FIXED POINTS OF SINGLE AND MULTIVALUED MAPPINGS

By

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DEPARTMENT OF MATHEMATICS

INTERNATIONAL ISLAMIC UNIVERSITY

ISLAMABAD, PAKISTAN

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***DOCTOR OF PHILOSOPHY IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD, PAKISTAN

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CERTIFICATE

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We accept this thesis as confirming to the required standard

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2010

DECLARATION

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DEDICATED TO....

My parents, teachers, friends and family for supporting and encouraging me.

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CONTENTS

Preface

"*Fixed point theorems deals with the assurance that a mapping T on a set X has one or more fixed points, i.e., the functional equation $x = Tx$ has one or more solutions. A large variety of the problems of analysis and applied mathematics relate to finding solutions of nonlinear functional equations which can be formulated in terms of finding the fixed points of a nonlinear mappings. In fact, fixed point theorems are extremely substantial tools for proving the existence and uniqueness of the solutions to various mathematical models (differential, integral and partial differential equations and variational inequalities etc.) exhibiting phenomena arising in broad spectrum of fields, such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, financial analysis, epidemics, biomedical research and flow of fluids etc. They are also used to study the problems of optimal control related to these systems. Fixed point theorems concerning ordered Banach spaces help us in finding exact or approximate solutions of boundary value problems, for details see Amann [8], Belluce and Kirk [35], Franklin [66], Karamardian [101], Lakshmikantham [107], Lions [111], Martin [112], Pathak and Shahzad [129], Robinson [148], Smart [157], Swaminathan [161], Tartar [162] and Waltman [172] etc.*

The Banach fixed point theorem is commonly known as Banach contraction principle, which states that if X is a complete metric space and T a single valued contractive self mapping on X, then T has a unique fixed point in X. This theorem looks simple but plays a fundamental role in the field of fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer.

For single valued self mappings, a general existence theory of fixed points was constructed over the period of decades (associated with the names of Brouwer, Browder, Fan, Lofschefz, Schauder, Tychonoff, and others). Afterwards, Agarwal [4], Edelstein [52, 53], Fisher [61, 62, 63, 64], Jungck [89], Kannan [99], Kirk [103], Lakshmikantham [107], Rhoades [145], Wong [176, 177] and many others proved remarkable fixed point theorems. In 1963, Ghaler [68], generalized the idea of metric space and introduced 2-metric space which was followed by a number of papers dealing with this generalized space. A plenty of material is available in other generalized metric spaces, such as, semi metric spaces, Quasi semi metric spaces and D-metric *spaces. Huang and Zhang [78] introduced the concept of cone metric spaces in 2007, by replacing the set of real numbers with a Banach space. Abbas and Jungck [1], Abbas and Rhoades [2], Raja and Vaezpour [134], Rezapour and Hamlbarani [141] and Vetro [166] proved fixed point theorems in cone metric spaces.*

A multivalued function is a set valued function. In the last thirty years, the theory of multivalued functions has advanced in a variety of ways. In 1969, The systematic study of Banach type fixed theorems of multivalued mappings had been started with the work of Nadler [118], who proved that a multivalued contractive mapping of a complete metric space X into the family of closed bounded subsets of X has a fixed point. He also established that every $(\varepsilon, \lambda)-$uniformly locally contractive mappings of an $\varepsilon-$chainable metric space X into the family of compact subsets of X has a fixed point. His findings were followed by Aubin and Siegel [13], Beg and Azam [30], Hu [76], Hussain and Tarafdar [80], Itoh and Takahashi [85], Kaneko [97], Massa [113] and Rhoades [142] and many others.

In 1965, Zadeh [178] introduced the notion of a fuzzy subset of a (usual) set as a method for representing uncertainty. Fuzzy set theory was mathematically formulated by the assumption that classical sets were not appropriate or natural in describing the real life problems. Fuzzy set theory has greater richness and scope in applications than the ordinary set theory. The field grew enormously; finding applications in areas as diverse as economics, engineering, information technology, defence, medical etc., for more details one can see Dubois and Prade [50], Li and Yen [110], Nguyen and Walker [120], Pedrycz and Gomide [130] and Zimmermann [180]. After the discovery of fuzzy sets, a lot of importance has been given in extending the fundamental concepts of classical analysis and thus developing the fuzzy fixed point theory. Albrycht and Maltoka [5], Beg [32], Butnariu [38], Heilpern [73], Papageorgiou [123] and Tsiporkova-Hristoskova, De Bates and Kerre [163, 164, 165] have investigated several properties of fuzzy multivalued functions and established some basic concepts. This dissertation consists of three chapters. Each chapter begins with a brief introduction which acts as a summery to the material there in.

Chapter 1 is a survey aimed at clarifying the terminology to be used and recalls basic definitions and facts.

Chapter 2 is devoted to study the results regarding the coincidence and common fixed points of mappings satisfying generalized contractive conditions. Some fixed point theorems have been *established in the frame work of cone metric spaces with normality and without normality. As an application, we prove an existence theorem for the common solutions of two Urysohn integral equations. Moreover, we initiate the study of rectangular cone metric spaces and prove Banach contraction principle.*

Chapter 3 deals with the multivalued and fuzzy set valued contractive mappings. A theorem on common fixed points of a sequence of multivalued locally contractive mappings in a $\varepsilon-$*chainable metric space is also established. We extend the results of Edelstein [52] for contractive and locally contractive mappings to fuzzy contractive and fuzzy locally contractive mappings. We investigate the existence of fixed points of fuzzy mappings under* $\varphi-$*contraction conditions on a metric space with* $d_{\infty}-$*metric on the family of fuzzy sets. We improve and rectify a significant fixed point theorem for fuzzy mappings due to Vijayaraju and Marudai [169].*

Muhammad Arshad

January, 2010

Islamabad, Pakistan."

Chapter 1

Preliminaries

"The aim of this chapter is to present some basic concepts and to explain the terminology used throughout this dissertation. Some previously known results are given without proof. Section 1.1 is concerned with the introduction of single valued and multivalued contractions. Section 1.2 is devoted to the introductory material on the notions of commuting and compatible single valued and multivalued mappings. In Section 1.3, we present the concept of cone metric spaces which is a natural generalization of metric spaces. Section 1.4, introduces the basic concepts related to fuzzy mappings."

1.1 *Contraction mappings*

"The contraction mappings are a special type of uniformly continuous functions defined on a metric space. Fixed point results for such mappings play an important role in analysis and applied mathematics."

1.1.1 *Definition [1]*

"A point $x \in X$ is said to be a fixed point of the mapping $T : X \longrightarrow X$ if image Tx coincides with x (i.e., $Tx = x$). A point $x \in X$ is said to be common fixed point of the pair (S, T) of self-mappings on X if $Sx = Tx = x$. A point $x \in X$ is said to be coincidence point of the pair (S, T) if $Sx = Tx$. A point $y \in X$ is called a point of coincidence of the pair (S, T) if there exists a point $x \in X$ such that $y = Sx = Tx$."

1.1.2 *Definition*

"*Let* (X, d) *be a metric space. A mapping* $T : X \longrightarrow X$ *is called a contraction (or Banach contraction) on* X, *if there is a positive real number* $0 < \lambda < 1$, *such that for all* $x, y \in X$,

$$d(Tx, Ty) \leq \lambda d(x, y).$$

The mapping T *is called contractive (Edelstein contractive) if*

$$d(Tx, Ty) < d(x, y) \text{ for } x \neq y, \; x, y \in X.$$

T *is called non-expansive if for all* $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y).$$

T *is called expansive mapping if*

$$d(Tx, Ty) \geq \eta d(x, y), \text{ for all } x, y \in X \text{ where } \eta > 1.$$

The concept of multivalued mappings has proven to be useful for generalizing in the context of metric fixed point theory (see [13, 15, 28, 30, 44, 47, 76, 80, 91, 93, 97, 98])."

1.1.3 *Definition*

"*Let* X *be a nonempty set. Then* $T : X \longrightarrow 2^X$ *is called multivalued mapping. A point* $x \in X$ *is said to be a fixed point of* T *if* $x \in Tx$. *A point* $x \in X$ *is said to be a coincidence point of a pair of multivalued mappings* (T, S) *if* $Tx \cap Sx \neq \emptyset$ *and* x *is called a common fixed point of the pair* (T, S) *if* $x \in Tx \cap Sx$.

Let (X, d) *be a metric space and*

$2^X = \{A : A$ *is nonempty subset of* $X\}$;

$CB(X) = \{A : A$ *is nonempty closed and bounded subset of* $X\}$;

$C(X) = \{A : A$ *is nonempty compact subset of* $X\}$.

In order to make the family $CB(X)$ *into metric space, we need to have a measure of "dis-*

tance" between two sets A and B of $CB(X)$. One such notion of distance is

$$d(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}.$$

This definition fails to discriminate sufficiently between sets. We would like the distance between two sets to be zero only if the two sets are the same, both in shape and position. For this purpose, the following concept is useful (cf., [102])."

1.1.4 *Definition*

"Let (X,d) be a metric space. For $A, B \in CB(X)$ and $\varepsilon > 0$ the sets $N(\varepsilon, A)$ and $E_{A,B}$ are defined as follows:

$$N(\varepsilon, A) = \{x \in X : d(x,A) < \varepsilon\},$$

$$E_{A,B} = \{\varepsilon : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A)\},$$

where $d(x,A) = \inf\{d(x,y) : y \in A\}$. The distance function H on $CB(X)$ induced by d is defined as

$$H(A,B) = \inf E_{A,B},$$

which is known as Hausdorff metric on X."

1.1.5 *Definition [75]*

"Let (X,d) be a metric space and a sequence $\{A_n\}$ in $CB(X)$ is said to converge to a set A if $\lim_{n \longrightarrow \infty} H(A_n, A) = 0$. A sequence $\{A_n\}$ in $CB(X)$ is said to be a Cauchy sequence if $H(A_n, A_m) \longrightarrow 0$ as $n,\ m \longrightarrow \infty$."

1.1.6 *Remark [13]*

"The completeness of (X,d) implies that $(CB(X), H)$ is complete."

1.2 *Commuting and Compatible mappings*

"*Sessa [152] generalized the concept of commuting mappings as follows:*"

1.2.1 *Definition*

"*Let* (X, d) *be a metric space then two mappings* $f, g : X \longrightarrow X$ *are said to be weakly commuting if* $d(fgx, gfx) \leq d(fx, gx)$, *for all* $x \in X$."

1.2.2 *Remark*

"*Definitely, commuting mappings are weakly commuting but the converse is not true in general (see [152]). Many authors obtained nice fixed point theorems utilizing this concept. However, since elementary functions as similar as* $fx = x^3$, $gx = 2x^3$ *are not weakly commutative. Jungck [87] introduced a less restrictive concept of compatible mappings. He also pointed out in [88, 89] the potential of compatible mappings for generalized fixed point theorems.*"

1.2.3 *Definition [87]*

"*Mappings* $f, g : X \longrightarrow X$ *are said to be compatible if, whenever there is a sequence* $\{x_n\} \subset X$ *satisfying* $\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_n = u$, *then* $\lim_{n \longrightarrow \infty} d(fgx_n, gfx_n) = 0$."

1.2.4 *Definition [30]*

"*A single valued mapping* $f : X \longrightarrow X$ *is compatible with multivalued mapping* $T : X \longrightarrow CB(X)$, *if and only if* $fTx \in CB(X)$ *for all* $x \in X$ *and* $H(fTx_n, Tfx_n) \longrightarrow 0$ *whenever* x_n *is a sequence in* X *such that* $Tx_n \longrightarrow M \in CB(X)$ *and* $fx_n \longrightarrow t \in M$."

1.2.5 *Definition [91]*

"*A pair* (f, T) *of self-mappings on* X *are said to be weakly compatible if they commute at their coincidence point (i.e.* $fTx = Tfx$ *whenever* $fx = Tx$ *).*

Junck [87] improved the Banach contraction principle for commuting mappings as follows:"

1.2.6 *Theorem [87]*

"Let (X,d) be a complete metric space and $f,g : X \longrightarrow X$ be two commuting mappings. If there exists a constant α, $0 \leq \alpha < 1$, such that $gX \subseteq fX$, $d(gx,gy) \leq \alpha d(fx,fy)$, then f and g have a unique common fixed point."

1.3 *Cone metric spaces*

1.3.1 *Definition [78]*

"Let E be a real Banach space and P be a subset of E. Then P is called an ordered cone, whenever

(i) P is non-empty, closed and $P \neq \{0\}$;

(ii) For all $a,\ b \geq 0 \implies ax + by \in P$ for all $x,\ y \in E$;

(iii) $P \cap (-P) = \{0\}$.

For a given ordered cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in intP$, where $intP$ denotes the interior of P. The ordered cone P is called normal if there is a number $\kappa \geq 1$ such that for all $x,\ y, \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq \kappa \|y\|. \tag{1.1}$$

The least number κ satisfying (1.1) is called the normal constant of P. For details we refer [78, 141].

Through this chapter, we always suppose that E is a real Banach space and P is an ordered cone in E with $intP \neq \emptyset$ and $\leqslant$ is a partial ordering with respect to P. For the sake of simplicity we will be calling P to be a cone instead of an ordered cone."

1.3.2 *Definition [78]*

"Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$, satisfies:

1. $0 \leq d(x,y)$, *for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;*

2. $d(x,y) = d(y,x)$ *for all* $x, y \in X$;

3. $d(x,y) \leq d(x,z) + d(z,y)$, *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) *is called a cone metric space."*

1.3.3 *Definition [78]*

"Let (X,d) *be a cone metric space. Let* x_n *be a sequence in* X *and* $x \in X$. *If for every* $c \in X$, *with* $0 \ll c$ *there is* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$, $d(x_n, x) \ll c$, *then* $\{x_n\}$ *is said to be convergent,* $\{x_n\}$ *converges to* x *and* x *is the limit of* $\{x_n\}$. *We denote this by* $lim_n x_n = x$, *or* $x_n \longrightarrow x$, *as* $n \rightarrow \infty$."

1.3.4 *Definition [78]*

"Let (X,d) *be a cone metric space. If for every* $c \in X$ *with* $0 \ll c$ *there is* $n_0 \in \mathbb{N}$, *such that, for all* $n, m > n_0$, $d(x_n, x_m) \ll c$, *then* $\{x_n\}$ *is called a Cauchy sequence in* X. *If every Cauchy sequence is convergent in* X, *then* X *is called a complete cone metric space."*

1.3.5 *Lemma [78]*

"Let (X,d) *be a cone metric space. If* P *is a normal cone with normal constant* κ, *then* $x_n \in X$ *converges to* $x \in X$, *if and only if* $d(x_n, x) \rightarrow 0$ *as* $n \rightarrow \infty$."

1.3.6 *Definition [78]*

"Let (X,d) *be a cone metric space. If* P *is normal cone then* $x_n \in E$ *is a Cauchy sequence if and only if* $d(x_n, x_m) \rightarrow 0$ *as* $n, m \rightarrow \infty$."

1.3.7 *Lemma [78]*

"Let (X,d) *be a cone metric space,* P *be normal cone with normal constant* κ. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in* X *and* $x_n \longrightarrow x$, $y_n \longrightarrow y$ *as* $n \rightarrow \infty$. *Then* $d(x_n, y_n) \rightarrow d(x,y)$ *as* $n \rightarrow \infty$."

1.4 *Fuzzy mappings.*

1.4.1 *Definition*

"*Let X be a a nonempty set. A fuzzy set in X is a real valued function with domain X and values in* $[0,1]$.

I^X *is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A.*"

1.4.2 *Definition [3, 73]*

"*The α -level set of a fuzzy set A in X is denoted by $[A]_\alpha$ and is defined as*

$$[A]_\alpha = \{x : A(x) \geqslant \alpha\} \text{ if } \alpha \in (0,1],$$

$$[A]_0 = \text{closure of the set } \{x : A(x) > 0\},$$

$$\widehat{A} = \{x : A(x) = \max_{y \in X} A(y)\}."$$

1.4.3 *Definition [3, 73]*

"*Let X, Y be two arbitrary non-empty sets. A mapping T is called fuzzy mapping if T is a mapping from X into I^Y. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$. The family of all mappings from X into I^Y is denoted by $(I^Y)^X$. For the sake of convenience, we denote α -level set of $T(x)$ by $[Tx]_\alpha$ instead of $[T(x)]_\alpha$.*

Heilpern [73] was the first who gave a contraction theorem (see corollary 3.3.16) for fuzzy mappings which is an analogue of Banach contraction principle for single valued mappings and Nadler [118] contraction theorem for multivalued mappings."

Chapter 2

Fixed points of single valued mappings

"Since the appearance of celebrated Banach contraction principle in 1932, several generalizations and improvements of this theorem have been obtained. We refer to Kirk [103], Murthy [117], Park [127, 128] and Rhoade [142, 145], for a complete survey of this subject. Junck [86] generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting mappings. He also pointed out in [88, 89] the significance of commuting mappings for generalizing fixed point theorems. Subsequently, a variety of extensions, generalizations and applications of this followed e.g., see [6, 40, 47, 71, 79, 82, 95, 114].

Huang and Zhang [78] have introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space, and they have established some fixed point theorems for contractive type mappings in a normal cone metric space. Afterwards, some other authors [1, 2, 45, 46, 81, 82, 92, 93, 132, 133, 134, 141, 166, 171, 175] have studied the fixed point results in cone metric spaces. In [1, 2, 45, 81, 82, 132], the authors have established the existence of common fixed points of contractive type mappings in the frame work of normal cone metric space.

Soon after, Rezapour and Hamlbarani [141] came up with a remarkable modification to the results of Huang and Zhang [78] and proved that there are no normal cones with normal constant $c < 1$ *and for each* $k > 1$, *there are cones with normal constant* $c > k$ *(We have defined normal* *cone accordingly, see definition 1.3.1). Also by omitting the assumption of normality, they obtain generalizations of some results of Huang and Zhang [78]. Afterwards, in [46, 92, 133] the authors obtained results on point of coincidence and common fixed points in cone metric spaces without the assumptions of normality. Recently [93, 171, 175] have initiated the idea of fixed point of set valued contractions in normal cone metric spaces.*

In this chapter, we continue these investigations and explore the fixed point and common fixed point results in cone metric spaces. In section 2.1, we deal with the normal cone metric spaces and prove the existence of common fixed points of a pair of self mappings satisfying a contractive type condition. Some theorems on points of coincidence along with common fixed points of three single valued self mappings in a normal cone metric space have also been established. Section 2.2 and 2.3 deal with cone metric spaces which may not be normal. In section 2.2, we establish similar results by omitting the assumption of normality. In section 2.3, we are concerned with the points of coincidence and common fixed points for three self mappings satisfying $\varphi-$contractive conditions. In section 2.4, we introduce the concept of cone rectangular metric space and prove Banach contraction principle in this setting."

2.1 Fixed points in normal cone metric spaces

Results of this section will appear in [22, 24].

"In this section, Theorem 2.1.1 proves the existence of common fixed points for a pair of mappings satisfying a generalized contractive condition in a complete normal cone metric space. The theorem thus establishes and extends the results of Abbas and Rhoades [2] and Haung and Zhang [78].

By providing theorem 2.1.12, we state some corollaries as a generalization of certain results in [1]. Theorem 2.1.16 involves a generalized contraction condition along with a condition of weak compatibility to prove the existence of coincidence and common fixed points of three self mappings in complete normal cone metric space."

2.1.1 Theorem

Suppose (X, d) be a cone metric space having completeness. Let P be a normal cone with normal constant κ, the mappings $S, T : X \to X$ satisfy:

$$d(Sx, Ty) \leq A\ d(x,\ y) + B\ d(x, Sx) + Cd(y, Ty) + D\ d(x, Ty) + E\ d(y, Sx) \tag{2.1}$$

for all $x, y \in X$ where A, B, C, D, E are non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. Then S and T have a unique common fixed point.

Proof

Let x_0 be an arbitrary point in X and define

$$\begin{aligned} x_{2k+1} &= Sx_{2k} \\ x_{2k+2} &= Tx_{2k+1}, \qquad k = 0, 1, 2, ... \end{aligned}$$

Then,

$$\begin{aligned} d(x_{2k+1},\ x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\leq Ad(x_{2k}, x_{2k+1}) + Bd(x_{2k}, Sx_{2k}) + Cd(x_{2k+1}, Tx_{2k+1}) \\ &\quad + Dd(x_{2k}, Tx_{2k+1}) + Ed(x_{2k+1}, Sx_{2k}) \\ &\leq [A + B]d(x_{2k}, x_{2k+1}) + Cd(x_{2k+1}, x_{2k+2}) + Dd(x_{2k}, x_{2k+2}) \\ &\leq [A + B + D]d(x_{2k}, x_{2k+1}) + [C + D]d(x_{2k+1}, x_{2k+2}). \end{aligned}$$

It implies that,

$$[1 - C - D]d(x_{2k+1}, x_{2k+2}) \leq [A + B + D]d(x_{2k}, x_{2k+1}).$$

That is,

$$d(x_{2k+1}, x_{2k+2}) \leqslant \left[\frac{A + B + D}{1 - C - D}\right] d(x_{2k},\ x_{2k+1}).$$

Similarly,

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\leq \left[\frac{A+C+E}{1-B-E}\right] d(x_{2k+1},\ x_{2k+2}). \end{aligned}$$

Now by induction, we obtain for each $k = 0, 1, 2, \ldots$

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\leqslant \left[\frac{A+B+D}{1-C-D}\right] d(x_{2k}, x_{2k+1}) \\ &\leqslant \left[\frac{A+B+D}{1-C-D}\right]\left[\frac{A+C+E}{1-B-E}\right] d(x_{2k-1},\ x_{2k}) \\ &\leqslant \left[\frac{A+B+D}{1-C-D}\right]\left[\frac{A+C+E}{1-B-E}\right]\left[\frac{A+B+D}{1-C-D}\right] d(x_{2k-2}, x_{2k-1}) \\ &\leq \ldots \leqslant \left[\frac{A+B+D}{1-C-D}\right]\left(\left[\frac{A+C+E}{1-B-E}\right]\left[\frac{A+B+D}{1-C-D}\right]\right)^k d(x_0,\ x_1) \end{aligned}$$

and

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &\leq \left[\frac{A+C+E}{1-B-E}\right] d(x_{2k+1}, x_{2k+2}) \\ &\leq \ldots \leq \left(\left[\frac{A+C+E}{1-B-E}\right]\left[\frac{A+B+D}{1-C-D}\right]\right)^{k+1} d(x_0,\ x_1). \end{aligned}$$

Let

$$F = \left[\frac{A+B+D}{1-C-D}\right], \quad G = \left[\frac{A+C+E}{1-B-E}\right].$$

In case $B = C$

$$FG = \left[\frac{A+B+D}{1-B-D}\right]\left[\frac{A+B+E}{1-B-E}\right] = \left[\frac{A+B+D}{1-B-E}\right]\left[\frac{A+B+E}{1-B-D}\right] < 1$$

and if $D = E$, then,

$$FG = \left[\frac{A+B+D}{1-C-D}\right]\left[\frac{A+C+D}{1-B-D}\right] < 1.$$

Now, for $p < q$ we have,

$$\begin{aligned} d(x_{2p+1},\ x_{2q+1}) &\leq d(x_{2p+1}, x_{2p+2}) + d(x_{2p+2}, x_{2p+3}) + d(x_{2p+3}, x_{2p+4}) \\ &\quad + \ldots + d(x_{2q},\ x_{2q+1}) \\ &\leq \left[F \sum_{i=p}^{q-1} (FG)^i + \sum_{i=p+1}^{q} (FG)^i \right] d(x_0, x_1) \\ &\leq \left[\frac{F(FG)^p}{1-FG} + \frac{(FG)^{p+1}}{1-FG} \right] d(x_0, x_1) \\ &\leq (1+G) \left[\frac{F(FG)^p}{1-FG} \right] d(x_0, x_1). \end{aligned}$$

Similarly, we obtain

$$d(x_{2p}, x_{2q+1}) \leq (1+F) \left[\frac{(FG)^p}{1-FG} \right] d(x_0, x_1),$$

$$d(x_{2p},\ x_{2q}) \leq (1+F) \left[\frac{(FG)^p}{1-FG} \right] d(x_0, x_1)$$

and

$$d(x_{2p+1},\ x_{2q}) \leq (1+G) \left[\frac{F(FG)^p}{1-FG} \right] d(x_0, x_1).$$

Hence, for $0 < n < m$ there exists $p > 0$ such that $p < n < m$ and

$$d(x_m, x_n) \leq \text{Max} \left\{ (1+G) \left[\frac{F(FG)^p}{1-FG} \right], (1+F) \left[\frac{(FG)^p}{1-FG} \right] \right\} d(x_0, x_1).$$

Since, P is a normal cone with normal constant κ, therefore,

$$\|d(x_n, x_m)\| \leq \kappa \left[\text{Max} \left\{ (1+G) \left[\frac{F(FG)^p}{1-FG} \right], (1+F) \left[\frac{(FG)^p}{1-FG} \right] \right\} \right] \|d(x_0, x_1)\|.$$

Thus,

$$\text{Max} \left\{ (1+G) \left[\frac{F(FG)^p}{1-FG} \right], (1+F) \left[\frac{(FG)^p}{1-FG} \right] \right\} \to 0 \ \text{ as } \ p \to \infty.$$

Therefore, $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$.

Now,

$$
\begin{aligned}
d(u,Su) &\leq d(u,x_{2n}) + d(x_{2n},Su) \\
&\leq d(u,x_{2n}) + d(Tx_{2n-1},Su) \\
&\leq d(u,x_{2n}) + Ad(u,x_{2n-1}) + Bd(u,Su) + Cd(x_{2n-1},Tx_{2n-1}) \\
&\quad + Dd(u,Tx_{2n-1}) + Ed(x_{2n-1},Su) \\
&\leq d(u,x_{2n}) + Ad(u,x_{2n-1}) + Bd(u,Su) + Cd(x_{2n-1},x_{2n}) \\
&\quad + Dd(u,x_{2n}) + Ed(x_{2n-1},u) + Ed(u,Su)] \\
&\leq (1+D)\, d(u,x_{2n}) + (A+E)d(u,x_{2n-1}) + Cd(x_{2n-1},x_{2n}) \\
&\quad + (B+E)d(u,Su).
\end{aligned}
$$

It further implies that

$$
\begin{aligned}
d(u,Su) - (B+E)d(u,Su) &\leq (1+D)d(u,x_{2n}) + (A+E)d(u,x_{2n-1}) \\
&\quad + Cd(x_{2n-1},x_{2n}).
\end{aligned}
$$

That is,

$$
\begin{aligned}
d(u,Su) \leq& \left[\frac{1+D}{1-B-E}\right] d(u,x_{2n}) + \left[\frac{A+E}{1-B-E}\right] d(u,x_{2n-1}) \\
&+ \left[\frac{C}{1-B-E}\right] d(x_{2n-1},x_{2n}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|d(u,Su)\| \leq \kappa \left[\frac{1+D}{1-B-E}\right] \|d(u,x_{2n})\| + \kappa \left[\frac{A+E}{1-B-E}\right] \|d(u,x_{2n-1})\| \\
&+\kappa \left[\frac{C}{1-B-E}\right] \|d(x_{2n-1},x_{2n})\|.
\end{aligned}
$$

Letting $n \to \infty$, we have $\|d(u,Su)\| = 0$. It implies that $d(u,Su) = \mathbf{0}$ and hence $u = Su$. Similarly, by using

$$d(u,Tu) \leq d(u,x_{2n+1}) + d(x_{2n+1},Tu),$$

we can show that $u = Tu$. It implies that u is a common fixed point of S, T. Next, we show that S and T have unique common fixed point. For this suppose that, there exists another point u^* in X such that $u^* = Su^* = Tu^*$.

Now,

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\leq Ad(u, u^*) + Bd(u, Su) + Cd(u^*, Tu^*) + Dd(u, Tu^*) + Ed(u^*, Su) \\ &\leq Ad(u, u^*) + Bd(u, u) + Cd(u^*, u^*) + Dd(u, u^*) + Ed(u, u^*) \\ &\leq (A + D + E)d(u, u^*). \end{aligned}$$

It implies that

$$u = u^*,$$

which completes the proof of the theorem.

2.1.2 Corollary[2]

"*Let (X, d) be a complete cone metric space, P be a normal cone with normal constant κ. Suppose the mappings $S, T : X \to X$ satisfy:*

$$d(Sx, Ty) \leq \alpha\ d(x, y) + \beta\,[d(x, Sx) + d(y, Ty)] + \gamma\,[d(x, Ty) + d(y, Sx)]$$

for all $x, y \in X$, where, α, β, γ are non negative real numbers with $\alpha + 2\beta + 2\gamma < 1$. Then S and T have a unique common fixed point."

2.1.3 Example

Let $X = \{1, 2, 3\}$, $E = R^2$ and $P = \{(x, y) \in E \mid x, y \geq 0\} \subset R^2$. Define $d : X \times X \to R^2$ as follows:

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y \\ (\frac{5}{7}, 5) & \text{if } x \neq y \text{ and } x, y \in X - \{2\} \\ (1, 7) & \text{if } x \neq y \text{ and } x, y \in X - \{3\} \\ (\frac{4}{7}, 4) & \text{if } x \neq y \text{ and } x, y \in X - \{1\}\ . \end{cases}$$

Define the mappings $S, T : X \to X$ as follows:

$$S(x) = 1 \text{ for each } x \in X,$$

$$T(x) = \begin{cases} 1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2. \end{cases}$$

Note that

$$d(S(3), T(2)) = (\frac{5}{7}, 5).$$

Now,

$$\begin{aligned}
& \alpha\, d(3,2) + \beta\,[d(3,S(3)) + d(2,T(2))] + \gamma\,[d(3,T(2)) + d(2,S(3))] \\
= \;& \alpha(\frac{4}{7}, 4) + \beta\,[d(3,1) + d(2,3)] + \gamma\,[d(3,3) + d(2,1)] \\
= \;& \alpha(\frac{4}{7}, 4) + \beta\left[(\frac{5}{7}, 5) + (\frac{4}{7}, 4)\right] + \gamma\,[0 + (1,7)\,] \\
= \;& \left(\frac{4\alpha + 9\beta + 7\gamma}{7}, 4\alpha + 9\beta + 7\gamma\right) \\
< \;& \left(\frac{5\alpha + 10\beta + 10\gamma}{7}, 5\alpha + 10\beta + 10\gamma\right) \\
< \;& \left(\frac{5\,(\alpha + 2\beta + 2\gamma)}{7}, 5\,(\alpha + 2\beta + 2\gamma)\right) \\
< \;& (\frac{5}{7}, 5) = d(S(3), T(2) \quad \text{as} \quad \alpha + 2\beta + 2\gamma < 1.
\end{aligned}$$

It follows that the mappings S and T do not satisfy the conditions of Corollary 2.1.2. For

$$A = B = C = D = 0, E = \frac{5}{7},$$

all the conditions of Theorems 2.1.1 are satisfied.

2.1.4 Corollary

Let (X, d) be a complete cone metric space, P be a normal cone with normal constant κ. Suppose the mapping $T : X \to X$ satisfies:

$$d(Tx, Ty) \le \alpha\ d(x, y) + \beta\ [d(x, Tx) + d(y, Ty)] + \gamma\, [d(x, Ty) + \ d(y, Tx)]$$

for all $x, y \in X$ where $\alpha,\ \beta,\ \gamma$ are non negative real numbers with $\alpha + 2\ \beta + 2\gamma < 1$. Then T has a unique fixed point.

Proof

Set $S = T$, in Corollary 2.1.2.

2.1.5 Corollary [2]

"*Let (X, d) be a complete cone metric space, P be a normal cone with normal constant κ. Suppose that the mapping $T : X \to X$ satisfies:*

$$d(Tx, Ty) \le A\ d(x,\ y) + B\ d(x, Tx) + Cd(y, Ty) + D\ d(x, Ty) + E\ d(y, Tx) \qquad (2.2)$$

for all $x, y \in X$, where, A, B, C, D, E are non negative real numbers with $A{+}B{+}C{+}D{+}E < 1$. Then T has a unique fixed point."

Proof

Set $S = T$, in Theorem 2.1.1.

2.1.6 Corollary [2]

"*Let (X, d) be a complete cone metric space, P be a normal cone with normal constant κ. Suppose the mapping $T : X \to X$ satisfies:*

$$d(Tx, Ty) \le \alpha\ d(x,\ y) + \beta\ \ [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $\alpha, \beta \ge 0$ with $\alpha + 2\ \beta < 1$. Then T has a unique fixed point."

2.1.7 Corollary [2, 78]

"*Let (X, d) be a complete cone metric space, P be a normal cone with normal constant κ. Suppose the mapping $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leq \alpha \, d(x, \ y)$$

for all $x, y \in X$ where $0 \leq \alpha < 1$. Then T has a unique fixed point."

2.1.8 Corollary [78]

"*Let (X, d) be a complete cone metric space, P be a normal cone with normal constant κ. Suppose the mapping $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leq \beta \ [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ where $0 \leqslant \beta < \frac{1}{2}$. Then T has a unique fixed point."

2.1.9 *Corollary [78]*

"*Let (X, d) be a complete cone metric space, P be a normal cone with normal constant κ. Suppose the mapping $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leq \gamma \ [d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$, where, $0 \leqslant \gamma < \frac{1}{2}$. Then T has a unique fixed point."

2.1.10 Example

Let $X = \{1, 2, 3\}$, $E = R^2$ and $P = \{(x, y) \in E \mid x, y \geq 0\} \subset R^2$. Define $d : X \times X \to R^2$ as follows:

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y \\ (\frac{4}{7}, \frac{2}{7}) & \text{if } x \neq y \text{ and } x, y \in X - \{2\} \\ (1, \frac{1}{2}) & \text{if } x \neq y \text{ and } x, y \in X - \{3\} \\ (\frac{1}{2}, \frac{1}{4}) & \text{if } x \neq y \text{ and } x, y \in X - \{1\}. \end{cases}$$

Define a mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} 3 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$

Note that

$$d(T(3), T(2)) = d(3, 1) = (\frac{4}{7}, \frac{2}{7}).$$

Now,

$$\begin{aligned}
& \alpha\, d(3,2) + \beta\,[d(3,T(3)) + d(2,T(2))] + \gamma\,[d(3,T(2)) + d(2,T(3))] \\
= \; & \alpha\, d(3,2) + \beta\,[d(3,3) + d(2,1)] + \gamma\,[d(3,1) + d(2,3)] \\
= \; & \alpha(\frac{1}{2}, \frac{1}{4}) + \beta\left[0 + (1, \frac{1}{2})\right] + \gamma\left[(\frac{4}{7}, \frac{2}{7}) + (\frac{1}{2}, \frac{1}{4})\right] \\
= \; & \left(\frac{\alpha}{2} + \beta + \frac{15\gamma}{14}, \frac{\alpha}{4} + \frac{\beta}{2} + \frac{15\gamma}{28}\right) \\
= \; & \left(\frac{7\alpha + 14\beta + 15\gamma}{14}, \frac{7\alpha + 14\beta + 15\gamma}{28}\right) \\
< \; & \left(\frac{\frac{15}{2}\alpha + 15\beta + 15\gamma}{14}, \frac{\frac{15}{2}\alpha + 15\beta + 15\gamma}{28}\right) \\
< \; & \left(\frac{\frac{15}{2}(\alpha + 2\beta + 2\gamma)}{14}, \frac{\frac{15}{2}(\alpha + 2\beta + 2\gamma)}{28}\right) \\
< \; & \left(\frac{15}{28}, \frac{15}{56}\right) < \left(\frac{4}{7}, \frac{2}{7}\right) = d(T(3), T(2)) \;\; \text{as} \;\; \alpha + 2\beta + 2\gamma < 1.
\end{aligned}$$

Therefore, Corollaries (2.1.2, 2.1.4-2.1.8) are not applicable to obtain fixed point of T.

In order to apply common fixed point result (i.e.,Theorem 2.1.1), define a constant mapping $S : X \to X$ by $Sx = 3$. Then,

$$d(Sx, Ty) = \begin{cases} (0,0) & \text{if } y \neq 2 \\ (\frac{4}{7}, \frac{2}{7}) & \text{if } y = 2 \end{cases}$$

and for $A = B = D = E = 0, C = \frac{4}{7}$, we have

$$A\ d(x,y) + B\ d(x,Sx) + Cd(y,Ty) + D\ d(x,Ty) + E\ d(y,Sx) = \left(\frac{4}{7}, \frac{2}{7}\right) \quad \text{if } y = 2.$$

It follows that S and T satisfy all the conditions of Theorem2.1.1 and we obtain $T(3) = 3$.

"Now, we obtain points of coincidence and common fixed points for three self mappings satisfying generalized contractive type condition in a complete normal cone metric space."

2.1.11 Lemma

Let X be a non-empty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and $(T, f)$are weakly compatible, then S, T and f have a unique common fixed point.

Proof

Let v be the point of coincidence of S, T and f. Then, $v = fu = Su = Tu$ for some $u \in X$. By weakly compatibility of (S, f) and (T, f), we have

$$Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv.$$

It implies that $Sv = Tv = fv = w$ (say). Thus, w is a point of coincidence of S, T and f. Therefore, $v = w$ by uniqueness. Hence, v is the unique common fixed point of S, T and f.

"Here by providing the next result, we state the following generalization of some recent results."

2.1.12 Theorem

Let $(X,\ d)$ be a cone metric space, P be a normal cone with normal constant κ. Suppose the mappings $T, f : X \to X$ satisfy:

$$d(Tx, Ty) \leq \alpha\left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx,\ fy)$$

for all $x, y \in X$, where $\alpha, \gamma \in [0, 1)$ with $2\alpha + \gamma < 1$. Also, suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Then, T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

2.1.13 Corollary

Let $(X,\ d)$ be a cone metric space, P be a normal cone with normal constant κ. Suppose the mappings $T,\ f : X \to X$ satisfy:

$$d(Tx, Ty) \leq \alpha d(fx, Ty) + \beta d(fy, Tx) + \gamma d(fx,\ fy) \tag{2.3}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$. Also, suppose that $T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X. Then T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

Proof

In (2.3) interchanging the roles of x and y and adding the resultant inequality to (2.3), we obtain

$$d(Tx, Ty) \leq \frac{\alpha + \beta}{2}\,[d(fx, Ty) + d(fy, Tx)] + \gamma d(fx,\ fy).$$

Now, by using Theorem 2.1.12, we obtain the required result.

2.1.14 Corollary[1]

"Let (X, d) be a cone metric space, P be a normal cone with normal constant κ and the mappings $T,\ f : X \to X$ satisfy:

$$d(Tx, Ty) \leq \gamma\ d(fx, fy),$$

for all $x, y \in X$, where $0 \leq \gamma < 1$. If $T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point."

2.1.15 Corollary[1]

"Let (X, d) be a cone metric space, P be a normal cone with normal constant κ and the mappings $T, f : X \to X$ satisfy:

$$d(Tx, Ty) \leq \alpha\,[d(fx, Ty) + d(fy, Tx)]$$

for all $x, y \in X$, where $0 \leq \alpha < \frac{1}{2}$. Also, suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Then, T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point."

Here, we further improve Theorem 2.1.12 as follows:

"For convenience, firstly, we define a notion of S-T-sequence with initial point $x_0 \in X$.

Let (X, d) be a cone metric space, S, T, f be self-mappings in X and $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Sx_0$. This can be done since $S(X) \subseteq f(X)$. Successively, choose a point x_2 in X such that $fx_2 = Tx_1$. Continuing this process having chosen $x_1, \ldots, x_{2k}$, we choose x_{2k+1} and x_{2k+2} in X such that

$$\begin{aligned} fx_{2k+1} &= Sx_{2k}, \\ fx_{2k+2} &= Tx_{2k+1}, \quad k = 0, 1, 2, ... \end{aligned}$$

The sequence $\{fx_n\}$ is called a S-T-sequence with initial point x_0."

2.1.16 Theorem

Let (X, d) be a cone metric space, P be a normal cone with normal constant κ. Suppose the mappings $S, T, f : X \to X$ satisfy:

$$d(Sx, Ty) \leq \alpha d(fx, Ty) + \beta d(fy, Sx) + \gamma d(fx, fy) \tag{2.4}$$

for all $x, y \in X$, where α, β, γ are non negative real numbers with

$$\alpha + \beta + \gamma < 1.$$

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof

Let x_0 be an arbitrary point in X and $\{fx_n\}$ be a *S-T-sequence* with initial point x_0.

Then,

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\leq \alpha d(fx_{2k}, Tx_{2k+1}) + \beta d(fx_{2k+1}, Sx_{2k}) + \gamma d(fx_{2k}, fx_{2k+1}) \\ &\leq [\alpha+\gamma] d(fx_{2k}, fx_{2k+1}) + \alpha d(fx_{2k+1}, fx_{2k+2}). \end{aligned}$$

It implies that

$$[1-\alpha] d(fx_{2k+1}, fx_{2k+2}) \leq [\alpha+\gamma]\ d(fx_{2k}, fx_{2k+1}).$$

That is,

$$d(fx_{2k+1},\ fx_{2k+2}) \leq \left[\frac{\alpha+\gamma}{1-\alpha}\right] d(fx_{2k},\ fx_{2k+1}).$$

Similarly,

$$\begin{aligned} d(fx_{2k+2},\ fx_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\leq \alpha d(fx_{2k+2}, Tx_{2k+1}) + \beta d(fx_{2k+1}, Sx_{2k+2}) + \gamma d(fx_{2k+2}, fx_{2k+1}) \\ &\leq \alpha d(fx_{2k+2}, fx_{2k+2}) + \beta d(fx_{2k+1}, fx_{2k+3}) + \gamma d(fx_{2k+2}, fx_{2k+1}) \\ &\leq [\beta+\gamma] d(fx_{2k+1}, fx_{2k+2}) + \beta d(fx_{2k+2}, fx_{2k+3}). \end{aligned}$$

Hence,

$$d(fx_{2k+2}, fx_{2k+3}) \leq \left[\frac{\beta+\gamma}{1-\beta}\right] d(fx_{2k+1}, fx_{2k+2}).$$

Now by induction, we obtain

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &\leq \left[\frac{\alpha+\gamma}{1-\alpha}\right] d(fx_{2k},\ fx_{2k+1}) \\ &\leq \left[\frac{\alpha+\gamma}{1-\alpha}\right] \left[\frac{\beta+\gamma}{1-\beta}\right] d(fx_{2k-1},\ fx_{2k}) \\ &\leq \left[\frac{\alpha+\gamma}{1-\alpha}\right] \left[\frac{\beta+\gamma}{1-\beta}\right] \left[\frac{\alpha+\gamma}{1-\alpha}\right] d(fx_{2k-2},\ fx_{2k-1}) \\ &\leq \ \dots \leq \left[\frac{\alpha+\gamma}{1-\alpha}\right] \left(\left[\frac{\beta+\gamma}{1-\beta}\right] \left[\frac{\alpha+\gamma}{1-\alpha}\right]\right)^k d(fx_0, fx_1) \end{aligned}$$

and

$$\begin{aligned} d(fx_{2k+2}, fx_{2k+3}) &\leq \left[\frac{\beta+\gamma}{1-\beta}\right] d(fx_{2k+1}, fx_{2k+2}) \\ &\leq \ \ldots \leq \left(\left[\frac{\beta+\gamma}{1-\beta}\right] \left[\frac{\alpha+\gamma}{1-\alpha}\right]\right)^{k+1} d(fx_0, fx_1), \end{aligned}$$

for each $k \geq 0$. Let

$$\lambda = \left[\frac{\alpha+\gamma}{1-\alpha}\right], \ \mu = \left[\frac{\beta+\gamma}{1-\beta}\right].$$

Then, $\lambda\mu < 1$. Now, for $p < q$ we have

$$\begin{aligned} d(fx_{2p+1}, fx_{2q+1}) &\leq d(fx_{2p+1}, fx_{2p+2}) + d(fx_{2p+2}, fx_{2p+3}) \\ &\quad + d(fx_{2p+3}, fx_{2p+4}) + \ \ldots + d(fx_{2q}, fx_{2q+1}) \\ &\leq \left[\lambda \sum_{i=p}^{q-1} (\lambda\mu)^i + \sum_{i=p+1}^{q} (\lambda\mu)^i\right] d(fx_0, fx_1) \\ &\leq \left[\frac{\lambda(\lambda\mu)^p[1-(\lambda\mu)^{q-p}]}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}[1-(\lambda\mu)^{q-p}]}{1-\lambda\mu}\right] d(fx_0, fx_1) \\ &\leq \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}}{1-\lambda\mu}\right] d(fx_0, fx_1) \\ &\leq (1+\mu)\left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1) \end{aligned}$$

$$d(fx_{2p}, fx_{2q+1}) \leq (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1),$$

$$d(fx_{2p}, \ fx_{2q}) \leq (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1),$$

and

$$d(fx_{2p+1}, fx_{2q}) \leq (1+\mu)\left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1).$$

Hence for $0 < n < m$, there exists $p < n < m$ such that $p \to \infty$ as $n \to \infty$ and

$$d(fx_n, fx_m) \leq Max\left\{(1+\mu)\left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right], (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right]\right\} d(fx_0, fx_1).$$

Since P is a normal cone with normal constant κ, we have

$$\|d(fx_n, fx_m)\| \leq \kappa \left[Max \left\{ (1+\mu) \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} \right], (1+\lambda) \left[\frac{(\lambda\mu)^p}{1-\lambda\mu} \right] \right\} \right] \|d(fx_0, fx_1)\|.$$

Thus, if $m, n \to \infty$, then

$$Max \left\{ (1+\mu) \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} \right], (1+\lambda) \left[\frac{(\lambda\mu)^p}{1-\lambda\mu} \right] \right\} \to 0,$$

and so $d(fx_n, fx_m) \to 0$. Hence, $\{fx_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, there exist $u, v \in X$ such that $fx_n \to v = fu$. Since

$$\begin{aligned} d(fu,\ Su) &\leq d(fu, fx_{2n}) + d(fx_{2n}, Su) \\ &\leq d(v, fx_{2n}) + d(Tx_{2n-1}, Su) \\ &\leq d(v, fx_{2n}) + \alpha d(fu, Tx_{2n-1}) \\ &\quad +\beta \left[d(fx_{2n-1}, fu) + d(fu, Su) \right] + \gamma d(fu, fx_{2n-1}). \end{aligned}$$

It implies that

$$\begin{aligned} d(fu, Su) &\leq \frac{1}{1-\beta} [d(v, fx_{2n}) + \alpha d(v, fx_{2n}) + \beta d(fx_{2n-1}, v) + \gamma d(v, fx_{2n-1})] \\ &\leq \frac{1}{1-\beta} \left[(1+\alpha)\, d(v, fx_{2n}) + \beta d(fx_{2n-1}, v) + \gamma d(v, fx_{2n-1}) \right]. \end{aligned}$$

Hence,

$$\|d(fu, Su)\| \leq \frac{\kappa}{1-\beta} \left\| (1+\alpha)\, d(v, fx_{2n}) + (\beta+\gamma)\, d(v, fx_{2n-1}) \right\|.$$

If $n \to \infty$, then we obtain $\|d(fu, Su)\| = 0$. Hence, $fu = Su$. Similarly, by using the inequality

$$d(fu, Tu) \leq d(fu, fx_{2n+1}) + d(fx_{2n+1}, Tu),$$

we can show that $fu = Tu$. This implies that v is a common point of coincidence of S, T and f, that is

$$v = fu = Su = Tu.$$

Now, we show that f, S and T have unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^* = Tu^*$ for some u^* in X. Now,

$$\begin{aligned} d(v, v^*) &= d(Su, Tu^*) \\ &\leq \alpha d(fu, Tu^*) + \beta d(fu^*, Su) + \gamma d(fu, fu^*) \\ &\leq (\alpha + \beta + \gamma)\, d(v, v^*). \end{aligned}$$

Hence, $v = v^*$. If (S, f) and (T, f) are weakly compatible, then

$$Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv.$$

It implies that $Sv = Tv = fv = w$ (say). Hence, w is a point of coincidence of S, T and f, and so $v = w$ by uniqueness. Thus, v is the unique common fixed point of S, T and f.

2.1.17 Example

Let $X = \{1, 2, 3\}, E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Define $d : X \times X \to E$ as follows:

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y \\ (\frac{5}{7}, 5) & \text{if } x \neq y \text{ and } x, y \in X - \{2\} \\ (1, 7) & \text{if } x \neq y \text{ and } x, y \in X - \{3\} \\ (\frac{4}{7}, 4) & \text{if } x \neq y \text{ and } x, y \in X - \{1\}. \end{cases}$$

Define the mappings $T, f : X \to X$ as follows:

$$T(x) = \begin{cases} 1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases} \quad \text{and} \quad fx = x.$$

Then, $d(T(3), T(2)) = (\frac{5}{7}, 5)$. Now, for $2\alpha + \gamma < 1$ we have

$$\begin{aligned}
& \alpha\,[d(f(3), T(2)) + d(f(2), T(3))] + \gamma d(f(3), f(2)) \\
= & \ \alpha\,[d(3, T(2)) + d(2, T(3))] + \gamma d(3, 2) \\
= & \ \gamma(\frac{4}{7}, 4) + \alpha\,[d(3,3) + d(2,1)] \\
= & \ \alpha\,[0 + (1,7)] + \gamma(\frac{4}{7}, 4) = (\frac{7\alpha + 4\gamma}{7}, 7\alpha + 4\gamma) \\
< & \ (\frac{8\alpha + 4\gamma}{7}, 8\alpha + 4\gamma) = (\frac{4\,(2\alpha + \gamma)}{7}, 4\,(2\alpha + \gamma)) \\
< & \ (\frac{4}{7}, 4) < (\frac{5}{7}, 5) = d(T(3), T(2)).
\end{aligned}$$

It follows that the mappings T and f do not satisfy the conditions of Theorem 2.1.12. Hence, Theorem 2.1.12 and its corollaries 2.1.13, 2.1.14 and 2.1.15 are not applicable here. Now, define the mapping $S : X \to X$ by $Sx = 1$ for all $x \in X$. Then,

$$d(Sx, Ty) = \begin{cases} (0,0) & if\ y \neq 2 \\ (\frac{5}{7}, 5) & if\ y = 2 \end{cases}$$

and

$$\alpha d(fx, Ty) + \beta d(fy, Sx) + \gamma d(fx, fy) = (\frac{5}{7}, 5)$$

if $y = 2,\ \alpha = \gamma = 0$ and $\beta = \frac{5}{7}$. It follows that all conditions of Theorems 2.1.16 are satisfied for $\alpha = \gamma = 0,\ \beta = \frac{5}{7}$ and one can obtain the unique common fixed point 1 for S, T and f.

2.2 Coincidence and common fixed point results in non-normal cone metric spaces

Results given in this section have been published in [11, 18].

"*This section deals with the results in cone metric spaces without the assumption of normality. In [141] Rezapour and Hamlbarani, established that there are non-normal cones.*

Let (X, d) *be a complete cone metric space,* P *be a normal cone with normal constant* κ*.*

Suppose that the mappings T , $f : X \to X$ satisfy:

$$
\begin{aligned}
d(Tx, Ty) \leq\ & A\, d(fx,\ fy) + B\ d(fx, Tx) + C d(fy, Ty) \\
& + D\ d(fx, Ty) + E\ d(fy, Tx),
\end{aligned} \tag{2.5}
$$

for all $x,\ y \in X$ where A, B, C, D, E are non-negative real numbers.

Huang and Zhang [78] proved that T has a unique fixed point if

(a) $f = I$, where I is the identity mapping on X (see [72, 140, 142])

and

(b) one of the following is satisfied:

(i) $B = C = D = E = 0$ *with* $A < 1$ *([78], theorem 1),*

(ii) $A = D = E = 0$ *with* $B = C < \frac{1}{2}$ *([78], theorem 3),*

(iii) $A = B = C = 0$ *with* $D = E < \frac{1}{2}$ *([78], theorem 4).*

Abbas and Jungck [1] proved that f and T have a unique point of coincidence and unique common fixed point if one of the following is satisfied:

(i) $B = C = D = E = 0$ *with* $A < 1$ *([1], theorem 2.1),*

(ii) $A = D = E = 0$ *with* $B = C < \frac{1}{2}$ *([1], theorem 2.3),*

(iii) $A = B = C = 0$ *with* $D = E < \frac{1}{2}$ *([1], theorem 2.4).*

Rezapour and Hamlbarani [141] generalized some results of [78] by omitting the assumption of normality on X.' We have the following improvement/generalization of these results."

2.2.1 Theorem

Let $(X,\ d)$ be a cone metric space. Suppose the mappings $T,\ f : X \to X$ satisfy:

$$
\begin{aligned}
d(Tx, Ty) \leq\ & A\, d(fx,\ fy) + B\ [d(fx, Tx) + d(fy, Ty)] \\
& + C\ [d(fx, Ty) + d(fy, Tx)]
\end{aligned} \tag{2.6}
$$

for all $x, y \in X$ where A, B, C are non-negative real numbers with

$$A + 2B + 2C < 1.$$

If

$$T(X) \subseteq f(X)$$

and $f(X)$ or $T(X)$ is a complete subspace of X, then T and f have a unique point of coincidence.

Proof

Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $fx_1 = Tx_0$. This can be done since $T(X) \subseteq f(X)$. Similarly, choose a point x_2 in X, such that $fx_2 = Tx_1$.Continuing this process and having chosen x_n in X, we obtain x_{n+1} in X such that

$$fx_{k+1} = Tx_k, k = 0, 1, 2, ...$$

Then

$$\begin{aligned} d(fx_{k+1}, fx_{k+2}) &= d(Tx_k, Tx_{k+1}) \\ &\leq A\ d(fx_k, fx_{k+1}) + B\ [d(fx_k, Tx_k) + d(fx_{k+1}, Tx_{k+1})] \\ &\quad +C\ [d(fx_k, Tx_{k+1}) + d(fx_{k+1}, Tx_k)] \\ &\leq [A + B]\ d(fx_k, fx_{k+1}) + B\ d(fx_{k+1}, fx_{k+2}) \\ &\quad +C\ d(fx_k, fx_{k+2}) \\ &\leq [A + B + C]\ d(fx_k, fx_{k+1}) + [B + C]\ d(fx_{k+1}, fx_{k+2}). \end{aligned}$$

It implies that

$$[1 - B - C] d(fx_{k+1}, fx_{k+2}) \leq [A + B + C]\ d(fx_k, fx_{k+1}).$$

That is

$$d(fx_{k+1}, fx_{k+2}) \leq \left[\frac{A + B + C}{1 - B - C}\right] d(fx_k, fx_{k+1}).$$

Moreover,

$$\begin{aligned} d(fx_{k+1},\ fx_{k+2}) &\leq \left[\frac{A+B+C}{1-B-C}\right]^2 d(fx_{k-1},\ fx_k) \\ &\leq \ ... \leq \left[\frac{A+B+C}{1-B-C}\right]^{k+1} d(fx_0,\ fx_1). \end{aligned}$$

Putting $y_n = fx_n$ and $\lambda = \left[\frac{A+B+C}{1-B-C}\right]$.

We have,

$$d(y_n, y_{n+1}) \leq \lambda^n d(y_0, y_1).$$

For $n > m$

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + ... + d(y_{m+1}, y_m) \\ &\leq \left(\lambda^{n-1} + \lambda^{n-2} + ... + \lambda^m\right)\ d(y_0, y_1) \\ &\leq \frac{\lambda^m}{1-\lambda} d(y_0, y_1). \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that

$$c + \{x \in Z : \|x\| < \delta\} \subseteq P.$$

Also choose a natural number N_1 such that

$$\frac{\lambda^m}{1-\lambda} d(y_0, y_1) \in \{x \in Z : \|x\| < \delta\}, \text{ for all } m \geq N_1.$$

Then

$$\frac{\lambda^m}{1-\lambda} d(y_0, y_1) \ll c, \text{ for all } m \geq N_1.$$

Thus,

$$n > m \Rightarrow d(y_n, y_m) \leq \frac{\lambda^m}{1-\lambda} d(y_0, y_1) \ll c,$$

which implies that $\{ y_n\}$ is a Cauchy sequence. We assume that $f(X)$ is complete, then there exist $u, v \in X$ such that $y_n \to v = fu$.

Choose a natural number N_2 such that for all $n \geq N_2$

$$d(y_{n-1}, y_n) \ll \left[\frac{c(1-B-C)}{3B}\right], \quad d(y_{n-1}, v) \ll \left[\frac{c(1-B-C)}{3(A+C)}\right]$$

and

$$d(y_n, v) \ll \left[\frac{c(1-B-C)}{1+C}\right].$$

Now, inequality (2.6) implies that

$$\begin{aligned} d(fu, Tu) &\leq d(fu,\ y_n) + d(y_n, Tu) \\ &\leq d(v, y_n) + d(Tx_{n-1}, Tu) \\ &\leq d(v, y_n) + Ad(fu,\ fx_{n-1}) + B[d(fu, Tu) + d(fx_{n-1}, Tx_{n-1})] \\ &\quad +C[d(fu, Tx_{n-1}) + d(fx_{n-1}, Tu)] \\ &\leq d(v, y_n) + Ad(v, y_{n-1}) + B[d(fu, Tu) + d(y_{n-1},\ y_n)] \\ &\quad +C[d(v, y_n) + d(y_{n-1}, v) + \ d(fu, Tu)] \\ &\leq (1+C)d(v,\ y_n) + (A+C)d(v,\ y_{n-1}) + Bd(y_{n-1},\ y_n) \\ &\quad +(B+C)\ d(fu, Tu) \end{aligned}$$

Consequently,

$$\begin{aligned} d(fu, Tu) \ \leq\ & \left[\frac{1+C}{1-B-C}\right] d(v, y_n) + \left[\frac{A+C}{1-B-C}\right] d(v,\ y_{n-1}) \\ & + \left[\frac{B}{1-B-C}\right] d(y_{n-1},\ y_n). \end{aligned}$$

It further implies that

$$d(fu, Tu) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

Thus,

$$d(fu, Tu) \ll \frac{c}{m}, \text{ for all } \ m \geq 1.$$

So, $\frac{c}{m} - d(fu, Tu) \in P$, for all $m \geq 1$. Since $\frac{c}{m} \to 0$ $(as\ m \to \infty)$ and P is closed, $-d(fu, Tu) \in$

P. But $P \cap (-P) = \{0\}$. Therefore, $d(fu, Tu) = 0$. Hence $v = fu = Tu$. Next we show that f and T have unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Tu^*$ for some u^* in X. Now

$$\begin{aligned} d(v, v^*) &= d(Tu, Tu^*) \\ &\leq Ad(fu,\ fu^*) + B[d(fu, Tu) + d(fu^*, Tu^*)] \\ &\quad +C[d(fu, Tu^*) + d(fu^*, Tu)] \\ &\leq Ad(v, v^*) + C[d(v, v^*) + d(v^*, v)] \\ &\leq (A + 2C)\ d(v, v^*), \end{aligned}$$

hence $v = v^*$.

On the other hand, if we assume that $T(X)$ is complete, then the Cauchy sequence

$$y_n = fx_n = Tx_{n-1}$$

converges to $v \in TX$. But $TX \subseteq fX$ which allows us to obtain $u \in fX$ such that $v = fu$. The rest of the proof is similar to previous case.

2.2.2 Theorem.

If in addition to the hypotheses of Theorem 2.2.1 the mappings $T,\ f : X \to X$ are weakly compatible, then T and f have a unique common fixed point.

Proof

As in the proof of Theorem 2.2.1, there is a unique point of coincidence v of f and T. Now T, f are weakly compatible, therefore

$$Tv = Tfu = fTu = fv.$$

It implies that $Tv = fv = w$ (say). Then w is a point of coincidence of T and f, therefore $v = w$ by uniqueness. Thus, v is a unique common fixed point of T and f.

2.2.3 Theorem

Let $(X,\ d)$ be a cone metric space. Suppose that the mappings $T,\ f : X \to X$ satisfy (2.5), for all $x, y \in X$ where A, B, C, D and E are non-negative real numbers with

$$A + B + C + D + E < 1.$$

If

$$T(X) \subseteq \ f(X)$$

and $f(X)$ *or* $T(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if $T,\ f$ are weakly compatible, then T and f have a unique common fixed point.

Proof

By hypothesis for all $x, y \in X$, we get,

$$d(Ty, Tx) \leq A\ d(fy,\ fx) + B\ d(fy, Ty) + Cd(fx, Tx) + D\ d(fy, Tx) + Ed(fx, Ty).$$

It follows that,

$$\begin{aligned} d(Tx, Ty) \ &\leq \ A\ d(fx,\ fy) + \left(\frac{B+C}{2}\right)[d(fx, Tx) + d(fy, Ty)] \\ &+ \left(\frac{D+E}{2}\right)\ [d(fx, Ty) + Ed(fy, Tx)]\,. \end{aligned}$$

The required result follows from Theorems 2.2.1 and 2.2.2.

2.2.4 Example

Let $X = R\ , E = R^2$,

$$d(x, y) = (|x - y|\,, \beta\,|x - y|)\,,\ \beta > 0$$

$$P = \{(x, y) : x, y \geq 0\},$$

$$T\ (x) = 2x^2 + 4x + 3 \text{ and } \ fx = 3x^2 + 6x + 4.$$

Then

$$TX = fX = [1, \infty)$$

and all the conditions of Theorem 2.2.1 are satisfied for

$$A \in \left[\frac{2}{3}, 1\right), \ B = c = 0.$$

as we obtain $1 \in X$ as a unique point of coincidence

$$1 = f(-1) = T(-1).$$

2.2.5 Remark

(i) Note that in Example 2.2.4

$$Tf(-1) = T(1) = 9 \text{ and } fT(-1) = f(1) = 13.$$

Thus T and f are not weakly compatible. It follows that except the weak compatibility of T and f, all other hypotheses of Theorem 2.2.2 are satisfied but

$$1 \neq f(1) \neq T(1).$$

It shows that the weak compatibility for T and f in Theorem 2.2.2 is an essential condition.

(ii) In example 2.2.4 if we assume $T(x) = 2x^2 + 4x + 1$and $f(x) = 3x^2 + 6x + 2$, then T and f become weakly compatible and all conditions of Theorems 2.2.1, 2.2.2 and 2.2.3 are satisfied to obtain a unique point of coincidence and a unique common fixed point $-1 = f(-1) = T(-1)$.

Our next example demonstrates the crucial role of the condition $T(X) \subseteq f(X)$ in our results.

2.2.6 Example

Let $X = R^+$(the set of all non-negative real numbers), $E = R^2$

$$d(x, y) = (|x - y|, e|x - y|),$$

$$P = \{(x, y) : x, y \geq 0\},$$

$$T\ (x) = e^x \text{ and } fx = e^{x+1}.$$

Then,

$$TX = (0, \infty) \not\subseteq [e, \infty) = fX,$$

$$\begin{aligned} d(Tx, Ty) &= (|e^x - e^y|, |e^{x+1} - e^{y+1}|) \\ &= \frac{1}{e}(|e^{x+1} - e^{y+1}|, |e^{x+2} - e^{y+2}|). \\ &= \frac{1}{e} d(fx, fy). \end{aligned}$$

It follows that all the assumptions of Theorem 2.2.1 except $TX \subseteq fX$ are satisfied for $A = \frac{1}{e}$, $B = C = 0$, but T and f do not have a point of coincidence in X.

" The next theorem involves a contraction condition stronger than that of Theorems 2.2.1, 2.2.2, 2.2.3 and proves the existence of unique points of coincidence and unique common fixed points for three self mapping S, T, f on X. First we prove that, under certain conditions every S-T-sequence with initial point $x_0 \in X$ is a Cauchy sequence. "

2.2.7 Proposition

Let (X, d) be a cone metric space and P be a cone. Let $S, T, f : X \to X$ be such that $S(X) \cup T(X) \subseteq f(X)$. Assume that the following conditions hold:

(i) $d(Sx, Ty) \leqslant \alpha d(fx, Sx) + \beta d(fy, Ty) + \gamma\ d(fx, fy)$, for all $x, y \in X$, with $x \neq y$, where α, β, γ are non negative real numbers with $\alpha + \beta + \gamma < 1$;

(ii) $d(Sx, Tx) < d(fx, Sx) + d(fx, Tx)$, for all $x \in X$, whenever $Sx \neq Tx$.

Then every *S-T-sequence* with initial point $x_0 \in X$ is a Cauchy sequence.

Proof

Let x_0 be an arbitrary point in X and $\{fx_n\}$ be a *S-T-sequence* with initial point x_0. First, we assume that $fx_n \neq fx_{n+1}$ for all $n \in \mathbb{N}$. It implies that $x_n \neq x_{n+1}$ for all n. Then,

$$\begin{aligned} d(fx_{2k+1},\ fx_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\leqslant \alpha\ d(fx_{2k}, Sx_{2k}) + \beta\ d(fx_{2k+1}, Tx_{2k+1}) + \gamma d(fx_{2k}, fx_{2k+1}) \\ &\leqslant [\alpha+\gamma]\ d(fx_{2k}, fx_{2k+1}) + \beta d(fx_{2k+1}, fx_{2k+2}). \end{aligned}$$

It implies that

$$[1-\beta] d(fx_{2k+1},\ fx_{2k+2}) \leqslant [\alpha+\gamma]\ d(fx_{2k}, fx_{2k+1}),$$

so,

$$d(fx_{2k+1},\ fx_{2k+2}) \leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right] d(fx_{2k},\ fx_{2k+1}).$$

Similarly, from

$$\begin{aligned} d(fx_{2k+2},\ fx_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\leqslant \alpha\ d(fx_{2k+2}, Sx_{2k+2}) + \beta d(fx_{2k+1}, Tx_{2k+1}) + \gamma\ d(fx_{2k+2}, fx_{2k+1}) \\ &\leqslant \alpha\ d(fx_{2k+2},\ fx_{2k+3}) + \beta d(fx_{2k+1},\ fx_{2k+2}) + \gamma\ d(fx_{2k+2}, fx_{2k+1}) \\ &\leqslant [\beta+\gamma]\ d(fx_{2k+1},\ fx_{2k+2}) + \alpha d(fx_{2k+2}, fx_{2k+3}), \end{aligned}$$

we obtain

$$d(fx_{2k+2},\ fx_{2k+3}) \leqslant \left[\frac{\beta+\gamma}{1-\alpha}\right]\ d(fx_{2k+1},\ fx_{2k+2}).$$

Now, by induction, for each $k = 0, 1, 2, \ldots$, we deduce

$$\begin{aligned} d(fx_{2k+1},\ fx_{2k+2}) &\leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right] d(fx_{2k},\ fx_{2k+1}) \\ &\leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right]\left[\frac{\beta+\gamma}{1-\alpha}\right] d(fx_{2k-1},\ fx_{2k}) \\ &\leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right]\left[\frac{\beta+\gamma}{1-\alpha}\right]\left[\frac{\alpha+\gamma}{1-\beta}\right] d(fx_{2k-2},\ fx_{2k-1}) \\ &\leqslant \cdots \leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right]\left(\left[\frac{\beta+\gamma}{1-\alpha}\right]\left[\frac{\alpha+\gamma}{1-\beta}\right]\right)^k d(fx_0,\ fx_1), \end{aligned}$$

$$
\begin{aligned}
d(fx_{2k+2},\ fx_{2k+3}) &\leqslant \left[\frac{\beta+\gamma}{1-\alpha}\right] d(fx_{2k+1},\ fx_{2k+2})\\
&\leqslant \cdots \leqslant \left(\left[\frac{\beta+\gamma}{1-\alpha}\right]\ \left[\frac{\alpha+\gamma}{1-\beta}\right]\right)^{k+1} d(fx_0,\ fx_1).
\end{aligned}
$$

Let

$$
\lambda = \left[\frac{\alpha+\gamma}{1-\beta}\right], \qquad \mu = \left[\frac{\beta+\gamma}{1-\alpha}\right].
$$

Then $\lambda\mu < 1$. Now, for $p < q$, we have

$$
\begin{aligned}
d(fx_{2p+1},\ fx_{2q+1}) &\leqslant d(fx_{2p+1},\ fx_{2p+2}) + d(fx_{2p+2},\ fx_{2p+3}) + d(fx_{2p+3},\ fx_{2p+4})\\
&\quad + \cdots + d(fx_{2q},\ fx_{2q+1})\\
&\leqslant \left[\lambda \sum_{i=p}^{q-1} (\lambda\mu)^i + \sum_{i=p+1}^{q} (\lambda\mu)^i\right] d(fx_0, fx_1)\\
&\leqslant \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}}{1-\lambda\mu}\right] d(fx_0, fx_1)\\
&\leqslant (1+\mu)\lambda\frac{(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1)\\
&\leqslant \frac{2\,(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1).
\end{aligned}
$$

In analogous way, we deduce

$$
d(fx_{2p},\ fx_{2q+1}) \leqslant (1+\lambda)\frac{(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1) \leq \frac{2\,(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1),
$$

$$
d(fx_{2p},\ fx_{2q}) \leqslant (1+\lambda)\frac{(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1) \leq \frac{2\,(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1)
$$

and

$$
d(fx_{2p+1},\ fx_{2q}) \leqslant (1+\mu)\lambda\frac{(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1) \leq \frac{2\,(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1).
$$

Hence, for $0 < n < m$

$$
d(fx_n, fx_m) \leqslant \frac{2\,(\lambda\mu)^p}{1-\lambda\mu},
$$

where p is the integer part of $\frac{n}{2}$. Fix $0 \ll c$ and choose $I(0,\delta) = \{x \in E : \|x\| < \delta\}$ such that $c + I(0,\delta) \subset \mathrm{Int}P$. Since

$$\lim_{p\to\infty} \frac{2\,(\lambda\mu)^p}{1-\lambda\mu}\, d(fx_0, fx_1) = 0,$$

there exists $n_0 \in \mathbb{N}$ be such that

$$\frac{2\,(\lambda\mu)^p}{1-\lambda\mu}\, d(fx_0, fx_1) \in I(0,\delta)$$

for all $p \geq n_0$. The choice of $I(0,\delta)$ assures

$$c - \frac{2\,(\lambda\mu)^p}{1-\lambda\mu}\, d(fx_0, fx_1) \in \mathrm{Int}P,$$

so,

$$\frac{2\,(\lambda\mu)^p}{1-\lambda\mu}\, d(fx_0, fx_1) \ll c.$$

Consequently, for all $n, m \in \mathbb{N}$, with $2\,n_0 < n < m$, we have

$$d(fx_n, fx_m) \ll c,$$

and hence $\{fx_n\}$ is a Cauchy sequence. Now, we suppose that $fx_m = fx_{m+1}$ for some $m \in \mathbb{N}$. If $x_m = x_{m+1}$ and $m = 2k$, by (ii) we have

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &< d(fx_{2k}, Sx_{2k}) + d(fx_{2k+1}, Tx_{2k+1}) \\ &= d(fx_{2k+1}, fx_{2k+2}), \end{aligned}$$

which implies $fx_{2k+1} = fx_{2k+2}$. If $x_m \neq x_{m+1}$, we use (i) to obtain $fx_{2k+1} = fx_{2k+2}$. Similarly, we deduce that $fx_{2k+2} = fx_{2k+3}$ and so $fx_n = fx_m$ for every $n \geq m$. Hence, $\{fx_n\}$ is a Cauchy sequence.

2.2.8 Theorem

Let $(X,\ d)$ be a cone metric space and P be a cone. Let $S, T, f : X \to X$ be such that $S(X) \cup T(X) \subseteq f(X)$. Assume that the following conditions hold:

(i) $d(Sx, Ty) \leqslant \alpha d(fx, Sx) + \beta d(fy, Ty) + \gamma\, d(fx,\ fy)$, for all $x, y \in X$, with $x \neq y$, where α, β, γ are non negative real numbers with $\alpha + \beta + \gamma < 1$;

(ii) $d(Sx, Tx) < d(fx, Sx) + d(fx, Tx)$, for all $x \in X$, whenever $Sx \neq Tx$.

If $f(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover, if $(S, f\ )$ and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof

Let x_0 be an arbitrary point in X. By Proposition 2.2.7, every S-Tsequence $\{fx_n\}$ with initial point x_0 is a Cauchy sequence. If $f(X)$ is a complete subspace of X, there exist $u, v \in X$ such that $fx_n \to v = fu$ (this holds also if $S(X) \cup T(X)$ is complete with $v \in S(X) \cup T(X)$). From

$$\begin{aligned} d(fu,\ Su) &\leqslant d(fu,\ fx_{2n}) + d(fx_{2n},\ Su) \\ &\leqslant d(v,\ fx_{2n}) + d(Tx_{2n-1},\ Su) \\ &\leqslant d(v,\ fx_{2n})\ + \alpha\ d(fu,\ Su)\ + \beta\ d(fx_{2n-1}, Tx_{2n-1}) + \gamma d(fu,\ fx_{2n-1}), \end{aligned}$$

we obtain

$$d(fu, Su) \leqslant \frac{1}{1-\alpha}\left[d(v,\ fx_{2n}) + \beta d(fx_{2n-1},\ fx_{2n}) + \gamma d(v,\ fx_{2n-1})\right].$$

Fix $0 \ll c$ and choose $n_0 \in \mathbb{N}$ be such that

$$d(v, fx_{2n}) \ll kc, \quad d(fx_{2n-1}, fx_{2n}) \ll kc, \quad d(v, fx_{2n-1}) \ll kc$$

for all $n \geq n_0$, where $k = (1-\alpha)/(1+\beta+\gamma)$. Consequently, $d(fu, Su) \ll c$ and hence, $d(fu, Su) \ll c/m$ for every $m \in \mathbb{N}$. From

$$\frac{c}{m} - d(fu, Su) \in \mathrm{Int}P,$$

being P closed, as $n \to \infty$, we deduce $-d(fu, Su) \in P$ and so $d(fu, Su) = 0$. This implies that $fu = Su$. Similarly, by using the inequality,

$$d(fu, Tu) \leqslant d(fu, fx_{2n+1}) + d(fx_{2n+1}Tu),$$

we can show that $fu = Tu$. It implies that v is a point of coincidence of S, T and f, that is

$$v = fu = Su = Tu.$$

Now, we show that S, T and f have a unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^* = Tu^*$, for some u^* in X. From

$$\begin{aligned} d(v, v^*) &= d(Su, Tu^*) \\ &\leqslant \alpha d(fu, Su) + \beta d(fu^*, Tu^*) + \gamma d(fu, fu^*) \\ &\leqslant \alpha d(v, v) + \beta d(v^*, v^*) + \gamma d(v, v^*) \\ &\leqslant \gamma d(v, v^*), \end{aligned}$$

we deduce $v = v^*$. Moreover, if (S, f) and (T, f) are weakly compatible, then

$$Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv,$$

which implies $Sv = Tv = fv = w$ (say). Then w is a point of coincidence of S, T and f therefore, $v = w$, by uniqueness. Thus, v is a unique common fixed point of S, T and f. From Theorem 2.2.8, if we choose $S = T$, we deduce the following theorem.

2.2.9 Theorem

Let $(X,\, d)$ be a cone metric space, P be a cone and $T, f : X \to X$ be such that $T(X) \subseteq f(X)$. Assume that the following inequality holds:

$$d(Tx,Ty) \leqslant \alpha\, d(fx,Tx) + \beta\, d(fy,Ty) + \gamma\; d(fx,\; fy) \tag{2.7}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$. If $f(X)$ or $T(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

Theorem 2.2.9 generalizes Theorem 1 of [166].

2.2.10 Remark

In Theorem 2.2.9, the condition 2.7 can be replaced by

$$d(Tx,Ty) \leqslant \alpha[d(fx,Tx) + d(fy,Ty)] + \gamma\; d(fx,\; fy) \tag{2.8}$$

for all $x, y \in X$, where $\alpha, \gamma \in [0,1)$ with $2\alpha + \gamma < 1$. (2.8) $\Longrightarrow$ (2.7) is obvious. (2.7) $\Longrightarrow$ (2.8). If in (2.7) interchanging the roles of x and y and adding the resultant inequality to (2.7), we obtain

$$d(Tx,Ty) \leqslant \frac{\alpha+\beta}{2}\,[d(fx,Tx) + d(fy,Ty)] + \gamma\; d(fx,\; fy).$$

From Theorem 2.2.9, we deduce the followings corollaries.

2.2.11 Corollary

Let (X, d) be a cone metric space, P be a cone and the mappings $T,\; f : X \to X$ satisfy:

$$d(Tx,Ty) \leqslant \gamma\; d(fx,\; fy)$$

for all $x, y \in X$ where, $0 \leqslant \gamma < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover if $(T, f\,)$ are weakly compatible, then T and f have a unique common fixed point.

Corollary 2.2.11 generalizes Theorem 2.1 of [1], Theorem 1 of [78] and Theorem 2.3 of [141].

2.2.12 Corollary

Let (X, d) be a cone metric space, P be a cone and the mappings $T,\ f : X \to X$ satisfy:

$$d(Tx, Ty) \leqslant \alpha\,[d(fx, Tx) + d(fy, Ty)]$$

for all $x, y \in X$, where $0 \leqslant \alpha < \frac{1}{2}$. If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

Corollary 2.2.12 generalizes Theorem 2.3 of [1], Theorem 3 of [78] and Theorem 2.6 of [141].

2.2.13 Example

Let $X = \{a, b, c\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) \in E \mid x, y \geqslant 0\}$. Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y \\ (\frac{5}{7}, 5) & \text{if } x \neq y \text{ and } x, y \in X - \{b\} \\ (1,7) & \text{if } x \neq y \text{ and } x, y \in X - \{c\} \\ (\frac{4}{7}, 4) & \text{if } x \neq y \text{ and } x, y \in X - \{a\} \end{cases}$$

Define mappings $f, T : X \to X$ as follows:

$$f\ (x) = x,$$

$$T(x) = \begin{cases} c & \text{if } x \neq b, \\ a & \text{if } x = b. \end{cases}$$

2.2.15 Theorem

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t, s, x(s))ds + g(t),$$

$$x(t) = \int_a^b K_2(t, s, x(s))ds + h(t),$$

where $t \in [a, b] \subset \mathbb{R}$, $x, g, h \in X$.

Assume that $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that

(i) F_x, $G_x \in X$, for each $x \in X$, where

$$F_x(t) = \int_a^b K_1(t, s, x(s))ds,$$

$$G_x(t) = \int_a^b K_2(t, s, x(s))ds \quad \forall\, t \in [a, b], \tag{2.9}$$

(ii) there exist $\beta, \gamma, \delta \geq 0$, such that

$$\left(|F_x(t) - G_y(t) + g(t) - h(t)|, p\,|F_x(t) - G_y(t) + g(t) - h(t)|\right)$$

$$\begin{aligned} \leq\ & \alpha\left(|F_x(t) + g(t) - x(t)|, p\,|F_x(t) + g(t) - x(t)|\right) \\ & + \beta\left(|G_y(t) + h(t) - y(t)|, p\,|G_y(t) + h(t) - y(t)|\right) \\ & + \gamma\left(|x(t) - y(t)|, p\,|x(t) - y(t)|\right), \end{aligned}$$

where $\alpha + \beta + \gamma < 1$, for every $x, y \in X$ with $x \neq y$ and $t \in [a, b]$.

(iii) whenever $F_x + g \neq G_x + h$

$$\begin{aligned} & \sup_{t\in[a,b]} \left(|F_x(t) - G_x(t) + g(t) - h(t)|, p\,|F_x(t) - G_x(t) + g(t) - h(t)|\right) \\ < \ & \sup_{t\in[a,b]} \left(|F_x(t) + g(t) - x(t)|, p\,|F_x(t) + g(t) - x(t)|\right) \\ & + \sup_{t\in[a,b]} \left(|G_x(t) + h(t) - x(t)|, p\,|G_x(t) + h(t) - x(t)|\right), \end{aligned}$$

for every $x \in X$. Then, the system of integral equations (2.9) have a unique common solution.

Proof

Define $S, T : X \to X$ by $S(x) = F_x + g,\ T(x) = G_x + h$. It is easily seen that

$$\begin{aligned}(\|S-T\|_\infty, p\|S-T\|_\infty) \;\leq\; & \alpha(\|S(x)-x\|_\infty, p\|S(x)-x\|_\infty) \\ & +\beta(\|T(y)-y\|_\infty, p\|T(y)-y\|_\infty) \\ & +\gamma(\|x-y\|_\infty, p\|x-y\|_\infty),\end{aligned}$$

for every $x, y \in X$, with $x \neq y$ and if $S(x) \neq T(x)$.

$$\begin{aligned}(\|S-T\|_\infty, p\|S-T\|_\infty) \;<\; & (\|S(x)-x\|_\infty, p\|S(x)-x\|_\infty) \\ & +(\|T(x)-x\|_\infty, p\|T(x)-x\|_\infty)\end{aligned}$$

for every $x, y \in X$, with $x \neq y$ and if $S(x) \neq T(x)$

$$d(S(x), T(x)) < d(S(x), x) + d(T(x), x)$$

for every $x \in X$. By Theorem 2.2.8, if f is the identity map on X, the Urysohn integral equations (2.9) have a unique common solution.

2.3 Weakly $\varphi-$pairs and common fixed points in cone metric spaces

Results given in this section will appear in [167].

"In this section Theorem 2.4.4, establishes the existence of unique point of coincidence of three self mapping $S, T,\ f : X \to X$ satisfying a φ- contraction condition with $S(X) \cup T(X) \subseteq f(X)$. Moreover, if (S, f) and (T, f) are weakly compatible then S, T and f have a unique common fixed point. Our results in this section improve and generalize some results in [1, 78, 134, 141, 166]."

which is a contradiction and so $z_k = d(fx_{2k}, fx_{2k+1})$. Therefore,

$$d(fx_{2k+1},\ fx_{2k+2}) \leq \varphi(d(fx_{2k}, fx_{2k+1})), \quad \text{for all } n \in \mathbb{N}.$$

Similarly, we obtain

$$d(fx_{2k+2},\ fx_{2k+3}) \leq \varphi(d(fx_{2k+1}, fx_{2k+2})), \quad \text{for all } n \in \mathbb{N}.$$

We deduce that

$$d(fx_n, fx_{n+1}) \leq \varphi(d(fx_{n-1}, fx_n)), \quad \text{for all } n \in \mathbb{N},$$

and consequently,

$$d(fx_n, fx_{n+1}) \leq \varphi^n(d(fx_0, fx_1)), \quad \text{for all } n \in \mathbb{N}. \tag{2.10}$$

Fix $\mathbf{0} \ll c$. We choose a positive real number δ such that $(c - \varphi(c))/2 + I(\mathbf{0}, \delta) \subset \mathrm{Int}P$, where $I(\mathbf{0}, \delta) = \{y \in E : \|y\| < \delta\}$. By (iii) of Definition 2.3.1, there exists a natural number N such that $\varphi^m(d(fx_0, fx_1)) \in I(\mathbf{0}, \delta)$, for all $m \geq N$. Then

$$\varphi^m(d(fx_0, fx_1)) \ll (c - \varphi(c))/2,$$

for all $m \geq N$. Consequently, $d(fx_m, fx_{m+1}) \ll (c - \varphi(c))/2$, for all $m \geq N$. Fix $m \geq N$. Now we prove

$$d(fx_{2m+1}, fx_{2n+2}) \ll c \tag{2.11}$$

for all $n \geq m$. Note that (2.11) holds when $n = m$. Assume that (2.11) holds for some $n = k \geq m$. Then we have

$$d(fx_{2m+1}, fx_{2k+2}) \ll c$$

and

$$d(fx_{2m+2}, fx_{2k+3}) = d(Tx_{2m+1}, Sx_{2k+2}) \leq \varphi(z),$$

for some

$$z \in M(x_{2k+2}, x_{2m+1}) = \{d(fx_{2k+2}, fx_{2k+3}), d(fx_{2m+1}, fx_{2m+2}), d(fx_{2k+2}, fx_{2m+1})\}.$$

It implies that

$$d(f(x_{2m+2}), f(x_{2k+3})) \leq \varphi(c)$$

and consequently,

$$\begin{aligned} d(fx_{2m+1}, fx_{2k+4}) &\leq d(fx_{2m+1}, f(x_{2m+2}) + d(fx_{2m+2}, fx_{2k+3}) + d(fx_{2k+3}, fx_{2k+4}) \\ &\ll \frac{c - \varphi(c)}{2} + \varphi(c) + \frac{c - \varphi(c)}{2} = c. \end{aligned}$$

Therefore, (2.11) holds when $n = k + 1$. By induction, we deduce (2.11) holds for all $n \geq m$. This is sufficient to conclude that $\{fx_n\}$ is a Cauchy sequence.

2.3.3 Theorem

Let $(X,\ d)$ be a cone metric space and P be a cone. Let $S, T, f : X \rightarrow X$ be such that $S(X) \cup T(X) \subseteq f(X)$. Assume that there exists a φ−map such that

$$d(Sx, Ty) \leqslant \varphi(z), \quad \textit{for some } z \in M(x, y),$$

for all $x, y \in X$. If $f(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover, if $(S, f\ )$ and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof

Let x_0 be an arbitrary point in X. By Proposition 2.3.2 every S-T-sequence $\{fx_n\}$ with initial point x_0 is a Cauchy sequence. If $f(X)$ is a complete subspace of X, there exist $u, v \in X$ such that $fx_n \rightarrow v = fu$ (this holds also if $S(X) \cup T(X)$ is complete with $v \in S(X) \cup T(X)$).

Then

$$\begin{aligned} d(fu,\ Su) &\leqslant d(fu,\ fx_{2n}) + d(fx_{2n},\ Su) \\ &\leqslant d(v,\ fx_{2n}) + \varphi(z_n), \end{aligned}$$

with $z_n \in \{d(fu, Su), d(fx_{2n-1}, fx_{2n}), d(fu, fx_{2n-1})\}$. Fix $0 \ll c$ and choose $L \in \mathbb{N}$ be such that

$$d(v, fx_{2n}) \ll c/2, \quad d(fx_{2n-1}, fx_{2n}) \ll c/2, \quad d(v, fx_{2n-1}) \ll c/2$$

for all $n \geq L$. Now for infinitely many $n \in \mathbb{N}$ one of the following conditions holds:

(i) $d(fu, Su) \ll c/2 + \varphi(c/2) \ll c/2 + c/2 = c$,

(ii) $d(fu, Su) - \varphi(d(fu, Su)) \ll c/2$.

Furthermore, either (i) or (ii) implies $fu = Su$. Similarly, by using the inequality,

$$d(fu, Tu) \leqslant d(fu, fx_{2n+1}) + d(fx_{2n+1}Tu)$$

we can show that $fu = Tu$. It implies that v is a point of coincidence of S, T and f, that is

$$v = fu = Su = Tu.$$

Now, we show that S, T and f have a unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^* = Tu^*$, for some u^* in X. From

$$d(v, v^*) = d(Su, Tu^*) \leqslant \varphi(z),$$

with $z \in \{d(fu, Su), d(fu^*, Tu^*), d(fu, fu^*)\}$ we deduce $v = v^*$. Moreover, if (S, f) and (T, f) are weakly compatible, then

$$Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv,$$

which implies $Sv = Tv = fv = w$ (say). Then w is a point of coincidence of S, T and f therefore, $v = w$, by uniqueness. Thus, v is a unique common fixed point of S, T and f.

2.3.4 Remark

From Theorem 2.3.3 if we choose $S = T$, we obtain Theorem 1 of [46] and if we choose $f = I_X$, the identity map on X, we obtain Theorem 4 of [46].

2.4 Banach contraction principle in rectangular cone metric spaces

Results given in this section have been published in [23].

"In this section, we introduce the notion of cone rectangular metric space and prove that a self mapping on complete cone rectangular metric space satisfying the Banach contraction condition has a unique fixed point."

2.4.1 Definition

"Let X be a nonempty set. Suppose, the mapping $d : X \times X \to E$ satisfies:

1. $0 \leq d(x, y)$, *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;
2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
3. $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ *for all* $x, y, \in X$ *and for all distinct points* $w, z \in X - \{x, y\}$ *[rectangular property].*

Then, d is called a cone rectangular metric on X, and (X, d) is called a cone rectangular metric space."

2.4.2 Definition

"Let x_n be a sequence in (X, d) and $x \in (X, d)$. If for every $c \in E$, with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $lim_n x_n = x$, or $x_n \longrightarrow x$, as $n \to \infty$."

2.4.3 Definition

"*If for every* $c \in E$ *with* $0 \ll c$, *there is* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$, $d(x_n, x_{n+m}) \ll c$, *then* $\{x_n\}$ *is called a Cauchy sequence in* (X, d). *If every Cauchy sequence is convergent in* (X, d), *then* (X, d) *is called a complete cone rectangular metric space.*"

2.4.4 Example

Let $X = \mathbb{N}$, $E = R^2$ and

$$P = \{(x, y) : x, y \geq 0\}.$$

Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ (3,9) & \text{if } x \text{ and } y \text{ are in } \{1,2\},\ x \neq y. \\ (1,3) & \text{otherwise.} \end{cases}$$

Now (X, d) is a cone rectangular metric space but (X, d) is not a cone metric space because it lacks the triangular property:

$$(3,9) = d(1,2) > d(1,3) + d(3,2) = (1,3) + (1,3) = (2,6)$$

as $(3,9) - (2,6) = (1,3) \in P$.

2.4.5 Lemma

Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant κ. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\|d(x_n, x)\| \to 0$ *as* $n \to \infty$.

Proof

Suppose that $\{x_n\}$converges to x. For a given real number $\epsilon > 0$, one has $c \in E$, with $0 \ll c$ such that $\kappa \|c\| < \epsilon$. Then for this $0 \ll c$, there is a natural number N, such that

$$d(x_n, x) \ll c \text{ for all } n > N.$$

Since P is a normal cone with normal constant κ, therefore,

$$\|d(x_n, x)\| \leq \kappa \|c\| < \epsilon \text{ for all } n > N.$$

It follows that

$$\|d(x_n, x)\| \to 0 \text{ as } n \to \infty.$$

Conversely, suppose that $\|d(x_n, x)\| \to \mathbf{0}$ as $n \to \infty$. For given $c \in E$ with $\mathbf{0} \ll c$, one has $\delta > 0$, such that

$$c - B(\mathbf{0}; \delta) \subseteq intP,$$

where $B(0; \delta) = \{x \in E : \|x\| < \delta\}$. For this δ, there is a natural number N such that

$$\|d(x_n, x)\| < \delta \text{ for all } n > N.$$

That is $d(x_n, x) \in B(\mathbf{0}; \delta)$ for all $n > N$. This means $c - d(x_n, x) \in intP$ for all $n > N$. Hence, $d(x_n, x) \ll c$ for all $n > N$. Therefore, $\{x_n\}$ converges to x.

2.4.6 Lemma

Let (X, d) be a cone rectangular metric space, P be a normal cone with normal constant κ. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\|d(x_n, x_{n+m})\| \to 0$ *as* $n \to \infty$.

Proof

Suppose that $\{x_n\}$ is a Cauchy sequence. For given $\epsilon > 0$, one can choose $c \in E$ with $\mathbf{0} \ll c$ and $\kappa \|c\| < \epsilon$. Then for this $\mathbf{0} \ll c$, there is a natural number N such that

$$d(x_n, x_{n+m}) \ll c \text{ for all } n > N.$$

Since P is a normal cone with normal constant κ, therefore,

$$\|d(x_n, x_{n+m})\| \leq \kappa \|c\| < \epsilon \text{ for all } n > N.$$

It follows that

$$\|d(x_n, x_{n+m})\| \to 0 \text{ as } n \to \infty.$$

Conversely, suppose that $\|d(x_n, x_{n+m}) \to \mathbf{0}\|$ as $n \to \infty$. For given $c \in E$ with $\mathbf{0} \ll c$, one has $\delta > 0$, such that:

$$c - B(\mathbf{0}; \delta) \subseteq intP.$$

For this δ, there is a natural number N such that:

$$\|d(x_n, x_{n+m})\| < \delta \text{ for all } n > N.$$

That is $d(x_n, x_{n+m}) \in B(\mathbf{0}; \delta)$ for all $n > N$. This means that $c - d(x_n, x_{n+m}) \in intP$, for all $n > N$. Hence, $d(x_n, x_{n+m}) \ll c$ for all $n > N$. Therefore, $\{x_n\}$ is a Cauchy sequence.

2.4.7 Theorem

Let (X, d) be a cone rectangular metric space, P be a normal cone with normal constant κ and the mapping $T : X \to X$ satisfies:

$$d(Tx, Ty) \leq \lambda\, d(x, y)$$

for all $x, y \in X$, where $0 \leq \lambda < 1$. Then T has a unique fixed point.

Proof

Let x_0 be an arbitrary point in X. Define a sequence of points in X as follows:

$$x_{n+1} = Tx_n = T^{\,n+1}x_0, \;\; n = 0, 1, 2, ...$$

We can suppose that x_0 is not a periodic point, in fact if $x_n = x_0$, then,

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) = d(T^{\,n}x_0, T^{\,n+1}x_0) \leq \lambda d(T^{\,n-1}x_0, T^{\,n}x_0) \\ &\leq \lambda^2 d(T^{\,n-2}x_0, T^{\,n-1}x_0) \leq ... \leq \lambda^n d(x_0, Tx_0). \end{aligned}$$

It follows that

$$[\lambda^n - 1]\ d(x_0, Tx_0) \in P.$$

It further implies that

$$\left[\frac{\lambda^n - 1}{1-\lambda^n}\right]\ d(x_0, Tx_0) \in P.$$

Hence $-d(x_0, Tx_0) \in P$ and $d(x_0, Tx_0) = \mathbf{0}$, this means x_0 is a fixed point of T. Thus, in this sequel of proof we can suppose that $x_m \neq x_n$ for all distinct $m, n \in \mathbb{N}$. Now, by using rectangular property for all $y \in X$, we have,

$$\begin{aligned} d(y, T^{4}y) &\leq d(y, Ty) + d(Ty, T^{2}y) + d(T^{2}y, T^{4}y) \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, T^{2}y). \end{aligned}$$

Similarly,

$$\begin{aligned} d(y, T^{6}y) &\leq d(y, Ty) + d(Ty, T^{2}y) + d(T^{2}y, T^{3}y) + d(T^{3}y, T^{4}y) \\ &\quad + d(T^{4}y, T^{6}y) \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \lambda^3 d(y, T\ y) \\ &\quad + \lambda^4 d(y, T^{2}y) \\ &\leq \sum_{i=0}^{3} \lambda^i d(y, Ty) + \lambda^4\ d(y, T^{2}y), \text{for all } y \in X. \end{aligned}$$

Now by induction, we obtain for each $k = 2, 3, 4, \ldots,$

$$d(y, T^{2k}y) \leq \sum_{i=0}^{2k-3} \lambda^i d(y, Ty) + \lambda^{2k-2}\ d(y, T^{2}y). \tag{2.12}$$

Moreover, for all $y \in X$,

$$\begin{aligned} d(y, T^{5}y) &\leq d(y, Ty) + d(Ty, T^{2}y) + d(T^{2}y, T^{3}y) + d(T^{3}y, T^{4}y) \\ &\quad + d(T^{4}y, T^{5}y) \\ &\leq \sum_{i=0}^{4} \lambda^i d(y, Ty). \end{aligned}$$

By induction, for each $k = 0, 1, 2, \ldots$ we have,

$$d(y, T^{2k+1}y) \leq \sum_{i=0}^{2k} \lambda^i d(y, Ty). \tag{2.13}$$

Using inequality (2.12), for $k = 1, 2, 3, \ldots$ we have,

$$\begin{aligned}
d(T^{n}x_0, T^{n+2k}x_0) &\leq \lambda^n d(x_0, T^{2k}x_0) \\
&\leq \lambda^n \left[\sum_{i=0}^{2k-3} \lambda^i \left(d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right) \right. \\
&\quad \left. + \lambda^{2k-2} \left(d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right) \right] \\
&\leq \lambda^n \sum_{i=0}^{2k-2} \lambda^i \left[d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right] \\
&\leq \frac{\lambda^n (1 - \lambda^{2k-1})}{1 - \lambda} \left[d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right] \\
&\leq \frac{\lambda^n}{1 - \lambda} \left[d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right].
\end{aligned}$$

Similarly, for $k = 0, 1, 2, \ldots$, inequality (2.13) implies that

$$\begin{aligned}
d(T^{n}x_0, T^{n+2k+1}x_0) &\leq \lambda^n d(x_0, T^{2k+1}x_0) \\
&\leq \lambda^n \sum_{i=0}^{2k} \lambda^i d(x_0, Tx_0) \\
&\leq \frac{\lambda^n}{1 - \lambda} \left[d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right].
\end{aligned}$$

Thus,

$$d(T^{n}x_0, T^{n+m}x_0) \leq \frac{\lambda^n}{1 - \lambda} \left[d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right].$$

Since P is a normal cone with normal constant κ, therefore,

$$\left\| d(T^{n}x_0, T^{n+m}x_0) \right\| \leq \frac{\lambda^n}{1 - \lambda} \kappa \left\| \left[d(x_0, Tx_0) + d(x_0, T^{2}x_0) \right] \right\|.$$

Therefore, $\|d(x_n, x_{n+m})\| \to 0$ as $n \to \infty$. Now Lemma 2.4.6, implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$. By Lemma 2.4.5, we have

$$\|d(T^{\,n}x_0, u)\| \to 0 \text{ as } n \to \infty.$$

Since $x_n \neq x_m$ for $n \neq m$, therefore by rectangular property, we have

$$\begin{aligned} d(Tu, u) &\leq \lambda d(u, T^{\,n-1}x_0) + d(T^{\,n}x_0, T^{\,n+1}x_0) + d(T^{\,n+1}x_0, u) \\ &\leq \lambda d(u, T^{\,n-1}x_0) + \lambda^n d(x_0, Tx_0) + d(T^{\,n+1}x_0, u). \end{aligned}$$

Thus,

$$\|d(Tu, u)\| \leq \kappa \left[\lambda \left\|d(u, T^{\,n-1}x_0)\right\| + \lambda^n \|d(x_0, Tx_0)\| + \left\|d(T^{\,n+1}x_0, u)\right\|\right].$$

Letting $n \to \infty$,we have

$$\|d(u, Tu)\| = 0.$$

Hence $u = Tu$. Now, we show that T has a unique fixed point. For this, let's assume that there exists another point v in X such that $v = Tv$. Now,

$$\begin{aligned} d(v, u) &= d(Tv, Tu) \\ &\leq \lambda\, d(v, u). \end{aligned}$$

Hence, $u = v$.

2.4.8 Example

Let $X = \{1, 2, 3, 4\}$, $E = R^2$ and

$$P = \{(x, y) : x, y \geq 0\}$$

is a normal cone in E. Define $d : X \times X \to E$ as follows:

$$\begin{aligned} d(1,2) &= d(2,1) = (3,6) \\ d(2,3) &= d(3,2) = d(1,3) = d(3,1) = (1,2) \\ d(1,4) &= d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = (2,4). \end{aligned}$$

Then (X, d) is a complete rectangular cone metric space but (X, d) is not a cone metric space because it lacks the triangular property:

$$(3,6) = d(1,2) > d(1,3) + d(3,2) = (1,2) + (1,2) = (2,4)$$

as $(3,6) - (2,4) = (1,2) \in P$. Now define a mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} 3 & if\ x \neq 4, \\ 1 & if\ x = 4. \end{cases}$$

Note that

$$d(T(1), T(2)) = d(T(1), T(3)) = d(T(2), T(3)) = 0$$

and in all other cases

$$d(Tx, Ty) = (1,2),\ d(x,y) = (2,4).$$

Hence, for $\lambda = \frac{1}{2}$, all conditions of Theorem 2.4.7 are satisfied to obtain a unique fixed point 3 of T.

2.4.9 Remark

In example 2.4.8, results of Huang and Zhang [78] are not applicable to obtain the fixed point of the mapping T on X, since (X, d) is not a cone metric space.

Chapter 3

Fixed points of multi-valued mappings

"*Nadler [118] was the first to combine the ideas of multivalued mappings and contractions. He proved some remarkable fixed point results for multivalued contractions. He also introduced the idea of multivalued locally contractions and generalized a fixed point theorem of Edelstein [52]. Afterwards, Dube and Singh [48], Iseki [84], Ray [136], Itoh and Takahashi [85], Aubin and Siegel [13], Hu [76], Massa [113], Kaneko [95, 97] and many others have studied fixed theorems for multivalued contractive type mappings. Kaneko [97], Naimpally, Singh and Whitfield [119], Rhoade, Singh and Kulshrestha [144] and several other authors succeeded to extend the fundamental contraction theorem of Banach to a pair of mappings. Beg and Azam [28] extended a result of [118] for a pair of mappings.*

The results of Razani and Fouladgar [138], motivated us to study the fixed points of sequence of locally contractive multivalued maps in an $\varepsilon-$chainable metric space. This area was not in focus of recent research for a while (since Waters [173] for single valued and Beg and Azam [28] for multivalued mappings) so we thought to build on the basic work of Nadler (1969). We believe that our work will highlight this area and research focus will again tilt back into fixed points of locally contractive mappings in $\varepsilon-$chainable metric spaces.

Fixed point theorems in fuzzy mathematics are emerging with varying hope and vital trust. Weiss [174] and Butnariu [38] initiated the study of fixed point theorems in fuzzy mathemat-

ics. Heilpern [73] first used the concept of fuzzy mappings to prove a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem for multivalued mappings of Nadler [118]. Afterwords, Arora and Sharma [10], Azam and Beg [20], Bose and Sahani [37], Lee and Cho [109], Park and Jeong [126], Rashwan and Ahmad [135], Rhoades [147], Som and Mukherjees [159], Vijayaraju and Marudai [169], among others studied fixed point theorems for fuzzy generalized contractive mappings. This chapter deals with the study of fixed point and common fixed point results for multivalued and fuzzy mappings.

In section 3.1, the existence of fixed points of sequence of locally contractive multivalued maps have been established. As an application, common fixed points of sequence of single valued expansive type mappings have been obtained.

In section 3.2, we establish some fixed point theorems for fuzzy contractive and fuzzy locally contractive mappings on a compact metric space using d_∞-metric for fuzzy sets.

In section 3.3, we establish common fixed point theorems for fuzzy mappings under $\varphi-$contraction condition on a metric space with the d_∞ metric on the family of fuzzy sets.

In section 3.4, a result on a common fixed point theorem for a pair of contractive type fuzzy mappings in a metric space is established which improves/rectifies the result of Vijayaraju and Marudai [169]."

3.1 Fixed points of a sequence of locally contractive multivalued maps

Results given in this section have been published in [15].

"In this section we prove the existence of common fixed points for a sequence of locally contractive multivalued maps in a $\varepsilon-$ chainable metric space and use it to obtain a result regarding common fixed points of a sequence of single valued uniformly locally expansive mappings

We recall the following concepts due to Edelstein [52, 53]."

3.1.1 Definition

"Let (X,d) be a metric space, $\varepsilon > 0$, $0 \leq \lambda < 1$, and $x, y \in X$. A strict $\varepsilon-$ chain from x to y is a finite set of points $x_1, x_2, x_3, ..., x_n$ such that $x = x_1$, $x_n = y$, and $d(x_{j-1}, x_j) < \varepsilon$ for

all $j = 1, 2, 3..., n$. A metric space (X, d) is said to be strict $\varepsilon-$ chainable if and only if given $x, y \in X$, there exists an $\varepsilon-$ chain from x to y.

In the section, for the sake of convenience, we will be using the term $\varepsilon-$ chain ($\varepsilon-$ chainable) instead of strict $\varepsilon-$ chain (strict $\varepsilon-$ chainable)."

3.1.2 Definition

"A mapping $T : X \to X$ is called a (ε, λ) uniformly locally contractive mapping if $0 < d(x, y) < \varepsilon$ implies $d(Tx, Ty) \leq \lambda d(x, y)$. For $\eta > 0$, $T : X \to X$ is called an (ε, η) uniformly locally expansive mapping if $0 < d(x, y) < \varepsilon$ implies $d(Tx, Ty) \geq \eta d(x, y)$."

3.1.3 Remark

"We observe that a globally contractive (contractive) mapping can be regarded as (∞, λ) uniformly locally contractive mapping and for some special spaces, every locally contractive mapping is globally contractive. For details (cf. [76, 116, 138])."

3.1.4 Theorem [52]

"Let (X, d) be a complete $\varepsilon-$ chainable metric space. If $T : X \to X$ is an (ε, λ) uniformly locally contractive mapping. Then T has a unique fixed point."

3.1.5 Definition

"A mapping $T : X \to CB(X)$ is called globally non-expansive multivalued mapping if for all $x, y \in X$, $H(Tx, Ty) \leq d(x, y)$. $T : X \to CB(X)$ is called $\varepsilon-$ non-expansive multivalued mapping if $x, y \in X$, $0 < d(x, y) < \varepsilon$ implies $H(Tx, Ty) \leq d(x, y)$.

A mapping $K : (0, \varepsilon) \to [0, 1)$ is said to have property (p) if for $t \in (0, \varepsilon)$ there exists $\delta(t) > 0$, $s(t) < 1$ such that $0 \leq r - t < \delta(t)$ implies $K(r) \leq s(t) < 1$ (cf. [28, 76, 116, 140]).

Nadler [118] extended the theorem 3.1.4 to multivalued mappings as follows:"

3.1.6 Theorem [118]

"Let (X, d) be a complete $\varepsilon-$ chainable metric space and $T : X \to C(X)$ be a mapping satisfying the following condition:

$x, y \in X$ and $0 < d(x,y) < \varepsilon$ implies $H(Tx, Ty) \leq \lambda d(x,y)$. Then T has a fixed point.

As an improvement and generalization of the above theorem we have the following result:"

3.1.7 Theorem

Let (X, d) be a complete $\varepsilon-$ chainable metric space and $\{T_n\}_{n=1}^{\infty}$be a sequence of mappings from X to $CB(X)$ satisfying the following condition:

$x, y \in X$ and $0 < d(x,y) < \varepsilon$ implies

$$H(T_n x, T_m y) \leq K(d(x,y))d(x,y) \tag{3.1}$$

for $n, m = 1, 2...$, where $K : (0, \varepsilon) \to [0, 1)$ is a function having property (p). Then there exists a point $y^* \in X$ such that $y^* \in \cap_{n=1}^{\infty} T_n y^*$.

Proof

Let y_0 be an arbitrary, but fixed element of X. We shall construct a sequence $\{y_n\}$ of points of X as follows. Let $y_1 \in X$ be such that $y_1 \in T_1 y_0$. Also let $y_0 = x_{(1,0)}, x_{(1,1)}, x_{(1,2)}, ..., x_{(1,m)} = y_1 \in T_1 y_0$ be an arbitrary $\varepsilon-$ chain from y_0 to y_1. Rename y_1 as $x_{(2,0)}$. Since $x_{(2,0)} \in T_1 x_{(1,0)}$,

$$\begin{aligned} H(T_1 x_{(1,0)}, T_2 x_{(1,1)}) &\leq K(d(x_{(1,0)}, x_{(1,1)}))d(x_{(1,0)}, x_{(1,1)}) \\ &< \sqrt{[K(d(x_{1,0)}, x_{1,1)}))]}d(x_{(1,0)}, x_{(1,1)}) \\ &< d(x_{(1,0)}, x_{(1,1)}) \\ &< \varepsilon. \end{aligned}$$

Using Lemma 1.1.9, we obtain $x_{(2,1)} \in T_2 x_{(1,1)}$ such that

$$\begin{aligned} d(x_{(2,0)}, x_{(2,1)}) &< \sqrt{[K(d(x_{(1,0)}, x_{(1,1)}))]}d(x_{(1,0)}, x_{(1,1)}) \\ &< d(x_{(1,0)}, x_{(1,1)}) \\ &< \varepsilon. \end{aligned} \tag{3.2}$$

Since $x_{(2,1)} \in T_2 x_{(1,1)}$ and

$$\begin{aligned} H(T_2 x_{(1,1)}, T_2 x_{(1,2)}) &\leq K(d(x_{(1,1)}, x_{(1,2)}))d(x_{(1,1)}, x_{(1,2)}) \\ &< \sqrt{[K(d(x_{(1,1)}, x_{(1,2)}))]}d(x_{(1,1)}, x_{(1,2)}) \\ &< d(x_{(1,1)}, x_{(1,2)}) \\ &< \varepsilon. \end{aligned}$$

We may choose an element $x_{(2,2)} \in T_2 x_{(1,2)}$ such that

$$\begin{aligned} d(x_{(2,1)}, x_{(2,2)}) &< \sqrt{[K(d(x_{(1,1)}, x_{(1,2)}))]}d(x_{(1,1)}, x_{(1,2)}) \\ &< d(x_{(1,1)}, x_{(1,2)}) \\ &< \varepsilon. \end{aligned}$$

Thus we obtain a finite set of points $x_{(2,0)}, x_{(2,1)}, x_{(2,2)}, ..., x_{(2,m)}$ such that $x_{(2,0)} \in T_1 x_{(1,0)}$ and $x_{(2,j)} \in T_2 x_{(1,j)}$, for $j = 1, 2, 3..., m$, with

$$\begin{aligned} d(x_{(2,j)}, x_{(2,j+1)}) &< \sqrt{[K(d(x_{(1,j)}, x_{(1,j+1)}))]}d(x_{(1,j)}, x_{(1,j+1)}) \\ &< d(x_{(1,j)}, x_{(1,j+1)}) \\ &< \varepsilon, \end{aligned}$$

for $j = 0, 1, 2..., m-1$. Let $x_{(2,m)} = y_2$, then the set of points $y_1 = x_{(2,0)}, x_{(2,1)}, x_{(2,2)}, ..., x_{(2,m)} = y_2 \in T_2 y_1$ is an $\varepsilon-$ chain from y_1 to y_2. Rename y_2 as $x_{(3,0)}$, then by the same procedure we obtain an ε- chain $y_2 = x_{(3,0)}, x_{(3,1)}, x_{(3,2)}, ..., x_{(3,m)} = y_3 \in T_3 y_2$ from y_2 to y_3. Inductively, we obtain $y_n = x_{(n+1,0)}, x_{(n+1,1)}, x_{(n+2,2)}, ..., x_{(n+1,m)} = y_{n+1} \in T_{n+1} y_n$ with

$$\begin{aligned} d(x_{(n+1,j)}, x_{(n+1,j+1)}) &< \sqrt{[K(d(x_{(n,j)}, x_{(n,j+1)}))]}d(x_{(n,j)}, x_{(n,j+1)}) \\ &< d(x_{(n,j)}, x_{(n,j+1)}) \\ &< \varepsilon, \end{aligned} \tag{3.3}$$

for j = 0,1,2...,m - 1 and n = 0,1,2,.... Consequently, we obtain a sequence $\{y_n\}$ of points of X

with

$$
\begin{aligned}
y_1 &= x_{(1,m)} = x_{(2,0)} \in T_1 y_0 \\
y_2 &= x_{(2,m)} = x_{(3,0)} \in T_2 y_1 \\
y_3 &= x_{(3,m)} = x_{(4,0)} \in T_3 y_2,
\end{aligned}
$$

and so on. That is,

$$y_{n+1} = x_{(n+1,m)} = x_{(n+2,0)} \in T_{n+1} y_n, \text{ for } n = 0, 1, 2, \ldots.$$

Then from (3.3), we see that $\{d(x_{(n,j)}, x_{(n,j+1)}) : n \geq 0\}$ is a decreasing sequence of non-negative real numbers and therefore, tends to a limit $t \geq 0$. We claim $t = 0$. For if $t > 0$, the inequality (3.3) yields $t < \varepsilon$. Then by the property (p) of K there exists $\delta(t) > 0$, $s(t) < 1$ such that

$$0 \leq r - t < \delta(t) \text{ implies } K(r) \leq s(t) < 1.$$

For this $\delta(t) > 0$, there exists an integer N such that

$$0 \leq d(x_{(n,j)}, x_{(n,j+1)}) - t < \delta(t) \text{ for } n \geq N.$$

Hence,

$$K(d(x_{(n,j)}, x_{(n,j+1)})) \leq s(t) < 1 \text{ whenever } n \geq N.$$

Then

$$
\begin{aligned}
d(x_{(n,j)}, x_{(n,,j+1)}) &< \sqrt{[K(d(x_{(n-1,j)}, x_{(n-1,j+1)}))]} d(x_{(n-1,j)}, x_{(n-1,j+1)}) \\
&< \max\left\{ \max_{i=1}^{N} \sqrt{[K(d(x_{(i,j)}, x_{(i,j+1)}))]}, s(t) \right\} d(x_{(n-1,j)}, x_{(n-1,j+1)}) \\
&< \left(\max\left\{ \sqrt{\max_{i=j}^{N} [K(d(x_{(i,j)}, x_{(i,j+1)}))], s(t)} \right\} \right)^n d(x_{(0,j)}, x_{(0,j+1)}).
\end{aligned}
$$

On letting $n \to \infty$ and in the view of $\max\left\{ \sqrt{\max_{i=j}^{N} [K(d(x_{(i.j)}, x_{(i.j+1)}))], s(t)} \right\} < 1$, the above inequality yields $d(x_{(n,j)}, x_{(n,j+1)}) \to 0$, a contraction. Hence $t = 0$ and

$$d(y_{n-1}, y_n) = d(x_{(n,0)}, x_{(n,m)}) \leq \sum_{j-0}^{m-1} d(x_{(n,j)}, x_{(n,j+1)}) \to 0.$$

Now, we prove that $\{y_n\}$ is a Cauchy sequence. Assume that $\{y_n\}$ is not a Cauchy sequence. Then there exists a number $t > 0$ (we may assume $t < \varepsilon$ without loss of generality) and two sequences $\{n_j\}, \{m_j\}$ of natural number with $n_j < m_j$ such that

$$d(x_{n_j}, x_{m_j}) \geq t,\ d(x_{n_j}, x_{m_{j-1}}) < t \text{ for } j = 1, 2, 3, ...$$

Then,

$$\begin{aligned} t &\leq d(x_{n_j}, x_{m_j}) \leq d(x_{n_j}, x_{m_{j-1}}) + d(x_{m_{j-1}}, x_{m_j}) \\ &< t + d(x_{m_{j-1}}, x_{m_j}). \end{aligned}$$

It follows that $\lim_{j\to\infty} d(x_{n_j}, x_{m_j}) = t \in (0, \varepsilon)$. For this $t > 0$, by property (p) of K, we can find $\delta(t) > 0$, $s(t) < 1$ such that $0 \leq r - t < \delta(t)$ implies $K(r) \leq s(t) < 1$. Now for this $\delta(t) > 0$, there exists an integer N such that $j \geq N$ implies $0 \leq d(x_{n_j}, x_{m_j}) - t < \delta(t)$ and hence,

$$K(d(x_{n_j}, x_{m_j})) < s(t) \text{ if } j \geq N. \tag{3.4}$$

Thus,

$$\begin{aligned} d(x_{n_j}, x_{m_j}) &\leq d(x_{n_j}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_{m_{j+1}}) + d(x_{m_{j+1}}, x_{m_j}) \\ &\leq d(x_{n_j}, x_{n_{j+1}}) + K(d(x_{n_j}, x_{m_j}))d(x_{n_j}, x_{m_j}) + d(x_{m_{j+1}}, x_{m_j}). \end{aligned}$$

This in the view of inequality 3.4 implies that $t \leq s(t)t < t$, a contradiction. It follows that $\{y_n\}$ is a Cauchy sequence. Since X is complete, therefore $y_n \to y^* \in X$. Hence, there exists an integer$M > 0$ such that $n > M$ implies $d(y_n, y^*) < \varepsilon$. This in the view of inequality 3.1 implies

$$H(T_{n+1}y_n, T_j y^*) \leq K(d(y_n, y^*)d(y_n, y^*).$$

Consequently,

$$H(T_{n+1}y_n, T_j y^*) \to 0.$$

Since $y_{n+1} \in T_{n+1}y_n$ with $d(y_{n+1}, y^*) \to 0$, now Lemma 1.1.7 implies that $y^* \in T_j y^*$, therefore, $y^* \in \cap_{n=1}^{\infty} T_n y^*$. This completes the proof.

3.1.8 Corollary

Let (X, d) be a complete ε- chainable metric space and $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from .X to $CB(X)$ satisfying the following condition:

$x, y \in X$ and $0 < d(x, y) < \varepsilon$ implies $H(T_n x, T_m y) \leq \lambda(d(x, y)$

for $n, m = 1, 2, 3....$Then there exists a point $y^* \in X$ such that $y^* \in \cap_{n=1}^{\infty} T_n y^*$.

"*Nadler [118] used Theorem 3.1.6 and obtained some results regarding fixed points of single valued (not necessarily one to one) uniformly locally expansive mapping T : $dom(T) \to X$ by placing some conditions on the inverse of T (e.g., $T^{-1}x \in (domT)$ and $T^{-1}$is ε−non-expansive). We use corollary 3.1.8 to improve and generalize corresponding results of [118] as follows:*"

3.1.9 Theorem(An application to single valued expansive mappings)

Let (X, d) be a complete ε- chainable metric space, $\eta \geq 1, \varnothing \neq A \subseteq X$ and $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from A onto X satisfying the following condition:

$x, y \in X$ and $0 < d(x, y) < \varepsilon$ implies $d(T_n x, T_m y) \geq \eta(d(x, y)$, for $n, m = 1, 2....$ If for each $n = 1, 2...$ and $x \in X$, $T_n^{-1}x \in CB(A)$ and $0 < d(x, y) < \varepsilon$ implies $H(T_n^{-1}x, T_m^{-1}y) < \varepsilon$, for $n, m = 1, 2, ...,$ then there exists a point $y^* \in X$, such that $y^* = T_n y^*$ for each $n = 1, 2, ...$

Proof

Let $x, y \in X$ such that $0 < d(x, y) < \varepsilon$ and choose $\beta > 0$. Let $u \in T_n^{-1}x$. Since $H(T_n^{-1}x, T_m^{-1}y) < \varepsilon$. Hence, by Lemma 1.1.9, there exists a point $v \in T_m^{-1}x$ such that $d(u, v) < \varepsilon$. Therefore,

$$d(T_n u, T_m v) \geq \eta d(u, v)$$

That is,

$$d(u, v) < \left[\frac{1}{\eta} + \beta\right] d(x, y).$$

It follows that

$$d(u, T_m^{-1}y) < \left[\frac{1}{\eta} + \beta\right] d(x, y).$$

This further implies that

$$u \in N\left(\left[\frac{1}{\eta} + \beta\right] d(x, y), T_m^{-1}y\right).$$

This proves that

$$T_n^{-1}x \subseteq N\left(\left[\frac{1}{\eta} + \beta\right] d(x, y), T_m^{-1}y\right).$$

Similarly, it can be shown that

$$T_m^{-1}y \subseteq N\left(\left[\frac{1}{\eta} + \beta\right] d(x, y), T_n^{-1}x\right).$$

Since β is arbitrary, it now follows that

$$H(T_n^{-1}x, T_m^{-1}y) \leq \frac{1}{\eta} d(x, y).$$

We may now apply Corollary 3.1.8 to conclude that there is a point $y^* \in X$ such that. $y^* \in \cap_{n=1}^{\infty} T_n^{-1} y^*$. Clearly $y^* = T_n y^*$, for each $n = 1, 2, ...$

3.2 Fixed points of fuzzy contractive and fuzzy locally contractive maps

Results given in this section have been published in [19].

"*In this section, we consider two metrices d and d^* on a set X, therefore Hausdorff metrics induced by d and d^* are respectively denoted by d_H and d_H^*. We establish some fixed point theorems for fuzzy contractive and fuzzy locally contractive mappings on compact metric space with $d_\infty-$metric for fuzzy sets.*

We recall the following notions from [3, 65, 73]:

Let X be a metric space. We define

$$C(X) = \left\{A \in I^X : \ [A]_\alpha \in C(X), \text{ for each } \alpha \in [0, 1]\right\},$$

$$K(X) = \{A \in I^X : \widehat{A} \in C(X)\},$$

$$\mathcal{F}(X) = \{A \in I^X, [A]_\alpha \in C(X) \text{ for some } \alpha \in [0,1]\},$$

$$E(X) = \left\{A \in I^X : [A]_\alpha \in CB(X), \text{ for each } \alpha \in [0,1]\right\}.$$

For $A, B \in I^X$, $A \subset B$ means $A(x) \leqslant B(x)$ for each $x \in X$. If there exists an $\alpha \in [0,1]$ such that $[A]_\alpha, [B]_\alpha \in CB(X)$, then define

$$\begin{aligned} p_\alpha(A,B) &= \inf_{x \in [A]_\alpha,\ y \in [B]_\alpha} d(x,y), \\ D_\alpha(A,B) &= d_H([A]_\alpha, [B]_\alpha). \end{aligned}$$

If $[A]_\alpha, [B]_\alpha \in CB(X)$ for each $\alpha \in [0,1]$, then define $P(A,B), d_\infty(A,B)$ as follows:

$$\begin{aligned} p(A,B) &= \sup_\alpha p_\alpha(A,B), \\ d_\infty(A,B) &= \sup_\alpha D_\alpha(A,B). \end{aligned}$$

We note that [65, 73] p_α is non-decreasing function of α, d_∞ is a metric on $E(X)$ and

$$(X,d) \mapsto (CB(X), d_H) \mapsto (E(X), d_\infty)$$

are isometrics embeddings by means $x \to \{x\}$ (crisp set) and $A \to \chi_A$ respectively."

3.2.1 Definition [173]

"Let (X,d) be a metric space. For $x, y \in X$, an ε-chain from x to y is a finite set of points $x_0, x_1, x_2, ..., x_n$ such that $x = x_0, x_n = y$ and $d(x_j, x_{j+1}) \leq \varepsilon$ for all $j = 0, 1, 2, ..., n-1$."

3.2.2 Definition

"A mapping $T : X \to C(X)$ is called fuzzy (globally) contraction [73] if there exists $\lambda \in [0,1)$ such that

$$d_\infty(T(x), T(y)) \leq \lambda d(x,y),$$

for all $x, y \in X$.

Mapping T is said to be (ε, λ) uniformly fuzzy locally contraction [20] if

$$x, y \in X,\ d(x,y) < \varepsilon\ \Rightarrow d_\infty(T(x), T(y)) \leq \lambda d(x,y).$$

Mapping T is said to be fuzzy (globally) contractive (see [53, 150]) if for all $x, y \in X, x \neq y$

$$d_\infty(T(x), T(y)) < d(x,y). \tag{3.5}$$

Mapping T is known as fuzzy locally contractive (see [53, 150]) if each x of X belongs to an open set U so that if $y, z \in U,\ y \neq z$,

$$d_\infty(T(y), T(z)) < d(y,z). \tag{3.6}$$

One very useful and significant fixed point theorem, due to Edelstein [53] is:"

3.2.3 Theorem

"If (X,d) is a compact metric space and $T : X \to X$ is a contractive mapping (i.e. $d(Tx,Ty) < d(x,y)$ for each $x \neq y,\ x, y \in X$), then there exists a unique fixed point of T.

Subsequently, Beg [31], Daffer and Kaneko [44], Grabiec [70], Hu and Rosen [77], Mihet [115], Park [124], Razani [137], Rosenholtz [150] and Smithson [158] among others studied some extensions (generalizations) and applications of this result.

We establish the following fixed point theorem for fuzzy contractive mappings on a compact metric space with the d_∞-metric for fuzzy sets, which extend the above result to fuzzy mappings."

3.2.4 Theorem (Edelstein Theorem for Fuzzy Contractive Maps)

Let (X, d) be a compact metric space and $T : X \to \mathfrak{C}(X)$ be a fuzzy (globally) contractive mapping. Then T has a fuzzy fixed point.

Proof

For each $x \in X$, $[Tx]_1$ is nonempty and compact. Define a real valued function $g : X \to \mathbb{R}$ by

$$g(x) = P_1(x, T(x)).$$

It implies that

$$\begin{aligned} g(x) &= P_1(x, T(x)) \\ &\leq d(x,y) + P_1(y, T(x)) \\ &\leq d(x,y) + P_1(y, T(y)) + d_H([Tx]_1, [Ty]_1) \\ &\leq d(x,y) + P_1(y, T(y)) + D_1(T(x), T(y)) \\ &\leq d(x,y) + P_1(y, T(y)) + \sup_{\alpha} D_\alpha(T(x), T(y)) \\ &\leq d(x,y) + g(y) + d_\infty(T(x), T(y)). \end{aligned}$$

It further implies that

$$g(x) - g(y) \leq d(x,y) + d_\infty(T(x), T(y)).$$

By symmetry, we obtain

$$|g(x) - g(y)| \leq d(x,y) + d_\infty(T(x), T(y)).$$

Using condition (3.5) along with the above inequality, it follows that g is continuous. By compactness, this function attains a minimum, say at x^*. Now, by compactness of $[T(x^*)]_1$, we can choose $x_1 \in X$, such that $\{x_1\} \subset T(x^*)$ and $d(x^*, x_1) = P_1(x^*, T(x^*)) = g(x^*)$. Then

$\{x^*\} \subset T(x^*)$, otherwise, $g(x_1) = P_1(x_1, T(x_1))$ implies that

$$\begin{aligned} g(x_1) &\leq d_H([Tx^*]_1, [Tx_1]_1) \\ &\leq d_\infty(T(x^*), T(x_1)) \\ &< d(x^*, x_1) = P_1(x^*, T(x^*)) = g(x^*), \end{aligned}$$

which is a contradiction to the fact that $g(x)$ is minimal at x^*. This completes the proof.

Another remarkable theorem of Edelstein [53] is:

3.2.5 Theorem

"*Let (X, d) be a compact and connected metric space and $T : X \to X$ is a locally contractive mapping (that is each x of X belongs to an open set U such that if y and z are distinct points of U, then $d(Ty, Tz) < d(y, z)$). Then T has a unique fixed point.*"

"*Recently, Ciric [42] obtained fixed points of locally contractive mappings in fuzzy metric spaces and established a fuzzy version of the above theorem.*

Theorem 3.2.7 is proved for fuzzy locally contractive mappings, which extends the above result. We shall make use of following lemma, which is noted in Rosenholtz [150] and Waters [173]."

3.2.6 Lemma

"*Let (X, d) be a compact connected metric space. Then for each $\varepsilon > 0$ and $x, y \in X$ there exists an ε-chain from x to y and the mapping $d^\varepsilon : X \times X \to \mathbb{R}$ defined by*

$$d^\varepsilon(x, y) = \inf \left\{ \sum_{j=0}^{n-1} d(x_j, x_{j+1}) : x_0, x_1, x_2, ..., x_n \text{ is an } \varepsilon - \text{chain from } x \text{ to } y \right\}$$

is a metric on X equivalent to d. Furthermore, for $x, y \in X$ and $\varepsilon > 0$ there exists an ε-chain $x = x_0, x_1, x_2, ..., x_n = y$ such that

$$d^\varepsilon(x, y) = \sum_{j=0}^{n-1} d(x_j, x_{j+1})."$$

3.2.7 Theorem(Edelstein Theorem for Fuzzy Locally Contractive Maps)

Let (X, d) be a compact connected metric space and $T : X \rightarrow \mathfrak{C}(X)$ be a fuzzy locally contractive mapping. Then T has a fuzzy fixed point.

Proof

First, inequality (3.6) implies that each x of X belongs to an open set U so that if $y, z \in U$, $y \neq z$,

$$d_H([Ty]_1, [Tz]_1) < d(y, z)). \tag{3.7}$$

Next, by Lemma 3.2.6, for each $\varepsilon > 0$ and each pair of points $u, v \in X$ there exists an ε-chain $u = x_0, x_1, x_2, ..., x_n = v$ from u to v. Now, use compactness of X to find $\delta > 0$ such that if $x \neq y$ and $d(x, y) < \delta$, then

$$d_H([Tx]_1, [Ty]_1) < d(x, y).$$

Define $d^* : X \times X \rightarrow R$ as follows:

$$d^*(u, v) = \inf \left\{ \sum_{j=0}^{n-1} d(x_j, x_{j+1}) : x_0, x_1, x_2, ..., x_n \text{ is an } \frac{\delta}{2}\text{-chain from } u \text{ to } v \right\},$$

that is $d^* = d^{\frac{\delta}{2}}$. By Lemma 3.2.6 d^* is a metric on X equivalent to d and there exists a $\frac{\delta}{2}$-chain $u = x_o, x_1, x_2, ..., x_n = v$ from u to v such that

$$d^*(u, v) = \sum_{j=0}^{n-1} d(x_j, x_{j+1}). \tag{3.8}$$

Now, $d(x_j, x_{j+1}) \leq \frac{\delta}{2} < \delta$ implies that

$$d_H([Tx_j]_1, [Tx_{j+1}]_1) < d(x_j, x_{j+1}) < \delta.$$

It further implies that

$$d(x_j, x_{j+1}) - d_H([Tx_j]_1, [Tx_{j+1}]_1) > 0.$$

Assume that

$$M_j = d(x_j, x_{j+1}) - d_H([Tx_j]_1, [Tx_{j+1}]_1),$$

for $j = 0, 1, 2, ..., n-1$. It implies that $M_j > 0$ and

$$d_H([Tx_j]_1, [Tx_{j+1}]_1) < d(x_j, x_{j+1}) - \frac{M_j}{2}, \tag{3.9}$$

for $j = 0, 1, 2, ..., n-1$. To show

$$[Tx_0]_1 \subset N^{d^*}(k, [Tx_n]_1) \tag{3.10}$$

for some $k > 0$, consider an arbitrary element $y_0 \in [Tx_0]_1$. In view of inequality (3.9), we may choose $y_1 \in [Tx_1]_1$such that

$$d(y_0, y_1) < d(x_0, x_1) - \frac{M_0}{2}.$$

Similarly, we can find $y_2 \in [Tx_2]_1$ such that

$$d(y_1, y_2) < d(x_1, x_2) - \frac{M_1}{2}.$$

Continuing in this fashion, we produce a set of points $y_0, y_1, y_2, ..., y_n$ where $y_j \in [Tx_j]_1$ such that

$$d(y_{j-1}, y_j) < d(x_{j-1}, x_j) - \frac{M_{j-1}}{2},$$

for $j = 0.1, 2, ..., n-1$. Obviously, $y_0, y_1, y_2, ..., y_n$ is a $\frac{\delta}{2}$- chain from y_0 to y_n.Thus,

$$\begin{aligned} d^*(y_0, y_n) &= \inf \left\{ \begin{array}{c} \sum_{j=0}^{n-1} d(x_j, x_{j+1}) : x_0, x_1, x_2, ..., x_n \text{ is} \\ \text{a } \frac{\delta}{2}\text{-chain from } y_0 \text{ to } y_n \end{array} \right\} \\ &\leq \sum_{j=0}^{n-1} d(y_j, y_{j+1}) \\ &< \sum_{j=0}^{n-1} \left(d(x_j, x_{j+1}) - \frac{M_j}{2} \right) \\ &= \sum_{j=0}^{n-1} d(x_j, x_{j+1}) - \sum_{j=0}^{n-1} \frac{M_j}{2}. \end{aligned}$$

This, in view of (3.8), we have

$$d^*(y_0, y_n) < d^*(u, v) - \sum_{j=0}^{n-1} \frac{M_j}{2}.$$

Suppose that

$$k = d^*(u, v) - \sum_{j=0}^{n-1} \left(\frac{M_j}{2}\right),$$

then $k > 0$ and $y_0 \in N^{d^*}(k, [Tx_n]_1)$. Hence, (3.10) holds. Now, we show that

$$[Tx_n]_1 \subset N^{d^*}(k, [Tx_0]_1). \tag{3.11}$$

Consider an arbitrary element $z_n \in [Tx_n]_1$. Again in view of inequality (3.9) along with Lemma 1.1.9, we may choose $z_{n-1} \in [Tx_{n-1}]$ such that

$$d(z_{n-1}, z_n) < d(x_0, x_1) - \frac{M_{n-1}}{2}.$$

Then in a similar way, we obtain a $\frac{\delta}{2}$- chain $z_0, z_1, z_2, ..., z_n$ from z_0 to z_n, where

$$d^*(z_0, z_n) \leq d^*(u, v) - \sum_{j=0}^{n-1} \left(\frac{M_j}{2}\right) = k.$$

Thus, $z_n \in N^{d^*}(k, [Tx_0]_1)$. Hence, (3.11) holds. In view of inequalities (3.10) and (3.11), it follows that $k \in E^{d^*}_{[Tx_0]_1, [Tx_n]_1}$. Thus,

$$d^*_H([Tx_0]_1, [Tx_n]_1) < k.$$

It implies that

$$d^*_H([Tu]_1, [Tv]_1) < d^*(u, v) - \sum_{j=0}^{n-1} \left(\frac{M_j}{2}\right) < d^*(u, v).$$

Hence, for all $x, y \in X$

$$D^*_1(T(x), T(y)) < d^*(x, y). \tag{3.12}$$

Define a real valued function g on X as follows:

$$g(x) = P_1^*(x, T(x)).$$

It implies that

$$\begin{aligned} g(u) &\leq d^*(u,v) + P_1^*(v, T(u)) \\ &\leq d^*(u,v) + P_1^*(v, T(v)) + D_1^*(T(u), T(v)). \end{aligned} \tag{3.13}$$

Now using (3.12) and (3.13), we obtain

$$g(u) \leq d^*(u,v) + P_1^*(v, T(v)) + d^*(u,v).$$

Therefore,

$$g(u) - g(v) \leq 2d^*(u,v).$$

By symmetry, we obtained

$$|g(u) - g(v)| \leq 2d^*(u,v).$$

It follows that g is continuous. By compactness, this function attains a minimum value say m, at a point $x^* \in X$. Then $m = 0$, otherwise, by compactness of $[T(x^*)]_1$, we can choose $u_1 \in [T(x^*)]_1$ such that $d^*(x^*, u_1) = g(x^*) = m$ and

$$\begin{aligned} g(u_1) &= P_1^*(u_1, T(u_1)) \\ &\leq D_1^*(T(x^*), T(u_1)). \end{aligned}$$

Now, using (3.12) along with above inequality, we obtain

$$g(u_1) < d^*(x^*, u_1) = m,$$

which is a contradiction to the fact that m is minimal at x^*. Hence $m = 0$, it implies that $\{x^*\} \subset [Tx^*]$. This completes the proof.

It is quite easy to exhibit spaces which admit fuzzy locally contractive mappings or even uniformly fuzzy locally contractions which are not (globally) contractions. The following is a simple example:

3.2.8 Example

$$\begin{aligned} X &= \{z_t = (\cos t, \sin t) : \ 0 \leq t \leq \frac{4\pi}{3}\}, \\ T(z_t) &= \chi_{\overline{B}\left(z_{\frac{t}{2}};1\right)}, \end{aligned}$$

where,

$$\overline{B}\left(z_{\frac{t}{2}};1\right) = \left\{z_u \in X : d(z_u, z_{\frac{t}{2}}) \leq 1\right\}$$

and X is taken with Euclidean metric of the plane.

3.2.9 Conclusion

"*Let* (X, d) *be a compact metric space, then we get the fractal space* $(C(X), d_H)$ *and the fuzzy fractal space* $(C(X), d_\infty)$ *(see [131]). Let* $T : X \to C(X)$ *be such that either for all* $x, y \in X, x \neq y$

$$d_\infty(T(x), T(y)) < d(x, y),$$

or X *is connected and each* x *of* X *belongs to an open set* U *so that if* $y, z \in U, \ y \neq z$,

$$d_\infty(T(y), T(z)) < d(y, z).$$

Then there exists an element $x \in X$ such that $\{x\} \subset T(x)$."

3.3 On a pair of fuzzy ϕ- contractive mappings

Results given in this section will appear in [25].

3.3.1 Definition [20]

"*A real linear space* V *along with a metric* d *is metric linear space if* $d(x + z, y + z) = d(x, y)$ *and* $\alpha_n \longrightarrow \alpha, x_n \longrightarrow x \implies \alpha_n x_n \longrightarrow \alpha x$.

A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0,1]$ and $\sup_{x\in V} A(x) = 1$. The family of all approximate quantities on V is denoted by $W(V)$.

In [10, 65, 108, 109, 126], the authors generalized Heilpern's [73] result by using more general contractive type conditions for the same class i.e., $(W(X))^X$ of fuzzy mappings using the d_∞-metric for fuzzy sets. Recently, Abu-Donia [3] (see also [94]) studied an important role of Hausdorff metric between fuzzy subsets and studied common fixed points of a pair fuzzy mappings $S,T \in (K(X))^X$ under a φ-contraction condition on $\widehat{S},\widehat{T}$. Of course, Abu-Donia's results do not require linearity on X and convexity on $\widehat{S}(x)(t)$, $\widehat{T}(x)(t)$ but Kamran [94] showed that Abu-Donia's [3] results hold only when $\widehat{S}(x)(t)$, $\widehat{T}(x)(t)$ are compact instead of closed and bounded. Therefore, Abu-Donia's results need some further adjustments/corrections (see corollary 3.3.8). In this section, we extend Abu-Donia's main result to a wider class $(\mathcal{F}(X))^X$ of fuzzy mappings. Furthermore, some other results of the literature are obtained as corollaries.

We recall the following lemmas which are required for our onward discussion."

3.3.2 Lemma [10]

"Let (V,d) be a complete metric linear space, $T: V \longrightarrow W(V)$ be a fuzzy mapping and $x_o \in V$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_o)$, that is $T(x_o)(x_1) = 1$."

3.3.3 Lemma [3]

"Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function satisfying the following conditions:

(i) φ is continuous from right,

(ii) $\sum_{i=0}^{\infty} \varphi^i(t) < \infty$ for all $t > 0$ (φ^i denotes the it iterative function of φ).

Then $\varphi(t) < t$.

In the rest of this section we, always suppose that φ is a function satisfying the conditions of lemma 3.3.3 and $\widehat{T}$ is the mapping induced by fuzzy mappings T i.e.,

$$\widehat{T}(x)(t) = \{y \in X : T(x)(y) = \max_{t\in X} T(x)(t)\}."$$

3.3.4 Lemma [3]

"*Let* (X,d) *be a metric space,* $x^* \in X$ *and* $T : X \to I^X$ *be fuzzy mappings such that* $\widehat{T}x \in C(X)$ *for all* $x \in X$*. Then* $x^* \in \widehat{T}(x^*)$ *iff* $T(x^*)(x^*) \geqslant T(x^*)(x)$ *for all* $x \in X$.

First, we furnish an example to show that the family $(\mathcal{F}(X))^X$ *of fuzzy mappings is a wider class than that of* $(K(X))^X$ *[3].*"

3.3.5 Example

Let $X = [0,\infty)$, $d(x,y) = |x-y|$, whenever $x, y \in X$ and $\mu \in \left(I^X\right)^{(0,\infty)}$ be defined as follows:

$$\mu(x)(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < \frac{x}{6}, \\ \frac{1}{2} & \text{if } \frac{x}{6} \leqslant t \leqslant \frac{x}{3}, \\ \frac{1}{3} & \text{if } \frac{x}{3} < t \leqslant \frac{x}{2}, \\ 0 & \text{if } \frac{x}{2} < t < \infty. \end{cases}$$

Now, define $T : X \longrightarrow I^X$ as follows:

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \mu(x) & \text{if } x \neq 0. \end{cases}$$

Then $T \in (\mathcal{F}(X))^X$ but $T \notin (K(X))^X$ as if $x \neq 0$,

$$\widehat{T}(x) = \{t \in X : T(x)(t) = 1\} = \left[0, \frac{x}{6}\right),$$

$$[T(x)]_{\frac{1}{2}} = \left\{t \in X : T(x)(t) = \frac{1}{2}\right\} = \left[0, \frac{x}{3}\right].$$

"*Let* (X,d) *be a metric space,* $S, T : X \to I^X$*. For every* $x, y \in X$ *and* $\alpha, \beta \in (0,1]$*, we put*

$$M(x,y,\alpha,\beta) = \max\{d(x,y),\ d(x,[Sx]_\alpha), d(y,[Ty]_\beta), \frac{1}{2}[d(x,[Ty]_\beta) + \ d(y,[Sx]_\alpha)]\}."$$

3.3.6 Theorem

Let (X,d) be a complete metric space and $S,T : X \to I^X$. Assume that for every $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0,1]$ such that $[Sx]_{\alpha S(x)}, [Tx]_{\alpha T(x)} \in C(X)$. If for all $x, y \in X$

$$H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) \leqslant \varphi(M(x,y,\alpha_S(x),\alpha_T(y))), \tag{3.14}$$

then there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Proof

Choose $x_0 \in X$, by hypothesis, there exists $\alpha_S(x_0) \in (0,1]$ such that $[Sx_0]_{\alpha(x_0)} \in C(X)$. For convenience, we denote $\alpha_S(x_0)$ by α_1. By compactness of $[Sx_0]_{\alpha_1}$, we can find $x_1 \in [Sx_0]_{\alpha_1}$ such that $d(x_0, x_1) = d\left(x_0, [Sx_0]_{\alpha_1}\right)$. Again by hypothesis, there exists $\alpha_T(x_1) \in (0,1]$ such that $[Tx_1]_{\alpha_T(x_1)} \in C(X)$, denote $\alpha_T(x_1)$ by α_2 and by compactness of $[Tx_1]_{\alpha_2}$, choose $x_2 \in [Tx_1]_{\alpha_2}$ such that $d(x_1, x_2) = d\left(x_1, [Tx_1]_{\alpha_2}\right)$. By induction, we produce a sequence $\{x_n\}$ of points of X, with

$$\begin{aligned} x_{2k+1} &\in [Sx_{2k}]_{\alpha_{2k+1}}, \\ x_{2k+2} &\in [Tx_{2k+1}]_{\alpha_{2k+2}}, \quad k = 0,1,2,\ldots, \end{aligned}$$

and such that

$$\begin{aligned} d(x_{2k}, x_{2k+1}) &= d(x_{2k}, [Sx_{2k}]_{\alpha_{2k+1}}), \\ d(x_{2k+1}, x_{2k+2}) &= d(x_{2k+1}, [Tx_{2k+1}]_{\alpha_{2k+2}}), \quad k = 0,1,2,\ldots. \end{aligned}$$

By Lemma 1.1.10 and the above equations, we have

$$\begin{aligned} d(x_{2k}, x_{2k+1}) &\leqslant H\left([Tx_{2k-1}]_{\alpha_{2k}}, [Sx_{2k}]_{\alpha_{2k+1}}\right), \\ d(x_{2k+1}, x_{2k+2}) &\leqslant H([Sx_{2k}]_{\alpha_{2k+1}}, [Tx_{2k+1}]_{\alpha_{2k+2}}), \ k = 0,1,2,\ldots. \end{aligned}$$

Assume that $x_{2k} = x_{2k+1}$ for some $k \geqslant 0$, then

$$\begin{aligned} M(x_{2k}, x_{2k+1}, \alpha_{2k+1}, \alpha_{2k+2}) &= \max\{d(x_{2k+1}, x_{2k+2}), \\ &\quad \frac{1}{2}[d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})]\} \\ &= d(x_{2k+1}, x_{2k+2}). \end{aligned}$$

Consequently (by using inequality 3.1.4),

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\leqslant H([Sx_{2k}]_{\alpha_{2k+1}}, [Tx_{2k+1}]_{\alpha_{2k+2}}) \\ &\leqslant \varphi(M(x_{2k}, x_{2k+1}, \alpha_{2k+1}, \alpha_{2k+2})) \\ &\leqslant \varphi(d(x_{2k+1}, x_{2k+2})). \end{aligned}$$

Since $\varphi(t) < t$ for all $t > 0$, we deduce that $d(x_{2k+1}, x_{2k+2}) = 0$, which further implies that

$$x_{2k} = x_{2k+1} \in [Sx_{2k}]_{\alpha_{2k+1}},\ x_{2k} = x_{2k+1} = x_{2k+2} \in [Tx_{2k+1}]_{\alpha_{2k+2}} = [Tx_{2k}]_{\alpha_{2k+2}}.$$

It follows that $x_{2k} \in [Sx_{2k}]_{\alpha_{2k+1}} \cap [Tx_{2k}]_{\alpha_{2k+2}}$. Thus, in this sequel of the proof, we can suppose that $x_{n+1} \neq x_n$ for $n = 0, 1, 2 \ldots$. Again by using inequality (3.14), we have

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\leqslant H([Sx_{2k}]_{\alpha_{2k+1}}, [Tx_{2k+1}]_{\alpha_{2k+2}}) \\ &\leqslant \varphi(M(x_{2k}, x_{2k+1}, \alpha_{2k+1}, \alpha_{2k+2})), \end{aligned}$$

where $M(x_{2k}, x_{2k+1}, \alpha_{2k+1}, \alpha_{2k+2}) = \max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\})$. If $M\{x_{2k}, x_{2k+1}, \alpha_{2k+1}, \alpha_{2k+2}) = d(x_{2k+1}, x_{2k+2})$, then the above inequality implies that

$$d(x_{2k+1}, x_{2k+2}) \leqslant \varphi(d(x_{2k+1}, x_{2k+2})) < d(x_{2k+1}, x_{2k+2}),$$

which is a contradiction, since $x_{n+1} \neq x_n$ for $n = 0, 1, 2, \ldots$. It follows that

$$M(x_{2k}, x_{2k+1}, \alpha_{2k+1}, \alpha_{2k+2}) = d(x_{2k}, x_{2k+1}),$$

and hence, $d(x_{2k+1}, x_{2k+2}) \leqslant \varphi\left(d(x_{2k}, x_{2k+1})\right)$. In a similar way, we prove that

$$d(x_{2k+1}, x_{2k}) \leqslant \varphi\left(d(x_{2k}, x_{2k-1})\right).$$

Consequently,

$$d(x_{n+1}, x_n) \leq \varphi\left(d(x_n, x_{n-1})\right) \leqslant \varphi^2\left(d(x_{n-1}, x_{n-2})\right) \leqslant \cdots \leqslant \varphi^n\left(d(x_1, x_0)\right),$$

for each $n \geqslant 1$. Now, for each positive integer m, n $(n > m)$, we have

$$\begin{aligned} d(x_m,\ x_n) &\leqslant d(x_m,\ x_{m+1}) + d(x_{m+1},\ x_{m+2}) + \cdots + d(x_{n-1},\ x_n) \\ &\leqslant \varphi^m\left(d(x_1, x_0)\right) + \varphi^{m+1}\left(d(x_1, x_0)\right) + \cdots + \varphi^{n-1}\left(d(x_1, x_0)\right) \\ &\leqslant \sum_{i=m}^{n-1} \varphi^i\left(d(x_1, x_0)\right) \leqslant \sum_{i=m}^{\infty} \varphi^i\left(d(x_1, x_0)\right). \end{aligned}$$

Since $\sum_{i=1}^{\infty} \varphi^i(t) < \infty$ for each $t > 0$, it yields that $\{x_n\}$ is a Cauchy sequence. As X is complete, there exists $u \in X$ such that $x_n \to u$. Now, Lemma 1.1.10 implies that

$$\begin{aligned} d(u, [Su]_{\alpha_S(u)}) &\leqslant d(u, x_{2n}) + d(x_{2n}, [Su]_{\alpha_S(u)}) \\ &\leqslant d(u, x_{2n}) + H([Tx_{2n-1}]_{\alpha_{2n}}, [Su]_{\alpha_S(u)}) \\ &\leqslant d(u, x_{2n}) + \varphi(M(u, x_{2n-1}, \alpha_S(u), \alpha_{2n})), \end{aligned}$$

where

$$\begin{aligned} M(u, x_{2n-1}, \alpha_S(u), \alpha_{2n}) &= \max\{d(u, x_{2n-1}), d(u, [Su]_{\alpha_S(u)}), d(x_{2n-1}, x_{2n}), \\ &\quad \frac{1}{2}[d(u, x_{2n}) + d(x_{2n-1}, [Su]_{\alpha_S(u)})]\} \\ &\geq d(u, [Su]_{\alpha_S(u)}). \end{aligned}$$

Now, $\lim_{n\to\infty} M(u, x_{2n-1}, \alpha_S(u), \alpha_{2n}) = d(u, [Su]_{\alpha_S(u)})$. Since φ is continuous from the right, as $n \to \infty$, we obtain

$$d(u, [Su]_{\alpha_S(u)}) \leqslant \varphi\left(d(u, [Su]_{\alpha_S(u)})\right).$$

Then there exists point $x^* \in X$ such that $T(x^*)(x^*) \geqslant T(x^*)(x)$ and $S(x^*)(x^*) \geqslant S(x^*)(x)$ for all $x \in X$."

Proof

By Theorem 3.3.7, there exists $x^* \in X$ such that $x^* \in \widehat{S}x^* \cap \widehat{T}x^*$. Now Lemma 3.3.4 implies that

$$T(x^*)(x^*) \geqslant T(x^*)(x), \quad S(x^*)(x^*) \geqslant S(x^*)(x)$$

for all $x \in X$.

3.3.9 Example

Let $X = [0, \infty)$, $d(x,y) = |x - y|$, whenever $x, y \in X$, $\alpha, \beta \in (0,1]$ and $S, T : X \longrightarrow I^X$ be fuzzy mappings such that $T(x)$, $S(x) : X \longrightarrow I = [0,1]$ are defined as follows:

$$\text{if } x = 0, \quad T(x)(t) = S(x)(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0, \end{cases}$$

if $x \in (0,1]$,

$$S(x)(t) = \begin{cases} \alpha & \text{if } 0 \leqslant t < \frac{x}{2} - \frac{x^2}{4} \\ \frac{\alpha}{2} & \text{if } \frac{x}{2} - \frac{x^2}{4} \leqslant t \leqslant x - \frac{x^2}{2} \\ \frac{\alpha}{3} & \text{if } x - \frac{x^2}{2} < t < x \\ 0 & \text{if } x \leqslant t < \infty, \end{cases}$$

$$T(x)(t) = \begin{cases} \beta & \text{if } 0 \leqslant t < \frac{x}{4} - \frac{x^2}{8} \\ \frac{\beta}{4} & \text{if } \frac{x}{4} - \frac{x^2}{8} \leqslant t \leqslant x - \frac{x^2}{2} \\ \frac{\beta}{10} & \text{if } x - \frac{x^2}{2} < t < x \\ 0 & \text{if } x \leqslant t < \infty, \end{cases}$$

if $x > 1$,

$$T(x)(t) = S(x)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{2} \\ 0 & \text{if } t \neq \frac{1}{2}. \end{cases}$$

Define $\phi : [0,\infty) \to [0,\infty)$ as follows:

$$\phi(t) = \begin{cases} t - \frac{t^2}{2} & \text{if } t \in [0,1] \\ \frac{1}{2} & \text{if } t > 1. \end{cases}$$

It is obvious that $\phi(t) < t$ for $t > 0$, but $\widehat{S}(x), \widehat{T}(x)$ are not compact for $x \in (0,1]$. Therefore, Corollary 3.3.8 is not applicable, whereas assumptions of Theorem 3.3.6 are satisfied. In fact, for any $x \in X$, there exists $\alpha_S(x) = \frac{\alpha}{2}, \alpha_T(x) = \frac{\beta}{4} \in (0,1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)}$ are compact. If $x = y = 0$ or $x, y > 1$, then $[Sx]_{\frac{\alpha}{2}} = [Ty]_{\frac{\beta}{4}}$ and

$$H\left([Sx]_{\frac{\alpha}{2}}, [Ty]_{\frac{\beta}{4}}\right) = 0 \leqslant \varphi\left(M\left(x, y, \frac{\alpha}{2}, \frac{\beta}{4}\right)\right).$$

If $x = 0,\ y \in (0,1]$, then $[S0]_{\frac{\alpha}{2}} = \{0\}$, $[Ty]_{\frac{\beta}{4}} = \left[0, y - \frac{y^2}{2}\right]$ and

$$\begin{aligned} H\left([S0]_{\frac{\alpha}{2}}, [Ty]_{\frac{\beta}{4}}\right) &= \left|y - \frac{y^2}{2}\right| = \phi(|y-0|) \\ &\leqslant \varphi\left(M\left(0, y, \frac{\alpha}{2}, \frac{\beta}{4}\right)\right) \end{aligned}$$

If $x, y \in (0,1]$, then

$$\begin{aligned} H\left([Sx]_{\frac{\alpha}{2}}, [Ty]_{\frac{\beta}{4}}\right) &= \left|x - \frac{x^2}{2} - y + \frac{y^2}{2}\right| = \left|(x-y) - \left(\frac{x^2 - y^2}{2}\right)\right| \\ &= \left|(x-y)\left(1 - \frac{(x+y)}{2}\right)\right| \\ &\leqslant |x-y|\left|1 - \frac{|x-y|}{2}\right| \\ &\leqslant |x-y| - \frac{|x-y|^2}{2} \\ &\leqslant \varphi\left(M\left(x, y, \frac{\alpha}{2}, \frac{\beta}{4}\right)\right). \end{aligned}$$

If $x \in (0,1]$, $y > 1$ then

$$\begin{aligned} H\left([Sx]_{\frac{\alpha}{2}}, [Ty]_{\frac{\beta}{4}}\right) &= H\left(\left[0, x - \frac{x^2}{2}\right], \left\{\frac{1}{2}\right\}\right) = \left|1 - \frac{1}{2} - \left(x - \frac{x^2}{2}\right)\right| \\ &= \left|(1-x) - \frac{1}{2}\left(1 - x^2\right)\right| = \left|(1-x)\left(1 - \frac{1}{2}(1+x)\right)\right| \\ &\leqslant |(1-x)|\left|\left(1 - \frac{1}{2}(1-x)\right)\right| = |(1-x)| - \frac{1}{2}|1-x|^2 \\ &= \phi(|1-x|) \leqslant \phi(|y-x|) \\ &\leqslant \varphi\left(M\left(x, y, \frac{\alpha}{2}, \frac{\beta}{4}\right)\right). \end{aligned}$$

"In [19, 20] the authors obtained fixed points of fuzzy contractive and fuzzy locally contractive mappings on a compact metric space with the d_∞-metric for fuzzy sets and established fuzzy extension of Edelstein's fixed point theorems [52, 53]. In the following, we establish some fixed point theorems for fuzzy ϕ-contractive mappings in connection with the d_∞-metric for fuzzy sets.

Let (X, d) be a metric space, $S, T : X \to I^X$. For every $x, y \in X$ and $\alpha, \beta \in (0,1]$, we put

$$m(x,y) = \max\{d(x,y),\ p(x, S(x)), p(y, T(y)), \frac{1}{2}[p(x, T(y)) +\ p(y, S(x))]\}."$$

3.3.10 Theorem

Let (X, d) be a complete metric space and $S, T : X \to \mathfrak{C}(X)$ be fuzzy mappings such that

$$d_\infty(S(x), T(y)) \leqslant \varphi(m(x,y)),$$

for all $x, y \in X$. Then there exists a point $u \in X$ such that $\{u\} \subset S(u)$, $\{u\} \subset T(u)$.

Proof

Pick $x \in X$, by assumptions $[Sx]_1, [Tx]_1$ are non-empty compact subsets of X. Now, for all $x, y \in X$

$$\begin{aligned} D_1(S(x), T(y)) &\leqslant d_\infty(S(x), T(y)) \\ &\leqslant \varphi(m(x,y)). \end{aligned}$$

Since $[Sx]_1 \subseteq [Sx]_\alpha$ for each $\alpha \in [0,1]$; therefore, $d(x,[Sx]_\alpha) \leqslant d(x,[Sx]_1)$ for each $\alpha \in [0,1]$ and it implies that $p(x,S(x)) \leqslant d(x,[Sx]_1)$. This further implies that

$$H([Sx]_1,[Ty]_1) \leqslant \varphi(M(x,y,1,1)).$$

Now, by Theorem 3.3.6 there exists $u \in X$ such that $u \in [Su]_1 \cap [Tu]_1$.

The following theorem improves/generalizes the results of [10, 65, 73, 108, 109, 126, 174].

3.3.11 Theorem

Let (X,d) be a complete metric linear space and $S,T : X \to \mathcal{W}(X)$ be fuzzy mappings and for all $x,y \in X$

$$d_\infty(S(x),T(y)) \leqslant \varphi(m(x,y)).$$

Then there exists a point $u \in X$ such that $\{u\} \subset S(u)$, $\{u\} \subset T(u)$.

Proof

Let $x \in X$, by Lemma 3.3.2 there exist $y,z \in X$ such that $y \in [Sx]_1$ and $z \in [Tx]_1$. It follows that for each $x \in X$, $[Sx]_1,[Tx]_1 \in C(X)$. The remaining part of the proof is similar as that of the previous theorem.

If in Theorem 3.3.10 we choose $\phi(t) = kt$, where $k \in [0,1)$ is a constant, we obtain the following corollary.

3.3.12 Corollary [126]

"*Let (X,d) be a complete metric linear space and $S,T : X \to W(X)$ be fuzzy mappings. Assume that there exists $k \in [0,1)$ such that for all $x,y \in X$.*

$$d_\infty(S(x),T(y)) \leqslant k\,m(x,y).$$

Then there exists a point $u \in X$ such that $\{u\} \subset S(u)$, $\{u\} \subset T(u)$.

Let $q \in \left[0,\frac{1}{2}\right)$. From

$$q\max\{d(x,y),\ p(x,S(x)),p(y,T(y)),p(x,T(y)),\ p(y,S(x))\} \leq 2q\,m(x,y)$$

and Corollary 3.3.12, we deduce the following result."

3.3.13 Corollary [10]

"Let (X, d) be a complete metric linear space and $S, T : X \to W(X)$ be fuzzy mappings. Assume that there exists $q \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$

$$d_\infty(S(x), T(y)) \leqslant q \max\{d(x,y), p(x, S(x)), p(y, S(y)), p(x, T(y)), p(y, S(x))\}.$$

Then there exists a point $u \in X$ such that $\{u\} \subset S(u)$, $\{u\} \subset T(u)$.

From Corollary 3.3.13, we deduce the following corollaries."

3.3.14 Corollary [108]

"Let (X, d) be a complete metric linear space and $S, T : X \to W(X)$ be fuzzy mappings. If

$$\begin{aligned} d_\infty(S(x), T(y)) &\leqslant a_1 p(x, S(x)) + a_2 p(y, T(y)) \\ &\quad + a_3 p(x, T(y)) + a_4 p(y, S(x) + a_5\, d(x,y)) \end{aligned} \tag{3.15}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non negative real numbers with $\sum_{i=1}^{5} a_i < 1$ and $a_3 \geqslant a_4$. Then there exists $u \in X$ such that $\{u\} \subset S(u), \{u\} \subset T(u)$."

3.3.15 Corollary [37]

"Let (X, d) be a complete metric linear space and $S, T : X \to W(X)$ be fuzzy mappings. If (3.15) is satisfied for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non negative real numbers with $\sum_{i=1}^{5} a_i < 1$ and $a_3 = a_4$ or $a_1 = a_2$. Then there exists $u \in X$ such that $\{u\} \subset S(u)$, $\{u\} \subset T(u)$."

3.3.16 Corollary [73]

"Let (X, d) be a complete metric linear space and $T : X \to W(X)$ be fuzzy mapping such that for all $x, y \in X$,

$$d_\infty(T(x), T(y)) \leqslant \lambda d(x, y)$$

where $0 \leqslant \lambda < 1$. *Then there exists* $u \in X$ *such that* $\{u\} \subset T(u)$."

3.4 On a fixed point theorem for fuzzy maps

Results given in this section will appear in [26].

"*We recall the Heilpern's [73] fuzzy contraction theorem which states that if* (X, d) *is a complete metric linear space and* $T : X \to W(X)$ *be a fuzzy mapping such that for all* $x, y \in X$,

$$\sup_{\alpha \in [0,1]} H([Sx]_\alpha, [Ty]_\alpha) \leqslant \lambda d(x, y)$$

where, $0 \leqslant \lambda < 1$, *then there exists* $u \in X$ *such that* $u \in [T(u)]_1$. *Several other authors generalized this result and studied the existence of fixed points and common fixed points of fuzzy (approximate quantity-valued) mappings satisfying a contractive type condition in a metric linear space. (e.g.,see [10, 19, 65, 108, 109, 126, 169] and references their in). Vijayaraju and Marudai [169, Theorem 3.1] studied a fixed point result for fuzzy (set-valued) mappings* X *to* $F(X)$ *in a metric space* X. *This result [169, Theorem 3.1] is significant as it does not require the condition of approximate quantity for* $T(x)$ *and linearity for* X. *However, its proof [169, Theorem 3.1] is incorrect and incomplete, therefore it needs some further adjustments and modifications. The aim of this section is to present the right version of this result.*

The following theorem is the main result of Vijayaraju and Marudai [169]."

3.4.1 Theorem

"*Let* (X, d) *be a complete metric space and let* F_1, F_2 *be fuzzy mappings from* X *to* $F(X)$ *satisfying the following conditions:*

(a) For each $x \in X$, *there exists* $\alpha(x) \in (0, 1]$ *such that* $[F_1x]_{\alpha(x)}, [F_2x]_{\alpha(x)}$ *are nonempty closed bounded subsets of* X *and*

(b)

$$\begin{aligned} H([F_1x]_{\alpha(x)}, [F_2y]_{\alpha(y)}) \leqslant\ & a_1\, d(x, [F_1y]_{\alpha(x)}) + a_2 d(y, [F_2y]_{\alpha(y)}) + \\ & + a_3\, d(x, [F_2y]_{\alpha(y)}) + a_4\, d(y, [F_1x]_{\alpha(x)}) \\ & + a_5\, d(x, y), \end{aligned}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non-negative real numbers and $\sum_{i=1}^{5} a_i < 1$ and either $\alpha_1 = a_2$ or $a_3 = a_4$. Then there exists $z \in X$ such that $z \in [F_1 z]_{\alpha(z)} \cap [F_2 z]_{\alpha(z)}$.

The following theorem is the right version of the above result."

3.4.2 Theorem

Let (X, d) be a complete metric space. Let $S, T : X \to \mathcal{F}(X)$ be fuzzy mappings. Suppose that for each $x \in X$, there exists $\alpha(x) \in (0, 1]$ such that $[Sx]_{\alpha(x)}, [Tx]_{\alpha(x)}$ are nonempty closed bounded subsets of X and

$$\begin{aligned} H([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leqslant\ & a_1 d(x, [Sx]_{\alpha(x)}) + a_2 d(y, [Ty]_{\alpha(y)}) \\ & + a_3 \left[d(x, [Ty]_{\alpha(y)}) + d(y, [Sx]_{\alpha(x)}) \right] \\ & + a_4 d(x, y), \end{aligned} \tag{3.16}$$

for all $x, y \in X$, where $a_1,\ a_2,\ a_3,\ a_4$, are non negative real numbers with $a_1 + a_2 + 2a_3 + a_4 < 1$. Then there exists $z \in X$ such that $z \in [Sz]_{\alpha(z)} \cap [Tz]_{\alpha(z)}$.

Proof

We consider the following three possible cases:

(i) $a_1 + a_3 + a_4 = 0$;

(ii) $a_2 + a_3 + a_4 = 0$;

(iii) $a_1 + a_3 + a_4 \neq 0,\ a_2 + a_3 + a_4 \neq 0$.

Case (i) $a_1 + a_3 + a_4 = 0$. Since $a_1, a_3, a_4 \geqslant 0$, therefore $a_1 = a_3 = a_4 = 0$. Let $x \in X$, then by assumptions there exists $\alpha(x) \in (0, 1]$ such that $[Sx]_{\alpha(x)}$ is nonempty closed bounded subset of X. Take $y \in [Sx]_{\alpha(x)}$ and $z \in [Ty]_{\alpha(y)}$. Then by Lemma 1.1.10, we obtain,

$$d(y, [Ty]_{\alpha(y)}) \leqslant H([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}).$$

Now, by inequality (3.16)

$$\begin{aligned} d(y,[Ty]_{\alpha(y)}) &\leqslant a_1 d(x,[Sx]_{\alpha(x)}) + a_2 d(y,[Ty]_{\alpha(y)}) \\ &\quad + a_3 \left[d(x,[Ty]_{\alpha(y)}) + d(y,[Sx]_{\alpha(x)})\right] \\ &\quad + a_4 d(x,y). \end{aligned}$$

Substituting, $a_1 = a_3 = a_4 = 0$, in the above inequality, we obtain

$$(1-a_2)\, d(y,[Ty]_{\alpha(y)}) \leqslant 0.$$

It follows that $y \in [Ty]_{\alpha(y)}$, which further implies that

$$d(y,[Sy]_{\alpha(y)}) \leqslant H([Ty]_{\alpha(y)}, [Sy]_{\alpha(y)}).$$

Again, inequality (3.16) yields $d(y,[Sy]_{\alpha(y)}) = 0$. Hence,

$$y \in [Sy]_{\alpha(y} \cap [Ty]_{\alpha(y)} .$$

Case(ii) If $a_2 + a_3 + a_4 = 0$, then, for $x \in X$, as in case (i), we can choose $y \in [Sx]_{\alpha(x)}$ and $z \in [Ty]_{\alpha(y)}$ such that

$$d(z,[Sz]_{\alpha(z)}) = d([Sz]_{\alpha(z)}, [Ty]_{\alpha(y)}),$$

which further implies that

$$z \in [Sz]_{\alpha(z)} \cap [Tz]_{\alpha(z)} .$$

Case(iii) Let $a_1 + a_3 + a_4 \neq 0$, $a_2 + a_3 + a_4 \neq 0$ and

$$q = \max\left\{\left(\frac{a_1 + a_3 + a_4}{1 - a_2 - a_3}\right), \left(\frac{a_2 + a_3 + a_4}{1 - a_1 - a_3}\right)\right\}.$$

Then $a_1 + a_2 + 2a_3 + a_4 < 1$ implies that $q < 1$. Choose $x_0 \in X$, by hypotheses there exists $\alpha(x_0) \in (0,1]$ such that $[Sx_0]_{\alpha(x_0)}$ is nonempty closed bounded subset of X. For convenience, we denote $\alpha(x_0)$ by α_1. Let $x_1 \in [Sx_0]_{\alpha_1}$, for this x_1 there exists $\alpha_2 \in (0,1]$ such that $[Tx_1]_{\alpha_2}$ is non-empty closed bounded subset of X. Since $a_1 + a_3 + a_4 > 0$, by Lemma

1.1.8, there exists $x_2 \in [Tx_1]_{\alpha_2}$ such that

$$d(x_1, x_2) \leqslant H([Sx_0]_{\alpha_1}, [Tx_1]_{\alpha_2}) + a_1 + a_3 + a_4.$$

By the same argument, we can find $\alpha_3 \in (0,1]$ and $x_3 \in [Sx_2]_{\alpha_3}$ such that

$$d(x_2, x_3) \leqslant H([Sx_2]_{\alpha_3}, [Tx_1]_{\alpha_2}) + q(a_2 + a_3 + a_4).$$

By induction, we produce a sequence $\{x_n\}$ of points of X,

$$\begin{aligned} x_{2k+1} &= [Sx_{2k}]_{\alpha_{2k+1}}, \\ x_{2k+2} &= [Tx_{2k+1}]_{\alpha_{2k+2}}, \quad k = 0, 1, 2, ..., \end{aligned}$$

such that

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\leqslant H([Sx_{2k}]_{\alpha_{2k+1}}, [Tx_{2k+1}]_{\alpha_{2k+2}}) + q^{2k}(a_1 + a_3 + a_4), \\ d(x_{2k+2}, x_{2k+3}) &\leqslant H([Sx_{2k+2}]_{\alpha_{2k+3}}, [Tx_{2k+1}]_{\alpha_{2k+2}}) + q^{2k+1}(a_2 + a_3 + a_4). \end{aligned}$$

It implies that

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\leqslant H([Sx_{2k}]_{\alpha_{2k+1}}, [Tx_{2k+1}]_{\alpha_{2k+2}}) + q^{2k}(a_1 + a_3 + a_4) \\ &\leqslant a_1 d(x_{2k}, [Sx_{2k}]_{\alpha_{2k+1}}) + a_2 d(x_{2k+1}, [Tx_{2k+1}]_{\alpha_{2k+2}}) \\ &\quad + a_3 \left[d(x_{2k}, [Tx_{2k+1}]_{\alpha_{2k+2}}) + d(x_{2k+1}, [Sx_{2k}]_{\alpha_{2k+1}}) \right] \\ &\quad + a_4 d(x_{2k}, x_{2k+1}) + q^{2k}(a_1 + a_3 + a_4) \\ &\leqslant (a_4 + a_1) d(x_{2k}, x_{2k+1}) + a_2 d(x_{2k+1}, x_{2k+2}) \\ &\quad + a_3 d(x_{2k}, x_{2k+2}) + q^{2k}(a_1 + a_3 + a_4) \\ &\leqslant (a_1 + a_3 + a_4) d(x_{2k}, x_{2k+1}) + (a_2 + a_3)\ d(x_{2k+1}, x_{2k+2}) \\ &\quad + q^{2k}(a_1 + a_3 + a_4) \\ &\leqslant q d(x_{2k},\ x_{2k+1}) + q^{2k+1}. \end{aligned}$$

Similarly,

$$d(x_{2k+2}, x_{2k+3}) \leqslant qd(x_{2k+1},\ x_{2k+2}) + q^{2k+2}.$$

It follows that for each $n = 1, 2, ...,$

$$d(x_n, x_{n+1}) \leqslant q^n d(x_0,\ x_1) + nq^n.$$

Since $q < 1$, it follows from Cauchy's root test that $\sum nq^n$ is convergent and hence, $\{x_n\}$ is a Cauchy sequence in X . The remaining part of the proof is same as that of Vijayaraju and Marudai [169].

3.4.3 Remark

"*In connection with the proof [169, Theorem 3.1], consider the following equations:*

$$q = \max\left\{\frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}, \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4}\right\}. \tag{3.17}$$

$$(1 - a_2 - a_3)\, d(x_1, x_2) \leqslant (a_1 + a_3 + a_5)\ d(x_0, x_1) + q \tag{3.18}$$

$$\begin{aligned} d(x_1, x_2) &\leqslant \left(\frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}\right) d(x_0, x_1) + q \\ &\leqslant qd(x_0, x_1) + q. \end{aligned} \tag{3.19}$$

$$d(x_n, x_{n+1}) \leqslant q^n d(x_0,\ x_1) + nq^n. \tag{3.20}$$

$$d(x_1, x_2) = \left(\frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}\right) d(x_0, x_1) + \frac{q}{1 - a_2 - a_3} \tag{3.21}$$

$$> \left(\frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}\right) d(x_0, x_1) + q. \tag{3.22}$$

In [169, Theorem3.1], the authors constructed a sequence $\{x_n\}$ in X to achieve the requirements (3.19) *and* (3.20) *by using* (3.18) $\Longrightarrow$ (3.19), *an invalid assertion. In fact* (3.18) $\Longrightarrow$

(3.21) $\Longrightarrow$ (3.22) *as*

$$\frac{q}{1-a_2-a_3} > q \text{ (i.e., } 1-a_2-a_3 < 1).$$

Note that (3.18) $\not\Rightarrow$ (3.19) *as* (3.18) $\Longrightarrow$ (3.22) *which is negation of* (3.19). *This type of wrong derivation (i.e.*(3.18) $\Longrightarrow$ (3.19)*) has been performed at each step of induction to obtain requirement* (3.20). *Moreover, the authors used* (3.20) *to show,* $\{x_n\}$ *is a Cauchy sequence by claiming "* $q < 1$ *", whereas (under the given assumptions of [169]) it may happen that* $q \geqslant 1$*, e.g., let* $a_1 = a_2 = \frac{1}{10}$, $a_3 = \frac{1}{20}$, $a_4 = \frac{7}{10}$, $a_5 = \frac{1}{40}$, *then* $\sum_{i=1}^{5} a_i < 1$ *and* $a_1 = a_2$ *but*

$$q = \max\left\{\frac{a_1+a_3+a_5}{1-a_2-a_3}, \frac{a_2+a_4+a_5}{1-a_1-a_4}\right\} = \frac{33}{8} > 1.$$

Therefore, in Theorem 3.4.2 first, we have replaced the condition "either $a_1 = a_2$ *or* $a_3 = a_4$*" by "*$a_3 = a_4$*" In this way the statement of [169, Theorem 3.1] has been revised with the help of four constants* a_1, a_2, a_3, a_4 *instead of five. This modification has been made to achieve "* $q < 1$ *". Then we have removed incorrect derivations by using an appropriate technique of constructing the sequence* $\{x_n\}$. *Moreover, the proofs for the cases (i), (ii) (i.e.*$a_1 + a_3 + a_4 = 0$, $a_2 + a_3 + a_4 = 0$*) which were neglected in [169], have been established."*

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