omega^-)}$$

where $0 \leq \xi_k \leq \infty$ and $\bar{\omega}^+ = \omega^- = (0,0,1)$.

To inject this function into the interval $[1,0]$ we change it into the following function

$$\xi_k(\omega_k,\omega^+) = 1-\frac{D_k(\omega_k,\omega^+)}{D_k(\omega_k,\omega^-)+D_k(\omega_k,\omega^+)} \quad (4)$$
$$= \frac{D_k(\omega_k,\omega^-)}{D_k(\omega_k,\omega^-)+D_k(\omega_k,\omega^+)}$$

Finally, the vector of real numbers $\xi_k = (\xi_1, \xi_2, \xi_3, ..., \xi_l)^T$ are obtained as individual measure degrees.

Step 2: Integration to the group

We define following two operators for cubic sets to integrate individual opinions into group opinions.

(i) Integration of individual decision matrix

$$\bar{d}_{ij} = ([1-\underset{k=1}{\overset{l}{\pi}}(1-\beta_{ij}^{-(k)})^{\xi_k}, 1-\underset{k=1}{\overset{l}{\pi}}(1-\beta_{ij}^{+(k)})^{\xi_k}], \underset{k=1}{\overset{l}{\pi}}(\mu_{ij}^{(k)})^{\xi_k}) \quad (5)$$
$$= ([\bar{\beta}_{ij}^-, \bar{\beta}_{ij}^+], \bar{\mu}_{ij})$$

(ii) Integration of individual criterion weights

$$\bar{w}_j = ([\underset{k=1}{\overset{l}{\pi}}(\beta_j^{-(k)})^{\xi_k}, \underset{k=1}{\overset{l}{\pi}}\beta_j^{+(k)})^{\xi_k}], 1-\underset{k=1}{\overset{l}{\pi}}(1-\mu_j^{(k)})^{\xi_k}) \quad (6)$$
$$= ([\bar{\beta}_j^-, \bar{\beta}_j^+], \bar{\mu}_j)$$

Thus we obtain group-integrated decision matrix $\bar{d}_{ij} = ([\bar{\beta}_{ij}^-, \bar{\beta}_{ij}^+], \bar{\mu}_{ij})$ and the criterion weights $\bar{w}_j = ([\bar{\beta}_j^-, \bar{\beta}_j^+], \bar{\mu}_j)$.

Step 3: Formation of cubic set Superiority/ Inferiority matrix

(i) Confirmation of the performance function

We define $h(B_i)$ as the performance function. For cubic set we calculate $h(B_i)$ as

$$D_j(\bar{d}_{ij}, d^+) = \frac{1}{2}(|\bar{\beta}_{ij}^- - \beta^-| + |\bar{\beta}_{ij}^+ - \beta^+| + |\bar{\mu}_{ij} - \mu^+|) \tag{7}$$

$$h(B_i) = \frac{D_j(\bar{d}_{ij}, d^-)}{D_j(\bar{d}_{ij}, d^-) + D_j(\bar{d}_{ij}, d^+)} \tag{8}$$

where $0 \leq h(B_i) \leq 1$.

(ii) Confirmation of the preference intensity $Q_k(B_i, B_t)$

We define $Q_k(B_i, B_t)$ as the preference intensity of alternative B_i over B_t, with respect to the kth criterion

$$Q_k(B_i, B_t) = \Phi_k(h_k(B_i) - h_k(B_t)) = \Phi_k(d) \tag{9}$$

where $1 \leq i, t \leq n$, $i \neq t$ and $1 \leq k \leq l$.Here $\Phi_k(d)$ is a threshold function whose range is $[0, 1]$. It may be one of generalized function or defined by the decision makers themselves.

(iii) Formation of Superiority and Inferiority matrices

For alternative B_i, we get cubic set Superiority/ Inferiority index and matrices as:

Superiority index (S-index):

$$S_k(B_i) = \sum_{t=1}^{n} Q_k(B_i, B_t) = \sum_{t=1}^{n} \Phi_k(h_k(B_i) - h_k(B_t)); S - matrix : S = (S_k(B_i))_{n \times m} \tag{10}$$

Inferiority index (I-index):

$$I_k(B_i) = \sum_{t=1}^{n} Q_k(B_t, B_i) = \sum_{t=1}^{n} \Phi_k(h_k(B_t) - h_k(B_i)); I - matrix : I = (I_k(B_i))_{n \times m} \tag{11}$$

Step 4: To get Superiority flow and inferiority flow: For cubic set we get the weighted Superiority flow and Inferiority flow as

$$\text{S-flow} \quad \Phi(B_i) = ([1 - \underset{j=1}{\overset{m}{\pi}}(1 - \bar{\beta}_j^-)^{S_j(B_i)}, 1 - \underset{j=1}{\overset{m}{\pi}}(1 - \bar{\beta}_j^+)^{S_j(B_i)}], \underset{j=1}{\overset{m}{\pi}}(\bar{\mu}_j^{S_j(B_i)})) \quad (12)$$

$$= ([\beta^{->}(B_i), \beta^{+>}(B_i)], \mu^{>}(B_i))$$

$$\text{I-flow} \quad \Psi(B_i) = ([1 - \underset{j=1}{\overset{m}{\pi}}(1 - \bar{B}_j^-)^{I_j(B_i)}, 1 - \underset{j=1}{\overset{m}{\pi}}(1 - \bar{B}_j^+)^{I_j(B_i)}], \underset{j=1}{\overset{m}{\pi}}(\bar{\mu}_j^{I_j(B_i)})) \quad (13)$$

$$= ([\beta^{-<}(B_i), \beta^{+<}(B_i)], \mu^{<}(B_i))$$

Obviously when we have higher S-flow $\Phi(B_i)$ and the lower I-flow $\Psi(B_i)$ then alternative B_i is better.Thus we obtain S-flow and I-flow of alternative B_i as

$B_i(\Phi(B_i), \Psi(B_i))$.

Step 5: The cubic set SIR rules:

(i) Confirmation of Superiority ranking and Inferiority ranking

The superiority ranking called (S-ranking) is obtained by descending order of $\Phi(B_i)$ as B_i $Q_>$ B_k iff $\Phi(B_i) > \Phi(B_k)$ and B_i $I_>$ B_k iff $\Phi(B_i) = \Phi(B_k)$.

In a similar manner the Inferiority ranking called (I-ranking) is obtained by using the ascending order of $\Psi(B_i)$ as $\Psi(B_i)$ as B_i $Q_<$ B_k iff $\Psi(B_i) > \Psi(B_k)$ and B_i $I_<$ B_k iff $\Psi(B_i) = \Psi(B_k)$.

(ii) Confirm the cubic set (CS) SIR

We combine both S-ranking and I-ranking into a partial ranking structure $R^* = \{Q, I, R\} = R^*_> \cap R^*_<$ of two alternatives B_i and B_k by applying the intersection principles.

(a) The Prefrence relation Q according to rule:

B_i Q B_k iff (B_i $Q_>$ B_k and B_i $Q_<$ B_k) or (B_i $Q_>$ B_k and B_i $I_<$ B_k) or (B_i $I_>$ B_k and B_i $Q_<$ B_k)

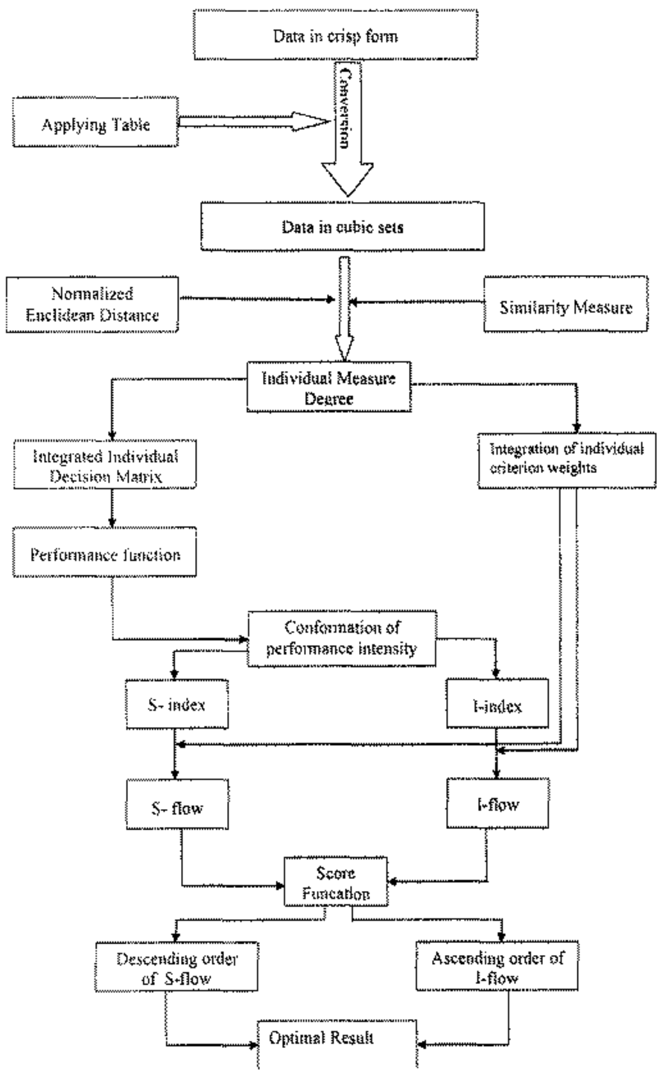
(b) The Indifference relation I according to rule:

B_i I B_k iff (B_i $I_>$ B_k and B_i $I_<$ B_k)

(c) The incomparability relation by rule: B_i R B_k iff (B_i $Q_>$ B_k and B_k $Q_<$ B_i) or (B_k $Q_>$ B_i and B_i $Q_<$ B_k).

Step 6: Completion of ranking map and making decision:

After obtaining every partial ranking structure of alternatives, to make the decision we complete the map of CS-SIR. We present the above algorithm by following flow chart.



Here we consider an example to illustrate the method.

2.2.2 Example

In Pakistan there are five different mobile network operators namely Mobilink, Telenor, Ufone, Zong and Warid which provide their services to the customers. If a government telecommunication company wants to make a business partnership with one of these mobile network operators, it will have to decide on the basis of many aspects. In simple words we can call it a multicriteria decision making (MCDM) problem in which decision makers evaluate each alternative from the alternatives; (1): (B_1) Mobilink; (2): (B_2) Telenor; (3): (B_3) Ufone; (4): (B_4) Zong; (5): (B_5) Warid. We consider that these alternatives are to be evaluated on the basis of following criterias; (C_1) Internet facility; (C_2) Call rates; (C_3) Coverage area; (C_4) Service performance.

Three telecommunication experts ($e_k; 1 \leq k \leq 3$) evaluate the alternative networks with the help of cubic set information. In the following, we apply the proposed cubic set SIR method to deal with this uncertainty group MCDM problem.

Step (a): The importance of the telecommuincation experts is described in the form of cubic set in Table 1. In the Table 2 we provide the weights assigned to the experts on the term importance in a cubic set form. Then individual measure degree is calculated with the help of Eq. 3 and Eq. 4 which is $\xi_k = (1.0000, 0.9353, 0.8750)$.

Step (b): Process of group integeration:

Individual decision matrix of experts has shown in Table 3 while Table 4 indicates the weights awarded to every criterion, these tables have cubic set values.

Expert	Importance Measure	CSV
e_1	Exteremly Important	$([1,1],0)$
e_2	Very Important	$([0.9,1]),0.1)$
e_3	Important	$([0.8,1]),0.1)$

Table 2. Theterm "Importance" weighted for experts.

Alternative networks	Expert	C_1	C_2	C_3	C_4
B_1	e_1	VVP	VP	VP	VVP
	e_2	VP	VVP	P	VVP
	e_3	VVP	VP	P	VP
B_2	e_1	VP	VVP	VVP	M
	e_2	M	VP	VPP	M
	e_3	P	P	VP	P
B_3	e_1	VP	M	VP	VVP
	e_2	M	P	P	VP
	e_3	VP	M	P	VP
B_4	e_1	VVP	P	VP	M
	e_2	VP	M	M	P
	e_3	VP	M	P	P
B_5	e_1	P	M	M	VP
	e_2	M	N	MP	VP
	e_3	M	M	P	P

Table 3. Individual decision matrix on term "Quality".

Expert	w_1	w_2	w_3	w_4
e_1	GI	VI	M	GI
e_2	GI	M	VI	I
e_3	VI	I	I	M

Table 4. Weights of criterion on term "Importance".

By using Table 1 and Table 3 we obtain the group-integrated decision matrix with the help of Eq. 5

$$\bar{d}_{ij} = \begin{bmatrix} ([0.9841, 0.9970], 0.0057) & ([0.9768, 0.9943], 0.0030) & ([0.9429, 0.9969], 0.0113) & ([0.9845, 0.9972], 0.0059) \\ ([0.9296, 0.9914], 0.0148) & ([0.9707, 0.9970], 0.0081) & ([0.9774, 0.9946], 0.0031) & ([0.8827, 0.9870], 0.0592) \\ ([0.9296, 0.9841], 0.0057) & ([0.8843, 0.9879], 0.0582) & ([0.9429, 0.9969], 0.0148) & ([0.9774, 0.9946], 0.0031) \\ ([0.9774, 0.9946], 0.0031) & ([0.8859, 0.9887], 0.0571) & ([0.9296, 0.9914], 0.0148) & ([0.9048, 0.9954], 0.0452) \\ ([0.8859, 0.9887], 0.0571) & ([0.8309, 0.9556], 0.0938) & ([0.8827, 0.9870], 0.0592) & ([0.9564, 0.9985], 0.0040) \end{bmatrix}$$

Using Table 1 and Table 4 we get the group-intergrated weights with the help of Eq. 6.

$\bar{w}_j = ([0.6709, 0.9119], 0.4659), ([0.3631, 0.6709], 0.4774),$

$([0.3564, 0.6709], 0.5181), ([0.4123, 0.7402], 0.5306))$

Step (c) Formation of cubic set S-matrix and I-matrix:

(i) With the help of Eq. 7 and Eq. 8 we obtain the performance function $h_k(B_i)$:

$$h_k(B_i) = \begin{bmatrix} \text{Alternatives} & C_1 & C_2 & C_3 & C_4 \\ B_1 & (0.9918) & (0.9893) & (0.9761) & (0.9919) \\ B_2 & (0.9687) & (0.9865) & (0.9890) & (0.9368) \\ B_3 & (0.9693) & (0.9380) & (0.9750) & (0.9896) \\ B_4 & (0.9896) & (0.9391) & (0.9687) & (0.9517) \\ B_5 & (0.9696) & (0.8975) & (0.9368) & (0.9836) \end{bmatrix}$$

(ii) We fix the threshold criterion function as

$$\Phi_k(d) = \begin{cases} 0.01 \text{ if d } > 0 \\ 0.00 \text{ if d } \leq 0 \end{cases}$$

Step (d): Calculation of S-flow and I-flow

To calculate S-flow we use Eq. 12 and to calculate I-flow we use Eq. 13. See the following table 5.

Alternatives	S-flow $\Phi(B_i)$	$M(\Phi(B_i))$	I-flow $\Psi(B_i)$	$M(\Psi(B_i))$
B_1	$([0.0924, 0.2046], 0.9001)$	0.0986	$([0.0097, 0.0243], 0.9872)$	0.0106
B_2	$([0.0307, 0.0748], 0.9527)$	0.0291	$([0.0678, 0.1497], 0.9387)$	0.0781
B_3	$([0.0396, 0.0934], 0.9539)$	0.0435	$([0.0592, 0.1323], 0.9375)$	0.0645
B_4	$([0.0508, 0.1128], 0.9506)$	0.0571	$([0.0481, 0.1133], 0.9407)$	0.0506
B_5	$([0.0323, 0.0728], 0.9724)$	0.0388	$([0.0662, 0.1516], 0.9196)$	0.0687

Table 5. S-flow and I-flow of alternatives

Step (e). The CS-SIR

Keeping in view the data in Table 5, we use Eq. 2 and the CS-SIR rules to rank alternative partners. We obtain the descending order of S-flow as:

$$\Phi(B_1) > \Phi(B_4) > \Phi(B_3) > \Phi(B_5) > \Phi(B_2)$$

Therefore, we can obtain superiority ranking as: $R^*_> : \{B_1\} \longrightarrow \{B_4\} \longrightarrow \{B_3\} \longrightarrow \{B_5\} \longrightarrow \{B_2\}$. We obtain the ascending order of I-flow as:

$$\Psi(B_1) < \Psi(B_4) < \Psi(B_3) < \Psi(B_5) < \Psi(B_2)$$

Therefore, we can obtain inferiority ranking as: $R^*_< : \{B_1\} \longrightarrow \{B_4\} \longrightarrow \{B_3\} \longrightarrow \{B_5\} \longrightarrow \{B_2\}$.

Step (f) Completion of ranking and decision making:

Finally we combine the S-ranking and I-ranking into ranking pattern $R^{*} = \{Q, I, R\} = R^{*}_{\gtrless} \cap R^{*}_{\lessgtr}$, we map the complete CS-SIR from superior to inferior as:

$$\{B_1\} \longrightarrow \{B_4\} \longrightarrow \{B_3\} \longrightarrow \{B_5\} \longrightarrow \{B_2\}.$$

Therefore the alternative A_1 i.e. Mobilink should be choosen for partnership.

Chapter 3

Multicriteria decision making using TOPSIS method

In this chapter, we deal with the multicriteria decision making problems using a new approach, named as TOPSIS. The first section is comprising of a procedure to solve the multicriteria decision making problem through TOPSIS approach while using score function. We have observed that the aid of score function makes the use of TOPSIS technique very simple. The second section relates to the solution of the multicriteria group decision making problem without any aid of score function.

3.1 An approach to decision making based on cubic set TOPSIS method

This section suggests the technique of TOPSIS (technique for order preference by similarity to ideal solution) to deal with the multicriteria decision making problems in which all the available preference information provided by the decision makers is expressed as cubic set decision matrices in which all the elements are charaterized by cubic set values. we convert the cubic set decision matrix into a score matrix with the aid of proposed score function. The proposed score function makes the things much simpler for the application of TOPSIS technique. From score matrix we calculate the separation measures of each alternative from the positive ideal solutions (PIS) and negative ideal solutions (NIS).We propose formulae to calculate the separation measure of alternatives to find the relative closeness coefficients. In accordance with the values of closeness coefficients, the ranking of the alternatives can be done to get the most optimal one(s) during process of decision making. At the end, an illustrative example has been provided to show the application and effectiveness of the suggested decision making approach.

3.2 Cubic set TOPSIS method based on score function

TOPSIS has wide application in multicriteria decision making problems. In the current section to deal with multicriteria decision making problem, in which the whole preference information given by deciders is presented as cubic set decision matrices (where each element is characterized by cubic set value) and the weight of each criterion is known, we develope a TOPSIS technique. Here we propose an accuracy or score function to determine the separation measure of every alternative from the PIS and NIS. Later these separation measures will help us to calculate the closeness coefficients.

Score function:

In decision making a score function measures the accuracy of possible alternatives. It is the measure of the calibration of a set of possible options. For a cubic set $B = ([b,c], d)$ we propose a score function which is defined as

$$M(B) = \frac{b+c-1+d}{2}, \ \textit{where } M(B) \in [-\mathbf{1}, +\mathbf{1}] \tag{1}$$

Some of the properties of this score function are as follows:

$$\begin{aligned} \text{(i) } M(B) &= 0 \text{ implies } b+c+d=1 \\ \text{(ii) } M(B) &= 1 \text{ implies } b+c+d=3 \\ \text{(iii) } M(B) &= -1 \text{ implies } b+c+d=-1 \\ \text{(iv) if } M(B_1) &< M(B_2) \text{ then } B_1 < B_2 \end{aligned}$$

Algorithm:

Let us consider that there exists a set of alternatives $B = \{B_1, B_2, B_3, \ldots, B_m\}$ in a multi-criteria decision making problem. Assessment of each alternative is based on 'n' criteria, which are represented as $C = \{C_1, C_2, C_3, \ldots, C_n\}$. According to a criterion C_j , the characteristic of an alternative B_i can be represented by a cubic set value as $u_{ij} = ([b_{ij}, c_{ij}], d_{ij})$,where $1 \leq i \leq m$ and $1 \leq j \leq n$. This cubic set value represents the membership and non-membership degree of each alternative with respect to the provided criteria. The decision matrix based on cubic set values is defined as

$$D_{m\times n}(u_{ij}) = \begin{bmatrix} ([b_{11}, c_{11}], d_{11}) & ([b_{12}, c_{12}], d_{12}) & \cdots & ([b_{1n}, c_{1n}], d_{1n}) \\ ([b_{21}, c_{11}], d_{21}) & ([b_{22}, c_{22}], d_{22}) & \cdots & ([b_{2n}, c_{2n}], d_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ ([b_{m1}, c_{m1}], d_{m1}) & ([b_{m2}, c_{m2}], d_{m2}) & \cdots & ([b_{mn}, c_{mn}], d_{mn}) \end{bmatrix}$$

With the help of score function we convert above given cubic set decision matrix $D_{m\times n}(u_{ij})$ into the following score marix:

$$S_{m\times n}(t_{ij}(u_{ij})) = \begin{bmatrix} t_{11}(u_{11}) & t_{12}(u_{12}) & \cdots & t_{1n}(u_{1n}) \\ t_{21}(u_{21}) & t_{22}(u_{22}) & \cdots & t_{2n}(u_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1}(u_{m1}) & t_{m2}(u_{m2}) & \cdots & t_{mn}(u_{mn}) \end{bmatrix}$$

Now consider the weight of the criterion C_j $(1 \leq j \leq n)$ given by the decision makers is w_j, where $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.The positive ideal solution for the alternatives expressed as

$$B^{+} = \{([1,1],0) \mid C_j \in C\}$$

where $1 \leq j \leq n$.

The negative ideal solution for the alternatives is expressed as

$$B^{-} = \{([0,0],1) \mid C_j \in C\}$$

where $1 \leq j \leq n$.

Separation measures of each alternative based on score function from the PIS and NIS is given by the following formulae:

$$d_i^{+}\left(B^{+},\ B_i\right) = \sqrt{\sum_{j=1}^{n} \left[w_j\left(1 - t_{ij}\left(u_{ij}\right)\right)\right]^2} \qquad (2)$$

$$d_i^{-}\left(B^{-},\ B_i\right) = \sqrt{\sum_{j=1}^{n} \left[w_j\left(1 - t_{ij}\left(u_{ij}\right)\right)\right]^2} \qquad (3)$$

Now the relative nearness of an alternative B_i with respect to the PIS is given by

$$N_i\left(B_i\right) = \frac{d_i^{-}\left(B^{-},\ B_i\right)}{d_i^{-}\left(B^{-},\ B_i\right) + d_i^{+}\left(B^{-},\ B_i\right)} \qquad (4)$$

where $N_i\left(B_i\right)$ $(1 \leq i \leq m)$ is the closeness co-efficient of the alternative B_i with respect to the positive ideal solution B^{+} and $0 \leq N_i\left(B_i\right) \leq 1$.Hence we can rank the alternatives in accordance with the descending order of $N_i\left(B_i\right)$. The alternative with the greatest value of $N_i\left(B_i\right)$ is the best one.

3.2.1 Example

In this section, we present an example for multicriteria decision making problem as an illustration of effectiveness of the developed decision making algorithm. Suppose there is a man who wants to visit a foreign country to enjoy his spare time. We consider four possible alternatives for him which are as follows:

(1) B_1 is America; (2) B_2 is Australia; (3) B_3 is Paris: (4) B_4 is Dubai. He has to make a decision in accordance with these criteria: (1) C_1 is the visa access; (2) C_2 is the security situation; and (3) C_3 is the historical places. The criterion weights are given as $C_1 = 0.35$, $C_2 = 0.25$ and $C_3 = 0.40$. The alternative B_i $(1 \leq i \leq 4)$ is to be evaluated using cubic set values by the decider under the above mentioned criteria as provided in the next matrix.

$$D_{4\times 3}\left(u_{ij}\right) = \begin{bmatrix} ([0.4, 0.6], 0.2) & ([0.6, 0.7], 0.3) & ([0.3, 0.6], 0.4) \\ ([0.3, 0.4], 0.4) & ([0.2, 0.3], 0.1) & ([0.5, 0.9], 0.7) \\ ([0.2, 0.7], 0.3) & ([0.5, 0.7], 0.6) & ([0.8, 0.9], 0.1) \\ ([0.1, 0.3], 0.5) & ([0.6, 0.9], 0.2) & ([0.7, 0.8], 0.8) \end{bmatrix}$$

By using (1) we transform cubic set decision matrix $D_{4\times 3}\left(u_{ij}\right)$ in the following score matrix:

$$S_{4\times 3}\left(t_{ij}\left(u_{ij}\right)\right) = \begin{bmatrix} 0.1000 & 0.3000 & 0.1500 \\ 0.0500 & 0.2000 & 0.5500 \\ 0.1000 & 0.4000 & 0.4000 \\ 0.0500 & 0.3500 & 0.6500 \end{bmatrix}$$

By using (2) we can compute $d_i^+ (B^+, B_i)$ $(1 \leq i \leq 4)$ as:

$$\begin{aligned} d_1^+ \left(B^+, B_1\right) &= 0.4954 \\ d_2^+ \left(B^+, B_2\right) &= 0.4277 \\ d_3^+ \left(B^+, B_3\right) &= 0.4235 \\ d_4^+ \left(B^+, B_4\right) &= 0.3957 \end{aligned} \tag{5}$$

By using (3) we can compute $d_i^- (B^-, B_i)$ $(1 \leq i \leq 4)$ as:

$$\begin{aligned} d_1^- \left(B^-, B_1\right) &= 0.1022 \\ d_2^- \left(B^-, B_2\right) &= 0.2263 \\ d_3^- \left(B^-, B_3\right) &= 0.1919 \\ d_4^- \left(B^-, B_4\right) &= 0.2749 \end{aligned} \tag{6}$$

With the help of (4,5 and 6) we have the following closeness co-efficients:

$$\begin{aligned} N_1 (B_1) &= 0.1710 \\ N_2 (B_2) &= 0.3460 \\ N_3 (B_3) &= 0.3118 \\ N_4 (B_4) &= 0.4099 \end{aligned}$$

The descending order of four alternatives is B_4, B_2, B_3 and B_1; abviously B_4 is better choice.

3.3 Group decision making procedure for supplier selection with cubic set TOPSIS approach

Selection of supply chain partner is main issue of supply chain area that contributes strong impact on the whole performance of the supply chain management. The selection of supplier is not an easy job to do because it is typically a multicriteria group decision making problem which falls under the most complex and uncertain environment. The technique for order preference by similarity to ideal solution (TOPSIS) is a well known method to deal with multicriteria decision making problems. In this section we apply the TOPSIS technique for group decision making with cubic set values to deal with supplier selection problems. Unlike the other researchers, the weights for every decision maker and the weights of each criteria are not predetermined in this algorithm. Considering the decision matrix of each decision maker we have calculated different weights. We have defined aggregating operators to convert the individual weights and individual criteria into the group weight and group criteria resp. We have proposed the the formula to calculate the normalized Hamming distance between two cubic set values. In the end, to clarify the suggested technique for supplier selection, an illustrative example has been provided.

3.3.1 Proposed method

The basic objective is to choose the most optimal alternative among m alternatives $B = \{B_1, B_2, ..., B_m\}$ under n criteria $C = \{C_1, C_2, ..., C_n\}$. Consider the group of t decision makers $D = \{D_1, D_2, ..., D_t\}$ to identify the most appropriate supplier. Now algorithm for the cubic set valued TOPSIS method has been provided in the steps given below.

Step 1. Calculate the weight of decision makers:

To make an appropriate decision or conclusion, we have to fix the importance of decision of every decision maker. Thus we calculate the weight of decision makers. The decision maker D_t, $t = 1, 2, \ldots, k$ can not easily determine precise value to the alternative B_i according to the criterion C_j. Let $X^t = \left(u_{ij}^t\right)$ be a decision matrix formed by the tth decision maker, where $u_{ij}^t = \langle\ \left[a_{ij}^t,\ b_{ij}^t\right],\ c_{ij}^t\ \rangle$ is a cubic set value. Now consider that $X^* = u_{ij}^* = \langle\ \left[a_{ij}^*,\ b_{ij}^*\right],\ c_{ij}^*\ \rangle$ is an ideal matrix where $u_{ij}^* = \frac{1}{k}\sum_{t=1}^{k} u_{ij}^t$, and therefore we have

$$u_{ij}^* = \langle\ \left[1 - \pi_{t=1}^{k}\left(1 - a_{ij}^t\right)^{\frac{1}{k}},\ 1 - \pi_{t=1}^{k}\left(1 - b_{ij}^t\right)^{\frac{1}{k}}\right],\ \pi_{t=1}^{k}\left(c_{ij}^t\right)^{\frac{1}{k}}\ \rangle. \tag{1}$$

Obviously, $u_{ij}^* = \langle\ \left[a_{ij}^*,\ b_{ij}^*\right],\ c_{ij}^*\ \rangle$ is a cubic set value. Now, we define the similarity between ideal decision matrix and decision matrix of decision makers $D_t, t = 1, 2, \ldots, k$ as

$$S\left(X^t,\ X^*\right) = \frac{d\left(X^t,\ X^{*^c}\right)}{d\left(X^t,\ X^*\right) + d\left(X^t,\ X^{*^c}\right)} \tag{2}$$

Now using the similarity measure we can obtain the weight of each decision maker D_t as

$$\lambda_t = \frac{S\left(X^t,\ X^*\right)}{\sum_{i=1}^{k}\left(X^i,\ X^*\right)} \tag{3}$$

Obviously, $\lambda_t \geq 0$ and $\sum_{t=1}^{k}\lambda_t = 1$, where $1 \leq t \leq k$.

Step 2. Formation of the aggregated cubic set valued decision matrix:

For the transformation of the individual decisions into the single decision to construct a single group decision we have to form a aggregated cubic set valued decision matrix. We define and denote this decision matrix as

$$u_{ij} = \langle \left[1 - \underset{t=1}{\overset{k}{\pi}} \left(1 - a_{ij}^{t}\right)^{\lambda_t}, \ 1 - \underset{t=1}{\overset{k}{\pi}} \left(1 - b_{ij}^{t}\right)^{\lambda_t}\right], \ \underset{t=1}{\overset{k}{\pi}} \left(c_{ij}^{t}\right)^{\lambda_t} \rangle. \tag{4}$$

Step 3. Calculation of the weights of criteria:

We denote the weights of criteria by $W = (w_1, w_2, ..., w_n)$, where w_j corresponds to criteria C_j.These weights are expressed interms of cubic set values. To get the weight W, we integerate the decision makers opinion to get the aggregated cubic set valued weights of criteria. To calculate the weights of criteria we use the following formula:

$$w_j = \langle \left[1 - \underset{t=1}{\overset{k}{\pi}} \left(1 - \alpha_{ij}^{t}\right)^{\lambda_t}, \ 1 - \underset{t=1}{\overset{k}{\pi}} \left(1 - \beta_{ij}^{t}\right)^{\lambda_t}\right], \ \underset{t=1}{\overset{k}{\pi}} \left(\gamma_{ij}^{t}\right)^{\lambda_t} \rangle. \tag{5}$$

We can observe that w_j is a cubic set value.

Step 4. Formation of the weighted decision matrix:

Our next step is to construct a weighted decision matrix. Let r_{ij} be a decision matrix then

$$r_{ij} = w_j \otimes u_{ij} = \langle \ \left[\alpha_{ij}, \beta_{ij}\right], \gamma_{ij} \ \rangle. \tag{6}$$

Step 5. Determine the cubic set valued positive ideal solutions (PIS) and negative ideal solutions (NIS):

The application of TOPSIS technique here needs to be defined the positive ideal solution (PIS) and negative ideal solution (NIS). The solution which maximizes the benefit criteria and minimizes the cost criteria, is refered as PIS. In contrary NIS minimizes the benefit criteria and maximizes the cost criteria. The best alternative is that which is nearest to the positive ideal solution and the farthest from the negative ideal solution. We will denote the set of criteria of benefit by A and the set of criteria of cost by C. We calculate the cubic set PIS as $B^+ = \{r_1^+, r_2^+, ..., r_n^+\}$, where

$$r_j^+ = \begin{cases} \langle \left[\max_i \alpha_{ij},\ \max_i \beta_{ij}\right], \min_i \gamma_{ij} \rangle,\ j \in A \\ \langle \left[\min_i \alpha_{ij},\ \min_i \beta_{ij}\right], \max_i \gamma_{ij} \rangle,\ j \in C \end{cases} \tag{7}$$

Similarly, we can determine the cubic set NIS as $B^- = \{r_1^-, r_2^-, ..., r_n^-\}$,where

$$r_j^- = \begin{cases} \langle \left[\min_i \alpha_{ij},\ \min_i \beta_{ij}\right], \max_i \gamma_{ij} \rangle,\ j \in A \\ \langle \left[\max_i \alpha_{ij},\ \max_i \beta_{ij}\right], \min_i \gamma_{ij} \rangle,\ j \in C \end{cases} \tag{8}$$

Now without any loss of generality we can say that for $1 \leq j \leq n$, we have $r_j^+ = \langle \left[\alpha_j^+, \beta_j^+\right], \gamma_j^+ \rangle$ and $r_j^- = \langle \left[\alpha_j^-, \beta_j^-\right], \gamma_j^- \rangle$.

Step 6. Calculation of separation measures:

The degree of the separation measures between the alternative B_i and the cubic set PIS is determined by using normalized Hamming distance as

$$d_i^+ = \frac{1}{4m}\sum_{j=1}^{n} \{|\ a_{ij} - \alpha_j^+| + |\ b_{ij} - \beta_j^+| + |\ c_{ij} - \gamma_j^+|\},\ 1 \leq i \leq m \tag{9}$$

Similarly, the degree of the separation measure between the alternative B_i and the cubic set NIS is determined by using normalized Hamming distance as

$$d_i^- = \frac{1}{4m}\sum_{j=1}^{n} \{|\ a_{ij} - \alpha_j^-| + |\ b_{ij} - \beta_j^-| + |\ c_{ij} - \gamma_j^-|\},\ 1 \leq i \leq m \tag{10}$$

Step 7. Calculation of closeness co-efficient:

With respect to the cubic set value of PIS, the closeness co-efficient of the alternative B_i is defined as follows

$$C_i = \frac{d_i^-}{d_i^- + d_i^+},\ 0 \leq C_i \leq 1 \tag{11}$$

Step 8. Ranking of alternatives:

In accordance with the relative closeness, we rank all alternatives in the descending order of C_i's.

3.3.2 Application

To demonstrate the suggested method, we consider the example presented in [13] where the managing board of a university has to come out with a conclusion for the making of a new campus. There are four alternatives B_1, B_2, B_3 and B_4 as possible companies, and also there is a group of four experts D_1, D_2, D_3 and D_4 which has to make a decision in accordance with these criteria: (1) C_1: price, (2) C_2: quality, (3) C_3: delivery time, (4) C_4 is performance history, (5) C_5 is economic status and (6) C_6 is relation with industry. So, there is one cost criterion C_1 and five benefit criteria $C_2, C_3, ..., C_6$. In the following tables we have converted different linguistic evaluations into cubic set values.

Linguistic Term	Cubic set values
Exteremely Important (EI)	$\langle [0.91, 0.95], 0.1 \rangle$
Very Important (VI)	$\langle [0.80, 0.90], 0.10 \rangle$
Important (I)	$\langle [0.70, 0.75], 0.20 \rangle$
Medium (M)	$\langle [0.50, 0.60], 0.45 \rangle$
Unimportant (U)	$\langle [0.30, 0.40], 0.60 \rangle$
Very Unimportant (VU)	$\langle [0, 0], 1 \rangle$

Table1." Importance" weight as linguisticvariables.

Linguistic Term	Cubic set values (CSVs)
Extremely good (EG)	$\langle [1,1], 0 \rangle$
Very very good (VVG)	$\langle [0.80, 0.90], 0.10 \rangle$
Very good (VG)	$\langle [0.70, 0.80], 0.10 \rangle$
Good (G)	$\langle [0.60, 0.70], 0.20 \rangle$
Medium good (MG)	$\langle [0.50, 0.60], 0.30 \rangle$
Fair (F)	$\langle [0.40, 0.50], 0.40 \rangle$
Medium bad (MB)	$\langle [0.30, 0.40], 0.50 \rangle$
Bad (B)	$\langle [0.20, 0.30], 0.60 \rangle$
Very bad (VB)	$\langle [0.00, 0.10], 0.80 \rangle$
Very very bad (VVB)	$\langle [0.00, 0.00], 1 \rangle$

Table2. Linguistic terms for" rating" thealternatives.

Alternatives	Experts	Criteria C_1	C_2	C_3	C_4	C_5	C_6
B_1	D_1	MB	MB	G	F	VG	VG
	D_2	B	B	VB	F	G	VG
	D_3	MG	FB	VG	VVG	G	G
	D_4	B	VB	MB	VB	B	MB
B_2	D_1	VG	VG	VVG	G	F	MB
	D_2	G	VVG	VG	F	G	F
	D_3	VVG	VG	VG	G	G	F
	D_4	G	G	VB	VG	G	VB
B_3	D_1	F	VVG	VG	G	MB	B
	D_2	MB	VG	G	G	B	VB
	D_3	B	VG	VG	G	MB	VVG
	D_4	G	MB	VB	MB	VB	MB
B_4	D_1	VVG	VG	VG	VB	VVG	G
	D_2	VG	VVG	VB	B	VG	VG
	D_3	VVG	VG	VVG	MG	VG	G
	D_4	VG	VB	G	VB	B	VB

Table 3. Individual Decision matrix on term " Rating".

Expert	Weights of criteria					
	C_1	C_2	C_3	C_4	C_5	C_6
D_1	I	EI	VI	M	I	I
D_2	VI	VI	EI	VI	I	U
D_3	I	VI	VI	U	I	I
D_4	M	I	I	U	U	I

Table 4. Criteria weights on term " Rating".

To proceed furthur we have to adopt the following setps.

Step 1. Calculation of the weights of decision makers:

To determine the weights of decision makers we have to get the ideal matrix by using formula (1) as shown in the Table 5 below.

Criteria	Alternative B_1	Alternative B_2	Alternative B_3	Alternative B_4
C_1	$\langle[0.3120, 0.4144], 0.4821\rangle$	$\langle[0.6870, 0.7940], 0.1414\rangle$	$\langle[0.3945, 0.4990], 0.3936\rangle$	$\langle[0.7551, 0.8586], 0.1000\rangle$
C_2	$\langle[0.2087, 0.3099], 0.5867\rangle$	$\langle[0.7087, 0.8139], 0.1189\rangle$	$\langle[0.6650, 0.7787], 0.1495\rangle$	$\langle[0.6337, 0.7551], 0.1682\rangle$
C_3	$\langle[0.4616, 0.5757], 0.2991\rangle$	$\langle[0.6337, 0.7551], 0.1682\rangle$	$\langle[0.5644, 0.6776], 0.2000\rangle$	$\langle[0.6064, 0.7289], 0.2000\rangle$
C_4	$\langle[0.4820, 0.6127], 0.3364\rangle$	$\langle[0.5880, 0.6920], 0.2000\rangle$	$\langle[0.5399, 0.6432], 0.2515\rangle$	$\langle[0.2047, 0.3099], 0.5826\rangle$
C_5	$\langle[0.5573, 0.6650], 0.2213\rangle$	$\langle[0.5573, 0.6591], 0.2378\rangle$	$\langle[0.2087, 0.3099], 0.5886\rangle$	$\langle[0.6536, 0.7700], 0.1565\rangle$
C_6	$\langle[0.6016, 0.7087], 0.1778\rangle$	$\langle[0.2915, 0.3938], 0.5030\rangle$	$\langle[0.4215, 0.5591], 0.3936\rangle$	$\langle[0.5319, 0.6432], 0.2378\rangle$

Table 5. Ideal matrix

Now with the help of formulas (2) and (3), we get the weight of each decision maker which is shown in the Table 6.

Decision maker	D_1	D_2	D_3	D_4
Weight	0.2690	0.2670	0.2940	0.1700

Table 6. Decision makers's weights.

Table 6 indicates that the decision of third decision maker is more worthy than other decision makers.

Step 2. Construction of the aggregated cubic set valued decision matrix:

To aggregate the all individual decisions and to construct one group decision we use formula (4). The aggregated cubic set valued decision matrix is shown in Table 7.

Criteria	Company B_1	Company B_2	Company B_3	Company B_4
C_1	$\langle [0.3278, 0.4303], 0.4660 \rangle$	$\langle [0.6980, 0.8052], 0.1354 \rangle$	$\langle [0.3649, 0.4687], 0.4251 \rangle$	$\langle [0.7612, 0.8646], 0.1000 \rangle$
C_2	$\langle [0.2292, 0.3302], 0.5686 \rangle$	$\langle [0.7173, 0.8219], 0.1125 \rangle$	$\langle [0.6893, 0.7999], 0.1315 \rangle$	$\langle [0.6696, 0.7854], 0.1424 \rangle$
C_3	$\langle [0.4837, 0.5983], 0.2760 \rangle$	$\langle [0.6699, 0.7857], 0.1424 \rangle$	$\langle [0.6025, 0.7122], 0.1714 \rangle$	$\langle [0.6143, 0.7389], 0.1960 \rangle$
C_4	$\langle [0.5262, 0.6558], 0.2994 \rangle$	$\langle [0.5755, 0.6791], 0.2139 \rangle$	$\langle [0.5601, 0.6625], 0.2337 \rangle$	$\langle [0.2315, 0.3369], 0.5553 \rangle$
C_5	$\langle [0.5835, 0.6893], 0.2001 \rangle$	$\langle [0.5539, 0.6558], 0.2410 \rangle$	$\langle [0.2292, 0.3302], 0.5686 \rangle$	$\langle [0.6822, 0.7946], 0.1356 \rangle$
C_6	$\langle [0.6229, 0.7284], 0.1612 \rangle$	$\langle [0.3179, 0.4197], 0.4779 \rangle$	$\langle [0.4478, 0.5885], 0.3709 \rangle$	$\langle [0.5671, 0.6755], 0.2104 \rangle$

Table 7. Aggregated cubic set valued decision matrix.

Step 3. To determine the criteria weights:

By applying formula (5) on tables 3 and 6 we get the importance of the criteria as listed in Table 8 below.

Criteria	Weights
C_1	$\langle [0.7064, 0.7880], 0.1908 \rangle$
C_2	$\langle [0.8271, 0.9030], 0.1125 \rangle$
C_3	$\langle [0.8269, 0.9029], 0.1125 \rangle$
C_4	$\langle [0.5424, 0.6666], 0.3442 \rangle$
C_5	$\langle [0.6535, 0.7099], 0.2411 \rangle$
C_6	$\langle [0.6238, 0.6842], 0.2682 \rangle$

Table 8. Weights of criteria

Step 4. Formation of the weighted decision matrix:

We can form the weighted decision matrix by using formula (6). Table 9 is the weighted decision matrix.

Criteria	Alternative B_1	Alternative B_2	Alternative B_3	Alternative B_4
C_1	$\langle [0.2316, 0.3391], 0.5679 \rangle$	$\langle [0.4931, 0.6345], 0.3004 \rangle$	$\langle [0.2578, 0.3693], 0.5348 \rangle$	$\langle [0.5377, 0.6813], 0.2717 \rangle$
C_2	$\langle [0.1896, 0.2982], 0.6171 \rangle$	$\langle [0.5933, 0.7422], 0.2123 \rangle$	$\langle [0.5701, 0.7223], 0.2292 \rangle$	$\langle [0.5538, 0.7092], 0.2389 \rangle$
C_3	$\langle [0.4000, 0.5402], 0.3575 \rangle$	$\langle [0.5539, 0.7094], 0.2389 \rangle$	$\langle [0.4982, 0.6430], 0.2646 \rangle$	$\langle [0.5080, 0.6672], 0.2865 \rangle$
C_4	$\langle [0.2854, 0.4372], 0.5405 \rangle$	$\langle [0.3122, 0.4527], 0.4845 \rangle$	$\langle [0.3038, 0.4416], 0.4975 \rangle$	$\langle [0.1256, 0.2246], 0.7884 \rangle$
C_5	$\langle [0.3813, 0.4893], 0.3930 \rangle$	$\langle [0.3620, 0.4656], 0.4240 \rangle$	$\langle [0.1498, 0.2344], 0.6726 \rangle$	$\langle [0.4458, 0.5641], 0.3440 \rangle$
C_6	$\langle [0.3836, 0.4984], 0.3862 \rangle$	$\langle [0.1983, 0.2872], 0.5450 \rangle$	$\langle [0.2793, 0.4027], 0.5396 \rangle$	$\langle [0.3538, 0.4622], 0.4222 \rangle$

Table 9. Weighted decision matrix.

Step 5. Determine the cubic set valued positive ideal solution (PIS) and negative ideal solution (NIS):

By using formulas (7) and (8) we have determined the cubic set valued PIS and NIS as shown in the following Table 10.

Criteria	PIS	NIS
C_1	$\langle\,[0.5377, 0.6813], 0.2717\,\rangle$	$\langle\,[0.2316, 0.3391], 0.5679\,\rangle$
C_2	$\langle\,[0.5933, 0.7422], 0.2123\,\rangle$	$\langle\,[0.1896, 0.2982], 0.6171\,\rangle$
C_3	$\langle\,[0.5539, 0.7094], 0.2389\,\rangle$	$\langle\,[0.4000, 0.5402], 0.3575\,\rangle$
C_4	$\langle\,[0.3122, 0.4527], 0.4845\,\rangle$	$\langle\,[0.1256, 0.2246], 0.7084\,\rangle$
C_5	$\langle\,[0.4458, 0.5641], 0.3440\,\rangle$	$\langle\,[0.1498, 0.2344], 0.6726\,\rangle$
C_6	$\langle\,[0.3836, 0.4984], 0.3862\,\rangle$	$\langle\,[0.1983, 0.2872], 0.5450\,\rangle$

Table 10. Cubic set valued PIS and NIS.

Step 6. Construction of the separation measures:

We have calculated the separation measures of each alternative from cubic set valued PIS and NIS with the help of formulae (9) and (10) respectively. Table 11 shows how far each alternative is from cubic set valued positive ideal solution (PIS) and negative ideal solution (NIS)..

Alternatives	Distance from positive ideal solution (PIS)	Distance from positive negative ideal solution (NIS)
B_1	0.0842	0.0776
B_2	0.0159	0.1629
B_3	0.1178	0.0993
B_4	0.0406	0.1588

Table 11. Distance of each alternative from positive ideal solution (PIS) and negative ideal solution (NIS).

Step 7. Calculation of closeness co-efficient:

The closness co-efficient of every alternative can be calculated by using formula (11). The following Table 12 shows the closeness co-efficient of each alternative.

Alternatives	Closeness co-efficient	Ranking
B_1	0.4796	3
B_2	0.9111	1
B_3	0.4574	4
B_4	0.7964	2

Table 12. Closeness co-efficient of each alternative.

Step 8. In accordance to the closeness co-efficient of all alternatives, the order of the ranking is shown in the above Table 12. Clearly company B_2, is selected as best construction company.

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