Peristaltic Transport of Jeffrey Fluid in a Rectangular Duct Through a Porous Medium under the Effects of Partial Slip

By

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Fluid mechanics

Jeffrey fluid

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A Thesis
Submitted in the Partial Fulfillment of the
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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

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2018

Declaration

I hereby declare and affirm that this research work neither as a whole nor as a part has been copied out from any source. It is further declared that I have developed this research work entirely on the basis of my personal efforts made under the sincere guidance of my supervisor. No portion of the work presented in this thesis has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

(FEHID ISHTIAQ)
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Dedicated

To

The Holy Prophet
♡AZRAT MUHAMMAD
(صَلَّى اللهُ عَلَيهِ وَسَلَّمْ)

♡

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I feel proud to thank **the Almighty ALLAH** who created us as human beings with the power of reasoning and has showered countless blessings upon us. I praise the Holy Prophet **Hazrat MUHAMMAD** □ who guided us to the way of knowledge and research and whose life is the best role model for the whole of humanity.

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FEHID ISHTIAQ

Preface

Main natural resources of fresh water reservoir are springs, fountains, rivers, and lakes. All rivers do not maintain their aboveground passage throughout the flow journey, from their sources to estuaries. Sometimes, a river vanishes beneath the ground, at one place and emerges above the ground at other. One wonders, how does this underground flow take place? Flow of all sort of fluids (liquids and gases) beneath the crust through sands, soil's particles and aquifers are caused by the "Porosity of the matter" or "Void friction". Porosity of a rock or a matter is a measure of its ability to hold or allow fluids to pass through it. For, in the rocks there are tiny spaces that hold the oil or gas etc. Quantity or capacity of porosity for any material lies between 0% to 100%, for instance the sandstone consists of 8% of the void friction. This means 92% of its composition is solid rock and 8% is an open space, permitting fluids to pass through. Moreover, the characteristic in any material allowing fluids to leave or enter in any vacuum is known as its "Permeability" and denoted by "k". Permeability is affected by the pressure in a rock. A practical unit of measure of permeability is called the "Darcy (d)", named after Henry Darcy.

Moreover, it is adequate enough to state that peristaltic flows are caused by propagation of waves along flexible walls of a rectangular duct or any channel, in which the fluid (air, liquid and gas) is moving. The transport of fluids from one place to another in body follows the mechanism/principle of peristaltic pumping. Besides, peristaltic flows have a significant function in the practical applications of various biomedical apparatuses, like as heart-lung machines etc. Thorough investigations of the biofluids, help physicians and surgeons to diagnose various diseases that arise in a living body due to their flow behavior of non-Newtonian fluid such as blood.

Flow of non-Newtonian fluids through the porous medium has always fancied the geologists for their interesting behaviors. Therefore, a number of research works is easily available for the young and new learners, in which the veteran scientists and engineers have employed their unique innovation and skills, with their own perspective, to achieve their desired goals.

It is a well-known fact that Latham [1] is considered to be the pioneer and is held responsible for introducing the concept of peristalsis. To understand peristaltic motion [1] in diverse situations, several theoretical and experimental attempts have been made. Among all, the early literature is presented by Jaffrin and Shapiro [2]. Most recently, Mahmoud et al. [3] have successfully applied the effects of MHD on peristaltic motion of Jeffrey fluid passing through porous medium in an asymmetric channel by means of Adomian decomposition method. The influences of Hall current and slip condition on MHD flow induced by sinusoidal peristaltic wavy wall in two dimensional viscous fluid through a porous medium for moderately large Reynolds number is reported by Mekheimer et al. [4]. They obtained series solution by regular perturbation method. Vajravelu et al. [5] have analyzed the influence of heat transfer on peristaltic flow of Jeffrey fluid in porous stratum. Also Elmaboud and Mekheimer [6] have studied peristaltic motion of second-order fluid through porous medium. The system of governing nonlinear partial differential equations is solved by using perturbation method up-to second-order. In order to derive out the expression for the pressure gradient the analytic solution has been obtained in the form of a stream function. The peristaltic flow of Jeffrey fluid with variable viscosity through porous medium in an asymmetric channel is investigated by Ellahi et al. [7] whereas Hayat at et al. [8] have obtained an analytic solution for the problem having the impacts of MHD on peristaltic flow of Maxwell fluid in porous medium. Ramesh and Devakar [9] have examined the influences of heat and mass transfer on peristaltic transport of magnetohydrodynamic couple stress fluid through homogeneous porous medium bin a vertical asymmetric channel. The peristaltic fluid flow through porous medium in a cylindrical tube is presented by Shehawey and El Sebai [10]. The simultaneous effects of MHD and slip condition, on peristaltic flow of Newtonian fluid are examined by Ebaid [11]. Tripathi [12] has presented analytically and numerically results on transient peristaltic heat for porous channel of finite length. The peristaltic flow of Newtonian fluid with heat transfer in vertical asymmetric channel through porous medium has been studied by Srinivas and Gayathri [13]. The analytical solution has been obtained in the form of temperature from which an axial velocity, stream function and pressure gradient have been derived. Srinivas and Kothandapani [14] have obtained an analytic solution under the assumption of long wave length and low Reynolds. In the said study the effects of heat and mass transfer on peristaltic transport of an electrically conducting fluid passing through porous space with compliant walls are perceived. Elshehawey et al. [15] offered the solution of peristaltic flow of an incompressible viscous fluid in an asymmetric channel through a porous medium. Afifi and Gad [16] have reported the interaction of peristaltic flow with pulsatile fluid through porous medium. Some noteworthy investigations on the topic can be seen from the list of references [17-26].

In view of existing literature, a useful contribution to see the effects of partial slip condition on the peristaltic flow of Jeffrey fluid in a rectangular duct through a porous medium, which indeed, is a unique innovation since for all reasons, the available literature is an indicative due to omitted idea reported in this article. In order to bridge this gap a fruitful attempt is made to analyze the peristaltic flow of Jeffrey fluid by using the assumption of long wave length and low Reynolds. Eventually, exact solutions are obtained with the help of separation of variables method. Moreover, the expression for stream functions is obtained numerically. Graphical behavior of important parameters have also been discussed and displayed.

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Chapter 1

Preliminaries

This is an introductory chapter which presents some basic definitions of fluid mechanics and governing equations necessary for the subsequent chapters.

1.1 Definitions

1.2 Fluid mechanics

Fluid mechanics is a branch of engineering sciences, which deals with the behavior of the fluid under the conditions of both static and dynamic situations. Moreover, one can say that fluid mechanics is the study of fluids (i.e., liquids and gases) and involves various properties of the fluid, such as velocity, pressure, density and temperature, as the function of space and time. Fluid mechanics may be divided into three categories these are:

- Fluid static.
- Fluid kinematics.
- Fluid dynamics.

1.2.1 Fluid statics

A branch of fluid mechanics deals with the fluid behavior when the fluid is at rest.

1.2.2 Fluid kinematics

A branch of fluid mechanics deals with the fluid behavior at rest, neglecting forces causing the motion.

1.2.3 Fluid dynamics

A branch of fluid mechanics deals with the fluid behavior at motion under the influence of forces acting on it.

1.3 Velocity field

When dealing with the fluids, we are interested in the description of the velocity field. If we define a fluid particle as a small mass of fluid of fixed identity of volume δv, the velocity at point c is defined as the instantaneous velocity of the fluid particles which, at a given instant, is passing through a point c, at a given instant, the velocity field V is a function of space coordinates (x, y, z) the velocity at any point in the flow field might vary from one instant to another. Thus the entirely representation of velocity is given by

$$V=V(x, y, z, t). \tag{1.1}$$

1.4 Fluid

Fluid is defined as a substance which is capable of flowing. It has no defined shape, but it takes the shape of its container. Besides, a fluid offers very little or no resistance to the external force/stress, when applied on it. In simpler terms, one can also make it out that the fluid is a substance which offers no resistance subject to shear force. Fluids can be liquid, vapor or gas. Common examples of daily fluid include water, diesel, petrol, gas and air etc.

1.5 Types of fluid

Physical features and nature go a long way, in understanding the different types of fluid that exist in or on the planet earth. Therefore, it is classified into four basic kinds. These are:

1.5.1 Ideal fluid

An ideal fluid is the one which has no viscosity. Moreover, it is not compressible in nature. Physically, such a fluid does not exist in our universe.

1.5.2 Real fluid

A real (or viscous) fluid is one which has finite viscosity and thus can exert a tangential (or shearing) stress on a surface with which it is in contact. Common examples of real fluid in daily lives include: Kerosene oil, Petrol, Castor oil etc.

1.5.3 Newtonian fluid

Newtonian fluid can be defined as a linear relation between shear stress and rate of strain. It can also be defined as "Fluid which holds Newton's law of viscosity is called Newtonian fluid". Mathematically, it can be described as

$$\tau_{xy} \propto \frac{du}{dy}, \tag{1.2}$$

$$\tau_{xy} = \mu \frac{du}{dy}, \tag{1.3}$$

where τ_{xy} is shear stress, μ is the fluid viscosity, x is the direction of flow and y is perpendicular to the flow. Fluid that exhibits Newtonian behavior are water, gasoline, air and glycerin.

1.5.4 Non-Newtonian fluid

Non-Newtonian fluid can be described as the fluid for which shear stress is directly but non-linearly proportional to the rate of deformation.

Mathematically, it can be written as

$$\tau_{xy} = \mu \left(\frac{du}{dy}\right)^n, n \neq 1, \tag{1.4}$$

where n called the flow behavior index. For $n = 1$ the above equation reduce to Newtonian's law of viscosity. Examples include lubricants, paintes, slurries, toothpastes, drilling muds and biological fluids.

1.6 Jeffrey fluid

Jeffrey is the name of a scientist who gave a stress tensor for a non-Newtonian fluid, containing a ratio of times λ_1 (i.e., the ratio of relaxation time to retardation times) known as Jeffrey parameter. It is denoted by

$$\lambda_1 = \frac{\text{Relaxation time of the fluid}}{\text{Retardation time of fluid}}, \tag{1.5}$$

$$\lambda_1 = \frac{\text{Relaxation time of the fluid}}{\lambda_2}. \tag{1.6}$$

1.6.1 Relaxation time

It is describe as the time taken by the Jeffrey fluid to regain equilibrium state, after its deformed or perturbed state.

1.6.2 Retardation time

It is describe as the time taken by the Jeffrey fluid to get deformed subject to shear stress. It is denoted by λ_2.

1.7 Properties of fluid

Fluids, in general, may have many properties related to mechanics, thermodynamics, or other science fields. The following are some of the most important and basic properties of the fluid which are used in this dissertation.

1.7.1 Density

The density defines as mass per unit volume. It can also be termed as the ratio of mass of a fluid to its volume. Mathematically, it can be written as

$$\text{Density} = \frac{\text{mass}}{\text{volume}}. \tag{1.7}$$

Density is represented by the symbol ρ its unit is kgm^{-3}. Generally, the fluid density decreases by increasing the temperature of fluid. Similarly it increases by increasing the pressure of fluid. Moreover, the density of a standard liquid (i.e., water) is 1000 kgm^{-3}.

1.7.2 Dynamic viscosity

The coefficient of dynamic viscosity μ can be defined as the shear force per unit area (or shear stress) required to drag one layer of fluid with unit velocity past another layer a unit distance away from it in the fluid. Rearranging Eq. (1.3),

$$\mu = \frac{\tau_{xy}}{du/dy}. \tag{1.8}$$

Units of viscosity in SI system are Nsm^{-2} or Pascal- second (pas).

1.7.3 Kinematic viscosity

The ratio of absolute viscosity to the fluid density is known as kinematic viscosity.

Mathematically, it can be written as

$$\text{Kinematic viscosity} = \frac{\text{absolute viscosity}}{\text{fluid denisty}}, \tag{1.9}$$

$$\upsilon = \frac{\mu}{\rho}. \tag{1.10}$$

Kinematic viscosity is denoted by Greek symbol υ (nu). In SI units, kinematic viscosity measured in square meter per second (m^2s^{-1}) as well as in the metric system the well-known units are square centimeter per second (cm^2s^{-1}) also called the stoke (St).

1.7.4 Pressure

Pressure is basically a type of surface forces. This is defined as the force per unit area. In fact, it is the ratio of force to an area of the fluid on which the force acts (area is normal to the direction of the force acting upon it).

Mathematically, it can be represented as

$$\text{Pressure} = \frac{\text{Force exerted}}{\text{Area on which the force acts}}, \tag{1.11}$$

$$\text{P} = \frac{\text{Force exerted}}{\text{Area on which the force acts}}. \tag{1.12}$$

In SI units pressure is measured in Newton per square meter (Nm^{-2}) and also known as Pascal.

1.8 Flow

It is a phenomenon in which the deformation of material increases continuously without limit when different forces act upon it.

1.9 Flow types

1.9.1 Internal flow

Flows that are bounded by entirely by solid surface are called internal flows. Internal flow can be laminar, turbulent, compressible or incompressible. Examples are flow in duct and flow in rivers.

1.9.2 External flow

Flow over bodies immersed in an unbounded fluid is called external flows. Examples are flow over an airplanes, missiles and ships.

1.9.3 Steady flow

A flow is said to be steady or stationary when the velocity vector and other fluid characteristic (i.e., pressure, density, etc.) at every point in a fluid do not change with respect to time so that the flow pattern remains unchanged. Mathematically, it can be written as

$$\frac{\partial v}{\partial t} = 0. \tag{1.13}$$

1.9.4 Unsteady flow

A flow is said to be unsteady when fluid characteristics and conditions at any point in a fluid change with time. Mathematically it can be written as

$$\frac{\partial v}{\partial t} \neq 0. \tag{1.14}$$

1.9.5 Uniform flow

Flow is described as uniform if the velocity at a given instant is the same in magnitude and direction at every point in the fluid.

1.9.6 Non-uniform flow

Flow is described as non-uniform if the velocity at a given instant, the velocity changes from point to point.

1.9.7 Incompressible flow

The flow in which the density of the fluid does not change during the flow and viscosity of fluid decreases with temperature is known as incompressible flow. All liquids are compressible fluids.

1.9.8 Laminar flow

Flow is described as laminar if the fluid particles move along straight parallels paths in layers (or laminar). For example, a stream of dye of ink inserted in a laminar flow will trace out a thin line and always be composed of the same fluid particles.

1.9.9 One dimensional flow

Flow is described as one dimensional if the velocity is function of only one space coordinate and time. For examples, flow through a straight pipe.

1.9.10 Two dimensional flow

Flow is described as two dimensional if the velocity is function of two space coordinates and time. For examples, flow over a porous plate.

1.9.11 Three dimensional flow

Flow is described as three dimensional if the velocity is function of three space coordinates. For examples, flow past a porous plate in a rotating fluids.

1.9.12 Volumetric flow rate

The volume of fluid passing any cross-section in unit time is called the volumetric flow rate (or discharge). It is usually represented by the symbol Q in SI units, volumetric flow rate measured in cubic meters per second ($m^3 s^{-1}$).

1.10 Forces in the fluid

In fluid dynamics, a moving fluid often comes under the effect of various kinds of forces, acting upon it. Which have been categorized into two main types. These forces are given as follows:

1.10.1 Body force

Body force is a force which applies on per unit mass of the fluid. This kind of force acts throughout the volume of a body. Gravitational force, Centrifugal force, Electric force and Magnetic force, are the common examples of the body force. Furthermore, it is also termed as long range force or volume force.

1.10.2 Surface force

It is define as a force, which acts upon per unit area of the fluid. Whenever, a surface force applies on the surface of any fluid it acts normally over the area, whereas shear stress acts tangentially over an area. Pressure, shear stresses, resistance etc. are the common examples of surface force. Mathematically, is denoted by f_s.

1.11 Dimensionless number

In field of Fluid mechanics, in order to reach better results and conclusions, it is often preferred to ignore the dimensions of some parameters. Therefore, to meet this purpose a sort of numerical quantity/parameter is used which is known as "Dimensionless number" or "Dimensionless parameter". Basically, this number is the ratio of a pair of forces. This can be obtained if force of inertia is divided by any one of these forces i.e., viscous force, force of gravity, pressure force, force of surface, or elastic force. There are various dimensionless numbers in use, each depending upon its use and condition. Most commonly used non-dimensional parameters include Reynolds number, Hartman number, Froude's number, weber's number, Richardson number. Here is the mention of those dimensionless parameters which have been used in this dissertation.

1.11.1 Reynolds number (Re)

It is defined as the ratio of inertial force, to viscous force of the flowing fluid. The expression for Reynolds number is obtained and denoted as:

$$Re = \frac{\rho UL}{\mu}, \quad (1.15)$$

Eq. (1.15) can be rewritten as:

$$Re = \frac{UL}{\nu}, \quad (1.16)$$

where L is the characteristic length, and U is the typical velocity.

The significance of a Reynolds number is, to help one in describing the flow pattern of a fluid. It determines whether the fluid's flow is a laminar or a turbulent. For instant, a fluid is passing through a round pipe if:

- Re $\leq$ 2100, then the fluid flow is termed as laminar flow.
- Re $\geq$ 4000, then the fluid flow is termed as Turbulent flow.
- 2100 $\leq$ Re$\leq$ 4000, then the fluid flow is termed as transitional flow.

1.11.2 Hartmann number (Ha)

It is described as the ratio of electromagnetic force to the viscous force and has the expression:

$$M = BL\sqrt{\frac{\sigma}{\mu}}, \quad (1.17)$$

where

- B is the magnetic field.
- σ is the electrically conductivity.

1.12 Stream line

Streamlines are a family of curves that are instantaneously tangent to the velocity vector of the flow.

1.13 Boundary

It is defined as a condition, which is required to be satisfied, by the set of differential equations, at all parts of the boundary of a region, in which the given set of differential equations is to be solved.

1.13.1 Partial slip condition

In fluid mechanics, partial slip condition also known as slip condition of a great significance. For, in real life phenomenon, no-slip condition rarely exists or does not hold in all the situations. In some cases, the fluid tends to slip at stationary wall solid boundary. This gives rise in difference between the velocity of the fluid and the wall. The idea of partial slip was, originally proposed by Navier. This condition states that velocity u in x-direction is directly proportional to shear stress at the wall.

Mathematically, it is denoted as

$$V_{wall} \propto \tau_{xy}, \tag{1.18}$$

$$V_{wall} = \beta_1 \tau_{xy}, \tag{1.19}$$

where β_1 denotes the slip parameter. Eq. (1.19) is known as the slip condition of the fluid at wall.

1.14 Peristalsis

Peristalsis is one of the most essential and significant systems being carried out in a human body. It is a spontaneous, series of muscular contraction and relaxation, which takes place in human's digestive system or digestive tract, and in some organs that connect kidney to the bladder. Peristalsis is responsible for the movement of

- Urine from kidney to bladder.
- Food through digestive system.
- Bile from gallbladder to the duodenum.

1.15 Magnetohydrodynamics (MHD)

In 1942, a Swedish scientist and Nobel Prize laureate, Hannes Alfven for the very first time introduce the use of (MHD). MHD is the academic discipline which studies the dynamics of electrically conducting fluids. Examples of such fluid include plasmas, salt, water and liquid metals. The elementary idea behind MHD generates forces on the fluid and also changes the magnetic field itself. Equations which describe MHD are a combination of the Navier-stokes equations of fluid dynamics and Maxwell's equation of electromagnetism.

1.16 Darcy number

It is described as

$$K = \frac{k_1}{a^2}, \tag{1.20}$$

where k_1 is the permeability of the medium and a is a characteristics length. Darcy number presents the effect of permeability of the medium versus its cross- sectional area.

1.17 Porous medium

A porous medium is a continuous solid phase with intervening void or gas pockets. Natural porous media include soil, sand, sponge, wood and others. Synthetic porous media include paper, cloth filters, chemical reactions catalysts and membranes.

1.17.1 Darcy's law

Darcy's law hold for viscous fluid flows with low speed in an unbounded porous medium. This law relates the pressure drop include by was the fractional drag and velocity and ignores the boundary effects on the flow (i.e., invalid where there are boundaries of the porous medium). According to this law the induced pressure drop is directly proportional to the velocity. Mathematically it can expressed as

$$\nabla P_1 = -\frac{\mu}{k_1}\boldsymbol{V}, \tag{1.21}$$

where k_1 is a constant called permeability (ability of porous medium to transmit or pass fluid through the voids) and is independent of the nature of the fluid and depends upon the geometry of the medium and has dimension [L^2].

1.17.2 Porosity

It is also known as void fraction. It is measured of void spaces in a material and defined as the ratio of volume of voids to the total volume.

$$\varphi = \frac{V_V}{V_T}. \tag{1.22}$$

where V_V is the volume of void-space (such as fluids) and V_T is the total or bulk volume of material, including the solid and void components.

1.18 Basic equations

1.18.1 The equation of continuity

The equation of continuity is the mathematically expression of the principle of conservation of mass. The differential form of this principle is

$$\frac{\partial\rho}{\partial t} = -\nabla.(\rho \boldsymbol{V}). \tag{1.23}$$

For steady state situation

$$\frac{\partial\rho}{\partial t} = 0. \tag{1.24}$$

So the continuity equation for steady state fluid takes the form

$$\nabla.(\rho \boldsymbol{V}) = 0. \tag{1.25}$$

If density ρ is constant as in the case of incompressible fluid, the continuity equation simplified to

$$\nabla.\boldsymbol{V} = 0. \tag{1.26}$$

1.18.2 The equation of motions

Equations of motion describe the law of conservation of linear momentum. In fluid dynamics, vector differential form of momentum equations, is given by

$$\left.\begin{aligned} \frac{D\boldsymbol{V}}{Dt} &= \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}.\nabla)\boldsymbol{V}, \\ \Rightarrow \frac{D\boldsymbol{V}}{Dt} &= \boldsymbol{F} - \frac{1}{\rho}\nabla p + \upsilon\nabla^2\boldsymbol{V} \end{aligned}\right\}. \tag{1.27}$$

1.18.3 Maxwell's equations

Maxwell's equations are the set of four equations which relates the electric and magnetic field to their sources, charge density and current density. Individually, these equations are Gauss's law, Gauss's law for magnetism, Faraday's law of induction and Ampere's law with Maxwell's correction.

These equation are described as

$$\nabla.\boldsymbol{E} = \frac{\rho_0}{\epsilon_0}, \tag{1.28}$$

$$\nabla . \boldsymbol{B} = 0 \,, \tag{1.29}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.30}$$

and

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \,. \tag{1.31}$$

In above equations, ρ_0 is charge density, $\boldsymbol{E}$ is the electrical field, ϵ_0 is the permittivity of the free space, μ_0 is the magnetic permeability, $\boldsymbol{B}$ is the magnetic field.

1.19 Separation of variables

In the solution of differential equations, separation of variables is one of several methods, which are used to solve both an ordinary and partial differential equations. It is also known as the Fourier method. Separation of variables is applicable in the solution of PDEs, if

- The given PDE is homogeneous (i.e., in the absence of forcing function).
- Boundary conditions are also homogeneous.
- Domain should be finite.

Chapter 2

Simultaneous effects of MHD and partial slip on peristaltic flow of Jeffrey fluid in a rectangular duct

This chapter contains the review work of [26]. In this chapter, we give a detailed calculation of the flow of the Jeffrey fluid under the effects of MHD and partial slip. The analytical solution of velocity and pressure gradient have been found under the supposition of long wave length and low Reynolds number. The expression for pressure rise in a rectangular duct has been evaluated numerically. The effects of related parameters, such as the Hartman number, slippage parameter, volume flow, aspect ratio and Jeffrey parameter, pressure gradient and pressure rise are graphically illustrated.

2.1 Formulation of the problem

Let us consider the flow of an incompressible, Jeffrey fluid in a duct of rectangular cross section having the channel width $2d$ and height $2a$. In the present geometry, the Cartesian coordinates system is considered in a way that $X - axis$ is taken along the axial direction, $Y - axis$ is taken along the lateral direction and $Z - axis$ is along the vertical direction of a rectangular duct.

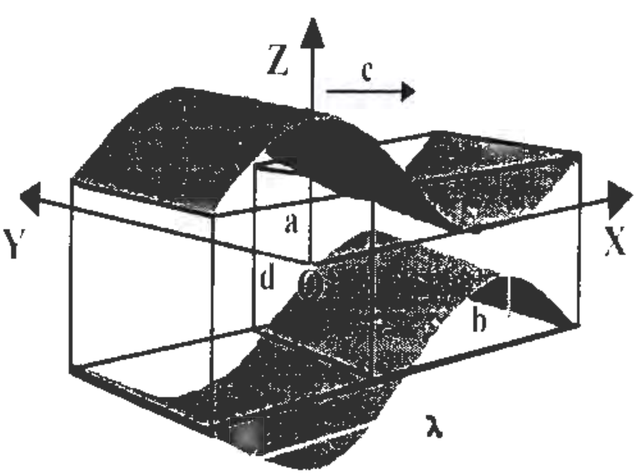

Fig. 2.1. Geometry of the problem.

The peristaltic wave on the wall is represented as:

$$Z = H(X,t) = \pm a \pm b\cos\left[\frac{2\pi}{\lambda}(X - ct)\right], \tag{2.1}$$

where a and b are the amplitudes of the waves, λ is the wave length, c is the velocity propagation, t is the time and X is the direction of wave propagation. The walls parallel to XZ plane remain undisturbed and are not to subject any peristaltic wave motion. It is assumed that there is no change in lateral direction of the duct cross section means the lateral velocity is zero. Let $V = (U, 0, W)$ be the velocity for a rectangular duct. The governing equations for the problem are given below.

(1). Equation of conservation of mass:

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0. \tag{2.2}$$

(2). Equation of momentum:

$$\rho\frac{DV}{Dt} = -\nabla P + \nabla.\boldsymbol{S} + \boldsymbol{J} \times \boldsymbol{B},$$

where $\boldsymbol{J}$ is the current density or Lorentz force which is the force required by a charge particle to move in magnetic and electric field. Mathematically, it is expressed as

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}),$$

in which $\boldsymbol{B}$ is the total magnetic field such that $\boldsymbol{B} = \boldsymbol{B_0} + \boldsymbol{b}$. $\boldsymbol{B}$ is the sum of applied and magnetic field $\boldsymbol{B_0}$ and induced magnetic field $\boldsymbol{b}$. Induced magnetic field is negligible compare with applied magnetic field. Moreover, the XZ-walls of the rectangular duct are electrically insulated and no energy or charge / electricity is added or extracted from the fluid by the electric field. Therefore, this implies that there is no electric field present in the fluid region. With the help of these assumptions the electromagnetic force $\boldsymbol{J} \times \boldsymbol{B}$ takes the following form

$$\boldsymbol{J} \times \boldsymbol{B} = \sigma(\boldsymbol{V} \times \boldsymbol{B}) \times \boldsymbol{B}.$$

Then, the above equation of momentum in the form of velocity components, becomes

$$\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial X}+W\frac{\partial U}{\partial Z}\right)=-\frac{\partial P}{\partial X}+\frac{\partial}{\partial X}S_{XX}+\frac{\partial}{\partial Y}S_{XY}+\frac{\partial}{\partial Z}S_{XZ}-\sigma B_0^2U, \tag{2.3}$$

$$0=-\frac{\partial P}{\partial Y}+\frac{\partial}{\partial X}S_{YX}+\frac{\partial}{\partial Y}S_{YY}+\frac{\partial}{\partial Z}S_{YZ}, \tag{2.4}$$

$$\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial X}+W\frac{\partial W}{\partial Z}\right)=-\frac{\partial P}{\partial Z}+\frac{\partial}{\partial X}S_{ZX}+\frac{\partial}{\partial Y}S_{ZY}+\frac{\partial}{\partial Z}S_{ZZ}-\sigma B_0^2W, \tag{2.5}$$

where U and W are the velocity components in fixed frame (X,Y) and, S denotes the shear stresses. Moreover, stress tensor for Jeffrey fluid is defined as

$$S=\frac{\mu}{1+\lambda_1}(\dot{\gamma}+\lambda_2\ddot{\gamma}), \tag{2.6}$$

$$\dot{\gamma}=\nabla V+\nabla V^{T}. \tag{2.7}$$

The boundary equations for the problem are:

$$U(X,Y,Z)=0,\ \text{at}\ Y=\pm d, \tag{2.8}$$

$$U(X,Y,Z)=U_{wall}-\frac{L}{1+\lambda_1}\frac{\partial U}{\partial Z},\quad at\ Z=H(X,t), \tag{2.9}$$

$$U(X,Y,Z)=U_{wall}+\frac{L}{1+\lambda_1}\frac{\partial U}{\partial Z},\quad at\ Z=-H(X,t), \tag{2.10}$$

where $H(X,t)=+a+b\cos\left[\frac{2\pi}{\lambda}(X-ct)\right]$ and $-H(X,t)=-a-b\cos\left[\frac{2\pi}{\lambda}(X-ct)\right]$.

Now, we define a wave frame (x,y) moving with the speed c away from the laboratory frame (X,Y) by using the following transformation:

$$\left.\begin{array}{c} x=X-ct, y=Y, z=Z, u=U-c,\\ w=W, p(x,z)=P(X,Z,t).\end{array}\right\}. \tag{2.11}$$

Selecting the following set of non-dimensional variables and parameters

$$\left.\begin{array}{c}\bar{x}=\frac{x}{\lambda},\bar{y}=\frac{y}{d},\bar{z}=\frac{z}{a},\bar{u}=\frac{u}{c},\bar{w}=\frac{w}{c\delta},\bar{t}=\frac{ct}{\lambda},\bar{h}=\frac{H}{a},p=\frac{a^2p}{\mu c\lambda},\\ \bar{S}_{\bar{x}\bar{x}}=\frac{a}{\mu c}S_{xx},\bar{S}_{\bar{x}\bar{y}}=\frac{d}{\mu c}S_{xy},\bar{S}_{\bar{x}\bar{z}}=\frac{a}{\mu c}S_{xz},\bar{S}_{\bar{y}z}=\frac{d}{\mu c}S_{yz},\bar{S}_{yy}=\frac{\lambda}{\mu c}S_{yy},\\ \phi=\frac{b}{a},\beta_1=\frac{L}{a},Re=\frac{\rho ac\delta}{\mu},M=\sqrt{\frac{\sigma}{\mu}}aB_0,\beta=\frac{a}{d},S_{zz}=\frac{\lambda}{\mu c}S_{zz},\delta=\frac{a}{\lambda}.\end{array}\right\}. \tag{2.12}$$

Eqs. (2.2) – (2.6) after dropping the signs of bar, take the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.13}$$

$$Re\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \delta\frac{\partial}{\partial x}S_{xx} + \beta^2\frac{\partial}{\partial y}S_{xy} + \frac{\partial}{\partial z}S_{xz} - M^2(u+1), \tag{2.14}$$

$$0 = -\frac{\partial p}{\partial y} + \delta^2\frac{\partial}{\partial x}S_{yx} + \delta^2\frac{\partial}{\partial y}S_{yy} + \delta\frac{\partial}{\partial z}S_{yz}, \tag{2.15}$$

$$Re\delta^2\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \delta^2\frac{\partial}{\partial x}S_{zx} + \delta\beta^2\frac{\partial}{\partial y}S_{zy} + \delta^2\frac{\partial}{\partial z}S_{zz} - \delta^2M^2w. \tag{2.16}$$

Using the supposition of long wave length and low Reynolds, terms of order δ and higher power are neglected. Then the above-mentioned Eqs. (2.13) – (2.16) will be reduce the following non-homogeneous, linear and second order partial differential equation (PDE):

$$\frac{\beta^2}{1+\lambda_1}\frac{\partial^2 u}{\partial y^2} + \frac{1}{1+\lambda_1}\frac{\partial^2 u}{\partial z^2} - M^2(u+1) = \frac{dp}{dx}. \tag{2.17}$$

Therefore, the corresponding boundary conditions at the walls are describes as:

$$u(x,y,z) = -1, \ at \quad y = \pm 1, \tag{2.18}$$

$$u(x,y,z) = -\frac{\beta_1}{1+\lambda_1}\frac{\partial u}{\partial z} - 1, \quad at \ z = +h(x) = 1 + \phi\cos 2\pi x, \tag{2.19}$$

$$u(x,y,z) = +\frac{\beta_1}{1+\lambda_1}\frac{\partial u}{\partial z} - 1, \quad at \ z = -h(x) = -(1 + \phi\cos 2\pi x). \tag{2.20}$$

2.2 Solution of the problem

It is observed that Eq. (2.17), is a linear, non-homogenous and second order partial differential equation, corresponding to the boundary conditions given in Eqs. (2.18) - (2.19). In order to solve Eq. (2.17), consider the following transformation

$$u(x,y,z) = v_1(x,y,z) + w_1(y), \tag{2.21}$$

which is suggested by Richard Haberman in his book ''Elementary applied partial Differential Equations''. This transformation is useful to convert non-homogeneous partial differential equations and boundary conditions into homogeneous partial differential equations and boundary conditions. By using Eq. (2.21), in Eq. (2.17). This gives

$$\frac{\beta^2}{1+\lambda_1}\frac{\partial^2 v_1}{\partial y^2}+\frac{1}{1+\lambda_1}\frac{\partial^2 v_1}{\partial z^2}-M^2 v_1=0 \tag{2.22}$$

and

$$\frac{\beta^2}{1+\lambda_1}\frac{d^2 w_1}{dy^2}-M^2 w_1-M^2=\frac{dp}{dx}. \tag{2.23}$$

Moreover, the boundary conditions are transformed into the following form

$$w_1(y)=-1,\ \ when\ \ y=\pm 1, \tag{2.24}$$

$$v_1(x,y,z)=0,\ \ when\ \ y=\pm 1, \tag{2.25}$$

$$v_1(x,y,z)=-\frac{\beta_1}{1+\lambda_1}\frac{\partial v_1}{\partial z}-1-w_1(y),\ \ when\ \ z=h(x), \tag{2.26}$$

$$v_1(x,y,z)=+\frac{\beta_1}{1+\lambda_1}\frac{\partial v_1}{\partial z}-1-w_1(y),\ \ when\ \ z=-h(x). \tag{2.27}$$

Here, it can be noted that the transformation yields, the main given problem into two systems of differential equation. First one is a partial differential equation (i.e., Eq. 2.22) and second one is a second order linear ordinary differential equation (i.e., Eq. 2.23) but non-homogeneous, corresponding to their boundary conditions. Now, we are going to find the solution of these two equations.

2.2.1 Solution of ordinary differential equation

By considering Eq. (2.23), with the boundary conditions Eq. (2.24) such that

$$\frac{\beta^2}{1+\lambda_1}\frac{d^2 w_1}{dy^2}-M^2 w_1-M^2=\frac{dp}{dx},$$

$$w_1(y)=-1\ \ at\ \ y=\pm 1.$$

The complementary function (C.F.) is the solution of homogeneous equation, for the above ODE, the complementary function and particular solutions are

$$w_{1_c}=c_1\cosh\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\,y\right)+c_2\sinh\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\,y\right) \tag{2.28}$$

and

$$w_{1_p} = -1 - \frac{1}{M^2}\frac{dp}{dx}. \tag{2.29}$$

Thus the general solution is determined by

$$w_1(y) = c_1 \cosh\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\, y\right) + c_2 \sinh\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\, y\right) - 1 - \frac{1}{M^2}\frac{dp}{dx}. \tag{2.30}$$

Applying the boundary conditions Eq. (2.24), in above equation to obtain the values of c_1 and c_2. Such that:

$$c_1 = \frac{1}{M^2}\frac{dp}{dx}\operatorname{sech}\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\right)$$

and

$$c_2 = 0.$$

Thus the general solution is

$$w_1(y) = -1 - \frac{1}{M^2}\frac{dp}{dx}\left[1 - \operatorname{sech}\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\right)\cosh\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\, y\right)\right]. \tag{2.31}$$

2.2.2 Solution of partial differential equation

Considering the PDE in Eq. (2.22), this is second order, linear and homogeneous partial differential equation with the corresponding boundary conditions from Eq. (2.25), such that

$$\frac{\beta^2}{1+\lambda_1}\frac{\partial^2 v_1}{\partial y^2} + \frac{1}{1+\lambda_1}\frac{\partial^2 v_1}{\partial z^2} - M^2 v_1 = 0,$$

$$v_1(x, y, z) = 0, \;\; at \;\; y = \pm 1.$$

Now we can use the method of separation of variable to obtain the solution of given PDE. It is assumed that

$$v_1(x, y, z) = Y(y) \times Z(z), \tag{2.32}$$

is one of the possible solutions of Eq. (2.22). Using the Eq. (2.32) in Eq. (2.22), it provides

$$\frac{Y''}{Y} = \frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{M^2}{\beta^2}(1 + \lambda_1) = -\alpha^2 \;\; \text{(say)}, \tag{2.33}$$

$$\Rightarrow \frac{Y''}{Y} = -\alpha^2 \qquad (2.34)$$

and

$$\frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{M^2}{\beta^2}(1 + \lambda_1) = -\alpha^2. \qquad (2.35)$$

Using the boundary conditions Eq. of (2.25), in Eq. (2.32). This yield

$$[Y(\pm 1)] \times [Z(z)] = 0,$$

$$\Rightarrow Z(z) \neq 0,$$

this implies that

$$Y(\pm 1) = 0. \qquad (2.36)$$

Then, there are two possible cases to attain the mandatory solution.

Case-I.

From Eq. (2.34)

$$\frac{Y''}{Y} = -\alpha^2,$$

$$\Rightarrow [D^2 + \alpha^2] \times Y(y) = 0,$$

$$\Rightarrow Y(y) \neq 0,$$

$$\Rightarrow [D^2 + \alpha^2] = 0,$$

$$D = \pm(i\alpha).$$

It is familiar that a trivial solution is attained for the values of $\alpha = 0$ and $\alpha < 0$. The only non-trivial solution is obtained, for the value of $\alpha > 0$. Which yields

$$Y(y) = c_3 \cos(\alpha y) + c_4 \sin(\alpha y), \qquad (2.37)$$

where c_3 and c_4 are the constants that are determined by using $Y(\pm 1) = 0$, in above equation. It gives

$$c_4 = 0, \qquad (2.38)$$

$$a_n = \left(\frac{(2n-1)\pi}{2}\right), \quad n = 1, 2, 3 \ldots \tag{2.39}$$

Thus the Eq. (2.37) becomes

$$Y(y) = c_3 \cos\left(\frac{(2n-1)\pi}{2} y\right), \quad n = 1, 2, 3 \ldots \tag{2.40}$$

Case-II

Similarly, from Eq. (2.35)

$$\frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{M^2}{\beta^2}(1+\lambda_1) = -\alpha^2,$$

$$\Rightarrow \frac{-1}{\beta^2}\frac{Z''}{Z} = -\alpha^2 - \frac{M^2}{\beta^2}(1+\lambda_1),$$

$$\Rightarrow D = \pm\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}.$$

This implies that

$$Z(z) = c_5 \cosh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) + c_6 \sinh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right). \tag{2.41}$$

By using the values of $Y(y)$ and $Z(z)$ in Eq. (2.32). This leads to be

$$\left.\begin{array}{c} v_1(x,y,z) = c_3\left[c_5 \cosh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) + c_6 \sinh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right)\right] \\ \times \cos\left(\frac{(2n-1)\pi}{2} y\right) \end{array}\right\}. \tag{2.42}$$

For suitability, it is supposed that

$$c_3 \times c_5 = c_7, \quad \text{(say)}$$

$$c_3 \times c_6 = c_8. \quad \text{(say)}$$

Consequently, Eq. (2.42) becomes

$$\left.\begin{array}{c} v_1(x,y,z) = \begin{bmatrix} c_7 \cosh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \\ +c_8 \sinh\left(z\sqrt{\alpha^2\beta^2 + M^2(1+\lambda_1)}\right) \end{bmatrix} \\ \times \cos\left(\frac{(2n-1)\pi}{2} y\right) \end{array}\right\}. \tag{2.43}$$

To calculate the values of the two constants given in the above equation, use the boundary condition Eqs. (2.26) - (2.27). This yields

$$\frac{-\beta_1}{1+\lambda_1}\frac{\partial v_1}{\partial z}-1- \ w_1(y)=\left.\begin{bmatrix} c_7\cosh\left(h\sqrt{\alpha^2\beta^2+M^2(1+\lambda_1)}\right) \\ +c_8\sinh\left(h\sqrt{\alpha^2\beta^2+M^2(1+\lambda_1)}\right) \end{bmatrix} \times \cos\left(\frac{(2n-1)\pi}{2}y\right)\right\}, \tag{2.44}$$

$$\frac{\beta_1}{1+\lambda_1}\frac{\partial v_1}{\partial z}-1- \ w_1(y)=\left.\begin{bmatrix} c_7\cosh\left(h\sqrt{\alpha^2\beta^2+M^2(1+\lambda_1)}\right) \\ -c_8\sinh\left(h\sqrt{\alpha^2\beta^2+M^2(1+\lambda_1)}\right) \end{bmatrix} \times \cos\left(\frac{(2n-1)\pi}{2}y\right)\right\}. \tag{2.45}$$

Substituting the value of $w_1(y)$ into the Eqs. (2.44) - (2.45) and integrating the both sides of the equations w.r.t "y" from '0' to "h" respectively.

It is attained

$$v_1(x,y,z)=\left\{\frac{32\frac{dp}{dx}(1+\lambda_1)^2\cos\left(\frac{\pi y}{2}\right)\cosh\left(z\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\right)}{\left[\begin{matrix}2\pi(1+\lambda_1)[\pi^2\beta^2+4M^2(1+\lambda_1)]\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right)+ \\ \beta_1\pi[\pi^2\beta^2+4M^2(1+\lambda_1)]^{\frac{3}{2}}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right)\end{matrix}\right]}\right\}. \tag{2.46}$$

Replacing the values of $v_1(y,z)$ and $w_1(y)$ in Eq. (2.21), it gives the desired velocity profile of the peristaltic wave, moving in the rectangular duct.

$$u(x,y,z) = \left\{ \begin{array}{c} -1-\frac{1}{M^2}\frac{dp}{dx}\left[1-\operatorname{sech}\left(\frac{M\sqrt{1+\lambda_1}}{\beta}\right)\cosh\left(\frac{My\sqrt{1+\lambda_1}}{\beta}\right)\right] \\ +\dfrac{32\frac{dp}{dx}(1+\lambda_1)^2\cos\left(\frac{\pi y}{2}\right)\cosh\left(z\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\right)}{\left|\begin{array}{c} 2\pi(1+\lambda_1)[\pi^2\beta^2+4M^2(1+\lambda_1)]\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right)+ \\ \beta_1\pi[\pi^2\beta^2+4M^2(1+\lambda_1)]^{\frac{3}{2}}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right) \end{array}\right|} \end{array} \right\}. \quad (2.47)$$

Also, volume flow rate average volume flow rate of the peristaltic wave are specified by

$$q = -\left[\frac{(1+\phi\cos 2\pi x)\times\left[\sqrt{(1+\lambda_1)}\left(M^3+M\frac{dp}{dx}\right)-\frac{dp}{dx}\beta\tanh\left(\frac{M\sqrt{(1+\lambda_1)}}{\beta}\right)\right]}{M^3\sqrt{(1+\lambda_1)}}\right]$$

$$+\frac{128\frac{dp}{dx}(1+\lambda_1)^2\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right)}{\pi^2\{\pi^2\beta^2+4M^2(1+\lambda_1)\}^{\frac{3}{2}}\times\left\{\begin{array}{c} 2(1+\lambda_1)\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right) \\ +\beta_1\sqrt{\pi^2\beta^2+4M^2(1+\lambda_1)}\sinh\sqrt{\frac{\pi^2\beta^2}{4}+M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x) \end{array}\right\}}.$$

(2.48)

Average volumetric flow rate over one period $T = \lambda/c$ of the peristaltic wave is stated as

$$Q = \frac{1}{T}\int_0^T \bar{Q}dt = q + 1, \quad (2.49)$$

where

$$\bar{Q} = \int_0^1 \int_0^{h(x)} (u(x,y,z)+1)dzdy = q + h\,. \quad (2.50)$$

Therefore, the average volumetric flow rate is given as

$$Q = \left[\frac{M^3\sqrt{(1+\lambda_1)} - (1+\phi\cos 2\pi x)\times\left[\sqrt{(1+\lambda_1)}\left(M^3 + M\frac{dp}{dx}\right) - \frac{dp}{dx}\beta\tanh\left(\frac{M\sqrt{(1+\lambda_1)}}{\beta}\right)\right]}{M^3\sqrt{(1+\lambda_1)}}\right]$$
$$+ \frac{128\frac{dp}{dx}(1+\lambda_1)^2\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+4M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right)}{\pi^2(\pi^2\beta^2+4M^2(1+\lambda_1))^{\frac{3}{2}}\times\left\{\begin{array}{c} 2(1+\lambda_1)\cosh\left(\sqrt{\frac{\pi^2\beta^2}{4}+4M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right) \\ +\beta_1\sqrt{\pi^2\beta^2+4M^2(1+\lambda_1)}\sinh\left(\sqrt{\frac{\pi^2\beta^2}{4}+4M^2(1+\lambda_1)}\,(1+\phi\cos 2\pi x)\right)\end{array}\right\}}.$$

(2.51)

Likewise, the pressure gradient of the above equation can be obtained, such that

$$\frac{dp}{dx} = \frac{(Q+\phi\cos 2\pi x)\times\left[M^3\sqrt{(1+\lambda_1)}\times\pi^2\{\pi^2\beta^2+4M^2(1+\lambda_1)\}^{\frac{3}{2}}\times\left(2(1+\lambda_1)\cosh(\theta)+\beta_1\sqrt{\pi^2\beta^2+4M^2(1+\lambda_1)}\sinh(\theta)\right)\right]}{-128M^3(1+\lambda_1)^{\frac{5}{2}}\sinh(\theta)+\left(2(1+\lambda_1)\cosh(\theta)+\beta_1\sqrt{\pi^2\beta^2+4M^2(1+\lambda_1)}\sinh(\theta)\right)\left[(1+\phi\cos 2\pi x)\left\{M+\sqrt{(1+\lambda_1)}-\beta\tanh\left(\frac{M\sqrt{(1+\lambda_1)}}{\beta}\right)\right\}\right]},$$

(2.52)

where

$$\theta = \sqrt{\frac{\pi^2\beta^2}{4} + 4M^2(1+\lambda_1)}\ (1+\phi\cos 2\pi x). \tag{2.53}$$

The fluid pressure rise can be calculated numerically by integration of Eq. (2.52) on wave length yields

$$\Delta p = \int_0^1 \frac{dp}{dx}dx. \tag{2.54}$$

2.3 Results and discussion

In this section, the relative changes in the fluid flow behavior caused by changes in different parameters graphically displayed and discussed. The parameters that gives the variation of the problem under consideration are aspect ratio β, Jeffrey's parameter λ_1, volumetric flow rate Q, slip parameter β_1 and Hartmann's number M.

Figure 2.1 and Figure 2.2 show the velocity profile u of the peristaltic wave, for dissimilar values of slip parameter β_1, Hartmanns's number M. It can be observed from these figures that the velocity profile, of wave propagates in a rectangular duct, is slowly down-turn with the passage of time, corresponding to increase the values of slip parameter β_1 and Hartmann number M. Figures 2.3 – 2.6 and 2.7 – 2.9 show the behavior of the pressure rise for different values of the different parameters in relation to the volumetric flow rate Q and the aspect ratio β, respectively.

The pressure rise Δp, in figures 2.3 – 2.4 above, as plotted relative to the volumetric flow rate Q. It has been noted that Δp is increasing corresponding to the increase in β and M. Moreover, in figures 2.5 – 2.6, the values of Δp decline for rise in numerical values of slip parameter β_1 and Jeffrey parameter λ_1. Similarly, the variation trend of Δp with respect to β, has also been displayed in figure 2.7. Therefore, it is noteworthy that Δp declines corresponding to the rise in Q. whereas in figurers 2.8 – 2.9, Δp increasingly converges to 0.2 and 0, for the grater values of slip parameter β_1 and Jeffrey parameter λ_1, respectively.

A very interesting phenomenon in the fluid transport is trapped. The formation of an internal circulating bolus of the fluid through the closed stream line is called trapping and this trapped bolus is continued along the peristaltic wave at the speed of wave. The bolus describes as volume of fluid bounded by closed stream lines in the wave frame, moving to the wave pattern. Figures 2.10 – 2.25, represents the streamlines for the fluid flow. The size of bolus varies; correspond to the variation in Jeffrey parameter λ_1, aspect ratio β, and Hartmann number M and slip parameter β_1 respectively. Figures 2.10 – 2.21, show that boluses are gradually decreases in size, with the increase in Jeffrey parameter λ_1, aspect ratio β and Hartmann number M, respectively, therefore, it can be deduced that fluid flow becomes passive to the variation of the these parameters.

It is inferred from the graphs, depicted in the figures 2.22 – 2.25, that fluid is making its way through consider rectangular duct, at ease. As the boluses get expanded corresponding to the increase in numerical values of slip parameter β_1.

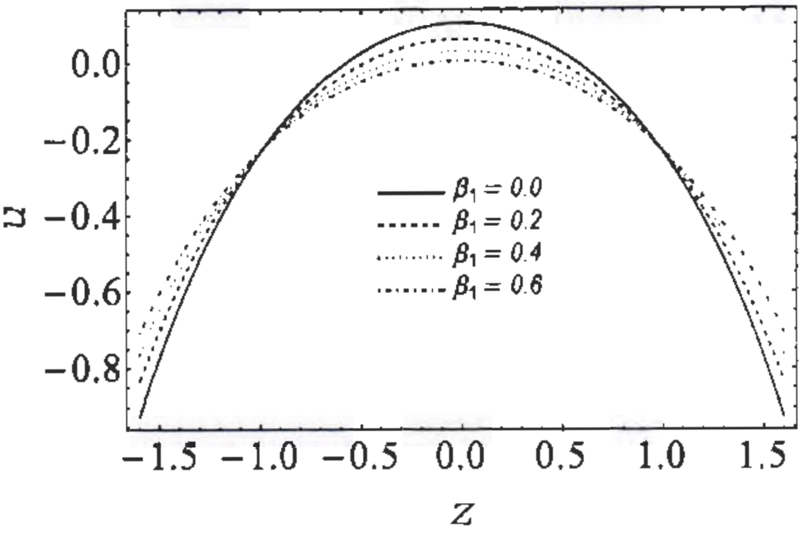

Figure. 2.1: Variation in the velocity trend for unlike values of β_1 when $\phi = 0.6, x = 0,$ $y = 0.5, Q = 0.5, M = 0.5, \beta = 0.5, \lambda_1 = 2$.

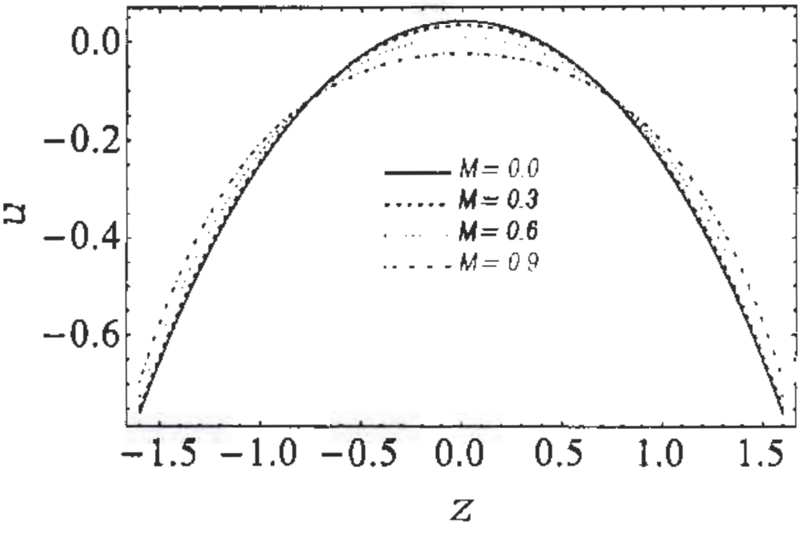

Figure. 2.2: Variation in the velocity trend for unlike values of M when $\phi = 0.6, x = 0,$ $y = 0.5, Q = 0.5, \beta_1 = 0.5, \beta = 0.5, \lambda_1 = 2$.

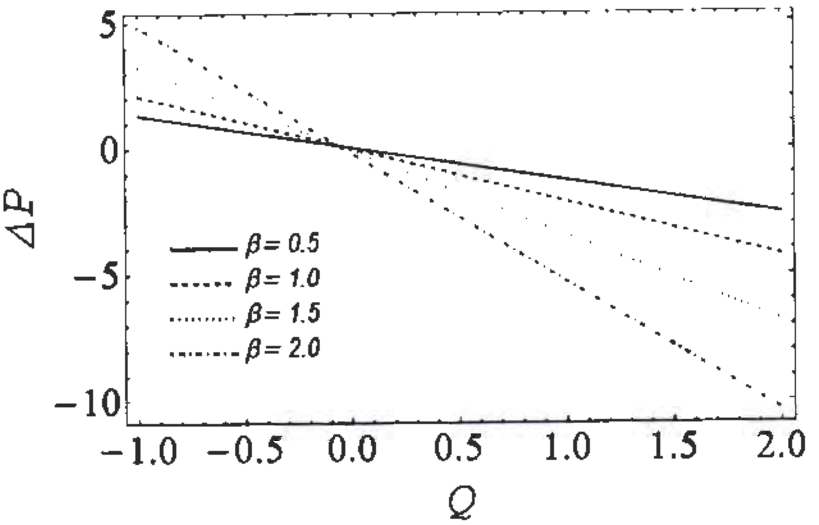

Figure 2.3: Variation of Δp with Q for unlike values of β when $\phi = 0.6$, $\beta_1 = 0.5$, $M = 0.5$, $\lambda_1 = 2$.

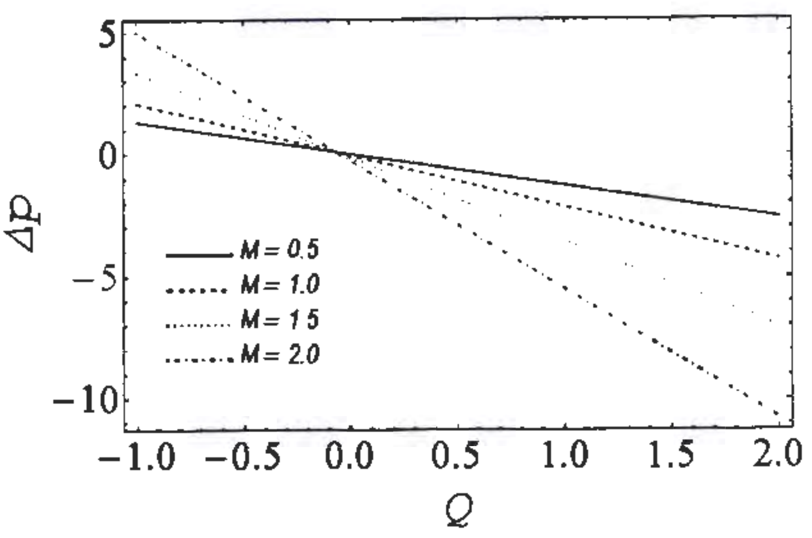

Figure 2.4: Variation of Δp with Q for unlike values of M when $\phi = 0.6$, $\beta_1 = 0.5$, $\beta = 0.5$, $\lambda_1 = 2$.

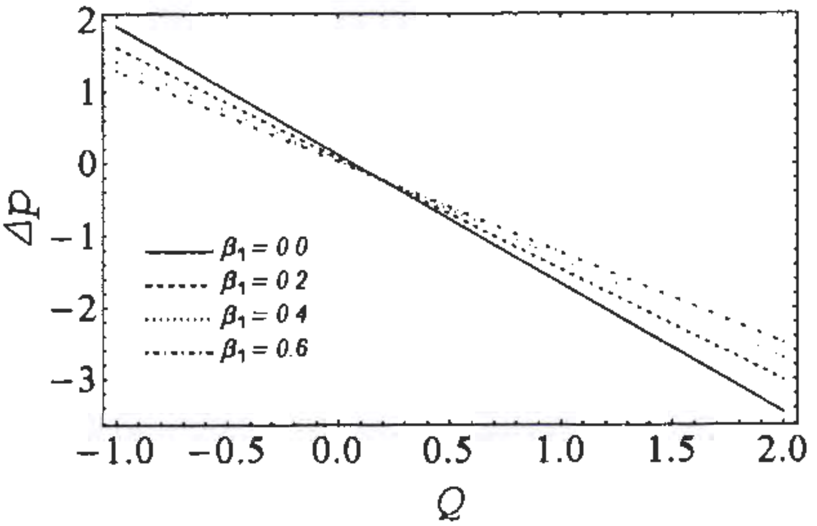

Figure 2.5: Variation of Δp with Q for unlike values of β_1 when $\phi = 0.6$, $M = 0.5$, $\beta = 0.5$, $\lambda_1 = 2$.

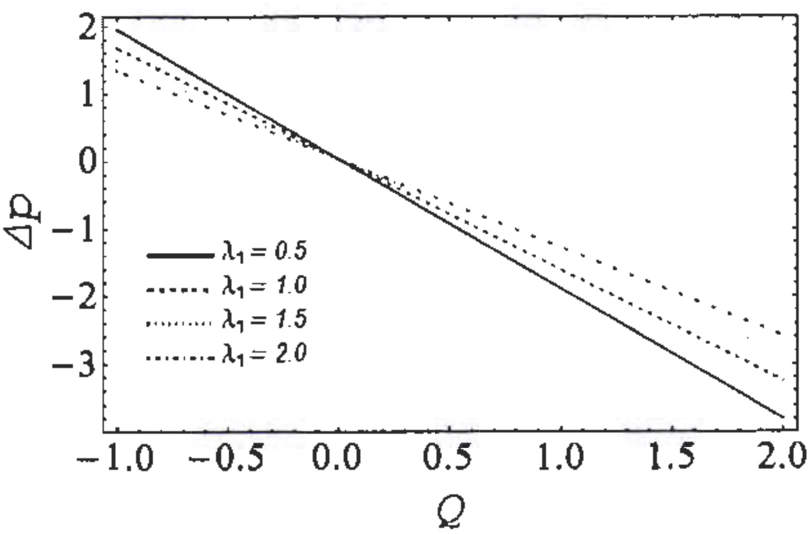

Figure 2.6: Variation of Δp with Q for unlike values of λ_1 when $\phi = 0.6$, $\beta = 0.5$, $\beta_1 = 0.5$, $M = 0.5$, $\lambda_1 = 2$.

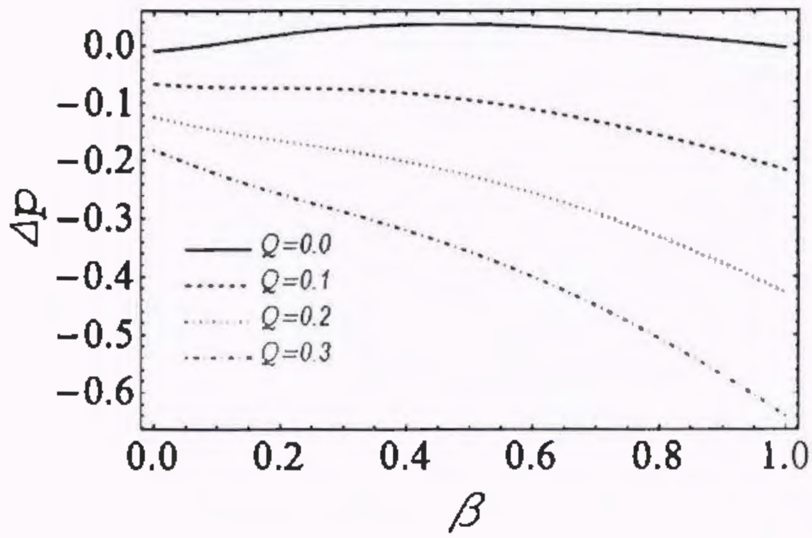

Figure 2.7: Variation of Δp with β for unlike values of Q when $\phi = 0.6$, $M = 0.5$, $\beta_1 = 0.5, \lambda_1 = 2$.

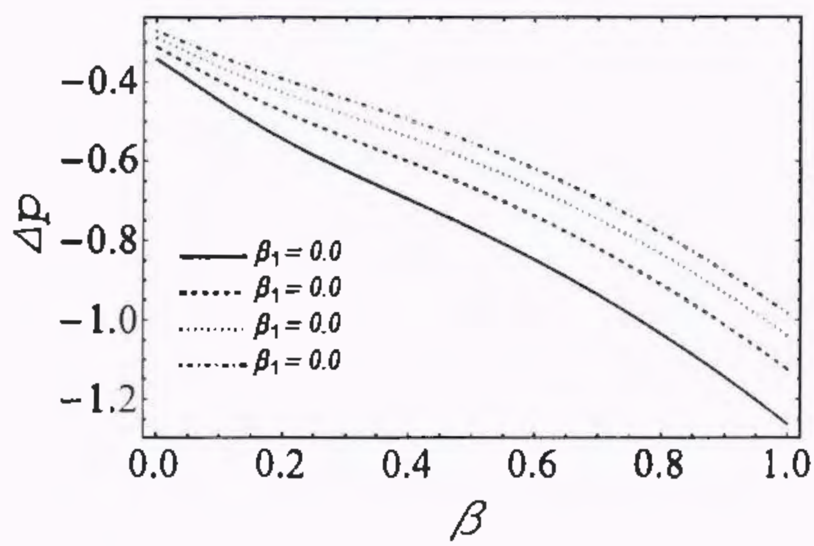

Figure 2.8: Variation of Δp with β for unlike values of β_1 when $\phi = 0.6$, $M = 0.5$, $Q = 0.5, \lambda_1 = 2$.

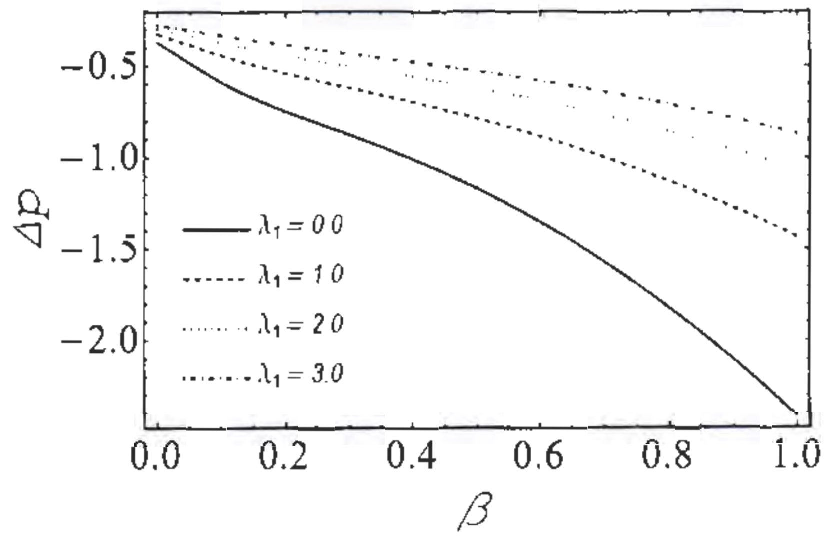

Figure 2.9: Variation of Δp with β for unlike values of λ_1 when $\phi = 0.6$, $\beta_1 = 0.5$, $Q = 0.5$, $M = 0.5$.

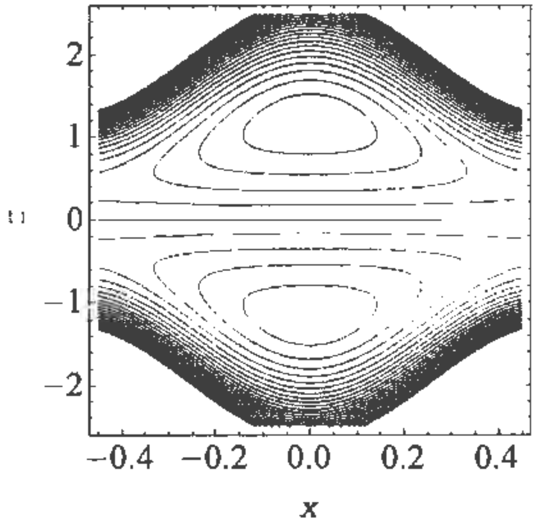

Figure 2.10: Stream lines for $\lambda_1 = 0$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5, \beta = 0.5, M = 0.5, Q = 1$.

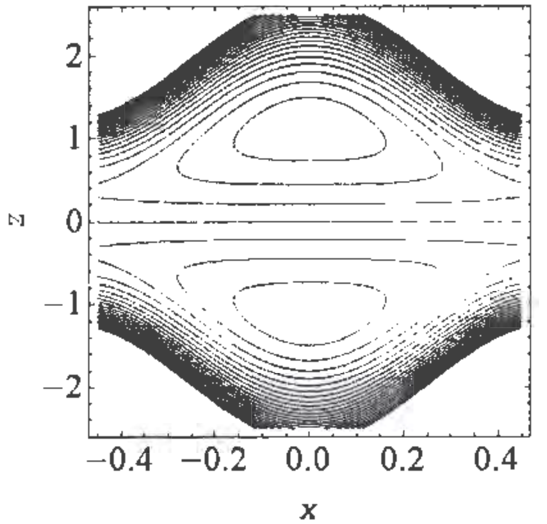

Figure 2.11 : Stream lines for $\lambda_1 = 1$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5, \beta = 0.5, M = 0.5, Q = 1$.

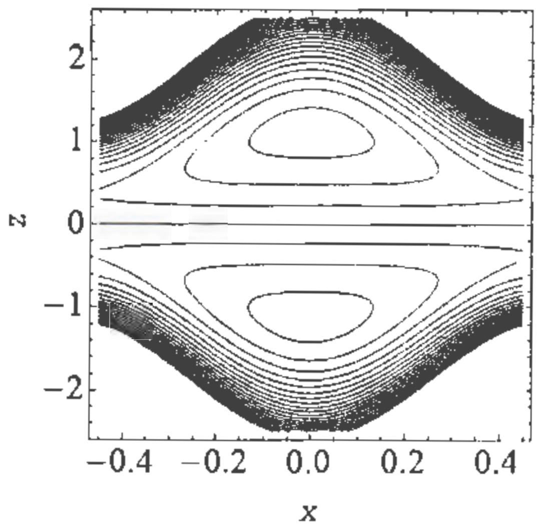

Figure 2.12 : Stream lines for $\lambda_1 = 2$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5, \beta = 0.5, M = 0.5, Q = 1$.

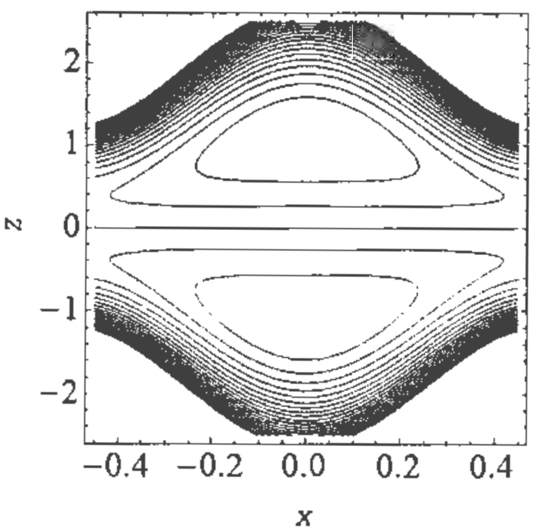

Figure 2.13: Stream lines for $\lambda_1 = 3$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta_1 = 0.5, \beta = 0.5, M = 0.5, Q = 1$.

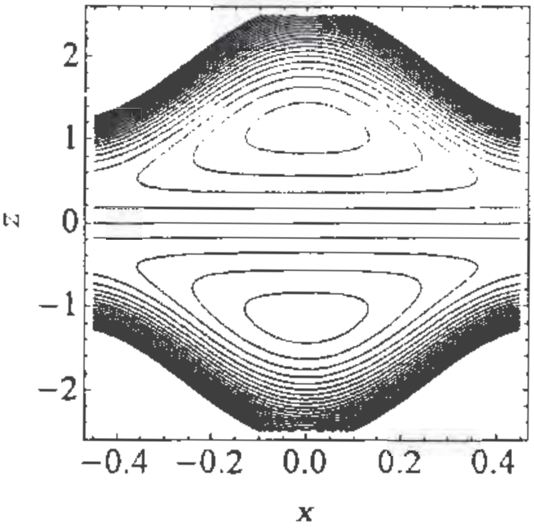

Figure 2.14: Stream lines for $\beta = 0.3$. The other parameters are $y = 0.5$, $\phi = 0.6$, $Q = 1$, $\beta_1 = 0.5$, $M = 0.5$.

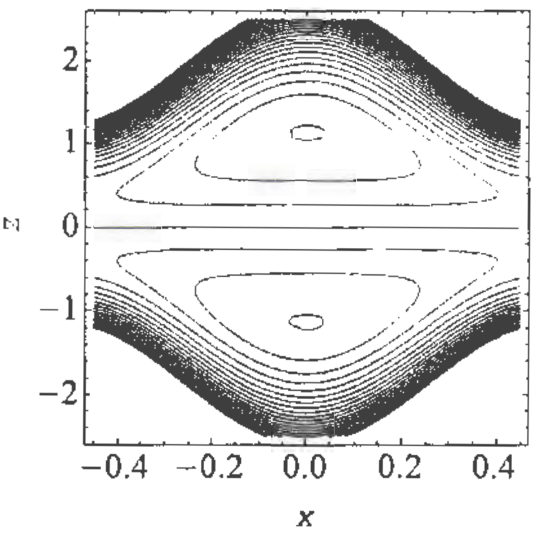

Figure 2.15: Stream lines for $\beta = 0.6$. The other parameters are $y = 0.5$, $\phi = 0.6$, $Q = 1$, $\beta_1 = 0.5$, $M = 0.5$.

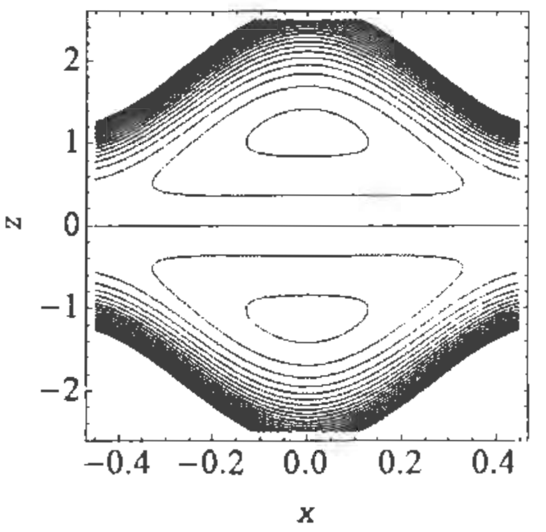

Figure 2.16: Stream lines for $\beta = 0.9$. The other parameters are $y = 0.5$, $\phi = 0.6$, $Q = 1$, $\beta_1 = 0.5$, $M = 0.5$.

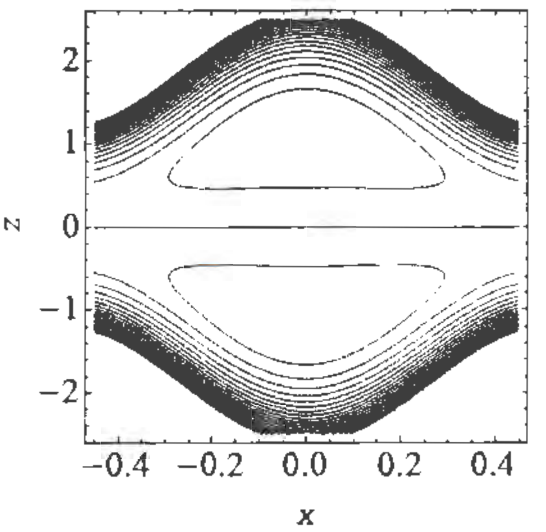

Figure 2.17: Stream lines for $\beta = 1.2$. The other parameters are $y = 0.5$, $\phi = 0.6$, $Q = 1$, $\beta_1 = 0.5$, $M = 0.5$.

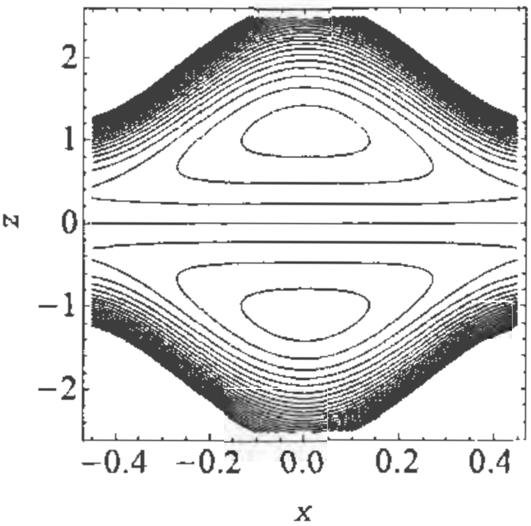

Figure 2.18: Stream lines for $M = 0.3$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta = 0.5$, $\beta_1 = 0.5$, $Q = 1.$.

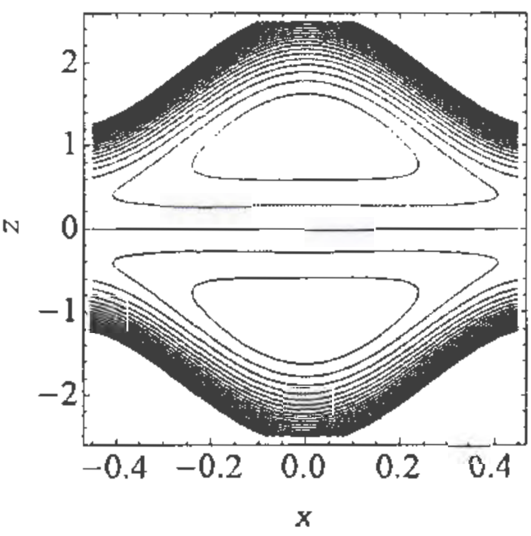

Figure 2.19: Stream lines for $M = 0.6$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta = 0.5$, $\beta_1 = 0.5$, $Q = 1$.

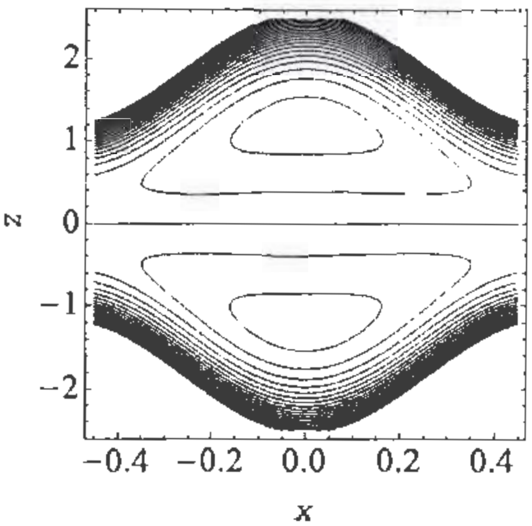

Figure 2.20: Stream lines for $M = 0.9$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta = 0.5$, $\beta_1 = 0.5$, $Q = 1$.

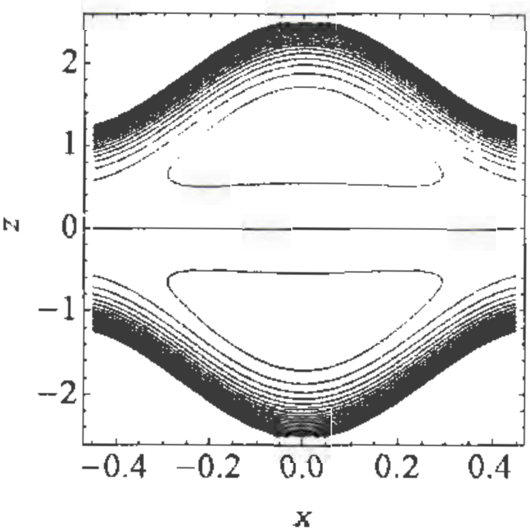

Figure 2.21: Stream lines for $M = 1.2$. The other parameters are $y = 0.5$, $\phi = 0.6$, $\beta = 0.5$, $\beta_1 = 0.5$, $Q = 1$.

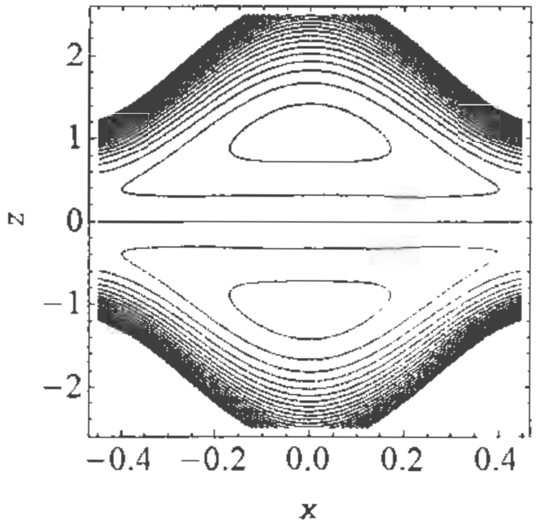

Figure 2.22: Stream lines for $\beta_1 = 0$. The other parameters are $y = 0.5$, $\phi = 0.6$, $M = 0.5$, $\beta = 0.5$, $Q = 1$. .

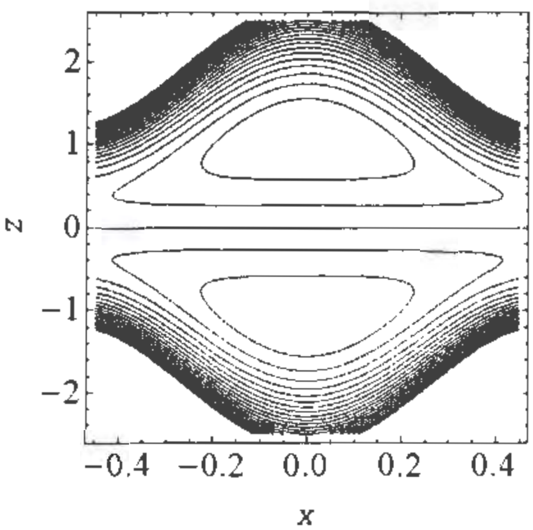

Figure 2.23: Stream lines for $\beta_1 = 0.2$. The other parameters are $y = 0.5$, $\phi = 0.6$, $Q = 1$, $\beta = 0.5$, $M = 0.5$.

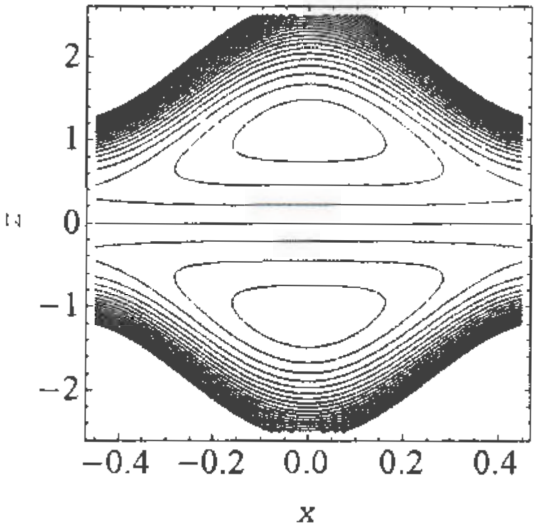

Figure 2.24: Stream lines for $\beta_1 = 0.4$. The other parameters are $y = 0.5$, $\phi = 0.6$, $Q = 1$, $\beta = 0.5$, $M = 0.5$.

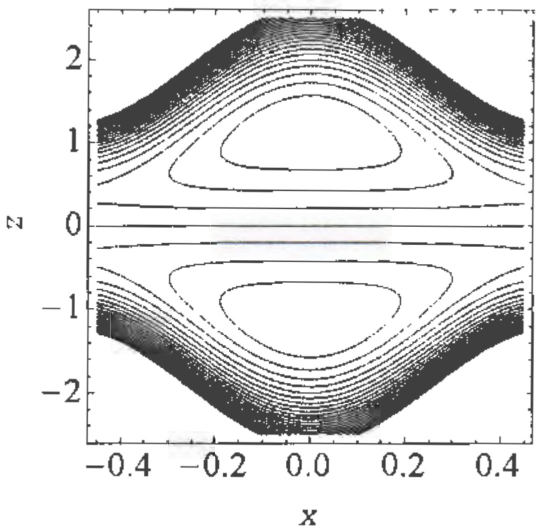

Figure 2.25: Stream lines for $\beta_1 = 0.6$. The other parameters are $y = 0.5$, $\phi = 0.6$, $Q = 1$, $\beta = 0.5$, $M = 0.5$.

Chapter 3

Peristaltic transport of Jeffrey fluid in a rectangular duct through a porous medium under the effects of partial slip

The objective of this chapter is to extend the wok of Ellahi and Hussain [26]. Leading equations along with associated boundary conditions are first made dimensionless using appropriate transformation and then exact solution is achieved viz the method of "Separation of Variables". Impact of related flow parameters against the velocity profile and pressure rise are offered graphically. The streamlines are also presented to examine the trapping phenomenon.

3.1 Formulation of the problem

An incompressible Jeffrey fluid in a duct of rectangular cross section having the channel width $2d$ and height $2a$ as shown in Fig. 3.1 is considered.

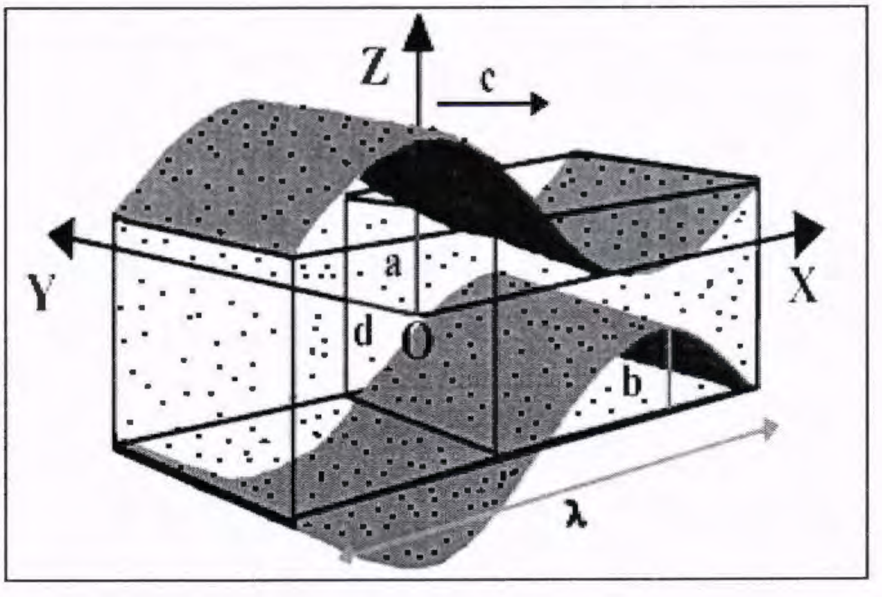

Fig. 3.1. Geometry of the problem.

Let $\boldsymbol{V} = (U, 0, W)$ be the velocity of Jeffrey fluid, moving through peristaltic wave under the influence of porous medium inside a rectangular duct then mathematically one can write it as:

$$Z = H(X,t) = \pm a \pm b \cos\left[\frac{2\pi}{\lambda}(X - ct)\right]. \tag{3.1}$$

(1). Equation of conservation of mass:

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0. \tag{3.2}$$

(2). Equation of momentum:

$$\rho \frac{DV}{Dt} = -\nabla P + \nabla . S - \frac{\mu}{k_1} V. \tag{3.3}$$

The above-mentioned equation of motion in the form of velocity components can be written as:

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + W\frac{\partial U}{\partial Z}\right) = -\frac{\partial P}{\partial X} + \frac{\partial}{\partial X}S_{XX} + \frac{\partial}{\partial Y}S_{XY} + \frac{\partial}{\partial Z}S_{XZ} - \frac{\mu}{k_1}U, \tag{3.4}$$

$$0 = -\frac{\partial P}{\partial Y} + \frac{\partial}{\partial X}S_{YX} + \frac{\partial}{\partial Y}S_{YY} + \frac{\partial}{\partial Z}S_{YZ}, \tag{3.5}$$

$$\rho\left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial X} + W\frac{\partial W}{\partial Z}\right) = -\frac{\partial P}{\partial Z} + \frac{\partial}{\partial X}S_{ZX} + \frac{\partial}{\partial Y}S_{ZY} + \frac{\partial}{\partial Z}S_{ZZ} - \frac{\mu}{k_1}W. \tag{3.6}$$

Jeffrey stress tensor and shear stresses will remain same, which are used in second chapter. By using the same transformation which are given in Eq. (2.11) of the second chapter to convert the given Lab/Fixed frame into the wave frame.

Select the following given variables and parameters in their non-dimensional form

$$\left.\begin{array}{c} \bar{x} = \frac{x}{\lambda}, \bar{y} = \frac{y}{d}, \bar{z} = \frac{z}{a}, \bar{u} = \frac{u}{c}, \bar{w} = \frac{w}{c\delta}, \bar{t} = \frac{ct}{\lambda}, \bar{h} = \frac{H}{a}, \bar{p} = \frac{a^2 p}{\mu c \lambda}, \\ \bar{S}_{\bar{x}\bar{x}} = \frac{a}{\mu c}S_{xx}, \bar{S}_{\bar{x}\bar{y}} = \frac{d}{\mu c}S_{xy}, \bar{S}_{\bar{x}\bar{z}} = \frac{a}{\mu c}S_{xz}, \bar{S}_{\bar{y}\bar{z}} = \frac{d}{\mu c}S_{yz}, \bar{S}_{\bar{y}\bar{y}} = \frac{\lambda}{\mu c}S_{yy}, \\ \phi = \frac{b}{a}, \beta_1 = \frac{L}{a}, Re = \frac{\rho a c \delta}{\mu}, \beta = \frac{a}{d}, \bar{S}_{\bar{z}\bar{z}} = \frac{\lambda}{\mu c}S_{zz}, \delta = \frac{a}{\lambda}, \bar{k} = \frac{k_1}{a^2}. \end{array}\right\}. \tag{3.7}$$

Having used the above given transformation, non-dimensional variables and parameters.

The Eqs. (3.2) to (3.6) after ignoring the signs of bar, take the following form

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{3.8}$$

$$Re\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\delta\frac{\partial}{\partial x}S_{xx}+\beta^2\frac{\partial}{\partial y}S_{xy}+\frac{\partial}{\partial z}S_{xz}-\frac{1}{k}(u+1), \tag{3.9}$$

$$0=-\frac{\partial p}{\partial y}+\delta^2\frac{\partial}{\partial x}S_{yx}+\delta^2\frac{\partial}{\partial y}S_{yy}+\delta\frac{\partial}{\partial z}S_{yz}, \tag{3.10}$$

$$Re\delta^2\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\delta^2\frac{\partial}{\partial x}S_{zx}+\delta\beta^2\frac{\partial}{\partial y}S_{zy}+\delta^2\frac{\partial}{\partial z}S_{zz}-\delta^2\frac{w}{k}. \tag{3.11}$$

Under the supposition of long wave length $\delta \leq 1$ and low Reynolds, ignoring the relations of order δ and higher. Then the above-mentioned Eqs. (3.8) – (3.11) will be reduced to the following non-homogeneous, linear and second order partial differential equation:

$$\frac{\partial^2 u}{\partial y^2}+\frac{1}{\beta^2}\frac{\partial^2 u}{\partial z^2}-\left(\frac{1+\lambda_1}{\beta^2 k}\right)(u+1)=\left(\frac{1+\lambda_1}{\beta^2}\right)\frac{dp}{dx}. \tag{3.12}$$

Therefore, the corresponding slip boundary conditions at the walls are described as:

$$u(x,y,z)=-1,\ at\quad y=\pm 1, \tag{3.13}$$

$$u(x,y,z)+1=-\frac{\beta_1}{1+\lambda_1}\frac{\partial u}{\partial z},\quad when\ z=+h(x)=1+\phi\cos 2\pi x, \tag{3.14}$$

$$u(x,y,z)+1=+\frac{\beta_1}{1+\lambda_1}\frac{\partial u}{\partial z},\quad when\ z=-h(x)=-(1+\phi\cos 2\pi x). \tag{3.15}$$

3.2 Solution of the problem

Solution of the above given problem, is attempt via using the following transformation in Eqs. (3.12) to (3.15).

$$u(x,y,z)=\overline{v_1}(x,y,z)+\overline{w_1}(y). \tag{3.16}$$

Thus Eq. (3.12) provide

$$\frac{\partial^2 \overline{v_1}}{\partial y^2} + \frac{1}{\beta^2}\frac{\partial^2 \overline{v_1}}{\partial z^2} - \left(\frac{1+\lambda_1}{k\beta^2}\right)\overline{v_1} = 0 \tag{3.17}$$

and

$$\frac{\beta^2}{1+\lambda_1}\frac{d^2 \overline{w_1}}{dy^2} - \frac{1}{k}\overline{w_1} = \frac{1}{k} + \frac{dp}{dx}. \tag{3.18}$$

Moreover, the boundary conditions are transformed into the following form

$$\overline{w_1}(y) = -1, \ \ when \ \ y = \pm 1, \tag{3.19}$$

$$\overline{v_1}(x,y,z) = 0, \ \ when \ \ y = \pm 1, \tag{3.20}$$

$$\overline{v_1}(x,y,z) = -\frac{\beta_1}{1+\lambda_1}\frac{\partial \overline{v_1}}{\partial z} - 1 - \overline{w_1}(y), \ \ when \ \ z = h(x), \tag{3.21}$$

$$\overline{v_1}(x,y,z) = +\frac{\beta_1}{1+\lambda_1}\frac{\partial \overline{v_1}}{\partial z} - 1 - \overline{w_1}(y), \ \ when \ \ z = -h(x). \tag{3.22}$$

Interestingly, the transformation which is considered again provide, the main given problem into two systems of differential equations. First one is an PDE (i.e., Eq. 3.17) whereas the second one is ODE (i.e., Eq. 3.18), corresponding to their boundary conditions. Thus the solution of the problem is obtained by solving the Eq. (3.18) and Eq. (3.17), respectively.

3.2.1 Solution of ordinary differential equation

It is noted that Eq. (3.18) is a second order non-homogeneous and linear ODE, which corresponds to non-homogeneous boundary conditions Eq. (3.19). So, the general solution is given

$$\overline{w_1}(y) = -1 - k\frac{dp}{dx}\left[1 - \operatorname{sech}\left(\sqrt{\frac{1+\lambda_1}{\beta^2 k}}\right)\cosh\left(\sqrt{\frac{1+\lambda_1}{\beta^2 k}}\, y\right)\right]. \tag{3.23}$$

3.2.2 Solution of partial differential equation

In order to attain the solution of Eq. (3.17), this is second order, homogeneous and linear partial differential equation (PDE), subject to the homogeneous boundary conditions Eq. (3.20). By implementing process of "Separation of Variables" it is supposed that

$$\overline{v_1}(x, y, z) = Y(y) \times Z(z), \tag{3.24}$$

is one of the possible solutions of Eq. (3.17). By using the Eq. (3.24) in Eq. (3.16) which provides

$$\frac{Y''}{Y} = \frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{(1+\lambda_1)}{k\beta^2} = -\alpha^2, \tag{3.25}$$

$$\Rightarrow \frac{Y''}{Y} = -\alpha^2 \tag{3.26}$$

and

$$\frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{(1+\lambda_1)}{k\beta^2} = -\alpha^2. \tag{3.27}$$

Using the boundary conditions of Eq. (3.20), in Eq. (3.24) yield

$$[Y(\pm 1)] \times [Z(z)] = 0, \tag{3.28}$$

$$\Rightarrow Z(z) \neq 0, \tag{3.29}$$

this implies that

$$Y(\pm 1) = 0. \tag{3.30}$$

Then, there are two possible cases to attain the mandatory solution.

Case-I

From Eq. (3.26)

$$\frac{Y''}{Y} = -\alpha^2, \tag{3.31}$$

$$\Rightarrow [D^2 + \alpha^2] \times Y(y) = 0, \tag{3.32}$$

$$\Rightarrow Y(y) \neq 0, \tag{3.33}$$

$$\Longrightarrow [D^2 + \alpha^2] = 0, \tag{3.34}$$

$$D = \pm(i\alpha). \tag{3.35}$$

It is familiar that a trivial solution is attained for the values of $\alpha = 0$ and $\alpha < 0$. The only non-trivial solution is obtained, for the value of $\alpha > 0$. Which yields

$$Y(y) = c_3 \cos(\alpha y) + c_4 \sin(\alpha y), \tag{3.36}$$

where c_3 and c_4 are the constants that are determined by using $Y(\pm 1) = 0$, in above equation. It provides

$$c_4 = 0,$$

$$\alpha_n = \left(\frac{(2n-1)\pi}{2}\right), \qquad n = 1, 2, 3 \ldots$$

Thus the Eq. (3.36) becomes

$$Y(y) = c_3 \cos\left(\frac{(2n-1)\pi}{2} y\right), \qquad n = 1, 2, 3 \ldots \tag{3.37}$$

Case-II

Similarly, from Eq. (3.27)

$$\frac{-1}{\beta^2}\frac{Z''}{Z} + \frac{(1+\lambda_1)}{k\beta^2} = -\alpha^2, \tag{3.38}$$

$$\Longrightarrow \frac{-1}{\beta^2}\frac{Z''}{Z} = -\alpha^2 - \frac{(1+\lambda_1)}{k\beta^2}, \tag{3.39}$$

$$\Longrightarrow D = \pm\sqrt{\alpha^2\beta^2 + \frac{(1+\lambda_1)}{k}}. \tag{3.40}$$

This implies that

$$Z(z) = c_5 \cosh\left(z\sqrt{\alpha^2\beta^2 + \frac{(1+\lambda_1)}{k}}\right) + c_6 \sinh\left(z\sqrt{\alpha^2\beta^2 + \frac{(1+\lambda_1)}{k}}\right). \tag{3.41}$$

Substituting the values of $Y(y)$ and $Z(z)$ in Eq. (3.24). This leads to be

$$\left.\begin{aligned}\overline{v_1}(x,y,z) = c_3\left[c_5\cosh\left(z\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)+c_6\sinh\left(z\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)\right]\\ \times\cos\left(\frac{(2n-1)\pi}{2}y\right)\end{aligned}\right\}. \tag{3.42}$$

For suitability, it is supposed that

$$c_7 = c_3 \times c_5, \text{ (say)}$$

$$c_8 = c_3 \times c_6. \text{ (say)}$$

Consequently, Eq. (3.42) becomes

$$\overline{v_1}(x,y,z) = \left.\begin{aligned}&\left[\begin{aligned}&c_7\cosh\left(z\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)\\ &+c_8\sinh\left(z\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)\end{aligned}\right]\\ &\times\cos\left(\frac{(2n-1)\pi}{2}y\right)\end{aligned}\right\}. \tag{3.43}$$

To calculate the values of the two constants given in the above equation, use the boundary condition Eqs. (3.21) - (3.22). This yields

$$-1-\overline{w_1}(y)-\frac{\beta_1}{1+\lambda_1}\frac{\partial\overline{v_1}}{\partial z} = \left.\begin{aligned}&\left[\begin{aligned}&c_7\cosh\left(h\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)\\ &+c_8\sinh\left(h\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)\end{aligned}\right]\\ &\times\cos\left(\frac{(2n-1)\pi}{2}y\right)\end{aligned}\right\}, \tag{3.44}$$

$$-1-\overline{w_1}(y)+\frac{\beta_1}{1+\lambda_1}\frac{\partial\overline{v_1}}{\partial z} = \left.\begin{aligned}&\left[\begin{aligned}&c_7\cosh\left(h\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)\\ &-c_8\sinh\left(h\sqrt{\alpha^2\beta^2+\frac{(1+\lambda_1)}{k}}\right)\end{aligned}\right]\\ &\times\cos\left(\frac{(2n-1)\pi}{2}y\right)\end{aligned}\right\}. \tag{3.45}$$

Substitute the value of $\overline{w_1}(y)$ into the Eqs. (3.44) to (3.45) and integrate both sides of the equations w.r.t 'y' from '0' to 'h' one get

$$\left.\overline{v_1}(x,y,z) = \frac{32k\frac{dp}{dx}(1+\lambda_1)^2\cos\left(\frac{\pi y}{2}\right)\cos\left(z\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)}{\pi\left(k(\pi\beta)^2+4\lambda_1+4\right)\times\left(\begin{array}{c}2(1+\lambda_1)\cosh\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)\\ +\beta_1\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\sinh\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)\end{array}\right)}\right\}. \quad (3.46)$$

Since the values of $\overline{v_1}(y,z)$ and $\overline{w_1}(y)$ are known. Thus, Eq. (3.16) changes

$$u(x,y,z) = \left(\begin{array}{c}-1-k\frac{dp}{dx}\left[1-\operatorname{sech}\left(\sqrt{\frac{1+\lambda_1}{\beta^2 k}}\right)\cosh\left(y\times\sqrt{\frac{1+\lambda_1}{\beta^2 k}}\right)\right]\\ +\frac{32k\frac{dp}{dx}(1+\lambda_1)^2\cos\left(\frac{\pi y}{2}\right)\cos\left(z\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)}{\pi\left(k(\pi\beta)^2+4\lambda_1+4\right)\times\left(\begin{array}{c}2(1+\lambda_1)\cosh\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)\\ +\beta_1\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\sinh\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)\end{array}\right)}\end{array}\right). \quad (3.47)$$

The above equation mentioned Eq. (3.47) shows the required velocity profile of the Jeffrey fluid peristaltically moving in the rectangular duct. Integrating Eq. (3.47) twice w.r.t z and y, respectively thus the Eq. (3.47) gives the volumetric flow rate of the peristaltic wave that is

$$q = \frac{128\frac{dp}{dx}(1+\lambda_1)}{\pi^2\left(\frac{k(\pi\beta)^2+4\lambda_1+4}{k}\right)^{\frac{3}{2}}\left\{2\coth\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)+\sqrt{\frac{\beta_1{}^2(k(\pi\beta)^2+4\lambda_1+4)}{k(1+\lambda_1)^2}}\right\}}$$

$$-\frac{(1+\phi\cos 2\pi x)\left\{\left(1+k\frac{dp}{dx}\right)\sqrt{1+\lambda_1}-k^{\frac{3}{2}}\frac{dp}{dx}\beta\tanh\left(\sqrt{\frac{1+\lambda_1}{k\beta^2}}\right)\right\}}{\sqrt{1+\lambda_1}}. \quad (3.48)$$

Average volumetric flow rate over one period ($T = \lambda/c$) of the peristaltic wave is defined as

$$Q = \frac{1}{T}\int_0^T \bar{Q}dt = q+1, \quad (3.49)$$

where

$$\bar{Q} = \int_0^1 \int_0^h (u(x,y,z)+1)\, dzdy = q + h. \tag{3.50}$$

Consequently, the average volumetric flow rate is given as

$$Q = \frac{128\frac{dp}{dx}(1+\lambda_1)}{\pi^2\left(\frac{k(\pi\beta)^2+4\lambda_1+4}{k}\right)^{\frac{3}{2}}\left\{2\coth\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)+\sqrt{\frac{\beta_1{}^2(k(\pi\beta)^2+4\lambda_1+4)}{k(1+\lambda_1)^2}}\right\}}$$

$$-\frac{\sqrt{1+\lambda_1}+(1+\phi\cos 2\pi x)\left\{\left(1+k\frac{dp}{dx}\right)\sqrt{1+\lambda_1}-k^{\frac{3}{2}}\frac{dp}{dx}\beta\tanh\left(\sqrt{\frac{1+\lambda_1}{k\beta^2}}\right)\right\}}{\sqrt{1+\lambda_1}}. \tag{3.51}$$

However, from the above Eq. (3.51) the expression for pressure gradient can be calculated as

$$\frac{dp}{dx} = \frac{\pi^2(Q+\phi\cos 2\pi x)\left(\frac{k(\pi\beta)^2+4\lambda_1+4}{k}\right)^{\frac{3}{2}}\left\{\sqrt{\frac{\beta_1{}^2(k(\pi\beta)^2+4\lambda_1+4)}{k(1+\lambda_1)^2}}+2\coth\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)\right\}}{\left\{\begin{array}{c}128(1+\lambda_1)+\pi^2(1+\phi\cos 2\pi x)\frac{\left((k(\pi\beta)^2+4\lambda_1+4)\right)^{\frac{3}{2}}}{\sqrt{k}}\times \\ \left\{\sqrt{\frac{\beta^2 k}{1+\lambda_1}}\tanh\left(\sqrt{\frac{1+\lambda_1}{k\beta^2}}\right)-1\right\}\left\{2\coth\left((1+\phi\cos 2\pi x)\sqrt{\frac{k(\pi\beta)^2+4\lambda_1+4}{4k}}\right)+\sqrt{\frac{\beta_1{}^2(k(\pi\beta)^2+4\lambda_1+4)}{k(1+\lambda_1)^2}}\right\}\end{array}\right\}}. \tag{3.52}$$

The pressure difference may be computed along the axial length by the expression

$$\Delta p = \int_0^1 \frac{dp}{dx}\, dx. \tag{3.53}$$

3.3 Results and discussion

In this segment, we purpose the approximate variations in the tendency of flow and discussed thorough graphs for different parameters. The most significant parameters are, aspect ratio β, volumetric flow rate Q, aspect ratio β, slippage parameter β_1, Jeffrey fluid parameter λ_1 and above all, the porosity parameter k.

Keeping the numerical values of some parameters, we realize from the figure 3.1, that the velocity profile, of the wave which propagates in the rectangular duct, is slowly increasing in reverse direction time being, corresponding to upturn the values of slip parameter β_1. On the contrary in figures 3.2 – 3.4, the velocity profile of the fluid through porous medium, keeps on increasing corresponding to the increase in porosity parameter, aspect ratio and Jeffrey parameter respectively.

Having witnessed the graphs presented in figure 3.5 - 3.7, it is noticed that pressure rise Δp is decreasing with the increase of slip parameter β_1, Jeffrey parameter λ_1 and porosity parameter k. Yet, in the figure 3.8 the variation trend of pressure rise is quite opposite as compare to figures 3.5 – 3.7.

Figures 3.9 – 3.24 display the change in stream lines as Jeffrey parameter, aspect ratio, porosity and slip parameter are varied respectively. Figures 3.9 – 3.16 show the streams lines for various values of Jeffrey parameter and aspect ratio. It is noticed that figure of boluses are reducing when we give higher values to Jeffrey parameter λ_1 and aspect ratio β. While, in figures 3.17 – 3.24, stream lines behave differently for both porosity and slip parameter, by causing enough resistance at the wall which is evinced by the emergence of new boluses.

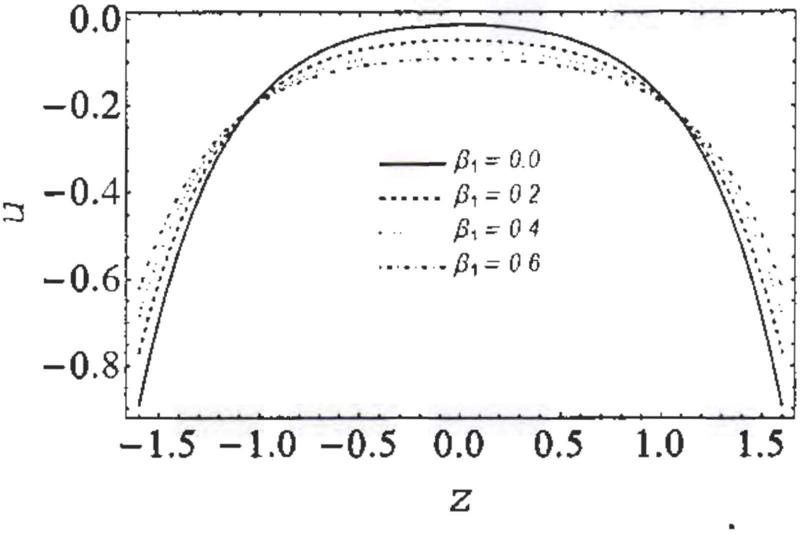

Figure 3.1: Variation in the velocity trend for unlike values of β_1 when $\phi = 0.6$, $x = 0$, $y = 0.5$, $Q = 0.5$, $k = 0.5$, $\lambda_1 = 2$, $\beta = 0.5$.

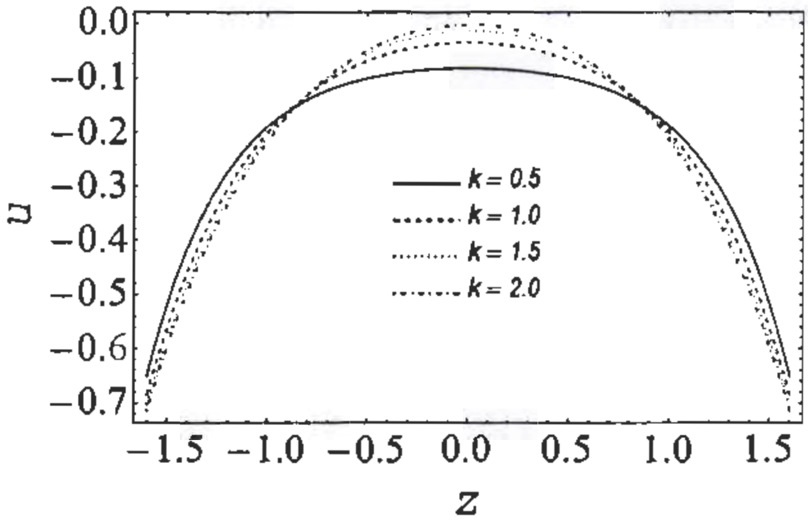

Figure 3.2: Variation in the velocity trend for unlike values of k when $\phi = 0.6$, $x = 0$, $y = 0.5$, $Q = 0.5$, $\lambda_1 = 2$, $\beta = 0.5, \beta_1 = 0.5$.

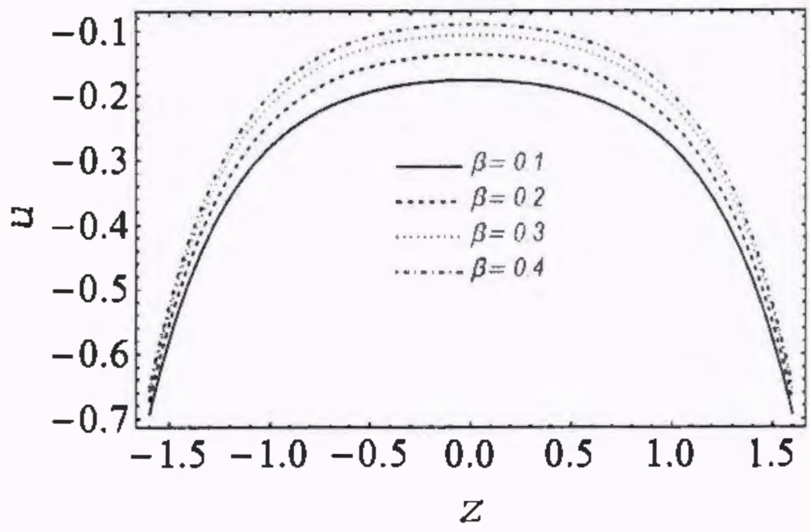

Figure 3.3: Variation in the velocity trend for unlike values of β when $\phi = 0.6$, $x = 0$, $y = 0.5$, $Q = 0.5$, $k = 0.5$, $\lambda_1 = 2$, $\beta_1 = 0.5$.

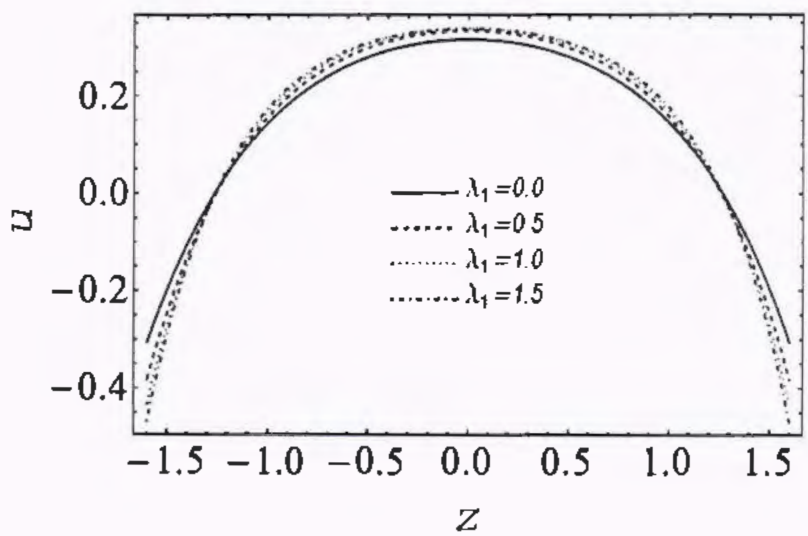

Figure 3.4: Variation in the velocity trend for unlike values of λ_1 when $\phi = 0.6$, $x = 0$, $y = 0.5$, $Q = 0.5$, $k = 0.5$, $\beta = 2$, $\beta_1 = 0.5$.

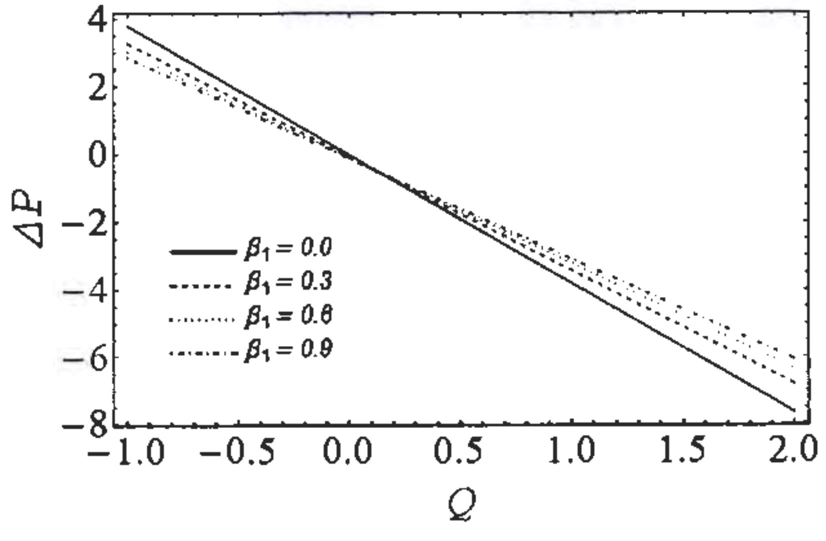

Figure 3.5: Change in Δp with Q for unlike values of β_1 when $\phi = 0.6$, $Q = 0.5$, $k = 0.5, \beta = 0.5, \lambda_1 = 2$.

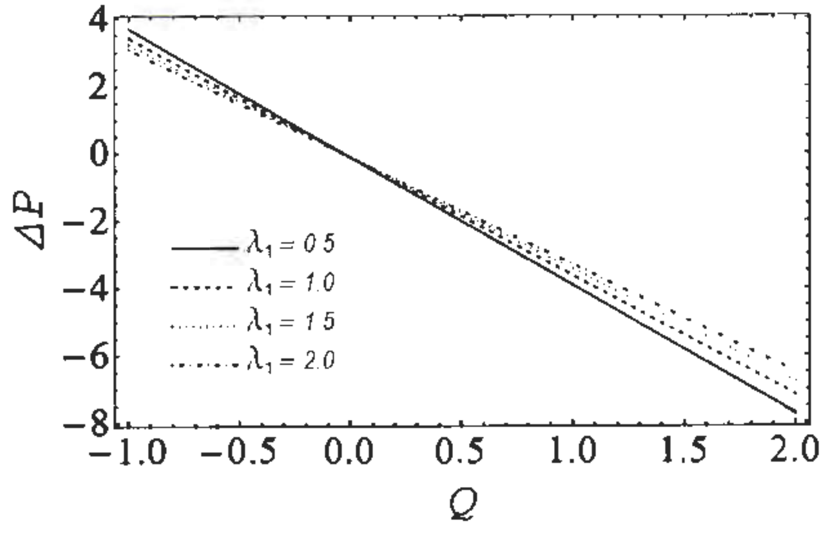

Figure 3.6: Change in Δp with Q for unlike values of λ_1 when $\phi = 0.6$, $Q = 0.5$, $k = 0.5, \beta = 0.5, \beta_1 = 0.5$.

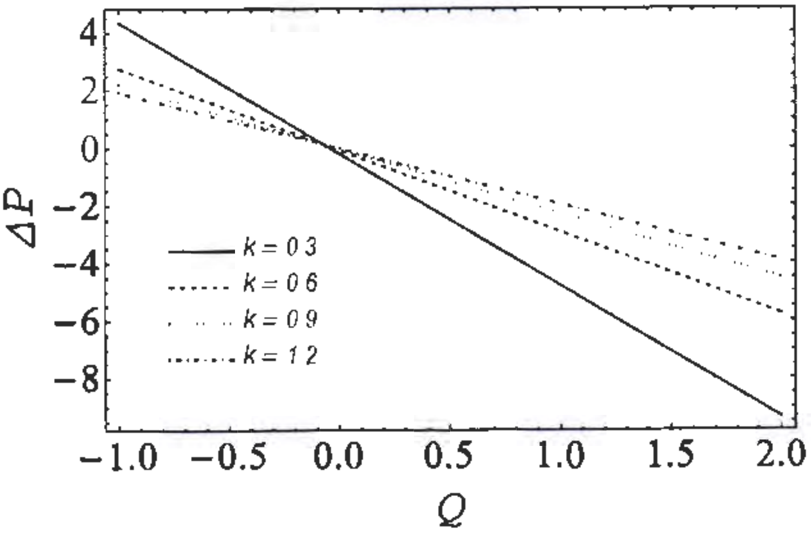

Figure 3.7: Change in Δp with Q for unlike values of k when $\phi = 0.6$, $Q = 0.5$, $\beta_1 = 0.5, \beta = 0.5, \lambda_1 = 2$.

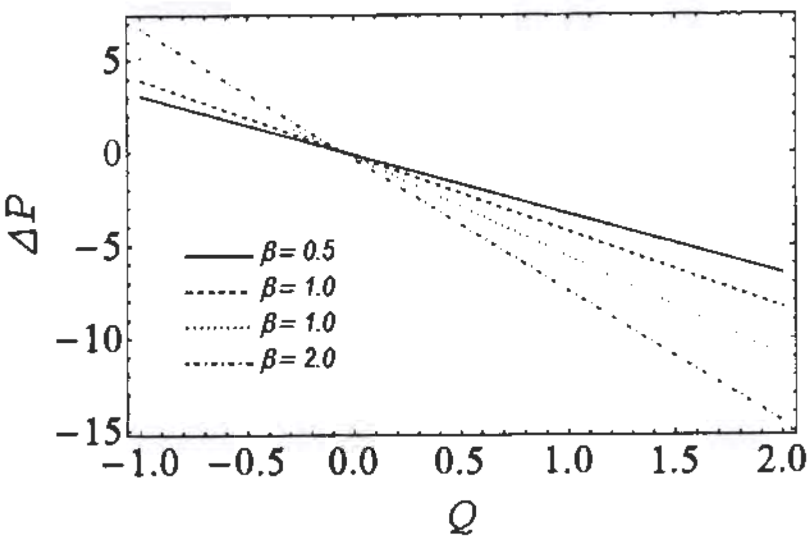

Figure 3.8: Change in Δp with Q for unlike values of β when $\phi = 0.6$, $Q = 0.5$, $\beta_1 = 0.5, k = 0.5, \lambda_1 = 2$.

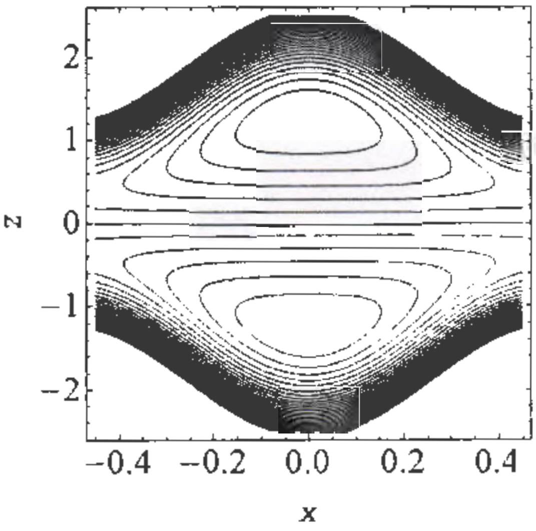

Figure 3.9: Display of stream lines for $\lambda_1 = 0$. The other parameters are $\phi = 0.6$, $\beta_1 = 0.5, \beta = 0.5, k = 0.5, Q = 0.5$.

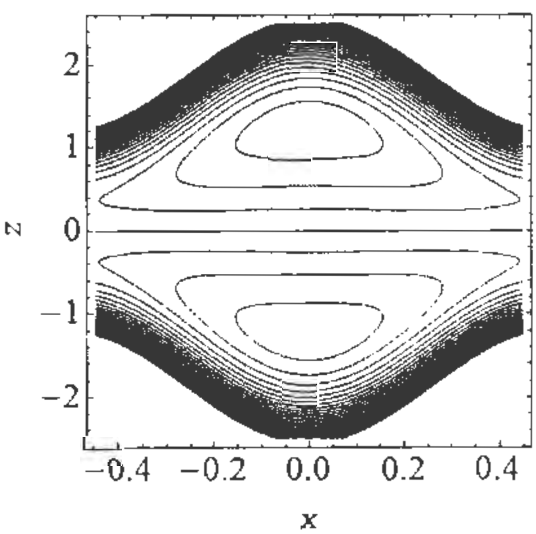

Figure 3.10: Display of stream lines for $\lambda_1 = 0.5$. The other parameters are $\phi = 0.6$, $\beta_1 = 0.5, \beta = 0.5, k = 0.5, Q = 1$.

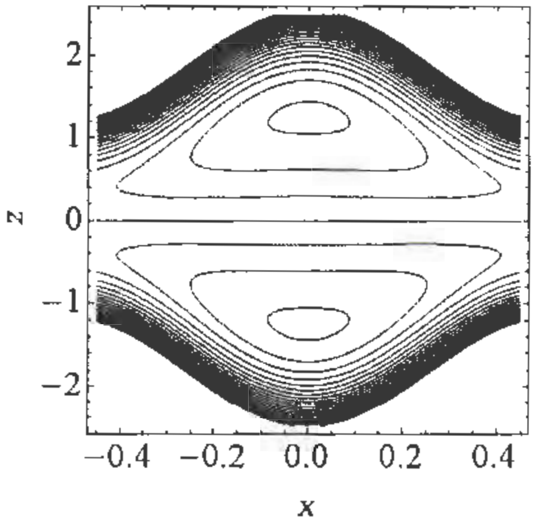

Figure 3.11: Display of stream lines for $\lambda_1 = 1$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, k = 0.5, \beta_1 = 0.5, \lambda_1 = 2$.

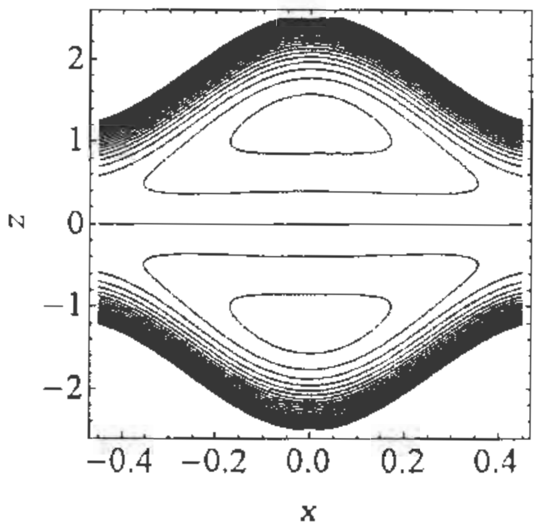

Figure 3.12: Display of stream lines for $\lambda_1 = 1.5$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, k = 0.5, \beta_1 = 0.5, \lambda_1 = 2$.

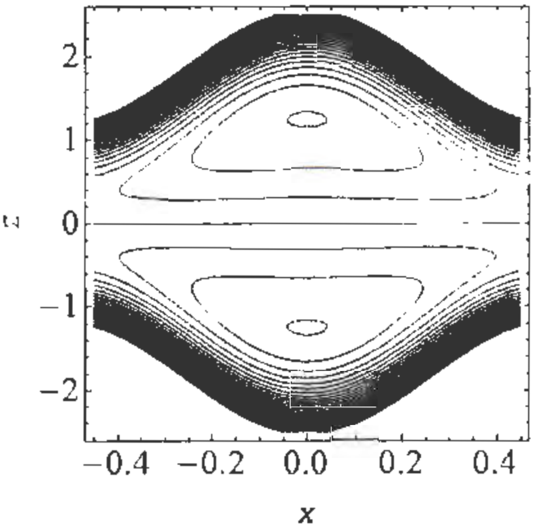

Figure 3.13: Display of stream lines for $\beta = 0.4$. The other parameters are $\phi = 0.6$, $Q = 1$, $\beta_1 = 0.5$, $k = 0.5$, $k = 0.5$, $\lambda_1 = 2$.

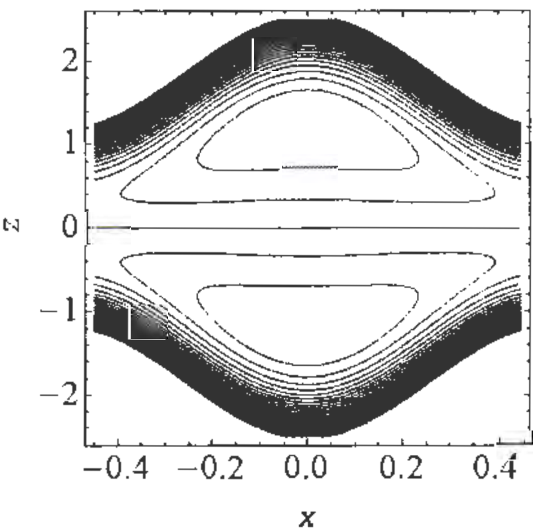

Figure 3.14: Display of stream lines for $\beta = 0.7$. The other parameters are $\phi = 0.6$, $Q = 1$, $\beta_1 = 0.5$, $k = 0.5$, $k = 0.5$, $\lambda_1 = 2$.

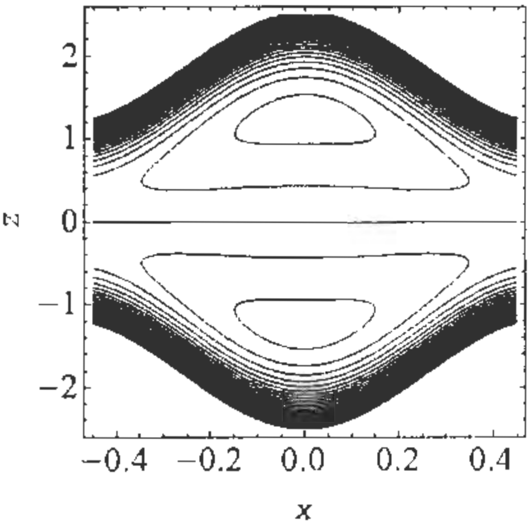

Figure 3.15: Display of stream lines for $\beta = 1$. The other parameters are $\phi = 0.6$, $Q = 1, \beta_1 = 0.5, k = 0.5, k = 0.5, \lambda_1 = 2$.

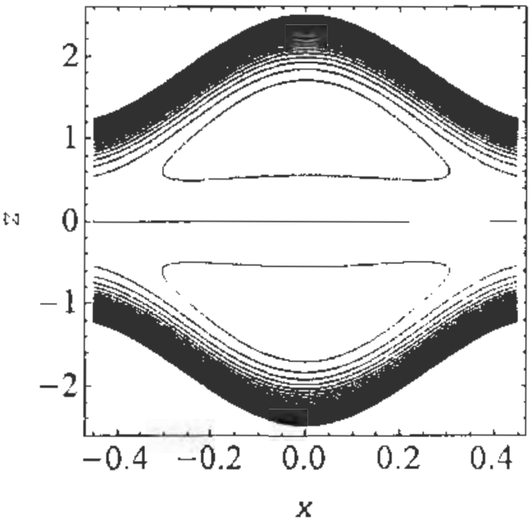

Figure 3.16: Display of stream lines for $\beta = 1.3$. The other parameters are $\phi = 0.6$, $Q = 1, \beta_1 = 0.5, k = 0.5, k = 0.5, \lambda_1 = 2$.

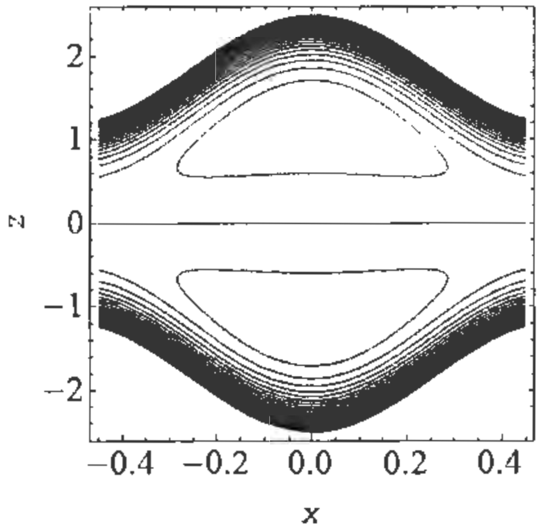

Figure 3.17: Display of stream lines for $k = 0.3$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, \beta_1 = 0.5, \lambda_1 = 2$.

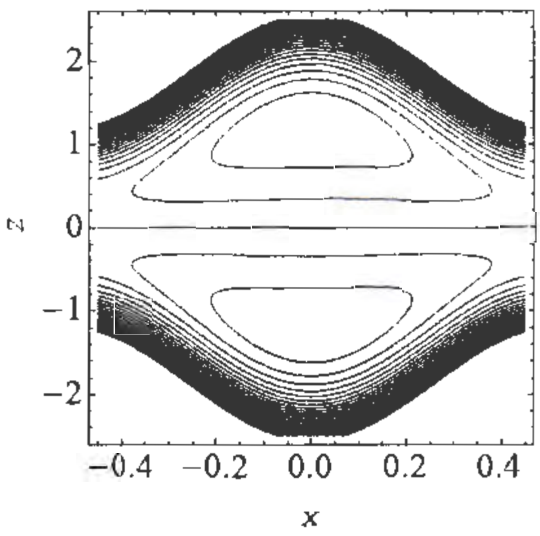

Figure 3.18: Display of stream lines for $k = 0.6$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, \beta_1 = 0.5, \lambda_1 = 2$.

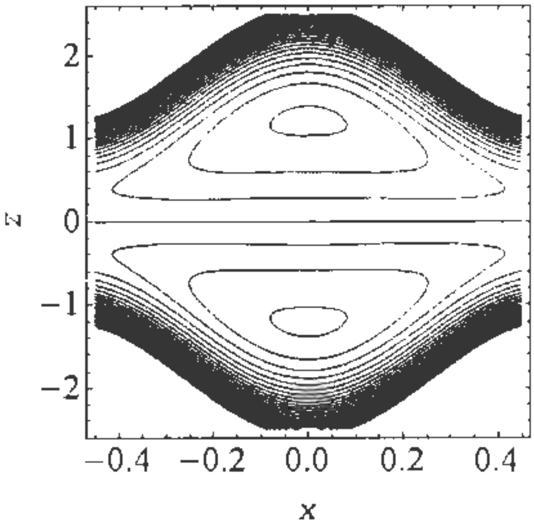

Figure 3.19: Display of stream lines for $k = 0.9$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, \beta_1 = 0.5, \lambda_1 = 2$.

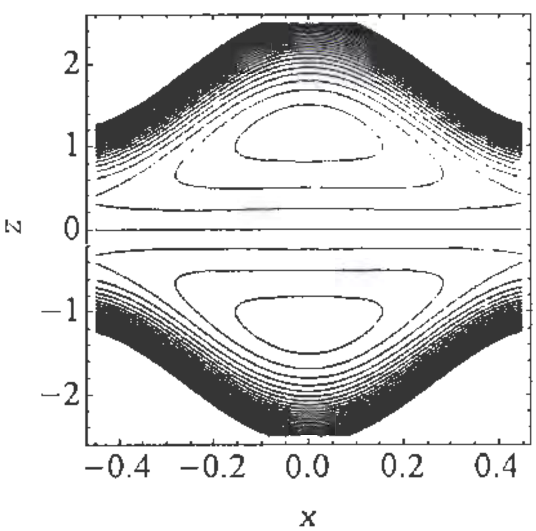

Figure 3.20: Display of stream lines for $k = 1.2$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, \beta_1 = 0.5, \lambda_1 = 2$.

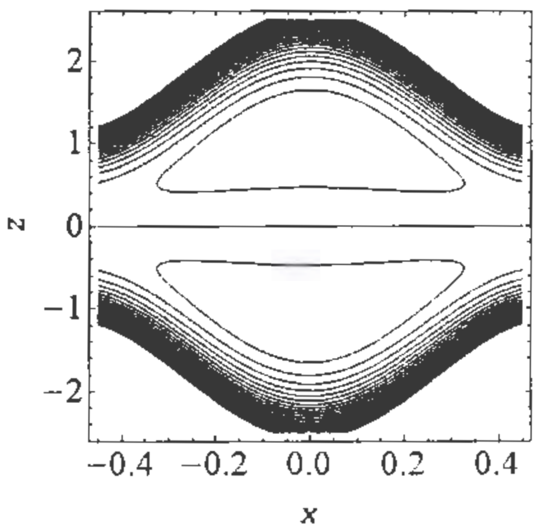

Figure 3.21: Display of stream lines for $\beta_1 = 0.0$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, k = 0.5, \lambda_1 = 2$.

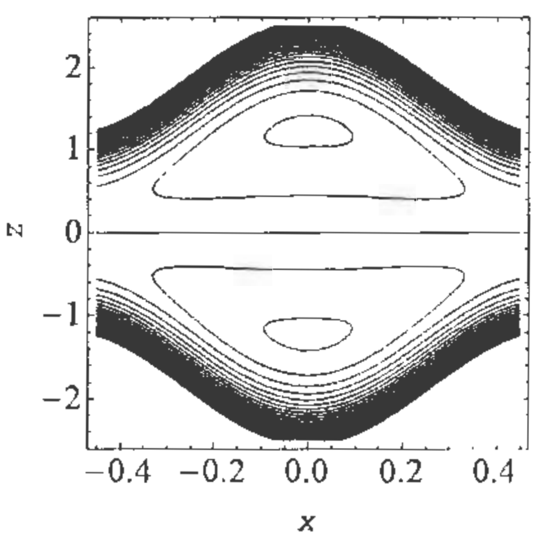

Figure 3.22: Display of stream lines for $\beta_1 = 0.3$. The other parameters are $\phi = 0.6$, $Q = 1, \beta = 0.5, k = 0.5, \lambda_1 = 2$.

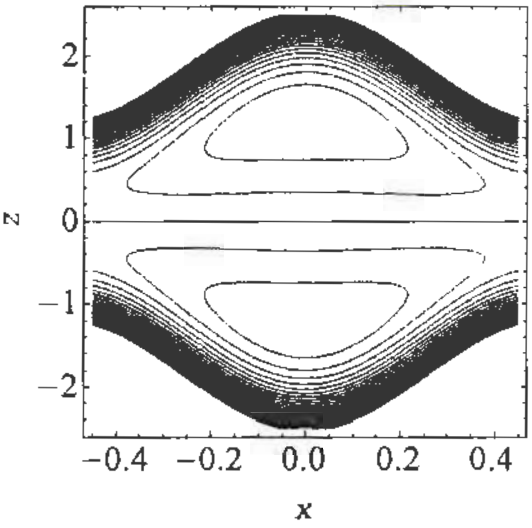

Figure 3.23: Display of stream lines for $\beta_1 = 0.6$. The other parameters are $\phi = 0.6$, $Q = 1$, $\beta = 0.5$, $k = 0.5$, $\lambda_1 = 2$.

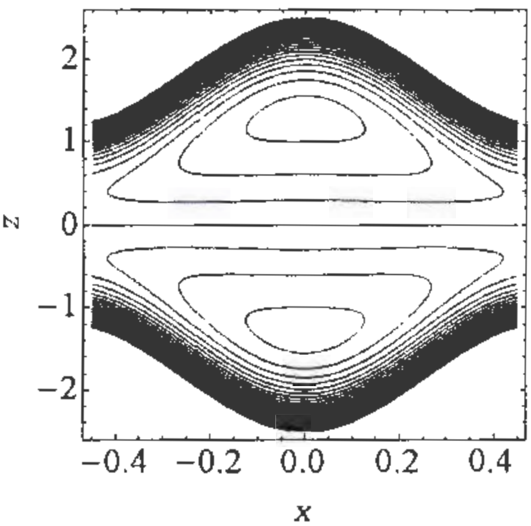

Figure 3.24: Display of stream lines for $\beta_1 = 0.9$. The other parameters are $\phi = 0.6$, $Q = 1$, $\beta = 0.5$, $k = 0.5$, $\lambda_1 = 2$.

3.4 Conclusion

An exact solution for the peristaltically moving under the application of slip and porosity has been done under the constrains of low Reynolds and long wave length. It may be noticed that variations in the velocity profile decreases by increasing the slip parameter while quit opposite behavior is noted for the case of porosity parameter, aspect ratio and Jeffrey parameter. It is also observed that a linear dependence of pressure rise per unit length, pressure rise increase for β, while decreases for slip, porosity and Jeffrey parameter. Furthermore, the trapped blouse above and below formed there is an increase with increase in k and β_1 while decrease for λ_1 and β. Also, Newtonian model of fluid can also be deduced by taking $\lambda_1 = 0$, as a special case of presented model.

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