Non-Newtonian flow in slider and journal bearing systems

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A Thesis
Submitted in the Partial Fulfillment of the
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MASTER OF SCIENCE IN MATHEMATICS

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We accept this dissertation as conforming to the required standard.

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Declaration

I hereby declare and affirm that this research work neither as a whole nor as a part has been copied out from any source. It is further declared that I have developed this research work entirely on the basis of my personal efforts.

Moreover, no portion of the work presented in this thesis has been submitted in support of an application for other degree of qualification in this or any other university or institute of learning.

Signature: ____________

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بسم الله الرحمن الرحيم

Dedication

Dedicated to

My Parents

And

All my teachers

Whose prayer has always been the reason of my

Success and prosperity in life.

Acknowledgement

The first person that I need to thank is **Dr. Nasir Ali**, under whose supervision I began my research **Non-Newtonian flow in slider and journal bearing systems**. In this research we discuss non-Newtonian flow for second and third grade fluids. This was my first hands-on project at International Islamic University, Islamabad. I owe gratitude to my kind supervisor for his open door policy that also facilitated the process of research and admit that I could not have had a better research experience without his guidance.

I owe a great deal to **Dr. Nasir Ali** for taking me under his scholarly wing when my research seemed to be going in circles. He suggested that I should review work of **Yurusoy** 'pressure distribution in a slider bearing lubricated with second and third grade fluids' and 'perturbation theory for visco-elastic fluid between rotating eccentric cylinders' based on **A. N. Baris, R. C. Armstrong and R. A. Brown**, which is presented in this thesis. He helped me combine the experimental and computational aspects of the topic, so that I could bring about contribution to the wealth of knowledge on the topic. **Dr. Nasir Ali** was always willing to answer any questions, and made sure that the answers were understood before leaving. His constant encouragement, support, and enthusiasm about the topic helped me keep going and finish my research making sure that I did not give up on my quest for knowledge.

I would not be so mean as to forget thanking my family and friends for their continued support through these years. I am grateful for all of the people I have met over specially **Mr. Khurram, Mr. Asif** and **Mr. Akbar Zaman**. Having the support and advice of so many people helped in making me who I am today and never let me forget where I came from. Special thanks are due to **Dr. Nasir Ali** who stood by me through the ups and downs of my life at IIUI. I could not have asked for a better friend. Words just cannot express the emotions behind them to do justice they deserve.

Asad Ullah Khan

Preface

The knowledge of flow in slider and journal bearing systems is quite useful for proper functioning of these instruments. Numerous studies are available where Newtonian model is used to analyze the flow. However, due to the use of additives in lubricating fluids, the flow in the bearing systems becomes non-Newtonian. In this dissertation idealized geometrics are used to study the flow in slider and journal bearing systems. The geometry of slider journal bearing system consist of a fixed surface inclined at an angle over a horizontal sliding surface, while the geometry of the journal bearing comprises of two eccentric cylinders one of which is rotating at constant angular speed. Some specialized literature on flow in a slider and journal bearing systems is reviewed in the next paragraph.

Some of the relevant studies on non –Newtonian lubrication in bearing are as follows: Ng and Saibel [1] used a special third grade fluid (second grade terms neglected) and studied the flow occurring in a slider bearing. Harnoy and Hanin [2] and Harnoy and Philippoff [3] studied the flow of a second grade fluid in a journal bearing. Bourgin and gay [4] used a similar model with that of Ng and Saibel [1] to investigate the behavior of flow in a journal bearing. Buckholz [5] used a power-law model as a non-Newtonian lubricant in a slider bearing. Kacou et al [6] studied the flow of a third grade fluid in a journal bearing and constructed a perturbative solution. The work is extended by the same author (Kacou et al [7]) by including thermal effects. Yürüsoy and Pakdemirli [8] studied the flow of a special third grade fluid in a slider bearing. Lot of work has also been done on various shapes of slider bearings. For instance Lin *et al.* [9] performed linear stability analysis of a wide inclined plane slider bearing. An analysis of dynamic characteristics for wide slider bearing with an exponential film profile was carried out by Lin and Hung [10]. The characteristics of parabolic slider bearing and tapered land slider bearing were also studied by Lin et al. [11,12].

Motivated by above studies, we in this dissertation review two important articles on flow in slider and journal bearing systems by Yürüsoy [13] and Beris et al. [14]. The dissertation is based on three chapters. Chapter 1 is introductory and includes some basic definitions and equations. In chapter 2, the work of Yürüsoy [13] is reproduced with full details. Here pressure distribution in a slider bearing is studied which lubricated with second and third grade fluids. In chapter 3, the flow in a journal bearing system is studied using constitutive equation of upper convected Maxwell fluid. This paper is review of work by Beris at al. [14]. The velocity and pressure field are computed using domain perturbation method and discussed in detail.

Contents

Chapter 1

Preliminaries

The purpose of this chapter is to introduce the readers with some of the basic terminologies of fluid mechanics. We start with some basic definitions and then present the classification of fluids and their flow. Different non-Newtonian models slider and journal bearing systems and some dimensionless numbers are also introduced in this chapter. The chapter ends up with a being account on solution methodology.

1.1 Some basic definitions

1.1.1 Fluid

A substance which deforms continuously under the application of shear stress is called fluid. It moves and flow under the action of the force. Fluids are conventionally classified as either liquids or gases.

1.1.2 Fluid mechanics

Fluid mechanics deals with the study of the motion of fluid and the forces acts on it, i.e. liquid and gases. Velocity and acceleration are the two basic field variables in the study of fluid mechanics.

1.1.3 Fluid dynamics

Fluid dynamics deals with the study of the effect of forces on fluid flow and the natural sciences of fluid.

1.1.4 Fluid kinematics

Fluid kinematics referred the study of quantities involving space and time only. It describes the motion of particles and objects.

1.1.5 Hydro-statics or Fluid statics

Fluid statics deals with the study of fluid at rest. Any force developed is only due to normal stresses i.e. pressure.

1.2 Flow

When a force is applied on a material, it goes under deformation, if the deformation exceeds continuously without limit, this phenomena is called flow. Flow is also described as movement of gases and liquids in a medium.

1.3 Types of flow

On the basis of the flow parameters the fluids flow can be classified as:

1.3.1 Uniform flow

Flow is said to be uniform, if velocity has the same magnitude and direction at any point in the fluid.

1.3.2 Non-uniform flow

Flow is said to be non-uniform if velocity is not the same at any point in the fluid (every fluid that flows near the boundary of a solid will be considered as non-uniform as the fluid speed is zero).

1.3.3 Steady flow

Flow is steady if the fluid's velocity and pressure at a particular fixed point remains constant with time. Cross section area is constant through the flow path.

1.3.4 Rotational flow

In a flow if every fluid's particle rotates about its own axes the flow is called rotational flow.

$$\nabla \times \mathbf{V} \neq 0, \tag{1.1}$$

where V is the velocity field.

1.3.5 Irrotational flow

In a flow if the fluid particles do not rotate about its axes, the flow is called irrotational flow.

$$\nabla \times \mathbf{V} = 0. \tag{1.2}$$

1.3.6 Turbulent flow

In turbulent flow, fluid particles do not flow parallel to the path in the form of layers. In this flow velocities vary erratically from point to point as well as time to time.

1.3.7 Inviscid flow

An ideal fluid flow showing no viscosity is called an inviscid flow.

1.3.8 Viscous flow

Viscous flows are always rotational because of shear stress that is exerted on the fluid element due to viscosity.

1.3.9 Compressible flow

Flow in which density of a fluid is not constant during the flow is called compressible flow.

1.3.10 Incompressible flow

Flow in which density remain constant is called incompressible flow. Flow of liquids usually be considered as incompressible.

1.4 Lubricant

A substance which reduces the friction, if introduced between moving surfaces, is called lubricant.

1.4.1 Lubrication

For preventing the friction and wear, introduction of a lubricant between moving surfaces is called lubrication.

1.5 Types of lubrication

Out of the many types of lubrication, we introduce the following three relevant types:

1.5.1 Hydro-dynamic lubrication

When moving loaded surfaces are separated by thick film of lubricant for prevention from contact is called hydrodynamics lubrication.

1.5.2 Hydro-static lubrication

At high pressure when a lubricant is introduced into a loaded bearing region, it is called hydrostatic lubrication.

1.5.3 Boundary lubrication

When lubricant is introduced at the boundary for increasing velocity, it is called boundary lubrication.

1.6 Fluid properties

1.6.1 Pressure

Pressure is the magnitude of normal force acting on a unit surface area. It is a scalar quantity.

$$P = \frac{F}{A}, \tag{1.3}$$

where, F is the magnitude of force acting in the direction perpendicular to the surface of the fluid and A is the area of the surface of the fluid.

1.6.2 Density

The mass of unit volume of the fluid at a certain temperature and pressure is called the density of the fluid.

1.6.3 Viscosity

The viscosity is a measurement of resistance to the motion of the fluid. Viscosity is the ratio of shear stress to the rate of shear strain.

1.6.4 Zero-shear viscosity

When shear rate approaches to zero, the measured viscosity is called zero-shear viscosity.

1.6.5 Kinematic viscosity

The kinematic viscosity (also called momentum diffusivity) is the ratio of the dynamic viscosity μ to the density of the fluid. It is usually denoted by the ν and is given by

$$\nu = \frac{\mu}{\rho}. \tag{1.4}$$

1.6.6 Visco-Elasticity

Viscoelasticity is the property of a material to demonstrate both viscous and elastic properties under the same conditions when it undergoes deformation. Viscous materials present resistance to shear flow and strain linearly with time when a stress is applied. Some common and well-known viscoelastic materials include paint, blood, ketchup, honey, mayonnaise, polymer melt, polymer solution and suspension, shampoo, and corn starch.

1.7 Classification of fluids

Following are the types of fluids

1) Ideal fluid.
2) Real fluid.
3) Newtonian fluid.
4) Non-Newtonian fluid.

1.7.1 Ideal fluid

An Ideal fluid is non-viscous and offers no resistance $\mu = 0$ whatsoever to a shearing force. An ideal fluid really does not exist. One example of this is the flow far from solid surfaces. In the region of the flow field far away from the boundaries the viscous effects can be neglected and the fluid is treated as inviscid (ideal fluid). In an ideal fluid, there is no existence of shear force because of vanishing viscosity.

1.7.2 Real fluid

All the fluids in the nature have viscosity $\mu > 0$ and hence they are known as real fluid and viscous flow is called their motion.

1.7.3 Newtonian fluid

Fluid is said to be Newtonian if shear stress linearly proportional to deformation rate. This fluid obeys Newtonian law of viscosity, molten semiconductors, pure water, mercury, molten metals, and many molten salts are the examples of Newtonian fluids. Let us consider a flow of Newtonian fluid between two parallel plates one of which is in motion with constant velocity and denote u as x-component of velocity of fluid and y the direction perpendicular to the flow, then the shear stress is related to the velocity gradient by the following equation.

$$\tau = -\mu \frac{du}{dy}. \tag{1.5}$$

Eq. (1.5) is also known as Newtonian law of viscosity. Viscosity may depend on temperature and pressure but not on the force acted upon it. Generally, the stress tensor **T** and strain rate tensor $\mathbf{A}_1$ are related by the following equation:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1.6}$$

where **I** is the identity tensor, $\mathbf{S} = \mu \mathbf{A}_1$ and $\mathbf{A}_1$ is given by

$$\mathbf{A}_1 = \nabla \mathbf{V} + \nabla \mathbf{V}^{\dagger}, \tag{1.7}$$

in which $\dagger$ denotes the transpose.

1.7.4 Non-Newtonian fluid

Fluids for which shear stress and deformation rate are non-linearly related. Non-Newtonian fluid viscosity can be change by adding additives, and flow properties are different from Newtonian fluid. The viscosity of non-Newtonian fluid is dependent on shear rate. Non-Newtonian fluid is a fluid whose flow curve (shear stress versus shear rate) is nonlinear. In practice, many fluid materials exhibit non-Newtonian fluid behavior such as, salt solutions, molten polymers, ketchup, custard, toothpaste, starch suspensions, paint, blood, and shampoo etc. For such fluids the analogue of Eq. (1.5) is

$$\tau = \eta \left(\frac{du}{dy} \right)^n, \tag{1.8}$$

where η is the consistency index and n is the flow behavior index.

1.8 Some non-Newtonian fluids models

1 Second grade fluid.

2 Third grade fluid.

3 Maxwell fluid.

1.8.1 Second grade fluid

A model (second-grade) for viscoelastic fluids were given by Rivlin and Erisksen with the following constitutive equation:

$$\mathbf{T}=-p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2, \tag{1.9}$$

where, μ is the dynamic viscosity, α_1 and α_2 are the non-Newtonian fluid parameters and $\mathbf{A}_2$ is defined by

$$\mathbf{A}_2=\frac{\mathbf{DA}_1}{\mathbf{Dt}}+\mathbf{A}_1(\nabla\mathbf{V})+(\nabla\mathbf{V})^{\dagger}\mathbf{A}_1. \tag{1.10}$$

1.8.2 Third grade fluid

The constitutive relation for a third grade fluid is

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S}, \tag{1.11}$$

where

$$\mathbf{S}=\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, \tag{1.12}$$

and is β called third grade parameter.

1.8.3 Upper convected Maxwell model

The constitutive equation for upper-convected Maxwell fluid from Eq. (1.11), where **S** satisfies

$$\mathbf{S}+\lambda_1\overset{\nabla}{\mathbf{S}}=\mu\mathbf{A}_1. \tag{1.13}$$

The upper convected derivative of **S** in above equation is defined as

$$\overset{\nabla}{\mathbf{S}}=\frac{\partial\mathbf{S}}{\partial t}+(\mathbf{V}.\nabla)\mathbf{S}-(\nabla\mathbf{V})^{\dagger}\mathbf{S}-\mathbf{S}(\nabla\mathbf{V}). \tag{1.14}$$

1.9 Some dimensionless numbers

1.9.1 Deborah number

A dimensionless number used in rheology, equal to the relaxation time for some process divided by the time it is observed. It characterizes the fluidity of materials under specific flow conditions. For smaller relaxation time all materials are fluids but for large time of observation all materials are solids. Deborah number is expressed as:

$$De = \frac{t_c}{t_0}, \tag{1.15}$$

where, t_c is the characteristics time, t_0 is the time of observation.

1.9.2 Weissenberg number

A dimensionless number used in the study of viscoelastic flows. The weissenberg number is the ratio of the relaxation time of the fluid and a specific process time. In simple steady shear, it is often abbreviated as *Wi* or *We*.

1.9.3 Reynold number

It is a dimensionless number that establishes the proportionality between the fluid inertia and the sheer stress as a result of viscosity. In the simplest form, it can be described as $Re = \rho V l / \mu$, where ρ is the density in kilograms per cubic meters (1.2250) for air at sea level), V is the velocity of the fluid in meters per second, l is the linear dimension of the body (chord length, in airfoils), and μ is the coefficient of the viscosity of the fluid.

1.10 Bearing

A bearing is a machine element that constrains relative motion to only the desired motion, and reduces friction between moving parts. The design of the bearing may, for example, provide for free linear movement of the moving part or for free rotation around a fixed axis.

1.10.1 Slider bearing

The hydro-dynamically lubricated bearing is able to separate two sliding surfaces working against load acting on the bearing. Slider bearing is a system of stationary block and a horizontal surface (moving with a velocity U_s) in positive x-direction, separated by a varying heights, (b_1, b, b_2) containing non-Newtonian fluid between the surfaces.

1.10.2 Journal bearing

Journal bearing is a system of circular cylinders, inner cylinder rotates while outer cylinder is stationary. The parallel axes of the two cylinders are separated by a distance e. Lubricant is used between the cylinder, either Newtonian or non-Newtonian.

1.11 Solution methodology

The most powerful technique to solve nonlinear partial differential equations is perturbation method. Perturbation method leads to an expression for desired solution in term of power series in some small parameter called perturbation series. The leading term in this power series is the solution of exact solvable problem. In this technique we assume a very small physical parameter, expend the dependent variables in power series of small parameter and then put this series into original equation(s) and conditions (boundary and initial). After equating the terms corresponding to powers of small parameter, one get system of linear differential equations. Solving such system sequentially one gets the solution of the original problem. In this work we non-dimensionalize the equations of motion, then construct the solution for velocities and the pressure distributions by perturbation technique and finally construct the parameters on various flow characteristics. Perturbation method is used to compute analytically, the two dimensional velocity and stress fields for the creeping flow between two slightly eccentric cylinders separated by a small gap. The flow associated with the constitutive equation of Maxwell fluid examined. Construct the graph of radial velocity, tangential velocity; also construct the contours of the tangential normal stress and for the absolute pressure for upper convected Maxwell model. For the construction of graphs and other difficult mathematics work we use "Mathematica" software, which is easy to handle and produces good results.

Chapter 2

Non-Newtonian flow in a slider bearing system

In this chapter, flow in a slider bearing system is analyzed using the constitutive equation of third grade fluid. We first derive the equation of motion for second and third grade fluid in a slider bearing under thin film or lubrication approximation. Perturbation solution will be developed assuming second and third grade effects to be small compared with the viscous effects. The pressure calculated on the base of non-Newtonian connection is explained through various plots. This chapter is a detailed review of work by Yürüsoy [13].

2.1 Formulation of the problem

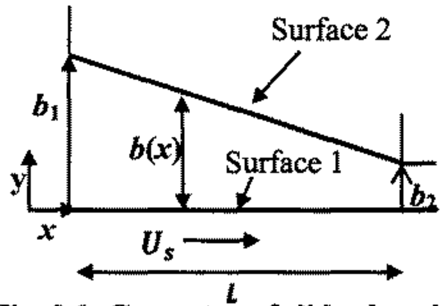

Fig. 2.1. Geometry of slider bearing system.

A typical slider bearing system is explained in Fig. 2.1. The equations governing the flow inside the slider bearing system are:

$$div\mathbf{V} = 0, \tag{2.1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \text{div}\mathbf{T}, \tag{2.2}$$

$$\mathbf{T} = -\text{p}\mathbf{I} + \mathbf{S}. \tag{2.3}$$

For a third grade fluid S is given through Eq. (1.13). In component form Eqs. (2.1) and (2.2) for a two dimensional flow can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.4}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y}, \tag{2.5}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}, \tag{2.6}$$

since flow is two-dimensional, therefore,

$$\mathbf{A}_1 = \begin{bmatrix} 2u_x & u_y + v_x \\ v_x + u_y & 2v_y \end{bmatrix}, \tag{2.7}$$

and

$$\mathbf{A}_2 = \begin{bmatrix} 2uu_{xx} + 2u_{xy}v + 2u_x{}^2 + u_y v_x + v_x{}^2 & uu_{xy} + vu_{yy} + vv_{xy} + uv_{xx} + 2u_x u_y \\ +2u_x{}^2 + u_y v_x + v_x{}^2 & +u_y v_y + v_x v_y + u_x u_y + u_x v_x + 2v_x v_y \\ uu_{xy} + vu_{yy} + uv_{xx} + vv_{xy} + u_x u_y + & 2uv_{xy} + 2vv_{yy} + 2u_y{}^2 + u_y v_x + 2v_y{}^2 \\ u_x v_x + 2v_x v_y + 2u_x u_y + u_y v_y + v_x v_y & +2u_y{}^2 + u_y v_x + 2v_x{}^2 \end{bmatrix}, \tag{2.8}$$

where u and v are velocity components in x-and y-directions, respectively and subscripts denote differentiation. Let us scale various quantities to be non-dimensionalize as

$$\bar{x} = \frac{x}{L},\ \bar{y} = \frac{y}{b_1},\ \bar{u} = \frac{u}{U},\ \bar{v} = \frac{Lv}{b_1 U},\ \bar{b} = \frac{b}{b_1},\ \bar{p} = \frac{pb_1}{\rho U^2 L},\ \bar{S}_{\bar{x}\bar{x}} = \frac{S_{xx} b_1 \mathrm{Re}}{\rho U^2 L} \tag{2.9}$$

with the help of above defined variables, Eqs. (2.4)-(2.6) after using the expression take the form of various stress components as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.10}$$

$$\begin{aligned} \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= \frac{-1}{\delta}\frac{\partial P}{\partial x} + \frac{1}{\delta^2 \mathrm{Re}}\frac{\partial^2 u}{\partial y^2} + \frac{\gamma_1}{\delta^2}\left(v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right) \\ &+2\frac{(2\gamma_1 + \gamma_2)}{\delta^2}\frac{\partial^2 u}{\partial x \partial y}\frac{\partial u}{\partial y} + 6\frac{\gamma_3}{\delta^4}\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2, \end{aligned} \tag{2.11}$$

$$\delta\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{-1}{\delta^2}\frac{\partial P}{\partial y} + 2\frac{(2\gamma_1 + \gamma_2)}{\delta^3}\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial y} + 2\frac{\gamma_3}{\delta^3}\left(\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}\right), \tag{2.12}$$

where

$$\mathrm{Re} = \frac{\rho UL}{\mu},\ \gamma_1 = \frac{\alpha_1}{\rho L^2},\ \gamma_2 = \frac{\alpha_2}{\rho L^2},\ \gamma_3 = \frac{\beta U}{\rho L^3},\ \frac{1}{\delta} = \frac{L}{b_1}, \tag{2.13}$$

and bars are removed for brevity. The subsequent analysis we consider that $1/\mathrm{Re}$, γ_1 and γ_2 is of order δ and γ_3 of order δ^3. With these assumptions the major terms in Eqs. (2.10)-(2.12) are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.14}$$

$$\frac{dp^*}{dx} = \frac{\partial^2 u}{\partial y^2} + K_1\left(v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right) + 6K_3\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2}, \tag{2.15}$$

$$\frac{dp^*}{dy}=0, \tag{2.16}$$

where $p^*=p-(2K_1+K_2)\left(\frac{\partial u}{\partial y}\right)^2$ and γ_1/δ, γ_2/δ and γ_3/δ^3 are replaced by K_1, K_2 and K_3 respectively. The above equations are subjected to the following boundary conditions

$$u(0)=1,\ u(b)=0, \tag{2.17}$$

$$v(0)=0,\ v(b)=0. \tag{2.18}$$

2.2 Solution of the problem

In this section, we shall employ perturbation method to solve the above equations for the case when second and third grade effects are smaller compared with the viscous effects. Therefore we shall assume

$$K_i=\varepsilon\bar{K}_i,\ i=1,2,3, \tag{2.19}$$

where ε is the small parameter. We now expand various quantities as follows:

$$u=u_0+\varepsilon u_1, \tag{2.20}$$

$$v=v_0+\varepsilon v_1, \tag{2.21}$$

$$p^*=p_0^*+\varepsilon p_1^*, \tag{2.22}$$

substituting Eqs. (2.19)-(2.22) into Eqs. (2.14) - (2.15), we get the following systems at various order of ε.

2.2.1 System of order 1:

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}=0, \tag{2.23}$$

$$\frac{\partial^2 u_0}{\partial y^2}=\frac{dP_0^*}{dx}, \tag{2.24}$$

$$u_0(0)=0,\ u_0(b)=0, \tag{2.25}$$

$$v_0(0)=0,\ v_0(b)=0. \tag{2.26}$$

2.2.2 System of order of ε:

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=0, \tag{2.27}$$

$$\frac{\partial^2 u_1}{\partial y^2}=\frac{dP_1^*}{dx}-\bar{K}_1\left(v_0\frac{\partial^3 u_0}{\partial y^3}+u_0\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial y^2}-\frac{\partial u_0}{\partial y}\frac{\partial^2 u_0}{\partial x\partial y}\right)-6\bar{K}_3\left(\frac{\partial^2 u_0}{\partial y^2}\left(\frac{\partial u_0}{\partial y}\right)^2\right), \tag{2.28}$$

$$u_1(0)=0,\ u_1(b)=0, \tag{2.29}$$

$$v_1(0)=0,\ v_1(b)=0. \tag{2.30}$$

It is noted that Eqs. (2.23)-(2.26) yield well known Newtonian problem whose solution is

$$u_0=\frac{1}{2}\frac{dP_0^*}{dx}\left(y^2-by\right)-\left(\frac{1}{b}y-1\right), \tag{2.31}$$

$$v_0=-\frac{1}{2}\frac{d}{dx}\left(\frac{dP_0^*}{dx}\left(\frac{y^3}{3}-\frac{by^2}{2}\right)\right)-\frac{db}{dx}\left(\frac{y^2}{2b^2}\right). \tag{2.32}$$

The differential equation of zeroth-order pressure is obtained by using the condition $\int_0^b u_0 dy=0$ i.e.

$$\frac{d}{dx}\left(\frac{dp_0^*}{dx}b^3\right)=6\frac{db}{dx}. \tag{2.33}$$

Eq. (2.33) is subjected to the following conditions:

$$p_0^*(0)=p_0^*(1)=0. \tag{2.34}$$

In a similar manner, the solution of first order system reads

$$u_1=\frac{dP_1^*}{dx}\left(\frac{y^2}{2}-\frac{yb}{2}\right)-\bar{K}_3\left(\left(\frac{dP_0^*}{dx}\right)^3\left(\frac{y^4}{2}-by^3+\frac{3}{4}b^2y^2-\frac{1}{4}b^3y\right)+\left(\frac{dP_0^*}{dx}\right)^2\left(-\frac{2y^3}{b}+3y^2-yb\right)+3\frac{dP_0^*}{dx}\left(\frac{y^2}{b^2}-\frac{y}{b}\right)\right)$$
$$-\bar{K}_1\left(\frac{d}{dx}\left(\frac{dP_0^*}{dx}\right)\left(\frac{y^2}{4}-\frac{y^2b^2}{8}\frac{dP_0^*}{dx}-\frac{yb}{4}+\frac{b^3y}{8}\frac{dP_0^*}{dx}\right)+\frac{db}{dx}\left(\left(\frac{dP_0^*}{dx}\right)^2\frac{b^2y}{8}+\frac{y^2}{2b^3}-\frac{y}{2b^2}-\left(\frac{dP_0^*}{dx}\right)^2\frac{by^2}{8}\right)\right). \tag{2.35}$$

.

$$v_1=\frac{d}{dx}\left(\frac{dP_1^*}{d\bar{x}}\left(\frac{y^2}{2}-\frac{yb}{2}\right)\right)-\bar{K}_1\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dP_0^*}{d\bar{x}}\right)\left(\frac{1}{2}-\frac{dP_0^*}{d\bar{x}}\frac{b^2}{4}\right)+\frac{db}{dx}\left(\frac{1}{b^3}-\left(\frac{dP_0^*}{d\bar{x}}\right)^2\frac{b}{4}\right)\right)\right)- \tag{2.36}$$
$$\bar{K}_3\left(\frac{d}{dx}\left(\left(\frac{dP_0^*}{d\bar{x}}\right)^2\left(\frac{y^2}{4}-y^3b+\frac{3y^2b^2}{4}-\frac{yb^3}{4}\right)\right)+\left(\frac{dP_0^*}{d\bar{x}}\right)^2\left(\frac{-2y^3}{b}-yb+3y^2\right)+3\left(\frac{dP_0^*}{d\bar{x}}\right)\left(\frac{y^2}{b^2}-\frac{y}{b}\right)\right).$$

Utilizing the constraint $\int_0^b u_1 dy=0$ the following differential equation evolves for the first order pressure

$$-\frac{dP_1^*}{d\bar{x}}\frac{b^3}{12}=-\bar{K}_1\left(\frac{d}{dx}\left(\frac{dP_0^*}{d\bar{x}}\right)\left(\frac{-b^3}{24}+\frac{dP_0^*}{d\bar{x}}\frac{b^5}{48}\right)+\left(\frac{db}{dx}\left(\left(\frac{dP_0^*}{d\bar{x}}\right)^2\frac{b^4}{48}-\frac{1}{12}\right)\right)\right)-\bar{K}_3\left(\left(\frac{b^5}{40}\right)\left(\frac{dP_0^*}{d\bar{x}}\right)^3-\left(\frac{dP_0^*}{d\bar{x}}\right)\frac{b}{2}\right). \quad (2.37)$$

The above equation is subjected to the conditions

$$p_1^*(0)=p_1^*(1)=0. \quad (2.38)$$

Inserting the expressions of u_0 and u_1 in Eq. (2.20) and making use of substitution $\varepsilon\bar{K}_i=K_i$,

we get

$$u=\left(\frac{y^2}{2}-\frac{by}{2}\right)\left(\frac{dP}{dx}\right)-\left(\frac{1}{b}y-1\right)-K_3\left(\left(\frac{y^4}{2}\right)\left(\frac{dP_0^*}{dx}\right)^3+\frac{3}{4}\left(\frac{dP_0^*}{dx}\right)^3b^2y^2+\left(\frac{3y^2}{b^2}\right)\left(\frac{dP_0^*}{dx}\right)\right)$$
$$-K_3\left(-\left(\frac{dP_0^*}{dx}\right)^3by^3+3\left(\frac{dP_0^*}{dx}\right)^2y^2-\left(\frac{2y^3}{b}\right)\left(\frac{dP_0^*}{dx}\right)^2+\left(\frac{yb^3}{2}\right)\left(\frac{dP_0^*}{dx}\right)^3-\frac{3}{4}\left(\frac{dP_0^*}{dx}\right)^3b^3y-\frac{3y}{b}\left(\frac{dP_0^*}{dx}\right)\right)$$
$$-K_1\left(\left(\frac{y^2}{2}\right)\frac{d}{dx}\left(\frac{dP_0^*}{dx}\right)-\left(\frac{y^2b^2}{8}\right)\left(\frac{dP_0^*}{dx}\right)\frac{d}{dx}\left(\frac{dP_0^*}{dx}\right)+\frac{y^2}{2b^3}\left(\frac{db}{dx}\right)-\left(\frac{y^2b}{8}\right)\left(\frac{dP_0^*}{dx}\right)^2\frac{db}{dx}-\left(\frac{y^2}{4}\right)\frac{d}{dx}\left(\frac{dP_0^*}{dx}\right)\right)$$
$$-K_1\left(\frac{b^3y}{8}\frac{d}{dx}\left(\frac{dP_0^*}{dx}\right)\frac{dP_0^*}{dx}-\left(\frac{yb}{4}\right)\frac{d}{dx}\left(\frac{dP_0^*}{dx}\right)-\frac{b^2y}{8}\frac{db}{dx}\left(\frac{dP_0^*}{dx}\right)^2-\left(\frac{y}{2b^2}\right)\frac{db}{dx}\right). \quad (2.39)$$

Similarly

$$v=\frac{by^2}{4}\frac{d}{dx}\left(\frac{dp_0^*}{dx}\right)-\frac{y^3}{6}\frac{d}{dx}\left(\frac{dp_0^*}{dx}\right)-\frac{\varepsilon yb}{2}\left(\frac{dp_1^*}{dx}\right)+\frac{\varepsilon y^2}{2}\left(\frac{dp_1^*}{dx}\right)+K_3\frac{3y}{b}\left(\frac{dp_0^*}{dx}\right)-3K_3\frac{y^2}{b^2}\left(\frac{dp_0^*}{dx}\right)$$
$$+3K_3by\left(\frac{dp_0^*}{dx}\right)^2-3K_3y^2\left(\frac{dp_0^*}{dx}\right)^2-2K_3yb\left(\frac{dp_0^*}{dx}\right)^2+2K_3\frac{y^3}{b}\left(\frac{dp_0^*}{dx}\right)^2+\frac{K_3yb^3}{2}\left(\frac{dp_0^*}{dx}\right)^3-\frac{3K_3y^4b^3}{2}$$
$$\left(\frac{dp_0^*}{dx}\right)^3-K_3yb^3\left(\frac{dp_0^*}{dx}\right)^3+K_3y^3b\left(\frac{dp_0^*}{dx}\right)^3+\frac{3K_3yb^3}{4}\left(\frac{dp_0^*}{dx}\right)^3-\frac{3K_3y^2b^2}{4}\left(\frac{dp_0^*}{dx}\right)^3+K_1\frac{y}{2b^2}\frac{db}{dx}- \quad (2.40)$$
$$K_1\frac{y^2}{2b^3}\frac{db}{dx}-K_1\frac{yb^2}{8}\frac{db}{dx}\left(\frac{dp_0^*}{dx}\right)^2+K_1\frac{y^2b}{8}\frac{db}{dx}\left(\frac{dp_0^*}{dx}\right)^2+K_1\frac{yb}{4}\frac{d}{dx}\left(\frac{dp_0^*}{dx}\right)-K_1\frac{y^2}{4}\frac{d}{dx}\left(\frac{dp_0^*}{dx}\right)-$$
$$K_1\frac{yb^3}{8}\left(\frac{dp_0^*}{dx}\right)\frac{d}{dx}\left(\frac{dp_0^*}{dx}\right)+K_1\frac{y^2b^2}{8}\left(\frac{dp_0^*}{dx}\right)\frac{d}{dx}\left(\frac{dp_0^*}{dx}\right)-\frac{db}{dx}\frac{y^2}{2b^2}.$$

2.3 Pressure field

The solution of Eq. (2.33) subjected to the boundary condition gives

$$P_0^*=\frac{6x(b-r)}{b^2(1+r)}, \quad (2.41)$$

where

$$b=\left(1-\left(1-r\right)x\right) \text{ and } r=b_2/b_1, \tag{2.42}$$

$$b=\left(r-1\right)x+1. \tag{2.43}$$

In a similar manner, one can obtained the expression of p_1^*. Finally, p_1^* is given through

$$p^*=p^*_0+\varepsilon p^*_1. \tag{2.44}$$

$$\begin{aligned}
p^*=&\left(\frac{6x(b-r)}{b_2(1+r)}\right)-\frac{8K_1}{(r-1)^2(1+r)^4}+\frac{8K_1}{b^2(r-1)^2(1+r)^4}+\frac{8rK_1}{(r-1)^2(1+r)^4}-\frac{8rK_1}{b^2(r-1)^2(1+r)^4}+\frac{8r^2K_1}{(r-1)^2(1+r)^4}\\
&+\frac{8r^2K_1}{b^2(r-1)^2(1+r)^4}-\frac{16r^3K_1}{(r-1)^2(1+r)^4}+\frac{16r^3K_1}{b^2(r-1)^2(1+r)^4}+\frac{8r^4K_1}{(r-1)^2(1+r)^4}-\frac{8r^4K_1}{b^2(r-1)^2(1+r)^4}+\frac{8r^5K_1}{(r-1)^2(1+r)^4}\\
&-\frac{8r^5K_1}{b^2(r-1)^2(1+r)^4}-\frac{8r^6K_1}{(r-1)^2(1+r)^4}+\frac{8r^6K_1}{b^2(r-1)^2(1+r)^4}-\frac{2K_1}{(r-1)(1+r)^3}+\frac{2K_1}{b^6(r-1)(1+r)^3}+\frac{8rK_1}{(r-1)(1+r)^3}-\\
&\frac{8rK_1}{b^6(r-1)(1+r)^3}-\frac{6r^2K_1}{(r-1)(1+r)^3}+\frac{6r^2K_1}{b^6(r-1)(1+r)^3}-\frac{8r^3K_1}{(r-1)(1+r)^3}+\frac{8r^3K_1}{b^6(r-1)(1+r)^3}+\frac{8r^4K_1}{(r-1)(1+r)^3}-\qquad(2.45)\\
&\frac{8r^4K_1}{b^6(r-1)(1+r)^3}-\frac{8xK_1}{b^6(r-1)(1+r)^3}+\frac{28rxK_1}{b^6(r-1)(1+r)^3}-\frac{20r^2xK_1}{b^6(r-1)(1+r)^3}-\frac{20r^3xK_1}{b^6(r-1)(1+r)^3}+\frac{28r^4xK_1}{b^6(r-1)(1+r)^3}\\
&-\frac{8r^5xK_1}{b^6(r-1)(1+r)^3}+\frac{12r^2xK_1}{b^6(r-1)(1+r)^3}-\frac{36rx^2K_1}{b^6(r-1)(1+r)^3}+\frac{18r^2x^2K_1}{b^6(r-1)(1+r)^3}+\frac{36r^3x^2K_1}{b^6(r-1)(1+r)^3}-\frac{36r^4x^2K_1}{b^6(r-1)(1+r)^3}\\
&+\frac{6r^6x^2K_1}{b^6(r-1)(1+r)^3}-\frac{8x^3K_1}{b^6(r-1)(1+r)^3}+\frac{20rx^3K_1}{b^6(r-1)(1+r)^3}-\frac{36r^3x^3K_1}{b^6(r-1)(1+r)^3}\frac{24r^4x^3K_1}{b^6(r-1)(1+r)^3}+\frac{12r^5x^3K_1}{b^6(r-1)(1+r)^3}\\
&-\frac{16r^6x^3K_1}{b^6(r-1)(1+r)^3}+\frac{4r^7x^3K_1}{b^6(r-1)(1+r)^3}+\frac{2x^4K_1}{b^6(r-1)(1+r)^3}-\frac{4rx^4K_1}{b^6(r-1)(1+r)^3}-\frac{4r^2x^4K_1}{b^6(r-1)(1+r)^3}+\frac{12r^3x^4K_1}{b^6(r-1)(1+r)^3}\\
&-\frac{12r^5x^4K_1}{b^6(r-1)(1+r)^3}+\frac{4r^6x^4K_1}{b^6(r-1)(1+r)^3}-\frac{648r^4K_3}{25(r-1)^2(1+r)^4}+\frac{648r^4K_3}{25b^2(r-1)^2(1+r)^4}-\frac{312r^5K_3}{25(r-1)^2(1+r)^4}+\\
&\frac{312r^5K_3}{25b^2(r-1)^2(1+r)^4}+\frac{168K_3}{5(r-1)(1+r)^3}-\frac{168K_3}{5b^6(r-1)(1+r)^3}-\frac{72rK_3}{5(r-1)(1+r)^3}+\frac{72rK_3}{5b^6(r-1)(1+r)^3}+\frac{648r^2K_3}{25(r-1)(1+r)^3}\\
&-\frac{648r^2K_3}{25b^6(r-1)(1+r)^3}-\frac{312r^3K_3}{25(r-1)(1+r)^3}+\frac{312r^3K_3}{25b^6(r-1)(1+r)^3}+\frac{504xK_3}{5b^6(r-1)(1+r)^3}-\frac{144rxK_3}{5b^6(r-1)(1+r)^3}-\\
&\frac{1872r^2xK_3}{25b^6(r-1)(1+r)^3}+\frac{144r^3xK_3}{5b^6(r-1)(1+r)^3}-\frac{648r^4xK_3}{25b^6(r-1)(1+r)^3}-\frac{504x^2K_3}{5b^6(r-1)(1+r)^3}+\frac{72rx^2K_3}{5b^6(r-1)(1+r)^3}+\\
&\frac{1008r^2x^2K_3}{5b^6(r-1)(1+r)^3}\frac{4r^7x^4K_1}{b^6(r-1)(1+r)^3}-\frac{2r^8x^4K_1}{b^6(r-1)(1+r)^3}+\frac{648K_3}{25(r-1)^2(1+r)^4}-\frac{648K_3}{25b^2(r-1)^2(1+r)^4}-\frac{312K_3}{25(r-1)^2(1+r)^4}\\
&+\frac{312K_3}{25b^2(r-1)^2(1+r)^4}-\frac{72rK_3}{5(r-1)^2(1+r)^4}+\frac{72rK_3}{5b^2(r-1)^2(1+r)^4}+\frac{72r^3K_3}{5(r-1)^2(1+r)^4}-\frac{72r^3K_3}{5b^2(r-1)^2(1+r)^4}-
\end{aligned}$$

$$\frac{144r^3x^2K_3}{5b^6(r-1)(1+r)^3}-\frac{504r^4x^2K_3}{5b^6(r-1)(1+r)^3}+\frac{72r^5x^2K_3}{5b^6(r-1)(1+r)^3}+\frac{168x^3K_3}{5b^6(r-1)(1+r)^3}-\frac{504r^2x^3K_3}{5b^6(r-1)(1+r)^3}+$$
$$\frac{504r^4x^3K_3}{5b^6(r-1)(1+r)^3}-\frac{168r^6x^3K_3}{5b^6(r-1)(1+r)^3}.$$

2.4 Graphical results

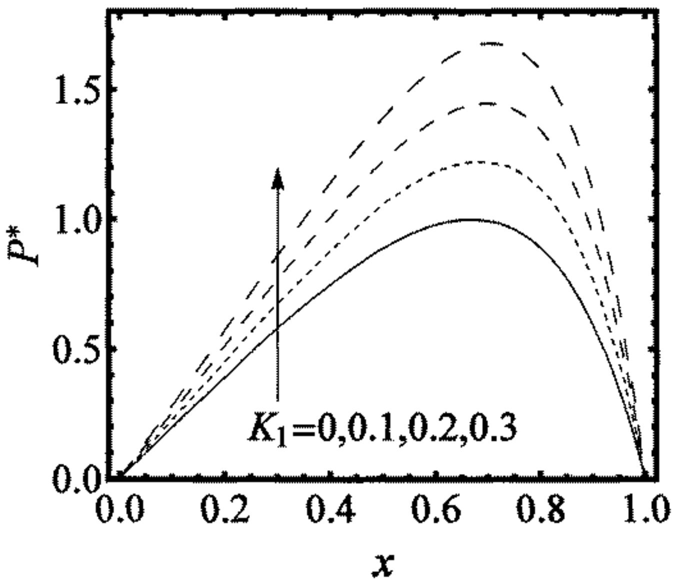

Fig. 2.2: Profiles of pressure versus x for $r=0.5$ when $K_3=0$. Solid line corresponds to Newtonian case ($K_1=K_3=0$).

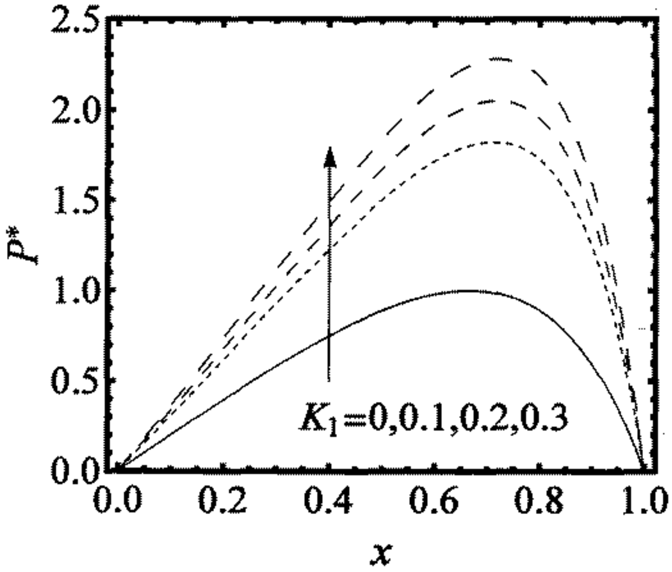

Fig. 2.3: Profiles of pressure versus x for $r=0.5$ when $K_3=0.04$. Solid line corresponds to Newtonian case ($K_1=K_3=0$).

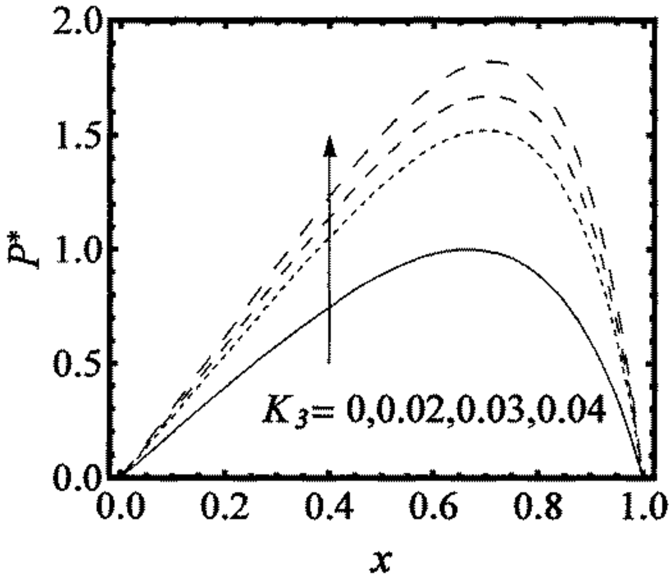

Fig. 2.4: Profiles of pressure versus x for $r = 0.5$ when $K_1 = 0.1$. Solid line corresponds to Newtonian case ($K_1 = K_3 = 0$).

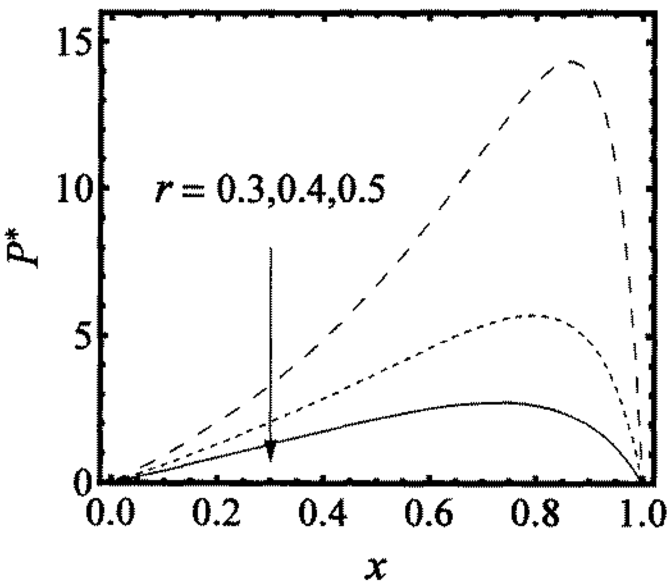

Fig. 2.5: Pressure distribution in the bearing corresponding to different clearance ratios for $K_1 = K_3 = 0.1$.

2.5 Results and discussion

The plots of pressure field for various values of second and third grade parameters are shown in Figs. 2.2-2.5. Fig. 2.2 shows the effects of second grade parameters on pressure distribution taking third grade parameters equal to zero. The corresponding curve of Newtonian fluid is also included for comparison purpose. It is observed that pressure increases with increase the second grade parameters. The manner in which pressure varies with K_1 for non-vanishing values of K_3 is illustrated in Fig. 2.3. It is observed that as before by increasing K_1 the pressure increases. However, the pressure for non-Newtonian third grade fluid attains significantly higher values than that for a second grade fluid. The plots of pressure for different values of K_3 are shown in Fig. 2.4 for a fixed value of $K_1 = 0.1$. Here again it is noted that pressure inside the bearing increases by increasing third grade fluid parameters. The effect of clearance ratio on magnitude of pressure inside the bearing is shown through Fig. 2.5. It is evident from Fig. 2.5 that pressure rises in the bearing for lower clearance ratios.

Chapter 3

Non-Newtonian flow in a journal bearing system

The famous model of the journal bearing consists of two eccentric cylinders with a stationary outer cylinder and a rotating inner cylinder. In this work, we discuss the flow of upper convected Maxwell fluid in the journal bearing and explore the influence of visco-elasticity on both the velocity field and pressure. We assume the flow to be steady and two-dimensional. To obtain an analytical solution both gap and eccentricity are considered to be small. Due to complicated nature of the Maxwell constitutive equation the velocity, stress and pressure variables are computed employing domain perturbation technique. The effects of Deborah number on radial and tangential velocity, tangential stress and pressure are illustrated graphically. This chapter is based on the paper by Beris at al. [14].

3.1 Geometry of the problem

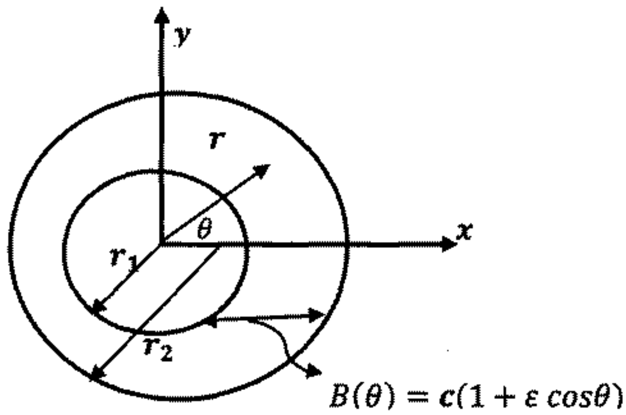

Fig. 3.1. Geometry of journal bearing system

Fig. 3.1 shows a journal bearing consisting of two cylinders of radii r_1 and r_2 respectively. Let the inner cylinder be rotating with angular velocity Ω so that the linear velocity is $V = r_1\Omega$. Let the center of the inner cylinder be the origin of cylindrical polar and Cartesian coordinates system. The local gap with between the cylinders is given by $B(\theta) = c(1+\varepsilon\cos\theta)$, where $c = r_2 - r_1$ is the average gap and $\varepsilon = a/c$ is the dimension less eccentricity. Here a is the distance between the axes of the cylinders. The gap inside the cylinders is filled with lubricant whose rheological behavior is represented by the constitutive equations of Maxwell fluid.

3.2 Formulation of the problem

Neglecting the gravitational and inertial forces, the governing, continuity, momentum and constitutive equations are given by

$$\nabla.\mathbf{V}=0, \tag{3.1}$$

$$\nabla \mathrm{p}+\nabla.\mathbf{S}=0, \tag{3.2}$$

$$\mathbf{S}+\lambda_1\left(\mathbf{V}.\nabla\mathbf{S}-\nabla\mathbf{V}^{\dagger}.\mathbf{S}-\mathbf{S}.\nabla\mathbf{V}\right)=-\eta_0\mathbf{A}_1. \tag{3.3}$$

Where $\dagger$ denotes transpose. Since the flow is two-dimensional and steady therefore, $v_r=v_r\left(r,\theta\right)$, $v_\theta=v_\theta\left(r,\theta\right)$, and $v_z=0$. With this choice of velocity field, the above set of equations can be written in components form as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r.v_r\right)+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}=0, \tag{3.4}$$

$$\frac{\partial p}{\partial r}+\frac{\partial S_{rr}}{\partial r}+\left(\frac{S_{rr}-S_{\theta\theta}}{r}\right)+\frac{1}{r}\frac{\partial S_{r\theta}}{\partial\theta}=0, \tag{3.5}$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta}+\frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta}+\frac{\partial S_{r\theta}}{\partial r}+\frac{2S_{r\theta}}{r}=0, \tag{3.6}$$

$$S_{rr}+\lambda_1\left(v_r\frac{\partial S_{rr}}{\partial r}+\frac{v_\theta}{r}\frac{\partial S_{rr}}{\partial\theta}-2\frac{v_\theta}{r}S_{r\theta}-\left(2k_{rr}S_{rr}+k_{r\theta}S_{\theta r}\right)\right)=-2\eta_0\frac{\partial v_r}{\partial r}, \tag{3.7}$$

$$S_{r\theta}+\lambda_1\left(v_r\frac{\partial S_{r\theta}}{\partial r}+\frac{v_\theta}{r}\frac{\partial S_{r\theta}}{\partial\theta}+\frac{v_\theta}{r}\left(S_{rr}-S_{r\theta}\right)-\left(k_{rr}S_{r\theta}+k_{r\theta}S_{r\theta}\right)-\left(k_{r\theta}S_{rr}+k_{\theta\theta}S_{\theta r}\right)\right)=-\eta_0\left(\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right), \tag{3.8}$$

$$S_{\theta\theta}+\lambda_1\left(v_r\frac{\partial S_{\theta\theta}}{\partial r}+\frac{v_\theta}{r}\frac{\partial S_{\theta\theta}}{\partial\theta}-2\frac{v_\theta}{r}S_{r\theta}-2\left(k_{\theta\theta}S_{\theta\theta}+k_{r\theta}S_{r\theta}\right)\right)=-2\eta_0\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right), \tag{3.9}$$

where we use the abbreviation $\mathbf{K}=\left(\nabla\mathbf{V}\right)^{\dagger}$. With this abbreviation $\left(\nabla\mathbf{V}\right)=\mathbf{K}^{\dagger}$ and for plane flow under consideration we have

$$\left(\nabla\mathbf{V}\right)=\mathbf{K}^{\dagger}=\begin{bmatrix}\frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} & 0\\ \left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right) & \left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right) & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{3.10}$$

The components of stress S_{rz}, $S_{\theta z}$ and S_{zz} are set to zero because of the invariance of the flow in z-direction. The flow problem is subjected to the following no slip boundary conditions

$$v_r = 0,\ v_\theta = V. \qquad \text{on} \qquad r = r_1, \tag{3.11}$$

$$v_r = 0,\ v_\theta = 0. \qquad \text{on} \quad r = r_1 + B(\theta). \tag{3.12}$$

The pressure and stresses are subject to the following periodicity conditions:

$$S_{ij}(r,\theta) = S_{ij}(r,\theta + 2\pi), \tag{3.13}$$

$$p(r,\theta) = p(r,\theta + 2\pi). \tag{3.14}$$

3.3 Solution of the problem

The set of Eqs. (3.4)-(3.9) subject to boundary conditions Eqs. (3.11)-(3.12) is solved by domain perturbation technique. In this technique, the original domain is mapped to some reference domain. For the present problem it is better to choose the reference domain to a concentric cylinders geometry with coordinates $(\overline{r},\overline{\theta})$. The mapping is illustrated in Fig. 3.2. The coordinates in eccentric domain (r,θ) and concentric domain $(\overline{r},\overline{\theta})$ are related by the transformation

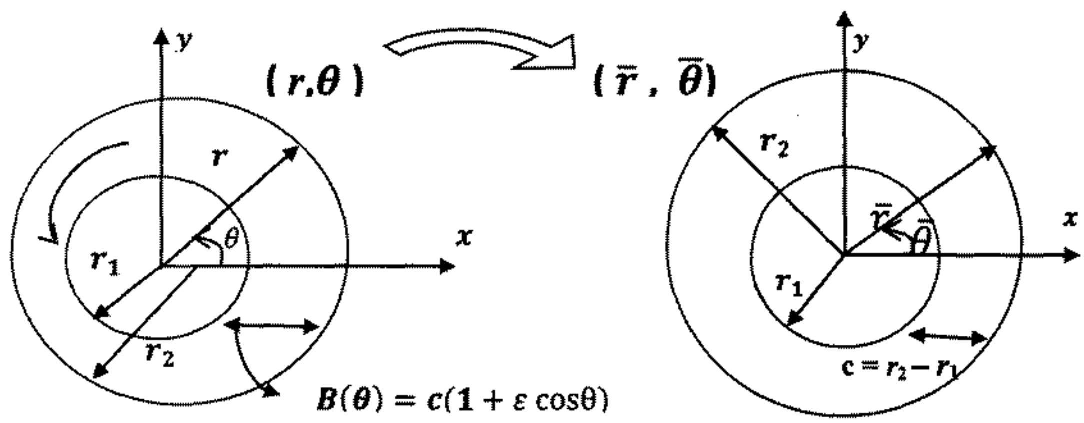

Figure. 3.2. Transformation of eccentric cylinder to concentric cylinders.

$$r = r^{[0]}(\overline{r},\overline{\theta};\varepsilon) = r = r_1 + (\overline{r} - r_1)(1 + \varepsilon\cos\overline{\theta}), \qquad (r_1 \le \overline{r} \le r_2), \tag{3.15}$$

$$\theta = \theta^{[0]}(\overline{r},\overline{\theta};\varepsilon) = \overline{\theta}, \qquad (0 \le \overline{\theta} \le 2\pi). \tag{3.16}$$

In above transformation the superscript square bracket [ ] denotes the function of coordinates $(\overline{r},\overline{\theta})$. In the later part we shall also use superscript angular bracket $\langle\ \rangle$ to denotes functions of the original coordinates (r,θ). The purpose of this change of variable is to transform the perturbation expression in eccentricity for the original eccentric problem to the concentric cylinder shape. Now a perturbation expression in either coordinates system involve the use of partial derivatives of dependent variables with respect to ε. The solution in the reference domain can be transformed back to original domain only by having the relation between two sets of partial derivatives. We now illustrate this relation for a function, $u(r,\theta;\varepsilon)\cong u^{\langle 0\rangle}(r,\theta;\varepsilon)=u^{[0]}(\bar{r},\bar{\theta};\varepsilon)$. For such a function the two partial derivatives are given by

$$u^{\langle n\rangle}(r,\theta;\varepsilon)\equiv\frac{\partial^n u^{\langle n\rangle}}{\partial\varepsilon^n},u^{[n]}(\bar{r},\bar{\theta};\varepsilon)\equiv\frac{\partial^n u^{[0]}}{\partial\varepsilon^n}. \tag{3.17}$$

Now making use of chain rule the following relation between them can be developed

$$u^{\langle n\rangle}(r,\theta;\varepsilon)\equiv\frac{\partial^n u^{\langle n\rangle}}{\partial\varepsilon^n},u^{[0]}(\bar{r},\bar{\theta};\varepsilon)\equiv\frac{\partial^n u^{[0]}}{\partial\varepsilon^n}, \tag{3.18}$$

$$u^{[1]}=\frac{\partial u^{\langle 0\rangle}}{\partial\varepsilon}+\frac{\partial r^{[0]}}{\partial\varepsilon}\frac{\partial u^{\langle 0\rangle}}{\partial r}=u^{\langle 1\rangle}+r^{[1]}\frac{\partial u^{\langle 0\rangle}}{\partial r}. \tag{3.19}$$

Eqn. (3.18), is analogous to the definition of substantial derivative. Thus $u^{[1]}$ can be interpreted as a "substantial derivative" following the mapping. In view of Eq. (3.15), factor $r^{[1]}$ can be computed as

$$r^{[1]}=\frac{\partial r^{[0]}}{\partial\varepsilon}=(\bar{r}-r_1)\cos\bar{\theta}, \tag{3.20}$$

$$r^{[1]}(\bar{r},\bar{\theta};0)=(r-r_1)\cos\theta, \tag{3.21}$$

where we have used the fact that $\bar{r}=r$ and $\bar{\theta}=\theta$ at $\varepsilon=0$. Eq. (3.20) will be used in the later analysis. Once the relation between partial derivatives is developed, the next step is to expand the dependent variables in original domain in series in ε calculated in the reference domain keeping $\bar{r},\bar{\theta}$ fixed:

$$\begin{pmatrix} v(r,\theta;\varepsilon) \\ S(r,\theta;\varepsilon) \\ p(r,\theta;\varepsilon) \end{pmatrix}=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\begin{pmatrix} v^{[n]}(\bar{r},\bar{\theta};\varepsilon) \\ S^{[n]}(\bar{r},\bar{\theta};\varepsilon) \\ p^{[n]}(\bar{r},\bar{\theta};\varepsilon) \end{pmatrix}_{\varepsilon=0}. \tag{3.22}$$

The choice of above expansion rather than the one in original domain is basically derive by fact that in reference domain the boundary conditions are much simpler. However, despite the simplicity in the boundary location, the expression for the "substantial" derivatives $v^{[1]}$, etc., are not easy to obtain. A remedy of this difficulty can be made by expressing the derivatives $v^{[n]}$ $(\bar{r},\bar{\theta};0)$ in terms of the $v^{\langle n\rangle}$ $(r,\theta;0)$ by using chain-rule as explained in Eq. (3.21). Now, taking the partial derivatives $\partial^n/\partial\varepsilon^n$ of the governing equations given by Eqs. (3.1)-

(3.3), keeping in mind that $v(r,\theta;\varepsilon) \equiv v^{\langle 0\rangle}(r,\theta;\varepsilon)$ and then evaluating these at $\varepsilon = 0$ yield the following determining equations for $v^{\langle n\rangle}(r,\theta;0)$.

$$\nabla.\mathbf{V}^{\langle n\rangle} = 0, \tag{3.23}$$

$$\nabla\frac{\partial^n p}{\partial\varepsilon^n} + \left(\nabla.\frac{\partial^n S}{\partial\varepsilon^n}\right) = 0, \tag{3.24}$$

$$\nabla p^{\langle n\rangle} + \left(\nabla.\mathbf{S}^{\langle n\rangle}\right) = 0, \tag{3.25}$$

$$\mathbf{S}^{\langle n\rangle} + \lambda_1\left(\left(\mathbf{V}.\nabla\mathbf{S}\right)^{\langle n\rangle} - \left(\left(\nabla\mathbf{S}\right)^{\dagger}.\mathbf{S}\right)^{\langle n\rangle} - \left(\mathbf{S}.\left(\nabla\mathbf{S}\right)\right)^{\langle n\rangle}\right) = -\eta_0\mathbf{A}_1{}^{\langle n\rangle}, \tag{3.26}$$

where all dependent variables are evaluated at $\varepsilon = 0$. It is pointed here the explicit form of these equations for a fixed value of n can be readily obtained from Eqs. (3.1)-(3.3) by substituting series expansion in ε for $v^{\langle n\rangle}$, $p^{\langle n\rangle}$ and $\mathbf{S}^{\langle n\rangle}$ and equating equal order in ε. In order to determine boundary conditions for $v^{\langle n\rangle}(r,\theta,0)$, we substitute the series expansion for v given through Eq. (3.22) into Eqs. (3.11)-(3.12). In this way at $r = \bar{r} = r_1$ we have:

$$v_r = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}v^{[n]}\left(\bar{r},\bar{\theta};0\right)\Big|_{\bar{r}=r_1},$$

$$v_r = v_r^{<0>}\left(r_1,\theta;0\right) + \varepsilon\left[v_r^{\langle 1\rangle}\left(r_1,\theta;0\right) + \left(r-r_1\right)\cos\theta\frac{\partial v_r^{\langle 0\rangle}\left(r_1,\theta;0\right)}{\partial r}\right]_{r=r_1},$$

$$\begin{aligned} v_r &= v_r^{<0>}\left(r_1,\theta;0\right) + \varepsilon v_r^{\langle 1\rangle}\left(r_1,\theta;0\right) + ..., \\ v_\theta &= v_\theta^{<0>}\left(r_1,\theta;0\right) + \varepsilon v_\theta^{\langle 1\rangle}\left(r_1,\theta;0\right) + ... \end{aligned} \tag{3.27}$$

From now onward all $v_i^{\langle n\rangle}$ are evaluated at $\varepsilon = 0$, therefore in favor of $v_i^{\langle n\rangle}(r,\theta)$. A comparison of v_r and v_θ in Eq. (3.27) with Eq. (2.11) gives

$$\left.\begin{aligned} v_r^{\langle 0\rangle}\left(r_1,\theta\right) &= 0, & v_\theta^{\langle 0\rangle}\left(r_1,\theta\right) &= V, \\ v_r^{\langle 1\rangle}\left(r_1,\theta\right) &= 0, & v_\theta^{\langle 1\rangle}\left(r_1,\theta\right) &= 0. \end{aligned}\right\} \tag{3.28}$$

In a similar fashion at the outer cylinder $r = r_1 + B(\theta)$ or $\bar{r} = r_2$, we have

$$v_r\left(r_2,\theta\right) = v_r^{\langle 0\rangle}\left(r_2,\theta\right) + \varepsilon\left(v_r^{\langle 1\rangle}\left(r,\theta\right) + \left(r-r_1\right)\cos\theta\frac{\partial v_r^{\langle<0>\rangle}\left(r,\theta\right)}{\partial r}\right)_{r=r_2} + ...,$$

$$v_r\left(r_2,\theta\right) = v_r^{\langle 0\rangle}\left(r_2,\theta\right) + \varepsilon\left(v_r^{\langle 1\rangle}\left(r_2,\theta\right) + c\cos\theta\frac{\partial v_r^{\langle<0>\rangle}\left(r_2,\theta\right)}{\partial r}\right) + ...,$$

$$v_\theta(r_2,\theta)=v_\theta^{\langle 0\rangle}(r_2,\theta)+\varepsilon\left(v_\theta^{\langle 1\rangle}(r_2,\theta)+C\cos\theta\frac{\partial v_\theta^{\langle 0\rangle}(r_2,\theta)}{\partial r}\right)+\dots \tag{3.29}$$

Comparison of Eq. (3.29) with Eq. (3.12) gives the following boundary conditions at $r=r_2$,

$$\left.\begin{array}{ll} v_r^{\langle 0\rangle}(r_2,\theta)=0, & v_\theta^{\langle 0\rangle}(r_2,\theta)=0, \\ v_r^{\langle 1\rangle}(r_2,\theta)=-c\cos\theta\dfrac{\partial v_r^{\langle 0\rangle}(r_2,\theta)}{\partial r}, & v_\theta^{\langle 1\rangle}(r_2,\theta)=-c\cos\theta\dfrac{\partial v_\theta^{\langle 0\rangle}(r_2,\theta)}{\partial r}. \end{array}\right\} \tag{3.30}$$

3.4 Solution of the zeroth-order problem

The explicit form of Eqs. (3.23)-(3.26) for $n=0$ is given by

$$\nabla.\mathbf{V}^{\langle 0\rangle}=0, \tag{3.31}$$

$$\nabla p^{\langle 0\rangle}+\left(\nabla.\mathbf{S}^{\langle 0\rangle}\right)=0, \tag{3.32}$$

$$\mathbf{S}^{\langle 0\rangle}+\lambda_1\left(\mathbf{V}^{\langle 0\rangle}.\nabla\mathbf{S}^{\langle 0\rangle}-\left(\nabla\mathbf{V}^{\langle 0\rangle}\right)^{\dagger}\mathbf{S}^{\langle 0\rangle}-\mathbf{S}^{\langle 0\rangle}\left(\nabla\mathbf{V}^{\langle 0\rangle}\right)^{\dagger}\right)=-\eta_0\mathbf{A}_1^{\langle 0\rangle}. \tag{3.33}$$

The component forms of these equations are given through Eqs. (3.4)-(3.9). In view of the prescribed boundary conditions on $v_r^{\langle 0\rangle}$ and $v_\theta^{\langle 0\rangle}$ we can assume, $v_r^{\langle 0\rangle}=0$, $v_\theta^{\langle 0\rangle}=v_\theta^{\langle 0\rangle}(r)$, $S_{ij}^{\langle 0\rangle}=S_{ij}^{\langle 0\rangle}(r)$, and $p^{\langle 0\rangle}=p^{\langle 0\rangle}(r)$. This choice of solution makes the continuity equation satisfied identically. The expression of velocity gradient takes the form

$$\nabla\mathbf{V}^{\langle 0\rangle}=\begin{bmatrix} 0 & \dfrac{\partial v_\theta^{\langle 0\rangle}}{\partial r} & 0 \\ \dfrac{-v_\theta^{\langle 0\rangle}}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\boldsymbol{\kappa}^{\langle 0\rangle^{\dagger}}. \tag{3.34}$$

The above equation of $\nabla\mathbf{V}^{\langle 0\rangle}$ in combination with the constitutive equations for $S_{rr}^{\langle 0\rangle}$, $S_{r\theta}^{\langle 0\rangle}$, and $S_{\theta\theta}^{\langle 0\rangle}$ gives

$$S_{rr}^{\langle 0\rangle}=0, \tag{3.35}$$

$$S_{r\theta}^{\langle 0\rangle}=-\eta_0 A_{1r\theta}^{\langle 0\rangle}=-\eta_0\left(\frac{\partial v_\theta^{\langle 0\rangle}}{\partial r}-\frac{v_\theta^{\langle 0\rangle}}{r}\right), \tag{3.36}$$

$$S_{\theta\theta}{}^{\langle 0\rangle} = -2\lambda_1\eta_0\left(A_{1r\theta}{}^{\langle 0\rangle}\right)^2 = 2\lambda_1\left(\frac{\partial v_\theta{}^{\langle 0\rangle}}{\partial r} - \frac{v_\theta{}^{\langle 0\rangle}}{r}\right)S_{r\theta}{}^{\langle 0\rangle}. \tag{3.37}$$

The component forms of equation of motion yield

r-component:
$$\frac{dp^{\langle 0\rangle}}{dr} = \frac{S_{\theta\theta}}{r}, \tag{3.38}$$

θ-component:
$$\frac{dS_{r\theta}{}^{\langle 0\rangle}}{dr} = -\frac{2S_{r\theta}{}^{\langle 0\rangle}}{r}. \tag{3.39}$$

In the subsequent analysis, we shall develop solutions to the above equation for small dimensionless gap μ, defined as:

$$\mu = c/r_1. \tag{3.40}$$

In this context it is appropriate to define the following dimensionless variables and groups:

$$\left.\begin{aligned} &\zeta = \left(\frac{r - r_1}{c}\right), \\ &T_{ij}{}^{\langle 0\rangle}(\zeta) = \frac{S_{ij}{}^{\langle 0\rangle}(r)}{(\eta_0 v/c)}, \\ &P^{\langle 0\rangle}(\zeta) = \frac{p^{\langle 0\rangle}(r)}{(\eta_0 v/c)}, \\ &\hat{v}_i{}^{\langle 0\rangle}(\zeta) = \frac{v_i{}^{\langle 0\rangle}(r)}{V}, \end{aligned}\right\} \tag{3.41}$$

$$\begin{aligned} &De = We \times \mu = \lambda_1 \frac{V}{r_1}, \\ &We = \lambda_1 \frac{V}{c}. \end{aligned} \tag{3.42}$$

In terms of the dimensionless variables the θ-component of the equation of motion can be written for small μ as:

$$\frac{dS_{r\theta}{}^{\langle 0\rangle}}{dr} + \frac{2S_{r\theta}{}^{\langle 0\rangle}}{r} = 0, \tag{3.43}$$

$$(\eta_0 v/c)\frac{1}{c}\frac{d}{d\zeta}T_{r\theta}{}^{<0>}(\zeta) = -(\eta_0 v/c)\frac{2}{(c\zeta + r_1)}T_{r\theta}{}^{<0>}(\zeta), \tag{3.44}$$

$$\frac{d}{d\zeta}T_{r\theta}{}^{\langle 0\rangle}(\zeta) = -\frac{2c}{(c\zeta + r_1)}T_{r\theta}{}^{\langle 0\rangle}(\zeta), \tag{3.45}$$

$$T_{r\theta}{}^{\langle 0\rangle}(\zeta) = C_1(r_1 - 2c\zeta + \ldots). \tag{3.46}$$

The solution of above equation turns out to be

$$T_{r\theta}^{\langle 0\rangle}(\zeta)=C_0+\mu(C_1-2C_0\zeta)+\ldots, \tag{3.47}$$

where C_0 and C_1 are constants. The above solution when substituted in the constitutive equations of $T_{r\theta}^{\langle 0\rangle}$ gives an ordinary differential equation for $\hat{v}_\theta^{\langle 0\rangle}$ i.e.

$$\frac{d\hat{v}_\theta^{\langle 0\rangle}(\zeta)}{d\zeta}\frac{1}{c}-\frac{\mu V\,\hat{v}_\theta^{\langle 0\rangle}(\zeta)}{c}(1-\mu\zeta)=-\frac{1}{c}\left(C_0+\mu(C_1-2C_0\zeta)\right). \tag{3.48}$$

The solution of above differential equation together with conditions $\hat{v}_\theta^{\langle 0\rangle}(0)=1$ and $\hat{v}_\theta^{\langle 0\rangle}(1)=0$ leads to:

$$\hat{v}_\theta^{\langle 0\rangle}=(1-\zeta)+\frac{\mu}{2}(\zeta^2-\zeta)+O(\mu^2). \tag{3.49}$$

Having $\hat{v}_\theta^{\langle 0\rangle}$ in hand, the other components of stress and velocity at leading order become

$$\left.\begin{aligned}
&\hat{v}_r^{\langle 0\rangle}=0,\\
&T_{rr}^{\langle 0\rangle}(\zeta)=0,\\
&T_{r\theta}^{\langle 0\rangle}(\zeta)=1+\mu\left(\frac{3}{2}-2\zeta\right)+O(\mu^2),\\
&T_{\theta\theta}^{\langle 0\rangle}(\zeta)=-2\frac{De}{\mu}(1+O(\mu)),\\
&P^{\langle 0\rangle}=P_0-2De\zeta+O(\mu),
\end{aligned}\right\} \tag{3.50}$$

where P_0 is an arbitrary reference pressure. It is noted from Eq. (3.50) that $T_{\theta\theta}^{\langle 0\rangle}$ is singular and thus the dominant terms for $\mu\to 0$.

3.5 Solution to the first order problem

The first order system results from Eq. (3.23)-(3.26) by taking n=1, we get:

$$\left(\nabla.\mathbf{V}^{\langle 1\rangle}\right)=0, \tag{3.51}$$

$$\nabla.p^{\langle 1\rangle}+\nabla.\mathbf{S}^{\langle 1\rangle}=0, \tag{3.52}$$

$$\mathbf{S}^{\langle 1\rangle}+\lambda_1\left(\mathbf{V}^{\langle 0\rangle}.\nabla\mathbf{S}^{\langle 1\rangle}+\mathbf{V}^{\langle 1\rangle}.\nabla\mathbf{S}^{\langle 0\rangle}-\left(\nabla\mathbf{V}^{\langle 0\rangle}\right)^{\dagger}\mathbf{S}^{\langle 1\rangle}-\mathbf{S}^{\langle 1\rangle}\left(\nabla\mathbf{V}^{\langle 0\rangle}\right)-\left(\nabla\mathbf{V}^{\langle 1\rangle}\right)^{\dagger}\mathbf{S}^{\langle 0\rangle}-\mathbf{S}^{\langle 0\rangle}\left(\nabla.\mathbf{V}^{\langle 1\rangle}\right)^{\dagger}\right)=-\eta_0\mathbf{A}_1^{\langle 1\rangle}. \tag{3.53}$$

The above equations are subjected to boundary conditions given by Eq. (3.28) and Eq. (3.30). In the conditions the term $\partial v_i^{\langle 0\rangle}/\partial r$ is evaluated using Eq. (3.30). Equating the terms which are second order in μ, we get

$$\left.\begin{array}{llll} \text{at} & r=r_1 & v_r^{(1)}=0, \\ & & v_\theta^{(1)}=0, \\ \text{at} & r=r_2 & v_r^{(1)}=0, \\ & & v_\theta^{(1)}=VCos\theta+O\left(\mu^2\right). \end{array}\right\} \tag{3.54}$$

Due to the periodic nature of the boundary condition, it is instructive to choose a solution of the form

$$v_r^{(1)}=Re\left\{iF(r)e^{i\theta}\frac{c}{r}\right\}, \tag{3.55}$$

$$v_\theta^{(1)}=Re\left\{-F'(r)e^{i\theta}c\right\}. \tag{3.56}$$

In view of Eqs. (3.55)-(3.56) the continuity equation is satisfied identically. With the help of Eq. (3.7), the *rr*-component of the Eq. (3.53) reduces to

$$S_{rr}^{(1)}+\lambda_1\left(\frac{v_r^{(0)}}{r}\frac{\partial S_{rr}^{(1)}}{\partial\theta}-\frac{2}{r}\frac{\partial v_r^{(1)}}{\partial\theta}S_{r\theta}^{(1)}\right)=-2\eta_0\frac{\partial v_r^{(1)}}{\partial r}, \tag{3.57}$$

which is view of Eq. (3.50) takes the form

$$S_{rr}^{(1)}+\lambda_1\left(\left(\frac{V(1-\zeta)}{r}\frac{\partial S_{rr}^{(1)}}{\partial\theta}\right)-\frac{2}{r}\frac{\partial v_r^{(1)}}{\partial\theta}\eta_0\frac{V}{c}\left(1+\mu(1-\zeta)+......\right)\right)=-2\eta_0\frac{\partial v_r^{(1)}}{\partial r}. \tag{3.58}$$

Let us define dimensionless variables through Eq. (3.50) and a dimensionless stream function f by

$$f(\zeta)=\frac{F(r)}{V}. \tag{3.59}$$

Moreover, assume the following periodic solutions for stresses and pressure:

$$S_{ij}^{(1)}=Re\left\{T_{ij}^{(1)}(\zeta)e^{i\theta}\right\}\left(\eta_0\frac{V}{c}\right), \tag{3.60}$$

$$p^{(1)}=Re\left\{P^{(1)}(\zeta)e^{i\theta}\right\}\left(\eta_0\frac{V}{c}\right). \tag{3.61}$$

In this way, stress and pressure are functions of ζ alone. Now Eq. (3.59), Eq. (3.55) and Eq. (3.56), we can write

$$\frac{d}{dr}F(r)=V\frac{d}{dr}f(\zeta), \tag{3.62}$$

this implies

$$\frac{d}{dr}F(r)=\frac{V}{c}f'(\zeta), \tag{3.63}$$

where prime denotes the derivative with respect to ζ. In view of Eq. (3.63), we can write

$$\frac{\partial v_r^{\langle 1\rangle}}{\partial r} = -c\sin\theta\left(\frac{d}{dr}F(r)\frac{1}{r} - \frac{1}{r^2}F(r)\right), \tag{3.64}$$

or

$$\frac{\partial v_r^{\langle 1\rangle}}{\partial r} = -\left(\frac{iVf'(\zeta)}{r_1\left(1+\dfrac{c\zeta}{r_1}\right)} - \frac{i\mu V}{r_1\left(1+\dfrac{c\zeta}{r_1}\right)^2}f(\zeta)\right), \tag{3.65}$$

similarly

$$\frac{\partial v_r^{\langle 1\rangle}}{\partial \theta} = \left(-\cos\theta\frac{cV}{r_1\left(1+\dfrac{c\zeta}{r_1}\right)}f(\zeta)\right), \tag{3.66}$$

and

$$\frac{\partial S_{rr}^{\langle 1\rangle}}{\partial \theta} = -\sin\theta T_{rr}^{\langle 1\rangle}\frac{\eta_0 V}{c}. \tag{3.67}$$

Using Eqs. (3.63)-(3.67) in Eq. (3.58), we get

$$\cos\theta\frac{\eta_0 V}{c}T_{rr}^{\langle 1\rangle} + \lambda_1\left(\frac{V}{r_1\left(1+\dfrac{c\zeta}{r_1}\right)}(1-\zeta)\left(-\sin\theta\frac{\eta_0 V}{c}T_{rr}^{\langle 1\rangle}\right) - (1+\mu-\mu\zeta+\ldots)\left(-fV\mu\cos\theta\frac{\eta_0 V}{c\left(1+\dfrac{c\zeta}{r_1}\right)}\right)\frac{2}{r_1\left(1+\dfrac{c\zeta}{r_1}\right)}\right)$$

$$= -2\eta_0\left(\frac{-iVf'}{r_1\left(1+\dfrac{c\zeta}{r_1}\right)} + \frac{f\mu Vi}{r_1\left(1+\dfrac{c\zeta}{r_1}\right)^2}\right). \tag{3.68}$$

A further simplification yield

$$T_{rr}^{\langle 1\rangle} + iDe(1-\zeta)\left(1-\frac{3}{2}\mu\zeta+\ldots\right)T_{rr}^{\langle 1\rangle} = -2\mu\left(if'+Def\right)+O(\mu^2). \tag{3.69}$$

From above equations $T_{rr}^{\langle 1\rangle}$ turns out to be

$$T_{rr}^{\langle 1\rangle} = -\frac{2\left(if'+Def\right)}{\left(1+iDe(1-\zeta)\right)}\mu + O\left(\mu^2\right). \tag{3.70}$$

Similarly

$$T_{r\theta}^{\langle 1\rangle}\left(1+iDe(1-\zeta)\right)=f''+2De^2 f+\frac{2De\left(if'+Def\right)}{\left(1+iDe(1-\zeta)\right)}+O(\mu), \quad (3.71)$$

$$T_{\theta\theta}^{\langle 1\rangle}\left(1+iDe(1-\zeta)\right)=\frac{2De}{\mu}\left(2iDef'-f''-T_{r\theta}^{\langle 1\rangle}\right)+O(1). \quad (3.72)$$

It is evident that $T_{rr}^{\langle 1\rangle}\sim\mu$, we can develop $T_{rr}^{\langle 1\rangle}$ as a series in μ with leading term. Moreover, $T_{r\theta}^{\langle 1\rangle}\sim 1$ and $T_{\theta\theta}^{\langle 1\rangle}\sim 1/\mu$. Thus similar to $T_{\theta\theta}^{\langle 0\rangle}$, $T_{\theta\theta}^{\langle 1\rangle}$ is singular and dominant. Now, we are in a position to combine the constitutive equations with equation of motion. From Eq. (3.52), we can write

r-component:

$$P^{\langle 1\rangle\prime}+T_{rr}^{\langle 1\rangle\prime}+\frac{\mu\left(T_{rr}^{\langle 1\rangle}-T_{\theta\theta}^{\langle 1\rangle}\right)}{(1+\mu\zeta)}+\frac{\mu}{(1+\mu\zeta)}iT_{r\theta}^{\langle 1\rangle}=0. \quad (3.73)$$

θ-component:

$$i\mu P^{\langle 1\rangle}+i\mu T_{rr}^{\langle 1\rangle}+(1+\mu\zeta)T_{r\theta}^{\langle 1\rangle\prime}+2\mu T_{r\theta}^{\langle 1\rangle}=0. \quad (3.74)$$

Eq. (3.73) indicates $P^{\langle 1\rangle\prime}\sim 1$, while Eq. (3.74) gives $P^{\langle 1\rangle}\sim 1/\mu$. Thus the series representation of $p^{\langle 1\rangle}$ is

$$p^{\langle 1\rangle}=\frac{p_0^{\langle 1\rangle}}{\mu}+O(1). \quad (3.75)$$

Eq. (3.75) implies that the pressure is constant across the gap at lowest order in μ. Elimination of pressure from Eq. (3.74) by taking $d/d\zeta$ of this equation results in the determining equations for stream function. Retaining the leading order terms in this equations, we get

$$i\mu T_{\theta\theta}^{\langle 1\rangle\prime}+T_{r\theta}^{\langle 1\rangle\prime\prime}=0. \quad (3.76)$$

Inserting Eqs. (3.71)-(3.72) and performing much tedious algebra, one find

$$\left(-2\zeta+i\left(1-\zeta^2\right)\right)g^{(iv)}-\left(-2\zeta^2+2i\zeta\right)g'''+2i\zeta^2 g''-4i\zeta g'+4ig=0, \quad (3.77)$$

Where

$$\xi=De(1-\zeta);\ g(\xi)=f(\zeta);\ g'(\xi)=-\frac{1}{De}f'(\zeta);\ and\ g''(\xi)=\frac{1}{De^2}f''(\zeta). \quad (3.78)$$

The general solution to Eq. (3.77) is

$$g(\xi)=Ae^{(1-i)\xi}+Be^{-(1+i)\xi}+C\xi^2+D\xi, \quad (3.79)$$

in which A, B, C and D are integration constants. The relevant boundary conditions on $g(\xi)$ are

$$g(0)=g'(De)=g(De)=0;\ g'(0)=\frac{1}{De}. \tag{3.80}$$

Imposition of above conditions on Eq. (3.79) yields

$$A=\frac{1}{De\left\{\left(1-i-\frac{2}{De}\right)e^{(1-i)De}+\left(1+i+\frac{2}{De}\right)e^{-(1+i)De}+2\right\}}, \tag{3.81}$$

$$B=-\left(\frac{1}{De\left\{\left(1-i-\frac{2}{De}\right)e^{(1-i)De}+\left(1+i+\frac{2}{De}\right)e^{-(1+i)De}+2\right\}}\right), \tag{3.82}$$

$$C=De^{-2}\left\{A\left(2De+e^{-(1+i)De}-e^{(1-i)De}\right)-1\right\}, \tag{3.83}$$

$$D=-2\left(\frac{1}{De\left\{\left(1-i-\frac{2}{De}\right)e^{(1-i)De}+\left(1+i+\frac{2}{De}\right)e^{-(1+i)De}+2\right\}}\right)+De^{-1}. \tag{3.84}$$

3.6 Graphical results

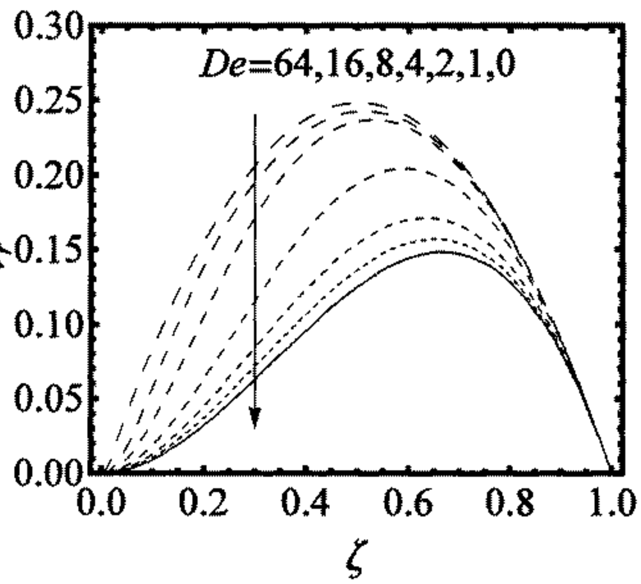

Fig. 3.3: Profiles of correction to the radial velocity $v_r^* = v_r^{(1)}/V\mu$ depends on the dimensionless radial coordinate $\zeta=(r-r_i)/c(1+\varepsilon cos\theta)$ at $\theta=3\pi/2$ for convected Maxwell model.

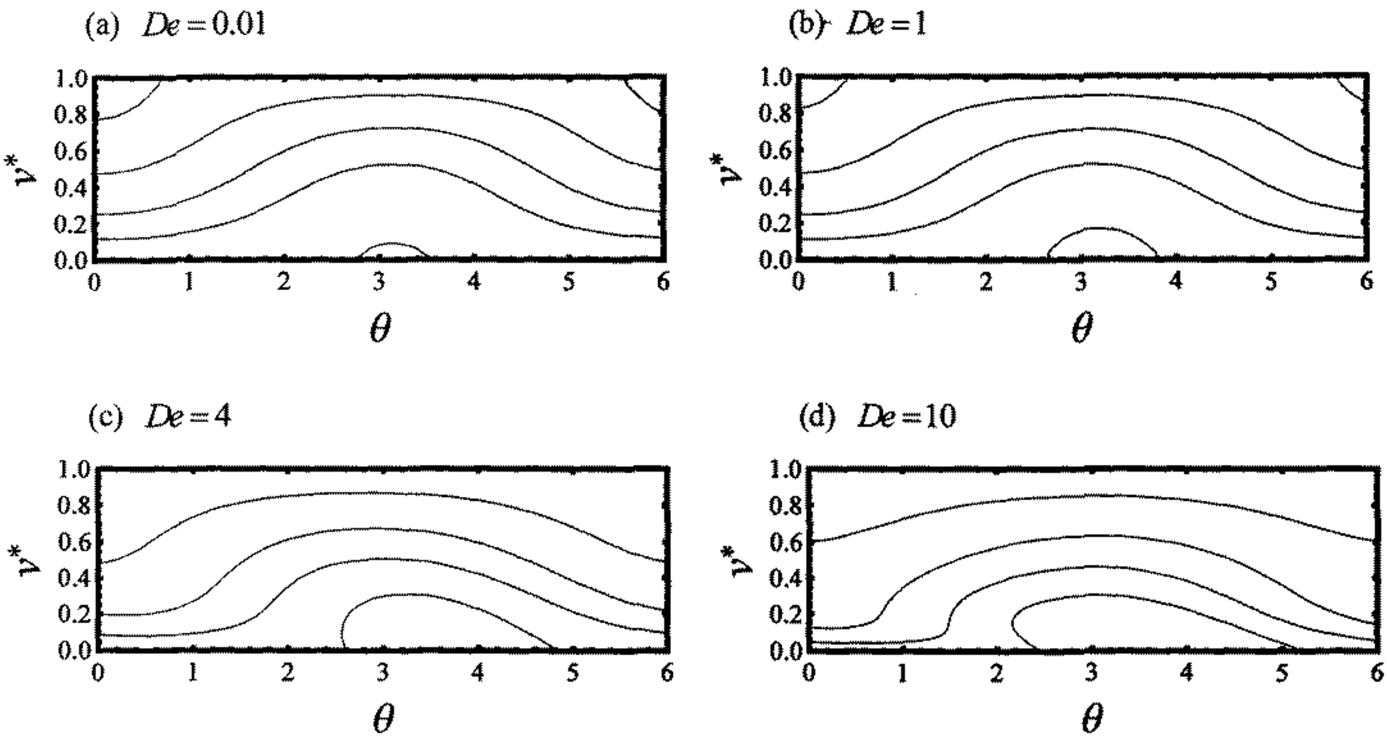

Fig. 3.4: Contours of tangential velocity $v_\theta^*(\zeta,\theta) = \left(S_{\theta\theta}^{\langle 1\rangle}\mu / \left(\eta^{\langle 0\rangle}(V/c)\right)\right) De$.

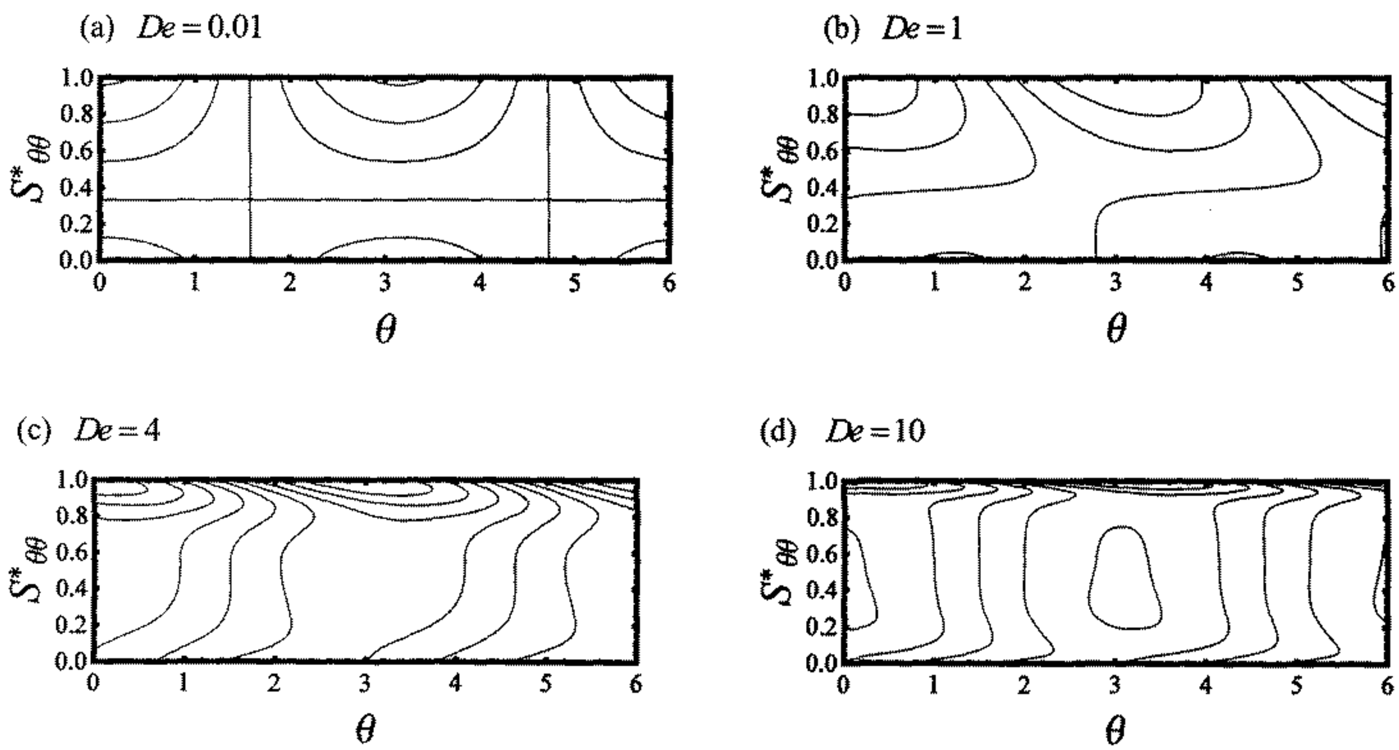

Fig. 3.5: Contours of correction to normal stress $S_{\theta\theta}^* = \left(S_{\theta\theta}^{\langle 1\rangle}\mu / \left(\eta^{\langle 0\rangle}(V/c)\right)\right) De$.

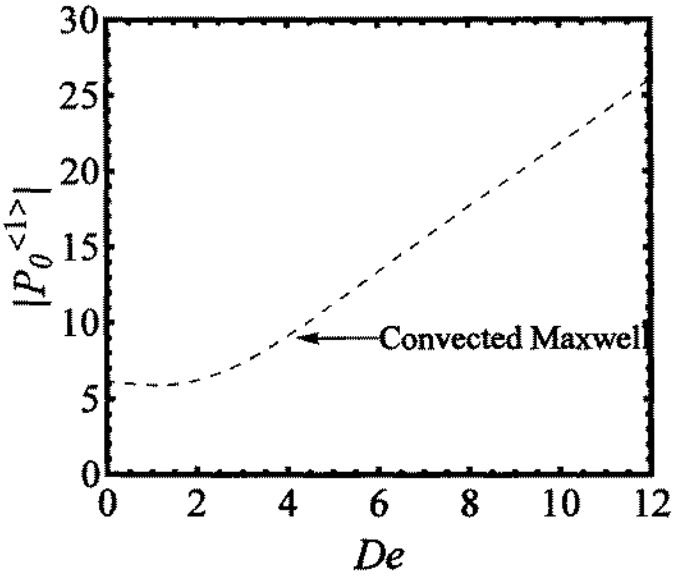

Fig. 3.6: Variation of $\left|P_0^{\langle 1\rangle}\right|$ with *De*.

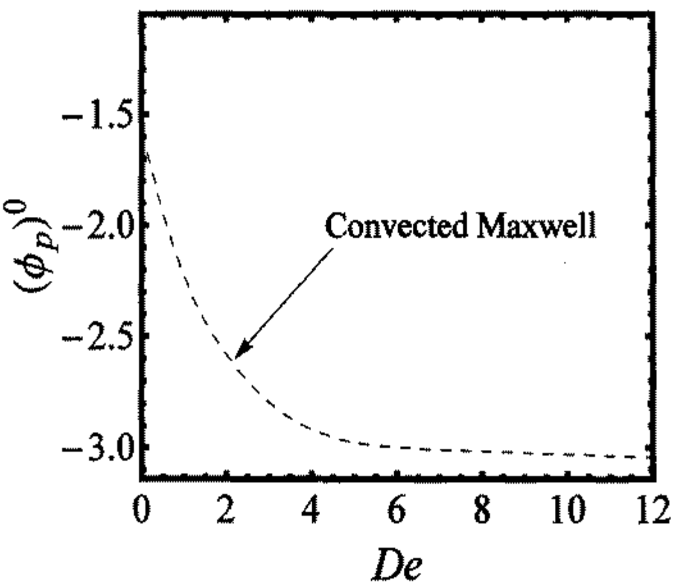

Fig. 3.7: Variation of ϕ_p with *De*.

3.7 Results and discussion

The profiles of correction to the radial velocity $v_r^* = v_r^{(1)}/V\mu$ at $\theta = 3\pi/2$ for different values of De are shown in Fig. 3.3. It is evident through Eqs. (3.55)-(3.56) that at this position $v_r^{(1)}$ is proportional to the real part of F. Moreover, since $v_r^{(0)} = 0$, therefore, v_r^* gives the total radial velocity up-to order ε. Fig. 3.3 shows that v_r^* increases with Deborah number with maximum in it shifting towards the inner cylinder.

The tangential velocity in reference domain includes contribution from both $v_\theta^{[0]}$ and $v_\theta^{[1]} = v_\theta^{(1)} - \zeta V cos\theta$. The contours of dimensionless total tangential velocity $v^* = v_\theta^* = v_\theta/V$ are shown in Fig. 3.4. It is observed that for De=0.01 there is a flow separation at the outer cylinder at $\theta = 0$. An increase in De causes this secondary flow to disappear.

The contours of correction to normal stress $S_{\theta\theta}^*$ for various values of De are shown in Fig. 3.5. The contours predict the development of a boundary layer near the outer and inner cylinders for large value of De. Moreover, away from the outer wall $S_{\theta\theta}^*$ is nearly independent of ζ. The amplitude of dimensionless pressure $P^* = P^{<1>}\mu/(\eta_0(V/c))$, $P^* = \left|P_\theta^{(1)}\right| Cos(\theta + \phi_p)$ is plotted against De in Fig. 3.6. This figure depicts that amplitude of pressure increases by increasing De. The variation of ϕ_p with De is shown in Fig. 3.7. The phase angle ϕ_p determines the steady-state position of journal relative to the outer cylinder. For Newtonian ϕ_p=-90^0. In convected Maxwell fluid the viscoelastic nature of the fluid is the responsible for the shifting of pressure force towards the wide part of the gap.

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