Common Fixed Point Results In Generalized Metric Like Spaces

MS/M.Phil. Thesis

By
Zubair Nisar

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University Islamabad, Pakistan

2015

1- Metric topology.
2- Topological spaces.

Common Fixed Point Results In Generalized Metric Like Spaces

MS/M.Phil. Thesis

By

Zubair Nisar

Reg. # 88-FBAS/MSMA/F12

Supervised By: Dr. Muhammad Arshad Zia

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University Islamabad, Pakistan

2015

بسم الله الرحمن الرحيم

Common Fixed Point Results In Generalized Metric Like Spaces

MS/M.Phil. Thesis

By
Zubair Nisar
Reg. # 88-FBAS/MSMA/F12

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE DEGREE OF **MASTER OF SCIENCES IN MATHEMATICS (MS)** AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULITY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD

Supervised By: Dr. Muhammad Arshad Zia

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University Islamabad, Pakistan
2015

DECLARATION

I hereby declare and affirm that this research work neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have developed this research work entirely on the basis of my personal efforts. If any part of this project is proved to be copied out or found to be a reproduction of some other source, I shall stand by the consequences.

Any portion of this presented work has not been submitted or used in support of an application for other degree or qualification of this or any other University or Institute of learning.

Zubair Nisar ____________
Reg. # 88-FBAS/MSMA/F12

DEDICATED TO....

"My parents, especially my respectable father NISAR AHMAD (Late), my family, Rabia Hashmi and all my well wishers"

ACKNOWLEDGEMENT

I owe special thanks to the supreme power, **Almighty Allah (S.W.T)** the most gracious, the most merciful for his uncountable blessings, strength and knowledge to complete this dissertation and uncountable blessings upon **Holy Prophet Muhammad (P.B.U.H)**, constant source of guidance for humanity.

I am thankful to my honorable supervisor **Assistant Professor Dr. Muhammad Arshad Zia** (Department of Mathematics & Statistics, International Islamic University Islamabad, Pakistan) for his inspiring guidance, helpful pieces of advice, motivation, keen interest and constant encouragement throughout the span of my research work. He will always remain a source of inspiration for me. I am also thankful to **The Head of department of mathematics and statistics IIUI** for ensuring and providing a great research environment.

I wish to express my gratitude to **Aftab Shah, Syed Husnain Kazmi, Jamshaid Jasim, Kafayat-ul-lal, Abdullah Shoaib and Rabia Hashmi** for their timely help and to encourage me throughout the research. May Allah bless all of them.

Finally I am very much grateful to my parents, especially my respectable father **Nisar Ahmad (Late)**, that they have blessed me so much and gave me their sincere support. May Allah (S.W.T) bless my parents and may the soul of my late father rest in heaven.

Zubair Nisar
Reg. # 88-FBAS/MSMA/F12

Certificate

COMMON FIXED POINT RESULTS IN GENERALISED METRIC LIKE SPACES

By

Zubair Nisar

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF SCIENCE IN MATHEMATICS***

We accept this dissertation as conforming to the required standard

1. Prof. Dr. Moiz Ud Din
(External Examiner)

2. Dr. Tahir Mehmood
(Internal Examiner)

3. Dr. Muhammad Arshad Zia
(Supervisor)

3. Dr. Rahmat Ellahi 27/8/15
(Chairman)

Department of Mathematics & Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan

2015

CONTENTS

Preface **1**
1. INTRODUCTION **4**
1.1 Basic Definitions 4
1.2 G-metric space 5
1.3 Some Classical fixed point results on closed ball 7
2. MODIFIED BANACH THEOREM IN G-METRIC SPACE **10**
2.1 Modified Banach Fixed Point Theorem 10
2.2 Error Bounds 19
3. COMMON FIXED POINT RESULTS IN G-METRIC LIKE SPACES **22**
3.1 Banach Common Fixed Point 22
3.1.1 Error Bounds 32
3.2 Kannan Common Fixed Point Result 35
3.2.1 Error Bounds 48
3.3 Chatterjea Common Fixed Point Result 51
3.3.1 Uniqueness of Chatterjea Common Fixed Point 60
3.3.2 Error Bounds 64
REFRENCES **68**

Preface

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. It is concerned with the result which states that under certain conditions a self map Γ on a set X admits one or more fixed points like functional equation $a = \Gamma a$ has one or more solutions. Fixed point theory started almost immidiately after the classical analysis began its rapid development. A large variety of the problems of analysis and applied mathematics relate to finding solutions of nonlinear functional equations which can be formulated in terms of finding the fixed points of a nonlinear mappings. In fact, fixed point theorems are extremely substantial tools for proving the existence and uniqueness of the solutions to various mathematical models (diferential, integral and partial diferential equations and variational inequalities etc), exhibiting phenomena arising in broad spectrum of fields, such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, financial analysis, epidemics, biomedical research and flow of fluids etc. They are also used to study the problems of optimal control related to these systems. Thus fixed point theory started as purely analytical theory. This field of mathematics can be divided into three major areas which are Metric fixed Point Theory, Topological Fixed Point Theory and Discrete Fixed point Theory. Classical and major results in these areas are Brouwer's Fixed Point Theorem, Banach Fixed Point Theorem and Tarski's Fixed Point Theorem.

A self mapping Γ on a metric space X is said to be a Banach contraction mapping if $d(\Gamma a, \Gamma b) \leq \rho d(a, b)$ holds for all $a, b \in X$ where $0 \leq \rho < 1$. This theorem plays a fundamental role in the field of fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. This principle has many applications and it has been extended by several authors. Taking this process much further Kannan, Chaterjea and Hardy Roger proved other fixed point theorems with better contractions, which also have many applications in fixed point theory. There has also been a lot of activity in different weakly contractive mappings, which are generalization of the existing contractive conditions.

Recently, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering $\preceq$. Ran and Reurings [30] proved an analogue of Banach's fixed point theorem in metric space endowed with a partial order and gave applications to matrix equations. In this way, they weakened the usual contractive condition. Subsequently, Nieto et. al. [37] extended the result in [30] for nondecreasing mappings and applied it to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. Indeed, they all deal with a monotone mappings (either order-preserving or order-reversing) mapping and such that for some $a_0 \in X$, either $a_0 \preceq \Gamma a_0$ or $\Gamma a_0 \preceq a_0$, where Γ is a self-map on metric space. To obtain unique solution an additional restriction that each pair of elements has a lower bound and an upper bound. Instead of monotone mapping, one can take dominated mapping, which is introduced in [2, 3, 4]. The dominated mapping which satisfies the condition $\Gamma a \preceq a$ occurs very naturally in several practical problems. For example if a denotes the total quantity of wheat produced over a certain period of time and $\Gamma(a)$ gives the quantity of wheat consumed over the same period in a certain village, then we must have $\Gamma a \preceq a$.

In 1963, Ghaler generalized the idea of metric space and introduce 2-metric space which followed by a number of papers dealing with this generalized space. A plenty of material is available in other generalized metric spaces, such as, semi metric spaces, Quasi semi metric spaces and D-metric spaces. Fixed points results of mappings satisfying certain contractive conditions on the entire domain has been at the centre of vigorous research activity, for example (see [2, 3, 5]). Thereafter, many work related to fixed point problems have also been considered in $G-metric\ spaces$ (see [7, 8, 10, 13]). Z. Mustafa and Sims introduced the concept of $G-metric\ Space$ in [12]. $G-metric\ Spaces$ have applications in theoretical computer science. Aydi [26] used the idea of partial metric space and partial order and gave some fixed point theorems for contractive condition on ordered partial metric spaces. Recently, Karapinar et. al. [28] introduced the concept of (Φ, Ψ) contractive $G-metric$ *space*. Azam et. al. [1, 5, 7] proved a significant result concerning the existence of fixed points of a mapping satisfying a contractive condition on closed ball of a complete metric space. Arshad et. al. [2] have submitted a paper related to fixed points of a pair of Banach type mappings on a closed ball in ordered partial metric spaces. For the last few decades, there has also been a lot of activity in weakly contractive type mappings and several well-known fixed point theorems have been extended by a number of authors in different directions (see, for example, [6, 10, 13, 15, 16, 17, 18, 20, 21, 22, 24, 25, 27, 30, 34, 35, 36]).

Chapter-1, is devoted for some essential and basic definitions and propositions, some classical fixed point results and their related examples.

Chapter-2, consist of modified Banach fixed point results for locally contractive mappings in $G-metric\ spaces$, which are proved by Erdal Karanpinar [39] for globally contractive mapping in $G-metric\ spaces$.

Chapter-3, is devoted for common fixed point results for locally contractive double mappings in $G-metric\ like\ spaces$.

CHAPTER I

INTRODUCTION

The purpose of this chapter is to present basic definitions and concepts about $G-metric$ $spaces$, some classical fixed point results and to explain the terminology used through out this thesis. Some previously known results are given without proof. Section 1.1 is about basic concepts, Section 1.2 is concerned with the introduction of $G-metric\ space$ and concept of closed ball. Section 1.3 deals with some classical fixed point results in $G-metric$ $spaces$ for single self dominated mappings and about error bounds.

1.1 Basic Concepts

Definition 1 *[33] Let (X, d) be a metric space. A point $a \in X$ is said to be a* **fixed point** *of mapping $\Gamma : X \to X$ if $a = \Gamma a$.*

Definition 2 *[4] Let $(X, \preceq)$ be a partial ordered set. Then elements $a, b \in X$ are called* **comparable elements** *if either $a \preceq b$ or $b \preceq a$ holds.*

Definition 3 *[4] Let $(X, \preceq)$ be a partially ordered set. A self mapping Γ on X is called* **dominated** *if $\Gamma a \preceq a$ for each $a \in X$.*

Definition 4 *[38] Let $(X, \preceq)$ be a partially ordered set. A self mapping Γ on X is called* **dominating** *if $a \preceq \Gamma a$ for each $a \in X$.*

Definition 5 *[37] If $(X, \preceq)$ is a partially ordered set and $\Gamma : X \to X$ we say that Γ is* **monotone non − decreasing** *if for $a, b \in X$,*

$$a \preceq b \Longrightarrow \Gamma a \preceq \Gamma b.$$

This definition coincides with the notion of a non decreasing function in the case where $X = R$ and $\preceq$ represents the usual total order in R."

1.2 G-metric Space

Definition 6 *[12] Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$, be a function satisfying the following properties:*

(G_1) $G(a, b, c) = 0$ if $a = b = c$,

(G_2) $0 < G(a, a, b)$; $\forall\ a, b \in X$, with $a \neq b$,

(G_3) $G(a, a, b) \leq G(a, b, c)$, $\forall\ a,\ b,\ c \in X$ with $b \neq c$,

(G_4) $G(a, b, c) = G(a, c, b) = G(b, a, c) = G(b, c, a) = G(c, a, b) = G(c, b, a)$, (symmetry in all three variables),

(G_5) $G(a, b, c) \leq G(a, d, d) + G(d, b, c)$, for all $a, b, c, d \in X$, (rectangle ineguality),

then the function G is called a **Generalized Metric**, more specifically a $G-$ *metric* on X and the pair (X, G) is a **G − metric space**. It is known that the function $G(x, y, z)$ on $G-$ *metric space* X is jointly continuous in all three of its variables, and $G(a, b, c) = 0$ if and only if $a = b = c$.

Definition 7 *[17] Let X be a nonempty set and let $G_d : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:*

(i) If $G_d(a, b, c) = G_d(a, c, b) = G_d(b, a, c) = G_d(b, c, a,) = G_d(c, a, b) = G_d(c, b, a) = 0$, then $a = b = c$,

(ii) $G_d(a, b, c) \leq G_d(a, a, d) + G_d(d, b, c)$ for all $a, b, c, d \in X$ (rectangle inequality).

Then the pair (X, G_d) is called the **dislocated quasi $\mathbf{G}_d$ − metric space.** It is clear that if

$G_d(a, b, c) = G_d(b, c, a) = G_d(c, a, b) = \cdots = 0$ then from (i) $a = b = c$. But if $a = b = c$ then $G_d(a, b, c)$ may not be 0. It is observed that if $G_d(a, b, c) = G_d(b, c, a) = G_d(c, a, b)$ for all $a, b, c \in X$, then (X, G_d) becomes a **$\mathbf{G}_d$ − metric like space.**

Example 8 *Let $X = R$ be a non empty set and $G : X \times X \times X \to [0, \infty)$ be a function defined by*

$$G(a, b, c) = d(a, b) + d(b, c) + d(a, c),$$

$\forall a,b,c \in X$ *where* $d : X \times X \longrightarrow [0,\infty)$ *is usual metric. Then clearly* $G : X \times X \times X \rightarrow [0,\infty)$ *is* $G-$ *metric Like Space.*

Definition 9 *[23] Let A G-metric space* (X,G) *is said to be* $\mathbf{G-complete}$ *if every G-Cauchy sequence in* (X,G) *is G-convergent in* X.

Definition 10 *[23] Let* (X,G) *be a G-metric space then for* $a_0 \in X$, $r > 0$, *the* $\mathbf{G-ball}$ *with centre* a_0 *and radius* $r \geq 0$ *is,*

$$B(a_0,r) = \{p \in X : G(a_0,p,p) < r\}.$$

Definition 11 *[28] Let* Γ *and* Δ *be self maps of set* X. *If* $b = \Gamma a = \Delta a$ *for some* $a \in X$, *then* a *is called a* **coincidence point** *of* Γ *and* Δ *and* b *is called* **point of coincidence** *of* Γ *and* Δ.

Note that if $a = b$ then $b \in X$ becomes common fixed point of self mappings Γ and Δ.

Definition 12 *[16] Two self mappings* Γ *and* Δ *are said to be* **weakly compatible** *if they commute at coincidence point.*

Proposition 13 *[12] Let* (X,G) *be a G-metric space, then the following are equivalent:*

(1) $\{a_n\}$ is G convergent to a,

(2) $G(a_n,a_n,a) \rightarrow 0$ as $n \rightarrow \infty$,

(3) $G(a_n,a,a) \rightarrow 0$ as $n \rightarrow \infty$,

(4) $G(a_n,a_m,a) \rightarrow 0$ as $m\ n \rightarrow \infty$.

Proposition 14 *[23] Let* (X,G) *be a G-metric space. Then the function* $G(a,b,c)$ *is jointly continuous in all three variables.*

Proposition 15 *[23] Let* (X,G) *be a G-metric Like space, then the following are equivalent:*

(1) The sequence $\{a_n\}$ is $G_d - cauchy$,

(2) For $\in > 0$,there exists $k \in \mathbb{N}$ such that $G_d(a_n,a_m,a) < \in$, for all $n,m \geq k$.

Definition 16 *[23] Let (X,G) be a G-metric Like space and let $\{a_n\}$ be a sequence of points in X. A point a in X is said to be the* **limit of the sequence** *$\{a_n\}$ if $\lim_{m,n\to\infty} G_d(a,a_n,a_m) = 0$, and one says that sequence $\{a_m\}$ is* **G − convergent** *to a. Thus, if $a_n \to a$ in a G-metric space (X,G), then for any $\in > 0$, there exists $n,m \in \mathbb{N}$ such that $G_d(a,a_n,a_m) < \in$, for all $n,m \geq N$. $a \in X$ be a non empty set.*

Definition 17 *Let (X,G_d) be a $G-$metric Like space then,*

(i) A sequence $\{a_n\}$ in (X,G_d) is called **Cauchy Sequence** if for all $\in > 0$, there exists $n_0 \in N$ such that $m,n,l \geq n_0$, $G_d(a_n,a_m,a_l) < \in$, $G_d(a_m,a_n,a_l) < \in$, and $G_d(a_l,a_n,a_m) < \in$.

(ii) A sequence $\{a_n\}$ in *$G-$metric Like space* (for short G_d converges) to a if $\lim_{n\to\infty} G_d(a_n,a_n,a) = \lim_{n\to\infty} G_d(a,a_n,a_n) = 0$. In this case a is called a **G_d − limit of** $\{a_n\}$.

(iii) (X,G_d) is **complete** if every Cauchy Sequence in it is $G_d - convergent$.

Lemma 18 *[39] Let (X,G) be a G-metric space then $\forall a,b \in X$*

$$G(a,a,b) \leq 2G(a,b,b).$$

Definition 19 *[41] Let (X,G) be a G-metric space then it is* ***symmetric G-metric space*** *if $\forall a,b \in X$*

$$G(a,b,b) = G(b,a,a).$$

1.3 Some Classical Fixed Point Results on Closed Ball

This section deals with some fixed point results on a closed ball for ordered complete metric spaces.

Theorem 20 *[33] Let (X,d) be a complete metric space, $S : X \to X$ be a mapping, $r > 0$ and a_0 be an arbitrary point in X. Suppose there exists $k \in [0,1)$ with*

$$d(Sa,Sb) \leq kd(a,b),$$

$\forall\ a,b \in Y = \overline{B(a_0,r)}$ and

$$d(a_0,Sa_0) < (1-k)r,$$

then there exists a unique point a^* in $\overline{B(a_0,r)}$, such that, $a^* = Sa^*$.

Theorem 21 *[8] Let $(X, \preceq, d_q)$ be an ordered complete dislocated quasi metric space, $S : X \to X$ be a dominated map and a_0 be an arbitrary point in X. Suppose there exists $\xi \in [0, 1)$ with*

$$d_q(Sa, Sb) \leq \xi d_q(a, b), \ \forall a, b \in \overline{B(a_0, r)}$$

$$\text{and } d_q(a_0, Sa_0) \leq (1 - \xi)r.$$

If, for a nonincreasing sequence $\{a_n\} \to u$ implies that $u \preceq a_n$. Then there exists a point a^* in $\overline{B(a_0, r)}$ such that $a^* = Sa^*$ and $d_q(a^*, a^*) = 0$. Moreover, if for any two points a, b in $\overline{B(a_0, r)}$ there exists a point $c \in \overline{B(a_0, r)}$ such that $c \preceq a$ and $c \preceq b$, that is, every pair of elements in $\overline{B(a_0, r)}$ has a lower bound, then, the point a^* is unique.

Theorem 22 *[8] Let $(X, \preceq, d_q)$ be an ordered complete dislocated quasi metric space, $S : X \to X$ be a dominated map and a_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with*

$$d_q(Sa, Sb) \leq \xi[d_q(a, Sa) + d_q(b, Sb)],$$

for all comparable elements a, b in $\overline{B(a_0, r)}$ and

$$d_q(a_0, Sa_0) \leq (1 - \theta)r,$$

where $\theta = \frac{\xi}{1-\xi}$. If for a nonincreasing sequence $\{a_n\} \to u$ implies that $u \preceq a_n$. Then there exists a point a^* in $\overline{B(a_0, r)}$ such that $a^* = Sa^*$ and $d_q(a^*, a^*) = 0$. Moreover, if for any two points a, b in $\overline{B(a_0, r)}$ there exists a point $c \in \overline{B(a_0, r)}$ such that $c \preceq a$ and $c \preceq b$,and

$$d_q(a_0, Sa_0) + d_q(c, Sc) \leq d_q(a_0, c) + d_q(Sa_0, Sc), \ \forall \ c \preceq a_0,$$

then, the point a^* is unique.

Theorem 23 *[9] Let $(X, \preceq, d_q)$ be an ordered complete dislocated quasi metric space, $a_0 \in X$, $r > 0$ and $S, T : X \to X$ be a two dominated mappings. Suppose there exists x, y, $z \in [0, 1)$ with $x + 2y + 2z < 1$ such that*

$$d_l(Sa, Tb) \leq x \ d_l(a, b) + y \ [d_l(a, Ta) + d_l(b, Tb)] + z[d_l(a, Tb) + d_l(b, Ta)],$$

for all compareable elements $a, b \in \overline{B(a_0, r)}$ *and*

$$d_l(a,b) \leq (1-\lambda)r,$$

where $\lambda = \frac{(x+y+z)}{1-y-z}$. *If for a non increasing sequence* $\{a_n\}$ *in* $\overline{B(a_0, r)}$. $a_n \longrightarrow u$ *implies that* $u \preceq a_n$ *then there exist a point* $a^\star$ *in* $\overline{B(a_0, r)}$ *such that* $d_l(a^\star, a^\star) = 0$ *and* $a^\star = Sa^\star = Ta^\star$.

Theorem 24 *[40] Let* (X, G) *be a complete* $G-$ *metric space and let* $\Gamma : X \to X$ *be a mapping satisfying te following condition* $\forall a, b, c \in X$

$$G(\Gamma a, \Gamma b, \Gamma c) \leq \xi G(a, b, c),$$

where $\xi \in [0, 1)$. *Then* Γ *has a unique fixed point* $a^* \in X$ *such that* $\Gamma a^* = a^*$.

Theorem 25 *[39] Let* (X, G) *be a complete* $G-$ *metric space and let* $\Gamma : X \to X$ *be a mapping satisfying te following condition* $\forall a, b, c \in X$,

$$G(\Gamma a, \Gamma b, \Gamma c) \leq \xi M(a, b, c),$$

and

$$\begin{aligned} M(a,b,c) = \max\{ & G(a,\Gamma a,b), G(b,\Gamma^2 a,\Gamma b), G(\Gamma a,\Gamma^2 a,\Gamma b), G(b,\Gamma a,\Gamma b), G(a,\Gamma a,c), \\ & G(c,\Gamma^2 a,\Gamma c), G(\Gamma a,\Gamma^2 a,\Gamma c), G(c,\Gamma a,\Gamma b), G(a,b,c), G(a,\Gamma a,\Gamma a), \\ & G(b,\Gamma b,\Gamma b), G(c,\Gamma c,\Gamma c), G(c,\Gamma a,\Gamma a), G(a,\Gamma b,\Gamma b), G(b,\Gamma c,\Gamma c)\}, \end{aligned}$$

where $\xi \in [0, \frac{1}{2})$.*Then* $\exists$ *unique* $a^* \in X$ *such that* $\Gamma a^* = a^*$.

CHAPTER II

MODIFIED BANACH THEOREM IN G-METRIC SPACE

Karapinar et al. [39] have proved Banach fixed point theorem for globally contractive mappings in $G-$*metric spaces*. In this chapter we will prove modified Banach theorem for locally contractive dominated mappings in $G-$*metric spaces* and the related examples are given to verify the results.

2.1 *Modified Banach Fixed Point Theorem*

Theorem 26 *Suppose for a $G-$ metric spac (X,G) if a defined dominated mapping $\Gamma : X \longrightarrow X$ satisfies,*

$$G(\Gamma a, \Gamma b, \Gamma c) \leq \xi W(a,b,c) \tag{2.1}$$

$\forall a, a_0, b, c \in \overline{B_G(a_0, r)} \subseteq X$ and $r > 0$, where $\xi \in [0, \frac{1}{2})$ and

$$\begin{aligned} W(a,b,c) &= \max\{G(b,\Gamma^2 a,\Gamma b), G(\Gamma a,\Gamma^2 a,\Gamma b), G(a,\Gamma a,b), \\ &\quad G(a,\Gamma a,c), G(c,\Gamma^2 a,\Gamma c), G(b,\Gamma a,\Gamma b), G(\Gamma a,\Gamma^2 a,\Gamma c), \\ &\quad G(c,\Gamma a,\Gamma b), G(a,b,c), G(a,\Gamma a,\Gamma a), G(b,\Gamma b,\Gamma b), \\ &\quad G(c,\Gamma c,\Gamma c), G(a,\Gamma b,\Gamma b), G(b,\Gamma c,\Gamma c), G(c,\Gamma a,\Gamma a)\} \end{aligned} \tag{2.2}$$

And

$$G(a_0, a_1, a_1) \leq (1-\rho) r \tag{2.3}$$

where $\rho \in \{\xi, \Upsilon = \frac{\xi}{1-\xi}\}$ and $\rho \in [0,1)$. Then $\exists$ unique $a \in \overline{B_G(a_0, r)}$ such that $\Gamma a = a$.

Proof. Consider a picard sequence $\{a_n\}$ with initial guess $a_0 \in X$ such that $a_{n+1} \neq a_n$,

$$a_{n+1} = \Gamma a_n, \ \forall n \in N \tag{2.4}$$

From (2.3) it is clear that

$$\begin{aligned} G(a_0, a_1, a_1) &\leq (1-\rho) r \\ G(a_0, a_1, a_1) &\leq r \end{aligned}$$

Then $a_1 \in \overline{B_G(a_0, r)}$. Now consider the relation

$$\begin{aligned} G(a_1, a_{2,} a_2) &= G(\Gamma a_0, \Gamma a_{1,} \Gamma a_1) \\ G(a_1, a_{2,} a_2) &\leq \xi W(a_0, a_{1,} a_1), \text{ From (2.1)} \end{aligned}$$

From (2.2),

$$\begin{aligned} W(a_0, a_{1,} a_1) &= \max\{G(a_0, a_{1,} a_1), G(a_1, a_{2,} a_2), G(a_1, a_{1,} a_2), \\ &\quad G(a_0, a_{2,} a_2)\} \end{aligned}$$

In first case if $W(a_0, a_{1,} a_1) = G(a_1, a_{2,} a_2)$ then,

$$\begin{aligned} G(a_1, a_{2,} a_2) &\leq \xi G(a_1, a_{2,} a_2) \\ (1-\xi) G(a_1, a_{2,} a_2) &\leq 0 \end{aligned}$$

$$\begin{aligned} G(a_1, a_{2,} a_2) &= 0 \\ a_1 &= a_2 \end{aligned}$$

It is contradiction because $a_1 \neq a_2$. In second case if $W(a_0, a_{1,} a_1) = G(a_1, a_{1,} a_2)$ then,

$$\begin{aligned} G(a_1, a_{2,} a_2) &\leq \xi G(a_1, a_{1,} a_2) \\ G(a_1, a_{2,} a_2) &\leq 2\xi G(a_1, a_{2,} a_2) \end{aligned}$$

$$\begin{aligned} (1-2\xi) G(a_1, a_{2,} a_2) &\leq 0 \\ G(a_1, a_{2,} a_2) &\leq 0 \end{aligned}$$

$$\begin{aligned} G(a_1, a_{2,} a_2) &= 0 \\ a_1 &= a_2 \end{aligned}$$

It is again contradiction because $a_1 \neq a_2$. In third case if $W(a_0, a_{1,} a_1) = G(a_0, a_{1,} a_1)$ then,

$$G(a_1, a_{2,} a_2) \leq \xi G(a_0, a_{1,} a_1)$$

It is true for $\xi \in [0, \frac{1}{2})$. In fourth case if $W(a_0, a_1, a_1) = G(a_0, a_2, a_2)$ then,

$$\begin{aligned} G(a_1, a_2, a_2) &\leq \xi G(a_0, a_2, a_2) \\ G(a_1, a_2, a_2) &\leq \xi G(a_0, a_1, a_1) + \xi G(a_1, a_2, a_2) \end{aligned}$$

$$\begin{aligned} (1-\xi) G(a_1, a_2, a_2) &\leq \xi G(a_0, a_1, a_1) \\ G(a_1, a_2, a_2) &\leq \frac{\xi}{1-\xi} G(a_0, a_1, a_1) \end{aligned}$$

It is again true for $\Upsilon = \frac{\xi}{1-\xi}$ and $0 \leq \Upsilon < 1$. Hence

$$G(a_1, a_2, a_2) \leq \rho G(a_0, a_1, a_1), \text{ For } \rho \in \{\xi, \Upsilon\}$$

Now by rectangular property,

$$G(a_0, a_2, a_2) \leq G(a_0, a_1, a_1) + G(a_1, a_2, a_2)$$

$$G(a_0, a_2, a_2) \leq (1+\rho) G(a_0, a_1, a_1)$$

$$\begin{aligned} G(a_0, a_2, a_2) &\leq (1+\rho)(1-\rho) r = (1-\rho^2) r \\ G(a_0, a_2, a_2) &\leq r \end{aligned}$$

Hence $a_2 \in \overline{B_G(a_0, r)}$. Now let $a_3, a_4, \ldots\ldots\ldots, a_i \in \overline{B_G(a_0, r)}$, by mathematical induction general inequality can be obtain far all even $i \in N$ as follows,

$$G(a_{i-1}, a_i, a_i) = \rho^{i-1} G(a_0, a_1, a_1) \tag{2.5}$$

Now consider the relation

$$\begin{aligned} G(a_i, a_{i+1}, a_{i+1}) &= G(\Gamma a_{i-1}, \Gamma a_i, \Gamma a_i) \\ G(a_i, a_{i+1}, a_{i+1}) &\leq \xi W(a_{i-1}, a_i, a_i) \end{aligned}$$

From (2.2),

$$\begin{aligned} W(a_{i-1}, a_i, a_i) = \max\{ & G(a_i, a_{i+1}, a_{i+1}), G(a_{i-1}, a_{i+1}, a_{i+1}), G(a_{i-1}, a_i, a_i), \\ & G(a_i, a_i, a_{i+1})\} \end{aligned}$$

In first case if $W(a_{i-1}, a_i, a_i) = G(a_i, a_{i+1}, a_{i+1})$ then,

$$\begin{aligned} G(a_i, a_{i+1}, a_{i+1}) &\leq \xi G(a_i, a_{i+1}, a_{i+1}) \\ (1-\xi)G(a_i, a_{i+1}, a_{i+1}) &\leq 0 \end{aligned}$$

$$\begin{aligned} G(a_i, a_{i+1}, a_{i+1}) &= 0 \\ a_i &= a_{i+1} \end{aligned}$$

It is contradiction because $a_i \neq a_{i+1}$. In second case if $W(a_{i-1}, a_i, a_i) = G(a_i, a_i, a_{i+1})$ then,

$$\begin{aligned} G(a_i, a_{i+1}, a_{i+1}) &\leq \xi G(a_i, a_i, a_{i+1}) \\ G(a_i, a_{i+1}, a_{i+1}) &\leq \xi G(a_i, a_{i+1}, a_{i+1}) + \xi G(a_{i+1}, a_i, a_{i+1}) \end{aligned}$$

By symmetry of $G-$*metric space*, as $G(a_i, a_{i+1}, a_{i+1}) = G(a_{i+1}, a_i, a_{i+1})$ then,

$$\begin{aligned} G(a_i, a_{i+1}, a_{i+1}) &\leq 2\xi G(a_i, a_{i+1}, a_{i+1}) \\ (1-2\xi)G(a_i, a_{i+1}, a_{i+1}) &\leq 0 \end{aligned}$$

$$\begin{aligned} G(a_i, a_{i+1}, a_{i+1}) &= 0 \\ a_i &= a_{i+1} \end{aligned}$$

It is again contradiction because $a_i \neq a_{i+1}$. In third case if $W(a_{i-1}, a_i, a_i) = G(a_{i-1}, a_{i+1}, a_{i+1})$ then,

$$\begin{aligned} G(a_i, a_{i+1}, a_{i+1}) &\leq \xi G(a_{i-1}, a_{i+1}, a_{i+1}) \\ G(a_i, a_{i+1}, a_{i+1}) &\leq \xi G(a_{i-1}, a_i, a_i) + \xi G(a_i, a_{i+1}, a_{i+1}) \end{aligned}$$

$$\begin{aligned} (1-\xi)G(a_i, a_{i+1}, a_{i+1}) &\leq \xi G(a_{i-1}, a_i, a_i) \\ G(a_i, a_{i+1}, a_{i+1}) &\leq \frac{\xi}{1-\xi} G(a_{i-1}, a_i, a_i) \end{aligned} \tag{2.6}$$

It is true for $\Upsilon = \frac{\xi}{1-\xi} \in [0,1)$. In fourth case if $W(a_{i-1}, a_i, a_i) = G(a_{i-1}, a_i, a_i)$ then,

$$G(a_i, a_{i+1}, a_{i+1}) \leq \xi G(a_{i-1}, a_i, a_i) \tag{2.7}$$

It is true for $\xi \in [0, \frac{1}{2})$. Hence in general from relations (2.6) and (2.7),

$$G(a_i, a_{i+1}, a_{i+1}) \leq \rho G(a_{i-1}, a_i, a_i) \tag{2.8}$$

where $\rho \in \{\xi, \Upsilon = \frac{\xi}{1-\xi}\}$. Therefore from relations (2.5), relation (2.8) gives,

$$G(a_i, a_{i+1}, a_{i+1}) \leq \rho^i G(a_0, a_1, a_1) \tag{2.9}$$

Now from rectangular property,

$$\begin{aligned} G(a_0, a_{i+1}, a_{i+1}) &\leq G(a_0, a_1, a_1) + G(a_1, a_2, a_2) + ... + G(a_i, a_{i+1}, a_{i+1}) \\ G(a_0, a_{i+1}, a_{i+1}) &\leq G(a_0, a_1, a_1) + \rho G(a_0, a_1, a_1) + \rho^2 G(a_0, a_1, a_1) \\ &+ ..... + \rho^i G(a_0, a_1, a_1) \end{aligned}$$

$$\begin{aligned} G(a_0, a_{i+1}, a_{i+1}) &\leq (1 + \rho + \rho^2 + ... + \rho^i) G(a_0, a_1, a_1) \\ G(a_0, a_{i+1}, a_{i+1}) &\leq (\frac{1 - \rho^{i+1}}{1 - \rho})(1 - \rho) r \end{aligned}$$

$$\begin{aligned} G(a_0, a_{i+1}, a_{i+1}) &\leq (1 - \rho^{i+1}) r \\ G(a_0, a_{i+1}, a_{i+1}) &\leq r \end{aligned}$$

Hence $a_{i+1} \in \overline{B_G(a_0, r)}$. Therefore picard sequence $\{a_n\} \in \overline{B_G(a_0, r)}$, $\forall n \in N \cup \{0\}$. Now to show that picard sequence $\{a_n\}$ is Cauchy sequence consider for $m, n \in N$ such that for $n < m$,

$$\begin{aligned} G(a_n, a_m, a_m) &\leq G(a_n, a_{n+1}, a_{n+1}) + G(a_{n+1}, a_{n+2}, a_{n+2}) + ... \\ &+ G(a_{m-2}, a_{m-1}, a_{m-1}) + G(a_{m-1}, a_m, a_m) \end{aligned}$$

$$\begin{aligned} G(a_n, a_m, a_m) &\leq \rho^n G(a_0, a_1, a_1) + \rho^{n+1} G(a_0, a_1, a_1) + \rho^{n+2} G(a_0, a_1, a_1) + \\ &...... + \rho^{m-1} G(a_0, a_1, a_1) \end{aligned}$$

$$G(a_n, a_m, a_m) \leq (1 + \rho + \rho^2 + ... + \rho^{m-n-1}) \rho^n G(a_0, a_1, a_1)$$

$$G(a_n, a_m, a_m) \leq (\frac{1 - \rho^{m-n}}{1 - \rho}) \rho^n G(a_0, a_1, a_1) \tag{2.10}$$

from relation (2.3),

$$G(a_n, a_m, a_m) \leq (1 - \rho^{m-n})\rho^n r$$

$$G(a_n, a_m, a_m) \leq \rho^n r \tag{2.11}$$

As $\rho \in [0, 1)$, then $\rho^n \to 0$ if $n \to \infty$. Hence $\rho^n r \to 0$ if $n \to \infty$. So for any $\widetilde{\epsilon} \in R$ however small, $\exists\ k \in R$ such that from (2.11),

$$G(a_n, a_m, a_m) \leq \rho^n r = \widetilde{\epsilon}, \text{ when } m, n > k$$

Therefore picard sequence $\{a_n\}$ is Cauchy sequence in close ball $\overline{B_G(a_0, r)}$. As close ball $\overline{B_G(a_0, r)}$ is close subset of set X, then the sequence $\{a_n\}$ is convergent in close ball $\overline{B_G(a_0, r)}$ and the point of convergence is $a \in \overline{B_G(a_0, r)}$. Hence $a_n \to a$ as $n \to \infty$. In general it is clear that,

$$\underset{n\to\infty}{Lim} G(a_n, a, a) = \underset{n\to\infty}{Lim} G(a, a_n, a_n) = 0 \tag{2.12}$$

To check $a \in \overline{B_G(a_0, r)}$ is either fixed point of $\Gamma : X \to X$ or not consider,

$$G(a, \Gamma a, \Gamma a) \leq G(a, a_{n+1}, a_{n+1}) + G(a_{n+1}, \Gamma a, \Gamma a)$$

$$G(a, \Gamma a, \Gamma a) \leq G(a, a_n, a_n) + \xi W(a_n, a, a) \tag{2.13}$$

From (2.2),

$$\begin{aligned} W(a_n, a, a) \ = \ & \max\{G(a, \Gamma^2 a_n, \Gamma a), G(a_n, \Gamma a_n, a), G(\Gamma a_n, \Gamma^2 a_n, \Gamma a), \\ & G(a, \Gamma a_n, \Gamma a), G(a_n, \Gamma a_n, a), G(a, \Gamma^2 a_n, \Gamma a), \\ & G(\Gamma a_n, \Gamma^2 a_n, \Gamma a), G(a, \Gamma a_n, \Gamma a), G(a_n, a, a), \\ & G(a_n, \Gamma a_n, \Gamma a_n), G(a, \Gamma a, \Gamma a), G(a, \Gamma a, \Gamma a), \\ & G(a, \Gamma a_n, \Gamma a_n), G(a, \Gamma a, \Gamma a), G(a_n, \Gamma a, \Gamma a)\} \end{aligned}$$

$$\begin{aligned} W(a_n, a, a) = \max\{& G(a, a_{n+2}, \Gamma a), G(a_n, a_{n+1}, a), G(a_{n+1}, a_{n+2}, \Gamma a), \\ & G(a, a_{n+1}, \Gamma a), G(a_n, a_{n+1}, a), G(a, a_{n+2}, \Gamma a), \\ & G(a_{n+1}, a_{n+2}, \Gamma a), G(a, a_{n+1}, \Gamma a), G(a_n, a, a), \\ & G(a_n, a_{n+1}, a_{n+1}), G(a, \Gamma a, \Gamma a), G(a, \Gamma a, \Gamma a), \\ & G(a, a_{n+1}, a_{n+1}), G(a, \Gamma a, \Gamma a), G(a_n, \Gamma a, \Gamma a)\} \end{aligned}$$

$$\begin{aligned} W(a_n,a,a) &= \max\{G(a,a_{n+2},\Gamma a),G(a_n,a_{n+1},a_{n+1}),G(a,a_{n+1},a_{n+1}), \\ &\quad G(a,a_{n+2},\Gamma a),G(a,a_{n+1},\Gamma a),G(a_{n+1},a_{n+2},\Gamma a),G(a_n,a,a), \\ &\quad G(a,\Gamma a,\Gamma a),G(a_n,a_{n+1},a),G(a_n,\Gamma a,\Gamma a)\} \qquad (2.14) \end{aligned}$$

For every selection of $W(a_n, a, a)$ from (2.14) and applying limit $n \to \infty$ on (2.13) gives,

$$\begin{aligned} G(a,\Gamma a,\Gamma a) &= 0 \\ \Gamma a &= a \end{aligned}$$

Hence $a \in \overline{B_G(a_0,r)}$ is fixed point of $\Gamma : X \to X$. For uniqueness of fixed point consider $a, b \in \overline{B_G(a_0,r)}$ are two distint fixed point of $\Gamma : X \to X$. So consider the relation,

$$\begin{aligned} G(a,b,b) &= G(\Gamma a,\Gamma b,\Gamma b) \\ G(a,b,b) &\leq \xi W(a,b,b) \qquad (2.15) \end{aligned}$$

Where

$$W(a,b,b) = \max\{G(a,b,b),G(b,a,b,),G(a,a,b),G(b,a,a)\} \qquad (2.16)$$

From (2.16) every choice of $W(a, b, b)$, relation (2.15) gives,

$$\begin{aligned} G(a,b,b) &= 0 \\ a &= b \end{aligned}$$

It is contradiction to our assumption ($\because a \neq b$). So our suposition is wrong. Hence fixed point of $\Gamma : X \to X$ is unique. ■

Example 27 *If for a set $X = [0, 2]$, a mapping $G : X \times X \times X \longrightarrow [0, \infty)$, $\forall a, b, c \in X$ defined by,*

$$G(a,b,c) = a + b + c \qquad (2.17)$$

then (X, G) is symmetric and complete $G-$metric like space. Let mapping $\Gamma : X \to X$ are defined by,

$$\Gamma a = \begin{cases} \frac{a}{8} & \text{if } a \in [0,1] \\ a + \frac{1}{8} & \text{if } a \in (1,2] \end{cases} \qquad (2.18)$$

Obviously Γ *dominated mapping inside of* $[0,1]$ *but not dominated outside of* $[0,1]$. *Let* $a_0 = \frac{2}{3}$ *and* $r = \frac{8}{3}$ *such that* $\overline{B_G(a_0, r)} = [0,1]$. *Also let* $\rho \in \{\xi = \frac{1}{3}, \Upsilon = \frac{\xi}{1-\xi} = \frac{1}{2}\} \subseteq [0,1)$ *such that,*

$$\textit{for } \rho = \frac{1}{3},\ (1-\rho)r = \frac{16}{9}$$

and

$$\textit{for } \rho = \frac{1}{2},\ (1-\rho)r = \frac{4}{3}$$

Also as

$$G(a_0, a_{1,} a_1) = \frac{2}{3} + 2\Gamma(\frac{2}{3}) = 1$$

Clearly

$$G(a_0, a_{1,} a_1) \le (1-\rho)r,\ \textit{For every } \rho \in \{\frac{1}{3}, \frac{1}{2}\} \tag{2.19}$$

To show contractive condition is locally contractive, for first case let $a, b, c \in [0,1]$ *then,*

$$\begin{aligned} G(\Gamma a, \Gamma b, \Gamma c) &= G(\frac{a}{8}, \frac{b}{8}, \frac{c}{8}) \\ G(\Gamma a, \Gamma b, \Gamma c) &= \frac{1}{8}(a+b+c) \end{aligned} \tag{2.20}$$

Also let

$$\begin{aligned} W(a,b,c) = \max\{ & \frac{9a+8b}{8}, \frac{a+72b}{64}, \frac{9a+8b}{64}, \frac{a+9b}{8}, \\ & \frac{9a+8c}{8}, \frac{a+72c}{64}, \frac{9a+8c}{64}, \frac{9a+b}{8}, a+b+c, \\ & \frac{5a}{4}, \frac{5b}{4}, \frac{5c}{4}, \frac{4a+b}{4}, \frac{4b+c}{4}, \frac{4c+a}{4}\} \end{aligned}$$

If $a, b, c \in [0,1]$, *then*

$$\begin{aligned} 0 &\le \frac{9a+8b}{64}, \frac{9a+8c}{64} \le \frac{17}{64}, 0 \le \frac{a+72b}{64}, \frac{a+72c}{64} \le \frac{73}{64} \\ 0 &\le \frac{a+9b}{8}, \frac{9a+b}{8}, \frac{5a}{4}, \frac{5b}{4}, \frac{5c}{4}, \frac{4a+b}{4}, \frac{4b+c}{4}, \frac{4c+a}{4} \le \frac{5}{4} \\ 0 &\le \frac{9a+8b}{8}, \frac{9a+8c}{8} \le \frac{17}{8}, 0 \le a+b+c \le 3 \end{aligned}$$

Clearly above inequalities shows that maximum value for $W(a,b,c)$ *is*

$$W(a,b,c) = a+b+c \tag{2.21}$$

From (2.20) *and* (2.21),

$$\begin{aligned} \frac{1}{8}(a+b+c) &\leq \frac{1}{3}(a+b+c) \\ G(\Gamma a, \Gamma b, \Gamma c) &\leq \xi W(a,b,c) \end{aligned}$$

Hence contractive condition is locally satisfied on $\overline{B_G(a_0,r)} = [0,1]$. *For the second case if* $a,b,c \in (1,2]$ *then,*

$$\begin{aligned} G(\Gamma a, \Gamma b, \Gamma c) &= G(a+\frac{1}{8}, b+\frac{1}{8}, c+\frac{1}{8}) \\ G(\Gamma a, \Gamma b, \Gamma c) &= (a+b+c)+\frac{3}{8} \end{aligned} \tag{2.22}$$

Also let

$$\begin{aligned} W(a,b,c) = \max\{& a+2b+\frac{3}{8}, 2a+b+\frac{1}{8}, a+2b+\frac{1}{4},, 2a+b+\frac{1}{2} \\ & 2a+c+\frac{1}{8}, a+2c+\frac{3}{8}, 2a+c+\frac{1}{2}, 2a+b+\frac{1}{4}, \\ & a+b+c+\frac{3}{8}, 3a+\frac{1}{4}, 3b+\frac{1}{4}, 3c+\frac{1}{4}, a+2b+\frac{1}{4}, \\ & b+2c+\frac{1}{4}, c+2a+\frac{1}{4}\} \end{aligned}$$

If $\forall\ a,b,c \in (1,2]$ *then*

$$\begin{aligned} \frac{25}{8} &\leq 2a+b+\frac{1}{8}, 2a+c+\frac{1}{8} \leq \frac{49}{8} \\ \frac{13}{4} &\leq a+2b+\frac{1}{4}, 2a+b+\frac{1}{4}, 3a+\frac{1}{4}, 3b+\frac{1}{4}, 3c+\frac{1}{4}, a+2b+\frac{1}{4}, b+2c+\frac{1}{4}, c+2a+\frac{1}{4} \leq \frac{25}{4} \\ \frac{27}{8} &\leq a+2b+\frac{3}{8}, a+2c+\frac{3}{8}, a+b+c+\frac{3}{8} \leq \frac{51}{8}, \frac{7}{2} \leq 2a+b+\frac{1}{2}, 2a+c+\frac{1}{2} \leq \frac{13}{2} \end{aligned}$$

Clearly above inequalities shows that maximum values for $W(a,b,c)$ *are,*

$$W(a,b,c) = 2a+b+\frac{1}{2} \text{ and } W(a,b,c) = 2a+c+\frac{1}{2}$$

Now as

$$\begin{aligned} (a+b+c)+\frac{3}{8} &\geq \frac{1}{3}(2a+b+\frac{1}{2}) \\ G(\Gamma a, \Gamma b, \Gamma c) &\geq \xi W(a,b,c) \end{aligned}$$

or

$$\begin{aligned} (a+b+c)+\frac{3}{8} &\geq \frac{1}{3}(2a+c+\frac{1}{2}) \\ G(\Gamma a, \Gamma b, \Gamma c) &\geq \xi W(a,b,c) \end{aligned}$$

Hence contractive condition is failed outside of $\overline{B_G(a_0,r)} = [0,1]$. *Therefore fixed point of* $\Gamma : X \to X$ *exists and is* $0 \in \overline{B_G(a_0,r)}$ *such that* $\Gamma 0 = 0$.

In above theorem, interval for contractive condition can be extended to [0,1) as shown by following corollary.

Corollary 28 *Suppose for a* $G-$*metric spac* (X,G) *if a defined mapping* $\Gamma : X \longrightarrow X$ *satisfies,*

$$G(\Gamma a, \Gamma b, \Gamma c) \leq \xi W(a,b,c) \tag{2.23}$$

$\forall a_0, a, b, c \in \overline{B_G(a_0,r)} \subseteq X$ *and* $r > 0$, *where* $\xi \in [0,1)$ *and*

$$\begin{aligned} W(a,b,c) \;=\; & \max\{G(b,\Gamma^2 a,\Gamma b), G(\Gamma a,\Gamma^2 a,\Gamma b), G(a,\Gamma a,b), G(c,\Gamma^2 a,\Gamma c), \\ & G(\Gamma a,\Gamma^2 a,\Gamma c), G(a,\Gamma a,c), G(a,b,c), G(a,\Gamma a,\Gamma a), \\ & G(b,\Gamma b,\Gamma b), G(c,\Gamma c,\Gamma c), G(c,\Gamma a,\Gamma a), G(b,\Gamma c,\Gamma c)\} \end{aligned} \tag{2.24}$$

And

$$G(a_0, a_1, a_1) \leq (1-\rho) r \tag{2.25}$$

where $\rho \in \{\xi, \Upsilon = \frac{\xi}{1-\xi}\}$ *and* $\rho \in [0,1)$. *Then* $\exists$ *unique* $a \in \overline{B_G(a_0,r)}$ *such that* $\Gamma a = a$.

2.2 Error Bounds

In this section errors approximations and their related example are discused.

Corollary 29 ***(Iteration, Error Bounds)*** *From Theorem 26, iterative sequence* (2.4), *with arbitrary* $a_0 \in \overline{B_G(a_0,r)} \subseteq X$, *converges to unique fixed point* $a \in \overline{B_G(a_0,r)}$ *of dominated mapping* $\Gamma : X \to X$. *Error estimates are the* ***prior estimate***

$$G(a_n, a, a) \leq \frac{\rho^n}{1-\rho} G(a_0, a_1, a_1) \tag{2.26}$$

and the posterior estimate

$$G(a_n, a, a) \leq \frac{\rho}{1-\rho} G(a_{n-1}, a_n, a_n) \tag{2.27}$$

Proof. As from relation (2.10) of theorem 1,

$$G(a_n, a_m, a_m) \leq \rho^n(\frac{1-\rho^{m-n}}{1-\rho})G(a_0, a_1, a_1)$$

As sequence $\{a_m\}$ is convergent to $a \in \overline{B_G(a_0, r)} \subseteq X$, then by taking $m \to \infty$ gives $a_m \to \infty$ and $\rho^{m-n} \to 0$. Therefore above relatioin leads to the *prior estimate* i.e.,

$$G(a_n, a, a) \leq \frac{\rho^n}{1-\rho}G(a_0, a_1, a_1)$$

Setting $n = 1$ and write b_o for a_0 and b_1 for a_1 in (2.26) gives,

$$G(b_1, a, a) \leq \frac{\rho}{1-\rho}G(b_0, b_1, b_1)$$

Letting $b_o = a_{n-1}$ then $b_1 = \Gamma b_o = \Gamma a_{n-1} = a_n$ in above relation leads to the *posterior estimate* i.e.,

$$G(a_n, a, a) \leq \frac{\rho}{1-\rho}G(a_{n-1}, a_n, a_n)$$

■

The prior bound (2.26) can be used at the beginning of the calculation for estimating the required number iterations to obtain the assumed accuracy. While posterior estimate(2.27) can be used at intermediate stages or at the end of the calculation. Posterior estimate(2.27) is at least as accurate as prior estimate(2.26).

Example 30 *If for a set $X = [0, 2]$, a mapping $G : X \times X \times X \longrightarrow X$, $\forall a, b, c \in X$ defined by,*

$$G(a, b, c) = a + b + c \tag{2.28}$$

then (X, G) is symmetric and complete $G-$metric like space. Let mapping $\Gamma : X \to X$ are defined by,

$$\Gamma a = \begin{cases} \frac{a}{8} & \text{if } a \in [0, 1] \\ a + \frac{1}{8} & \text{if } a \in (1, 2] \end{cases}$$

Obviously Γ dominated mapping inside of $[0, 1]$ but not dominated outside of $[0, 1]$. Let $a_0 = \frac{2}{3}$ and $r = \frac{8}{3}$ such that $\overline{B_G(a_0, r)} = [0, 1]$. Also let $\rho \in \{\xi = \frac{1}{3}, \Upsilon = \frac{\xi}{1-\xi} = \frac{1}{2}\} \subseteq [0, 1)$. Construct the picard iterative sequence taking $a_0 = \frac{2}{3} \in [0, 1]$ as initial guess as,

$$a_n = \Gamma a_{n-1} = \frac{a_0}{8^n}, \forall n \in N \cup \{0\} \tag{2.29}$$

Also as

$$G(a_0, a_1, a_1) = \frac{2}{3} + 2\Gamma(\frac{2}{3}) = \frac{5}{6}$$

and

$$G(a_n, a, a) = a_n + 2a$$

As picard sequence $\{a_n\}$ *satisfies all the conditions of modified Banach fixed point theorem 1 as in example 2, then If* $n \to \infty$ *so* $a_n \to a$ *i.e* $a_n \approx a$. *Then*

$$G(a_n, a, a) = 3a_n = \frac{3a_0}{8^n} = \frac{2}{8^n}$$

As from prior estimate

$$G(a_n, a, a) \leq \frac{\rho^n}{1-\rho} G(a_0, a_1, a_1) \tag{2.30}$$

If $\rho = \frac{1}{3}$ *then* (2.30) *gives,*

$$\begin{aligned} \frac{2}{8^n} &\leq \frac{3}{2.3^n} . \frac{5}{6} \\ \frac{\ln(\frac{8}{5})}{\ln(\frac{8}{3})} &\leq n \Longrightarrow 0.47919 \leq n \\ n &= 1, 2, 3, ... \textit{ being integer} \end{aligned}$$

If $\rho = \frac{1}{2}$ *then* (2.30) *gives,*

$$\begin{aligned} \frac{2}{8^n} &\leq \frac{2}{2^n} . \frac{5}{6} \\ \frac{\ln(\frac{6}{5})}{\ln(\frac{8}{2})} &\leq n \Longrightarrow 0.1315172 \leq n \\ n &= 1, 2, 3, ... \textit{ being integer} \end{aligned}$$

In either case if $\rho \in \{\frac{1}{3}, \frac{1}{2}\}$, *picard sequence* $\{a_n\}$ *converges for* $n = 1, 2, 3, ...$ *If* $n = 2$

$$a_2 = \Gamma a_1 = \frac{a_0}{8^2} = 0.0104166667$$

If $n = 3$ *then*

$$a_3 = \Gamma a_2 = \frac{a_0}{8^3} = 0.0013020833$$

Therefore

$$0.0013020833 \approx \Gamma 0.0104166667$$

This suggests, when integer $n \geq 1$ *goes on increasing, picard sequence moves towards fixed point of* Γ *which is* $a = 0 \in [0, 1]$, *i.e,* $\Gamma 0 = 0$.

CHAPTER III

COMMON FIXED POINT RESULTS IN G-METRIC LIKE SPACES

Banach [32], Kannan [35] and Chattergea [24] have independently proved fixed point theorems by using different contractive conditions in metric spaces. In this chapter we will generalize some common fixed point results for locally contractive dominated mappings in G-metric like spaces.

3.1 Banach Common Fixed Point Result

Theorem 31 *Suppose $(X;\leq;G)$ be a symmetric and ordered complete $G-$ metric like space and $\varphi,\delta : X \to X$ are any two dominated mappings and a_0, a, $b,c \in X$, $r > 0$. Suppose that $\exists\ \xi \in [0,1)$ such that,*

$$G_d(\varphi a, \delta b, \delta c) \leq \xi G_d(a,b,c),\ \forall\ a,b,c \in \overline{B_G(a_0,r)} \subseteq X \tag{3.1.1}$$

and

$$G_d(a_0,a_1,a_1) = G_d(a_0,\varphi a_0,\varphi a_0) \leq (1-\xi)r \tag{3.1.2}$$

If for non-increasing sequence $\{a_n\}$ in $\overline{B_G(a_0,r)}$, $\{a_n\} \to v$ then $\exists$ a unique $\overline{a} \in \overline{B_G(a_0,r)}$ such that $G_d(\overline{a},\overline{a},\overline{a}) = 0$ and $\varphi\overline{a} = \delta\overline{a} = \overline{a}$. Moreover $G_d(\overline{a},\overline{a},\overline{a}) = 0$.

Proof. With initially choosen guess $a_0 \in \overline{B_G(a_0,r)} \subseteq X$, consider picard sequence

$$a_{2n+1} = \delta a_{2n} \text{ and } a_{2n+2} = \varphi a_{2n+1} \tag{3.1.3}$$

As δ and φ are dominated mappings then,

$$.....a_n \leq a_{n-1} \leq a_{n-2} \leq ..... \leq a_3 \leq a_2 \leq a_1 \leq a_0$$

We know that,

$$\begin{aligned} G_d(a_0,a_1,a_1) &= G_d(a_0,\varphi a_0,\varphi a_0) \leq (1-\xi)r \\ G_d(a_0,a_1,a_1) &\leq r \end{aligned}$$

Clearly $a_1 \in \overline{B_G(a_0, r)}$. Now concider the relation

$$G_d(a_1, a_2, a_2) = G_d(\delta a_0, \varphi a_1, \varphi a_1)$$

As G-metric is symmetric then

$$\begin{aligned} G_d(a_1, a_2, a_2) &= G_d(\varphi a_1, \delta a_0, \delta a_0) \\ G_d(a_1, a_2, a_2) &\leq \xi G_d(a_0, a_1, a_1) \end{aligned} \tag{3.1.4}$$

Now for $a_2 \in X$, again consider by rectangular proprty of G-metric space,

$$G_d(a_0, a_2, a_2) \leq G_d(a_0, a_1, a_1) + G_d(x_1, x_2, x_2)$$

$$\begin{aligned} G_d(a_0, a_2, a_2) &\leq (1+\xi)G_d(a_0, a_1, a_1) \\ G_d(a_0, a_2, a_2) &\leq (1+\xi)(1-\xi)r \end{aligned}$$

$$\begin{aligned} G_d(a_0, a_2, a_2) &\leq (1-\xi^2)r \\ G_d(a_0, a_2, a_2) &\leq r \end{aligned}$$

$a_2 \in \overline{B_G(a_0, r)}$. Again consider the relation

$$\begin{aligned} G_d(a_2, a_3, a_3) &= G_d(\varphi a_1, \delta a_2, \delta a_2) \\ G_d(a_2, a_3, a_3) &\leq \xi G_d(a_1, a_2, a_2) \end{aligned}$$

$$G_d(a_2, a_3, a_3) \leq \xi^2 G_d(a_0, a_1, a_1) \tag{3.1.5}$$

Again consider for $a_3 \in X$

$$\begin{aligned} G_d(a_0, a_3, a_3) &\leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + G_d(a_2, a_3, a_3) \\ G_d(a_0, a_3, a_3) &\leq G_d(a_0, a_1, a_1) + \xi G_d(a_0, a_1, a_1) + \xi^2 G_d(a_0, a_1, a_1), \quad \because \text{ of } (3.1.5) \\ G_d(a_0, a_3, a_3) &\leq (1+\xi+\xi^2)G_d(a_0, a_1, a_1) \end{aligned}$$

$$G_d(a_0, a_3, a_3) \leq (1+\xi+\xi^2)(1-\xi)r, \quad \because \text{ of } (3.1.2)$$

$$\begin{aligned} G_d(a_0, a_3, a_3) &\leq (1-\alpha^3)r \\ G_d(a_0, a_3, a_3) &\leq r \end{aligned}$$

$a_3 \in \overline{B_G(a_0,r)}$. Now let a_4, $a_{5,........}$, $a_i \in \overline{B_G(a_0,r)}$, by mathematical induction general inequality can be obtain for all even $i \in N$ as follows,

$$
\begin{aligned}
G_d(a_i, a_{i+1}, a_{i+1}) &= G_d(\varphi a_{i-1}, \delta a_i, \delta a_i) \\
G_d(a_i, a_{i+1}, a_{i+1}) &\leq \xi G_d(a_{i-1}, a_i, a_i) = \xi G_d(\delta a_{i-2}, \varphi a_{i-1}, \varphi a_{i-1}) \\
&= \xi G_d(\varphi a_{i-1}, \delta a_{i-2}, \delta a_{i-2})
\end{aligned}
$$

$$G_d(a_i, a_{i+1}, a_{i+1}) \leq \xi^2 G_d(a_{i-1}, a_{i-2}, a_{i-2})$$

.....

.....

$$G_d(a_i, a_{i+1}, a_{i+1}) \leq \xi^{i-1} G_d(a_0, a_1, a_1) \tag{3.1.6}$$

Now consider by rectangular property of $G-metric\ space$ for $a_{i+1} \in X$,

$$
\begin{aligned}
G_d(a_0, a_i, a_i) &\leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + ... + G_d(a_{i-1}, a_i, a_i) \\
G_d(a_0, a_i, a_i) &\leq G_d(a_0, a_1, a_1) + \xi G_d(a_0, a_1, a_1) + ... + \xi^{i-1} G_d(a_0, a_1, a_1)
\end{aligned}
$$

$$
\begin{aligned}
G_d(a_0, a_i, a_i) &\leq (1 + \xi + \xi^2 + ... + \xi^{i-2} + \xi^{i-1}) G_d(a_0, a_1, a_1) \\
G_d(a_0, a_i, a_i) &\leq (1 + \xi + \xi^2 + ... + \xi^{i-2} + \xi^{i-1})(1-\xi) r, \ \because \text{of } (3.1.2)
\end{aligned}
$$

$$
\begin{aligned}
G_d(a_0, a_i, a_i) &\leq (1 - \xi^i) r \\
G_d(a_0, a_i, a_i) &\leq r
\end{aligned}
$$

Clearly $a_i \in \overline{B_G(a_0,r)}\ \forall\ i \in N$. Hence sequence $\{a_n\}$ is in the closed ball $\overline{B_G(a_0,r)}$. Now to show that picard sequence $\{a_n\}$ is Cauchy sequence consider for $m, n \in N$ such that $n < m$,

$$G_d(a_n, a_m, a_m) \leq G_d(a_n, a_{n+1}, a_{n+1}) + G_d(a_{n+1}, a_{n+2}, a_{n+2}) + ... + G_d(a_{m-1}, a_m, a_m)$$

$$G_d(a_n, a_m, a_m) \leq \xi^n (1 + \xi + \xi^2 + ... + \xi^{m-n-1}) G_d(a_0, a_1, a_1)$$

$$G_d(a_n, a_m, a_m) \leq \xi^n \left(\frac{1 - \xi^{m-n}}{1 - \xi}\right) G_d(a_0, a_1, a_1) \tag{3.1.7}$$

$$G_d(a_n, a_m, a_m) \leq \xi^n \left(\frac{1 - \xi^{m-n}}{1 - \xi}\right)(1-\xi) r, \ \because \text{of } (3.1.2)$$

$$
\begin{aligned}
G_d(a_n, a_m, a_m) &\leq \xi^n r - \xi^m r \\
G_d(a_n, a_m, a_m) &\leq \xi^n r
\end{aligned} \tag{3.1.8}
$$

As $\xi \in [0,1)$, then $\xi^n \to 0$ if $n \to \infty$. Hence $\xi^n r \to 0$ if $n \to \infty$. So for any $\widehat{\epsilon} \in R$ however small, $\exists\ j \in R$ such that from 1.6,

$$
G_d(a_n, a_m, a_m) \leq \xi^n r = \widehat{\epsilon}, \text{ when } m, n > j
$$

Hence picard sequence $\{a_n\}$ is Cauchy sequence in closed ball $\overline{B_G(a_0, r)}$. As closed ball $\overline{B_G(a_0, r)}$ is closed subset of set X, then the sequence $\{a_n\}$ is convergent in closed ball $\overline{B_G(a_0, r)}$ and the point of convergence is $\overline{a} \in \overline{B_G(a_0, r)}$. Hence $a_n \to \overline{a}$ as $n \to \infty$. In general it is clear that

$$
\lim_{n\to\infty} G_d(a_n, \overline{a}, \overline{a}) = \lim_{n\to\infty} G_d(\overline{a}, a_n, a_n) = 0 \tag{3.1.9}
$$

To check $\overline{a} \in \overline{B_G(a_0, r)}$ is either common fixed point of $\varphi, \delta : X \to X$ or not consider

$$
\begin{aligned}
G_d(\overline{a}, \varphi\overline{a}, \varphi\overline{a}) &\leq G_d(\overline{a}, a_{2n+1}, a_{2n+1}) + G_d(a_{2n+1}, \varphi\overline{a}, \varphi\overline{a}) \\
G_d(\overline{a}, \varphi\overline{a}, \varphi\overline{a}) &\leq G_d(\overline{a}, a_{2n+1}, a_{2n+1}) + G_d(\delta a_{2n}, \varphi\overline{a}, \varphi\overline{a})
\end{aligned}
$$

By symmetric condition of G-metric space

$$
\begin{aligned}
G_d(\overline{a}, \varphi\overline{a}, \varphi\overline{a}) &\leq G_d(\overline{a}, a_{2n+1}, a_{2n+1}) + G_d(\varphi\overline{a}, \delta a_{2n}, \delta a_{2n}) \\
G_d(\overline{a}, \varphi\overline{a}, \varphi\overline{a}) &\leq G_d(\overline{a}, a_{2n+1}, a_{2n+1}) + \xi G_d(\overline{a}, a_{2n}, a_{2n}) \\
G_d(\overline{a}, \varphi\overline{a}, \varphi\overline{a}) &\leq 0, \text{ when } n \to \infty, \ \because \text{ of } (3.1.9)
\end{aligned}
$$

As it is not possible that $G_d(\overline{a}, \varphi\overline{a}, \varphi\overline{a}) < 0$, so the only possibilty left is

$$
\begin{aligned}
G_d(\overline{a}, \varphi\overline{a}, \varphi\overline{a}) &= 0 \\
\varphi\overline{a} &= \overline{a}
\end{aligned}
$$

Again consider

$$
\begin{aligned}
G_d(a, \delta\overline{a}, \delta\overline{a}) &\leq G_d(\overline{a}, a_{2n+2}, a_{2n+2}) + G_d(a_{2n+2}, \delta\overline{a}, \delta\overline{a}) \\
G_d(a, \delta\overline{a}, \delta\overline{a}) &\leq G_d(\overline{a}, a_{2n+2}, a_{2n+2}) + G_d(\varphi a_{2n+1}, \delta\overline{a}, \delta\overline{a})
\end{aligned}
$$

$$\begin{aligned} G_d(a, \delta\overline{a}, \delta\overline{a}) &\leq G_d(\overline{a}, a_{2n+2}, a_{2n+2}) + \xi G_d(a_{2n+1}, \overline{a}, \overline{a}) \\ G_d(a, \delta\overline{a}, \delta\overline{a}) &\leq 0, \text{ when } n \to \infty, \because \text{of (3.1.9)} \end{aligned}$$

As it is not possible that $G_d(\overline{a}, \delta\overline{a}, \delta\overline{a}) < 0$, so the only possibilty left is

$$\begin{aligned} G_d(\overline{a}, \delta\overline{a}, \delta\overline{a}) &= 0 \\ \delta\overline{a} &= \overline{a} \end{aligned}$$

Hence $\overline{a} \in \overline{B_G(a_0, r)}$ is common fixed point of dominated mappings φ and δ, *i.e* $\varphi\overline{a} = \delta\overline{a} = \overline{a}$. For *uniqueness* of common fixed point, consider $\overline{a}, \overline{b} \in \overline{B_G(a_0, r)}$ are any two common fixed point of mappings φ and δ, such that $\overline{a} \neq \overline{b}$. Then there arises two cases for $\overline{a}, \overline{b} \in \overline{B_G(a_0, r)}$. In first case let $\overline{a}$, $\overline{b}$ are comparable say $\overline{a} \leq \overline{b}$. As $\overline{a}$ and $\overline{b}$ are common fixed point of dominated mappings φ and δ then,

$$\varphi(\overline{a}) = \overline{a},\ \delta(\overline{a}) = \overline{a},\ \varphi(\overline{b}) = \overline{b} \text{ and } \delta(\overline{b}) = \overline{b} \tag{3.1.10}$$

Now consider by relation,

$$G_d(\overline{a}, \overline{b}, \overline{b}) = G_d(\varphi\overline{a}, \delta\overline{b}, \delta\overline{b})$$

$$G_d(\overline{a}, \overline{b}, \overline{b}) \leq \xi G_d(\overline{a}, \overline{b}, \overline{b})$$

$$(1 - \xi) G_d(\overline{a}, \overline{b}, \overline{b}) \leq 0$$

As $\xi \in [0, 1)$, then $1 - \xi \neq 0$. Thus

$$G_d(\overline{a}, \overline{b}, \overline{b}) \leq 0$$

As by definition of G-metric space $G_d(\overline{a}, \overline{b}, \overline{b}) \nless 0$, then only possibilty left is

$$\begin{aligned} G_d(\overline{a}, \overline{b}, \overline{b}) &= 0 \\ \overline{a} &= \overline{b} \end{aligned}$$

It is contradiction ($\because$ $\overline{a} \neq \overline{b}$). So our supposition is wrong. In second case if $\overline{a}$, $\overline{b}$ are not comparable then $\exists$ a point $t_0 \in \overline{B_G(a_0, r)}$ such that $t_0 \leq \overline{a}$ and $t_0 \leq \overline{b}$. Then clearly $t_0 \in \overline{B_G(a_0, r)}$ is lower bound of both $\overline{a}$ and $\overline{b}$. Now construct an itterative picard sequence $\{t_j\} \subseteq X$ for $j \in N$ such that

$$t_{2j+1} = \delta t_{2j} \text{ and } t_{2j+2} = \varphi t_{2j+1} \tag{3.1.11}$$

As φ and δ are dominated mappings then clearly

$$...... \leq t_j \leq t_{j-1} \leq .... \leq t_3 \leq t_2 \leq t_1 \leq t_0$$

As by assumption $t_0 \in \overline{B_G(a_0, r)}$,

$$G_d(a_0, t_0, t_0) \leq r \tag{3.1.12}$$

Now consider the relation

$$\begin{aligned} G_d(a_1, t_1, t_1) &= G_d(\varphi a_0, \delta t_0, \delta t_0) \\ G_d(a_1, t_1, t_1) &\leq \xi G_d(a_0, t_0, t_0) \\ G_d(a_1, t_1, t_1) &\leq \xi r, \because \text{ of } (3.1.12) \end{aligned}$$

Consider for $t_1 \in X$ by rectangular property of G-metric space

$$\begin{aligned} G_d(a_0, t_1, t_1) &\leq G_d(a_0, a_1, a_1) + G_d(a_1, t_1, t_1) \\ G_d(a_0, t_1, t_1) &\leq (1-\xi)r + \xi r \end{aligned}$$

$$\begin{aligned} G_d(a_0, t_1, t_1) &\leq r - \xi r + \xi r \\ G_d(a_0, t_1, t_1) &\leq r \end{aligned}$$

Hence $t_1 \in \overline{B_G(a_0, r)}$. Again consider

$$G_d(a_2, t_2, t_2) = G_d(\delta a_1, \varphi t_1, \varphi t_1)$$

As G-metric is symmetric then

$$\begin{aligned} G_d(a_2, t_2, t_2) &= G_d(\varphi t_1, \delta a_1, \delta a_1) \\ G_d(a_2, t_2, t_2) &\leq \xi G_d(t_1, a_1, a_1) \\ G_d(a_2, t_2, t_2) &\leq \xi^2 r \end{aligned}$$

Again consider for $t_2 \in X$,

$$\begin{aligned} G_d(a_0, t_2, t_2) &\leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + G_d(a_2, t_2, t_2) \\ G_d(a_0, t_2, t_2) &\leq (1-\xi)r + \alpha(1-\xi)r + \xi^2 r \end{aligned}$$

$$G_d(a_0,t_2,t_2) \leq r-\xi r+\xi r-\xi^2 r+\xi^2 r$$
$$G_d(a_0,t_2,t_2) \leq r$$

Hence $t_2 \in \overline{B_G(a_0,r)}$. Now let t_3, t_4, $t_5, \ldots\ldots, t_j \in \overline{B_G(a_0,r)}$, then by mathematical induction consider for $t_{j+1} \in X$ such that $j+1 \in N$ is even,

$$G_d(a_{j+1},t_{j+1},t_{j+1}) = G_d(\varphi a_j,\delta t_j,\delta t_j)$$
$$G_d(a_{j+1},t_{j+1},t_{j+1}) \leq \xi G_d(a_j,t_j,t_j) = \xi G_d(\delta a_{j-1},\varphi t_{j-1},\varphi t_{j-1})$$
$$G_d(a_{j+1},t_{j+1},t_{j+1}) \leq \xi G_d(\varphi t_{j-1},\delta a_{j-1},\delta a_{j-1}) \leq \xi^2 G_d(t_{j-1},a_{j-1},a_{j-1})$$

.....

.....

$$G_d(a_{j+1},t_{j+1},t_{j+1}) \leq \xi^j G_d(a_0,t_0,t_0)$$
$$G_d(a_{j+1},t_{j+1},t_{j+1}) \leq \xi^{j+1} r$$

Hence $\forall\ j \in N$ following relation holds

$$G_d(a_j,t_j,t_j) \leq \xi^j r \tag{3.1.13}$$

To show sequence $\{t_j\} \subseteq X$ is in closed ball $\overline{B_G(a_0,r)}$. By mathematical induction for $j \in N$,consider the relations,

$$G_d(a_0,t_{j+1},t_{j+1}) \leq G_d(a_0,a_1,a_1)+G_d(a_1,a_2,a_2)+\ldots\ldots\ldots+G_d(a_j,a_{j+1},a_{j+1})$$
$$+G_d(a_{j+1},t_{j+1},t_{j+1})$$
$$G_d(a_0,t_{j+1},t_{j+1}) \leq (1-\xi)r+\xi(1-\xi)r+\ldots\ldots\ldots+\xi^j(1-\xi)r+\xi^{j+1}r$$

$$G_d(a_0,t_{j+1},t_{j+1}) \leq r-\xi r+\xi r-\xi^2 r+\xi^2 r+\ldots\ldots\ldots+\xi^j r-\xi^{j+1}r+\xi^{j+1}r$$
$$G_d(a_0,t_{j+1},t_{j+1}) \leq r$$

Hence $t_j \in \overline{B_G(a_0,r)}$. Hence in general $t_n \in \overline{B_G(a_0,r)}$, $\forall\ n \in N$. Now as $t_0 \leq \overline{a}$ and $t_0 \leq \overline{b}$, then $t_n \leq t_0 \leq \overline{a}$ and $t_n \leq t_0 \leq \overline{b}$. So clearly

$$t_n \leq \overline{a} = \varphi^n\overline{a},\ t_n \leq \overline{a} = \delta^n\overline{a},\ t_n \leq \overline{b} = \varphi^n\overline{b} \text{ and } t_n \leq \overline{b} = \delta^n\overline{b} \tag{3.1.14}$$

As $\forall\ n \in N$, $\delta^n\overline{a} = \overline{a} = \varphi^n\overline{a}$ and $\delta^n\overline{b} = \overline{b} = \varphi^n\overline{b}$. Then consider from (3.1.14),

$$\begin{aligned} G_d(\overline{a},\overline{b},\overline{b}) &= G_d(\varphi^n\overline{a},\delta^n\overline{b},\delta^n\overline{b}) \\ G_d(\overline{a},\overline{b},\overline{b}) &\leq \xi G_d(\varphi^{n-1}\overline{a},\delta^{n-1}\overline{b},\delta^{n-1}\overline{b}) \\ G_d(\overline{a},\overline{b},\overline{b}) &\leq \xi^2 G_d(\varphi^{n-2}\overline{a},\delta^{n-2}\overline{b},\delta^{n-2}\overline{b}) \\ &\cdots\cdots \\ &\cdots\cdots \end{aligned}$$

$$G_d(\overline{a},\overline{b},\overline{b}) \leq \xi^{n-1} G_d(\varphi\overline{a},\delta\overline{b},\delta\overline{b})$$

$$G_d(\overline{a},\overline{b},\overline{b}) \leq \xi^n G_d(\overline{a},\overline{b},\overline{b})$$

$$(1-\xi^n) G_d(\overline{a},\overline{b},\overline{b}) \leq 0$$

As $1-\xi^n \neq 0$, then

$$G_d(\overline{a},\overline{b},\overline{b}) \leq 0$$

By definition of $G-metric\ space$ $G_d(\overline{a},\overline{b},\overline{b}) \nless 0$. then the only possibility left is

$$\begin{aligned} G_d(\overline{a},\overline{b},\overline{b}) &= 0 \\ \overline{a} &= \overline{b} \end{aligned}$$

It is contradiction ($\because\ \overline{a} \neq \overline{b}$). So our supposition is wrong. Hence from both cases it is clear that common fixed point in closed ball $\overline{B_G(a_0,r)}$, of dominated mappings φ and δ is unique. ■

Example 32 *If for a set $X = [0,\infty)$, a mapping $G_d : X \times X \times X \longrightarrow X$ then $\forall a,b,c \in X$ defined by,*

$$G_d(a,b,c) = a+b+c \tag{3.1.15}$$

then $(X;\leq;G)$ is symmetric and complete G−metric like space. Let mappings $\varphi,\delta : X \to X$ are defined by,

$$\delta a = \begin{cases} \frac{a}{2} & \text{if } a \in [0,1] \\ a+\frac{1}{3} & \text{if } a \in (1,\infty) \end{cases}$$

$$\varphi a = \begin{cases} \frac{a}{3} & \text{if } a \in [0,1] \\ a+\frac{1}{4} & \text{if } a \in (1,\infty) \end{cases}$$

Obviously φ *and* δ *are dominated mappings inside of* $[0,1]$ *but not dominated outside of* $[0,1]$. *Let* $a_0 = \frac{1}{2}$ *and* $r = \frac{5}{2}$ *such that* $\overline{B_G(a_0, r)} = [0,1]$. *Also let* $\xi = \frac{3}{5} \in [0,1)$ *so to get,*

$$\begin{aligned} (1-\xi)r &= (1-\frac{3}{5})\frac{5}{2} \\ (1-\xi)r &= (\frac{2}{5})\frac{5}{2} \\ (1-\xi)r &= 1 \end{aligned} \tag{3.1.16}$$

Also as,

$$G_d(a_0, a_1, a_1) = \frac{1}{2} + 2\phi(\frac{1}{2})$$

$$\begin{aligned} G_d(a_0, a_1, a_1) &= \frac{1}{2} + \frac{1}{3} \\ G_d(a_0, a_1, a_1) &= \frac{5}{6} \end{aligned} \tag{3.1.17}$$

From (3.1.16) *and* (3.1.17),

$$\frac{5}{6} \leq 1$$

Also $\forall\ a,\ b, c \in [0,1]$

$$\begin{aligned} G_d(\varphi a, \delta b, \delta c) &= G_d(\frac{a}{3}, \frac{b}{2}, \frac{c}{2}) \\ G_d(\varphi a, \delta b, \delta c) &= \frac{a}{3} + \frac{b}{2} + \frac{c}{2} \quad \because \textit{From } 3.1.15 \end{aligned}$$

$$G_d(\varphi a, \delta b, \delta c) = \frac{1}{2}(a+b+c) - \frac{a}{6} \tag{3.1.18}$$

Also as,

$$\xi G_d(a,b,c) = \frac{3}{5}(a+b+c) \tag{3.1.19}$$

$\forall\ a, b, c \in [0,1]$ *it is clear that,*

$$\frac{1}{2}(a+b+c) - \frac{a}{6} \leq \frac{1}{2}(a+b+c) \leq \frac{3}{5}(a+b+c)$$

$$\begin{aligned} \frac{1}{2}(a+b+c) - \frac{a}{6} &\leq \frac{3}{5}(a+b+c), \quad \because \textit{From } (3.1.19) \textit{ and } (3.1.20) \\ G_d(\varphi a, \delta b, \delta c) &\leq \xi G_d(a,b,c) \end{aligned}$$

Hence contractive condition is satisfied inside of the closed ball $[0,1]$. *Now for* $a, b, c \in (1, \infty)$,

$$G_d(\varphi a, \delta b, \delta c) = \varphi a + \delta b + \delta c, \quad \because \textit{From } (3.1.15)$$

$$G_d(\varphi a, \delta b, \delta c) = (a+b+c) + \frac{7}{12} \tag{3.1.20}$$

From (3.1.19) *and* (3.1.20), $\forall\ a, b, c \in (1, \infty)$ *it is clear that,*

$$\begin{aligned} (a+b+c) + \frac{7}{12} &\geq \frac{3}{5}(a+b+c) \\ G_d(\varphi a, \delta b, \delta c) &\geq \xi G_d(a, b, c) \end{aligned}$$

Hence contractive condition is not satisfied outside of the closed ball $[0, 1]$. *It shows that all conditions of Banach fixed point theorem for double dominated mappings. Moreover* $0 \in [0, 1]$ *common fix point of mappings* φ *and* δ, *i.e* $\varphi 0 = \delta 0 = 0$.

Corollary 33 *Suppose* $(X; \leq; G)$ *be a symmetric and ordered complete* $G-$ *metric like space and* $\varphi, \delta : X \to X$ *are any two dominated mappings and* a_0, $a, b, c \in X$, $r > 0$. *Suppose that* $\exists\ \xi \in [0, 1)$ *such that,*

$$G_d(\varphi a, \varphi b, \varphi c) \leq \xi G_d(a, b, c) \tag{3.1.21}$$

$\forall$ *compareable elements* $a, b, c \in \overline{B_G(a_0, r)} \subseteq X$, *and*

$$G_d(a_0, a_1, a_1) = G_d(a_0, \varphi a_0, \varphi a_0) \leq (1-\xi) r \tag{3.1.22}$$

If for non-increasing sequence $\{a_n\}$ *in* $\overline{B_G(a_0, r)}$, $\{a_n\} \to v$ *then* $\exists$ *a unique* $\overline{a} \in \overline{B_G(a_0, r)}$ *such that* $\varphi \overline{a} = \delta \overline{a} = \overline{a}$. *Moreover* $G_d(\overline{a}, \overline{a}, \overline{a}) = 0$.

Proof. In the main result common fixed point of Banach mappings take $\varphi = \delta$ to get unique fixed point $\overline{a} = \varphi \overline{a}$. ■

Proof of the following corollary is similar to the proof of *Theorem* 31 but without discussing the ordered property of $G-$ *metric like spaces.*

Corollary 34 *Suppose* $(X; \leq; G)$ *be a symmetric and complete* $G-$ *metric like space and* $\varphi, \delta : X \to X$ *are any two dominated mappings and* a_0, a, $b, c \in X$, $r > 0$. *Suppose that* $\exists$ $\xi \in [0, 1)$ *such that,*

$$G_d(\varphi a, \delta b, \delta c) \leq \xi G_d(a, b, c),\ \forall\ a, b, c \in \overline{B_G(a_0, r)} \subseteq X \tag{3.1.23}$$

and

$$G_d(a_0, a_1, a_1) = G_d(a_0, \varphi a_0, \varphi a_0) \leq (1-\xi)r \tag{3.1.24}$$

If for non-increasing sequence $\{a_n\}$ *in* $\overline{B_G(a_0, r)}$, $\{a_n\} \to v$ *then* $\exists$ *a unique* $\overline{a} \in \overline{B_G(a_0, r)}$ *such that* $G_d(\overline{a}, \overline{a}, \overline{a}) = 0$ *and* $\varphi\overline{a} = \delta\overline{a} = \overline{a}$. *Moreover* $G_d(\overline{a}, \overline{a}, \overline{a}) = 0$.

3.1.1 Error Bounds

In this section errors approximations and their related example are discused.

Corollary 35 ***(Iteration, Error Bounds)*** *From Theorem* 31, *iterative sequence* (3.1.3), *with arbitrary* $a_0 \in \overline{B_G(a_0, r)} \subseteq X$, *converges to unique common fixed point* $\overline{a}$ *of dominated mappings* φ *and* δ. *Error estimates are the* ***prior estimate***

$$G_d(a_n, \overline{a}, \overline{a}) \leq \frac{\xi^n}{1-\xi} G_d(a_0, a_1, a_1) \tag{3.1.25}$$

and the ***posterior estimate***

$$G_d(a_n, \overline{a}, \overline{a}) \leq \frac{\xi}{1-\xi} G_d(a_{n-1}, a_n, a_n) \tag{3.1.26}$$

Proof. As from (3.1.7) of *Theorem* 31,

$$G_d(a_n, a_m, a_m) \leq \alpha^n \left(\frac{1-\alpha^{m-n}}{1-\alpha}\right) G_d(a_0, a_1, a_1)$$

As the sequence $\{a_m\}$ is convergent to $\overline{a} \in \overline{B_G(a_0, r)} \subseteq X$, then by taking $m \to \infty$ gives $a_m \to \overline{a}$ and $\alpha^{m-n} \to 0$. Therefore above relation leads to the *prior estimate, i.e,*

$$G_d(a_n, \overline{a}, \overline{a}) \leq \frac{\xi^n}{1-\xi} G_d(a_0, a_1, a_1)$$

Setting $n = 1$ and write b_0 for a_0 and b_1 for a_1 in (3.1.25)

$$G_d(b_1, \overline{a}, \overline{a}) \leq \frac{\xi}{1-\xi} G_d(b_0, b_1, b_1)$$

Letting $b_0 = a_{n-1}$ then $b_1 = \delta b_0 = \delta a_{n-1} = a_n$ in above relation leads to the *posterior estimate* (3.1.26), *i.e,*

$$G_d(a_n, \overline{a}, \overline{a}) \leq \frac{\xi}{1-\xi} G_d(a_{n-1}, a_n, a_n)$$

■

The prior error bound (3.1.25) can be used at the beginning of the calculation for estimating the required number of steps to obtain a assumed accuracy. While posterior error bound (3.1.26) can be used at intermediate stages or at the end of the calculation. Posterior error bound (3.1.26) is at least as accurate as prior error bound (3.1.25).

Example 36 *If for a set* $X = [0, \infty)$, *a mapping* $G_d : X \times X \times X \longrightarrow X$, $\forall a, b, c \in X$ *defined by,*

$$G_d(a, b, c) = a + b + c \tag{3.1.27}$$

then $(X; \leq; G)$ *is symmetric and complete* $G-$*metric like space. Let mappings* $\varphi, \delta : X \to X$ *are defined by,*

$$\delta a = \begin{cases} \frac{a}{2} & \text{if } a \in [0,1] \\ a + \frac{1}{3} & \text{if } a \in (1, \infty) \end{cases}$$

$$\varphi a = \begin{cases} \frac{a}{3} & \text{if } a \in [0,1] \\ a + \frac{1}{4} & \text{if } a \in (1, \infty) \end{cases}$$

Obviously φ *and* δ *are dominated mappings inside of* $[0,1]$ *but not dominated outside of* $[0,1]$. *Let* $a_0 = \frac{1}{2}$ *and* $r = \frac{5}{2}$ *such that* $\overline{B_G(a_0, r)} = [0,1]$. *Also let* $\xi = \frac{3}{5} \in [0,1)$. *Construct the picard iterative sequence taking* $a_0 = \frac{1}{2} \in [0,1]$ *as initial guess as,*

$$a_1 = \delta a_0 = \frac{1}{(2)^2.(3)^0} \text{ and } a_2 = \varphi a_1 = \frac{1}{(2)^2.(3)^1}$$

$$a_3 = \delta a_2 = \frac{1}{(2)^3.(3)^1} \text{ and } a_4 = \varphi a_3 = \frac{1}{(2)^3.(3)^2}$$

$$a_5 = \delta a_4 = \frac{1}{(2)^4.(3)^2} \text{ and } a_6 = \varphi a_5 = \frac{1}{(2)^4.(3)^3}$$

.....

.....

$$a_{2m-1} = \delta a_{2m-2} = \frac{1}{2^{m+1}.3^{m-1}} \text{ and } a_{2m} = \varphi a_{2m-1} = \frac{1}{2^{m+1}.3^m}$$

Letting $2m - 1 = n$ *then for every odd* $n \in N$ *gives,*

$$a_n = \frac{\sqrt{3}}{2\sqrt{2}.(\sqrt{6})^n} \text{ and } a_{n+1} = \frac{1}{2\sqrt{6}.(\sqrt{6})^n} \tag{3.1.28}$$

Now as,

$$G_d(a_0, a_1, a_1) = a_0 + 2\delta a_0 = \frac{1}{2} + 2(\frac{1}{4}) = 1$$

Also as,

$$G_d(a_n, \overline{a}, \overline{a}) = a_n + 2\overline{a}$$

As picard sequence $\{a_n\}$ *satisfies all the conditions of Banach fixed point theorem* 31 *as in example* 32*, then If* $n \to \infty$ *so* $a_n \to \overline{a}$ *i.e* $a_n \approx \overline{a}$*. Then*

$$G_d(a_n, \overline{a}, \overline{a}) = 3a_n$$

As prior error estimate is given by,

$$\begin{aligned} G_d(a_n, \overline{a}, \overline{a}) &\leq \frac{\xi^n}{1-\xi} G_d(a_0, a_1, a_1) \\ 3a_n &\leq \frac{5}{2}(\frac{3}{5})^n \\ \frac{3\sqrt{3}}{5\sqrt{2}} &\leq (\frac{3\sqrt{6}}{5})^n \\ 0.73484692283495 &\leq (1.46969384566991)^n \end{aligned}$$

$$\begin{aligned} \frac{\ln(0.73484692283495)}{\ln(1.46969384566991)} &\leq n \\ -0.8002076 &\leq n \end{aligned}$$

This shows that for odd integers $n \geq 1$*, picard sequence becomes convergent, i.e. for* $n = 3$ *relation* (3.1.28) *becomes,*

$$\begin{aligned} a_3 &= \frac{1}{2^{\frac{3+3}{2}}.3^{\frac{3-1}{2}}} = \frac{1}{24} \text{ and } a_{3+1} = \frac{1}{2^{\frac{3+3}{2}}.3^{\frac{3+1}{2}}} = \frac{1}{72} \\ a_3 &= 0.04166666666667 \text{ and } a_4 = 0.01388888888889 \end{aligned}$$

Also for $n = 5$,

$$\begin{aligned} a_5 &= \frac{1}{2^{\frac{5+3}{2}}.3^{\frac{5-1}{2}}} = \frac{1}{144} \text{ and } a_{5+1} = \frac{1}{2^{\frac{5+3}{2}}.3^{\frac{5+1}{2}}} = \frac{1}{432} \\ a_5 &= 0.00694444444444 \text{ and } a_6 = 0.0023148148481 \end{aligned}$$

Hence it is clear,

$$\begin{aligned} \delta(a_6) &\approx \varphi(a_5) \approx a_6 \approx 0.0023148148481 \\ \varphi(0.00694444444444) &\approx \delta(0.0023148148481) \approx 0.0000000005414 \end{aligned}$$

This suggests, when odd integer $n \geq 1$ goes on increasing, picard sequence moves towards common fixed point of δ and φ which is $\overline{a} = 0 \in [0,1]$, i.e, $\varphi 0 = \delta 0 = 0$.

Remark 37 *Above results not only holds for dominated mapping but also holds for dominating mappings.*

3.2 Kannan Common Fixed Point Result

Theorem 38 *Suppose $(X; \leq; G)$ be a symmetric and ordered complete $G-$metric like space and $\Omega, \Psi : X \to X$ are any two dominated mappings and a_0, a, $b, c \in X$, $r > 0$. Suppose that $\exists\ \theta \in [0, \frac{1}{7})$ such that the following conditions holds*

$$\begin{aligned} G_d(\Omega a, \Psi b, \Psi c) &\leq \theta[G_d(a, \Omega a, \Omega a) + G_d(b, \Psi b, \Psi b) \\ &\quad + G_d(c, \Psi c, \Psi c)] \end{aligned} \tag{3.2.1}$$

$\forall$ comparabe elements $a, b, c \in \overline{B_G(a_0, r)} \subseteq X$

$$G_d(a_0, a_1, a_1) = G_d(a_0, \Omega a_0, \Omega a_0) \leq (1-\gamma)r \tag{3.2.2}$$

where $\gamma = \frac{5\theta}{1-2\theta}$. If for non-increasing sequence $\{a_n\}$ in $\overline{B_G(a_0, r)}$, $\{a_n\} \to v$ then $\exists$ a unique $a^* \in \overline{B_G(a_0, r)}$ such that $G_d(a^*, a^*, a^*) = 0$ and $\Omega a^* = \Psi a^* = a^*$.

Proof. With initially choosen guess $a_0 \in \overline{B_G(a_0, r)} \subseteq X$, and consider picard sequence $\{a_n\}$ such that

$$a_{2n+1} = \Omega(a_{2n}) \text{ and } a_{2n+2} = \Psi(a_{2n+1}) \tag{3.2.3}$$

As Ω and Ψ are dominated mappings then,

$$.....a_n \leq a_{n-1} \leq a_{n-2} \leq ..... \leq a_3 \leq a_2 \leq a_1 \leq a_0$$

As from relation,

$$\begin{aligned} G_d(a_0, a_1, a_1) &= G_d(a_0, \Omega a_0, \Omega a_0) \leq (1-\gamma)r \\ G_d(a_0, a_1, a_1) &\leq r \end{aligned}$$

Hence $a_1 \in \overline{B_G(a_0, r)}$. Now consider the relation,

$$\begin{aligned} G_d(a_1, a_2, a_2) &= G_d(\Omega a_0, \Psi a_1, \Psi a_1) \\ G_d(a_1, a_2, a_2) &\leq \theta[G_d(a_0, \Omega a_0, \Omega a_0) + 2G_d(a_1, \Psi a_1, \Psi a_1)] \end{aligned}$$

$$G_d(a_1, a_2, a_2) \leq \theta G_d(a_0, a_1, a_1) + 2\theta G_d(a_1, a_2, a_2)$$

$$G_d(a_1, a_2, a_2) \leq \frac{\theta}{1-2\theta} G_d(a_0, a_1, a_1)$$

$$G_d(a_1, a_2, a_2) \leq \gamma G_d(a_0, a_1, a_1) \tag{3.2.4}$$

For $a_2 \in X$, consider

$$G_d(a_0, a_2, a_2) \leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2)$$

$$\begin{aligned} G_d(a_0, a_2, a_2) &\leq (1+\gamma) G_d(a_0, a_1, a_1) \\ G_d(a_0, a_2, a_2) &\leq (1+\gamma)(1-\gamma)r, \ \because \text{From (3.2.2)} \end{aligned}$$

$$\begin{aligned} G_d(a_0, a_2, a_2) &\leq (1-\gamma^2)r \\ G_d(a_0, a_2, a_2) &\leq r \end{aligned}$$

Hence $a_2 \in \overline{B_G(a_0, r)}$. Again consider the relation,

$$G_d(a_2, a_3, a_3) = G_d(\Psi a_1, \Omega a_2, \Omega a_2)$$

As G-metric is symmetric then

$$\begin{aligned} G_d(a_2, a_3, a_3) &= G_d(\Omega a_2, \Psi a_1, \Psi a_1) \\ G_d(a_2, a_3, a_3) &\leq \theta[G_d(a_2, \Omega a_2, \Omega a_2) + 2G_d(a_1, \Psi a_1, \Psi a_1)] \end{aligned}$$

$$G_d(a_2, a_3, a_3) \leq \theta G_d(a_2, a_3, a_3) + 2\theta G_d(a_1, a_2, a_2)$$

$$(1-\theta) G_d(a_2, a_3, a_3) \leq 2\theta G_d(a_1, a_2, a_2)$$

$$G_d(a_2, a_3, a_3) \leq (\frac{5\theta}{1-2\theta})^2 G_d(a_0, a_1, a_1), \ \because \frac{2\theta}{(1-\theta)} \leq \frac{5\theta}{(1-2\theta)}$$

Now for $a_3 \in X$ consider

$$G_d(a_0, a_3, a_3) \leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + G_d(a_2, a_3, a_3)$$

$$G_d(a_0, a_3, a_3) \leq (1+\gamma+\gamma^2)G_d(a_0, a_1, a_1)$$

$$G_d(a_0, a_3, a_3) \leq (1+\gamma+\gamma^2)(1-\gamma)r, \quad \because \text{ From (3.2.2)}$$

$$G_d(a_0, a_3, a_3) \leq (1-\gamma^3)r$$

$$G_d(a_0, a_3, a_3) \leq r$$

Therefore $a_3 \in \overline{B_G(a_0, r)}$. Now let $a_{4,}\ a_{5,.........,}\ a_j \in \overline{B_G(a_0, r)}$, then following relation holds as,

$$G_d(a_{j-1}, a_j, a_j) \leq (\frac{2\theta}{1-2\theta})^{j-1} G_d(a_0, a_1, a_1) \tag{3.2.5}$$

By mathematical induction for $j+1 \in N$ let,

$$\begin{aligned} G_d(a_j, a_{j+1}, a_{j+1}) &\leq G_d(\Omega a_{j-1}, \Psi a_j, \Psi a_j) \\ G_d(a_j, a_{j+1}, a_{j+1}) &\leq \theta[G_d(a_{j-1}, \Omega a_{j-1}, \Omega a_{j-1}) + 2G_d(a_j, \Psi a_j, \Psi a_j)] \end{aligned}$$

$$G_d(a_j, a_{j+1}, a_{j+1}) \leq \theta G_d(a_{j-1}, a_j, a_j) + 2\theta G_d(a_j, a_{j+1}, a_{j+1})$$

$$\begin{aligned} (1-2\theta)G_d(a_j, a_{j+1}, a_{j+1}) &\leq \theta G_d(a_{j-1}, a_j, a_j) \leq 5\theta G_d(a_{j-1}, a_j, a_j), \ \textit{because } \theta \leq 5\theta \\ (1-2\theta)G_d(a_j, a_{j+1}, a_{j+1}) &\leq 5\theta G_d(a_{j-1}, a_j, a_j) \end{aligned}$$

$$\begin{aligned} G_d(a_j, a_{j+1}, a_{j+1}) &\leq (\frac{5\theta}{1-2\theta})G_d(a_{j-1}, a_j, a_j) \\ G_d(a_j, a_{j+1}, a_{j+1}) &\leq (\frac{5\theta}{1-2\theta})(\frac{5\theta}{1-2\theta})^{j-1}G_d(a_0, a_1, a_1), \end{aligned}$$

$$G_d(a_j, a_{j+1}, a_{j+1}) \leq (\frac{5\theta}{1-2\theta})^j G_d(a_0, a_1, a_1)$$

$$G_d(a_j, a_{j+1}, a_{j+1}) \leq \gamma^j G_d(a_0, a_1, a_1)$$

Similarly by rectangular property of $G-\textit{metric like spaces}$, consider for $a_{j+1} \in X$

$$G_d(a_0, a_{j+1}, a_{j+1}) \leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + ... + G_d(a_j, a_{j+1}, a_{j+1})$$

$$G_d(a_0, a_{j+1}, a_{j+1}) \leq (1+\gamma+\gamma^2+.....+\gamma^j)G_d(a_0, a_1, a_1)$$

$$\begin{aligned} G_d(a_0, a_{j+1}, a_{j+1}) &\leq (1-\gamma^{j+1})r \\ G_d(a_0, a_{j+1}, a_{j+1}) &\leq r \end{aligned}$$

Hence $a_{j+1} \in \overline{B_G(a_0, r)}$, for $j+1 \in N$. Hence $\forall\, n \in N$, $a_n \in \overline{B_G(a_0, r)}$. Now to check that the sequence $\{a_n\} \subseteq \overline{B_G(a_0, r)}$ is Cauchy sequence consider $m, n \in Z$ and $m > n$ such that,

$$\begin{aligned} G_d(a_n, a_m, a_m) &\leq G_d(a_n, a_{n+1}, a_{n+1}) + G_d(a_{n+1}, a_{n+2}, a_{n+2}) \\ &\quad + ... + G_d(a_{m-1}, a_m, a_m) \end{aligned}$$

$$G_d(a_n, a_m, a_m) \leq (\gamma^n + \gamma^{n+1} + ... + \gamma^{m-1}) G_d(a_0, a_1, a_1)$$

$$\begin{aligned} G_d(a_n, a_m, a_m) &\leq \gamma^n (1 + \gamma + \gamma^2 + ... + \gamma^{m-n-1}) G_d(a_0, a_1, a_1) \\ G_d(a_n, a_m, a_m) &\leq \gamma^n \left(\frac{1 - \gamma^{m-n}}{1 - \gamma}\right) G_d(a_0, a_1, a_1) \end{aligned} \tag{3.2.6}$$

$$G_d(a_n, a_m, a_m) \leq \frac{\gamma^n}{1 - \gamma} G_d(a_0, a_1, a_1)$$

Now if $n \longrightarrow \infty$, then $\gamma^n \longrightarrow 0$ because $\gamma \in [0, 1)$.

$$G_d(a_n, a_m, a_m) \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

Hence $\{a_n\} \subseteq \overline{B_G(a_0, r)}$ is Cauchy sequence. Therefore $\exists$ a point $x^* \in \overline{B_G(x_0, r)}$ such that

$$\underset{n \longrightarrow \infty}{Lim} G_d(a_n, a^*, a^*) = \underset{n \longrightarrow \infty}{Lim} G_d(a^*, a_n, a_n) = 0$$

Therefore $a^* \in \overline{B_G(a_0, r)}$ is limit point of the sequence $\{a_n\} \subseteq \overline{B_G(a_0, r)}$. Now to show $a^* \in \overline{B_G(a_0, r)}$ is common fixed point of dominted mappings Ω and Ψ. Consider for dominated mapping Ω,

$$\begin{aligned} G_d(a^*, \Omega a^*, \Omega a^*) &\leq G_d(a^*, a_{2n}, a_{2n}) + G_d(a_{2n}, \Omega a^*, \Omega a^*) \\ G_d(a^*, \Omega a^*, \Omega a^*) &\leq G_d(a^*, a_{2n}, a_{2n}) + G_d(\Psi a_{2n-1}, \Omega a^*, \Omega a^*) \end{aligned}$$

$$\begin{aligned} G_d(a^*, \Omega a^*, \Omega a^*) &\leq G_d(a^*, a_{2n}, a_{2n}) + G_d(\Omega a^*, \Psi a_{2n-1}, \Psi a_{2n-1}) \\ G_d(a^*, \Omega a^*, \Omega a^*) &\leq G_d(a^*, a_{2n}, a_{2n}) + \theta[G_d(a^*, \Omega a^*, \Omega a^*) \\ &\quad + 2G_d(a_{2n-1}, \Psi a_{2n-1}, \Psi a_{2n-1})] \end{aligned}$$

$$\begin{aligned} G_d(a^*, \Omega a^*, \Omega a^*) &\leq G_d(a^*, a_{2n}, a_{2n}) + \theta G_d(a^*, \Omega a^*, \Omega a^*) \\ &\quad + 2\theta G_d(a_{2n-1}, a_{2n}, a_{2n}) \end{aligned}$$

$$
\begin{aligned}
(1-\theta)G_d(a^*,\Omega a^*,\Omega a^*) &\leq G_d(a^*,a_{2n},a_{2n})+2\theta G_d(a_{2n-1},a^*,a^*)\\
&+2\theta G_d(a^*,a_{2n},a_{2n})
\end{aligned}
$$

$$
(1-\theta)G_d(a^*,\Omega a^*,\Omega a^*) \leq (1+2\theta)G_d(a^*,a_{2n},a_{2n})+2\theta G_d(a_{2n-1},a^*,a^*)
$$

As $a^* \in \overline{B_G(a_0,r)}$ is limit point of the Cauchy sequnce $\{a_n\} \subseteq \overline{B_G(a_0,r)}$ as $n \longrightarrow \infty$, so

$$
\underset{n\longrightarrow\infty}{Lim}\, G_d(a^*,a_{2n},a_{2n}) = 0 = \underset{n\longrightarrow\infty}{Lim} G_d(a_{2n-1},a^*,a^*)
$$

So the relation becomes

$$
\begin{aligned}
(1-\theta)G_d(a^*,\Omega a^*,\Omega a^*) &\leq 0+0\\
(1-\theta)G_d(a^*,\Omega a^*,\Omega a^*) &\leq 0\\
G_d(a^*,\Omega a^*,\Omega a^*) &\leq 0
\end{aligned}
$$

As $G_d(a^*,\Omega a^*,\Omega a^*) \nless 0$, so the only possibility left is

$$
G_d(a^*,\Omega a^*,\Omega a^*) = 0
$$

By symmetric condtion of $G-metric\ space$

$$
\begin{aligned}
G_d(a^*,\Omega a^*,\Omega a^*) &= G_d(\Omega a^*,a^*,a^*) = 0\\
\Omega a^* &= a^*
\end{aligned}
$$

Therefore $a^* \in \overline{B_G(a_0,r)}$ is fixed point of dominated mapping $\Omega : X \longrightarrow X$. Now for second dominated mapping Ψ,

$$
G_d(a^*,\Psi a^*,\Psi a^*) \leq G_d(a^*,a_{2n-1},a_{2n-1}) + G_d(a_{2n-1},\Psi a^*,\Psi a^*)
$$

$$
\begin{aligned}
G_d(a^*,\Psi a^*,\Psi a^*) &\leq G_d(a^*,a_{2n-1},a_{2n-1}) + G_d(\Omega a_{2n-2},\Psi a^*,\Psi a^*)\\
G_d(a^*,\Psi a^*,\Psi a^*) &\leq G_d(a^*,a_{2n-1},a_{2n-1}) + \theta[G_d(a_{2n-2},\Omega a_{2n-2},\Omega a_{2n-2})\\
&+2G_d(a^*,\Psi a^*,\Psi a^*)]
\end{aligned}
$$

$$
\begin{aligned}
G_d(a^*,\Psi a^*,\Psi a^*) &\leq G_d(a^*,a_{2n-1},a_{2n-1}) + \theta G_d(a_{2n-2},\Omega a_{2n-2},\Omega a_{2n-2})\\
&+2\theta G_d(a^*,\Psi a^*,\Psi a^*)
\end{aligned}
$$

$$
\begin{aligned}
(1-2\theta)G_d(a^*,\Psi a^*,\Psi a^*) &\leq G_d(a^*,a_{2n-1},a_{2n-1})+\theta G_d(a_{2n-2,}a_{2n-1},a_{2n-1}) \\
(1-2\theta)G_d(a^*,\Psi a^*,\Psi a^*) &\leq G_d(a^*,a_{2n-1},a_{2n-1})+\theta G_d(a_{2n-2,}a^*,a^*) \\
&\quad +\theta G_d(a^*,a_{2n-1},a_{2n-1})
\end{aligned}
$$

$$
(1-2\theta)G_d(a^*,\Psi a^*,\Psi a^*) \leq (1+\theta)G_d(a_{2n-1},a^*,a^*)+\theta G_d(a_{2n-2,}a^*,a^*)
$$

As $a^* \in \overline{B_G(a_0,r)}$ is limit point of the Cauchy sequnce $\{a_n\} \subseteq \overline{B_G(a_0,r)}$ as $n \longrightarrow \infty$, so

$$
\underset{n\longrightarrow\infty}{Lim} G_d(a_{2n-1},a^*,a^*) = 0 = \underset{n\longrightarrow\infty}{Lim} G_d(a_{2n-2,}a^*,a^*)
$$

So the relation becomes

$$
\begin{aligned}
(1-2\theta)G_d(a^*,\Psi a^*,\Psi a^*) &\leq 0 \\
G_d(a^*,\Psi a^*,\Psi a^*) &\leq 0
\end{aligned}
$$

As $G_d(a^*,\Psi a^*,\Psi a^*) \nless 0$ so the only possibility left is,

$$
G_d(a^*,\Psi a^*,\Psi a^*) = 0
$$

By symmetric condtion of $G-metric\ spaces$,

$$
\begin{aligned}
G_d(a^*,\Psi a^*,\Psi a^*) &= G_d(\Psi a^*,a^*,a^*) = 0 \\
\Psi a^* &= a^*
\end{aligned}
$$

Hence $a^* \in \overline{B_G(a_0,r)}$ is fixed point of dominated mapping $\Psi : X \longrightarrow X$. As

$$
\Omega a^* = \Psi a^* = a^*
$$

Therefore $a^* \in \overline{B_G(a_0,r)}$ is common fixed point of dominated mapping $\Psi, \Omega : X \longrightarrow X$. Now to prove *uniqueness* of common fixed point $a^* \in \overline{B_G(a_0,r)}$. For this let us take another common fixed point $b^* \in \overline{B_G(a_0,r)}$ of dominated mapping $\Psi, \Omega : X \longrightarrow X$ such that $a^* \neq b^*$. Then there arises two cases for $a^*, b^* \in \overline{B_G(a_0,r)}$. In first case let a^* and b^* are comparable, that is either $a^* \leq b^*$ or $b^* \leq a^*$. As a^* and b^* are common fixed point of dominated mappings Ψ and Ω then,

$$
\Psi a^* = a^*,\ \Omega a^* = a^*,\ \Psi b^* = b^*,\ \Omega b^* = b^*
$$

Now consider the relation,

$$\begin{aligned} G_d(a^*, b^*, b^*) &= G_d(\Omega a^*, \Psi b^*, \Psi b^*) \\ G_d(a^*, b^*, b^*) &\leq \theta\{G_d(a^*, \Omega a^*, \Omega a^*) + 2G_d(b^*, \Psi b^*, \Psi b^*)\} \end{aligned}$$

$$\begin{aligned} G_d(a^*, b^*, b^*) &\leq \theta G_d(a^*, a^*, a^*) + 2\theta G_d(b^*, b^*, b^*) \\ G_d(a^*, b^*, b^*) &\leq 0 \end{aligned}$$

As by definition of $G-metric\ space$, $G_d(a^*, b^*, b^*) \nless 0$, then only possibility left is,

$$G_d(a^*, b^*, b^*) = 0$$

As $G-metric\ space$ is symmetric then,

$$\begin{aligned} G_d(b^*, b^*, a^*) &= G_d(a^*, b^*, b^*) = 0 \\ a^* &= b^* \end{aligned}$$

It is contradiction ($\because a^* \neq b^*$). So our suposition is wrong. Hence $a^* \in \overline{B_G(a_0, r)}$ is unique when a^* and b^* are comparable. For second case when a^* and b^* are not comparable then $\exists\, \overline{c_0} \in \overline{B_G(a_0, r)}$ such that $\overline{c_0} \leq a^*$ and $\overline{c_0} \leq b^*$, then $\overline{c_0}$ is lower bound of both a^* and b^*. Now consider a picard sequnce $\{\overline{c_k}\}$ such that,

$$\overline{c_{2k-1}} = \Psi\overline{c_{2k-2}} \text{ and } \overline{c_{2k}} = \Omega\overline{c_{2k-1}}$$

As Ψ and Ω are dominated mappings then,

$$\overline{c_{2k}} = \Omega\overline{c_{2k-1}} \leq \overline{c_{2k-1}} = \Psi\overline{c_{2k-2}} \leq \overline{c_{2k-2}}$$

It gives $\forall\ k \in N \cup \{0\}$,

$$\begin{aligned} \ldots\ldots &\leq \overline{c_k} \leq \ldots\ldots \leq \overline{c_2} \leq \overline{c_1} \leq \overline{c_0} \leq a^* \\ \ldots\ldots &\leq \overline{c_k} \leq \ldots\ldots \leq \overline{c_2} \leq \overline{c_1} \leq \overline{c_0} \leq b^* \end{aligned}$$

So,

$$\begin{aligned} \overline{c_k} &\leq a^* \text{ and } \overline{c_k} \leq b^* \\ \overline{c_k} &\leq \Psi a^*,\ \overline{c_k} \leq \Omega a^*,\ \overline{c_k} \leq \Psi b^*,\ \overline{c_k} \leq \Omega b^* \end{aligned}$$

Now to check picard sequence $\{\overline{c_k}\}$ is in closed ball $\overline{B_G(a_0,r)}$ consider the relation,

$$\begin{aligned} G_d(a_1,\overline{c_1},\overline{c_1}) &= G_d(\Omega a_0,\Psi\overline{c_0},\Psi\overline{c_0}) \\ G_d(a_1,\overline{c_1},\overline{c_1}) &\leq \theta\{G_d(a_0,\Omega a_0,\Omega a_0)+2G_d(\overline{c_0},\Psi\overline{c_0},\Psi c)\} \\ G_d(a_1,\overline{c_1},\overline{c_1}) &\leq \theta G_d(a_0,a_1,a_1)+2\theta G_d(\overline{c_0},\overline{c_1},c) \end{aligned}$$

$$\begin{aligned} G_d(a_1,\overline{c_1},\overline{c_1}) \leq\ & \theta G_d(a_0,a_1,a_1)+2\theta G_d(\overline{c_0},a_0,a_0) \\ & +2\theta G_d(a_0,a_1,a_1)+2\theta G_d(a_1,\overline{c_1},\overline{c_1}) \end{aligned}$$

$$\begin{aligned} (1-2\theta)G_d(a_1,\overline{c_1},\overline{c_1}) &\leq 3\theta G_d(a_0,a_1,a_1)+2\theta G_d(a_0,\overline{c_0},\overline{c_0}) \\ (1-2\theta)G_d(a_1,\overline{c_1},\overline{c_1}) &\leq 3\theta(1-\gamma)r+2\theta r=5\theta r \end{aligned}$$

$$\begin{aligned} G_d(a_1,\overline{c_1},\overline{c_1}) &\leq \frac{5\theta}{1-2\theta}r \\ G_d(a_1,\overline{c_1},\overline{c_1}) &\leq \gamma r \end{aligned}$$

Now let,

$$\begin{aligned} G_d(a_0,\overline{c_1},\overline{c_1}) &\leq G_d(a_0,a_1,a_1)+G_d(a_1,\overline{c_1},\overline{c_1}) \\ G_d(a_0,\overline{c_1},\overline{c_1}) &\leq (1-\gamma)r+\gamma r \\ G_d(a_0,\overline{c_1},\overline{c_1}) &\leq r \end{aligned}$$

Hence $\overline{c_1}\in\overline{B_G(a_0,r)}$. Again consider the relation,

$$\begin{aligned} G_d(a_2,\overline{c_2},\overline{c_2}) &= G_d(\Psi a_1,\Omega\overline{c_1},\Omega\overline{c_1}) \\ G_d(a_2,\overline{c_2},\overline{c_2}) &= G_d(\Omega\overline{c_1},\Psi a_1,\Psi a_1) \end{aligned}$$

$$\begin{aligned} G_d(a_2,\overline{c_2},\overline{c_2}) &\leq \theta\{G_d(\overline{c_1},\Omega\overline{c_1},\Omega\overline{c_1})+2G_d(a_1,\Psi a_1,\Psi a_1)\} \\ G_d(a_2,\overline{c_2},\overline{c_2}) &\leq \theta G_d(\overline{c_1},\overline{c_2},\overline{c_2})+2\theta G_d(a_1,a_2,a_2) \end{aligned}$$

$$\begin{aligned} G_d(a_2,\overline{c_2},\overline{c_2}) \leq\ & \theta G_d(\overline{c_1},a_1,a_1)+\theta G_d(a_1,a_2,a_2) \\ & +\theta G_d(a_2,\overline{c_2},\overline{c_2})+2\theta G_d(a_1,a_2,a_2) \end{aligned}$$

$$
\begin{aligned}
(1-\theta)G_d(a_2,\overline{c_2},\overline{c_2}) &\leq 3\theta G_d(a_1,a_2,a_2)+\theta G_d(a_1,\overline{c_1},\overline{c_1})\\
(1-\theta)G_d(a_2,\overline{c_2},\overline{c_2}) &\leq 3\theta.(\frac{5\theta}{1-2\theta})G_d(a_0,a_1,a_1)+\theta.(\frac{5\theta}{1-2\theta})r
\end{aligned}
$$

$$(1-\theta)G_d(a_2,\overline{c_2},\overline{c_2}) \leq (\frac{5\theta^2}{1-2\theta})r+(\frac{15\theta^2}{1-2\theta})(1-\gamma)r$$

$$G_d(a_2,\overline{c_2},\overline{c_2}) \leq (\frac{5\theta^2}{1-2\theta})r+(\frac{15\theta^2}{1-2\theta})(1-\frac{5\theta}{1-2\theta})r$$

$$G_d(a_2,\overline{c_2},\overline{c_2}) \leq (\frac{5\theta}{1-2\theta})^2 r, \ \because\ 20\theta^2 \leq 25\theta^2 \text{ and } \frac{1}{1-\theta} \leq \frac{1}{1-2\theta}$$

$$G_d(a_2,\overline{c_2},\overline{c_2}) \leq \gamma^2 r$$

Now let,

$$G_d(a_0,\overline{c_2},\overline{c_2}) \leq G_d(a_0,a_1,a_1)+G_d(a_1,a_2,a_2)+G_d(a_2,\overline{c_2},\overline{c_2})$$

$$G_d(a_0,\overline{c_2},\overline{c_2}) \leq (1-\gamma)r+\gamma G_d(a_0,a_1,a_1)+\gamma^2 r$$

$$
\begin{aligned}
G_d(a_0,\overline{c_2},\overline{c_2}) &\leq r-\gamma r+\gamma(1-\gamma)r+\gamma^2 r\\
G_d(a_0,\overline{c_2},\overline{c_2}) &\leq r-\gamma r+\gamma r-\gamma^2 r+\gamma^2 r\\
G_d(a_0,\overline{c_2},\overline{c_2}) &\leq r
\end{aligned}
$$

Hence $\overline{c_2} \in \overline{B_G(a_0,r)}$. Now let $\overline{c_3},\overline{c_4},\overline{c_5},.....,\overline{c_j} \in \overline{B_G(a_0,r)}$ for some $j \in N$. Then following relation holds,

$$G_d(a_j,\overline{c_j},\overline{c_j}) \leq (\frac{5\theta}{1-2\theta})^j r \tag{3.2.7}$$

Then by mathematical induction consider for even $j+1 \in N$,

$$
\begin{aligned}
G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &= G_d(\Omega a_j,\Psi\overline{c_j},\Psi\overline{c_j})\\
G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &\leq \theta\{G_d(a_j,\Omega a_j,\Omega a_j)+2G_d(\overline{c_j},\Psi\overline{c_j},\Psi\overline{c_j})\}
\end{aligned}
$$

$$
\begin{aligned}
G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &\leq \theta G_d(a_j,a_{j+1},a_{j+1})+2\theta G_d(\overline{c_j},\overline{c_{j+1}},\overline{c_{j+1}})\\
G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &\leq \theta G_d(a_j,a_{j+1},a_{j+1})+2\theta G_d(\overline{c_j},a_j,a_j)\\
&\quad +2\theta G_d(a_j,a_{j+1},a_{j+1})+2\theta G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}})
\end{aligned}
$$

$$(1-2\theta)G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) \leq 3\theta G_d(a_j,a_{j+1},a_{j+1})+2\theta G_d(\overline{c_j},a_j,a_j)$$

$$(1-2\theta)G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) \leq 3\theta(\frac{5\theta}{1-2\theta})^j r + 2\theta(\frac{5\theta}{1-2\theta})^j r$$

$$\begin{aligned}
(1-2\theta)G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &\leq 5\theta(\frac{5\theta}{1-2\theta})^j r\\
G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &\leq \frac{5\theta}{(1-2\theta)}(\frac{5\theta}{1-2\theta})^j r
\end{aligned}$$

$$\begin{aligned}
G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &\leq (\frac{5\theta}{1-2\theta})^{j+1} r\\
G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}}) &\leq \gamma^{j+1} r
\end{aligned}$$

Now let by rectangular property of $G-metric\ spaces$,

$$\begin{aligned}
G_d(a_0,\overline{c_{j+1}},\overline{c_{j+1}}) \leq\ & G_d(a_0,a_1,a_1)+G_d(a_1,a_2,a_2)+.....+G_d(a_j,a_{j+1},a_{j+1})\\
&+G_d(a_{j+1},\overline{c_{j+1}},\overline{c_{j+1}})
\end{aligned}$$

$$\begin{aligned}
G_d(a_0,\overline{c_{j+1}},\overline{c_{j+1}}) &\leq (1+\gamma+\gamma^2+.....+\gamma^j)G_d(a_0,a_1,a_1)+\gamma^{j+1}r\\
G_d(a_0,\overline{c_{j+1}},\overline{c_{j+1}}) &\leq (\frac{1-\gamma^{j+1}}{1-\gamma})(1-\gamma)r+\gamma^{j+1}r
\end{aligned}$$

$$\begin{aligned}
G_d(a_0,\overline{c_{j+1}},\overline{c_{j+1}}) &\leq (1-\gamma^{j+1})r+\gamma^{j+1}r\\
G_d(a_0,\overline{c_{j+1}},\overline{c_{j+1}}) &\leq r
\end{aligned}$$

Hence $\overline{c_{j+1}} \in \overline{B_G(a_0,r)}$. Therefore $\forall\ k \in N$, $\overline{c_k} \in \overline{B_G(a_0,r)}$. Now as $\overline{c_0} \leq a^*$ and $\overline{c_0} \leq b^*$ then $\overline{c_k} \leq \overline{c_0} \leq a^*$ and $\overline{c_k} \leq \overline{c_0} \leq b^*$. So clearly,

$$\overline{c_k} \leq a^* = \Omega^n a,\ \overline{c_k} \leq a^* = \Psi^n a^*,\ \overline{c_k} \leq b^* = \Omega^n b^* \text{and } \overline{c_k} \leq b^* = \Psi^n b^* \tag{3.2.8}$$

As $\forall\ k \in N$, $\Omega^n a^* = \Psi^n a^* = a^*$ and $\Omega^n b^* = \Psi^n b^* = b^*$. Then consider from (3.2.8),

$$\begin{aligned}
G_d(a^*,b^*,b^*) &= G_d(\Omega^n a^*,\Psi^n b^*,\Psi^n b^*)\\
G_d(a^*,b^*,b^*) &\leq \theta(G_d(a^*,\Omega^n a^*,\Omega^n a^*)+2G_d(b^*,\Psi^n b^*,\Psi^n b^*))\\
G_d(a^*,b^*,b^*) &\leq \theta(0+0)\\
G_d(a^*,b^*,b^*) &\leq 0
\end{aligned}$$

As by definition of $G-metric\ space$, $G_d(a^*,b^*,b^*) \nless 0$, then only possibility left is,

$$G_d(a^*,b^*,b^*) = 0$$

As $G - metric\ space$ is symmetric then,

$$\begin{aligned} G_d(b^*, b^*, a^*) &= G_d(a^*, b^*, b^*) = 0 \\ a^* &= b^* \end{aligned}$$

It is contradiction ($\because a^* \neq b^*$). So our supposition is wrong. Hence common fixed point is unique. So if a^* and b^* are not comparable then common fixed point is unique. Hence Kannan common fixed point for double self dominated mappings is unique. ∎

Example 39 *If for a set* $X = [0, \infty)$*, a mapping* $G_d : X \times X \times X \longrightarrow X$*,* $\forall\ a, b, c \in X$ *defined by,*

$$G_d(a, b, c) = a + b + c \tag{3.2.9}$$

then $(X; \leq; G)$ *is symmetric and complete* $G-metric\ like\ space$*. Let mappings* $\Omega, \Psi : X \to X$ *are defined by,*

$$\Omega a = \begin{cases} \frac{a}{9} & if\ a \in [0,1] \\ a + \frac{1}{5} & if\ a \in (1, \infty) \end{cases}$$

$$\Psi a = \begin{cases} \frac{a}{10} & if\ a \in [0,1] \\ a + \frac{1}{7} & if\ a \in (1, \infty) \end{cases}$$

Obviously Ω *and* Ψ *are dominated mappings inside of* $[0, 1]$ *but not dominated outside of* $[0, 1]$*. Let* $a_0 = \frac{1}{3}$ *and* $r = \frac{7}{3}$ *such that* $\overline{B_G(a_0, r)} = [0, 1]$*. Also let* $\theta = \frac{1}{10} \in [0, \frac{1}{7})$*, such that* $\gamma = \frac{5\theta}{1-2\theta} = \frac{5}{8}$ *so to get,*

$$\begin{aligned} (1-\gamma)r &= (1 - \frac{5}{8})\frac{7}{3} \\ (1-\gamma)r &= \frac{7}{8} \end{aligned} \tag{3.2.10}$$

Also as,

$$\begin{aligned} G_d(a_0, a_1, a_1) &= \frac{1}{3} + 2(\frac{1}{27}),\ \because\ From\ (3.2.9) \\ G_d(a_0, a_1, a_1) &= \frac{11}{27} \end{aligned} \tag{3.2.11}$$

From (3.210) *and* (3.2.11),

$$\begin{aligned} \frac{11}{27} &\leq \frac{7}{8} \\ G_d(a_0, a_1, a_1) &\leq (1-\gamma)r \end{aligned}$$

Now to check that contractive condition is locally satisfied on $\overline{B_G(a_0,r)} = [0,1]$ *but not satisfied outside of* $\overline{B_G(a_0,r)} = [0,1]$. *If* $\forall a,b,c \in [0,1]$ *then,*

$$G_d(\Omega a, \Psi b, \Psi c) = G_d(\frac{a}{9}, \frac{b}{10}, \frac{c}{10})$$

$$\begin{aligned} G_d(\Omega a, \Psi b, \Psi c) &= \frac{a}{9} + \frac{b}{10} + \frac{c}{10} \\ G_d(\Omega a, \Psi b, \Psi c) &= \frac{10a + 9b + 9c}{90} \\ G_d(\Omega a, \Psi b, \Psi c) &= \frac{1}{9}(a+b+c) - \frac{1}{90}(b+c) \end{aligned} \quad (3.2.12)$$

Also as,

$$\begin{aligned} \theta\{G_d(a,\Omega a,\Omega a) + G_d(b,\Psi b,\Psi b) + G_d(c,\Psi c,\Psi c)\} &= \frac{1}{10}\{G_d(a,\frac{a}{9},\frac{a}{9}) \\ &\quad +G_d(b,\frac{b}{10},\frac{b}{10}) \\ &\quad +G_d(c,\frac{c}{10},\frac{c}{10}) \end{aligned}$$

$$\begin{aligned} \theta\{G_d(a,\Omega a,\Omega a) + G_d(b,\Psi b,\Psi b) + G_d(c,\Psi c,\Psi c)\} &= \frac{1}{10}\{a + 2\frac{a}{9} + b \\ &\quad +2\frac{b}{10} + c + 2\frac{c}{10}\} \end{aligned}$$

$$\begin{aligned} \theta\{G_d(a,\Omega a,\Omega a) + G_d(b,\Psi b,\Psi b) + G_d(c,\Psi c,\Psi c)\} &= \frac{11}{90}(a+b+c) \\ &\quad -\frac{1}{450}(b+c) \end{aligned} \quad (3.2.13)$$

As $\forall\ a,b,c \in [0,1]$ *clearly from* (3.2.12) *and* (3.2.13),

$$\frac{1}{9}(a+b+c) \le \frac{11}{90}(a+b+c) \text{ and } -\frac{1}{90}(b+c) \le -\frac{1}{450}(b+c)$$

$$\frac{1}{9}(a+b+c) - \frac{1}{90}(b+c) \le \frac{11}{90}(a+b+c) - \frac{1}{450}(b+c)$$

$$G_d(\Omega a, \Psi b, \Psi c) \le \theta[G_d(a,\Omega a,\Omega a) + G_d(b,\Psi b,\Psi b) + G_d(c,\Psi c,\Psi c)]$$

Hence contractive condition is satisfied on the closed ball $[0,1]$. *Now for* $a,b,c \in (1,\infty)$,

$$G_d(\Omega a, \Psi b, \Psi c) = G_d(a + \frac{1}{5}, b + \frac{1}{7}, c + \frac{1}{7})$$

$$G_d(\Omega a, \Psi b, \Psi c) = (a+b+c) + \frac{17}{35} \quad (3.2.14)$$

Also as,

$$\begin{aligned}\theta\{G_d(a,\Omega a,\Omega a)+G_d(b,\Psi b,\Psi b)+G_d(c,\Psi c,\Psi c)\} &= \frac{1}{10}\{(a+b+c)\\ &+2(\Omega a+\Psi b\\ &+\Psi c)\}\end{aligned}$$

$$\begin{aligned}\theta\{G_d(a,\Omega a,\Omega a)+G_d(b,\Psi b,\Psi b)+G_d(c,\Psi c,\Psi c)\} &= \frac{1}{10}\{(a+b+c)\\ &+2(a+\frac{1}{5}+b\\ &+\frac{1}{7}+c+\frac{1}{7})\}\end{aligned}$$

$$\begin{aligned}\theta\{G_d(a,\Omega a,\Omega a)+G_d(b,\Psi b,\Psi b)+G_d(c,\Psi c,\Psi c)\} &= \frac{3}{10}(a+b+c)\\ &+\frac{17}{175} \qquad (3.2.15)\end{aligned}$$

As clearly from (3.2.14) *and* (3.2.15),

$$(a+b+c)+\frac{17}{35}\geq\frac{3}{10}(a+b+c)+\frac{17}{175}$$

$$G_d(\Omega a,\Psi b,\Psi c)\geq\theta\{G_d(a,\Omega a,\Omega a)+G_d(b,\Psi b,\Psi b)+G_d(c,\Psi c,\Psi c)\}$$

Hence contractive condition is not satisfied outside of the closed ball $[0,1]$. *It shows that all conditions of Banach fixed point theorem for double dominated mappings. Moreover* $0\in[0,1]$ *common fix point of mappings* Ψ *and* Ω, *i.e* $\varphi 0=\delta 0=0$.

Corollary 40 *Suppose* $(X;\leq;G)$ *be a symmetric and ordered complete* $G-$ *metric like space and* $\Omega,\Psi:X\to X$ *are any two dominated mappings and* a_0, $a,b,c\in X$, $r>0$. *Suppose that* $\exists\ \theta\in[0,\frac{1}{7})$ *such that the following conditions holds*

$$\begin{aligned}G_d(\Omega a,\Omega b,\Omega c) &\leq \theta[G_d(a,\Omega a,\Omega a)+G_d(b,\Omega b,\Omega b)\\ &+G_d(c,\Omega c,\Omega c)] \qquad (3.2.16)\end{aligned}$$

$\forall$ *comparabe elements* $a,b,c\in\overline{B_G(a_0,r)}\subseteq X$

$$G_d(a_0,a_1,a_1)=G_d(a_0,\Omega a_0,\Omega a_0)\leq(1-\gamma)r \qquad (3.2.17)$$

Where $\gamma=\frac{5\theta}{1-2\theta}$. *If for non-increasing sequence* $\{a_n\}$ *in* $\overline{B_G(a_0,r)}$, $\{a_n\}\to v$ *then* $\exists$ *a unique* $a^*\in\overline{B_G(a_0,r)}$ *such that* $G_d(a^*,a^*,a^*)=0$ *and* $\Omega a^*=\Psi a^*=a^*$.

Proof. In the main result Kannan common fixed point of dominated mappings take $\Psi = \Omega$ to get unique fixed point $\Omega a^* = a^*$. ■

Proof of the following corollary is similar to the proof of *Theorem* 38 but without discussing the ordered property of $G-$ *metric like spaces.*

Corollary 41 *Suppose* $(X; \leq; G)$ *be a symmetric and complete* $G-$ *metric like space and* $\Omega, \Psi : X \to X$ *are any two dominated mappings and* a_0, a, $b, c \in X$, $r > 0$. *Suppose that* $\exists$ $\theta \in [0, \frac{1}{7})$ *such that the following conditions holds*

$$\begin{aligned} G_d(\Omega a, \Psi b, \Psi c) &\leq \theta[G_d(a, \Omega a, \Omega a) + G_d(b, \Psi b, \Psi b) \\ &\quad + G_d(c, \Psi c, \Psi c)] \end{aligned} \tag{3.2.18}$$

$\forall$ *comparabe elements* $a, b, c \in \overline{B_G(a_0, r)} \subseteq X$

$$G_d(a_0, a_1, a_1) = G_d(a_0, \Omega a_0, \Omega a_0) \leq (1-\gamma)r \tag{3.2.19}$$

where $\gamma = \frac{5\theta}{1-2\theta}$. *If for non-increasing sequence* $\{a_n\}$ *in* $\overline{B_G(a_0, r)}$, $\{a_n\} \to v$ *then* $\exists$ *a unique* $a^* \in \overline{B_G(a_0, r)}$ *such that* $G_d(a^*, a^*, a^*) = 0$ *and* $\Omega a^* = \Psi a^* = a^*$.

3.2.1 Error Bounds

In this section errors approximations and their related example are discused.

Corollary 42 *From Theorem* 38, *iterative sequence* (3.2.3), *with arbitrary* $a_0 \in \overline{B_G(a_0, r)} \subseteq X$, *converges to unique common fixed point* a^* *of dominated mappings* Ω *and* Ψ. *Error estimates are the* ***prior estimate***

$$G_d(a_n, a^*, a^*) \leq \frac{\gamma^n}{1-\gamma} G_d(a_0, a_1, a_1) \tag{3.2.20}$$

and the ***posterior estimate***

$$G_d(a_n, a^*, a^*) \leq \frac{\gamma}{1-\gamma} G_d(a_{n-1}, a_n, a_n) \tag{3.2.21}$$

Proof. As from relation (3.2.6) of *Theorem* 38,

$$G_d(a_n, a_m, a_m) \leq \gamma^n (\frac{1-\gamma^{m-n}}{1-\gamma}) G_d(a_0, a_1, a_1)$$

As the sequence $\{a_m\}$ is convergent to $a^* \in \overline{B_G(a_0,r)} \subseteq X$, then by taking $m \to \infty$ gives $a_m \to a^*$ and $\gamma^{m-n} \to 0$. Therefore above relation leads to the *prior estimate*, *i.e*,

$$G_d(a_n, a^*, a^*) \leq \frac{\gamma^n}{1-\gamma} G_d(a_0, a_1, a_1)$$

Setting $n = 1$ and write b_0 for a_0 and b_1 for a_1 in (3.2.20)

$$G_d(b_1, a^*, a^*) \leq \frac{\gamma}{1-\gamma} G_d(b_0, b_1, b_1)$$

Letting $b_0 = a_{n-1}$ then $b_1 = \Psi b_0 = \Psi a_{n-1} = a_n$ in above relation leads to the *posterior estimate* (3.2.21), *i.e*,

$$G_d(a_n, a^*, a^*) \leq \frac{\gamma}{1-\gamma} G_d(a_{n-1}, a_n, a_n)$$

■

The prior error bound (3.2.20) can be used at the beginning of the calculation for estimating the required number of steps to obtain a assumed accuracy. While posterior error bound (3.2.21) can be used at intermediate stages or at the end of the calculation. Posterior error bound (3.2.21) is at least as accurate as prior error bound (3.2.20).

Example 43 *If for a set* $X = [0, \infty)$, *a mapping* $G_d : X \times X \times X \longrightarrow X$ *then* $\forall\, a, b, c \in X$ *defined by,*

$$G_d(a, b, c) = a + b + c, \tag{3.2.22}$$

then $(X; \leq; G)$ *is symmetric and complete* $G-$*metric like space. Let mappings* $\Omega, \Psi : X \to X$ *are defined by,*

$$\Omega a = \begin{cases} \frac{a}{9} & \text{if } a \in [0,1] \\ a + \frac{1}{5} & \text{if } a \in (1, \infty) \end{cases}$$

$$\Psi a = \begin{cases} \frac{a}{10} & \text{if } a \in [0,1] \\ a + \frac{1}{7} & \text{if } a \in (1, \infty) \end{cases}$$

Obviously Ω *and* Ψ *are dominated mappings inside of* $[0,1]$ *but not dominated outside of* $[0,1]$. *Let* $a_0 = \frac{1}{3}$ *and* $r = \frac{7}{3}$ *such that* $\overline{B_G(a_0,r)} = [0,1]$. *Also let* $\theta = \frac{1}{10} \in [0, \frac{1}{7})$, *such that* $\gamma = \frac{5\theta}{1-2\theta} = \frac{5}{8}$. *Construct the picard iterative sequence by taking* $a_0 = \frac{1}{3}$ *as initial guess as,*

$$a_{2m-1} = \Omega a_{2m-1} = \frac{a_0}{9^m.10^{m-1}} \text{ and } a_{2m} = \Psi a_{2m-1} = \frac{a_0}{9^m.10^m}$$

$$a_{2m-1} = \frac{a_0}{3^{2m}.10^{m-1}} \text{ and } a_{2m} = \frac{a_0}{3^{2m}.10^m}$$

Let $2m-1=n$ *then for odd* $n \in N$,

$$a_n = \frac{a_0}{3^{n+1}.10^{\frac{n-1}{2}}} \text{ and } a_{n+1} = \frac{a_0}{3^{n+1}.10^{\frac{n+1}{2}}} \tag{3.2.23}$$

Now as,

$$G_d(a_n, a^*, a^*) = a_n + 2a^*, \because \text{ From (3.2.22)}$$

As picard sequence $\{a_n\}$ *satisfies all the conditions of Kannan fixed point theorem* 38 *as in example* 39, *then If* $n \to \infty$ *so* $a_n \to a^*$ *i.e* $a_n \approx a^*$. *Then*

$$G_d(a_n, a^*, a^*) = 3a_n \tag{3.2.24}$$

Also as,

$$\begin{aligned} G_d(a_0, a_1, a_1) &= G_d(a_0, \Omega a_0, \Omega a_0) = a_0 + 2\Omega a_0 \\ G_d(a_0, a_1, a_1) &= \frac{1}{3} + 2\Omega\frac{1}{3} = \frac{1}{3} + \frac{2}{27} = \frac{11}{27} \end{aligned}$$

And finally as,

$$\begin{aligned} G_d(a_n, a^*, a^*) &\leq \frac{\gamma^n}{1-\gamma} G_d(a_0, a_1, a_1) \\ 3a_n &\leq \frac{8}{3}.(\frac{5}{8})^n.\frac{11}{27} = \frac{88}{81}.(\frac{5}{8})^n, \because \text{ From (3.2.24)} \end{aligned}$$

$$\begin{aligned} 2.76136364a_n &\leq (0.625)^n \\ \frac{2.76136364}{3^{n+2}.(\sqrt{10})^{n-1}} &\leq (0.625)^n, \because \text{ From (3.2.23)} \end{aligned}$$

$$\frac{2.76136364\sqrt{10}}{9.3^n.(\sqrt{10})^n} \leq (0.625)^n$$

$$0.97024428337 \leq (0.626 \times 3 \times \sqrt{10})^n = (5.92927061282)^n$$

$$\begin{aligned} \frac{\ln(0.97024428337)}{\ln(5.92927061282)} &\leq n \\ -0.01697139176 &\leq n \end{aligned}$$

As $n \in N$ *is odd and* $-0.01697139176 \leq n$ *then for* $n = 1, 3, 5, \ldots\ldots$ *picard sequence starts converging to its limit point i.e* $a^* = 0 \in [0,1]$ *such that let for* $n = 3$,

$$a_3 = \frac{1}{3^{3+2}.(\sqrt{10})^{3-1}} \text{ and } a_{3+1} = \frac{1}{3^{3+2}.(\sqrt{10})^{3+1}}$$

$$a_3 = \frac{1}{3^5.10^1} = 0.00041152263 \text{ and } a_4 = \frac{1}{3^5.10^2} = 0.000041152263$$

Again let for $n = 5$,

$$a_5 = \frac{1}{3^{5+2}.(\sqrt{10})^{5-1}} \text{ and } a_{5+1} = \frac{1}{3^{5+2}.(\sqrt{10})^{5+1}}$$

$$a_5 = \frac{1}{3^7.10^2} = 0.00000457247, \text{ and } a_6 = \frac{1}{3^7.10^3} = 0.000000457247$$

Thus,

$$a_4 = \Psi a_3 \text{ and } a_5 = \Omega a_4$$

$$0.000041152263 = \Psi(0.00041152263) \text{ and } 0.00000457247 = \Omega(0.000041152263)$$

Hence common fixed point approximation is,

$$\Omega(0.000041152263) \approx \Psi(0.00041152263) \approx 0.00000457247$$

Finally when $n \longrightarrow \infty$ *then,*

$$\Omega(0) = \Psi(0) = 0$$

Remark 44 *Above results not only holds for dominated mapping but also holds for dominating mappings.*

3.3 Chatterjea Common Fixed Point Result

Theorem 45 *Suppose* $(X; \leq; G)$ *be a symmetric and ordered complete G-metric like space and* $\Gamma, \Delta : X \rightarrow X$ *are any two dominated mappings and* a_0, a, $b, c \in X$, $r > 0$. *Suppose that* $\exists\ \sigma \in [0, \frac{1}{8})$ *such that the following conditions holds*

$$\begin{aligned} G_d(\Gamma a, \Delta b, \Delta c) \leq\ & \sigma[G_d(a, \Delta b, \Delta b) + G_d(a, \Delta c, \Delta c) \\ & + G_d(b, \Gamma a, \Gamma a) + G_d(b, \Delta c, \Delta c) \\ & + + G_d(c, \Gamma a, \Gamma a) + G_d(c, \Delta b, \Delta b)] \end{aligned} \quad (3.3.1)$$

$\forall$ *comparabe elements* $a, b, c \in \overline{B_G(a_0, r)} \subseteq X$

$$G_d(a_0, a_1, a_1) = G_d(a_0, \Gamma a_0, \Gamma a_0) \leq (1 - \mu) r \quad (3.3.2)$$

where $\mu = \frac{4\sigma}{1-4\sigma}$. *If for non-increasing sequence* $\{a_n\}$ *in* $\overline{B_G(a_0, r)}$, $\{a_n\} \rightarrow v$ *then* $\exists$ *a point* $a^* \in \overline{B_G(a_0, r)}$ *such that* $\Gamma a^* = \Delta a^* = a^*$ *and moreover* $G_d(a^*, a^*, a^*) = 0$.

Proof. With initially choosen guess $a_0 \in \overline{B_G(a_0, r)} \subseteq X$, consider picard sequence $\forall n \in N$,

$$a_{2n-1} = \Gamma(a_{2n-2}) \text{ and } a_{2n} = \Delta(a_{2n-1}) \tag{3.3.3}$$

As Γ and Δ are dominated mappings then,

$$.....a_n \leq a_{n-1} \leq a_{n-2} \leq ..... \leq a_3 \leq a_2 \leq a_1 \leq a_0$$

We know that,

$$\begin{aligned} G_d(a_0, a_1, a_1) &= G_d(a_0, \Gamma a_0, \Gamma a_0) \leq (1-\mu)r, \text{ From (3.3.2)} \\ G_d(a_0, a_1, a_1) &\leq r \end{aligned}$$

Clearly $a_1 \in \overline{B_G(a_0, r)}$. Now concider the relation

$$\begin{aligned} G_d(a_1, a_2, a_2) &= G_d(\Gamma a_0, \Delta a_1, \Delta a_1) \\ G_d(a_1, a_2, a_2) &\leq \sigma[2G_d(a_0, \Delta a_1, \Delta a_1) + 2G_d(a_1, \Gamma a_0, \Gamma a_0) \\ &\quad +2G_d(a_1, \Delta a_1, \Delta a_1)], \text{ From (3.3.1)} \end{aligned}$$

$$\begin{aligned} G_d(a_1, a_2, a_2) &\leq \sigma[2G_d(a_0, a_2, a_2) + 2G_d(a_1, a_1, a_1) \\ &\quad +2G_d(a_1, a_2, a_2)] \end{aligned}$$

$$\begin{aligned} G_d(a_1, a_2, a_2) &\leq 2\sigma G_d(a_0, a_1, a_1) + 2\sigma G_d(a_1, a_2, a_2) + 2(0) \\ &\quad +2\sigma G_d(a_1, a_2, a_2) \end{aligned}$$

$$\begin{aligned} (1-4\sigma)G_d(a_1, a_2, a_2) &\leq 2\sigma G_d(a_0, a_1, a_1) \\ G_d(a_1, a_2, a_2) &\leq \frac{2\sigma}{1-4\sigma} G_d(a_0, a_1, a_1) \end{aligned}$$

$$\begin{aligned} G_d(a_1, a_2, a_2) &\leq \frac{4\sigma}{1-4\sigma} G_d(a_0, a_1, a_1), \ \because\ 2\sigma \leq 4\sigma \\ G_d(a_1, a_2, a_2) &\leq \mu G_d(a_0, a_1, a_1) \end{aligned} \tag{3.3.4}$$

For $a_2 \in X$, consider by rectangular property,

$$G_d(a_0, a_2, a_2) \leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2)$$

$$G_d(a_0, a_2, a_2) \leq (1+\mu)G_d(a_0, a_1, a_1)$$

$$G_d(a_0, a_2, a_2) \leq (1+\mu)(1-\mu)r, \text{ From (3.3.2)}$$

$$\begin{aligned} G_d(a_0, a_2, a_2) &\leq (1-\mu^2)r \\ G_d(a_0, a_2, a_2) &\leq r \end{aligned}$$

Hence $a_2 \in \overline{B_G(a_0, r)}$. Again consider the relation,

$$G_d(a_2, a_3, a_3) = G_d(\Delta a_1, \Gamma a_2, \Gamma a_2) = G_d(\Gamma a_2, \Delta a_1, \Delta a_1)$$

$$\begin{aligned} G_d(a_2, a_3, a_3) &\leq \sigma[2G_d(a_2, \Delta a_1, \Delta a_1) + 2G_d(a_1, \Gamma a_2, \Gamma a_2) \\ &\quad +2G_d(a_1, \Delta a_1, \Delta a_1)], \text{ From (3.3.1)} \end{aligned}$$

$$\begin{aligned} G_d(a_2, a_3, a_3) &\leq 2\sigma G_d(a_2, a_2, a_2) + 2\sigma G_d(a_1, a_3, a_3) \\ &\quad +2\sigma G_d(a_1, a_2, a_2) \end{aligned}$$

$$\begin{aligned} G_d(a_2, a_3, a_3) &\leq 2\sigma(0) + 2\sigma G_d(a_1, a_2, a_2) + 2\sigma G_d(a_2, a_3, a_3) \\ &\quad +2\sigma G_d(a_1, a_2, a_2) \end{aligned}$$

$$(1-2\sigma)G_d(a_2, a_3, a_3) \leq 4\sigma G_d(a_1, a_2, a_2)$$

$$G_d(a_2, a_3, a_3) \leq \frac{4\sigma}{1-2\sigma}\mu G_d(a_0, a_1, a_1), \text{ From (3.3.4)}$$

$$\begin{aligned} G_d(a_2, a_3, a_3) &\leq \frac{4\sigma}{1-4\sigma}\mu G_d(a_0, a_1, a_1), \ \because \frac{1}{1-2\sigma} \leq \frac{1}{1-4\sigma} \\ G_d(a_2, a_3, a_3) &\leq \mu^2 G_d(a_0, a_1, a_1) \end{aligned} \quad (3.3.5)$$

Now for $a_3 \in X$ consider

$$G_d(a_0, a_3, a_3) \leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + G_d(a_2, a_3, a_3)$$

$$\begin{aligned} G_d(a_0, a_3, a_3) &\leq (1+\mu+\mu^2)G_d(a_0, a_1, a_1) \\ G_d(a_0, a_3, a_3) &\leq (1+\mu+\mu^2)(1-\mu)r, \ \because \text{ From (3.3.2)} \end{aligned}$$

$$G_d(a_0, a_3, a_3) \leq (1-\mu^3)r$$
$$G_d(a_0, a_3, a_3) \leq r$$

Therefore $a_3 \in \overline{B_G(a_0, r)}$. Now let $a_4,\ a_5,........,\ a_j \in \overline{B_G(a_0, r)}$, then foloowing relation holds for $j \in N$,

$$G_d(a_{j-1}, a_j, a_j) \leq \mu^{j-1} G_d(a_0, a_1, a_1) \tag{3.3.6}$$

By mathematical induction for $j+1 \in N$, and let $j \in N$ is odd then,

$$G_d(a_j, a_{j+1}, a_{j+1}) \leq G_d(\Gamma a_{j-1}, \Delta a_j, \Delta a_j)$$

$$\begin{aligned} G_d(a_j, a_{j+1}, a_{j+1}) \leq\ & \sigma[2G_d(a_{j-1}, \Delta a_j, \Delta a_j) + 2G_d(a_j, \Delta a_j, \Delta a_j) \\ & +2G_d(a_j, \Gamma a_{j-1}, \Gamma a_{j-1})] \end{aligned}$$

$$\begin{aligned} G_d(a_j, a_{j+1}, a_{j+1}) \leq\ & 2\sigma G_d(a_{j-1}, a_{j+1}, a_{j+1}) + 2\sigma G_d(a_j, a_{j+1}, a_{j+1}) \\ & +2\sigma G_d(a_j, a_j, a_j) \end{aligned}$$

$$\begin{aligned} G_d(a_j, a_{j+1}, a_{j+1}) \leq\ & 2\sigma G_d(a_{j-1}, a_j, a_j) + 2\sigma G_d(a_j, a_{j+1}, a_{j+1}) \\ & +2\sigma G_d(a_j, a_{j+1}, a_{j+1}) + 2\sigma(0) \end{aligned}$$

$$(1-4\sigma) G_d(a_j, a_{j+1}, a_{j+1}) \leq 2\sigma G_d(a_{j-1}, a_j, a_j)$$
$$G_d(a_j, a_{j+1}, a_{j+1}) \leq \frac{2\sigma}{1-4\sigma} G_d(a_{j-1}, a_j, a_j)$$
$$G_d(a_j, a_{j+1}, a_{j+1}) \leq \frac{4\sigma}{1-4\sigma} G_d(a_{j-1}, a_j, a_j),\ \because 2\sigma \leq 4\sigma$$

$$G_d(a_j, a_{j+1}, a_{j+1}) \leq \mu.\mu^{j-1} G_d(a_0, a_1, a_1), \text{ From (3.3.6)}$$
$$G_d(a_j, a_{j+1}, a_{j+1}) \leq \mu^j G_d(a_0, a_1, a_1) \tag{3.3.7}$$

Similarly by rectangular property of $G-metric\ space$ consider for $a_{j+1} \in X$

$$G_d(a_0, a_j, a_j) \leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + ... + G_d(a_j, a_{j+1}, a_{j+1})$$

$$G_d(a_0, a_j, a_j) \leq (1 + \mu + \mu^2 + ... + \mu^j) G_d(a_0, a_1, a_1)$$

$$G_d(a_0, a_j, a_j) \leq (1 + \mu + \mu^2 + ... + \mu^j)(1 - \mu)r, \quad \because \text{ From (3.3.2)}$$

$$G_d(a_0, a_j, a_j) \leq (1 - \mu^{j+1})r$$
$$G_d(a_0, a_j, a_j) \leq r$$

Hence $a_{j+1} \in \overline{B_G(a_0, r)}$, for $j + 1 \in N$. Hence $\forall\ n \in N$, $a_n \in \overline{B_G(a_0, r)}$. Now we check that the sequence $\{a_n\} \subseteq \overline{B_G(a_0, r)}$ is Cauchy sequence. For this we consider m, $n \in Z$ and $m > n$ such that,

$$G_d(a_n, a_m, a_m) \leq G_d(a_n, a_{n+1}, a_{n+1}) + G_d(a_{n+1}, a_{n+2}, a_{n+2})$$
$$+ ..... + G_d(a_{m-2}, a_{m-1}, a_{m-1}) + G_d(a_{m-1}, a_m, a_m)$$

$$G_d(a_n, a_m, a_m) \leq \mu^n(1 + \mu + \mu^2 + ... + \mu^{m-n-1})G_d(a_0, a_1, a_1)$$
$$G_d(a_n, a_m, a_m) \leq \mu^n(\frac{1 - \mu^{m-n}}{1 - \mu})G_d(a_0, a_1, a_1) \tag{3.3.8}$$
$$G_d(a_n, a_m, a_m) \leq \frac{\mu^n}{1 - \mu}G_d(a_0, a_1, a_1)$$

Now if $n \longrightarrow \infty$, then $\mu^n \longrightarrow 0$ because $\mu \in [0, 1)$,

$$G_d(a_n, a_m, a_m) \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

Hence $\{a_n\} \subseteq \overline{B_G(a_0, r)}$ is Cauchy sequence. Therefore $\exists$ a point $a^* \in \overline{B_G(a_0, r)}$ such that

$$\underset{n \longrightarrow \infty}{Lim} G_d(a_n, a^*, a^*) = \underset{n \longrightarrow \infty}{Lim} G_d(a^*, a_n, a_n) = 0$$

Therefore $a^* \in \overline{B_G(a_0, r)}$ is limit point of the sequnce $\{a_n\} \subseteq \overline{B_G(a_0, r)}$. Now to show $a^* \in \overline{B_G(a_0, r)}$ is common fixed point of dominted mappings Γ and Δ. Consider for dominated mapping Γ,

$$G_d(a^*, \Gamma a^*, \Gamma a^*) \leq G_d(a^*, a_{2n}, a_{2n}) + G_d(a_{2n}, \Gamma a^*, \Gamma a^*)$$

$$G_d(a^*, \Gamma a^*, \Gamma a^*) \leq G_d(a^*, a_{2n}, a_{2n}) + G_d(\Gamma a^*, \Delta a_{2n-1}, \Delta a_{2n-1})$$

$$G_d(a^*, \Gamma a^*, \Gamma a^*) \leq G_d(a^*, a_{2n}, a_{2n}) + \sigma[2G_d(a^*, \Delta a_{2n-1}, \Delta a_{2n-1})$$
$$+ 2G_d(a_{2n-1}, \Gamma a^*, \Gamma a^*) + 2G_d(a_{2n-1}, \Delta a_{2n-1}, \Delta a_{2n-1})]$$

$$G_d(a^*, \Gamma a^*, \Gamma a^*) \leq G_d(a^*, a_{2n}, a_{2n}) + 2\sigma G_d(a^*, a_{2n}, a_{2n}) + 2\sigma G_d(a_{2n-1}, \Gamma a^*, \Gamma a^*) + 2\sigma G_d(a_{2n-1}, a_{2n}, a_{2n})$$

$$G_d(a^*, \Gamma a^*, \Gamma a^*) \leq G_d(a^*, a_{2n}, a_{2n}) + 2\sigma G_d(a^*, a_{2n}, a_{2n})$$

$$G_d(a^*, \Gamma a^*, \Gamma a^*) \leq G_d(a^*, a_{2n}, a_{2n}) + 2\sigma G_d(a^*, a_{2n}, a_{2n}) + 2\sigma G_d(a_{2n-1}, a^*, a^*) + 2\sigma G_d(a^*, \Gamma a^*, \Gamma a^*) + 2\sigma G_d(a_{2n-1}, a^*, a^*) + 2\sigma G_d(a^*, a_{2n}, a_{2n})$$

$$(1 - 2\sigma) G_d(a^*, \Gamma a^*, \Gamma a^*) \leq 5\sigma G_d(a^*, a_{2n}, a_{2n}) + 4\sigma G_d(a_{2n-1}, a^*, a^*)$$

$$(1 - 2\sigma) G_d(a^*, \Gamma a^*, \Gamma a^*) \leq 0, \text{ When } n \longrightarrow \infty$$

$$G_d(a^*, \Gamma a^*, \Gamma a^*) \leq 0, \text{ When } n \longrightarrow \infty. \because (1 - 2\sigma) \neq 0$$

As $G_d(a^*, \Gamma a^*, \Gamma a^*) \nless 0$ then the only possibility left is,

$$G_d(a^*, \Gamma a^*, \Gamma a^*) = 0$$

Also as $G - metric\ space$ is symmetric then,

$$G_d(\Gamma a^*, a^*, a^*) = G_d(a^*, \Gamma a^*, \Gamma a^*) = 0$$

$$\Gamma a^* = a^*$$

Hence $a^* \in \overline{B_G(a_0, r)}$ is fixed point of dominated mapping $\Gamma : X \longrightarrow X$. Now consider for dominated mapping Δ,

$$G_d(a^*, \Delta a^*, \Delta a^*) \leq G_d(a^*, a_{2n-1}, a_{2n-1}) + G_d(a_{2n-1}, \Delta a^*, \Delta a^*)$$

$$G_d(a^*, \Delta a^*, \Delta a^*) \leq G_d(a^*, a_{2n-1}, a_{2n-1}) + G_d(\Gamma a_{2n-2}, \Delta a^*, \Delta a^*)$$

$$G_d(a^*, \Delta a^*, \Delta a^*) \leq G_d(a^*, a_{2n-1}, a_{2n-1}) + \sigma[2G_d(a_{2n-2}, \Delta a^*, \Delta a^*) + 2G_d(a^*, \Gamma a_{2n-2}, \Gamma a_{2n-2}) + 2G_d(a^*, \Delta a^*, \Delta a^*)]$$

$$G_d(a^*, \Delta a^*, \Delta a^*) \leq G_d(a^*, a_{2n-1}, a_{2n-1}) + 2\sigma G_d(a_{2n-2}, a^*, a^*) + 2\sigma G_d(a^*, \Delta a^*, \Delta a^*) + 2\sigma G_d(a^*, a_{2n-1}, a_{2n-1}) + 2\sigma G_d(a^*, \Delta a^*, \Delta a^*)$$

$$(1-4\sigma)G_d(a^*,\Delta a^*,\Delta a^*) \leq (1+2\sigma)G_d(a^*,a_{2n-1},a_{2n-1}) + 2\sigma G_d(a_{2n-2},a^*,a^*)$$

$$\begin{aligned} G_d(a^*,\Delta a^*,\Delta a^*) &\leq G_d(a^*,a_{2n-1},a_{2n-1}) + 2\sigma G_d(a_{2n-2},\Delta a^*,\Delta a^*) \\ &\quad +2\sigma G_d(a^*,a_{2n-1},a_{2n-1}) + 2\sigma G_d(a^*,\Delta a^*,\Delta a^*) \end{aligned}$$

$$\begin{aligned} (1-4\sigma)G_d(a^*,\Delta a^*,\Delta a^*) &\leq 0, \text{ When } n \longrightarrow \infty \\ G_d(a^*,\Delta a^*,\Delta a^*) &\leq 0, \text{ When } n \longrightarrow \infty. \ \because (1-4\sigma) \neq 0 \end{aligned}$$

As $G_d(a^*,\Delta a^*,\Delta a^*) \not< 0$ so the only possibility left is,

$$G_d(a^*,\Delta a^*,\Delta a^*) = 0$$

Also as $G-metric\ space$ is symmetric then,

$$\begin{aligned} G_d(\Delta a^*,a^*,a^*) &= G_d(a^*,\Delta a^*,\Delta a^*) = 0 \\ \Delta a^* &= a^* \end{aligned}$$

Hence $a^* \in \overline{B_G(a_0,r)}$ is fixed point of dominated mapping $\Delta : X \longrightarrow X$. As

$$\Gamma a^* = \Delta a^* = a^*$$

Therefore $a^* \in \overline{B_G(a_0,r)}$ is common fixed point of dominated mapping $\Gamma, \Delta : X \longrightarrow X$. ∎

Example 46 *If for a set $X = [0,\infty)$, a mapping $G_d : X \times X \times X \longrightarrow X$, $\forall\ a,b,c \in X$ defined by,*

$$G_d(a,b,c) = a+b+c \tag{3.3.9}$$

then $(X;\leq;G)$ is symmetric and complete $G-$metric like space. Let mappings $\Gamma, \Delta : X \to X$ are defined by,

$$\Gamma a = \begin{cases} \frac{a}{4} & if\ a \in [0,1] \\ a+\frac{1}{4} & if\ a \in (1,\infty) \end{cases}$$

$$\Delta a = \begin{cases} \frac{a}{8} & if\ a \in [0,1] \\ a+\frac{1}{8} & if\ a \in (1,\infty) \end{cases}$$

Obviously Γ *and* Δ *are dominated mappings on* $[0,1]$ *but not dominated outside of* $[0,1]$. *Let* $a_0 = \frac{1}{4}$ *and* $r = \frac{9}{4}$ *such that* $\overline{B_G(a_0, r)} = \overline{B_G(\frac{1}{4}, \frac{9}{4})} = [0,1]$. *Also let* $\sigma = \frac{1}{10} \in [0, \frac{1}{8})$ *and* $\mu = \frac{4\sigma}{1-4\sigma} = \frac{4}{6}$ *so to get,*

$$\begin{aligned}(1-\mu)r &= (1-\frac{4}{6})\frac{9}{4}\\ (1-\mu)r &= (\frac{2}{6})\frac{9}{4}\\ (1-\mu)r &= \frac{3}{4}\end{aligned} \tag{3.3.10}$$

Also as,

$$\begin{aligned}G_d(a_0, a_1, a_1) = \frac{1}{4} + 2\Gamma(\frac{1}{4})\\ G_d(a_0, a_1, a_1) = \frac{3}{8}\end{aligned} \tag{3.3.11}$$

From (3.3.10) *and* (3.3.11),

$$\begin{aligned}\frac{3}{8} &< \frac{3}{4}\\ G_d(a_0, a_1, a_1) &< (1-\mu)r\end{aligned}$$

Now to check contractive condition is either satisfied or not on the closed ball $[0,1]$, *let for* $a, b, c \in [0,1]$,

$$\begin{aligned}G_d(\Gamma a, \Delta b, \Delta c) &= G_d(\frac{a}{4}, \frac{b}{8}, \frac{c}{8})\\ G_d(\Gamma a, \Delta b, \Delta c) &= \frac{a}{4} + \frac{b}{8} + \frac{c}{8} \quad \because \text{ From } (3.3.9)\end{aligned}$$

$$G_d(\Gamma a, \Delta b, \Delta c) = \frac{1}{4}(a+b+c) - \frac{1}{8}(b+c) \tag{3.3.12}$$

Also let $R = \sigma[G_d(a, \Delta b, \Delta b) + G_d(a, \Delta c, \Delta c) + G_d(b, \Gamma a, \Gamma a) + G_d(b, \Delta c, \Delta c) + G_d(c, \Gamma a, \Gamma a) + G_d(c, \Delta b, \Delta b)]$ *then,*

$$R = \frac{1}{10}[a + 2\Delta b + a + 2\Delta c + b + 2\Gamma a + b + 2\Delta c + c + 2\Gamma a + c + 2\Delta b]$$

$$\begin{aligned}R = \frac{1}{10}[2(a+b+c) + 4(\frac{a}{4} + \frac{b}{8} + \frac{c}{8})]\\ R = \frac{3}{10}(a+b+c) - \frac{1}{20}(b+c)\end{aligned} \tag{3.3.13}$$

As clearly,

$$\frac{1}{4}(a+b+c) \leq \frac{3}{10}(a+b+c) \text{ and } -\frac{1}{8}(b+c) \leq -\frac{1}{20}(b+c)$$

Then

$$\frac{1}{4}(a+b+c)-\frac{1}{8}(b+c)\leq\frac{3}{10}(a+b+c)-\frac{1}{20}(b+c)$$

Therefore from (3.3.12) *and* (3.3.13),

$$G_d(\Gamma a,\Delta b,\Delta c)\leq R$$

Hence contractive condition is satisfied on the close ball $[0,1]$. *Now for* $a,b,c\in(1,\infty)$,

$$G_d(\Gamma a,\Delta b,\Delta c)=G_d(a+\frac{1}{5},b+\frac{1}{7}\ ,c+\frac{1}{7}\ )$$

$$G_d(\Gamma a,\Delta b,\Delta c)=(a+b+c)+\frac{1}{2} \tag{3.3.14}$$

Also as,

$$R = \sigma[2(a+b+c)+4(\Gamma a+\Delta b+\Delta c)]$$

$$R = \frac{1}{10}[2(a+b+c)+4(\Gamma a+\Delta b+\Delta c)]$$

$$R=\frac{1}{10}[2(a+b+c)+4(a+\frac{1}{4}+b+\frac{1}{8}+c+\frac{1}{8})]$$

$$R=\frac{1}{10}[6(a+b+c)+4(\frac{1}{4}+\frac{1}{8}+\frac{1}{8})]$$

$$R=\frac{3}{5}(a+b+c)+\frac{1}{5} \tag{3.3.15}$$

As clearly,

$$(a+b+c)\geq\frac{3}{5}(a+b+c)\ and\ \frac{1}{2}\geq\frac{1}{5}$$

Then

$$(a+b+c)+\frac{1}{2}\geq\frac{3}{5}(a+b+c)+\frac{1}{5}$$

Hence from (3.3.14) *and* (3.3.15),

$$G_d(\Gamma a,\Delta b,\Delta c)\geq R$$

Hence contractive condition is not satisfied on the close ball $[0,1]$. *It shows that all conditions of Chatterjea common fix point theorem satisfied. Moreover* $0\in[0,1]$ *common fix point of mappings* Γ *and* Δ, *i.e.* $\Delta 0=\Gamma 0=0$.

Corollary 47 *Suppose* $(X;\leq;G)$ *be a symmetric and ordered complete* $G-$ *metric like space and* $\Gamma,\Delta : X \to X$ *are any two dominated mappings and* a_0, a, $b,c \in X$, $r > 0$. *Suppose that* $\exists\ \sigma \in [0,\frac{1}{8})$ *such that,*

$$\begin{aligned} G_d(\Gamma a,\Delta b,\Delta c) &\leq \sigma[G_d(a,\Delta b,\Delta b)+G_d(a,\Delta c,\Delta c) \\ &+G_d(b,\Gamma a,\Gamma a)+G_d(b,\Delta c,\Delta c) \\ &++G_d(c,\Gamma a,\Gamma a)+G_d(c,\Delta b,\Delta b)] \end{aligned} \tag{3.3.16}$$

$\forall$ *comparabe elements* $a,b,c \in \overline{B_G(a_0,r)} \subseteq X$

$$G_d(a_0,a_1,a_1) = G_d(a_0,\Gamma a_0,\Gamma a_0) \leq (1-\mu)r \tag{3.3.17}$$

where $\mu = \frac{4\sigma}{1-4\sigma}$. *If for non-increasing sequence* $\{a_n\}$ *in* $\overline{B_G(a_0,r)}$, $\{a_n\} \to v$ *then* $\exists$ *a unique* $a^* \in \overline{B_G(a_0,r)}$ *such that* $\Gamma a^* = \Delta a^* = a^*$ *and moreover* $G_d(a^*,a^*,a^*) = 0$.

Proof. In the main result Chatterjea common fixed point of dominated mappings take $\Gamma = \Delta$ to get unique fixed point $\Gamma a^* = a^*$. ■

In the next theorem uniqueness of Chatterjea common fixed point in *Theorem* 13 is proved.

3.3.1 Uniqueness of Chatterjea Common Fixed Point

Theorem 48 *Suppose* $(X;\leq;G)$ *be an ordered and symmetric complete G-metric like space and* Γ *and* Δ *are any two self dominated mappings and* a_0, a, $b,c \in X$. *Suppose that* $\exists$ $\sigma \in [0,\frac{1}{8})$ *such that,*

$$\begin{aligned} G_d(\Gamma a,\Delta b,\Delta c) &\leq \sigma[G_d(a,\Delta b,\Delta b)+G_d(a,\Delta c,\Delta c) \\ &+G_d(b,\Gamma a,\Gamma a)+G_d(b,\Delta c,\Delta c) \\ &++G_d(c,\Gamma a,\Gamma a)+G_d(c,\Delta b,\Delta b)] \end{aligned} \tag{3.3.18}$$

$\forall$ *comparabe elements* $a,b,c \in X$

$$G_d(a_0,a_1,a_1) = G_d(a_0,\Gamma a_0,\Gamma a_0) \leq (1-\mu)r \tag{3.3.19}$$

where $\mu = \frac{4\sigma}{1-4\sigma}$. *If for non-increasing sequence* $\{a_n\}$ *in* X, $\{a_n\} \to v$ *then* $\exists$ *a unique* $a^* \in X$ *such that* $\Gamma a^* = \Delta a^* = a^*$ *and moreover* $G_d(a^*,a^*,a^*) = 0$.

Proof. To prove uniquness of Chatterjea common fixed point, consider a^*, $b^* \in X$ are any two common fixed points of self dominated mappings Γ and Δ, such that $a^* \neq b^*$. Then there arises two cases for a^*, $b^* \in X$. In first case let a^*, b^* are comparable say $a^* \leq b^*$. As $a^*$and b^* are common fixed point of dominated mappings φ and δ then,

$$\Gamma a^* = \Delta a^* = a^* \text{ and } \Gamma b^* = \Delta b^* = b^* \tag{3.3.20}$$

Now consider the relation,

$$G_d(a^*, b^*, b^*) = G_d(\Gamma a^*, \Delta b^*, \Delta b^*), \text{ From (3.3.20)}$$

$$G_d(a^*, b^*, b^*) \leq \sigma[2G_d(a^*, \Delta b^*, \Delta b^*) + 2G_d(b^*, \Delta b^*, \Delta b^*) + 2G_d(b^*, \Gamma a^*, \Gamma a^*)]$$

$$\begin{aligned} G_d(a^*, b^*, b^*) &\leq \sigma[2G_d(a^*, b^*, b^*) + 2G_d(b^*, b^*, b^*) \\ &\quad +2G_d(b^*, a^*, a^*)], \text{ From (3.3.20)} \end{aligned}$$

$$(1-4\sigma)G_d(a^*, b^*, b^*) \leq 0$$

$$G_d(a^*, b^*, b^*) \leq 0, \ \because 1 - 4\sigma \in (\frac{1}{2}, 1] \ \forall \sigma \in [0, \frac{1}{8})$$

As $G_d(a^*, b^*, b^*) \nless 0$, then only possibilty left is,

$$G_d(a^*, b^*, b^*) = 0$$

Also as $G - metric\ space$ is symmetric then,

$$\begin{aligned} G_d(a^*, b^*, b^*) &= G_d(b^*, a^*, a^*) = 0 \\ a^* &= b^* \end{aligned}$$

It is contradiction ($\because a^* \neq b^*$). So our supposition is wrong. Hence common fixed point is unique. Hence for comparable $a^*, b^* \in X$ commom fixed point is unique. Now in second case if $a^*, b^* \in X$ are not comparable then $\exists\ w_0 \in X$ such that $w_0 \leq a^*$ and $w_0 \leq b^*$. Then clearly $w_0 \in X$ is lower bound of both $a^*, b^* \in X$. Now construct an itterative sequence $\{w_q\} \subseteq X$ for $q \in N$ such that,

$$w_{2n-1} = \Delta w_{2n-2} \text{ and } w_{2n} = \Gamma w_{2n-1} \tag{3.3.21}$$

As Γ and Δ are dominated self mappingts then,

$$..... \leq w_{q+1} \leq w_q \leq w_{q-1} \leq .... \leq w_2 \leq w_1 \leq w_0 \leq a^*$$

And

$$..... \leq w_{q+1} \leq w_q \leq w_{q-1} \leq .... \leq w_2 \leq w_1 \leq w_0 \leq b^*$$

Consider the relation,

$$\begin{aligned} G_d(w_1, w_2, w_2) &= G_d(\Delta w_0, \Gamma w_1, \Gamma w_1), \text{ From (3.3.21)} \\ G_d(w_1, w_2, w_2) &= G_d(\Gamma w_1, \Delta w_0, \Delta w_0) \end{aligned}$$

$$\begin{aligned} G_d(w_1, w_2, w_2) &\leq \sigma[2G_d(w_1, \Delta w_0, \Delta w_0) + 2G_d(w_0, \Delta w_0, \Delta w_0) \\ &\quad +2G_d(w_0, \Gamma w_1, \Gamma w_1)] \end{aligned}$$

$$\begin{aligned} G_d(w_1, w_2, w_2) &\leq 2\sigma G_d(w_1, w_1, w_1) + 2\sigma G_d(w_0, w_1, w_1) + 2\sigma G_d(w_0, w_2, w_2) \\ G_d(w_1, w_2, w_2) &\leq 2\sigma G_d(w_0, w_1, w_1) + 2\sigma G_d(w_0, w_1, w_1) + 2\sigma G_d(w_1, w_2, w_2) \end{aligned}$$

$$(1 - 2\sigma)G_d(w_1, w_2, w_2) \leq 4\sigma G_d(w_0, w_1, w_1)$$

$$G_d(w_1, w_2, w_2) \leq \frac{4\sigma}{1 - 2\sigma} G_d(w_0, w_1, w_1)$$

$$G_d(w_1, w_2, w_2) \leq \frac{4\sigma}{1 - 4\sigma} G_d(w_0, w_1, w_1), \ \because \frac{1}{1 - 2\sigma} \leq \frac{1}{1 - 4\sigma}$$

$$G_d(w_1, w_2, w_2) \leq \mu G_d(w_0, w_1, w_1)$$

Following in same way for $q, q + 1 \in N$ let following relation holds,

$$G_d(w_q, w_{q+1}, w_{q+1}) \leq \mu^q G_d(w_0, w_1, w_1) \tag{3.3.22}$$

Consider for odd $q \in N$,

$$\begin{aligned} G_d(a^*, w_q, w_q) &= G_d(\Gamma a^*, \Delta w_{q-1}, \Delta w_{q-1}) \\ G_d(a^*, w_q, w_q) &\leq \sigma[2G_d(a^*, \Delta w_{q-1}, \Delta w_{q-1}) + 2G_d(w_{q-1}, \Gamma a^*, \Gamma a^*) \\ &\quad +2G_d(w_{q-1}, \Delta w_{q-1}, \Delta w_{q-1})] \end{aligned}$$

$$G_d(a^*, w_q, w_q) \leq 2\sigma G_d(a^*, w_q, w_q) + 2\sigma G_d(w_{q-1}, w_q, w_q) + 2\sigma G_d(w_{q-1}, a^*, a^*)$$

$$\begin{aligned} G_d(a^*, w_q, w_q) &\leq 2\sigma G_d(a^*, w_q, w_q) + 2\sigma G_d(w_{q-1}, w_q, w_q) + 2\sigma G_d(w_q, a^*, a^*) \\ &+2\sigma G_d(w_{q-1}, w_q, w_q) \end{aligned}$$

$$(1-4\sigma)G_d(a^*, w_q, w_q) \leq 4\sigma G_d(w_{q-1}, w_q, w_q)$$

$$\begin{aligned} G_d(a^*, w_q, w_q) &\leq \frac{4\sigma}{1-4\sigma} G_d(w_{q-1}, w_q, w_q) \\ G_d(a^*, w_q, w_q) &\leq \mu G_d(w_{q-1}, w_q, w_q) \end{aligned}$$

$$G_d(a^*, w_q, w_q) \leq \mu^q G_d(w_0, w_1, w_1), \text{ From (3.3.22)}$$

Taking limit $q \longrightarrow \infty$, gives $\mu^q \longrightarrow 0$, $\because \mu \in [0, 1)$ then,

$$G_d(a^*, w_q, w_q) \leq$$

As $G_d(a^*, w_q, w_q) \nless 0$ then,

$$G_d(a^*, w_q, w_q) = 0$$

As $G - metric\ space$ is symmetric then,

$$G_d(w_q, a^*, a^*) = G_d(a^*, w_q, w_q) = 0$$

Similarly it can be shown that,

$$G_d(w_q, b^*, b^*) = G_d(b^*, w_q, w_q) = 0$$

Now finally consider by rectangular property of $G - metric\ spaces$,

$$\begin{aligned} G_d(a^*, b^*, b^*) &\leq G_d(a^*, w_q, w_q) + G_d(w_q, b^*, b^*) \\ G_d(a^*, b^*, b^*) &\leq 0, \text{ as } q \longrightarrow \infty \end{aligned}$$

As by definition of $G - metric\ spaces$,

$$G_d(a^*, b^*, b^*) = 0, \text{ as } q \longrightarrow \infty$$

By symmetry $G - metric\ space$,

$$G_d(b^*, a^*, a^*) = G_d(a^*, b^*, b^*) = 0, \text{ as } q \longrightarrow \infty$$

$$a^* = b^*$$

It is contradiction ($\because a^* \neq b^*$). So our supposition is wrong. Hence common fixed point is unique. So if a^* and b^* are not comparable then common fixed point is unique. Hence Chatterjea common fixed point for double self dominated mappings is unique. ■

Proof of the following corollary is similar to the proof of *Theorem* 45 but without discussing the ordered property of $G-$*metric like spaces*.

Corollary 49 *Suppose* $(X;\leq;G)$ *be a symmetric and complete G-metric like space and* $\Gamma,\Delta : X \to X$ *are any two dominated mappings and* a_0, a, $b, c \in X$, $r > 0$. *Suppose that* $\exists$ $\sigma \in [0, \frac{1}{8})$ *such that the following conditions holds*

$$\begin{aligned} G_d(\Gamma a, \Delta b, \Delta c) \leq\ & \sigma[G_d(a, \Delta b, \Delta b) + G_d(a, \Delta c, \Delta c) \\ & + G_d(b, \Gamma a, \Gamma a) + G_d(b, \Delta c, \Delta c) \\ & + + G_d(c, \Gamma a, \Gamma a) + G_d(c, \Delta b, \Delta b)] \end{aligned} \tag{3.3.23}$$

$\forall$ *comparabe elements* $a, b, c \in \overline{B_G(a_0, r)} \subseteq X$

$$G_d(a_0, a_1, a_1) = G_d(a_0, \Gamma a_0, \Gamma a_0) \leq (1-\mu)r \tag{3.3.24}$$

where $\mu = \frac{4\sigma}{1-4\sigma}$. *If for non-increasing sequence* $\{a_n\}$ *in* $\overline{B_G(a_0, r)}$, $\{a_n\} \to v$ *then* $\exists$ *a point* $a^* \in \overline{B_G(a_0, r)}$ *such that* $\Gamma a^* = \Delta a^* = a^*$ *and moreover* $G_d(a^*, a^*, a^*) = 0$.

3.3.2 Error Bounds

In this section errors approximations and their related example are discused.

Corollary 50 *From Theorem* 45, *iterative sequence* (3.3.3), *with arbitrary* $a_0 \in \overline{B_G(a_0, r)} \subseteq X$, *converges to unique common fixed point* a^* *of dominated self mappings* Γ *and* Δ. *Error estimates are the* ***prior estimate***

$$G_d(a_n, a^*, a^*) \leq \frac{\mu^n}{1-\mu} G_d(a_0, a_1, a_1) \tag{3.3.25}$$

and the ***posterior estimate***

$$G_d(a_n, a^*, a^*) \leq \frac{\mu}{1-\mu} G_d(a_{n-1}, a_n, a_n) \tag{3.3.26}$$

Proof. As from relation (3.3.8) of *Theorem* 45,

$$G_d(a_n, a_m, a_m) \leq \mu^n(\frac{1-\mu^{m-n}}{1-\mu})G_d(a_0, a_1, a_1)$$

As the sequence $\{a_m\}$ is convergent to $a^* \in \overline{B_G(a_0, r)} \subseteq X$, then by taking $m \longrightarrow \infty$ gives $a_m \longrightarrow a^*$ and $\mu^{m-n} \longrightarrow 0$. Therefore above relation leads to the *prior estimate* (3.3.25), *i.e*,

$$G_d(a_n, a^*, a^*) \leq \frac{\mu^n}{1-\mu}G_d(a_0, a_1, a_1)$$

Setting $n = 1$ and write b_0 for a_0 and b_1 for a_1 in (3.3.25)

$$G_d(b_1, a^*, a^*) \leq \frac{\mu}{1-\mu}G_d(b_0, b_1, b_1)$$

Letting $b_0 = a_{n-1}$ then $b_1 = \Gamma b_0 = \Gamma a_{n-1} = a_n$ in above relation leads to the *posterior estimate* (3.3.26), *i.e*,

$$G_d(a_n, a^*, a^*) \leq \frac{\mu}{1-\mu}G_d(a_{n-1}, a_n, a_n)$$

■

The prior error bound (3.3.25) can be used at the beginning of the calculation for estimating the required number of steps to obtain a assumed accuracy. While posterior error bound (3.3.26) can be used at intermediate stages or at the end of the calculation. Posterior error bound (3.3.26) is at least as accurate as prior error bound (3.3.25).

Example 51 *If for a set* $X = [0, \infty)$, *a mapping* $G_d : X \times X \times X \longrightarrow X$, $\forall\ a, b, c \in X$ *defined by,*

$$G_d(a, b, c) = a + b + c, \tag{3.3.27}$$

then $(X; \leq; G)$ *is symmetric and complete* $G-$*metric like space. Let mappings* $\Gamma, \Delta : X \rightarrow X$ *are defined by,*

$$\Gamma a = \begin{cases} \frac{a}{4} & \text{if } a \in [0, 1] \\ a + \frac{1}{4} & \text{if } a \in (1, \infty) \end{cases}$$

$$\Delta a = \begin{cases} \frac{a}{8} & \text{if } a \in [0, 1] \\ a + \frac{1}{8} & \text{if } a \in (1, \infty) \end{cases}$$

Obviously Γ *and* Δ *are dominated mappings inside of* $[0,1]$ *but not dominated outside of* $[0,1]$. *Let* $a_0 = \frac{1}{4}$ *and* $r = \frac{9}{4}$ *such that* $\overline{B_G(a_0, r)} = \overline{B_G(\frac{1}{4}, \frac{9}{4})} = [0,1]$. *Also let* $\sigma = \frac{1}{10} \in [0, \frac{1}{8})$ *and* $\mu = \frac{4\sigma}{1-4\sigma} = \frac{2}{3}$. *Construct a picard iterative* $\{a_n\}$ *sequence by taking* $a_0 = \frac{1}{4}$ *as initial guess as,*

$$\begin{aligned} a_1 &= \Gamma a_0 = \frac{a_0}{4^1.8^0} \text{ and } a_2 = \Delta a_1 = \frac{a_0}{4^1.8^1} \\ a_3 &= \Gamma a_2 = \frac{a_0}{4^2.8^1} \text{ and } a_4 = \Delta a_3 = \frac{a_0}{4^2.8^2} \end{aligned}$$

....

....

$$\begin{aligned} a_{2m-1} &= \Gamma a_{2m-2} = \frac{a_0}{4^m.8^{m-1}} \text{ and } a_{2m} = \Delta a_{2m-1} = \frac{a_0}{4^m.8^m} \\ a_{2m-1} &= \Gamma a_{2m-2} = \frac{1}{4^{m+1}.8^{m-1}} \text{ and } a_{2m} = \Delta a_{2m-1} = \frac{1}{4^{m+1}.8^m} \end{aligned}$$

Consider $2m - 1 = n$ *then for odd* $n \in N$ *gives,*

$$\begin{aligned} a_n &= \Gamma a_{n-1} = \frac{1}{4^{\frac{n+3}{2}}.8^{\frac{n-1}{2}}} \text{ and } a_{n+1} = \Delta a_n = \frac{1}{4^{\frac{n+3}{2}}.8^{\frac{n+1}{2}}} \\ a_n &= \frac{1}{2\sqrt{2}.(4\sqrt{2})^n} \text{ and } a_{n+1} = \frac{1}{16\sqrt{2}.(4\sqrt{2})^n} \end{aligned} \tag{3.3.28}$$

Now as,

$$G_d(a_n, a^*, a^*) = a_n + 2a^*, \quad \because \text{ From } (3.3.27)$$

As picard sequence $\{a_n\}$ *satisfies all the conditions of Chattergea fixed point theorem* 45 *as in example* 46, *then If* $n \to \infty$ *so* $a_n \to a^*$ *i.e* $a_n \approx a^*$. *Then*

$$G_d(x_n, x^*, x^*) = 3a_n \tag{3.3.29}$$

Also as,

$$\begin{aligned} G_d(a_0, a_1, a_1) &= G_d(a_0, \Gamma a_0, \Gamma a_0) = a_0 + 2\Gamma a_0 \\ G_d(a_0, a_1, a_1) &= \frac{1}{4} + 2\Gamma\frac{1}{4} = \frac{1}{4} + \frac{2}{16} \\ G_d(a_0, a_1, a_1) &= \frac{3}{8} \end{aligned} \tag{3.3.30}$$

And finally as,

$$G_d(a_n, a^*, a^*) \leq \frac{\mu^n}{1-\mu} G_d(a_0, a_1, a_1)$$

$$3a_n \leq 3.(\frac{2}{3})^n.\frac{3}{8} = \frac{9}{8}.(\frac{2}{3})^n, \ \because \ \textit{From } (3.3.29) \textit{ and } (3.3.30)$$

$$\frac{8}{3}a_n \leq (\frac{2}{3})^n$$

$$1.88561808316413 \leq (3.77123616632825)^n$$

$$\frac{\ln(1.88561808316413)}{\ln(3.77123616632825)} \leq n$$

$$0.4778170138305 \leq n$$

As $n \in N$ *is odd and* $0.4778170138305 \leq n$ *then for* $n = 1, 3, 5, \ldots\ldots$ *picard sequence starts converging to its limit point i.e* $a^* = 0 \in [0,1]$ *such that let for* $n = 3$ (3.3.28) *gives,*

$$a_3 = \frac{1}{2\sqrt{2}.(4\sqrt{2})^3} = 0.001953125 \textit{ and } a_4 = \frac{1}{16\sqrt{2}.(4\sqrt{2})^3} = 0.000244140625$$

And if $n = 5$ *then* (3.3.28) *gives,*

$$a_5 = \frac{1}{2\sqrt{2}.(4\sqrt{2})^5} = 0.000006103515265 \textit{ and } a_6 = \frac{1}{16\sqrt{2}.(4\sqrt{2})^5} = 0.00000762939453125$$

Thus

$$a_5 = \Gamma a_4 \Longrightarrow \Gamma(0.000244140625) = 0.000006103515265$$

and

$$a_6 = \Delta a_5 \Longrightarrow \Delta(0.000006103515265) = 0.00000762939453125$$

Hence common fixed point approximation is,

$$\Gamma(0.000244140625) \approx \Delta(0.000006103515265) \approx 0.00000762939453125$$

Finally when $n \longrightarrow \infty$ *then,*

$$\Gamma(0) = \Delta(0) = 0$$

Remark 52 *Above results not only holds for dominated mapping but also holds for dominating mappings.*

REFERENCES

[1] M. Arshad, A. Azam and S. Hussain, Common Fixed Points of Kannan Type Fuzzy Mappings on closed balls, Appl. Math. Inf. Sci. Lett. 2, No. 1(2013), 7-10.

[2] M. Arshad, A. Shoaib, M. Abbas and A. Azam, Fixed points of pair of Kannan type mappinga on a closed ball in ordered partial metric spaces, Miskolc Mathematical Notes, 14(2013), 769–784.

[3] M. Arshad, A. Azam, M. Abbas, A. Shoaib, fixed poimts results of dominated mappings on a closed ball in ordered partial metric spaces without continuity, U.P.B. Sci. Bull., Series A, 76(2014), 123-134.

[4] M. Arshad, A. Shoaib and I. Beg, Fixed point of pair of contractive dominated mappings on a closed ball in an ordered dislocated metric space, Fixed Point Theory and Appl., 115(2013), 1-15.

[5] M. Arshad, A. Azam and S. Hussain, Common fixed points of Chitterjea type fuzzy mappings on closed balls, Neural Computing & Appl., 21(2011), 313-317.

[6] M. Arshad, A. azam and P. Wetro, Some common fixed point results in cone metric spaces, Fixed Point Theory and Appl., 2009(2009), 1-11.

[7] A. Azam, J. Ahmad and S. Saejung, Common fixed point results for contractive mappings in complex valued metric spaces, Fixed Point Theory and Appl., 67(2014), 1-11.

[8] M. Arshad, A. Shoaib, I. Beg, fixed point on a closed ball in ordered dislocated quasi metric spaces. (Accepted)

[9] M. Arshad, A. Shoaib, and P. Vetro, Common fixed points of a pair of Hardy Rogers Type Mappings on a closed ball in ordered dislocated metric spaces, Journal of Function Spaces and Appl., 2013(2013).

[10] M. Arshad, A. Shoaib and J. Ahmad, fixed point results of locally contractive mappings in ordered quasi-partial metric spaces, The Sci. World. J., 2013(2013).

[11] M. Jleli and B. Samet, Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory & appl., 2012(2012), 1-7.

[12] Z. Mustafa and B. Sims, A new approch to generalized metric spaces, Journal of Nonlinear and Convex Analysis, 7(2006), 289-297.

[13] Z. Mustafa, Common fixed points of weakly compatible mappings in G-metric spaces, Applied Mathematical Sciences, 6(2012), 4589-4600.

[14] Z. Mustafa, H. Obiedat, and F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory and Appl., 2008(2008), 1-12.

[15] Z. Mustafa and H. obiedat, Fixed point results on a non-symmetric G-metric spaces, Jordan Journal of Mathematics and Statictics, 3(2010), 65-79.

[16] Z. Mustafa, H. Aydi and E. Karapinar, On common fixed points in G-metric spaces using (E.A) property, Computers and Maths with Appl., 64(2012), 1944-1956.

[17] R. Shrivastave, Z. K. Ansari and M. Sharma, Some results on fixed points in dislocated and dislocated quasi metric spaces, Journal of Advanced Studies in Topology, 3(2012), 25-31.

[18] Z. Shu-hang and G. Feng, Some new fixed point theorems in G-metric spaces, Journal of Hangzhou Normal University, 11(2012), 47-50.

[19] R. K. Vats, S. Kumar and V. Sihag, Common Fixed Point Theorem for Expansive Mappings in G-Metric Spaces , Journal of Mathematics and Computer Science, 6(2013), 60-71.

[20] M. Abbas and B. E. Rhoades, Common fixed point results for noncommuting mappings without continuity in generalized metric spaces, Appl. Maths. and Computation, 215(2009), 262-269.

[21] M. Abbas, S. H. Khan and T. Nazir, Common fixed points of R-weakly commuting maps in generalized metric spaces, Fixed poin Theory and Appl., 41(2011), 1-11.

[22] M. Abbas, T. Nazir, and P. Vetro, Common fixed point results for three maps in G-metric spacees, Filomat, 25(2011), 1-17.

[23] M. Abbas and S. Z. Nemeth, Finding solutions of implicit complementarity problems by isotonicity of metric projection, Nonlinear Analysis, 75(2012), 2349-2361.

[24] S. K. Chatterjea, Fixed Point Theorems. C.R. Acad. Bulgare Sci. 6(1983), 89-94.

[25] Lj. Gajić and M. Stojaković, On Ćirić generalization of mappings with a contractive iterate at a point in G-metric spaces , Applied Mathematics and Computations, 219(2012), 435-441.

[26] H. Aydi, W. Shatanawi and C. vetro, On generalized week G-contraction mapping in G-metric spaces, Computers & mathematics with Applications, 62(2011), 4222-4229.

[27] W. Shatanawi, Fixed point theory for contractive mappings satisfying $\Phi - maps$ in G-metric spaces, Fixed Point Theory and Appl., 2010(2010), 1-9

[28] R. P. Agarwal and E. Karapinar, Remarks on some coupled fixed point theorems in G-metric spaces, Fixed Point Theory and Appl., 2013(2013), 1-33

[29] R. P. Agarwal, M. Meehan and D. O'Regan, Fixed point theory and applications, Cambridge university Press, Cambridge, 2001.

[30] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to metrix equations, Proc. Amer. Math. Soc., 132(2003), 1435-1443.

[31] B. Samet, C. Vetro, and F. Vetro, Remarks on G-metric spaces, Int. J. Anal., 2013(2013), 1-6.

[32] S. Banach, On operations in the abstract sets and their application to integral equations, Fund. Math., 3(1922), 133–181.

[33] E. Kryeyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, New York, 1978.

[34] A. Kaewcharoen, Common fixed points for four mappings in G-metric spaces, Int. J. Maths. Anal., 6(2012), 2345-2356.

[35] R. Kannan, Some Results on Fixed Points, Bull. Cal. Math. Soc., 60(1968), 71-76.

[36] H. K. Nashine, Coupled common fixed point results in ordered G-metric spaces, J. Nonlinear Sc. Appl., 1(2012), 1-13.

[37] J. J. Nieto and R. Rodrigguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Springer 2006, 22(2005), 223-239.

[38] M. Arshad, fixed point of a pair of dominating mappings on a closed ball in ordered partial metric spaces, J. Adv. Math. Stud., 7(2014), 123-134.

[39] E. Karapinar and Ravi P. Agarwal, Further fixed point results on G-metric spaces, Fixed Point Theorem and Applications, 154(2013), 1-14

[40] Z. Mustafa, A new structure for generalised metric spaces: with application to fixed point theory, Ph.d Thesis, The University of Newcastle Australia, 2005, 1-268

[41] Sumitra Dalal and Dimplekumar Chalishajar, Coupled fixed points results for W-Compatible Maps in symmetric G-Metric spaces, African Journal of Mathematics and Mathematical Sciences, 2(2013), 38-46.