Integral Method for Boundary Layer Flows

By

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
IN
MATHEMATICS

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF SCIENCE in MATHEMATICS***

We accept this dissertation as conforming to the required standard.

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2015

Dedication

To my family,
For the endless support and patience.

To my Teachers,
For the constant source of Knowledge and Inspiration.

To my friends,
The ones that are close and the ones that are far.

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All the praises to Almighty **ALLAH** the most gracious, the most merciful and the creator of all the creature in the creature. Thanks to **ALLAH** and countless Darood and Salaams to His **Prophet HAZRAT MUHAMMAD (P.B.U.H)**. There are few people who made this journey easier for me with their encouragement and ideas.

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Haleem Afsar

DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor **Dr. Ahmer Mehmood**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

Signature: ____________________

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Preface

The study of magnetohydrodynamic (MHD) natural convection flow and heat transfer of an electrically conducting fluid past a heated semi-infinite vertical solid surface is important from application point of view. It is applicable in many engineering problems such as magnetohydrodynamic generator, nuclear reactors, geothermal extractions and the boundary layer control in the field of aeronautics and aerodynamics. It serves as the basis for understanding some of the important phenomena occurring in heat exchanger devices as well. Since thermal radiation effects are important in context of space technology and processes involving high temperatures therefore Ozisik [1], Sparrow and Cess [2], Elbarbary [3] and Cess [4] and Arpaci [5] first studied the interaction of thermal radiation and natural convection but their analysis was confined to the case of a vertical semi-infinite plate.

The aforementioned problems are modelled mathematically through Navier-Stokes equations. The governing partial differential equations are solved directly or by transforming them into ordinary differential equations. There are mainly two approaches, namely, the analytic and the numerical in literature to tackle the arising differential equations. Most problems are so complicated that their analytic exact solution is hard to find. Therefore many researchers are always interested in approximate methods, such as perturbation methods, numerical methods and integral methods etc. In order to obtain approximate closed form analytic solution one best choice is the integral method because it have much less limitations in term of their geometry and boundary conditions. It can also be applied to both laminar and turbulent flow situations. It easily provides an accurate approximate solution to complex problems. Using the integral method, one usually integrates the conservative differential boundary layer equation over the boundary layer thickness by assuming a profile for velocity, temperature and concentration as needed. The better the approximate shape of the profile is, such as velocity and temperature, better is the prediction of drag force and heat transfer (friction coefficient or heat transfer coefficient). The integral methodology has been applied to a variety of configurations to solve transport phenomena problems.

This dissertation is divided into three chapters. Chapter 1 includes some basic definitions. In chapter 2 integral solution for laminar natural convection flow for similar and non-similar cases is discussed whereas chapter 3 considers the natural convection flow or viscous fluid near a vertical plate with sinusoidal magnetic field applied normal to the plate by taking into account the radiative heat transfer. Integral method has been applied to obtain the approximate solution and the results are discussed through graphs.

Contents

Chapter 1

Introduction and Preliminaries

The objective of this chapter is to give brief introduction of the work presented in this dissertation. Relevant definitions and fundamental governing laws are introduced here which will provide background for the subsequent chapters. The frequently used dimensionless numbers are also defined.

1.1 Introduction

In 1905 Prandtl gave the first description of the boundary layer concept. Prandtl showed that the flow past a body can be divided into two regions, a very thin layer close to the boundary where the viscosity is important and the remaining region where the viscosity can be neglected. Outside the boundary layer the flow was essentially that which had already been studied for the previous two centuries. With the help of this concept he explained the boundary layer theory. Initially boundary layer theory was develop mainly for the laminar flow of an incompressible fluid where stokes law of friction could be used as an ansatz for the various forces. The objects might be of different shapes on which boundary layer develops. The simplest geometry on which the boundary layer is formed is an infinitely long plate along which a viscous incompressible fluid flows. If the surface is curved then the boundary layer structure would be more complex. The thickness of the boundary layer grows along the surface from the leading edge to the rear. At the leading edge of a flat plate the thickness is zero and it grows continuously towards rear end of plate.

The study of magnetohydrodynamic (MHD) natural convection flow and heat transfer of an electrically conducting fluid past a heated semi-infinite vertical solid surface is important from application point of view. It is applicable in many engineering problems such as magnetohydrodynamic generator, nuclear reactors, geothermal extractions and the boundary layer control in the field of aeronautics and aerodynamics. It serves as the basis for understanding some of the important phenomena occurring in heat exchanger devices as well. Since thermal radiation effects are important in context of space technology and processes involving high temperatures therefore Ozisik [1], Sparrow and Cess [2], Elbarbary [3] and Cess [4] and Arpaci [5] first studied the interaction of thermal radiation and natural convection but their analysis was confined to the case of a vertical semi-infinite plate.

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1.2 Basic Definitions

1.2.1 Fluid

A fluid is a substance that deforms continuously when subjected to a shear stress (i.e., tangential forces) no matter how small that shear stress may be. In simple words, a fluid is a substance which is capable of flowing and which conforms to the shape of containing vessel. Fluids are usually divided into two groups, namely, liquids and gases.

1.2.2 Fluid Mechanics

Fluid mechanics is a well known branch of continuum mechanics. It usually deals with the behavior of fluids (liquids and gases) in the states of rest and motion. Fluid mechanics may be divided into three categories: fluid statics, fluid kinematics and fluid dynamics. Fluid statics deals with the study of fluids at rest, while fluid kinematics is the study of fluids in motion without considering the forces which cause or accompany the motion. On the other hand, fluid dynamics is the study of fluids in motion considering the forces acting on the fluids.

1.2.3 Flow

Since the deforms continuously when different forces act upon it. If the deformation continuously increases without limit, this phenomenon is known as flow.

1.2.4 Deformation

It is a relative change in position or length of the fluid particles.

1.2.5 Velocity Field

Velocity is the rate of change of the position of an object. It is the vector quantity whose magnitude is the speed and whose direction is the direction of motion. The velocity of the fluid at a given point is defined as the instantaneous velocity of the particle, which is passing through that point at a given instant. It is denoted by $\mathbf{V} = \mathbf{V}(x, y, z, t)$. In vector form $\mathbf{V} = [u, v, w]$, where u, v and w are three scalar components of the velocity in x, y and z directions and t is the time.

1.3 Characteristics of Fluid

1.3.1 Density

The density of the fluid is the mass of unit volume of the fluid at given temperature and pressure. If the density of the fluid varies then the density at the point is given as

$$\rho = \lim_{\delta V \to \delta V'} \left(\frac{\delta m}{\delta V'} \right), \tag{1.1}$$

where $\delta V'$ is the small volume over which the substance can be considered as a continuum.

1.3.2 Stress

The stress is defined as the force per unit area of the surface on which it acts. The stress at any point P in the fluid is defined as

$$\text{Stress at any point } P = \lim_{\Delta S \to 0} \frac{\Delta \overrightarrow{F}}{\Delta S}, \tag{1.2}$$

where $\Delta \overrightarrow{F}$ is the force acting on an element of surface area ΔS enclosing the point P. The strèss in the direction of the normal to the surface at P is said to be the normal stress, while the stress in the direction of the tangent to the surface at P is called the shearing or tangential stress. For a closed surface, the stress exerted by the inner fluid on the outer (surrounding) fluid is taken as positive, where the stress exerted by the outer fluid on the inner fluid is taken as negative.

1.3.3 Absolute or Dynamic Viscosity

Viscosity is defined as the measure of resistance of a fluid to deformation by shear stress or tensile stress. It is usually taken as "thickness or resistance to flow" and is related to the internal friction of a moving fluid. For unidirectional motion of a fluid between two long parallel plates one of which is at rest, the other moving with a constant velocity parallel to itself under the action of a constant force, it can be shown that

$$\tau = \mu \frac{du}{dy}, \tag{1.3}$$

here u is velocity component in the direction of flow and y is distance measured normal from the lower plate. The law of fluid friction given by (1.3) is known as Newton's law of viscosity. From (1.3), we can write

$$\mu = \frac{\tau}{du/dy} = \frac{shear\ stress}{rate\ of\ shear\ strain}, \tag{1.4}$$

eqs. (1.4) can be regarded as the definition of viscosity.

1.3.4 Kinematic Viscosity

Kinematic viscosity is stated as the ratio of absolute viscosity μ to the density ρ and is given as

$$\upsilon = \frac{\mu}{\rho}. \tag{1.5}$$

The units of kinematic viscosity is m^2/s or Stoke (St).

1.4 Classification of Fluid

Generally, fluid are classified in two main branches.

1.4.1 Newtonian Fluid

Newtonian fluid are those which possess a linear relation between shear stress and rate of strain. Generally, they are defined as fluids for which Newton's law of viscosity holds. The graphical representation of shear stress versus velocity gradient is a straight line which passes through the origin. Fluids that exhibit Newtonian behavior are water, gasoline, air and glycrine, etc.

1.4.2 Non-Newtonian Fluid

Non-Newtonian fluids are those fluids in which shear stress is directly but non-linearly proportional to deformation rate. For such fluid shear stress and rate of deformation satisfy power-law model i.e,

$$\tau_{xy} = k\left(\frac{du}{dy}\right)^n, \ n \neq 1, \tag{1.6}$$

where k is called the consistency index which is a measure of the consistency of the fluid, and n is called the flow behaviour index and is a measure of how the fluid deviates from a Newtonian fluid. For $k = \mu$ and $n = 1$ it reduces to Newton's law of viscosity. The shear stress versus velocity gradient plots for non-Newtonian fluids are non-linear, in general, some of them pass through the origin while others do not. Thus the viscosity μ of a non-Newtonian fluid at a given temperature and pressure is a function of the velocity gradient. Examples of non-Newtonian fluids includes toothpaste, blood, honey, ketchup, shampoo, gels, paint, drilling muds and biological fluids etc. While studying flow of non-Newtonian fluid the following two characteristics are technically important.

Shear Thinning Effect

Shear thinning is an effect where viscosity deceases with increasing rate of shear stress. Materials that execute shear thinning are called pseudoplastic. There are certain complex solutions such as lava, ketchup, whipped cream, blood, paint and nail polish, which describe such effects.

Shear Thickening Effect

A shear thickening effect is one in which viscosity of a fluid increases with the rate of shear stress. Fluids which describe such effects are termed as dilatent. Mixture of cornstarch and water can easily be seen to understand this effect.

1.5 Types of Flow

1.5.1 Internal Flows

Flows that are bounded completely by solid surface are known as internal flows. Internal flow can be compressible, incompressible, laminar or turbulent. Examples are flow in pipe or in duct.

1.5.2 External Flows

Flows over bodies immersed in an unbounded fluid are known as external flows. Examples are flow over an aeroplane, cars and ships.

1.5.3 Uniform Flow

Uniform flow is defined as flow in which the velocity of the fluid is of the same magnitude and direction at every point in the fluid.

1.5.4 Non-Uniform Flow

If the velocity of the fluid does not have the same magnitude and direction at every point in the fluid is termed as non-uniform flow.

1.5.5 Steady Flow

Steady flow is defined as the type of flows in which fluid characteristics like velocity, pressure, density etc. at a point do not change with respect to time.

1.5.6 Unsteady Flow

If at any point in the fluid, the conditions change with time, the flow is described as unsteady.

1.5.7 Compressible Flow

Compressible flow is the flow in which the density of the fluid changes during the flow. Flow of gases is usually considered as compressible.

1.5.8 Incompressible Flow

The flow in which the density of the fluid does not change during the flow is known as incompressible flow. Liquids flow are generally incompressible.

1.5.9 Laminar Flow

When fluid flows in parallel layers such that there is no disruption then flow is said to be laminar. In laminar flow, the velocity of the fluid at each point does not change in magnitude as well as in direction. Examples include flow of air over an aircraft wing.

1.5.10 Turbulent Flow

It is a flow in which fluid undergoes irregular fluctuations as compared to laminar flow. In turbulent flow, the velocity of fluid at each point continuously changes both in magnitude and direction. Examples are flow over a golf ball and smoke rising from cigarette.

1.5.11 Inviscid Flow

An inviscid flow is the flow of an ideal fluid that is assumed to have no viscosity.

1.5.12 Rotational and Irrotational Flow

A flow is said to be rotational if the fluid particles go on rotating about their own axes during the flow, i.e. the particles have some angular velocity. Mathematically, for such flow

$$\nabla \times \mathbf{V} \neq 0, \tag{1.7}$$

On the other hand, a flow is said to be irrotational if the fluid particles does not rotate about their own axes during the flow. i.e., $\nabla \times \mathbf{V} = 0$.

1.6 Dimensionless Numbers

A dimensionless quantity is a quantity without an associated physical dimension. It is thus a "pure" number and as such always has a dimension of **1**. Dimensionless quantities are widely used in mathematics, physics, engineering and economics. Definitions of some dimensionless quantities that are involved in our work are given here.

1.6.1 Reynolds Number

Reynolds number is defined as the ratio between inertial forces and viscous forces. Reynolds number is usually denoted by Re. When the Reynold number is small then flow is treated as creeping flow and when it is large it becomes turbulent. Mathematically, it can be defined as

$$\mathrm{Re} = \frac{\rho UL}{\mu} = \frac{UL}{\upsilon}, \tag{1.8}$$

where U is the characteristic velocity and L is a characteristic linear dimension, associated with the flow under consideration.

1.6.2 Prandtl Number

The Prandtl number Pr is a dimensionless number, named after the German physicist Ludwig Prandtl, defined as the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity. That is, the Prandtl number is given as:

$$\mathrm{Pr} = \frac{\upsilon}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{C_p\mu}{k},$$

Where:

υ =kinematic viscosity,$\upsilon = \frac{\mu}{\rho}$, (SI units: m^2/s)

α =thermal diffusivity, $\alpha = \frac{k}{(\rho C_p)}$,(SI units : m^2/s)

μ =dynamic viscosity, (SI units : $Pas = Ns/m^2$)

k =thermal conductivity, (SI units : $W/(mK)$)

C_p =specific heat, (SI units : $J/(kgK)$)

ρ = density, (SI units : kg/m^3).

1.6.3 Grashof Number

The Grashof number (Gr) is a dimensionless number frequently used in the areas of fluid dynamics and heat transfer which approximates the ratio of the buoyancy to viscous force acting on a fluid. It frequently arises in the study of situations involving natural convection. It is named after the German engineer Franz Grashof.

$$Gr_l = \frac{g\beta(T - T_\infty)L^3}{\upsilon^2}, \tag{1.9}$$

where L indicate the length scale basis for the Grashof Number,

g = acceleration due to Earth's gravity,

β = volumetric thermal expansion coefficient,

T = surface temperature,

T_∞ = Temperature of the ambient fluid,

L = characteristic length,

υ = kinematic viscosity.

1.6.4 Rayleigh Number

The product of the Grashof number and the Prandtl number gives the Rayleigh number, a dimensionless number that characterizes convection problems in heat transfer.

$$Ra_l = (Gr_l)(\mathrm{Pr}), \tag{1.10}$$

1.7 Basic Governing Equations

In this section the general form of equations governing the flow and heat transfer phenomena are presented in usual notations.

1.7.1 Continuity Equation:

Continuity equation is the mathematical expression of law of conservation of mass and it is described as

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{V}) = 0 \tag{1.11}$$

If density remains constant with respect to time and space then flow becomes incompressible and eq. (1.11) reduce to

$$\operatorname{div} \mathbf{V} = 0, \tag{1.12}$$

where div denotes the divergence operator.

1.7.2 Momentum Equation

The momentum equation or the law of coservation of momentum under the influence of magnetic field and bouncy effects is given by

$$\mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \upsilon \nabla^2 \mathbf{V} + \frac{1}{\rho}(\mathbf{j} \times \mathbf{B}) + \mathbf{g}\beta(T - T_\infty). \tag{1.13}$$

The utilization of magnetic field in the momentum equation also requires the satisfication of following physical laws:

Ohm's law

$$\mathbf{j} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}], \tag{1.14}$$

Maxwell's equation

$$\nabla \times \mathbf{E} = 0, \nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{1.15}$$

in which

$$\mathbf{V} = (u, v, w), \tag{1.16}$$

is the velocity vector, $\mathbf{j}$ the electric current density, $\mathbf{B}$ represents the magnetic induction vector, ρ the density of the fluid, υ the kinematic coefficient of viscosity, $\mathbf{g}$ identifies the gravtational vector β is the coefficient of thermal expansion and ∇ is the gradiant operator.

1.7.3 Energy Equation

The energy equation is given by

$$\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla)T = \alpha(\nabla^2 T - \frac{1}{k} \nabla \cdot q_r), \tag{1.17}$$

where T is the temperature, α is thermal diffusivity, k is thermal conductivity and q_r radiative heat flux is given by

$$q_r = \frac{4\sigma_0}{3(a+\sigma_s)} \nabla T^4,$$

in which σ_0 is the Stefan-Boltzmann constant, σ_s is the scattering coefficient and a is the Rosseland mean absorption coefficient.

Chapter 2

Similar and Non-Similar Solutios to Natural Convection Flow

2.1 Introduction

In this chapter we consider steady two dimensional boundary layer flow of a viscous incompressible fluid near a vertical wall. The wall is supposed to be at rest and the flow is due to temperature gradiants. The problem has already been studied by Bejan and Lage [6]. Many other authers have studied the problem like McAdams [7], Warner Arpaci [8], Jackson [9], and Churchill and Chu[10]. They used integral method to solve the governing system of equations by assuming suitable forms of the velocity and temperature functions. They made a selection of the arbitrary free stream of self similar form and accordingly boundary layer thickness $\delta(x)$, we also make a selection of the same form, that is,

$$U = c_1 x^m, \delta = c_2 x^n, \tag{2.1}$$

where c_1 and c_2 are independent of x. Using these transformation they calculated the boundary layer thickness, Nusselt number and skin friction. The apporach they adopted is quite specific to the self similar form of the free stream velocity. However, we have also dealt this problem for the non-similar flow by making no prior assumptions for the functions $U(x)$ and $\delta(x)$. In this way this chapter includes the review work concerning the self-similar flow and also includes the analysis for non-similar flow.

2.2 General Equations for Natural Convection Flow

The two-dimensional boundary layer equations as considerd by Bejan and Lage [6] are given as:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.2}$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty), \tag{2.3}$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{2.4}$$

Subjected to the boundary conditions

$$\begin{aligned} y &= 0; u = v = 0, T = T_w, \\ u &= v = 0; T = T_\infty, y \to \infty, \end{aligned}$$

where u and v are the x and y component of the velocity vector, v is the kinematics viscosity, g is acceleration due to Earth's gravity. Integrating the continuity equation with respect to y in the interval $(0, Y)$, where $Y > \delta$ & δ_t, one obtains:

$$\int_0^y \frac{\partial u}{\partial x} dy + \int_0^y \frac{\partial v}{\partial y} dy = 0, \tag{2.6}$$

$$\int_0^y \frac{\partial u}{\partial x} dy + v|_0^y = 0,$$

$$\int_0^y \frac{\partial u}{\partial x} dy + v|_y = 0,$$

$$v|_y = -\int_0^y \frac{\partial u}{\partial x} dy. \tag{2.7}$$

This is known as integral form of continuity equation.

2.3 Integral Form of Momentum and Energy Equations

In order to obtain the integral form of momentum and energy equations, we integrate momentum and energy equations over the boundary layer thickness δ and δ_t respectiely. First we Consider the momentum equation which can be rewritten as:

$$\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} - u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = v\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty), \tag{2.8}$$

or

$$\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} = v\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty). \tag{2.9}$$

Integrating the momentum equation with respect to y in the interval of $(0, Y)$, where Y is greater than both δ and δ_t, one obtains:

$$\int_0^y \frac{\partial u^2}{\partial x} dy + \int_0^y \frac{\partial (uv)}{\partial y} dy = v\int_0^y \frac{\partial^2 u}{\partial y^2} dy + g\beta \int_0^y (T - T_\infty) dy. \tag{2.10}$$

After integrating the 2nd term on L.H.S. and using boundary condition ($u = v = 0$), eq. (2.10) yields

$$\int_0^y \frac{\partial u^2}{\partial x} dy = -v\frac{\partial u}{\partial y}|_{y=0} + g\beta \int_0^y (T - T_\infty) dy, \tag{2.11}$$

or

$$\frac{\partial}{\partial x}\int_0^y u^2 dy = -v\frac{\partial u}{\partial y}|_{y=0} + g\beta \int_0^y (T - T_\infty) dy,$$

or

$$\frac{\partial}{\partial x}\int_0^y U^2 (\frac{u}{U})^2 dy = -v\frac{\partial u}{\partial y}|_{y=0} + g\beta \int_0^y (T - T_\infty) dy,$$

since

$$v\frac{\partial u}{\partial y}|_{y=0} = \frac{\tau_w}{\rho}, \tag{2.11a}$$

Therefore

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2dy=-\frac{\tau_w}{\rho}+g\beta\int_0^y(T-T_\infty)dy. \tag{2.12}$$

The eq. (2.12) is known as integral form of momentum equation.

The energy equation (2.4) can also be rewritten in the following form

$$\frac{\partial(uT)}{\partial x}+\frac{\partial(vT)}{\partial y}-T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{2.13}$$

By incorporating the continuity equation it takes the form

$$\frac{\partial(uT)}{\partial x}+\frac{\partial(vT)}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{2.14}$$

Let us integrate the above equation with respect to y in the interval $(0,y)$ to get

$$\int_0^y\frac{\partial(uT)}{\partial x}dy+\int_0^y\frac{\partial(vT)}{\partial y}dy=\alpha\int_0^y\frac{\partial^2 T}{\partial y^2}dy, \tag{2.15}$$

or

$$\int_0^y\frac{\partial(uT)}{\partial x}dy+vT|_0^y=\alpha\frac{\partial T}{\partial y}|_0^y,$$

or

$$\int_0^y\frac{\partial(uT)}{\partial x}dy+vT|_y=-\alpha\frac{\partial T}{\partial y}|_0.$$

As from continuity equation we have

$$vT|_y=-T\int_0^y\frac{\partial u}{\partial x}dy,$$

subsituting this in above equation we have

$$\int_0^y\frac{\partial(uT)}{\partial x}dy-T_\infty\int_0^y\frac{\partial u}{\partial x}dy=-\alpha\frac{\partial T}{\partial y}|_0. \tag{2.16}$$

Following a similar procedure as above the integral form of energy equation is given below

$$\frac{d}{dx}\int_0^y u(T-T_\infty)dy = -\alpha\frac{\partial T}{\partial y}|_0. \tag{2.17}$$

For obtaining the solution of a natural convection flow, the appropriate velocity and temperature profiles must be utilized together with the integral equations (2.12) and (2.17). The velocity and temperature profiles depend on the momentum and thermal boundary layers thickness, which in turn depends on the Prandtl number. Assuming the velocity profile as the third degree polynomial function of y in the boundary layer and using the boundary conditions to determine the unspecified quantities. The velocity profile is assumed of the form

$$\frac{u}{U} = \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2, \tag{2.18}$$

where U is the characteristic velocity that is a function of x. Similarly, the temperature profile can be obtained by assuming a second degree polynomial function of the form

$$\frac{T-T_\infty}{T_w-T_\infty} = \left(1-\frac{y}{\delta_t}\right)^2. \tag{2.19}$$

Substitution of velocity and temperature profiles into eq. (2.12) we have

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2 dy = -\frac{\tau_w}{\rho} + g\beta\int_0^y (T-T_\infty)dy,$$

or

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2 dy = \frac{d}{dx}U^2\int_0^y\left(\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2\right)^2 dy. \tag{2.20}$$

Let us make a substitution

$$\frac{y}{\delta} = \eta \Longrightarrow y = \delta\eta; dy = d\eta\delta,$$

So that

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2 dy = \frac{d}{dx}U^2\int_0^y\left(\eta\left(1-\eta\right)^2\right)^2 d\eta\delta, \tag{2.21}$$

which further simplifies to

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2dy=\frac{1}{105}\frac{d}{dx}(U^2\delta).$$

Following the same procedure for integral energy equation

$$\int_0^y\left(\frac{T-T_\infty}{T_w-T_\infty}\right)dy=\int_0^y\left(1-\frac{y}{\delta_t}\right)^2dy,$$

and assuming that

$$\frac{y}{\delta_t}=n; y=n\delta_t\Longrightarrow dy=dn\delta_t,$$

the integral form of energy equation takes the form

$$\int_0^y\left(\frac{T-T_\infty}{T_w-T_\infty}\right)dy=\int_0^y(1-n)^2\,dn\delta_t.$$

Upon further integrationthe the above equation simplifies to

$$\int_0^y\left(\frac{T-T_\infty}{T_w-T_\infty}\right)dy=\frac{1}{3}\delta_t. \tag{2.23}$$

Since the shear stress is defined as

$$\tau_w=\mu\frac{\partial u}{\partial y}|_{y=0},$$

which due to eq. (2.18) comes out to be

$$\tau_w=\frac{\mu U}{\delta}. \tag{2.24}$$

Substituting (2.22), (2.23) and (2.24) in momentum integral equation (2.12) we get

$$\frac{1}{105}\frac{d}{dx}(U^2\delta)=-\frac{\upsilon U}{\delta}+\frac{1}{3}g\beta(T_w-T_\infty)\delta. \tag{2.25}$$

Now consider the eq. (2.17) energy integral equation

$$\frac{d}{dx}\int_0^y u(T-T_\infty)dy=-\alpha\frac{\partial T}{\partial y}|_0,$$

which can also be written as

$$\frac{d}{dx}U_\infty \int_0^y \frac{u}{U_\infty}(T - T_\infty)dy = -\alpha \frac{\partial T}{\partial y}|_0. \tag{2.26}$$

Here we assume that

$$\frac{y}{\delta_t} = n; \frac{\delta_t}{\delta} = \varphi \Longrightarrow \left(\frac{y}{\delta_t}\right)\left(\frac{\delta_t}{\delta}\right) = n, \frac{y}{\delta} = n\varphi,$$

and substitute it into eq. (2.26) to get

$$\frac{d}{dx}U_\infty \int_0^y \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 (T_w - T_\infty)\left(1 - \frac{y}{\delta_t}\right)^2 dn\delta_t = -\frac{k}{\rho c_p}\left[\frac{-2(T_w - T_\infty)}{\delta_t}\right], \tag{2.27}$$

or

$$\frac{d}{dx}U_\infty \int_0^y [n\varphi(1 + n^2\varphi^2 - 2n\varphi)(1 + n^2 - 2n)]dn\delta_t = \frac{2\alpha}{\delta_t},$$

and finally

$$\frac{d}{dx}U_\infty \int_0^y [(n + n^3 - 2n^2)\varphi + (n^3 + n^5 - 2n^4)\varphi^3 - (2n^2 + 2n^4 - 4n^3)\varphi^2]dn\delta_t = \frac{2\alpha}{\delta_t}.$$

Upon further integration we obtain the simplified result as under

$$\frac{1}{30}\frac{d}{dx}(U\delta) = \frac{2\alpha}{\delta_t}. \tag{2.28}$$

At the leading edge of the vertical plate, the boundary layer thickness is zero and the characteristic velocity U is also zero

$$U = \delta = 0, x = 0. \tag{2.29}$$

Equation (2.29) serves as the initial condition for eqs. (2.25) and (2.28).

2.3.1 Self Similar Solution

Substituting eq. (2.1) in eq. (2.25) we get

$$\frac{1}{105}\frac{d}{dx}(c_1^2x^{2m}c_2x^n) = -\frac{\upsilon c_1 x^m}{c_2x^n} + \frac{1}{3}(T_w - T_\infty)c_2x^n,$$

or

$$\frac{c_1^2c_2}{105}\frac{d}{dx}\left(x^{2m+n}\right) = -\frac{\upsilon c_1}{c_2}x^{m-n} + \frac{1}{3}(T_w - T_\infty)c_2x^n,$$

or

$$\frac{c_1^2c_2}{105}(2m+n)(x^{2m+n-1}) = -\frac{\upsilon c_1}{c_2}x^{m-n} + \frac{1}{3}(T_w - T_\infty)c_2x^n. \tag{2.31}$$

Substitution of eq. (2.1) into eq. (2.28) gives

$$\frac{1}{30}\frac{d}{dx}(c_1x^mc_2x^n) = \frac{2\alpha}{c_2x^n},$$

or

$$\frac{c_1c_2}{30}\frac{d}{dx}(x^{m+n}) = \frac{2\alpha}{c_2}x^{-n},$$

or

$$\frac{c_1c_2}{30}(m+n)\left(x^{m+n-1}\right) = \frac{2\alpha}{c_2}x^{-n}. \tag{2.32}$$

The above two relations (2.31) and (2.32) can be true for all x only if the indices of x for all terms in the same equations are the alike, that is, when the following equations hold:

$$2m+n-1 = m-n = n, \tag{2.33}$$

and

$$m+n-1 = -n. \tag{2.34}$$

Solving eqs. (2.33) and eq. (2.34) we get the values of m and n as

$$m = \frac{1}{2};\ n = \frac{1}{4}. \tag{2.35}$$

This suggests that this is an agreement with the result of the scaling analysis. Substituting back the values of m and n into eqs. (2.31) and (2.32), one obtains

$$\frac{c_1^2 c_2}{105}\left(2\left(\frac{1}{2}\right)+\frac{1}{4}\right)\left(x^{2(\frac{1}{2})+\frac{1}{4}-1}\right) = -\frac{\upsilon c_1}{c_2}x^{\frac{1}{2}-\frac{1}{4}} + \frac{1}{3}(T_w - T_\infty)c_2 x^{\frac{1}{4}},$$

which after further simplification yields

$$\frac{c_1^2 c_2}{84} = -\frac{\upsilon c_1}{c_2} + \frac{1}{3}(T_w - T_\infty)c_2. \tag{2.36}$$

Similarly eq. (2.32) becomes

$$\frac{c_1 c_2}{30}\left(\frac{1}{2}+\frac{1}{4}\right)\left(x^{\frac{1}{2}+\frac{1}{4}-1}\right) = \frac{2\alpha}{c_2}x^{-\frac{1}{4}},$$

and finally simplifies to

$$\frac{c_1 c_2}{40} = \frac{2\alpha}{c_2}. \tag{2.37}$$

Solving for C_1 and C_2 we have

$$c_1^2 c_2 = -\frac{84\upsilon c_1}{c_2} + 28g\beta(T_w - T_\infty)c_2, \tag{2.38}$$

and

$$c_1^2 c_2 = \frac{80\alpha c_1}{c_2}. \tag{2.39}$$

Comparing eqs. (2.38) and (2.39) to get

$$\frac{80\alpha c_1}{c_2} = -\frac{84\upsilon c_1}{c_2} + 28g\beta(T_w - T_\infty)c_2,$$

or

$$80\alpha = -84\upsilon + 28g\beta(T_w - T_\infty)\frac{c_2^2}{c_1}.$$

Upon further manipulation we get

$$3\alpha\left(\frac{80\alpha}{c_2^2}\right)\left(\frac{20}{21} + \frac{\upsilon}{\alpha}\right) = g\beta(T_w - T_\infty)c_2^2,$$

or

$$c_2^4 = 240\alpha^2\left(\frac{20}{21} + \frac{\upsilon}{\alpha}\right)\left[g\beta(T_w - T_\infty)\right]^{-1},$$

or

$$c_2 = 4\left(\frac{15}{16}\right)^{\frac{1}{4}}\left(\frac{20}{21} + \frac{\upsilon}{\alpha}\right)^{\frac{1}{4}}\left[\frac{g\beta(T_w - T_\infty)}{\upsilon^2}\right]^{-\frac{1}{4}}\left(\frac{\upsilon}{\alpha}\right)^{-\frac{1}{4}}. \tag{2.40}$$

Since from eq. (2.39), we have

$$c_1 = \frac{80\alpha}{c_2^2}. \tag{2.41}$$

Therefore substitution of eq. (2.40) in eq. (2.41), gives

$$c_1 = 4\left(\frac{5}{3}\right)^{\frac{1}{2}}\upsilon\left(\frac{20}{21} + \frac{\upsilon}{\alpha}\right)^{-\frac{1}{2}}\left[\frac{g\beta(T_w - T_\infty)}{\upsilon^2}\right]^{\frac{1}{2}}. \tag{2.42}$$

Thus from eq. (2.1)

$$\delta = c_2 x^n, \tag{2.43}$$

where $n = \frac{1}{4}$. Inserting eqs. (2.35) and (2.40) in eq. (2.43), we have the expression for boundary layer thickness

$$\delta = 4\left(\frac{15}{16}\right)^{\frac{1}{4}}\left(\frac{20}{21} + \frac{\upsilon}{\alpha}\right)^{\frac{1}{4}}\left[\frac{g\beta(T_w - T_\infty)}{\upsilon^2}\right]^{-\frac{1}{4}}\left(\frac{\upsilon}{\alpha}\right)^{-\frac{1}{4}}x^{\frac{1}{4}},$$

or

$$\delta = 4\left(\frac{15}{16}\right)^{\frac{1}{4}}\left(\frac{20}{21} + \frac{\upsilon}{\alpha}\right)^{\frac{1}{4}}\left[\frac{g\beta(T_w - T_\infty)x^3}{\upsilon^2}\right]^{-\frac{1}{4}}\left(\frac{\upsilon}{\alpha}\right)^{-\frac{1}{4}}x. \tag{2.44}$$

2.3.2 Boundary Layer Thickness, Nusselt Number and Skin Friction

The boundary layer thickness in its compact form is given as

$$\frac{\delta}{x} = 3.93\left(\frac{0.952 + \mathrm{Pr}}{\mathrm{Pr}^2}\right)^{\frac{1}{4}} Gr_x^{-\frac{1}{4}}. \tag{2.45}$$

The local heat transfer coefficient at the surface of the vertical plate can be obtained by

$$h_x = -\frac{k}{T_w - T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{2.46}$$

which reduces to

$$h_x = \frac{2k}{\delta}. \tag{2.47}$$

The local Nusselt number is defined as

$$Nu_x = \frac{h_x x}{k} = \frac{2x}{\delta}, \tag{2.48}$$

or

$$Nu_x = 0.508\left(\frac{\mathrm{Pr}^2}{0.952 + \mathrm{Pr}}\right)^{\frac{1}{4}} Gr_x^{\frac{1}{4}}.$$

Since

$$Gr_x^{\frac{1}{4}} = Ra_x^{\frac{1}{4}}\left(\frac{1}{\mathrm{Pr}}\right)^{\frac{1}{4}},$$

therefore

$$Nu_x = 0.508\left(\frac{\mathrm{Pr}}{0.952 + \mathrm{Pr}}\right)^{\frac{1}{4}} Ra_x^{\frac{1}{4}}. \tag{2.49}$$

The skin friction is defined as:

$$c_f = \frac{2\tau_w}{\rho U^2}. \tag{2.50}$$

Since from eq. (2.24)

$$\tau_w = \frac{\mu U}{\delta},$$

and finally,

$$\tau_w = 0.2544 \left(\frac{0.952 + \mathrm{Pr}}{\mathrm{Pr}^2}\right)^{-\frac{1}{4}} Gr_x^{\frac{1}{4}} \frac{\rho \upsilon U}{x}. \tag{2.51}$$

Substituting eq. (2.51) in eq. (2.50), we get

$$c_f = \frac{2(0.2544)\left(\frac{0.952+\mathrm{Pr}}{\mathrm{Pr}^2}\right)^{-\frac{1}{4}} Gr_x^{\frac{1}{4}} \rho \upsilon U}{\rho U^2 x},$$

or

$$c_f = 0.5088 \left(\frac{0.952 + \mathrm{Pr}}{\mathrm{Pr}^2}\right)^{-\frac{1}{4}} Gr_x^{\frac{1}{4}} \,\mathrm{Re}_x\,. \tag{2.52}$$

2.4 Non-Similar Solution (General Solution)

In this section we again consider the eqs. (2.25) and (2.28) without taking into account the assumption made in eq. (2.1)

$$\frac{1}{105}\frac{d}{dx}(U^2\delta) = -\frac{\upsilon U}{\delta} + \frac{1}{3} g\beta (T_w - T_\infty)\delta,$$

or

$$\frac{d}{dx}(U^2\delta) = -\frac{105 \upsilon U}{\delta} + \frac{105}{3} g\beta (T_w - T_\infty)\delta.$$

where U is characteristic velocity and in general we assume that $U = U(x)$, thus

$$2U\delta\frac{dU}{dx} + U^2\frac{d\delta}{dx} = -\frac{105\upsilon}{U\delta} + \frac{35}{U^2} g\beta(T_w - T_\infty)\delta,$$

or

$$\delta\frac{d\delta}{dx} = -\frac{105\upsilon}{U} + \frac{35}{U^2} g\beta(T_w - T_\infty)\delta^2 - \frac{2}{U}\frac{dU}{dx}\delta^2. \tag{2.53}$$

Since

$$\frac{d(\delta)^2}{dx} = 2\delta\frac{d\delta}{dx},$$

and

$$\frac{1}{2}\frac{d(\delta)^2}{dx} = \delta\frac{d\delta}{dx}, \tag{2.54}$$

using eq. (2.54) in eq. (2.53), we have

$$\frac{d(\delta)^2}{dx} = -\frac{210\upsilon}{U} + \frac{70}{U^2}g\beta(T_w - T_\infty)\delta^2 - \frac{4}{U}\frac{dU}{dx}\delta^2, \tag{2.55}$$

similarly equation (2.28) takes the form

$$\frac{d(\delta)^2}{dx} = \frac{120\alpha}{U} - \frac{2\delta^2}{U}\frac{dU}{dx}. \tag{2.56}$$

Comparing eq. (2.55) and eq. (2.56) we have

$$\frac{120\alpha}{U} - \frac{2\delta^2}{U}\frac{dU}{dx} = -\frac{210\upsilon}{U} + \frac{70}{U^2}g\beta(T_w - T_\infty)\delta^2 - \frac{4}{U}\frac{dU}{dx}\delta^2,$$

which after some rearrangements takes the form

$$\left[g\beta(T_w - T_\infty) - \frac{U}{35}\frac{dU}{dx}\right]\delta^2 = 3U\alpha\left(\frac{4}{7} + \frac{\upsilon}{\alpha}\right),$$

and finally we have

$$\delta = 1.7321\left(\frac{0.571 + \mathrm{Pr}}{\mathrm{Pr}}\right)^{\frac{1}{2}} \mathrm{Re}_x^{\frac{1}{2}}(x)\left[Gr_x - \frac{1}{35}\mathrm{Re}_x\left(\frac{x^2}{\upsilon}\right)\frac{dU}{dx}\right]^{-\frac{1}{2}}, \tag{2.57}$$

where Pr is the Prandtl number.

2.4.1 Boundary Layer Thickness, Nusselt Number and Skin Friction

The boundary layer thickness (2.57) in dimensionless form read as

$$\frac{\delta}{x} = 1.7321\left(\frac{0.571 + \mathrm{Pr}}{\mathrm{Pr}}\right)^{\frac{1}{2}}\left[\frac{Gr_x}{\mathrm{Re}_x} - \frac{1}{35}\frac{x^2}{\upsilon}\frac{dU}{dx}\right]^{-\frac{1}{2}}, \tag{2.58}$$

where

$$\mathrm{Re}_x = \frac{Ux}{\upsilon},\ Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{\upsilon^2}.$$

The local heat transfer coefficient at the surface of the vertical plate can be obtained from eqs. (2.46), (2.47) and (2.48), we get the local Nusselt number of the form

$$Nu_x = \frac{h_x x}{k} = \frac{2x}{\delta}.$$

After substituting the necessary information in above equation we have

$$Nu_x = 1.1547\left(\frac{0.571+\mathrm{Pr}}{\mathrm{Pr}}\right)^{-\frac{1}{2}}\left[\frac{Gr_x}{\mathrm{Re}_x} - \frac{x^2}{35\upsilon}\frac{dU}{dx}\right]^{\frac{1}{2}}. \tag{2.59}$$

Skin Friction in this case becomes of the form

$$c_f = \frac{2(0.5747)\left(\frac{0.571+\mathrm{Pr}}{\mathrm{Pr}}\right)^{-\frac{1}{2}}\left[\frac{Gr_x}{\mathrm{Re}_x} - \frac{x^2}{35\upsilon}\frac{dU}{dx}\right]^{\frac{1}{2}}\rho\upsilon U}{\rho U^2 x},$$

which in the form of dimensionless parameters takes the form

$$c_f = 1.1494\left(\frac{0.571+\mathrm{Pr}}{\mathrm{Pr}}\right)^{-\frac{1}{2}}\left[\frac{Gr_x}{\mathrm{Re}_x} - \frac{x^2}{35\upsilon}\frac{dU}{dx}\right]^{\frac{1}{2}}\mathrm{Re}_x^{-1}. \tag{2.60}$$

It is important to mention here that the results presented in eqs. (2.58) – (2.60) are valid for all forms of the function $U(x)$ and the results presented in the previous section can also be recovered if one considers eq. (2.1).

2.5 Graphical Results and Discussion

In this section the result have been presented graphically. A comparision between integral solution and similarity solution is also presented through graphs in figure 2.1. It can be seen that the integral solution under predicts the local Nusselt number for low Prandtl number but over predicts the local Nusselt number for high Prandtl number. At $Pr = 10^{-4}$, the integral solution yields $Nu_x/Ra_x^{\frac{1}{4}} = 0.051$, which is 13% lower than the value of 0.059 obtained from the similarity solution. At $Pr = 0.72$, which is the Prandtl number for air, the result obtained by the integral solution ($Nu_x/Ra_x^{\frac{1}{4}} = 0.412$) is 6.8% higher than the similarity solution. It can also be seen that the agreement between the integral and similarity solutions is the best at high Prandtl number values. When $Pr = 10^4$, the difference between the integral and similarity solution is only 1.5%. In fig. 2.2 the skin friction is plotted against Pr. By increasing the Prandtl number Pr the skin friction increases but at 10^0 and after this value the skin friction becomes smooth and there are no variations in the skin friction curve. Figure 2.3 shows the boundary layer thickness behaviour on increasing the value of Pr. The boundary layer thickness decreases but at the point 10^0 the boundary layer thickness becomes smooth and no further significant variations are obscened in the boundary layer thickness plot.

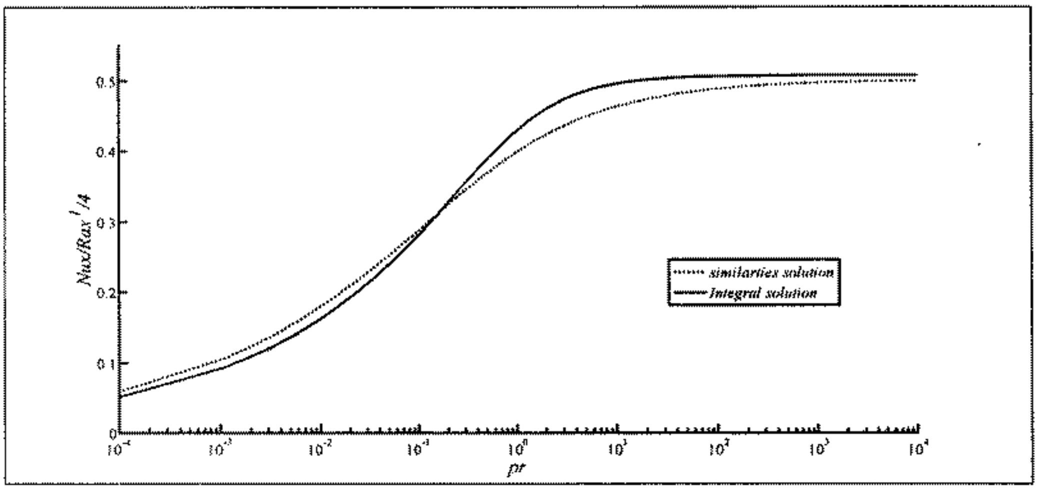

Fig. 2.1: Comparision between integral and similarity solution.

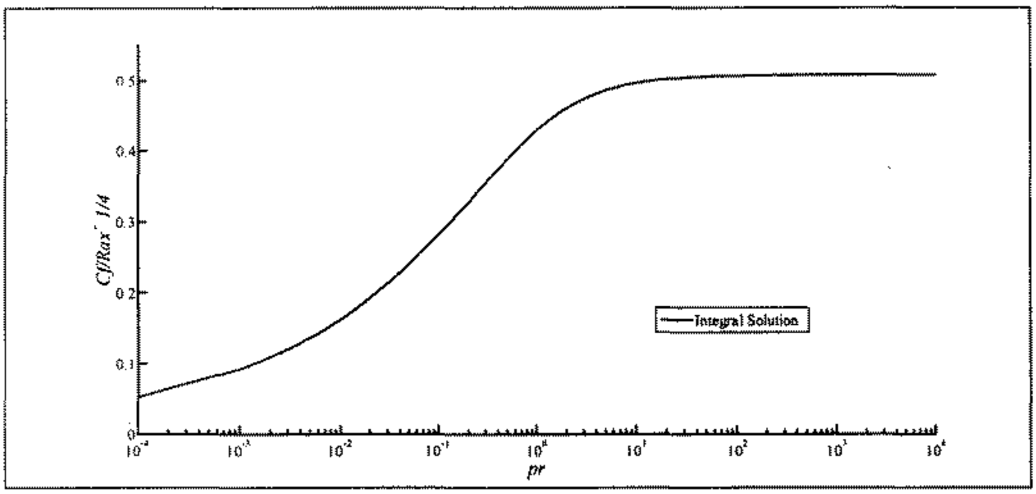

Fig. 2.2: Skin friction plotted against Prandtl number.

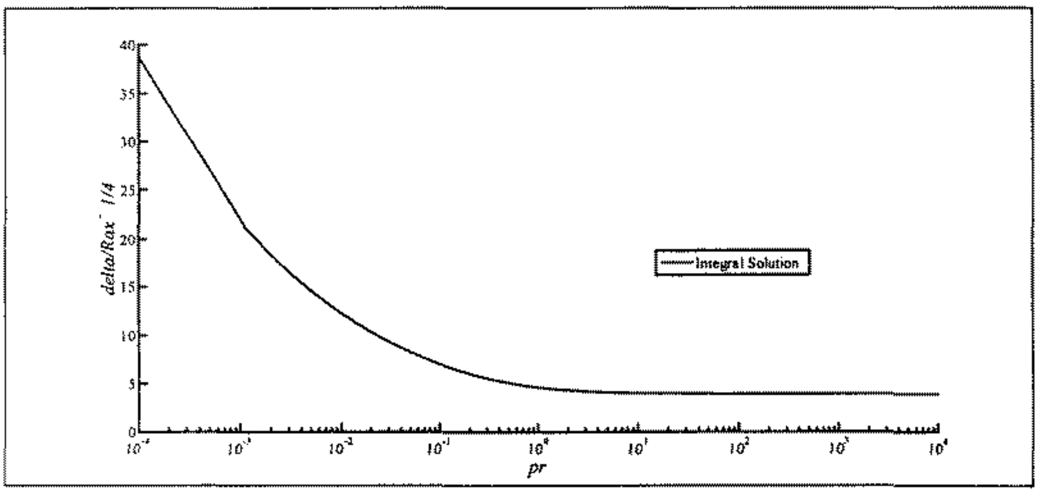

Fig. 2.3: Boundary layer thickness plotted against Prandtl number.

2.5.1 Conclusion

This chapter represents an approximate solution (integral solution) for laminar natural convection flow. The governing laws are transformed into integral form and velocity and temperature profiles are assumed. Self similar integral solution is obtained by assuming special form (scaling type) of the free stream velocity and the boundary layer thickness. A general approach is adopted to get the integral solution that is useful for all types of the free stream velocity. The integral method enabled us to obtain the analytic expression for boundary layer thickness. With the help of boundary layer thickness expressions the expressions for local nusselt number as well as for local skin friction coefficient are also obtained.

Chapter 3

Natural Convection Boundary Layer Flow Under the Influence of Sinusoidal Magnetic Field

Magnetohydrodynamic natural convection periodic boundary layer flow of an electrically conducting and optically dense gray viscous fluid near a heated vertical plate has been studied by various authors. Although this problem was discussed in detail by several investigators such as Sparrow and Cess [11], Riley [12], Kuiken [13], Wilks [14] and Hunt and Wilks [15]. Various other authors analyzed the influence of MHD in connection with a number of important physical phenomenon. Very recently, Siddiqa et. al. [16], Hossain et. al. [17] and Makinde [18] investigated the problem of conjugate effects of heat and mass transfer on magnetohydrodynamic free convection fluid flow in a strong cross field. For instance, Pop and Watanabe [19] investigated the effects of Hall current on the MHD free convection flow about a semi-infinite vertical flat plate.

In this chapter we revisit the problem studied by Siddiqa et. al. [16]. They considered natural convection boundary layer flow of an electrically conducting fluid near a vertical plate. The magnetic field is assumed to be sinusoidal in longitudnal direction. Siddiqa et. al. [16] obtained numerical solution of the problem using finite difference method direct numerical approach. We consider the same problem and obtain analytic solution using the integral method as it is done in chapter 2.

3.1 Mathematical Formulation

We consider steady two dimensional natural convection flow of a viscous, electrically conducting and optically dense gray fluid along a semi-infinite vertical heated surface in the presence of magnetic field prescribed in the streamwise direction $B_y(x)$ of the form (see [20]):

$$B_x = 0, B_y = B_0 \sin\left(\frac{\pi x}{\lambda}\right),$$

where B_0 and λ are the constants related to transverse magnetic field and wavelength of the applied magnetic field respectively. We proceed to find governing equation under the usual approximation for steady magnetohydrodynamic flow with Ohm'slaw and Maxwell's equations associated with thermal radiation may now be written as continuity:

$$\nabla \cdot \mathbf{V} = 0, \tag{3.1}$$

Momentum equation:

$$\mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \upsilon \nabla^2 \mathbf{V} + \frac{1}{\rho}(\mathbf{j} \times \mathbf{B}) + g\beta(T - T_\infty), \tag{3.2}$$

Energy equation:

$$\mathbf{V} \cdot \nabla T = \alpha \left(\nabla^2 T - \frac{1}{k} \nabla \cdot q_r\right),$$

Ohm's law

$$\mathbf{j} = \sigma[\mathbf{E} + \mathbf{v} \times \mathbf{B}],$$

Maxwell's equation

$$\nabla \times \mathbf{E} = 0, \nabla \cdot \mathbf{B} = 0, \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j},$$

where $\mathbf{V} = (u, v, 0)$ is the velocity vector, $\mathbf{j}$ the electric current density, $\mathbf{B} = (0, B_y, 0)$ represents the magnetic induction vector, ρ the density of the fluid, υ the kinematic coefficient of viscosity,

$g = (-g_x, 0, 0)$ identifies the gravtational vector, β the coefficient of thermal expansion, q_r is the Radiative heat flux defined by

$$\mathbf{q}_r = \frac{4\sigma_0}{3(a+\sigma_s)}\nabla T^4, \tag{3.3}$$

in which σ_0 is the Stefan-Boltzmann constant, σ_s is Scattering coefficient, a is the Rosseland mean absorption coefficient. The boundary layer equations for the conservation of mass, momentum and energy for the flow past a heated vertical platein the presence of sinusoidal magnetic field become

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.4}$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}\sin^2(\pi x)u + g_x\beta(T - T_\infty), \tag{3.5}$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial y^2} - \frac{1}{k}\frac{\partial q_r}{\partial y}\right), \tag{3.6}$$

Boundary conditions are:

$$\begin{aligned} y &= 0; u = v = 0, \\ T &= T_w; y = 0, \\ u &= v = 0; y \to \infty, \\ T &= T_\infty, y \to \infty, \end{aligned} \tag{3.7}$$

where α is thermal diffusivity, T_w temperature at the surface and T_∞ temperature of the ambient fluid.

3.2 The Integral Method Solution

3.2.1 Integral Form of Governing Equations

The momentum equation can be written in the following form:

$$\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}-u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)u+g\beta(T-T_\infty), \tag{3.8}$$

or

$$\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)u+g\beta(T-T_\infty). \tag{3.9}$$

Integrating eq. (3.9) with respect to y in the interval of $(0,Y)$, where Y is greater than both δ and δ_t, we obtain

$$\int_0^y \frac{\partial u^2}{\partial x}dy+\int_0^y \frac{\partial (uv)}{\partial y}dy=\upsilon\int_0^y \frac{\partial^2 u}{\partial y^2}dy-\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y udy+g\beta\int_0^y(T-T_\infty)dy, \tag{3.10}$$

or

$$\int_0^y \frac{\partial u^2}{\partial x}dy+uv\mid_0^y=\upsilon\frac{\partial u}{\partial y}\mid_0^y-\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y udy+g\beta\int_0^y(T-T_\infty)dy, \tag{3.11}$$

Implementing the boundary data, we get

$$\int_0^y \frac{\partial u^2}{\partial x}dy=-\upsilon\frac{\partial u}{\partial y}\mid_{y=0}-\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y udy+g\beta\int_0^y(T-T_\infty)dy, \tag{3.12}$$

which can also be rewritten as:

$$\frac{\partial}{\partial x}\int_0^y U^2\left(\frac{u}{U}\right)^2dy=-\upsilon\frac{\partial u}{\partial y}\mid_{y=0}-\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y udy+g\beta\int_0^y(T-T_\infty)dy. \tag{3.13}$$

Since the shear stress is defined by

$$\upsilon\frac{\partial u}{\partial y}\mid_{y=0}=\frac{\tau_w}{\rho}, \tag{3.14}$$

due to which eq. (3.13) takes the form

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2dy=-\frac{\tau_w}{\rho}-\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y udy+g\beta\int_0^y(T-T_\infty)dy. \quad (3.15)$$

This is known as integral form of momentum equation.

Following the same procedure, the integral energy equation can be obtained by rewriting the energy equation (3.6) as

$$\frac{\partial(uT)}{\partial x}+\frac{\partial(vT)}{\partial y}-T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\alpha\left(\frac{\partial^2T}{\partial y^2}-\frac{1}{k}\frac{\partial \mathbf{q}_r}{\partial y}\right), \quad (3.16)$$

which on incorporating the continuity equation (3.4) simplifies to

$$\frac{\partial(uT)}{\partial x}+\frac{\partial(vT)}{\partial y}=\alpha\left(\frac{\partial^2T}{\partial y^2}-\frac{1}{k}\frac{\partial q_r}{\partial y}\right). \quad (3.17)$$

Let us integrate eq. (3.17) with respect to y in the interval $(0,y)$

$$\int_0^y\frac{\partial(uT)}{\partial x}dy+\int_0^y\frac{\partial(vT)}{\partial y}dy=\alpha\int_0^y\frac{\partial^2T}{\partial y^2}dy-\frac{\alpha}{k}\int_0^y\frac{\partial \mathbf{q}_r}{\partial y}dy. \quad (3.18)$$

The term $\mathbf{q}_r$ represents radiative heat flux in the $y-$direction. In order to reduce the complexity of the problem and to provide a means of comparison with further studies that might employ a more detailed representation for the radiative heat flux, here the optically thick radiation limit, known as Rosseland diffusion approximation (see [12]), is considered. Due to this assumption, the radiative heat flux $\mathbf{q}_r$ is given as

$$\mathbf{q}_r=\frac{4\sigma_0}{3(a+\sigma_s)}\nabla T^4, \quad (3.19)$$

therefore eq. (3.18) takes the form

$$\int_0^y\frac{\partial(uT)}{\partial x}dy+vT|_y=-\alpha\frac{\partial T}{\partial y}|_0-\frac{\alpha}{k}\frac{16T^3\sigma_0}{3(a+\sigma_s)}\int_0^y\frac{\partial^2T}{\partial y^2}dy. \quad (3.20)$$

As from continuity equation we have

$$vT|_y=-T\int_0^y\frac{\partial u}{\partial x}dy. \quad (3.21)$$

Substituting this in equation (3.20) we have

$$\int_0^y \frac{\partial(uT)}{\partial x}dy - T_\infty \int_0^y \frac{\partial u}{\partial x}dy = -\alpha\frac{\partial T}{\partial y}|_0 - \frac{\alpha}{k}\frac{16T^3\sigma_0}{3(a+\sigma_s)}\frac{\partial T}{\partial y}|_{y=0}, \tag{3.22}$$

which further simplifies to

$$\frac{d}{dx}\int_0^y u(T-T_\infty)dy = -\alpha\frac{\partial T}{\partial y}|_0 - \frac{\alpha}{k}\frac{16T^3\sigma_0}{3(a+\sigma_s)}\frac{\partial T}{\partial y}|_{y=0}. \tag{3.23}$$

This is known as energy integral equation.

3.2.2 Velocity and Temperature Profiles

For obtaining the solution of a considered problem, the appropriate velocity and temperature profiles must be utilized together with the integral equations (3.15) and (3.23). The velocity and temperature profiles depend upon the thicknesses of the momentum and thermal boundary layers, which in turn depend on the Prandtl number. We assume the velocity profile as a third degree polynomial of the form

$$\frac{u}{U} = \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2, \tag{3.24}$$

where U is a characteristic free stream velocity that is a function of x and $\delta(x)$ is the momentum boundary layer thickness. Similarly, the temperature profile can be obtained by assuming a second degree polynomial function as

$$\frac{T-T_\infty}{T_w-T_\infty} = \left(1-\frac{y}{\delta_t}\right)^2, \tag{3.25}$$

where $\delta_t(x)$ is the thermal boundary layer thickness. Here we evaluate every term of the momentum eq. (3.15) separately, therefore first consider the first term on the L.H.S of eq. (3.15)

$$\frac{d}{dx}U^2\int_0^y \left(\frac{u}{U}\right)^2 dy = \frac{d}{dx}U^2\int_0^y \left(\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2\right)^2 dy, \tag{3.26}$$

and assume that

$$\frac{y}{\delta} = \eta \Longrightarrow y = \delta\eta; dy = d\eta\delta,$$

so that

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2dy=\frac{d}{dx}U^2\int_0^y\left(\eta\left(1-\eta\right)^2\right)^2d\eta\delta, \tag{3.27}$$

which finally simplifies to

$$\frac{d}{dx}U^2\int_0^y\left(\frac{u}{U}\right)^2dy=\frac{1}{105}\frac{d}{dx}(U^2\delta). \tag{3.28}$$

Second term on the R.H.S of eq. (3.15) evaluates as

$$\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y udy=\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y\left[\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2\right]Udy,$$

and finally

$$\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\int_0^y udy=\frac{1}{12}\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)U\delta. \tag{3.29}$$

Further we consider the

$$\int_0^y\left(\frac{T-T_\infty}{T_w-T_\infty}\right)dy=\int_0^y\left(1-\frac{y}{\delta_t}\right)^2dy,$$

and assume that

$$\frac{y}{\delta_t}=n;y=n\delta_t\Longrightarrow dy=dn\delta_t,$$

so that we have

$$\int_0^y\left(\frac{T-T_\infty}{T_w-T_\infty}\right)dy=\frac{1}{3}\delta_t. \tag{3.30}$$

Since

$$\tau_w=\mu\frac{\partial u}{\partial y}|_{y=0},$$

which after consideration of eq. (3.24) takes the form

$$\tau_w=\frac{\mu U}{\delta}. \tag{3.31}$$

Inserting eqs. (3.28), (3.29), (3.30) and (3.31) in eq. (3.15) we get

$$\frac{1}{105}\frac{d}{dx}(U^2\delta) = -\frac{\upsilon U}{\delta} - \frac{1}{12}\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)U\delta + \frac{1}{3}g\beta(T_w - T_\infty)\delta. \tag{3.32}$$

Following the same procedure as we did above, the term by term evaluation of the energy equation is given as

$$\frac{d}{dx}U_\infty\int_0^y \frac{u}{U_\infty}(T - T_\infty)dy = -\alpha\frac{\partial T}{\partial y}|_0 - \frac{\alpha}{k}\frac{16T^3\sigma_0}{3(a+\sigma_s)}\frac{\partial T}{\partial y}|_{y=0}. \tag{3.33}$$

In order to make the further calculation simple we make new assumption

$$\frac{y}{\delta_t} = n; \frac{\delta_t}{\delta} = \varphi \Longrightarrow \left(\frac{y}{\delta_t}\right)\left(\frac{\delta_t}{\delta}\right) = n\varphi; \frac{y}{\delta} = n\varphi,$$

due to which we have

$$\begin{aligned}\frac{d}{dx}U_\infty\int_0^y \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2(T_w - T_\infty)\left(1-\frac{y}{\delta_t}\right)^2 dn\delta_t &= -\frac{k}{\rho c_p}\left[\frac{-2(T_w - T_\infty)}{\delta_t}\right] \\ &\quad -\frac{16\alpha T^3}{3k}\left(\frac{\sigma_0}{a+\sigma_s}\right)\left[\frac{-2(T_w - T_\infty)}{\delta_t}\right],\end{aligned} \tag{3.34}$$

which further reduces to

$$\frac{d}{dx}U_\infty\int_0^y[(n+n^3-2n^2)\varphi + (n^3+n^5-2n^4)\varphi^3 - (2n^2+2n^4-4n^3)\varphi^2]dn\delta_t = \frac{2\alpha}{\delta_t} + \frac{8\alpha}{3k}\frac{4\sigma_0 T^3}{(a+\sigma_s)\delta},$$

and finally we get

$$\frac{1}{30}\frac{d}{dx}(U\delta) = \frac{2\alpha}{\delta} + \frac{2\alpha}{\delta}\frac{4}{3}R_d. \tag{3.35}$$

We now need to solve the eqs. (3.32) and (3.35) simultaneously in order to obtain δ. Since U is a reference velocity in this case and is assumed to be constant. Therefore eqs. (3.32) and (3.35) simplifying as

$$\delta\frac{d\delta}{dx} = -\frac{105\upsilon}{U} - \frac{35}{4}\frac{\sigma B_0^2}{\rho U}\sin^2(\pi x)\delta^2 + 35U^2 g\beta(T_w - T_\infty)\delta^2, \tag{3.36}$$

and

$$\delta\frac{d\delta}{dx} = \frac{60}{U}\left(\alpha + \frac{4}{3}R_d\right). \tag{3.37}$$

Since

$$\delta\frac{d\delta}{dx} = \frac{1}{2}\frac{d(\delta)^2}{dx}. \tag{3.38}$$

Therefore the above two equations take the form

$$\frac{d(\delta)^2}{dx} = -\frac{210\upsilon}{U} - \frac{35}{2}\frac{\sigma B_0^2}{\rho U}\sin^2(\pi x)\delta^2 + 70U^2 g\beta(T_w - T_\infty)\delta^2, \tag{3.39}$$

and

$$\frac{d(\delta)^2}{dx} = \frac{120\alpha}{U}\left(1 + \frac{4}{3}R_d\right). \tag{3.40}$$

Equating eq. (3.39) with eq. (3.40) we get

$$\frac{120\alpha}{U}\left(1 + \frac{4}{3}R_d\right) = -\frac{210\upsilon}{U} - \frac{35}{2}\frac{\sigma B_0^2}{\rho U}\sin^2(\pi x)\delta^2 + 70U^2 g\beta(T_w - T_\infty)\delta^2, \tag{3.41}$$

or

$$30\left(4\alpha\left(1 + \frac{4}{3}R_d\right) + 7\upsilon\right) = \frac{35}{U}\left[2g\beta(T_w - T_\infty)\delta^2 - \frac{U}{2}\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\delta^2\right],$$

which upon further simplification takes the form

$$3\left(\frac{4}{7}\left(1 + \frac{4}{3}R_d\right) + \frac{\upsilon}{\alpha}\right) = \frac{\upsilon^2}{U\alpha\lambda^3}\left[\frac{g\beta(T_w - T_\infty)\lambda^3}{\upsilon^2} - \frac{U\lambda^3}{4\upsilon^2}\frac{\sigma B_0^2}{\rho}\sin^2(\pi x)\right]\delta^2,$$

and finally

$$\begin{aligned}\delta^2 &= 3\left(0.571\left(1 + \frac{4}{3}R_d\right) + \mathrm{Pr}\right) \\ &\quad \left(\frac{\upsilon}{U}\right)^{-1}\left(\frac{\upsilon}{\alpha}\right)^{-1}\left(\frac{1}{\lambda^3}\right)^{-1}\left[\frac{g\beta(T_w - T_\infty)\lambda^3}{\upsilon^2} - \frac{1}{4}\frac{U^2\lambda^2}{\upsilon^2}\frac{\sigma B_0^2\lambda}{\rho U}\sin^2(\pi x)\right]^{-\frac{1}{2}}.\end{aligned} \tag{3.42}$$

Let us denote

$$M^2 = \frac{\sigma B_0^2 \lambda}{\rho U}. \tag{3.43}$$

Invoking eq. (3.43) in eq. (3.42) we have

$$\delta = 1.74 \left(0.571 \left(1 + \frac{4}{3} R_d\right) + \Pr\right)^{\frac{1}{2}} \left(\frac{\upsilon}{U}\right)^{-\frac{1}{2}} \left(\frac{\upsilon}{\alpha}\right)^{-\frac{1}{2}} (\lambda)^{\frac{1}{2}} \lambda \left[\frac{g\beta(T_w - T_\infty)\lambda^3}{\upsilon^2} - \frac{1}{4}\frac{U^2\lambda^2}{\upsilon^2} M^2 \sin^2(\pi x)\right]^{-\frac{1}{2}}. \tag{3.44}$$

3.2.3 Boundary Layer Thickness

The boundary layer thickness in dimensionless form read as

$$\frac{\delta}{\lambda} = 1.74 \left(\frac{0.571}{\Pr} \left(1 + \frac{4}{3} R_d\right) + \Pr\right)^{\frac{1}{2}} \mathrm{Re}_\lambda^{\frac{1}{2}} \left[Gr_\lambda - \frac{1}{4} \mathrm{Re}_\lambda^2 M^2 \sin^2(\pi x)\right]^{-\frac{1}{2}}, \tag{3.45}$$

where

$$\mathrm{Re}_\lambda = \frac{U\lambda}{\upsilon}, \tag{3.46}$$

is the Reynolds number based upon the wavelength λ

$$\frac{\delta}{\lambda} = 1.74 \left(\frac{0.571}{\Pr} \left(1 + \frac{4}{3} R_d\right) + \Pr\right)^{\frac{1}{2}} \mathrm{Re}_\lambda^{-\frac{1}{2}} \left[\frac{Gr_\lambda}{\mathrm{Re}_\lambda^2} - \frac{1}{4} M^2 \sin^2(\pi x)\right]^{-\frac{1}{2}}. \tag{3.47}$$

3.2.4 Nusselt Number

The local heat transfer coefficient at the surface of the vertical plate is defined as

$$h_x = -\frac{k}{T_w - T_\infty} \left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{3.48}$$

which under assumption (3.25) takes the form

$$h_x = \frac{2k}{\delta}.$$

The local Nusselt number is defined as

$$Nu_x = \frac{h_x \lambda}{k} = \frac{2\lambda}{\delta}, \tag{3.49}$$

which upon insertion of the value of δ takes the form

$$Nu_x = 1.1494 \left(\frac{0.571}{\Pr} \left(1 + \frac{4}{3} R_d \right) + \Pr \right)^{-\frac{1}{2}} \mathrm{Re}_\lambda^{\frac{1}{2}} \left[\frac{Gr_\lambda}{\mathrm{Re}_\lambda^2} - \frac{1}{4} M^2 \sin^2(\pi x) \right]^{\frac{1}{2}}. \tag{3.50}$$

3.2.5 Skin Friction

The coefficient of skin friction is defined by

$$c_f = \frac{2\tau_w}{\rho U^2}, \tag{3.51}$$

where the shear stress is

$$\tau_w = \frac{\mu U}{\delta}. \tag{3.52}$$

Inserting the value of δ in eq.(3.52) we have

$$\tau_w = \frac{1}{1.74} \frac{\mu U}{\lambda} \left(\frac{0.571}{\Pr} \left(1 + \frac{4}{3} R_d \right) + \Pr \right)^{-\frac{1}{2}} \mathrm{Re}_\lambda^{\frac{1}{2}} \left[\frac{Gr_\lambda}{\mathrm{Re}_\lambda^2} - \frac{1}{4} M^2 \sin^2(\pi x) \right]^{\frac{1}{2}}. \tag{3.53}$$

Finally the coefficient of skin friction is obtained after substitution of eq. (3.53) in eq. (3.51)

$$c_f = \frac{2}{1.74} \left(\frac{0.571}{\Pr} \left(1 + \frac{4}{3} R_d \right) + \Pr \right)^{-\frac{1}{2}} \mathrm{Re}_\lambda^{-\frac{1}{2}} \left[\frac{Gr_\lambda}{\mathrm{Re}_\lambda^2} - \frac{1}{4} M^2 \sin^2(\pi x) \right]^{\frac{1}{2}}. \tag{3.57}$$

3.3 Graphical Results and Discussion

In this section we have displayed graphical results in order to understand the physics of free convection flow of electrically conducting and optically dense gray fluid near a vertical plate in the presence of sinusoidal magnetic field. Solutions of the governing equations are obtained by integral method for the entire range of X, for different values of the physical parameters, i.e., for magnetic field parameter (or Hartmann number) M, thermal radiation parameter (or Planck constant) R_d. The influence of magnetic field parameter $M = 0.0, 0.05, 0.10, 0.15, 0.20$ is illustrated in figs.(1) – (3) for $R_d = 0.0$ and $Pr = 0.7$ on coefficient of local skin friction, $C_f \mathrm{Re}_\lambda^{\frac{1}{2}}$, boundary layer thickness and the local Nusselt number, $Nu_x \mathrm{Re}_\lambda^{-\frac{1}{2}}$ respectively. It can be seen from these figures that when the magnetic field parameter effects are absent, i.e. $M = 0$ then a straight line is obtained. That is the values of skin friction coefficient, boundary layer thickness and the Nusselt number become constant which is a self similar solution. As the values of M are increased the wavy nature of these quantities become visible. This is because of the reason that M represents the amplitude of applied wavy magnetic field and as the amplitude of the wavy magnetic field increases its effects become prominent as shown in figs. (3.1) – (3.6). The same graphs as plotted in figs. (3.1) – (3.3) are again plotted in figs. (3.4) – (3.6) for $R_d = 2.0$.It is obvious that in the presence of thermal radiation the skin friction is decreased and the boundary layer thickness is increased as shown in figs. (3.4) and (3.5) respectively. Further more for $R_d = 2.0$ the Nusselt number is also decreased. This is because of the reason that the thermal radiations enhances the temperature with in the boundary layer which in turn increases the layer thickness. On increasing the magnetic field strengths the skin friction decrease and become wavy as shown in figs.(3.1) and (3.4). On increasing the values of M the layer thickness increases and shown wavy pattern. The Nusselt number decreases by increasing the values of M. The wavy nature of the Nusselt number gets stronger for large values of the parameter M. The influence of Prandtl number on boundary layer thickness, nusselt number and skin friction is shown in figs. (3.7) – (3.9). From the graphs it can be seen that boundary layer thickness stays periodic on increasing Pr and the amplitude of the wavy patterns does not vary too much. However the layer thickness grows signigicantly by increasing the Prandtl numer. Same effects of Pr on Nusselt number and skin friction are observed in figs. (3.8) and (3.9) respectively. Radiation effects on boundary layer thickness, Nusselt number and skin friction are plotted in figs.(3.10) – (3.12). The boundary layer thickness is strongly effected by radiation parameter R_d and increases for larger radiation conditions. Large values of the radition parameter also lead towards increased nusselt number as well as skin friction.

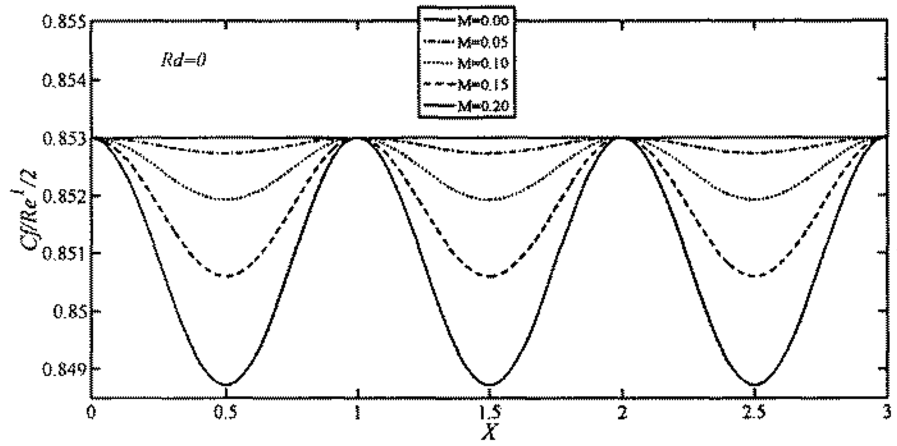

Fig. 3.1: Variation of coefficient of local skin friction with X.

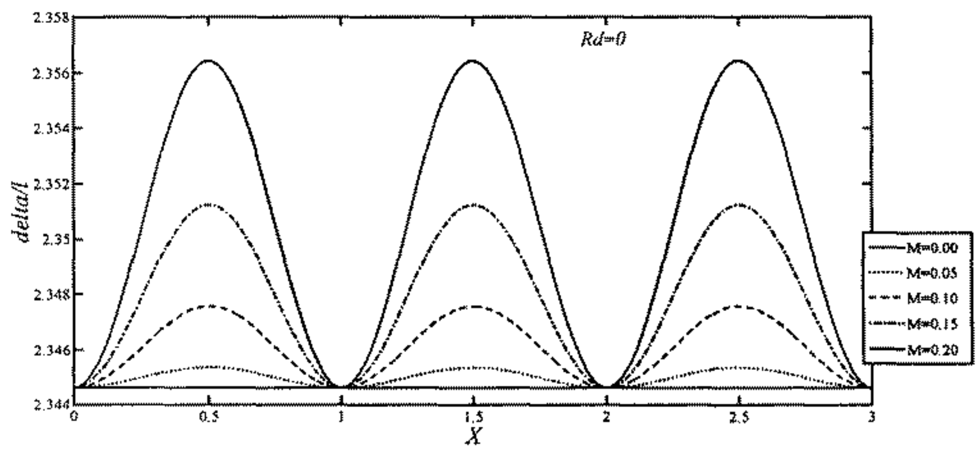

Fig. 3.2: Boundary layer thickness plotted against X.

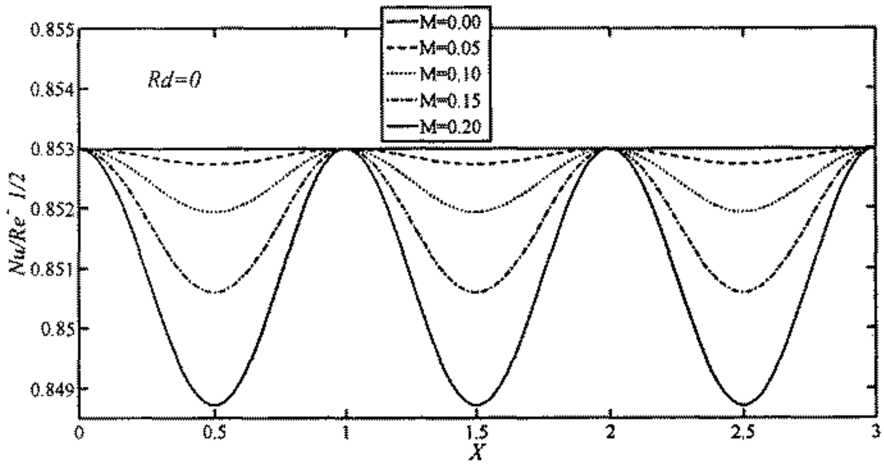

Fig. 3.3: Nusselt number graph at $R_d = 0.0$ for varied values of M.

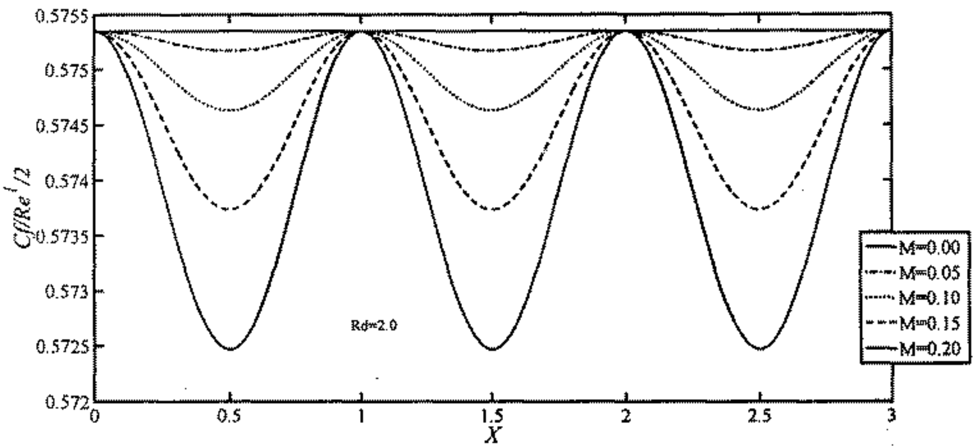

Fig. 3.4: Skin friction plotted against x at $R_d = 2.0$ different M.

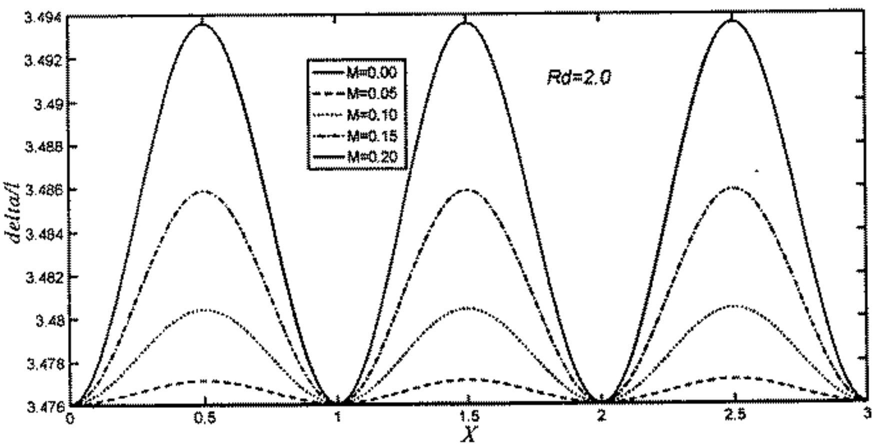

Fig. 3.5: Graph of boundary layer thickness for $M = 0.0, 0.05, 0.10, 0.15, 0.20$.

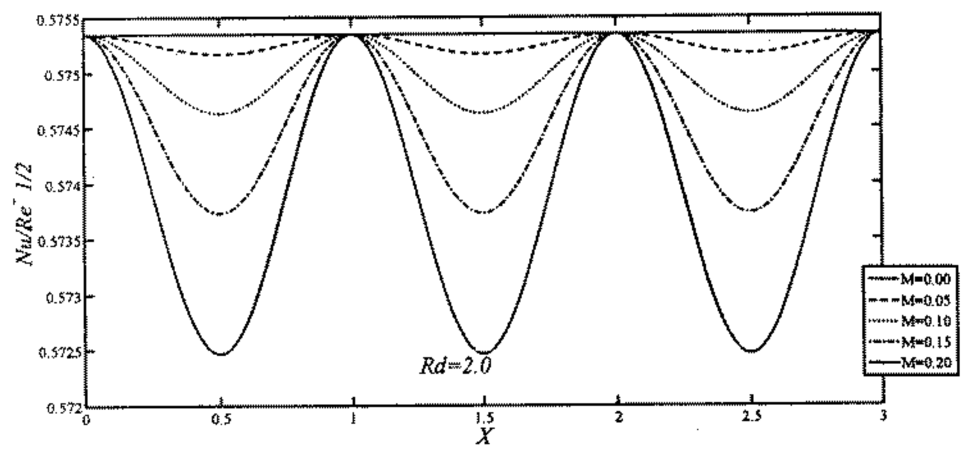

Fig. 3.6: Nusselt number graph at $R_d = 2.0$ and different values of M.

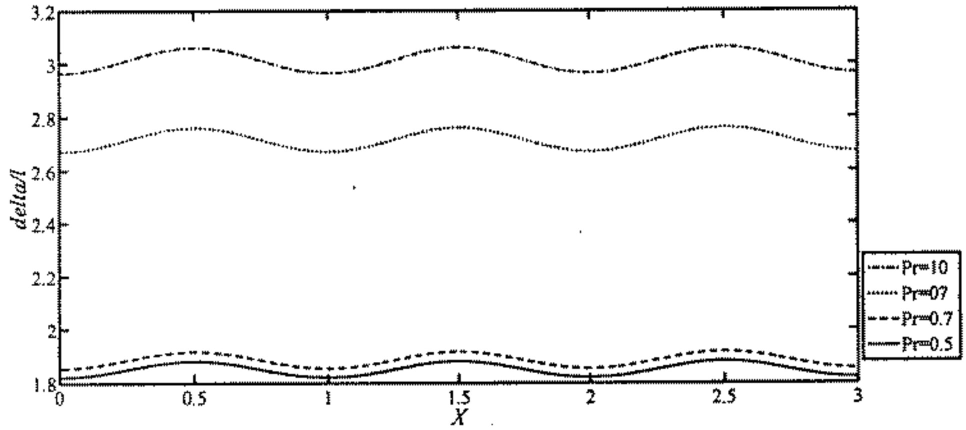

Fig. 3.7: Boundary layer thickness for $Rd = 0.5$,$M = 0.5$ and different values of Pr.

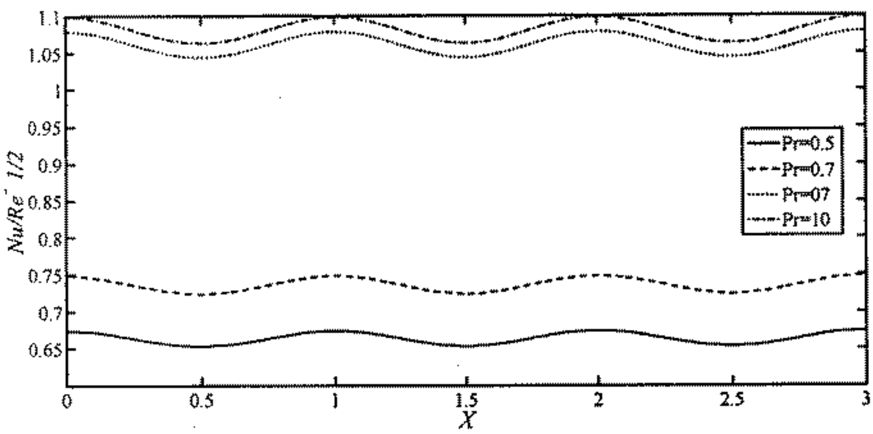

Fig. 3.8: Nusselt number for $R_d = 0.5$, $M = 0.5$ and different values of Pr.

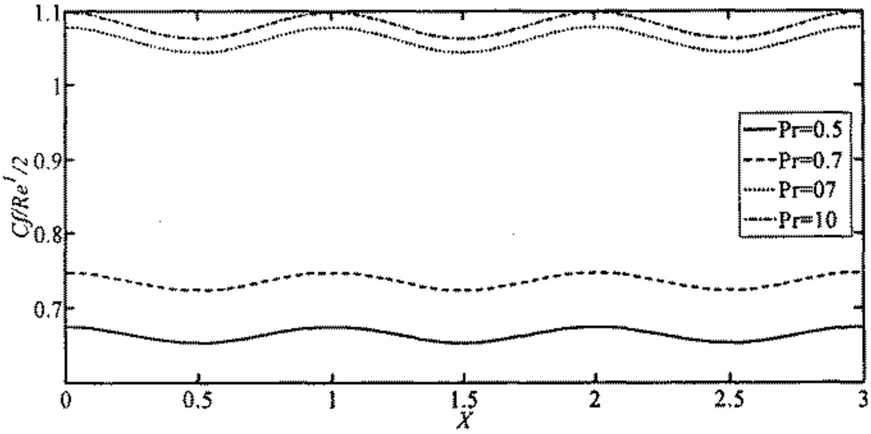

Fig. 3.9: Skin friction for $R_d = 0.5$, $M = 0.5$ and different values of Pr.

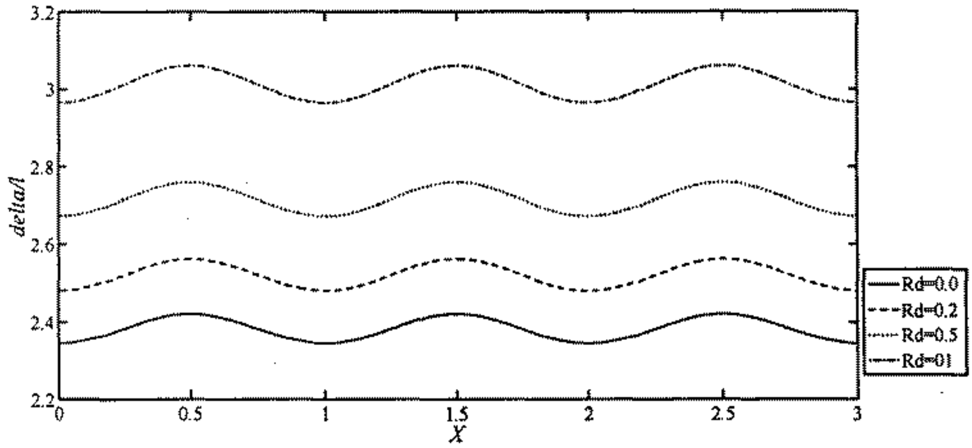

Fig. 3.10: Boundary layer thickness for $Pr = 0.7, M = 0.5$ and different values of R_d.

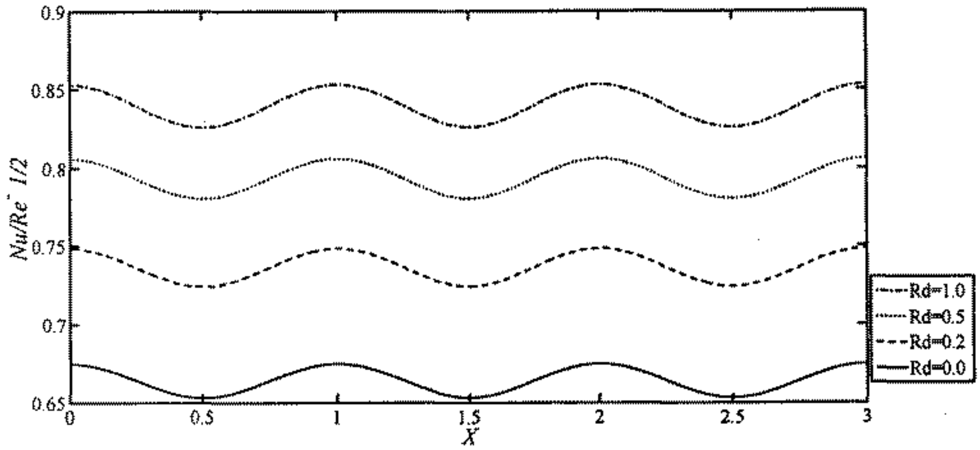

Fig. 3.11: Nusselt number for $Pr = 0.7$, $M = 0.5$ and different values of R_d.

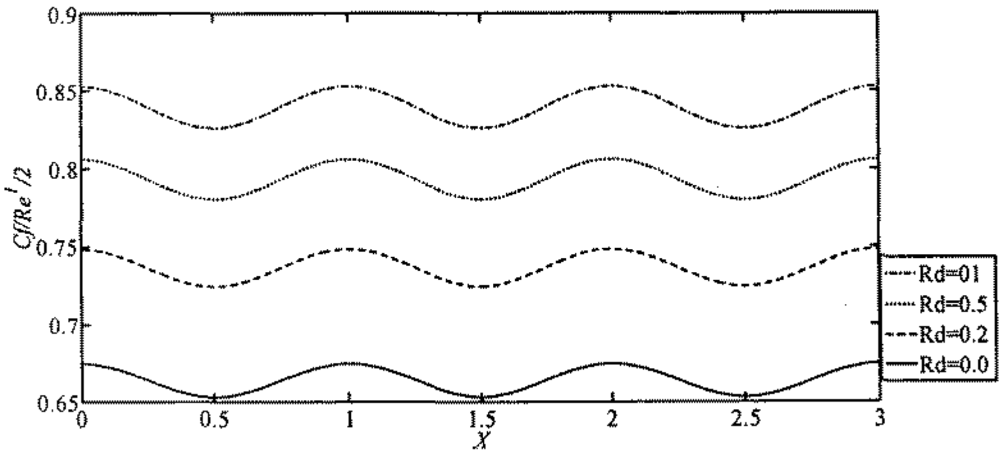

Fig. 3.12: Skin friction for $Pr = 0.7$, $M = 0.5$ and different values of R_d.

3.3.1 Conclusion

In this chapter, we have studied the non-similar magnetohydrodynamics flow with thermal radiation effects near a vertical wall. The integral method has been utilized to solve the governing non-linear system. The obtained solution is valid for all values of the involved parameter. At $M = 0.0$ the self similar solution is recovered. The effects of magnetic field is quite significant upon the skin friction and the Nusselt number. The radiation effects are noted to be enhance the rate of heat exchange.

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