Prediction of multiplicity of solutions of nonlinear boundary value problems using homotopy analysis method

By

Mirza Adnan Baig

114-FBAS/MSMA/F12

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan

2015

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Boundary value problems.
Nonlinear boundary

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
In
MATHEMATICS

Supervised by

Dr. Tariq Javed
(Assistant Professor Mathematics)

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2015

بسم الله الرحمن الرحيم

Certificate

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By

Mirza Adnan Baig

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE *MASTER OF SCIENCE in MATHEMATICS*

We accept this dissertation as conforming to the required standard.

1. Prof. Dr. Tasawar Hayat (External Examiner)
2. Dr. Nasir Ali (Internal Examiner)
3. Dr. Tariq Javed (Supervisor)
4. Dr. Rehmat Ellahi (Chairman) 21/8/15

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2015

Dedication

With due respect all my research work is dedicated to

My Spiritual Teacher and Guide

Shaykh ul Islam Dr Muhammad Tahir ul Qadri

"Whose lectures have always been a great source of Divine Pleasure for me"

My Parents

"Without whom none of my success would be possible"

Specially to my beloved and dearest

Father

Mirza Javed Baig

(May his soul rests in Heavens in peace)

&

My ever best friend and my beloved

Brother

Mirza Zeeshan Baig

(May his soul rests in Heavens in peace)

&

All of my Family members

"Who always had been strengthening me whenever I felt disappointed"

Acknowledgements

I surrender all of my efforts, energies and bow myself to **Almighty Allah** Who is the most Beneficent, the most Merciful, the Master of life and death and the Master of whole universe. Who provided me the strength and ability to learn and achieve the objective of my dissertation. I offer millions and trillions of Salutations to our beloved the **Holy Prophet Hazrat Muhammad (PBUH)** who was the greatest educator, reformer and teacher of all the teachers to mankind and whose teachings became the torch to bring the mankind from darkness to brightness.

With due respect I express my gratitude to my spiritual teacher and guide **Shaykh ul Islam Dr Muhammad Tahir ul Qadri** whose lectures have always been a great source of divine pleasure for me in the light of the Holy Quran and Sunnah of the **Holy Prophet Hazrat Muhammad (PBUH)**. I learnt a lot from him about leading a peaceful life and dare to speak truthfulness whatever the situation it is! I just would say that he is a calm voice in the sea of extremism and terrorism. May ALLAH Almighty bless him with success in his life to make Pakistan centre of peace, love and tranquility and a Model for all the nations of the entire globe.

I pay my best regards to all my teachers **Dr Nasir Ali, Dr Ahmed Zeeshan, Dr Ahmer Mehmood, Dr Rehmat Ellahi, Sir Niaz Ahmed, Dr Arshad Zia** and **Dr. Irshad Ahmad Arshad** (Ex- Chairman Maths and Stats) whose utmost efforts brought me to the level of academic excellence. Particularly, I would like to put across my sincere gratitude to my affectionate, kind natured, competent, dedicated, prominent and devoted supervisor **Dr. Tariq Javed** who supported me with many encouraging suggestions and knowledgeable comments. He guided me to advance level academic standards. His many valuable comments, suggestions and instructions greatly improved the transparency of this document. I pray to **Almighty Allah** to keep showering His countless blessings on my supervisor **Dr.Tariq Javed** forever.

I am very thankful to my very very-beloved friends **Syed Saqib Shah, Engr. Faisal Nisar, Ghulam Mustafa, Hafiz Usman Ghani, Muhammad Adil Shehzad, Ch. Irfan Ashraf, Arif Ali Sandhu, Hafiz Muhammad Atif, Waqas Nazir, Mubbashar Nazeer Mehar and Haleem Afsar** for their continuous support and sincere suggestions throughout my career at IIUI. Their Love, respect and affection will always be a part of my melodious memories. They all are unforgettable. May they all prosper in all the fields of life and stay successful ever. My friends definitely are my PRIDE and HONOUR for me. They all are my identity. May countless showers of the blessings of ALLAH Almighty be on them all. Aameen

I also am grateful to my respected seniors and research fellows **Sir Saleem Iqbal, Ziafat Ali, Bilal Ahmed, Sir Hussain Ahmad, Sir Abid Majeed, Abuzar Ghaffari, Irfan Mustafa, Abdul Haleem** and **Abaid ur Rehman** for their friendly environment in research room and outside it and very special thanks to **Arshad Siddiqui** and **Sami ullah Khan** for their support and guidance during my research work brought me to write this dissertation. Mr. Arshad Siddiqui's cooperation was remarkable. May he stay blessed, happy and successful. Aameen

Mirza Adnan Baig (114-FBAS/MSMA/F12)

DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor **Dr. Tariq Javed**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

Mirza Adnan Baig

MS (Mathematics)

Reg. No. 114-FBAS/MSMA/F12

Department of Mathematics and Statistics,

Faculty of Basic and Applied Sciences,

International Islamic University Islamabad, Pakistan.

Preface

Many of the mathematical modeling of the practical phenomena in science and engineering often leads to nonlinear differential equations. There are a lot of methods from the wide range of numerical methods to approximate analytical methods [1-17] available in literature used for the solution purpose. The important question which arises in this situation is that, do the approximate methods enable to predict multiplicity of the solutions of nonlinear differential equations? In other words can we predict the existence of multiple solutions of the nonlinear differential equations and further can we obtain all branches of the solutions through any analytical method? It is somehow difficult to answer the above question due to the reason that, it is our conventional belief that the approximate methods usually converge to one solution by one using initial guess.

The homotopy analysis method has been developed by Liao [18-22] to obtain series solution of various nonlinear problems. This method has further gained remarkable improvements and achievements by solving different nature of problems [23-43]. E. Shivanian and S. Abbasbandy provide predictor homotopy analysis method and its relevant several proof in the study [44]. In another paper, S. Abbasbandy and E. Shivanian proved a novel application of homotopy analysis method [45]. In this paper, the procedure to predict the existence of multiplicity of solution and further calculate all branches of the solution is provided. Motivated from the above two studies this dissertation is arranged as follows

Chapter 1 contains some basic definitions related to the homotopy analysis method, Adomian polynomial, Cauchy sequence and Lipschitz function.

In chapter 2, the predictor homotopy analysis method [44] is explained in detail. The proofs of relevant theorem are made parts of this chapter. The essence of the method is explained by using nonlinear reaction-diffusion model and obtained two branches of the solution.

Chapter 3 covers the study related to prediction of multiplicity of the solutions of nonlinear boundary value problems [45]. The core idea is implemented on three exactly solvable problems namely nonlinear heat transfer equation [46], strongly nonlinear Bratu equation [36, 40] and nonlinear reaction-diffusion equation [48]. The existence of multiplicity of solution of all problems can be proved very conveniently and further all branches of the solutions are calculated by using predictor homotopy analysis method within good accuracy.

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Chapter 1

Preliminaries

In this chapter, some important definitions related to the studies in next chapters are provided for the better understanding of the readers.

1.1 Homotopic functions

Two continuous functions from one topological space to another are called homo-topic if one can be continuously deformed into the other, such a deformation is called a homotopy between the two functions. More precisely, we have the following definition.

Let X, Y be two topological spaces, and $f, g : X \to Y$ continuously maps. A homotopy from f to g is a continuous function $\mathbf{F} : \mathbf{X}[0, 1] \to \mathbf{Y}$ satisfying

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x), \quad \text{for all } x \in X.$$

If such a homotopy exists, we say that f is homotopic to g, and denote this by $\mathbf{f} \simeq \mathbf{g}$.

Example

Let $\mathbf{f}, \mathbf{g} : \mathbf{R} \to \mathbf{R}$ be any two continuous, real functions. Then define a function $\mathbf{F} : \mathbb{R}[\mathbf{0}, \mathbf{1}] \to \mathbb{R}$ by

$$F(x, t) = (1 - t).f(x) + t\ g(x).$$

Clearly, F is continuous, being a composite of continuous functions. Moreover,

$$F(x,0) = (1-0).f(x) + 0.g(x) = f(x)$$

and

$$F(x,1) = (1-1).f(x) + 1.g(x) = g(x)$$

Thus, F is a homotopy between f and g and $\mathbf{f} \simeq \mathbf{g}$.

1.2 Homotopy Analysis Method

The homotopy analysis method (HAM) is an analytic approximation method for solving highly nonlinear equations arising in science, finance and all branches of engineering. It was first proposed by Dr. Shijun Liao in 1992 in his Ph.D. dissertation [18].

1.3 Motivations

Perturbation techniques are very famous and old and widely applied to obtain analytic approximations of nonlinear differential equations. However, these techniques are essentially based on small/large physical parameters (called perturbation quantity), but unfortunately there exist many nonlinear problems with no such kind of small/large physical parameters at all. In addition, neither perturbation techniques nor the traditional non-perturbation techniques (such as Lyapunov artificial small-parameter method, Adomian decomposition method, delta-expansion method and so on) can provide a way to guarantee the convergence of approximation series. Therefore, both perturbation techniques and the traditional non-perturbation methods mentioned above remained valid only for weakly nonlinear problems.

1.4 Advantages of the homotopy analysis method

Based on a generalized concept of the homotopy in topology, the HAM has the following advantages embedded in its framework

∘ The HAM is always valid,no matter whether there exist small/large physical parameters or not.

∘ The HAM provides a convenient way to control and adjust convergence region of the approximation series solution.

∘ The HAM provides great freedom to choose the base functions, initial guess and linear operator for the solutions.

As a result, the HAM overcomes the restrictions on all other analytic approximation methods mentioned above, and is valid for highly nonlinear problems.

1.5 Adomian Polynomial

Adomian polynomials decompose a function $u(x,t)$ into a sum of components

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

for a nonlinear operator F as

$$F(u(x,t)) = \sum_{n=0}^{\infty} A_n.$$

The Adomian Polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda_n}\left[F\left(\sum_{n=0}^{\infty}\lambda_i u_i\right)\right]_{\lambda=0} \qquad n = 0, 1, 2, \ldots.$$

One possible set of polynomials is given by

$$A_0 = F(u_0),$$

$$A_1 = u_1 F'(u_0),$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0),$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0), \ldots .$$

These polynomials have the property that A_n depends only on $u_0, u_1, ..., u_n$, and that the sum of subscripts for the component u_n is equal to n.

1.6 Calculations of Adomian Polynomials A_n for nonlinear functions

The Adomian polynomials of some important functions are calculated as follows

1.6.1 Nonlinear Polynomials

Case 1: $F(u) = u^2$

The Adomian polynomials can be obtained as follows

$$A_0 = F(u_0) = u_0^2,$$

$$A_1 = u_1 F'(u_0) = 2u_0u_1,$$

$$A_2 = u_2F'(u_0) + \frac{1}{2!}u_1^2F''(u_0) = 2u_0u_2 + u_1^2,$$

$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^2F'''(u_0) = 2u_0u_3 + 2u_1u_2, \ \dots \ .$$

1.6.2 Nonlinear Derivative Functions

Case 1: $F(u) = (u_x)^2$

The Adomian polynomials can be obtained as follows

$$A_0 = F(u_0) = u_{0_x}^2,$$

$$A_1 = u_1F'(u_0) = 2u_{0_x}u_{1_x},$$

$$A_2 = u_2F'(u_0) + \frac{1}{2!}u_1^2F''(u_0) = 2u_{0_x}u_{2_x} + u_{1_x}^2,$$

$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^2F'''(u_0) = 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x},$$

$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^2F'''(u_0) = 2u_0u_3 + 2u_1u_2, \dots \ .$$

Case 2: $F(u) = u_x^3$

The Adomian polynomials can be obtained as follows

$$A_0 = F(u_0) = u_{0_x}^3,$$

$$A_1 = u_1 F'(u_0) = 3u_{0_x}^2 u_{1_x},$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) = 3u_{0_x}^2 u_{2_x} + 3u_{0_x} u_{1_x}^2,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^2 F'''(u_0) = 3u_{0_x}^2 u_{3_x} + 6u_{0_x} u_{1_x} u_{2_x} + u_{1_x}^3, \dots .$$

Case 3: $F(u) = uu_x = \frac{1}{2}L_x(u^2)$

The Adomian polynomials can be obtained as follows

$$A_0 = F(u_0) = u_0 u_{0_x},$$

$$A_1 = \frac{1}{2}L_x(2u_0u_1) = u_{0_x}u_1 + u_0 u_{1_x},$$

$$A_2 = \frac{1}{2}L_x(2u_0u_2 + u_1^2) = u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0,$$

$$A_3 = \frac{1}{2}L_x(2u_0u_3 + 2u_1u_2) = u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0, \dots .$$

1.6.3 Trigonometric Nonlinearity

Case 1: $F(u) = sin(u)$

The Adomian polynomials can be obtained as follows

$$A_0 = \sin(u_0),$$

$$A_1 = u_1 \cos(u_0),$$

$$A_1 = u_2 \cos(u_0) - \frac{1}{2!}u_1^2 \sin(u_0),$$

$$A_3 = u_3 \cos(u_0) - u_1 u_2 \sin(u_0) - \frac{1}{3!}u_1^3 \cos(u_0), \dots .$$

Case 2: $F(u) = \cos(u)$

The Adomian polynomials can be obtained as follows

$$A_0 = \cos(u_0),$$

$$A_1 = -u_1 \sin(u_0),$$

$$A_1 = -u_2 \sin(u_0) - \frac{1}{2!}u_1^2 \cos(u_0),$$

$$A_3 = -u_3 \sin(u_0) - u_1 u_2 \cos(u_0) + \frac{1}{3!}u_1^3 \sin(u_0), \dots .$$

1.6.4 Hperbolic Nonlinearity

Case 1: $F(u) = \sinh(u)$

The Adomian polynomials can be obtained as follows

$$A_0 = \sinh(u_0),$$

$$A_1 = u_1 \cosh(u_0),$$

$$A_1 = u_2 \cosh(u_0) + \frac{1}{2!}u_1^2 \sinh(u_0),$$

$$A_3 = u_3 \cosh(u_0) + u_1 u_2 \sinh(u_0) + \frac{1}{3!}u_1^3 \cosh(u_0), \dots .$$

1.6.5 Exponential Nonlinearity

Case 1: $F(u) = e^u$

The Adomian polynomials can be obtained as follows

$$A_0 = e^{u_0},$$

$$A_1 = u_1 e^{u_0},$$

$$A_1 = (u_2 + \frac{1}{2!}u_1^2)e^{u_0},$$

$$A_3 = (u_3 + u_1 u_2 + \frac{1}{3!}u_1^3)e^{u_0}, \dots .$$

1.6.6 Logarithmic Nonlinearity

Case 1: $F(u) = \ln(u), \quad u > 0$

The Adomian polynomials can be obtained as follows

$$A_0 = \ln(u_0),$$

$$A_1 = \frac{u_1}{u_0},$$

$$A_1 = \frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0^2},$$

$$A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0} + \frac{1}{3}\frac{u_1^3}{u_0^3}, \cdots .$$

Case 2: $F(u) = \ln(1+u), \quad -1 < u \leq 1$

The Adomian polynomials can be obtained as follows

$$A_0 = \ln(1+u_0),$$

$$A_1 = \frac{u_1}{1+u_0},$$

$$A_1 = \frac{u_2}{1+u_0} - \frac{1}{2}\frac{u_1^2}{(1+u_0)^2},$$

$$A_3 = \frac{u_3}{1+u_0} - \frac{u_1 u_2}{(1+u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1+u_0)^3}, \cdots .$$

1.7 Cauchy Sequence

Definition

A sequence is called a Cauchy sequence if the terms of the sequence eventually all become arbitrarily close to one another. That is, given $\epsilon > 0$ there exists N such that if $m, n > N$ then $|a_m - a_n| < \epsilon$.

1.8 Lipschitz

1.8.1 Lipschitz Function

A function f such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y, where C is a constant independent of x and y, is called a Lipschitz function. For example, any function with a bounded first derivative must be Lipschitz.

1.8.2 Lipschitz Condition

A function $f(x)$ satisfies the Lipschitz condition of order β at $x = 0$ if

$$|f(h) - f(0)| \leq B|h|^{\beta}$$

for all $|h| < \epsilon$, where B and β are independent of h, $\beta > 0$, and alpha is an upper bound for all beta for which a finite B exists.

1.8.3 Lipschitz's Integral

$$\int_0^{\infty} e^{-ax} J_0(bx) = \frac{1}{\sqrt{a^2 + b^2}}$$

where $J_0(z)$ is the zeroth order Bessel function of the first kind.

1.8.4 Lipschitz domain

In mathematics, a Lipschitz domain (or domain with Lipschitz boundary) is a domain in Euclidean space whose boundary is "sufficiently regular" in the sense that it can be thought of as locally being the graph of a Lipschitz continuous function. The term is named after the German mathematician Rudolf Lipschitz.

Definition

Let $n \in N$, and let Ω be an open subset of R^n. Let $\partial\Omega$ denote the boundary of Ω. Then Ω is said to have Lipschitz boundary, and is called a Lipschitz domain, if, for every point $p \in \partial\Omega$, there exists a radius $r > 0$ and a map $h_p : B_r(p) \rightarrow Q$ such that

(i) h_p is a bijection

(ii) h_p and $h_p - 1$ are both Lipschitz continuous functions

(iii) $h_p(\partial\Omega \cap B_r(p)) = Q_0$

(iv) $h_p(\Omega \cap B_r(p)) = Q_+$,

where

$$B_r(p) := \{x \in \mathbb{R}^n | \, \|x - p\| < r\}$$

denotes the n-dimensional open ball of radius r about p, Q denotes the unit ball $B_1(0)$, and

$$Q_0 := \{(x_1, ..., x_n) \in Q | x_n = 0\};$$

$$Q_+ := \{(x_1, ..., x_n) \in Q | x_n = 0\},$$

1.8.5 Applications of Lipschitz domains

Many of the Sobolev embedding theorems require that the domain of study be a Lipschitz domain. Consequently, many partial differential equations and variational problems are defined on Lipschitz domains.

Hillam's Theorem

If $f : [a, b] \to [a, b]$ (where $[a, b]$ denotes the closed interval from a to b on the real line) satisfies a Lipschitz condition with constant K, i.e., if

$$|f(x) - f(y)| \leq K|x - y|$$

for all x, y in $[a, b]$, then the iteration scheme is

$$x_{n+1} = (1 - \lambda)x_n + f(x_n),$$

where $\lambda = 1/(K + 1)$, converges to a fixed point of f.

Picard's Existence Theorem

If f is a continuous function that satisfies the Lipschitz condition

$$|f(x,t) - f(y,t)| \leq L|x-y|$$

in a surrounding of $(x_0, t_0) \in \Omega \subset \mathbb{R}^n \times \mathbb{R} = \{(x,t) : |x - x_0| < b, |t - t_0| < a\}$, then the differential equation

$$\frac{dx}{dt} = f(x,t) \qquad x(t_0) = x_0$$

has a unique solution $x(t)$ in the interval $|t - t_0| < d$, where $d = min(a, b/B)$, min denotes the minimum, $B = sup|f(t,x)|$, and sup denotes the supremum.

1.9 Hilbert Space

A vector space H over the field of complex (or real) numbers, together with a complex-valued (or real-valued) function (x,y) defined on $H \times H$, with the following properties:

1) (x,x) if and only if $x = 0$

2) $(x,x) \geq 0$ for all $x \in H$

3) $(x+y,z) = (x,y) + (x,z)$, $x,y,z \in H$

4) $(\alpha x, y) = \alpha(x,y)$, $x,y \in H$, α a complex (or real) number

5) $(x,y) = (y,x)$, $x,y \in H$

6) if $x_n \in H$, $n = 1,2,3,...$, and if

$$\lim_{n,m \to \infty} (x_n - x_m, x_m - x_n) = 0,$$

then there exists an element $x \in H$ such that

$$\lim_{n \to \infty} (x - x_n, x - x_n) = 0,$$

the element x is called the limit of the sequence (x_n)

7) H is an infinite-dimensional vector space.

Definition

A Hilbert space is a vector space **H** with an inner product $< f, g >$ such that the norm defined by

$$|f| = \sqrt{< f, f >}$$

turns H into a complete metric space. If the metric defined by the norm is not complete, then H is instead known as an inner product space.

Examples of finite-dimensional Hilbert spaces include

1. The real numbers $\mathbb{R}^n$ with $< v, u >$, the vector dot product of v and u.

2. The complex numbers $\mathbb{C}^n$ with $< v, u >$, the vector dot product of v and the complex conjugate of u.

An example of an infinite-dimensional Hilbert space is L^2, the set of all functions $f : R \rightarrow R$ such that the integral of f^2 over the whole real line is finite. In this case, the inner product is

$$< f, g >= \int_{-\infty}^{\infty} f(x)g(x)dx.$$

A Hilbert space is always a Banach space, but the converse need not hold.

Chapter 2

Predictor homotopy analysis method: Two points second order boundary value problems

2.1 Introduction

Prediction of multiplicity of the solutions of nonlinear boundary value problems has always been extremely difficult job for the researchers in last five years. After it, calculation of multiple solutions analytically is another hard job. To ease these difficulties, predictor homotopy analysis method (PHAM) is recently developed to predict the multiplicity of solutions of two points second order boundary value problems by E. Shivanian and S. Abbasbandy [44]. The main objective of this chapter is to completely understand the predictor homotopy analysis method and its applications. The proof of the basic theorems behind the predictor homotopy analysis method is also studied which are given in [21]. To further illustrate this method, nonlinear reaction-diffusion model will be analyzed for the purpose of prediction of multiplicity of its solutions and calculating all multiple solutions of it.

2.2 Description of the method

Consider a second order two point boundary value problem of the following form

$$v'' = g(y, v, v'), \qquad \eta < y < \xi \tag{2.1}$$

subject to the boundary conditions

$$d_1 v(\eta) + d_2 v'(\eta) = d \quad \text{and} \quad c_1 v(\xi) + c_2 v'(\xi) = c \tag{2.2}$$

Introducing an unknown parameter δ in the boundary condition so that problem (2.1) and (2.2) reduced to equivalent initial value problem as follows

$$v(\eta) = f_1(\delta), \quad v'(\eta) = f_2(\delta), \text{ and } c_1 v(\xi) + c_2 v'(\xi) = c \tag{2.3}$$

where $c_1 v(\xi) + c_2 v'(\xi) = c$ is the forcing condition arising from the original conditions (2.2), and $f_1(\delta)$ and $f_2(\delta)$ satisfy $d_1 f_1(\delta) + d_2 f_2(\delta) = d$. Now we apply usual HAM as follows to the following reduced initial value problem

$$v'' = g(y, v, v'), \quad \eta < y < \xi \tag{2.4}$$

$$v(\eta) = f_1(\delta) \quad \text{and} \quad v'(\eta) = f_2(\delta). \tag{2.5}$$

2.2.1 Zero-order deformation equation

Let $v = v(y)$ be the solution of the problem (2.4) then a set of base functions $\{\omega_i(x), i = 0, 1, 2, ...\}$ can be expressed in term of solution as

$$v = v(y) = \sum_{i=0}^{+\infty} a_i \omega_i(x) \tag{2.6}$$

where a_i are unknowns to be calculated. Let $v_0(y, \delta)$ satisfies the initial conditions (2.5) be an initial guess and let auxiliary parameter$\hbar \neq 0$, auxiliary function $H(x) \neq 0$ and $\mathcal{L}$ a second order auxiliary linear operator. For zero-order deformation equation, we take $q \in [0, 1]$ as an embedding parameter. The zeroth order deformation equation takes the following form

$$(1 - q)\mathcal{L}[\varphi(y, \delta; q) - v_0(y, \delta)] = q\hbar H(y)\mathcal{N}\,[\varphi(y, \delta; q)] \tag{2.7}$$

subject to initial conditions

$$v(\eta) = f_1(\delta), \qquad v'(\eta) = f_2(\delta)$$

where $\varphi(y, \delta; q)$ is the solution of nonlinear boundary value problem to be calculated and

$$\mathcal{N}\left[\varphi(y, \delta; q)\right] = \frac{\partial^2 \varphi(y, \delta; q)}{\partial y^2} - f\left(y, \varphi(y, \delta; q), \frac{\partial \varphi(y, \delta; q)}{\partial y}\right)$$

Here we have two cases.

Case 1 When $q = 0$, Eq. (2.7) becomes

$$\mathcal{L}[\varphi(y, \delta; q) - v_0(y, \delta)] = 0 \tag{2.9}$$

which results

$$\varphi(y, \delta; 0) = v_0(y, \delta)$$

i.e. the initial guess is the solution of the equation.

Case 2 When $q = 1$, Eq. (2.7) becomes

$$\mathcal{N}\left[\varphi(y, \delta; 1)\right] = 0 \tag{2.10}$$

which is exactly the same as the original equation (2.1), provided that

$$\varphi(y, \delta; 1) = v(y, \delta).$$

Expanding $\varphi(y, \delta; q)$ in a Taylor series as a function of q, we have

$$\varphi(y, \delta; q) = v_0(y, \delta) + \sum_{m=1}^{m=+\infty} v_m(y, \delta) q^m, \tag{2.11}$$

where

$$v_m(y,\delta) = \frac{1}{m!}\frac{\partial \varphi(y,\delta;q)}{\partial q^m}, \qquad m = 0,1,2,\ldots . \tag{2.12}$$

It is established in the frame of homotopy analysis method when the linear operator $\mathcal{L}$, the initial approximation $v_0(y,\delta)$, the auxiliary parameter $\hbar \neq 0$, and the auxiliary function $H(y) \neq 0$ are chosen properly, the series solution (2.11) is convergent for $q = 1$, and thus

$$v(y,\delta) = v_0(y,\delta) + \sum_{m=1}^{m=+\infty} v_m(y,\delta) = \sum_{n=0}^{n=+\infty} a_n \omega_n(y) \tag{2.13}$$

will be solution of nonlinear problem (2.4) and (2.5).

2.2.2 High-order deformation equation

We define the vector $\overrightarrow{v}_n = \{v_0(y), v_1(y), ..., v_n(y)\}$, differentiate Eq. (2.7) m times w.r.t. 'q', and dividing it by $m!$, the mth-order deformation equation will take the form

$$\mathcal{L}[v_m(y,\delta) - \chi_m v_{m-1}(y,\delta)] = \hbar H(y) R_m(\overrightarrow{v}_{m-1}, y, \delta), \tag{2.14}$$

subject to the conditions

$$v(\eta) = 0, \qquad v'(\eta) = 0, \tag{2.15}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{2.16}$$

and

$$R_m(\overrightarrow{v}_{m-1}, y, \delta) = \frac{1}{m-1!}\frac{\partial^{m-1}\mathcal{N}\left[\varphi(y,\delta;q)\right]}{\partial q^{m-1}}\bigg|_{q=0} = \frac{1}{m-1!}\frac{\partial^{m-1}\mathcal{N}\left[\sum_{n=0}^{n=+\infty} v_n(y,\delta)q^n\right]}{\partial q^{m-1}}\bigg|_{q=0} \tag{2.17}$$

We can use computer software like Mathematica or Maple to solve mth order approximate Eq. (2.14) subject to the conditions (2.15) . Accordingly, the mth order approximate solution of the problem (2.4) and (2.5) is given by

$$v(y,\delta) \approx v_M(y,\delta,\hbar) = v_0(y,\delta) + \sum_{m=1}^{M} v_m(y,\delta) = \sum_{m=1}^{m'} a_n\omega_n(y). \tag{2.18}$$

2.2.3 Forecasting the multiplicity of solution

The rule of multiplicity of solutions is a method to determine the number of solutions admitted by the boundary value problem. For this purpose, it is required to calculate the value of δ as a function of $\hbar$. The forcing condition $c_1v(\xi) + c_2v'(\xi) = c$ plays a vital role in the existence of unique or multiple solutions. We can apply it as follows:

Consider the mth order approximation (2.18) and put it in the forcing condition (2.3) to derive the following expression

$$c_1V_m(\xi,\delta,\hbar) + c_2V_m'(\xi,\delta,\hbar) = c. \tag{2.19}$$

According to HAM, Eq. (2.18) converges at $y = \xi$ only in that range of $\hbar$, without changing δ and variating $\hbar$. This shows that a horizontal range appears implicitly in the plot of δ as function of $\hbar$ in $V_m(\xi,\delta,\hbar)$. The number of these horizontal plateaus, where $\delta(\hbar)$ becomes constant, gives us multiplicity of the solutions. We can say that the number of horizontal plateaus in the plot of $\delta(\hbar)$ are directly connected to the existences and number of multiple solutions. Let us now discuss the importance of $\hbar$-curve in relation to Taylor's series.

2.3 Theorem 2.1

Suppose the specifics values for δ and $\hbar$, obtained from rule of multiplicity of solutions, as long as in the series (2.18), i.e.

$$V_k(y,\delta,\hbar) = \sum_{k=1}^{K} V_k(y,\delta)$$

K goes to infinity, the $V_k(y,\delta,\hbar)$ tends to exact solution of the problems (2.1) and (2.2), i.e. $v(y)$.

Proof Let

$$s(y,\delta) = v_0(y,\delta) + \sum_{k=1}^{K} V_k(y,\delta) \tag{2.20}$$

be the convergent series. From the Eq. (2.14), we have

$$\hbar H(y,\delta)R_k(\overrightarrow{v}_{k-1},y,\delta) = \sum_{k=1}^{+\infty}\mathcal{L}\left[v_k(y,\delta) - \sum_{i=1}^{k-1}\beta_i v_{k-i}(y,\delta)\right]$$

or

$$= \mathcal{L}\left[\sum_{k=1}^{+\infty} v_k(y,\delta) - \sum_{k=1}^{+\infty}\sum_{i=1}^{k-1}\beta_i v_{k-i}(y,\delta)\right]$$

multiplying $\sum_{k=1}^{+\infty}$ with in the the brackets

$$= \mathcal{L}\left[\sum_{k=1}^{+\infty} v_k(y,\delta) - \sum_{i=1}^{+\infty}\sum_{k=i+1}^{+\infty}\beta_i v_{k-i}(y,\delta)\right]$$

as $\sum_{k=1}^{+\infty}$ is a linear operator, so it will multiply term by term as follows

$$= \mathcal{L}\left[\sum_{k=1}^{+\infty} v_k(y,\delta) - \left(\sum_{k=1}^{+\infty}\beta_i\right)\sum_{i=1}^{+\infty} v_k(y,\delta)\right]$$

taking $\sum_{k=1}^{+\infty} v_k(y,\delta)$ as common, we obtain

$$= \mathcal{L}\left[\left((1 - \sum_{i=1}^{+\infty}\beta_i\right)\sum_{k=1}^{+\infty} v_k(y,\delta)\right]$$

(2.20) implies

$$= \mathcal{L}\left[\left(1 - \sum_{i=1}^{+\infty}\beta_i\right)s(y,\delta)\right]$$

which gives, since$\hbar \neq 0$, $H(y,\delta) \neq 0$,

$$\sum_{i=1}^{+\infty} \alpha_i = 1 \quad , \quad \sum_{i=1}^{+\infty} \beta_i = 1 \tag{2.21}$$

and

$$\mathcal{L}[f(y,\delta)] = 0 \quad , \quad f(y,\delta) = 0 \tag{2.22}$$

this implies that

$$\sum_{k=1}^{+\infty} R_k(\overrightarrow{v}_{k-1}, y, \delta) = 0$$

But we also have

$$\sum_{k=1}^{+\infty} R_k(\overrightarrow{v}_{k-1}, y, \delta) = \sum_{k=1}^{+\infty} \sum_{i=1}^{k} \alpha_i \delta_{k-i}(y,\delta)$$

or

$$= \left(\sum_{i=1}^{n} \alpha_i \right) \sum_{k=1}^{+\infty} \delta_k(y,\delta)$$

using the Eqs. $(2.20 - 2.22)$, we have

$$\sum_{k=1}^{+\infty} R_k(\overrightarrow{v}_{k-1}, y, \delta) = \sum_{k=1}^{+\infty} \delta_k(y,\delta)$$

so the Adomian polynomial in this case while setting $q = 0$

$$= \sum_{k=1}^{+\infty} \frac{1}{k!} \frac{\partial^k \mathcal{N}[\Phi(y,\delta;q)]}{\partial q^k} \bigg|_{q=0} = 0$$

Commonly, $\Phi(y,\delta;q)$ does not satisfy the original nonlinear Eq. (2.1). So, Let

$$\varepsilon(y,\delta,q) = \mathcal{N}[\Phi(y,\delta;q)]$$

be the residual error as

$$\varepsilon(y,\delta,q) = 0$$

So, from above, the Maclaurin's series of the residual error $\varepsilon(y,\delta;q)$ about q is given as

$$\sum_{k=1}^{+\infty} \frac{p^k}{K!} \frac{\partial^k \varepsilon(y,\delta;q)}{\partial q^k}\bigg|_{q=0} = \sum_{k=1}^{+\infty} \frac{p^k}{K!} \frac{\partial^k N[\Phi(y,\delta;q)]}{\partial q^k}\bigg|_{q=0}$$

for $q=1$,

$$\varepsilon(y,\delta,1) = \sum_{k=1}^{+\infty} \frac{1}{K!} \frac{\partial^k \varepsilon(y,\delta;q)}{\partial q^k}\bigg|_{q=0} = 0$$

hence, we obtain the exact solution of the original equation when $q=1$. Thus the series

$$v_0(y,\delta) + \sum_{k=1}^{+\infty} v_k(y,\delta)$$

converges to the original Eq. (2.1). Which is required to prove.

2.4 Theorem 2.2

Suppose that $f(\hbar)$ be a continuous function onto interval $[c,d]$ and all derivatives of $g:[c,d]\to\mathbb{R}$ exist and have a common P so that

$$\max_{x\in[c,d]} \left|g^{(k)}(x)\right| \leq P \qquad \text{for all } k. \tag{2.23}$$

Furthermore, assume that $F_m(y,\alpha)$ be the Taylor polynomial of degree m for $g(y)$ about some $\alpha \in (a,b)$, say $\alpha = f(\hbar)$, then $\forall \varepsilon > 0$ and $\beta \in (c,d)$ there exist $N \in \mathbb{N}$ and interval (a,b) so that

$$\forall \hbar \in (a,b) \quad \text{and} \quad m \geq N : |g(\beta) - F_m(\beta, f(\hbar))| < \varepsilon$$

Proof Let $\gamma \in [c,d]$ be the point at which we want to calculate the error. We assume that $\gamma > \alpha$ and let

$$s(y) = g(\gamma) - g(y) - \frac{(\gamma-y)}{1!}g'(y) - \frac{(\gamma-y)^2}{2!}g''(y) - \ldots - \frac{(\gamma-y)^m}{m!}g^m(y), \tag{2.24}$$

then for $y \in (c,d)$, there exists $s'(y)$ and

$$s'(y) = -\frac{(\gamma - y)^m}{m!} g^{(m+1)}(y).$$

Now, assume a function of the form

$$V(y) = s(y) - \left(\frac{\gamma - y}{\gamma - \alpha}\right)^{m+1} s(\alpha),$$

then

$$V(\alpha) = V(\gamma) = 0.$$

From the differentiability of $s(y)$ and $\left(\frac{\gamma-y}{\gamma-\alpha}\right)^{m+1}$, we get $V(y)$ is differentiable on any subinterval of (c,d). Then there exists $\eta_\gamma \in (\alpha,\gamma)$ such that

$$V'(\eta_\gamma) = 0,$$

which results

$$-\frac{(\gamma - \eta_\gamma)^{m+1}}{(m+1)!} g^{(m+1)}(\eta_\gamma) + (m+1)\frac{(\gamma - \eta_\gamma)^m}{(\gamma - \alpha)^{m+1}} s(\alpha) = 0.$$

Since $\gamma \neq \eta_\gamma$, then

$$s(\alpha) = \frac{(\gamma - \alpha)^{m+1}}{(m+1)!} g^{(m+1)}(\eta_\gamma).$$

From Eq. (2.24), we get

$$s(\alpha) = g(\gamma) - F_m(\gamma, \alpha) = \frac{(\gamma - \alpha)^{m+1}}{(m+1)!} g^{(m+1)}(\eta_\gamma), \quad \eta_\gamma \in (\alpha, y).$$

Since we have chosen γ as an arbitrary, then

$$\forall y \in [\alpha, d], \quad \alpha \in [c,d] : g(y) - F_m(y,\alpha) = \frac{(y-\alpha)^{m+1}}{(m+1)!} g^{(m+1)}(\eta_y), \quad \eta_y \in (\alpha, y). \qquad (2.25)$$

Suppose that $\beta \in (c,d)$ and $\varepsilon > 0$, let $\alpha \in (c, \beta)$ then there exists N such that

$$\forall m \geq N : \frac{(\beta - \alpha)^{m+1}}{(m+1)!} < \frac{\varepsilon}{P},$$

then from Eqs. (2.23) and (2.25), we get

$$|g(\beta) - F_m(\beta, \alpha)| = \frac{(\beta - \alpha)^{m+1}}{(m+1)!} \left|g^{(m+1)}(\xi_\beta)\right| < \frac{\varepsilon}{P}.P = \varepsilon. \tag{2.26}$$

Thus, we have proved

$$\forall \beta \in (c, d), \qquad \alpha \in (c, \beta), \varepsilon > 0, \qquad \exists N \Rightarrow \forall m \geq N : |f(\beta) - F_m(\beta, \alpha)| < \varepsilon.$$

Since $f(\hbar)$ is continuous function onto interval $[c, d]$ then there exists interval (a, b) such that $f\{(a, b)\} = (c, \beta)$. Hence (2.26) can be rewritten as

$$\forall \beta \ \in (c, d), \quad \hbar \in (a, b), \varepsilon > 0, \qquad \exists N \Rightarrow \forall m \geq N : |g(\beta) - F_m(\beta, g(\hbar))| < \varepsilon$$

and the proof is completed.

2.5 Theorem 2.3

Suppose that $f(\hbar)$ be a continuous function onto interval $[c', d']$ and $F_m(y, \gamma) = \sum_{k=0}^{m} c_k(\gamma)(y-\gamma)^k$ be the Taylor polynomial of degree m for $g(y)$ about some $\gamma \in (c', d')$, say $\alpha = f(\hbar)$. Moreover, assume that

$$\forall k \in N, \gamma \in [c', d'], \quad y \in [c, d] : |y - \gamma| \leq \left|\frac{c_k(\gamma)}{c_{k+1}(\gamma)}\right|. \tag{2.27}$$

Then for $\varepsilon > 0$ and $\beta \in [c, d]$, there exist $N \in N$ and interval (a, b) so that

$$\forall \hbar \in (a, b) \quad \text{and} \quad m \geq N : |g(\beta) - F_m(\beta, f(\hbar))| < \varepsilon \tag{2.28}$$

Proof From the statement (2.27), it implies that there exist $0 < \phi < 1$ such that

$$\forall k \in N, \gamma \in [c', d'], \quad y \in [c, d] : |c_{k+1}(\gamma)||y - \gamma| \leq \phi |c_k(\gamma)| \tag{2.29}$$

We are to prove the statement (2.28), for this we will show that for a fixed $\beta \in [c', d']$,

$$F_m(\beta, \gamma), \gamma \in (c', d')$$

is convergent first we prove that $\{F_m(\beta,\gamma)\}_{m=0}^{\infty}$ is Cauchy sequence in the Hilbert space R. So, (2.29) gives us

$$\| F_{m+1}(\beta,\gamma) - F_m(\beta,\gamma) \| = |c_{m+1}(\gamma)| \, \| (\gamma-\gamma) \|^{m+1} \le \phi |c_m(\gamma)| \, \| (\beta-\gamma) \|^m$$

$$\le \phi^2 |c_{m-1}(\gamma)| \, \| (\beta-\gamma) \|^{m-1} \le \cdots \le \phi^{m-q+1} |c_q(\gamma)| \, \| (\beta-\gamma) \|^q$$

now, for every $m, n \in \mathbb{N}$, $m \ge n > q$, we have

$$\| F_m(\beta,\gamma) - F_m(\beta,\gamma) \|$$

$$= \| (F_m(\beta,\gamma) - F_{m-1}(\beta,\gamma)) + (F_{m-1}(\beta,\gamma) - F_{m-2}(\beta,\gamma)) + \ldots + (F_{n+1}(\beta,\gamma) - F_n(\beta,\gamma)) \|$$

$$\le \| F_m(\beta,\gamma) - F_{m-1}(\beta,\gamma) \| + \| F_{m-1}(\beta,\gamma) - F_{m+1}(\beta,\gamma) \| + \ldots + \| F_{n+1}(\beta,\gamma) - F_n(\beta,\gamma) \|$$

$$\le \phi^{m-q} \mid c_q(\gamma)| \, \| (\beta-\gamma) \|^q + \phi^{m-q-1} |c_q(\gamma)| \, \| (\beta-\gamma) \|^q + \cdots + \phi^{n-q+1} |c_q(\gamma)| \, \| (\beta-\gamma) \|^q$$

$$= \frac{1-\phi^{m-n}}{1-\phi} \phi^{n-q+1} \mid c_q(\gamma)| \, \| (\beta-\gamma) \|^q$$

Thus, we have

$$\lim_{n,m\to\infty} \| F_m(\beta,\gamma) - F_n(\beta,\gamma) \| = 0$$

and hence $\{F_m(\beta,\gamma)\}_{m=0}^{\infty}$ is convergent. Or we can say that $F_m(\beta,\gamma)$ is the Taylor polynomial of $g(y)$ at $y=\beta$.

Thus $\forall\, \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\forall m \ge N: \ |g(\beta) - F_m(\beta,\gamma)| < \varepsilon$$

from (c', d') we arbitrarily choose γ, so for each $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and interval (a,b) so that

$$\forall \hbar \in (a,b) \ \text{ and } \ \ m \ge N : |g(\beta) - F_m(\beta, f(\hbar))| < \varepsilon$$

This proves the theorem.

2.6 Analysis of Convergence

2.7 Theorem 2.4

Let $0 < \beta < 1$ and the solution components $v_0(y,\delta), v_1(y,\delta), v_2(y,\delta), \ldots$ obtained by (2.14) satisfy the following condition

$$\exists t_0 \in N, \quad \forall t \geq t_0 : \quad \| v_{t+1}(y,\delta) \| \leq \beta \| v_t(y,\delta) \|$$

then the series solution $\sum_{t=0}^{+\infty} v_t(y,\delta)$ is convergent.

Proof Let us assume a sequence $\{S_u\}_{u=0}^{+\infty}$ of the following form,

$$\left.\begin{array}{c} S_0 = v_0(y,\delta) \\ S_1 = v_0(y,\delta) + v_1(y,\delta) \\ S_2 = v_0(y,\delta) + v_1(y,\delta) + v_2(y,\delta) \\ . \\ . \\ . \\ S_u = v_0(y,\delta) + v_1(y,\delta) + \ldots + v_u(y,\delta) \end{array}\right\}.$$

We would like to show that $\{S_u\}_{u=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space $\mathbb{R}$. So let us assume that

$$\| S_{u+1} - S_u \| = \| v_{u+1}(y,\delta) \| \leq \alpha \| v_u(y,\delta) \|$$

$$\leq \beta^2 \| v_{u-1}(y,\delta) \| \leq \cdots \leq \beta^{u-k0+1} \| v_{k_0}(y,\delta) \|,$$

$\forall\, u, v \in N, u \geq v > t_0$, we have

$$\| S_u - S_v \| \quad \leq \quad \| (S_u - S_{u-1}) + (S_{u-1} - S_{u-2}) + \cdots + (S_{v+1} - S_v) \|$$

$$\leq \| (S_u - S_{u-1}) + (S_{u-1} - S_{u-2}) + \cdots + (S_{v+1} - S_v) \|$$

$$\leq \beta^{u-t_0} \|v_{t_0}(y,\delta)\| + \beta^{u-t_0-1} \|v_{t_0}(y,\delta)\| + \ldots + \beta^{v-t_0+1} \|v_{t_0}(y,\delta)\|$$

$$= \frac{1-\beta^{u-v}}{1-\beta} \beta^{v-t_0+1} \|v_{t_0}(y,\delta)\| \tag{2.30}$$

but $0 < \beta < 1$, so we have

$$\lim_{u,v \to \infty} \| S_u - S_v \| = 0$$

Thus, $\{S_u\}_{u=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space $\mathbb{R}$ which means the series solution (2.18) is convergent. hence proved.

2.8 Theorem 2.5

Assume that the series solution $\sum_{t=0}^{+\infty} v_t(y,\delta)$ defined in (2.18), is convergent to the solution $v(y)$. If the truncated series

$$V_K(y,\delta,\hbar) = \sum_{t=0}^{K} v_t(y,\delta)$$

is used as an approximation to the solution $v(y)$ of the problem (2.4) − (2.5), then the maximum absolute truncated error is estimated as,

$$\| v(y) - V_K(y,\delta,\hbar) \| \leq \frac{1}{1-\beta} \beta^{K-t_0+1} \|v_{t_0}(y,\delta)\|$$

Proof From the expression (2.30), we have

$$\| S_m - S_K \| \leq \frac{1-\beta^{m-K}}{1-\beta} \beta^{K-t_0+1} \|v_{t_0}(y,\delta)\|$$

for $m \geq K$.

As $m \to \infty$ then $S_m \to v(y)$ and $\beta^{m-K} \to 0$. So,

$$\| v(y) - V_K \| \leq \frac{1}{1-\beta} \beta^{K-t_0+1} \|v_{t_0}(y,\delta)\|$$

The above two theorems 2.3 and 2.4 show that horizontal plateaus occur in $\hbar$-curve where the series solution (2.18) converges which means $V_K(y,\delta,\hbar)$ is convergent. Hence proof is complete.

2.9 Theorem 2.6

Consider the boundary value problem (2.1) – (2.2) and suppose that the conditions of the convergence theorem hold for the initial value problem (2.4) – (2.5) and more, the conditions of the Theorem 2.4 hold for the series

$$A_K(y,\delta,\hbar)=\sum_{k=0}^{K}a_k(y,\delta)$$

if the number of M horizontal plateaus occur in the plane of $(\hbar,\delta)$ where the Eq.(2.19) is plotted implicitly, then the problem (2.1) – (2.2) admits the number of K multiple solutions in terms of the basis functions (2.6).

Proof Let M in the plane $(\hbar,\delta)$ be number of horizontal plateaus such as

$$\delta_1(\hbar_1),\delta_2(\hbar_2),...,\delta_M(\hbar_M)$$

where $(\hbar_i,\delta_i)$, $i=1,2,...,M$ are proper ordered pair which chosen from $(\hbar,\delta)$. We conclude from theorem 2.4 and by uniqueness of the Taylor's series that all the series $\sum_{k=0}^{+\infty}u_k(y,\delta_{i,}(\hbar_i))$, $i=1,2,...,M$ converge. Assume that

$$\left.\begin{aligned}
s_1(y)&=\sum_{k=0}^{+\infty}a_k(y,\delta_1(\hbar_1))\\
s_2(y)&=\sum_{k=0}^{+\infty}a_k\,(y,\delta_2(\hbar_2))\\
&\;\;.\\
&\;\;.\\
&\;\;.\\
s_M(y)&=\sum_{k=0}^{+\infty}a_k(y,\delta_M(\hbar_M))
\end{aligned}\right\}$$

We would like to prove that the above series are the solutions of the problem (2.1) – (2.2). For this, we assume that

$$s_i(y) = \sum_{k=0}^{+\infty} a_k(y, \delta_i(\hbar_i))$$

by high order-deformation equation (3.14) – (3.16), we obtain

$$\hbar_i H(y) \sum_{k=0}^{+\infty} R_k(\overrightarrow{u}_{k-1}, y, \delta_i(\hbar_i)) = \sum_{k=0}^{+\infty} \mathcal{L}[a_k(y, \delta_i(\hbar_i)) - \chi_k u_{k-1}(y, \delta_i(\hbar_i))]$$

$$= \mathcal{L}\{\sum_{k=0}^{+\infty}[a_k(y, \delta_i(\hbar_i)) - \chi_k u_{k-1}(y, \delta_i(\hbar_i))]\}$$

$$= \mathcal{L}\{a_1(y, \delta_i(\hbar_i))\} + \mathcal{L}\{a_2(y, \delta_i(\hbar_i)) - a_1(y, \delta_i(\hbar_i))\}$$

$$+ \mathcal{L}\{a_3(y, \delta_i(\hbar_i)) - a_2(y, \delta_i(\hbar_i))\}$$

$$+ \ldots \mathcal{L}\{a_j(y, \delta_i(\hbar_i)) - a_{j-1}(y, \delta_i(\hbar_i))\} + \ldots$$

$$= \{\lim_{j \to \infty} a_j(y, \delta_i(\hbar_i))\} = \mathcal{L}\{0\} = 0$$

$\mathcal{L}$ is Lipschitz in the above equations because $\mathcal{L} = \frac{\partial^2}{\partial y^2}$ also

$$\| \mathcal{L}(a-b) \| = \| \mathcal{L}a - \mathcal{L}b \| = \left\| g(y, a, a') - g(y, b, b') \right\|$$

$$\leq \left\| g(y, a, a') \right\| + \left\| g(y, b, b') \right\| \leq 2N \leq L \left\| a - b \right\|, \qquad L = \frac{2N}{\|a-b\|}$$

As we know that $\hbar_i \neq 0$ and $H(y) \neq 0$ are non-zero, then

$$\sum_{k=0}^{+\infty} R_k(\overrightarrow{u}_{k-1}, y, \delta_i(\hbar_i)) = 0$$

thus

$$\sum_{k=0}^{+\infty} \frac{1}{k!} \frac{\partial^k \mathcal{N}[\Phi(y, \delta_i(\hbar_i); p)]}{\partial p^k}\bigg|_{p=0} = \sum_{k=0}^{+\infty} \frac{1}{(k-1)!} \frac{\partial^{k-1} \mathcal{N}[\Phi(y, \delta_i(\hbar_i); p)]}{\partial p^{k-1}}\bigg|_{p=0} = 0$$

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Consider the residual of the original equation (2.1) as

$$\Re(y,\delta_i(\hbar_i);p)=\mathcal{N}[\Phi(y,\delta_i(\hbar_i);p)].$$

Now,we are to prove that $\Re(y,\delta_i(\hbar_i);p)=0$, this implies

$$\mathcal{N}[\Phi(x,\delta_j(\hbar_j);p)]=0,$$

but we also know that $\Phi(y,\delta_i(\hbar_i);1)=s_i(x)$. The Taylor's series of $\Re(y,\delta_i(\hbar_i);p)$ takes the form

$$\Re(y,\delta_i(\hbar_i);p)=\sum_{k=0}^{+\infty}\frac{1}{k!}\frac{\partial^k\Re(y,\delta_i(\hbar_i);p)}{\partial p^k}\bigg|_{p=0}p^k$$

which results

$$\Re(y,\delta_i(\hbar_i);p)=\sum_{k=0}^{+\infty}\frac{1}{k!}\frac{\partial^k\Re(y,\delta_i(\hbar_i);p)}{\partial p^k}\bigg|_{p=0}$$

$$=\sum_{k=0}^{+\infty}\frac{1}{k!}\frac{\partial^k\mathcal{N}(y,\delta_i(\hbar_i);p)}{\partial p^k}\bigg|_{p=0}=0$$

hence the proof is completed.

2.10 Nonlinear reaction-diffusion model

To illustrate the method, we here consider the problem of nonlinear reaction diffusion model as follows

$$v''v^{0.75}-0.64=0, \tag{2.31}$$

with boundary conditions

$$v'(0)=0, \qquad v(1)=1. \tag{2.32}$$

where the primes denote differentiation w.r.t. 'x', where $0\leq x\leq 1$ and v is the dimensionless concentration of the reactant. The above problem (2.31) and (2.32) can be reduced to an initial value problem to introduce δ as follows:

$$v^{''}v^{0.75} - 0.64 = 0 \tag{2.33}$$

subject to the conditions

$$v^{'}(0) = 0 \quad \text{and} \quad v(0) = \delta, \tag{2.34}$$

with additional forcing condition

$$v(1) = 1. \tag{2.35}$$

In which δ is an unknown parameter to be determined. Now, we apply HAM on (2.33) and (2.34) as follows. For this, choose the set of base functions

$$\{x^{2k} \mid k = 0, 1, 2, ...\}, \tag{2.36}$$

We choose $H(x) = 1$ as auxiliary function, $v_0(x) = \delta$ as initial guess satisfy the above conditions, and choose auxiliary linear operator $\mathcal{L}$ as second order operator as follows

$$\mathcal{L}[\varphi(x,\gamma;q)] = \frac{\partial^2 \varphi(x,\gamma;q)}{\partial x^2} \tag{2.37}$$

which satisfy the property

$$\mathcal{L}[c_1 + c_2 x] = 0. \tag{2.38}$$

The mth-order deformation equation for $M \geq 1$ becomes

$$\mathcal{L}[v_m(x,\delta) - \chi_m v_{m-1}(x,\delta)] = \hbar R_m(\overrightarrow{v}_{m-1}, x, \delta). \tag{2.39}$$

From (2.17) and (2.37),we have

$$R_m(\overrightarrow{v}_{m-1}, x, \delta) = \sum_{j=o}^{m-1} v_{m-1-j}^{''}(x) u_j(x) - 0.64(1 - \chi_m). \tag{2.40}$$

when $m = 1$ and $j = 0$, we have

$$R_0(\overrightarrow{v}_0, x, \delta) = v_0''(x)u_0(x),$$

where

$$u_0(x) = [v_0(x)]^{0.75}.$$

Which now has becomes the original equation (2.41) as

$$R_0 = v''v^{0.75} - 0.64,$$

and $u_n(x)$ is found by another additional Adomian polynomial as follows

$$u_n(x) = \frac{1}{n!}\frac{\partial^n[\varphi(x,\gamma;q)]^{0.75}}{\partial q^n}\bigg|_{q=0} = \frac{1}{n!}\frac{\partial^n[\sum_{k=0}^{k=+\infty} v_k(x,\delta)q^k]^{0.75}}{\partial q^n}\bigg|_{q=0}, \tag{2.41}$$

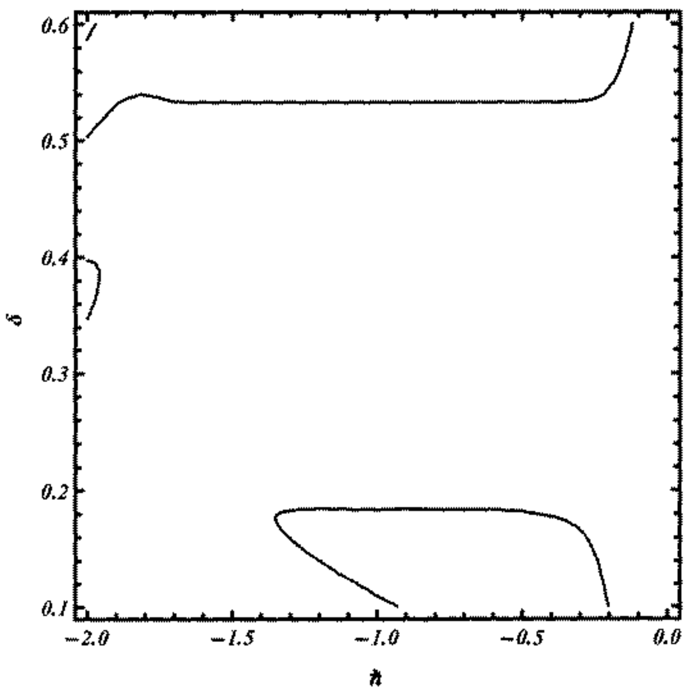

Fig.2.1 : Graph of δ as a function of $\hbar$

which for different values of $n = 0, 1, 2, ...$ implies that

$$u_0(x) = [v_0(x)]^{0.75}, \quad u_1(x) = \frac{0.75v_1(x)}{[v_0(x)]^{1.25}}, \tag{2.42}$$

$$u_2(x) = -\frac{0.09375[v_1(x)]^2}{[v_0(x)]^{1.25}} + \frac{0.75v_2(x)}{[v_0(x)]^{0.25}}. \tag{2.43}$$

The conditions for the higher-order deformation equation (2.41) becomes $v_m(0) = 0$, $v_m'(0) = 0$.Starting from $v_0(x, \delta) = \delta$ we, successively, can find the functions $v_m(x, \delta)$ for $m = 1, 2, 3, ...$The mth-order approximate HAM solution can be expressed as

$$V_M(x, \delta, \hbar) = \sum_{m=0}^{M} v_m(x, \delta). \tag{2.44}$$

Thus, when $v(1) = 1$, Eq. (2.19) takes the following form

$$v(1) \approx V_M(1, \delta, \hbar) = 1 \tag{2.45}$$

By using symbolic software Mathematica, we computed upto $M = 25th$ order approximate solution. The solution is later used in forcing condition (2.45) and drew the graph of δ as a function of $\hbar$ a shown in figure (2.1). It is seen that two horizontal lines namely $\delta = 0.1836$ in $\hbar$-range$[1.3, 0.4]$ and $\delta = 0.5330$ in $\hbar$-range $[1.7, 0.3]$ occur. Thus there exist two solutions for the problem (2.31) and (2.32). It is concluded that Predictor Homotopy analysis method provides convenient way to calculate dual solutions satisfying the exact dual solutions of the original problem.

2.11 Conclusion

In this paper we have shown how to predict the existence of unique or multiple solutions of the problems. For this, a new parameter δ is introduced to the problem. An additional forcing condition and $\hbar$ play vital role in the prediction of multiplicity. In the plot of δ as function of $\hbar$ existence of multiple solutions can be calculated corresponding to the horizontal ranges appearing in the graph.

Chapter 3

Prediction of multiplicity of solutions of nonlinear boundary value problems

Computing the solution of nonlinear boundary value problems analytically have been a difficult job for the researchers. Since the last two decades homotopy analysis method is being considered to be the best tool for this purpose as compared to other tools [3, 4]. This is due to its many advantages and freedom embedded in the solution procedure of HAM. Till to date many types of boundary value problems in science and engineering namely viscous flow of non-newtonian fluids [23], the KdV-type equation [26, 47], nonlinear heat transfer [46, 49], problem related to shallow water equations have been solved successfully by employing this method [30].

On the other hands, researchers have used many numerical methods time to time which are quite useful for this purpose. Few of them are Euler method, Runge–Kutta explicit and implicit methods, different multistep methods, Taylor series method, Hybrid methods, family of finite difference methods [1, 2] and spectral methods [29, 30] etc.. Many other methods are also there to give approximate solutions analytically like for example perturbation methods [3, 6], the artificial small parameter method [4], the δ-expansion method [5], the Adomian decomposition method [7], the variation iteration method [8] and so on. These methods do not give us a convenient way to control and adjust the convergence region and rate of given approximate solution which are guaranteed in the framework of homotopy analysis method through an auxiliary parameter called convergence control parameter $\hbar$.

By using above cited analytical methods, it is observed that the prediction or forecasting of multiplicity or solutions of the nonlinear boundary value problems is very much difficult task by using these above cited analytical methods. Since we assume that approximate method usually converge to one solution by one initial guess is the exactly means the convergence. The convergence control parameter $\hbar$ embedded in homotopy analysis method has found a key role in forecasting the multiplicity of the solutions of nonlinear problems investigated in [44]. Moreover, homotopy analysis method provides a convenient way to further control the multiple solutions (if exists) of the problems will be revisited in this chapter.

3.1 Description of the method

Let us consider the following nonlinear differential equation

$$\mathcal{N}\left[v(y)\right] = 0, \quad y \in \Omega \tag{3.1}$$

subject to the boundary condition

$$\mathfrak{B}\left(v, \frac{\partial v}{\partial n}\right) = 0, \qquad y \in \Gamma \tag{3.2}$$

where $\mathcal{N}$ is nonlinear operator, $\mathfrak{B}$ is boundary operator and Γ is boundary of the domain Ω. The key step of the procedure is that the boundary value problems (3.1) and (3.2) be replaced by an equivalent problem in such a way that the condition (3.2) must involves an unknown parameter like δ as follows

$$\mathfrak{B}\left(v, \delta, \frac{\partial v}{\partial n}\right) = 0, \quad \text{and} \quad v\left(\beta\right) = \gamma \qquad y \in \Gamma \tag{3.3}$$

where $v\left(\beta\right) = \gamma$ is the forcing condition, which comes from condition (3.2). Now applying the HAM on the problem (3.1) subject to the boundary conditions (3.3)

$$\mathcal{N}\left[v(y)\right] = 0, \quad y \in \Omega \tag{3.4}$$

$$\mathfrak{B}'(v, \delta, \frac{\partial v}{\partial n}) = 0, \qquad y \in \Gamma \tag{3.5}$$

3.1.1 Zero-order deformation equation

We assume that all the solutions $v = v(y)$ of problem (3.4) can be expressed by the set of base functions $\{w_j(y), j = 0, 1, 2, ...\}$ such that

$$v = v(y) = \sum_{n=0}^{+\infty} a_n w_n(y), \tag{3.6}$$

where a_n are unknown coefficients to be determined. We let $v_0(y, \delta)$ be an initial guess of $v(y)$ that satisfies conditions (3.5). The general zero-order deformation equation takes the following form

$$(1-q)\mathcal{L}[\varphi(y,\delta;q) - v_0(y,\delta)] = q\hbar H(y)\mathcal{N}[\varphi(y,\delta;q)] \tag{3.7}$$

$$\mathfrak{B}'\left(\varphi(y,\delta;q), \delta, \frac{\partial\varphi(y,\delta;q)}{\partial n}\right) = 0, \qquad y \in \Gamma \tag{3.8}$$

where $\hbar \neq 0$ be the convergence-controller parameter, $H(x) \neq 0$ be an auxiliary function, and $\mathcal{L}$ be an auxiliary linear operator and $\varphi(y, \delta; q)$ is to be determined. Here $q \epsilon [0, 1]$ as an embedding parameter, in which two cases occur:

Case 1 When $q = 0$, (3.7) leads us to

$$\mathcal{L}[\varphi(y,\delta;0) - v_0(y,\delta)] = 0 \tag{3.9}$$

which implies

$$\varphi(r,\delta;0) = v_0(y,\delta).$$

Case 2 When $q = 1$, (3.7) takes the form

$$\mathcal{N}[\varphi(y,\delta;q)] = 0, \tag{3.10}$$

which is the same nonlinear boundary value problem given in Eq. (3.1) such that $\varphi(y,\delta;1) = v(y,\delta)$. Now expanding $\varphi(y,\delta;q)$ in a Taylor series to the embedding parameter q as follows

$$\varphi(y,\delta;q) = v_0(y,\delta) + \sum_{m=1}^{m=+\infty} v_m(y,\delta)q^m, \tag{3.11}$$

where

$$v_m(y,\delta) = \frac{1}{m!}\frac{\partial\varphi(y,\delta;q)}{\partial q^m}. \qquad m = 0,1,2,... \tag{3.12}$$

It is well known in frame of HAM, after choosing auxiliary linear operator $\mathcal{L}$, suitable initial guess $v_0(y,\delta)$, auxiliary parameter $\hbar \neq 0$, and auxiliary function $H(y) \neq 0$, the series given in Eq. (3.11) converges for $q = 1$. Hence

$$v(y,\delta) = v_0(y,\delta) + \sum_{m=1}^{m=+\infty} v_m(y,\delta) = \sum_{n=0}^{n=+\infty} a_n w_n(y) \tag{3.13}$$

is one of solutions of (3.4) and (3.5).

3.1.2 High-order deformation equation

After properly choosing the initial guess function $v_0(y,\delta)$, the linear operator $\mathcal{L}$ and auxiliary function $H(y)$, we can calculate $v_m(y,\delta)$ in Eq. (3.13) with the help of the high-order deformation equations as follows. Note importantly that the value of convergence control parameter $\hbar$ will be chosen later. We first define the vector $\overrightarrow{v}_m = \{v_0(y), v_1(y), ..., v_m(y)\}$ then differentiating Eq. (3.7) m-times with respect to q, divide it by $m!$ and setting $q = 0$, with (3.8). We get mth-order deformation equation in the following form

$$\mathcal{L}[v_m(v,\delta) - \chi_m v_{m-1}(y,\delta)] = \hbar H(y) R_m(\overrightarrow{v}_{m-1}, y, \delta) \tag{3.14}$$

with boundary conditions

$$\frac{\partial^m}{\partial q^m}\mathfrak{B}'\left(\varphi(y,\delta;q),\delta,\frac{\partial\varphi(y,\delta;q)}{\partial n}\right)\bigg|_{q=0} = 0, \quad y\ \in \Gamma \tag{3.15}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{3.16}$$

and

$$R_m(\overrightarrow{v}_{m-1}, y, \delta) = \frac{1}{m-1!} \frac{\partial^{m-1} \mathcal{N}\left[\varphi(y, \delta; q)\right]}{\partial q^{m-1}}\bigg|_{q=0}$$

$$= \frac{1}{m-1!} \frac{\partial^{m-1} \mathcal{N}\left[\sum_{n=0}^{n=+\infty} v_n(y, \delta) q^n\right]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{3.17}$$

The mth-order deformation Eq. (3.14) with boundary condition (3.15) can easily be solved with the aid of any computer symbolic software like Mathematica or Maple. After starting from $v_0(y, \delta)$, we can calculate the functions $v_m(y, \delta)$ for $m = 1, 2, 3, \ldots$from Eqs. (3.14) and (3.15) successively. Thus, the mth-order approximate solution of the problems (3.4) and (3.5) has the following form

$$v(y, \delta) \approx v_M(y, \delta, \hbar) = v_0(y, \delta) + \sum_{m=1}^{M} v_m(y, \delta) = \sum_{n=0}^{m'} a_n w_n(y). \tag{3.18}$$

3.1.3 Forecasting the multiple solutions

The rule of multiplicity is a procedure to know the number of solutions admitted by the boundary value problem (3.1). Uptill now we have chosen the initial guess $v_0(y, \delta)$,the linear operator $\mathcal{L}$ and auxiliary function $H(y) \neq 0$ properly such that the series solutions (3.18) converges. But still we are to find out δ and $\hbar$ in the series (3.18). The existence of unique or multiple solution depends on whether the forcing condition $(v(\beta) = \gamma)$ as given in Eq. (3.3) admits unique or multiple solutions for formally introduced parameter δ.This stage is called rule of multiplicity of solutions. In simple words the criterion is order to know the existence of unique or multiple solutions of the boundary value problem is called rule of multiplicity of solutions. The rule of so called multiplicity of solutions is applied as follows: Consider the mth-order approximate solution (3.18) and set (3.3) in it to derive the following equation

$$v(\beta) \approx V_m(\beta, \delta, \hbar) = \gamma. \tag{3.19}$$

It is noted that the above Eq. (3.19) has two unknown parameters namely δ and convergence controlling auxiliary parameter $\hbar$ in (3.18) the series solution (2.18).Since it is clear from the frame of HAM that the series solution (3.18) is convergent for $y = \beta$ for the values of $\hbar$ only in that range of $\hbar$.However δ will not change with the value of $\hbar$. Thus, implicitly, we get the unique or multiplicity of the solutions when a unique or multiple horizontal plateaus occur in the plot of δ as function of $\hbar$ in the convergence range of the series $v(\beta)$. Now we are going to test or check the multiplicity of solutions by applying HAM on three different cases as given below.

3.2 Nonlinear Heat transfer problem

3.2.1 Equation and exact solutions

Let us consider a straight fin of uniform cross-section area A and length L. At temperature T_a the surface of fin is placed in a convective environment at temperature T_a. It is further assumed that the local heat transfer coefficient h exhibit a power-law-type dependence along fin on the local temperature difference between the fin and the ambient fluid as

$$h = (T - T_a)^n, \tag{3.20}$$

where the exponent n depends on the heat transfer mode, T be the local temperature on the fin surface and a be dimensional constant. The value of n varies in the range between -4 and 5. For one dimensional steady state heat conduction equation in dimensionless form is

$$x = \frac{X}{L}, \tag{3.21}$$

and

$$h = \frac{T - T_a}{T_b - T_a}, \tag{3.22}$$

can be written as

$$\frac{d^2\theta}{dx^2} + \psi^2\theta^{n+1} = 0, \tag{3.23}$$

x can be calculated from fin tip, T_b be temperature of fin base, and ψ be convective–conductive parameter of fin and is defined as

$$h = \left(\frac{h_b P L^2}{kA}\right)^{\frac{1}{2}} = \left(\frac{aPL^2}{kA}(T_b - T_a)^n\right)^{\frac{1}{2}}.$$

In the above expression k be the conductivity of the fin, P be the periphery of fin cross-section and $h_b$the heat transfer coefficient at fin base. It is assumed that fin tip is insulated and the boundary conditions of the problem (3.23) are

$$\frac{d\theta}{dx}(0) = 0, \quad \theta(1) = 1. \tag{3.24}$$

We can show that for $-1 \leq n \leq 5$, the Eqs.(3.23) and (3.24) admits unit solution, for $-2 < n < -1$ it has both unit and dual solutions, but only dual solutions for $-4 \leq n \leq -2$. Therefore for our purpose we assume $n = -3$ so that Eq. (3.23) is transformed as

$$\frac{d^2\theta}{dx^2} - \psi^2\theta^{-2} = 0. \tag{3.25}$$

after using the transformation $\frac{d\theta}{dx} = y$, we get

$$\frac{d^2\theta}{dx^2} = \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = y\frac{dy}{d\theta}. \tag{3.26}$$

Now, Eq. (3.25) takes the form

$$y\frac{dy}{d\theta} - \psi^2\theta^{-2} = 0,$$

which is separable type first-order ordinary differential equation, which may be written in the following form

$$ydy - \psi^2\theta^{-2}d\theta = 0.$$

Applying the integral operation, we get

$$\int ydy \; - \; \psi^2 \int \theta^{-2}d\theta = 0,$$

we obtain

$$\frac{y^2}{2} - \left(\psi^2 \frac{\theta^{-1}}{-1}\right) = c.$$

by using transformation $\frac{d\theta}{dx} = y$,we get

$$\frac{1}{2}\left(\frac{d\theta}{dx}\right)^2 + \psi^2\theta^{-1} = c, \tag{3.27}$$

where c is constant of integration. Now Eq. (3.27) after using the first boundary condition (3.24), retrieves as

$$\psi^2(\theta(0))^{-1} = c,$$

where $\theta(0) = \delta$ is the temperature on fin tip at $x = 0$. The value of constant c is obtained as

$$c = \psi^2\delta^{-1}. \tag{3.28}$$

Thus

$$\frac{1}{2}\left(\frac{d\theta}{dx}\right)^2 + \psi^2\theta^{-1} = \psi^2\delta^{-1}$$

which is again a separable equation. After separating the variables again, we have

$$(d\theta)^2 = 2\psi^2(\delta^{-1} - \theta^{-1})(dx)^2.$$

Taking the square root on both sides , we get

$$dx = \frac{d\theta}{\sqrt{2\psi^2(\delta^{-1} - \theta^{-1})}}.$$

After replacing θ by τ, we get

$$dx = \frac{d\tau}{\sqrt{2\psi^2(\delta^{-1} - \tau^{-1})}} \tag{3.29}$$

and taking the integration of above expression by using Mathematica, we get

$$x=\int_{\delta}^{\theta}\frac{d\tau}{\sqrt{2\psi^2(\delta^{-1}-\tau^{-1})}}=\frac{2\sqrt{\theta(\theta-\delta)}+\delta\left(-\log(\delta)+2\log\left(1+\sqrt{1-\frac{\delta}{\theta}}\right)+\log(\theta)\right)}{2\sqrt{\frac{2\psi^2}{\delta^2}}} \tag{3.30}$$

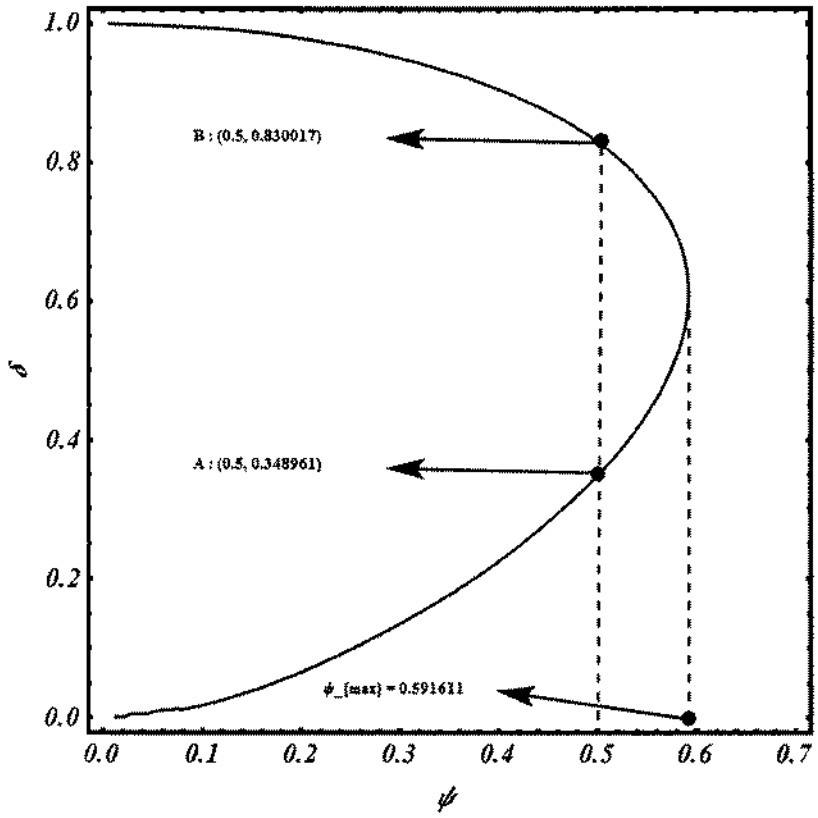

Fig.1 : Graph of δ as a functions of ψ.

The expression of the parameter δ can be obtained after using the boundary condition $\theta(1)=1$ from (3.30) as given below

$$\frac{\delta}{2\sqrt{2}\sqrt{\delta\psi^2}}\left(2\sqrt{1-\delta}+\delta(2\log(1+\sqrt{1-\delta})-\log(\delta)\right)=1. \tag{3.31}$$

In Fig. 1, δ as a function of ψ has been plotted from Eq. (3.31). In this figure, we can see that there exist two δ against ψ for $0\leq\psi\leq\psi_{\max}=0.591611$. So we can say that dual solutions occur. i.e. these values of $\delta=0.348961$ and $\delta=0.830017$ have been used to draw the exact dual solutions of Eqs. (3.25) subject to the boundary condition (3.25) as shown in Fig. A, for $\psi = 0.5$, we get $\delta = 0.348961$ and $\delta = 0.830017$ as mentioned by points A and B in Fig.1 respectively.

3.2.2 Forecasting the dual solutions by using HAM

The boundary value problem Eq. (3.25) subject to the boundary condition (3.24) and forcing condition for $\psi = 0.5$ are

$$\frac{d^2\theta}{dx^2} - 0.25\theta^{-2} = 0, \tag{3.32}$$

$$\frac{d\theta}{dx}(0) = 0, \qquad \theta(0) = \delta, \tag{3.33}$$

with additional forcing condition $\theta(1) = 1$, where δ is the temperature of the fin tip that we will determine later on. So, we apply HAM on the problems (3.32) and (3.33) as follows. we first assume a set of base functions

$$\{x^{2k} \mid k = 0, 1, 2, ...\}. \tag{3.34}$$

After choosing the initial guess $\theta_0(x) = x^2 + \delta$ satisfying the boundary conditions (3.33). Let $H(x) = 1$ and $\mathcal{L}$ be a second order operator defined as

$$\mathcal{L}[\varphi(x,\delta;p)] = \frac{\partial^2 \varphi(x,\delta;p)}{\partial x^2}, \tag{3.35}$$

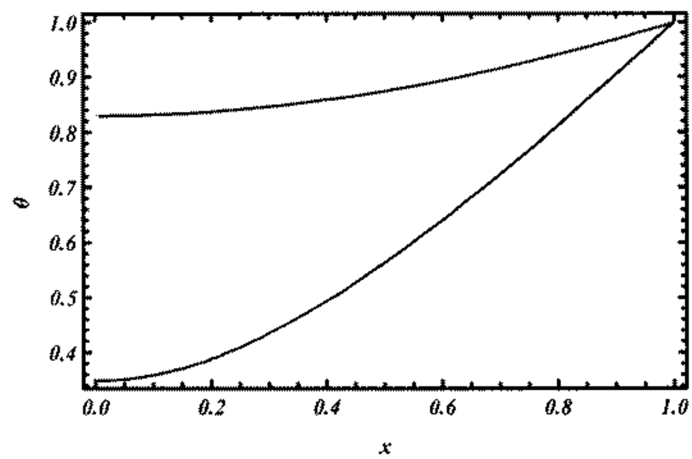

Fig. A : Exact dual functions of heat transfer Eq.(3.25) subject to conditions (3.24) which satisfies

$$\mathcal{L}[c_1 + c_2 x] = 0. \tag{3.36}$$

The mth-order deformation Eq. (3.14) after two subsequent integrations for $M \geq 1$ takes the following form

$$\theta_m(x,\delta) = x_m \theta_{m-1}(x,\delta) + \hbar \int_0^x \int_0^s R_m(\overrightarrow{\theta}_{m-1}, \tau, \delta) d\tau ds + c_1 + c_2 x, \tag{3.37}$$

where from Eqs. (3.17) and (3.32) implies that

$$R_m(\overrightarrow{\theta}_{m-1}, \tau, \delta) = \sum_{j=0}^{m-1} \theta''_{m-1-j}(x) \sum_{i=0}^{j} \theta_i(x)\theta_{j-i}(x) - 0.25(1 - \chi_m). \tag{3.38}$$

For $m = 1$, $i = 0$ and $j = 0$, we have

$$R_1(\overrightarrow{\theta}_{1-1}, \tau, \delta) = \theta''_{1-1-0}(x)\theta_0(x)\theta_{0-0}(x) - 0.25(1-0),$$

or

$$R_1(\overrightarrow{\theta}_0, \tau, \delta) = \theta''_0(x)(\theta_0(x))^2 - 0.25.$$

After dividing the above equation by $(\theta_0(x))^2$, we have

$$R_1(\overrightarrow{\theta}_0, \tau, \delta) = \theta''_0(x) - 0.25(\theta_0(x))^{-2}$$

which is almost the same as original equation. The mth-order boundary conditions are used to calculate the integration constants c_1 and c_2 are as follows

$$\theta_m(0) = 0, \qquad \theta'_m(0) = 0 \tag{3.39}$$

and thus $c_1$and c_2 become zero. In this way, we get the functions $\theta_m(0,\delta)$ from Eq. (3.37) for $m = 1, 2, 3, \ldots$ to get mth-order approximate solution.

$$\Theta_M(x, \delta, \hbar) = \sum_{m=0}^{M} \theta_m(x, \delta). \tag{3.40}$$

After using additional forcing condition $\theta_m(1) = 1$, Eq. (3.19) becomes

$$\theta(1) \approx \Theta_M(1, \delta, \hbar) = 1. \tag{3.41}$$

The mth-order solution ($M = 35$) has been calculated by using symbolic software Mathematica. The obtained solution is used in Eq. (3.41) and drawn the parameter are of δ as a function of convergence-controller parameter $\hbar$ as shown in Fig. 2. The admissible range of $\hbar$ is noted as $[-1.1, -0.1]$. It is seen that two δhorizontal ranges namely $\delta = 0.3489$ in the $\hbar$-range $[0.95, 0.3]$ and $\delta = 0.8300$ in the $\hbar$-range $[0.45, 0.25]$ has been plotted in Fig. 2. Thus, the HAM provides dual solutions satisfying the exact solutions completely.

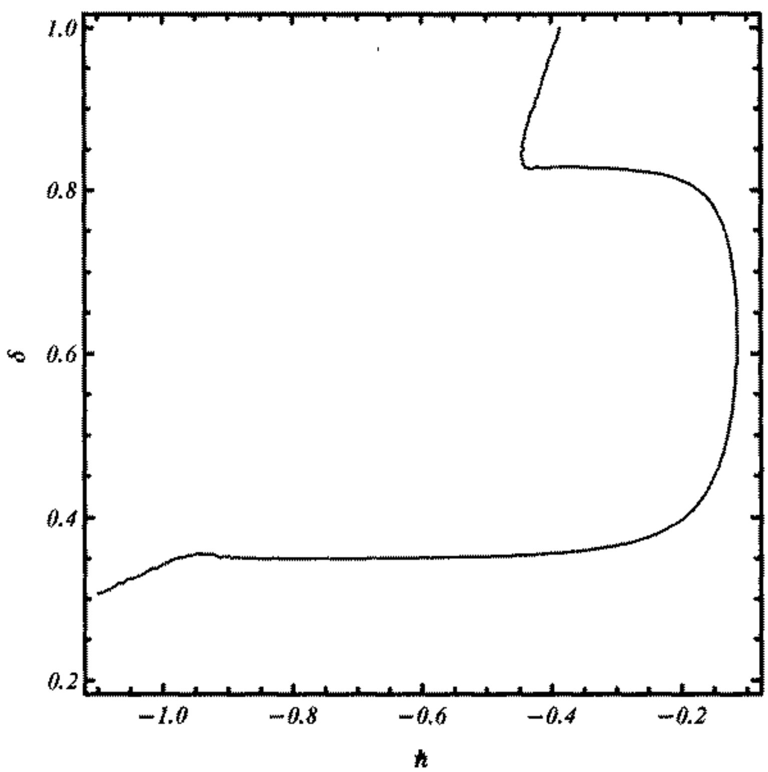

Fig.2 : graph of δ as a function of $\hbar$

3.2.3 Two branches of solution

The rule of multiplicity is a procedure to determine the number of solutions admitted by the boundary value problem. As pointed by points A and B in Fig. 1, we now are going to determine the dual solution for $\delta = 0.3489$ and $\delta = 0.8300$ explicitly and then we will compare it with the exact solutions (3.30) Both the lower branch and upper branch of solutions are determined simultaneously with various δ and $\hbar$ only by Eq. (3.40) as shown in Fig. 2.

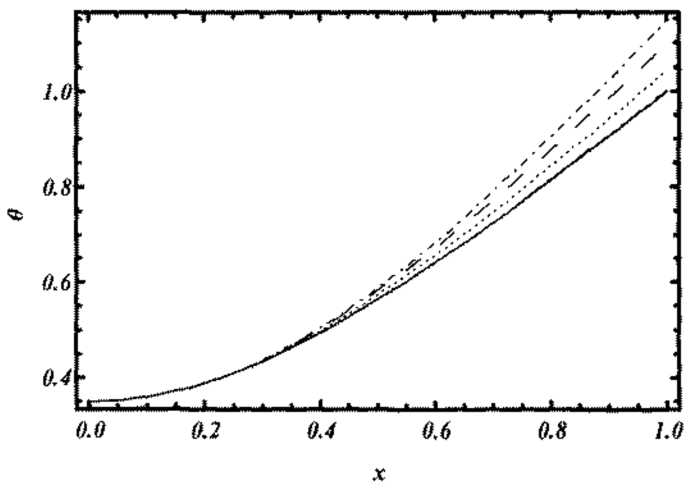

Fig.3 : Convergence of approximate lower solutions towards the exact one: $\Theta_3(x)$ – dot-dashed, $\Theta_5(x)$ – dashed and $\Theta_{10}(x)$ – dotted; and the exact lower solution - solid line

The approximate HAM solutions given by Eq. (3.40) on various values of m i.e. $\Theta_3(x, 0.3489, -0.6)$ - dotdashed line, $\Theta_5(x, 0.3489, -0.6)$- dashed line and $\Theta_{10}(x, 0.3489, -0.6)$ - dotted line where $\delta = 0.3489$ and $\hbar = -0.6$ are compared to the exact lower branch solution $\theta(x)$ as shown with solid line in Fig.3,and given by Eq. (3.30) for $\delta = 0.348961$ and $\psi = 0.5$ as plotted by point A of Fig. 1. Similarly, in Fig. 4, the approximate HAM solutions given by Eq.(3.40) on various values of m i.e. $\Theta_3(x, 0.8300, -0.4)$- dotdashed line, $\Theta_5(x, 0.8300, -0.4)$- dashed line and $\Theta_{10}(x, 0.8300, -0.4)$ - dotted line where $\delta = 0.8300$ and $\hbar = 0.4$ are compared to the exact upper branch solution $\theta(x)$ as shown with solid line given by Eq. (3.30) for $\delta = 0.830017$ and $\psi = 0.5$ as shown by point B of Fig. 1.

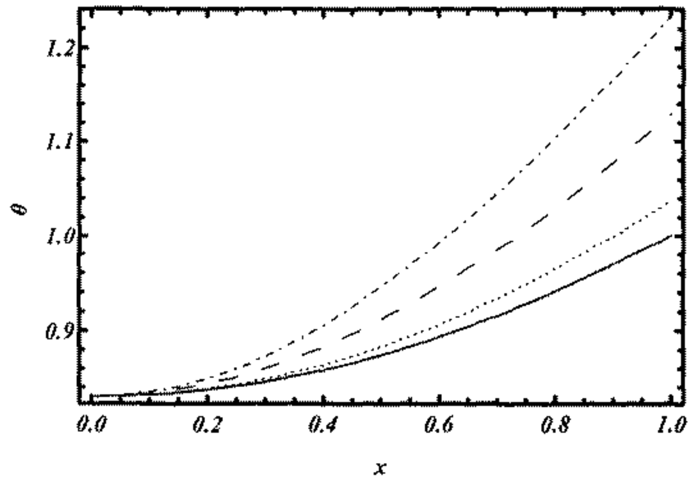

Fig.4 : Convergence of approximate upper solutions towards the exact one: $\Theta_3(x)$ – dot-dashed, $\Theta_5(x)$ – dashed and $\Theta_{10}(x)$ – dotted; and the exact upper solution - solid line

We need not to use more than one auxiliary linear operator, one initial approximation guess, and one auxiliary function used for convergence to dual solution. As we increase the order M the approximate solutions $\Theta_M(x)$ approaches the exact solution.

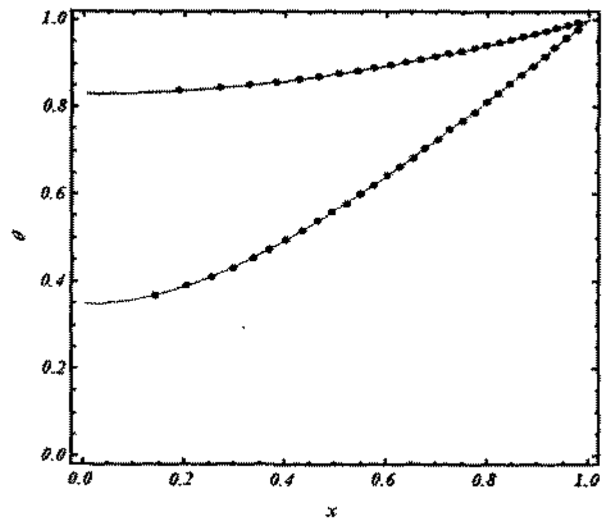

Fig.5 : Comparison of approximate dual solutions($\Theta_{35}(x, 0.8300, -0.4)$ – black bold dots and $\Theta_{35}(x, 0.3489, -0.6)$ – black bold dots) with the exact dual solutions - solid line

In Fig. 5 comparison of approximate dual HAM solutions $\Theta_M(x)$ of order $M = 35$ is shown by dotted lines with the exact dual solutions plotted with solid lines. We finally come to conclude that the HAM provides dual solutions which exactly match the exact solutions.

3.3 Strongly nonlinear Bratu equation

3.3.1 Equation and the exact solutions

The famous Bratu problem subject to the boundary conditions are as follows

$$v'' + \lambda e^v = 0 , \qquad x \in (0,1) \tag{3.42}$$

and

$$v(0) = v(1) = 0. \tag{3.43}$$

The exact solution of the above problem using reduction of order is as follows

$$v(x) - \log\left[\frac{a^2}{2\lambda\cosh^2[-\frac{a}{2}(x+b)]}\right]. \tag{3.44}$$

We Introduce the boundary conditions to calculate two arbitrary constants a, b such that from condition (3.43), we have $b = -\frac{1}{2}$ and a is

$$a^2 = 2\lambda\cosh^2\left(\frac{a}{4}\right) \tag{3.45}$$

let us consider that

$$4\alpha = a,$$

so expression (3.45) takes the following form

$$\lambda = 8\alpha\ \alpha\ \cosh^{-1}[\alpha]$$

and solution (3.44) becomes

$$v(x) = 2\log\left[\frac{\cosh\alpha}{\cosh[\alpha(1-2x)]}\right]. \tag{3.46}$$

Where α satisfies

$$\cosh\alpha = \frac{4}{\sqrt{2\lambda}}\alpha \tag{3.47}$$

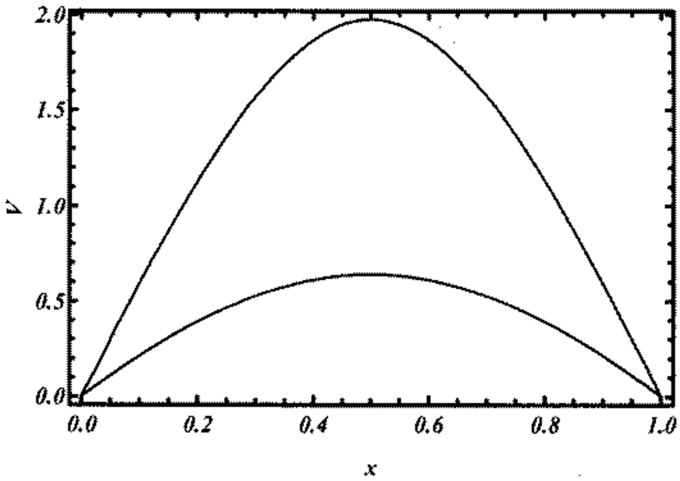

Fig. B : Exact dual functions of Bratu Eq. (3.42) given by Eq. (3.46)

This is the major task we need to accomplish. Actually we are to converge the approximate dual solutions using HAM to the Upper and Lower Branches of the exact dual solutions as given in the graph above. The dependence of α as function of λ is shown in Fig. 6. It is noted that there is no solution for $\lambda > \lambda_{\max}$, only one solution for the case $\lambda = \lambda_{\max}$ and dual solutions for $\lambda < \lambda_{\max}$. Let us consider the value of $\lambda = 3$ for discussion. We noted that two points C and D namely $\alpha = 0.84338$ and $\alpha = 1.64414$ exist against $\lambda = 3$, here two solutions exist for $\lambda = 3$. Differentiating (3.46) w.r.t. 'x' and setting $x = 0$, we get

$$v'(0) = 4\alpha\tanh\alpha \tag{3.48}$$

In Fig. 7 we have plotted $v'(0)$ as function of α from Eq. (3.48). Thus we have dual solutions as pointed (E, G) i.e. $v'(0) = 2.3196$ for the first solution and $v'(0) = 6.1034$ for the second solution.

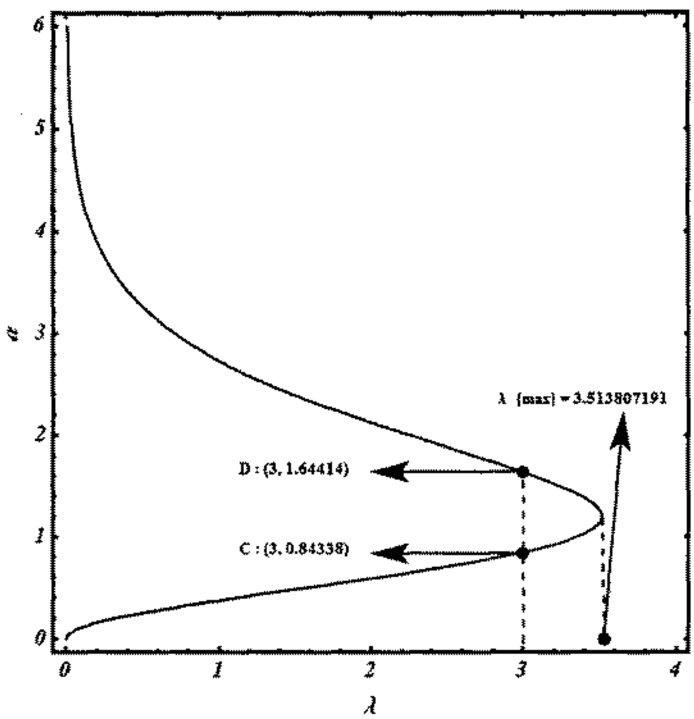

Fig.6 : Graph of α as function of λ

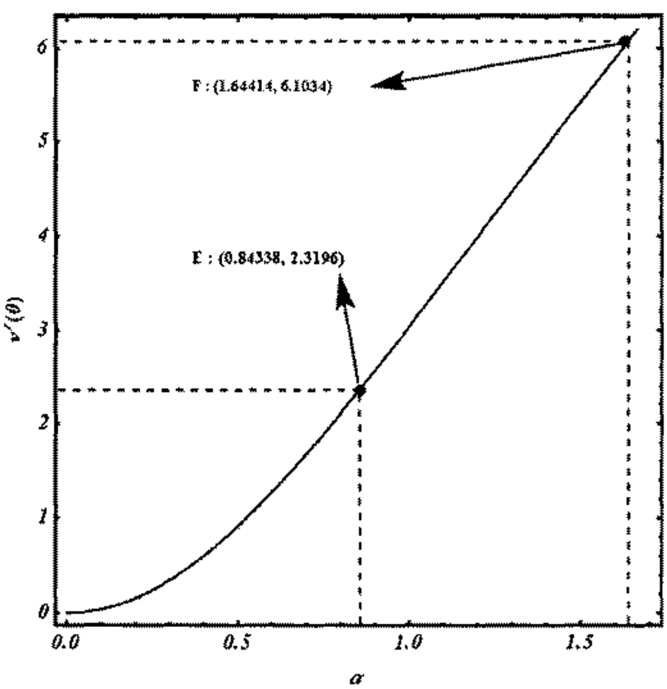

Fig.7 : graph of $v'(0)$ as function of α given in expression (3.48)

3.3.2 Forecasting the dual solutions by using HAM

Assuming $\lambda = 3$, we can reduce two-point boundary value problem given in Eqs. (3.42) and (3.43) to an initial value problem as follows

$$v^{''} + 3e^{v} = 0, \qquad x \in (0,1) \tag{3.49}$$

$$v(0) = 0, \quad v^{'}(0) = \delta, \tag{3.50}$$

with an additional forcing condition

$$v(1) = 0. \tag{3.51}$$

We can transform the above problems $(3.49 - 3.51)$ equivalently to new ones by assuming

$$f(x) = e^{-v(x)}, \tag{3.52}$$

taking log on both sides, we get

$$v(x) = -\log[f(x)].$$

Taking the derivative of above expression w.r.t. 'x', we get

$$v'(x) = -\frac{f'(x)}{f(x)},$$

again differentiating above expression w.r.t. 'x', we get

$$v''(x) = \frac{[f'(x)]^2 - f(x)\ f''(x)}{f^2(x)}.$$

$$f(x)f^{''}(x) - [f^{'}(x)]^2 - 3f(x) = 0, \qquad x \in (0,1) \tag{3.53}$$

From Eq. (3.52)

$$f^{'}(x) = -v^{'}(x)e^{-v(x)}$$

so, boundary conditions (3.50) now takes the following form

$$f(0) = 1, \quad f^{'}(0) = \gamma = -\delta, \tag{3.54}$$

with additional forcing condition by Eq. (3.52) is

$$f(1) = 1. \tag{3.55}$$

Now, We apply HAM on the problems (3.53) – (3.55) to determine the parameter γ. For this, we first take the set of base functions

$$\{x^k \mid k = 0, 1, 2, ...\}, \tag{3.56}$$

and choose the auxiliary function $H(x) = 1$, $f_0(x) = \gamma x + 1$ as initial guess of solution $f(x)$, and $\mathcal{L}$ to be second order linear operator of the form

$$\mathcal{L}[\varphi(x, \gamma; p)] = \frac{\partial^2 \varphi(x, \gamma; p)}{\partial x^2}, \tag{3.57}$$

which satisfies

$$\mathcal{L}[c_1 + c_2 x] = 0. \tag{3.58}$$

After two consecutive integrations for $M \geq 1$ by using equation (3.14), we arrive at

$$f_m(x, \gamma) = \chi_m f_{m-1}(x, \gamma) + \hbar \int_0^x \int_0^s R_m(\overrightarrow{f}_{m-1}, \tau, \gamma) d\tau ds + c_1 + c_2 x \tag{3.59}$$

where

$$R_m(\overrightarrow{f}_{m-1}, \tau, \gamma) = \sum_{j=0}^{m-1} f''_{m-1-j}(x) f_j(x) - \sum_{j=0}^{m-1} f'_{m-1-j}(x) f'_j(x) - 3 f_{m-1}(x). \tag{3.60}$$

For $m = 1$ and $j = 0$, we have

$$R_1(\overrightarrow{f}_0, \tau, \gamma) = f''_0(x) f_0(x) - [f'_0(x)]^2 - 3 f_0(x)$$

which is almost the same as original equation (3.53). After using the mth-order boundary conditions

$$f_m(0) = 0, \qquad f'_m(0) = 0, \tag{3.61}$$

the values of unknown constants c_1 and c_2 become zero. Finally, the mth-order approximate solution can be obtained from

$$F_M(x,\gamma,\hbar) = \sum_{m=0}^{M} f_m(x,\gamma). \tag{3.62}$$

Applying the forcing condition $f(1) = 1$, Eq. (3.19) becomes

$$f(1) \approx F_M(1,\gamma,\hbar) = 1. \tag{3.63}$$

$M = 40th$ order solution have been computed using Mathematica. Expression (3.40) at $M = 40$, in which γ as a function of convergence-controller parameter $\hbar$ in the $\hbar$-range $[2,0]$ has been plotted in Fig. 8. We can identify two γ-plateaus i.e. $\gamma = -6.1034$ or $\delta = (6.1034)$ in the $\hbar$-range $[-0.6,-0.4]$ and $\gamma = -2.3196$ or $(\delta = 2.3196)$ in the $\hbar$-range $[-0.8,-0.3]$. Here exist two solutions for $\lambda = 3$ i.e. $v^{'}(x) = \delta = -\gamma = 2.3196$ for the first solution and $v^{'}(x) = \delta = -\gamma = 6.1034$ for the second solution as shown in Figs. 6 and 7. Thus, we finally conclude that the HAM provides dual solutions satisfying the exact results.

3.3.3 Two branches of solution

To determine the dual solutions for $\delta = 2.3196$ and $\delta = 6.1034$ explicitly as mentioned with point pairs C and D of Fig. 6. The mth-order approximate solution by HAM by Eqs. (3.52) and (3.62) is given as

$$V_M(x,\delta,\hbar) = -\log[F_M(x,\gamma,\hbar)]. \tag{3.64}$$

Both the lower branch and upper branch of solutions are determined simultaneously with various δ and $\hbar$ only by Eq. (3.64).Here again, we need not to use more than one auxiliary linear operator, one initial approximation guess, and one auxiliary function used to convergence to dual solutions.

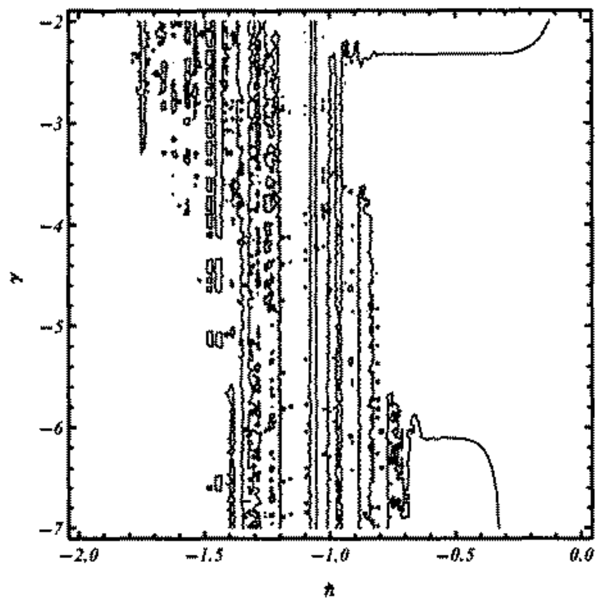

Fig.8 : The graph of γ as function of $\hbar$

In Fig. 9 the approximate HAM solutions given by Eq. (3.64) on various values of m i.e. $V_5(x, 2.3196, -0.5)$- dotdashed line, $V_8(x, 2.3196, -0.5)$- dashed line and $V_{10}(x, 2.3196, -0.5)$- dotted line where $\delta = 2.3196$ and $\hbar = -0.5$ are compared to the exact lower branch solution $v(x)$ shown by solid line given by Eq. (3.46) for $\alpha = 0.84338$ as mentioned by point C in Fig. 6.

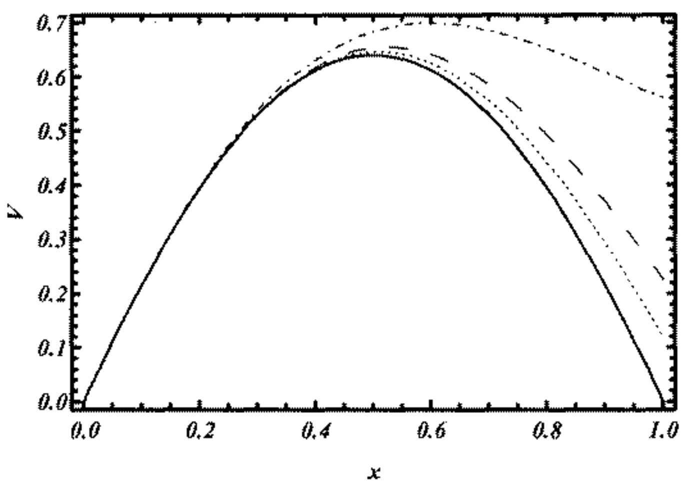

Fig.9 : Convergence of approximate lower solutions towards the exact one: $V_5(x)$ – dot-dashed, $V_8(x)$ – dashed and $V_{10}(x)$ – dotted; and the exact lower solution - solid line

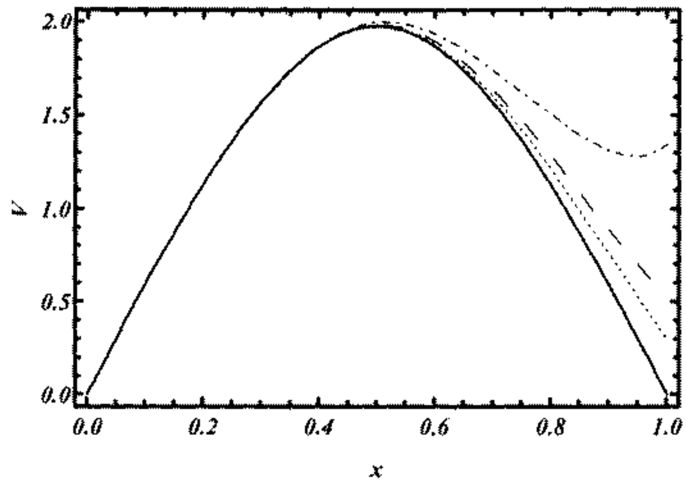

Fig.10 : Convergence of approximate upper solutions towards the exact one: $V_{17}(x)$ – dot-dashed, $V_{20}(x)$ – dashed and $V_{22}(x)$ – dotted; and the exact upper solution - solid line

Similarly, in Fig. 10 the approximate HAM solutions on different values of m i.e. $V_{17}(x, 6.1034, -0.5)$ - dotdashed line, $V_{20}(x, 6.1034, -0.5)$ - dashed line and $V_{22}(x, 6.1034, -0.5)$ - dotted line where $\delta = 6.1034$ and $\hbar = -0.5$ are compared to the exact upper branch solution $v(x)$ shown by solid line given by Eq. (3.46) for $\alpha = 1.64414$ as mentioned by point D in Fig. 6. It is observed that as we increase the order M the approximate solutions $\Theta_M(x)$ approaches the exact solution.

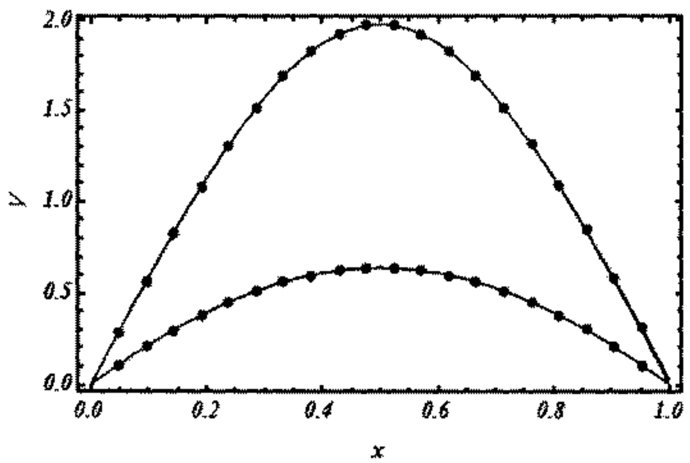

Fig.11 : Comparison of approximate dual solutions($V_{20}(x, 2.3196, -0.5)$ – black bold dots and $V_{30}(x, 6.1034, -0.5)$ – black bold dots) with the exact dual solutions - solid line

In Fig. 11, the comparison of dual solutions obtained by homotopy analysis method of order $M = 20$ and $M = 30$ and the exact solution is presented. In this figure; solid line represents the exact solution and dot represent the solutions calculated by HAM. Hence, we finally conclude that the HAM provides us dual solutions in a convenient way accurately.

3.4 Nonlinear reaction-diffusion model

3.4.1 Equation and the exact solutions

Let us consider here a special case of that problem of nonlinear reaction-diffusion model which is already discussed in previous chapter, when the model has -0.75 for reaction-order and 0.8 for Thiele modulus, as follows

$$v^{''} v^{0.75} - 0.64 = 0, \tag{3.65}$$

with boundary conditions

$$v^{'}(0) = 0, \qquad v(1) = 1. \tag{3.66}$$

The primes denote differentiation w.r.t. 'x', where $0 \leq x \leq 1$ and v is the dimensionless concentration of the reactant. Eqs. (3.65) and (3.66) satisfy the following solutions for different values of $v(x)$, the upper solution takes the form

$$22.6539x = 2\sqrt{2 - 2\left(\frac{0.1836}{v}\right)^{\frac{1}{4}}} \left(5 - 6\left(\frac{0.1836}{v}\right)^{\frac{1}{4}} + 8\sqrt{\frac{0.8136}{v}} + 16\left(\frac{0.1836}{v}\right)^{\frac{3}{4}}\right)\left(\frac{0.1836}{v}\right)^{\frac{1}{8}} v \tag{3.67}$$

in which the lower solution takes the form

$$25.8821x = 2\sqrt{2 - 2\left(\frac{0.5330}{v}\right)^{\frac{1}{4}}} \left(5 + 6\left(\frac{0.5330}{v}\right)^{\frac{1}{4}} + 8\sqrt{\frac{0.5330}{v}} + 16\left(\frac{0.5330}{v}\right)^{\frac{3}{4}}\right)\left(\frac{0.5330}{v}\right)^{\frac{1}{8}} v \tag{3.68}$$

the above two solutions are plotted in *Fig.C.* as given below. From the lower and upper branch solution from Eqs. (3.67) and (3.68) respectively, one can find

$$v(0) = 0.1836 \tag{3.69}$$

and

$$v(0) = 0.5330 \tag{3.70}$$

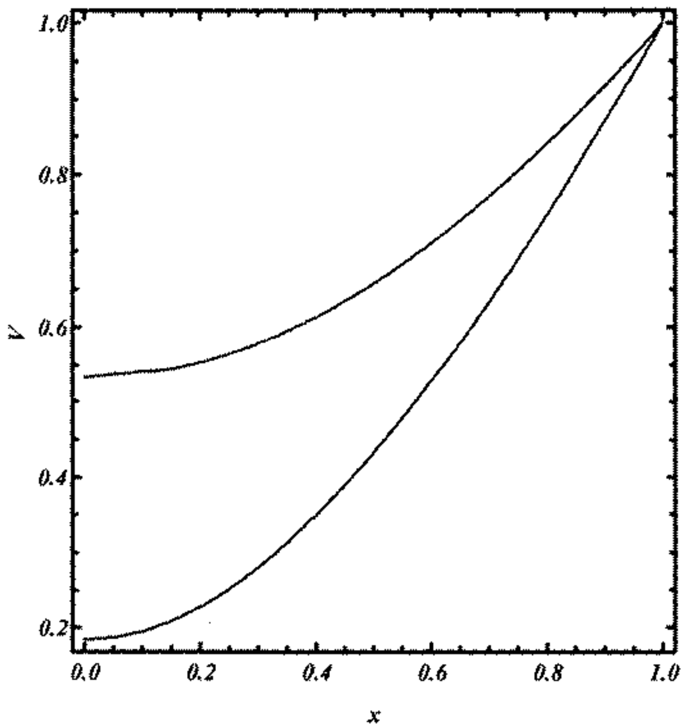

Fig.C : The graph of Exact dual functions of diffusion reaction

3.4.2 Forecasting the dual solutions by using HAM

Consider the following problem

$$v^{''}v^{0.75} - 0.64 = 0 \tag{3.71}$$

subject to the boundary conditions

$$v^{'}(0) = 0, \quad v(0) = \delta, \tag{3.72}$$

with additional forcing condition

$$v(1) = 1. \tag{3.73}$$

Where δ is an embedded parameter to be determined. Now, we apply HAM on (3.71) and (3.72) as follows. For this we first take a set of base functions

$$\{x^{2k} \mid k = 0, 1, 2, ...\}, \tag{3.74}$$

We choose auxiliary function $H(x) = 1$, initial guess $v_0(x) = \delta$, and choose linear operator $\mathcal{L}$ of second order as follows

$$\mathcal{L}[\varphi(x, \gamma; p)] = \frac{\partial^2 \varphi(x, \gamma; p)}{\partial x^2}, \tag{3.75}$$

which satisfies the property

$$\mathcal{L}[c_1 + c_2 x] = 0. \tag{3.76}$$

The mth-order deformation equation becomes, $M \geq 1$

$$\mathcal{L}[v_m(x, \delta) - \chi_m v_{m-1}(x, \delta)] = \hbar R_m(\overrightarrow{v}_{m-1}, x, \delta). \tag{3.77}$$

In which

$$R_m(\overrightarrow{v}_{m-1}, x, \delta) = \sum_{j=o}^{m-1} v''_{m-1-j}(x) u_j(x) - 0.64(1 - \chi_m), \tag{3.78}$$

which is different from the traditional problem. When $m = 1$ and $j = 0$, we have

$$R_0(\overrightarrow{v}_0, x, \delta) = v''_0(x) u_0(x),$$

where

$$u_0(x) = [v_0(x)]^{0.75}.$$

Which now has becomes the original equation (3.65)

$$R_0 = v'' v^{0.75} - 0.64,$$

and $u_n(x)$ is found by another additional Adomian polynomial as follows

$$u_n(x) = \frac{1}{n!} \frac{\partial^n [\varphi(x, \gamma; q)]^{0.75}}{\partial q^n}\bigg|_{q=0} = \frac{1}{n!} \frac{\partial^n [\sum_{k=0}^{k=+\infty} v_k(x, \delta) q^k]^{0.75}}{\partial q^n}\bigg|_{q=0}, \tag{3.79}$$

which for different values of $n = 0, 1, 2, \ldots$ implies that

$$u_0(x) = [v_0(x)]^{0.75}, \tag{3.80}$$

$$u_1(x) = \frac{0.75 v_1(x)}{[v_0(x)]^{1.25}}, \tag{3.81}$$

$$u_2(x) = -\frac{0.09375[v_1(x)]^2}{[v_0(x)]^{1.25}} + \frac{0.75 v_2(x)}{[v_0(x)]^{0.25}}. \tag{3.82}$$

With initial conditions, the high-order deformation equation (3.77) becomes

$$v_m(0) = 0, \quad v_m'(0) = 0, \tag{3.83}$$

starting from $v_0(x, \delta) = \delta$. We, successively, can find the functions $v_m(x, \delta)$ for $m = 1, 2, 3, \ldots$ and mth-order approximate solution

$$V_M(x, \delta, \hbar) = \sum_{m=0}^{M} v_m(x, \delta). \tag{3.84}$$

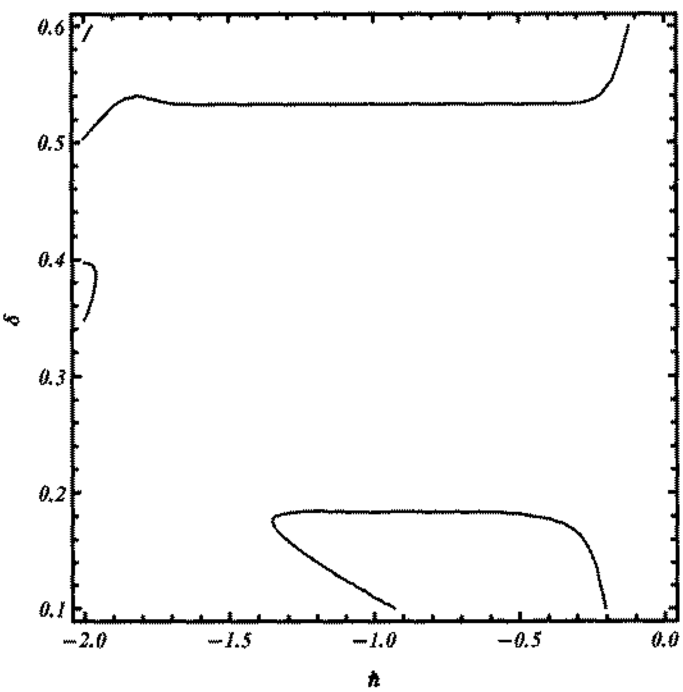

Fig.12 : The graph of δ as function of $\hbar$ by using $25th$ order HAM solution

Thus, when $v(1) = 1$, Eq. (3.84) takes the following form

$$v(1) \approx V_M(x, \delta, \hbar) = 1. \tag{3.85}$$

We compute the solutions upto $25th$-order of approximation by using symbolic software Mathematica. Then using Eq. (3.85), δ as function of $\hbar$ is drawn in Fig. 12 for the $\hbar$-range $[-2, 0]$. Two δ-plateaus namely $\delta = 0.1836$ in $\hbar$-range$[1.3, 0.4]$ and $\delta = 0.5330$ in $\hbar$-range $[1.7, 0.3]$ can be seen in the Figure, which clearly exhibits the reasons for the existence of dual solutions.

3.4.3 Two branches of solution

In Fig. 13, the approximate HAM solutions given by Eq. (2.84) on various values of m i.e. $V_3(x, 0.1836, -0.8)$- dotdashed line,, $V_5(x, 0.1836, -0.8)$- dashed line and $V_7(x, 0.1836, -0.8)$ - dotted line where $\delta = 0.1836$ and $\hbar = -0.8$ are compared to the exact lower branch solution $v(x)$ shown by solid line given by Eq. (2.67).

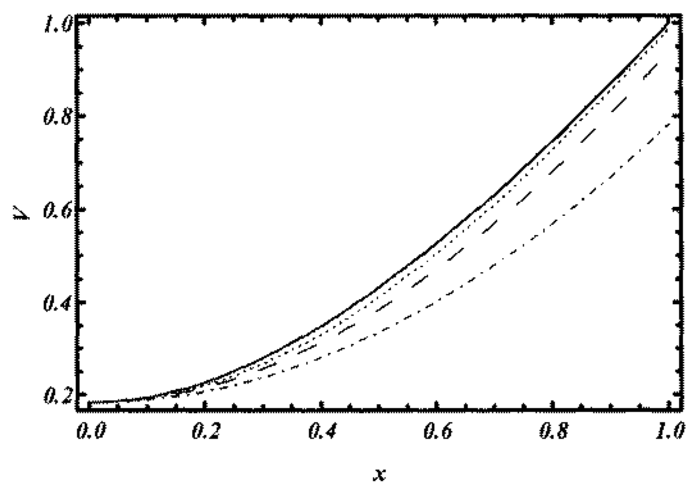

Fig. 13 : Convergence of approximate lower solutions towards the exact one: $V_3(x)$ – dot-dashed, $V_5(x)$ – dashed and $V_7(x)$ – dotted; and the exact lower solution - solid line

Similarly, in Fig. 14 the approximate HAM solutions on different values of m i.e. $V_0(x, 0.5330, -1)$ $= 0.5330$ - dotdashed line, $V_1(x, 0.5330, -1)$- dashed line and $V_2(x, 0.5330, -1)$ - dotted line where $\delta = 0.5330$ and $\hbar = -1$are compared to the exact upper branch solution $v(x)$ shown by solid line given by Eq. (3.68). As we increase the order M the approximate solutions $\Theta_M(x)$

approaches the exact solution. In Fig. 15 comparison of approximate dual HAM solutions $V_M(x)$ of order $M = 15$ with the exact solutions presented.

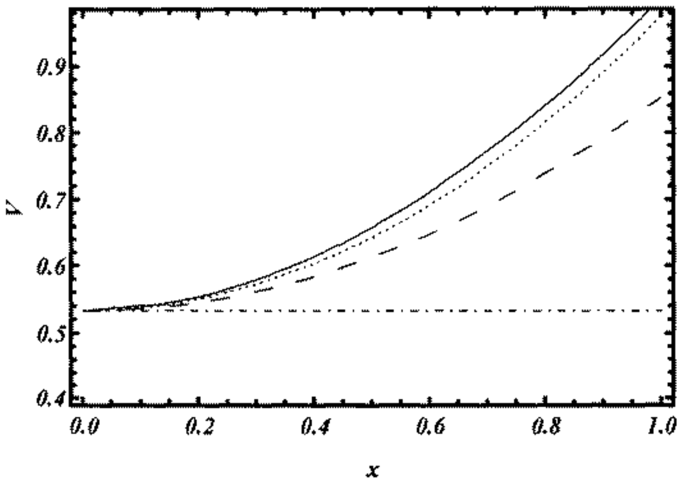

Fig.14 : Convergence of approximate upper solutions towards the exact one: $V_0(x)$ – dot-dashed, $V_1(x)$ – dashed and $V_2(x)$ – dotted; and the exact upper solution - solid line

The dotted line shows the approximate, while the solid line gives exact dual solutions as shown in Fig. 15. We finally conclude that the HAM provides dual solutions satisfying the exact solutions exactly.

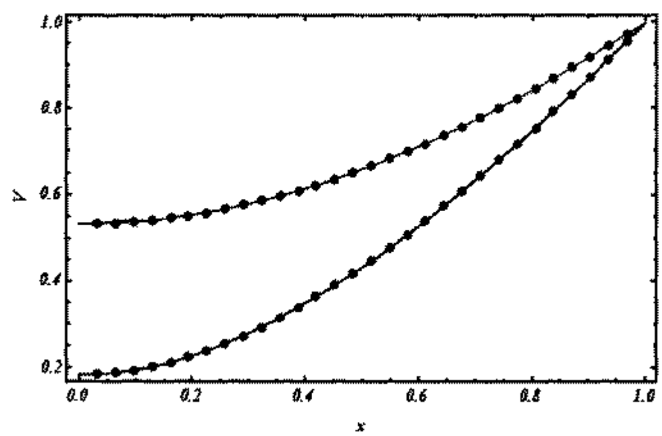

Fig.15 : Comparison of approximate dual solutions($V_{15}(x, 0.1836, -0.8)$ – black bold dots and $V_{15}(x, 0.5330, -1)$ – black bold dots) with the exact dual solutions - solid line

3.5 Conclusions

Without loss of generality it really is a very difficult task to forecast the multiplicity of solutions for a given nonlinear boundary value problem. So, perhaps for the first time, the rule of multiplicity of solutions have introduced for this purpose. We successfully revisited the applicability of this procedure by its applications to different important boundary value problems from the field of science and engineering which admit multiple solutions. Here the dual solutions of nonlinear equations such as nonlinear heat transfer equation, strongly nonlinear Bratu equation and nonlinear reaction-diffusion model in porous catalysts have been predicted and calculated by using homtopy analysis method. It is importantly mentioned that, we need not to use more than one initial guess, one auxiliary function, one auxiliary linear operator to find all the branches of the solutions.

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