Bipolar valued fuzzy h-ideals in hemirings

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INTERNATIONAL ISLAMIC UNIVERSITY
ISLAMABAD, PAKISTAN

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A THESIS SUBMITTED IN THE PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
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SUPERVISED BY
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ISLAMABAD, PAKISTAN

2015

DEDICATION

This work is dedicated
To
My Beloved Parents, Brothers and
Valued Teacher

بسم الله الرحمن الرحيم

Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MS IN STATISTICS

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DECLARATION

I, hereby declare that this thesis neither as a whole nor as a part of thesis has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor.

No portion of the work presented in this thesis has been submitted in support of an application for any degree or qualification of this or any other Institute of learning.

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Preface

Semirings, which are regarded as generalizations of associative rings, were first introduced by Vandiver [44] in 1934. Semirings have been used for studying optimization, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, theory of automata, formal language theory, coding theory, analysis of computer programmes. Additively commutative semirings with zero element are called hemirings. Hemirings, appears in a natural manner, in some applications to the theory of automata, the theory of formal languages and in computer sciences [5,6,10,11,12].

Ideals play an important role in the study of semirings and are very useful for many purposes. But they don't coincide with ring ideals. Thus many results of ring theory have no analogues in semirings using only ideals. In order to overcome this deficiency, Henriksen [13] defined a class of ideals in semirings, called k-ideals. These ideals have the property that if a semiring R is a ring then a subset of R is a k-ideal if and only if it is a ring ideal. A more restricted class of ideals in semiring is defined by Iizuka [17], called h-ideals. La Torre [23] thoroughly studied h-ideals and k-ideals and established some analogues ring results for hemirings.

The concept of fuzzy subset introduced by Zadeh [50], is a useful tool to describe situation in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. The concept of fuzziness is widely used in the theory of automata, studying matrices, determinants, set theory, group theory, optimization theory, measure theory, coding theory and topology [2,22,34,35,39,45,46,52]. Rosenfeld [40] initiated and defined fuzzy subgroups. In [1] J. Ahsan initiated the study of fuzzy semirings (see also [2]). Many researchers worked on fuzzy ideals of semirings, for example [41,42]. Fuzzy h-ideals in hemirings are studied in [8, 9, 20, 47,48, 51].

On the other hand Biswas [7] introduced the concept of anti fuzzy subgroup of a group. Hong and Jun [14] modified Biswasis idea and applied it to BCK-algebras. They defined anti fuzzy ideals of a BCK-algebra. In [4], Akram and Dar defined anti fuzzy h-ideals in hemirings.

Soft set theory is a generality of fuzzy set theory, that was determined by Molodtsov [31], to deal with precariousness in a non-parametric style. Furthermore, soft sets have been applied in several fields [32, 33, 36]. In 2001 Maji, et. al. [29], proposed the concept of the fuzzy soft sets. In 2002 Maji, et. al. [28], applied soft set theory in decision making. In recent years many researchers applied soft set in several fields and notably in decision making.

The membership degree for a fuzzy set expresses the degree of belongingness of elements to a fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [25]). Keeping in view the satisfaction degree, the membership degree 0 is assigned to those elements which do not satisfy some property. In the usual fuzzy set representation the elements with membership degree 0 are usually regarded as having the same characteristic. However it is interesting to note that among such elements some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. Consider a fuzzy set "young" defined on the age domain [0, 100]. Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "young", we may say that age 95 is more apart from the property rather than age 50 (see [24,26]). In such cases the usual fuzzy set does not help to differentiate between irrelevant elements and contrary elements. Hence if a set representation could express this kind of difference, then it would be more informative and helpful than the usual fuzzy set.

Jun et al. [19] introduced the notion of bipolar fuzzy subalgebra and bipolar fuzzy ideal in BCH-algebras. In [18] Y. B. Jun, C. H. Park, introduced Filters of BCH-Algebras Based on Bipolar Valued Fuzzy Sets. In [3] Akram et. all introduced bipolar fuzzy K-algebras by using bipolar valued fuzzy sets. In [26] K. J. Lee, used the notion of bipolar-valued fuzzy sets and worked on bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI- algebras. In [27] T. Mahmood et al. discuss bipolar fuzzy subgroup. In [30] Min Zhou and Shenggang introduced applications of bipolar fuzzy theory to hemirings. In [38] homomorphism and anti homomorphism on a bipolar anti fuzzy subgroup are introduced. Nagarajan et.al. [37], presented a socialistic decision making approach for bipolar fuzzy soft h-ideals over hemirings.

In 1981 a multi-criteria decision analysis method known as "technique for order preference by similarity to ideal solution (TOPSIS)" was established by Hwang and Yoon [15]. The main concept of TOPSIS technique is that the chosen alternative should have the shortest distance from the positive ideal solution and the longest distance from the negative ideal solution [16,49]. In [43], Comparative analysis of SAW and TOPSIS based on interval-valued fuzzy sets, discussed by Ting-Yu.

Chapter 1

Preliminaries

In this chapter, we recall some basic definitions and notions. These definitions will help us in later chapters. For undefine terms and notions, we reffer to [2, 11, 12, 13, 17, 41, 47, 48, 50, 51, 52].

1.1 Hemirings

In this section, we review some defnitions and notations regarding hemirings.

1.1.1 Definition

A set $R \neq \phi$ together with two binary operations addition "$+$" and multiplication "$\cdot$" is called semiring if $(R, +)$ and $(R, \cdot)$ are semigroups and multiplication distributes from both sides over addition. An element $0 \in R$ satisfying the condition, $0.x = x.0 = 0$ and $0 + x = x + 0 = x$ for all $x \in R$, is called zero of the semiring $(R, +, \cdot)$. An elememt $1 \in R$ satisfying the condition, $1.x = x.1 = x$, for all $x \in R$, is called identity of R.

1.1.2 Definition.

A semiring with commutative multiplication is called commutative semiring.

1.1.3 Definition

A semiring with zero element and commutative addition is called hemiring.

1.1.4 Examples

1. 1. All rings are hemirings.

2. The set of non-negative rational numbers are commutative hemirings under usual addition and multiplication.

3. Let $\mathbb{R}^+$ be the set of all positive real numbers. Then $\mathbb{R}^+$ is a commutative hemiring with identity under the binary operations of ordinary addition and multiplication of numbers.

4. Unit interval $[0,1]$ of real numbers is a semiring with $+ = max$ and $. = min$.

5. Let $B = \{0,1\}$, Define "+" and "." on B as follows:

+	0	1
0	0	1
1	1	1

,

.	0	1
0	0	0
1	0	1

.

Then $(B,+,.)$ is a semiring called Boolean semiring.

6. The set $R = \{0,x,1\}$ with the following binary operations

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

is a commutative hemiring.

1.1.5 Definition

A non-empty subset A of a hemiring R is called subhemiring of R if $a+b \in A$, $ab \in A$, for all a, $b \in R$ and $0 \in A$.

1.1.6 Examples

1. All rings are hemirings with subrings as subhemirings. The set of non-negative rational numbers are commutative hemirings under usual addition and multiplication. The set of whole numbers is a subhemiring of the set of non-negative rational numbers.

2. Let $\mathbb{R}^+$ be the set of all positive real numbers. Then $\mathbb{R}^+$ is a commutative hemiring with identity under the binary operations of ordinary addition and multiplication of numbers. Then $\mathbb{N}_0$ the set of non-negative integers is a subhemiring of $\mathbb{R}^+$ with identity under the binary operations of ordinary addition and multiplication of numbers.

3. Let $(S,\cdot)$ be a semigroup and $P(S)$ the power set of S. Then for $A, B \in P(S)$, $A \cup B$ may be considered as an addition on $P(S)$ and $A{\cdot}B = \{a{\cdot}b : a \in A \text{ and } b \in B\}$, as a multiplication on $P(S)/\emptyset$, where $\emptyset$ denotes the empty set. Then it is easy to check

that $(P(S), \cup, \cdot)$ is a semiring. This semiring has an identity E if and only if $(S, \cdot)$ has an identity e, namely, $E = \{e\}$. If one applies the rule $A \cdot B = \{a \cdot b : a \in A, b \in B\}$ also to empty sets, one obtains $A.B = \emptyset$ for $A = \emptyset$ or $B = \emptyset$. Then the system $(P(S), \cup, \cdot)$ is a semiring with $\emptyset$ as an absorbing zero. The finite subsets of S form a subsemiring of $(P(S), \cup, \cdot)$.

1.2 Ideals in hemirings

Ideals play a vital role in the theory of rings and it is therefore natural to study them also in the theory of hemirings. In this section, we defined ideals in hemirngs.

1.2.1 Definition

A non-empty subset I of a hemiring R is called left (resp., right) ideal of R if I is closed under addition and $RI \subsetneq I\,(IR \subseteq I)$. Furthermore, I is called an ideal of R if it is both left and right ideal of R and $I \neq R$.

1.2.2 Definition

A non-empty subset I of R is called interior ideal of R if it is closed under addition and $rar \in R$, for all $a \in I$, $r \in R$.

1.2.3 Definition

An ideal P of a commutative hemiring R with unity is called prime ideal if $ab \in P \Longrightarrow a \in P$ or $b \in P$, for all a, $b \in R$.

1.2.4 Definition

An ideal S of a commutative hemiring R with unity is called semiprime ideal if $a^2 \in S \Longrightarrow a \in S$, for all $a \in R$.

1.2.5 Definition

A non-empty subset I of a hemiring R is called bi-ideal of R if I is closed under addition and satisfying $IRI \subseteq I$.

1.2.6 Definition

A non-empty subset I of a hemiring R is called quasi-ideal of R if I is closed under addition and $RI \cap IR \subseteq I$.

1.3 h-ideals of hemirings

In this section, we review some defnitions regarding h-ideals.

1.3.1 Definition

A subhemiring (resp., left ideal, right ideal, interior ideal, prime ideal, semiprime ideal, bi-ideal) I of a hemiring R is called a h-subhemiring (resp., left h-ideal, right h-ideal, interior h-ideal, prime h-ideal, semiprime h-ideal, h-bi-ideal) if for all $x, z \in R$ and for any $a, b \in I$, from $x + a + z = b + z$ it follows $x \in I$.

It is not necessary that every left (right) ideal of R is a left (right) h-ideal of R.

1.3.2 Example

Let $R = \{0, a, b\}$ be a hemiring with addition "+" and multiplication ". " defined by the following table:

+	0	a	b
0	0	a	b
a	a	0	b
b	b	b	0

,

.	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

.

Then $I = \{0, b\}$ is an ideal of R but it is not an h-ideal of R, since $a+0+b = 0+b$ because $a \notin I$.

1.3.3 Lemma

The intersection of any number of left (right) h-ideals of a hemiring R is a left (right) h-ideal of R.

1.3.4 Definition[47]

The h-closure $\overline{A}$ of a non-empty subset A of hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a + z = b + z \text{ for some } a,\ b \in A,\ z \in R\}$$

1.3.5 Definition[47]

A quasi-ideal I of a hemiring R is called an h-quasi-ideal if $\overline{RI} \cap \overline{RI} \subseteq I$ and for all x, $z \in R$ and for any a, $b \in I$, from $x + a + z = b + z$ it follows $x \in I$.

1.3.6 Remark

Every left (right) h-ideal of R is an h-quasi-ideal of R and every h-quasi-ideal of R is an h-bi-ideal of R but the converse is not true.

1.3.7 Example[47]

The set R of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a hemiring with usual addition and multiplication of matrices, where $a_{ij} \in N_0, N_0$ is the set of all non-negative integers. Consider the set Q of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ $(a \in N_0)$. Evidently Q is an h-quasi-ideal of R but not a left (right) h-ideal of R.

1.3.8 Lemma[47]

Let R be a hemiring. Then for any left (right) h-ideal, h-bi-ideal and h-quasi-ideal, we have $A = \overline{A}$.

1.3.9 Definition[48]

A subset "A" of a hemiring R, is called h-idempotant if $A = \overline{A^2}$.

1.4 h-hemiregular and h-hemisimple hemirings

In this section, we review h-hemiregular hemirings and h-hemisimple hemirings, see[41, 47, 48].

1.4.1 Definition[48]

A hemiring R, is called h-semisimple if every h-ideal of R is h-idempotant.

1.4.2 Lemma[48]

A hemiring R, is h-semisimple if and only if one of the following holds:

(i) For all $x \in R$, there exist c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, $f'_j \in R$ such that $x + \Sigma_{i=1}^{m} c_i x d_i e_i x f_i + z = \Sigma_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z$.

(ii) For all $x \in R$, $x \in \overline{RxRxR}$.

(iii) For all $A \subseteq R$, $A \subseteq \overline{RARAR}$.

1.4.3 Example

Let Q_0 denotes the set of all non-negative rational numbers. Then $(Q_0, +, .)$is an h-hemisimple hemiring.

1.4.4 Definition[47]

A hemiring R, is called h-hemiregular if for each $x \in R$, there exist a, b, $z \in R$, such that $x + xax + z = xbx + z$.

1.4.5 Example

Let $R = \{0, a, b\}$ be a hemiring with addition "+" and multiplication ". " defined by the following table:

+	0	a	b
0	0	0	0
a	a	a	b
b	b	b	b

,

.	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then R is h-hemiregular hemiring.

1.4.6 Lemma[47]

A hemiring R, is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R, we have $\overline{IL} = I \cap L$.

1.4.7 Lemma[47]

The following conditions for a hemiring R are equivalent:

(*i*) R is h-hemiregular.

(*ii*) $\overline{MRM} = M$ for every h-bi-ideal M of R.

(*iii*) $\overline{LRL} = L$ for every h-quasi-ideal L of R.

1.5 Fuzzy sets

The theory of fuzzy sets was popularized by L. A. Zadeh in [50], as a generalization of the conceptual set theory. In this section, we will give a review of some basic concepts of fuzzy sets.

1.5.1 Definition

Let X be a non-empty subset. Then for any $A \subseteq X$ the characteristic function of A is denoted by C_A defined by $C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

for $x \in X$.

1.5.2 Definition

Let X be a nonempty set. A fuzzy subset λ of the set X is a fuction $\lambda : X \longrightarrow [0, 1]$.

A fuzzy subset $\lambda : X \longrightarrow [0, 1]$ is non-empty if λ is not a constant function forever taking the value zero.

1.5.3 Definition

A fuzzy subset of X of the form

$$\lambda(z) = \begin{cases} t & \text{if } z = x \\ 0 & \text{if } z \neq x \end{cases},$$

is called the fuzzy point with support x and value t, where $t \in (0, 1]$. It is usually denoted by x_t.

1.5.4 Remarks

1. Two fuzzy subsets λ and μ of a set X are said to be disjoint if there is no $x \in X$ such that $\lambda(x) = \mu(x)$. If $\lambda(x) = \mu(x)$ for each $x \in X$, then we say that λ and μ are equal and write $\lambda = \mu$.

2. Let λ and μ be two fuzzy subsets of non-empty set X. Then λ is said to be included in μ *i.e.*, $\lambda \subseteq \mu$ if and only if $\lambda(x) \leq \mu(x)$ for all $x \in X$.

3. Let λ and μ be two fuzzy subsets of non-empty set X. Then λ is said to be properly included in μ *i.e.*, $\lambda \subset \mu$ if and only if $\lambda(x) < \mu(x)$ for all $x \in X$.

4. The union of any family $\{\lambda_i : i \in \Omega\}$ of fuzzy subsets λ_i of a non-empty set X is denoted by $(\bigcup_{i\in\Omega} \lambda_i)$ and defined by $(\bigcup_{i\in\Omega} \lambda_i)(x) = \sup_{i\in\Omega} \lambda_i(x) = \bigvee_{i\in\Omega} \lambda_i(x)$, for all $x \in X$. Moreover $(\bigcup_{i\in\Omega} \lambda_i)$ is smallest fuzzy subset which containing λ_i.

5. The intersection of any family $\{\lambda_i : i \in \Omega\}$ of fuzzy subsets λ_i of a non-empty set X is denoted by $(\bigcap_{i\in\Omega} \lambda_i)$ and defined by $(\bigcap_{i\in\Omega} \lambda_i)(x) = \inf_{i\in\Omega} \lambda_i(x) = \bigwedge_{i\in\Omega} \lambda_i(x)$, for all $x \in X$. Moreover $(\bigcap_{i\in\Omega} \lambda_i)$ is largest fuzzy subset which is contained in λ_i.

1.6 Fuzzy hemirings

The concept of fuzzy set has been applied by many authers to generalize some of the basic notions of algebra. In this section, we will give a review of some basic concepts of fuzzy ideals of hemirings.

1.6.1 Definition

Let λ and μ be two fuzzy subsets of a hemiring R. Then product of λ and μ is defined as $(\lambda\mu)(x) = \sup_{x=x_1+x_2} \{\lambda(x_1) \wedge \mu(x_2)\}$ for all $x \in R$.

1.6.2 Definition[47]

Let λ and μ be two fuzzy subsets of hemiring R. Then the h-intrinsic product of λ and μ is defined as

$$(\lambda \odot_h \mu)(x) = \left\{ \begin{array}{l} \sup\limits_{x+\sum_{j=1}^{m} x_j y_j + z = \sum_{i=1}^{n} x'_i y'_i + z} (\min\{\lambda(x_j), \mu(y_j), \lambda(x_i), \mu(y_i) \mid j = 1,2,..m; i = 1,2,..n\}) \\ 0 \quad \text{if } x \text{ cannot be expressed as } x + \sum_{j=1}^{m} x_j y_j + z = \sum_{i=1}^{n} x'_i y'_i + z \end{array} \right\}$$

for all $x \in R$.

1.6.3 Proposition[47]

Let R be a hemiring and λ, μ, ν and ξ be any fuzzy subsets of R. If $\lambda \leq \mu$ and $\nu \leq \xi$ then $\lambda \odot_h \nu \leq \mu \odot_h \xi$.

1.6.4 Lemma[47]

Let R be a hemiring and A, $B \subseteq R$. Then we have

(i) $A \subseteq B$ if and only if $C_A \leq C_B$.

(ii) $C_A \wedge C_B = C_{A \cap B}$.

(iii) $C_A \odot_h C_B = C_{\overline{AB}}$.

1.6.5 Definition.

A fuzzy subset λ of a hemiring R is called fuzzy h-subhemiring of R if for all x, $y \in R$

(*i*) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$,

(*ii*) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$.

(iii) $x + a + z = b + z \longrightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$ for all $x, z, a, b \in R$.

1.6.6 Definition[20]

A fuzzy subset λ of a hemiring R is called fuzzy left (resp., right) h-ideal of R if for all $x, y \in R$

(i) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$

(ii) $\lambda(xy) \geq \lambda(y)$ (resp., $\lambda(xy) \geq \lambda(x)$)

(iii) $x + a + z = b + z \longrightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$ for all $x, z, a, b \in R$.

A fuzzy subset "λ" of a hemiring R is called a fuzzy h-ideal of R if it is both fuzzy left and right h-ideal of R.

1.6.7 Remark

If λ is a fuzzy left (right) h-ideal of a hemiring R, then $\lambda(0) \geq \lambda(x)$ for all $x \in R$.

1.6.8 Example

Let λ be a fuzzy subset of the hemiring N_0 defined by

$$\lambda(n) = \begin{cases} 1 \text{ if } n \text{ is even,} \\ 0.2 \ \text{ otherwise,} \end{cases}$$

Then λ is a fuzzy h-ideal of the hemiring N_0.

1.6.9 Proposition

Let λ and μ be two fuzzy left(resp., right) ideal of a hemiring R. Then $\lambda \cap \mu$ and $\lambda\mu$ is fuzzy left (resp., right) h-ideals of R.

1.6.10 Definition

A fuzzy h-ideal of a commutative hemiring R with unity is called fuzzy prime h-ideal if $\lambda(x) \vee \lambda(y) \geq \lambda(xy)$.

1.6.11 Definition

A fuzzy h-ideal of a commutative hemiring R with unity is called fuzzy prime h-ideal if $\lambda(x) \geq \lambda(x^2)$.

1.6.12 Example

In Example 1.6.8, λ is a fuzzy prime h-ideal of R.

1.6.13 Definition[47]

A fuzzy subset λ of a hemiring a R is called fuzzy h-bi-ideal of R if for all $x, y \in R$

(*i*) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$,

(*ii*) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$,

(*iii*) $\lambda(xzy) \geq \lambda(x) \wedge \lambda(y)$,

(*iv*) $x + a + z = b + z \longrightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$ for all $x, z, a, b \in R$.

1.6.14 Definition[47]

A fuzzy subset λ of a hemiring a R is called fuzzy h-quasi-ideal of R if for all $x, y \in R$

(*i*) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$,

(*ii*) $(\lambda \odot_h C_R) \wedge (C_R \odot_h \lambda) \leq \lambda$,

(iii) $x + a + z = b + z \longrightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$ for all x, z, a, $b \in R$.

Note that if λ is a fuzzy left h-ideal (right h-ideal, h-bi-ideal, h-quasi-ideal), then $\lambda(0) \geq \lambda(x)$ for all $x \in R$.

1.6.15 Example

Consider the hemiring N_0:

Let $r, s \in [0, 1]$ be such that $r \leq s$. Define a fuzzy subset λ of N_0 by

$$\lambda(x) = \begin{cases} s & if\ x \in \langle 3 \rangle, \\ r & otherwise \end{cases}$$

for all $x \in N_0$. Then λ is both a fuzzy h-bi-ideal and a fuzzy h-quasi-ideal of N_0.

1.6.16 Definition[4]

For any fuzzy set μ in R and any $\alpha \in [0, 1]$ we define the set $L(\mu; \alpha) = \{x \in X | \mu(x) \leq \alpha\}$, which is called lower level cut of μ.

1.6.17 Definition

A fuzzy subset λ of a hemiring a R is called anti fuzzy h-subhemiring of R if for all x, $y \in R$

(i) $\lambda(x + y) \leq \lambda(x) \vee \lambda(y)$,

(ii) $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$.

(iii) $x + a + z = b + z \longrightarrow \lambda(x) \leq \lambda(a) \vee \lambda(b)$ for all x, z, a, $b \in R$.

1.6.18 Definition[4]

A fuzzy subset λ of a hemiring a R is called anti fuzzy left (resp., right) h-ideal of R if for all x, $y \in R$

(i) $\lambda(x + y) \leq \lambda(x) \vee \lambda(y)$

(ii) $\lambda(xy) \leq \lambda(y)$ (resp., $\lambda(xy) \leq \lambda(x)$)

(iii) $x + a + z = b + z \longrightarrow \lambda(x) \leq \lambda(a) \vee \lambda(b)$ for all x, z, a, $b \in R$.

A fuzzy subset "λ" of a hemiring R is called a anti fuzzy h-ideal of R if it is both anti fuzzy left and right h-ideal of R.

1.6.19 Example

Consider the hemiring N_0:

Let $r, s \in [0; 1]$ be such that $r \geq s$. Define a fuzzy subset λ of N_0 by

$$\lambda(x) = \begin{cases} s & \text{if } x \in \langle 3 \rangle, \\ r & \text{if otherwise,} \end{cases}$$

for all $x \in N_0$. Then λ is a anti fuzzy λ-ideal of R.

1.7 Fuzzy soft sets

Throughout this thesis, U refers to an initial universe, E is a set of parameters, $L, M \subseteq E$, and $P(U)$ is the set of all fuzzy sets of U.

1.7.1 Definition[29]

A pair (G, L) is called a fuzzy soft set over U, where $G : L \longrightarrow P(U)$ is a mapping from L into $P(U)$.

1.7.2 Definiton[29]

Let U be a universe and E a set of attributes. Then the pair (U, E) denotes the collection of all fuzzy soft sets on U with attributes from E and is called a fuzzy soft class.

1.7.3 Definition[29]

For two fuzzy soft sets (G, L) and (G, L) in a fuzzy soft class (U, E), we say that (G, L) is a fuzzy soft subset of (H, M), if

(i) $L \subseteq M$,

(ii) For all $e \in G$, $G(e) \leq H(e)$,

and is written as $(G, L) \subseteq (H, M)$.

1.7.4 Definition[29]

The complement of a fuzzy soft set (G, L) is denoted $(G, L)^c$ and is denoted by $(G, L)^c = (G^c, \mid L)$ where $G^c :\mid L \longrightarrow \widetilde{P}(U)$ is mapping given by $G^c(\grave{u}) = (G(\grave{u}))^c$, for all $\grave{u} \in L$.

Union of two fuzzy soft sets is defined by Maji et al. [29] as follows.

1.7.5 Definition[29]

Union of two fuzzy soft sets (G, L) and (H, M) in a soft class (U, E) is a fuzzy soft set (Γ, N), where $N = L \cup M$, and for all $e \in N$,

$$\Gamma(e) = \begin{cases} G(e) & \text{if } e \in L - M \\ H(e) & \text{if } e \in M - L \\ G(e) \cup H(e) & \text{if } x \in M \cap L \end{cases}$$

and it is written as $(G, L) \tilde{\vee} (H, M) = (\Gamma, N)$.

1.8 Bipolar-valued fuzzy sets

In this section, we will give a review of bipolar-valued fuzzy sets and some basic definitions about bipolar-valued fuzzy sets.

1.8.1 Definition[25]

Let X be a universe. Then a bipolar-valued fuzzy subset B of X is an object having the form

$$B = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in X\}$$

where $\mu^+ : X \longrightarrow [0, 1]$ and $\mu^- : X \longrightarrow [-1, 0]$.

The postive membership degree function $\mu^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to the bipolar-valued fuzzy subset $B = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in X\}$ and the negative membership degree function $\mu^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to the bipolar-valued fuzzy subset $B = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in X\}$.

If $\mu^{+}(x) \neq 0$, $\mu^{-}(x) = 0$, then it is the situation that x is regarded as having only positive satisfaction for the bipolar-valued fuzzy subset $B = \{\langle x,\ \mu^{+}(x),\ \mu^{-}(x)\rangle : x \in X\}$. If $\mu^{+}(x) = 0$, $\mu^{-}(x) \neq 0$, then it is the situation that x does not satisfy property of bipolar-valued fuzzy subset $B = \{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle : x \in X\}$, but some what satisfies the counter-property of $B = \{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle : x \in X\}$. It is possible for an element x to be $\mu^{+}(x) \neq 0$, $\mu^{-}(x) \neq 0$, when the membership function of the property overlaps that of its counter-property over some portion of X [24]. From now to onward for the sake of simplicity, we shall use the symbols B or $B = (\mu^{+}, \mu^{-})$, for the BVF subset $B = \{\langle x, \mu^{+}(x), \mu^{-}(x)\rangle : x \in X\}$. The set of all bipolar-valued fuzzy subsets of R is denoted by $F(B)$. We use BVF set in place of bipolar-valued fuzzy set in rest of chapters.

1.8.2 Definition[27]

Let X be a universe and $A \subseteq X$. Then BVF characteristic function is given by $C_A = (C_A^{+}, C_A^{-})$, where

$$C_A^{+}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases},$$

$$C_A^{-}(x) = \begin{cases} -1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

1.8.3 Definition

Let x be an element of a non-empty set X and $t \in (0,1]$. Then a BVF subset $B = (\lambda^{+}, \lambda^{-})$ of X of the form

$$\lambda^{+}(z)=\begin{Bmatrix} t^{+} & if & z=x \\ 0 & if & z\neq x \end{Bmatrix},$$
$$\lambda^{-}(z)=\begin{Bmatrix} t^{-} & if & z=x \\ 0 & if & z\neq x \end{Bmatrix},$$

is called BVF point with value $t'=(t^{+},t^{-})$ where $t'\in(0,1]\times[-1,0)$ and support x or BVF singleton subset of X. It is denoted by $x_{t'}=(x_{t}^{+},x_{-t}^{-})$. A BVF point $x_{t'}$ is said to belong to BVF subset B, written as $x_{t'}\in B$ if $B(x)\geq t'$ *i.e.*, $\lambda^{+}(x)\geq t^{+}$ and $\lambda^{-}(x)\leq t^{-}$. A BVF point $x_{t'}$ is said to not belong to BVF subset B, written as $x_{t'}\overline{\in}B$ if $B(x)\leq t'$ *i.e.*, $\lambda^{+}(x)\leq t^{+}$ and $\lambda^{-}(x)\geq t^{-}$.

1.8.4 Definition

Let $B_1=(\lambda^{+},\lambda^{-})$, $B_2=(\mu^{+},\mu^{-})$ be two BVF sets of R. Then we write $B_1\leq B_2$ if $\lambda^{+}(x)\leq\mu^{+}(x)$ and $\lambda^{-}(x)\geq\mu^{-}(x)$ for all $x\in R$.

1.8.5 Definition

Let R be a hemiring and $B_1=(\lambda^{+},\lambda^{-})$ and $B_2=(\mu^{+},\mu^{-})$ be two BVF subsets of R. Then we have $B_1\wedge B_2=(\lambda^{+}\wedge\mu^{+},\lambda^{-}\vee\mu^{-})$ and $B_1\vee B_2=(\lambda^{+}\vee\mu^{+},\lambda^{-}\wedge\mu^{-})$.

1.8.6 Definition

A BVF subset $B=(\lambda^{+},\lambda^{-})$ of a hemiring R is called BVF subhemiring of R if it satisfies

$(E1)$ $\lambda^{+}(x+y)\geq\lambda^{+}(x)\wedge\lambda^{+}(y)$,

$(E2)$ $\lambda^{-}(x+y)\leq\lambda^{-}(x)\vee\lambda^{-}(y)$,

$(E3)$ $\lambda^{+}(xy)\geq\lambda^{+}(x)\wedge\lambda^{+}(y)$,

$(E4)$ $\lambda^{-}(xy) \leq \lambda^{-}(x) \vee \lambda^{-}(y)$,

for all $x, y \in R$.

1.8.7 Definition

A BVF subset $B = (\lambda^{+}, \lambda^{-})$ of a hemiring R is called BVF left(resp., right) ideal of R if it satisfies $(E1)$, $(E2)$ and

$(E5)$ $\lambda^{+}(xy) \geq \lambda^{+}(y)$ (resp., $\lambda^{+}(xy) \geq \lambda^{+}(x)$),

$(E6)$ $\lambda^{-}(xy) \leq \lambda^{-}(y)$ (resp., $\lambda^{-}(xy) \leq \lambda^{-}(x)$).

1.8.8 Definition

Let $B = (\mu^{+}, \mu^{-})$ be BVF subset of a hemiring R and $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, then

(1) The set $B_{\beta}^{+} = \{x \in R : \mu^{+}(x) \geq \beta\}$ is called positive β-cut of B .

(2) The set $B_{\alpha}^{-} = \{x \in R : \mu^{-}(x) \leq \alpha\}$ is called negative α-cut of B.

(3)The set $B_{(\alpha,\beta)} = \{x \in R : \mu^{-}(x) \leq \alpha \text{ and } \mu^{+}(x) \geq \beta\}$ is called (α, β)-cut of B.

1.8.9 Remark

(3) $B_{\gamma}^{+} \cap B_{-\gamma}^{-}$ is called γ-cut of B.

(4) If $\gamma \neq (0,0)$ then $B_{\gamma}^{+} \cap B_{-\gamma}^{-}$ is called BVF point of B.

1.9 Bipolar-valued fuzzy soft set

In this section, we define some basic definition about BVF soft set.

1.9.1 Definition[37]

Let U be an initial universe, E be the set of parameters, L is subset of E . Define $G : L \rightarrow BVFU$, where BVFU is the collection of all BVF subsets of U. Then (G, L) is said to be a BVF soft set over a universe U. It is defined by $(G, L) = \{(x, \lambda_e^+(x), \lambda_e^-(x)$: for all $x \in U$ and $e \in L\}$.

1.9.2 Example[37]

Let $U = \{c_1, c_2, c_3, c_4\}$ be the set of four houses under consideration and $E = \{e_1 = cheap, e_2 = beautiful, e_3 = good\ location, e_4 = modern\ \}$ be the set of parameters and $L = \{e_1, e_2, e_3\}$ is subset of E. Then

$$\left\{\begin{array}{l} (G,L) = F(e_1) = \left\{\begin{array}{c} (c_1,\ 0.2,\ -0.4),\ (c_2,\ 0.4,\ -0.5), \\ (c_3,\ 0.1,\ -0.4),\ (c_4,\ 0.4,\ -0.7) \end{array}\right\} \\ (G,L) = F(e_2) = \left\{\begin{array}{c} (c_1,\ 0.5,\ -0.7),\ (c_2,\ 0.3,\ -0.1), \\ (c_3,\ 0.9,\ -0.3),\ (c_4,\ 0.7,\ -0.8) \end{array}\right\} \\ (G,L) = F(e_3) = \left\{\begin{array}{c} (c_1,\ 0.4,\ -0.3),\ (c_2,\ 0.4,\ -0.6), \\ (c_3,\ 0.4,\ -0.7),\ (c_4,\ 0.8,\ -0.2) \end{array}\right\} \end{array}\right\}.$$

1.9.3 Definition[37]

Let U be a universe and E a set of attributes. Then, (U, E) is the collection of all BVF soft sets on U with attributes from E and is said to be BVF soft class.

1.9.4 Definition[37]

A bipolar fuzzy soft set (G, L) is said to be a null BVF soft set denoted by empty set Φ, if for all $e \in L, G(e) = \Phi$.

1.9.5 Definition[37]

A bipolar fuzzy soft set (G, L) is said to be an absolute BVF soft set, if for all $e \in L$, $G(e) = BVFU$.

1.9.6 Definition

The complement of a BVF soft set (G, L) is denoted $(G, L)^c$ and is denoted by $(G, L)^c = \{(x, 1 - \lambda_L^+(x), -1 - \lambda_L^-(x) \ \forall \ x \in U\}$.

1.9.7 Remark

Throught the thesis, we will use BVFS sets in place of bipolar-valued fuzzy soft sets and we denote hemiring R by simply R.

Chapter 2

Bipolar-valued fuzzy h-ideals of hemirings

In this chapter, we define BVF h-subhemiring . In [30], M. Zhou and Shenggang popularized BVF h-ideals. We analysed some basic definitions of BVF h-ideals and some basic properties of BVF h-ideals.

2.1 Bipolar-valued fuzzy h-ideal of hemirings

2.1.1 Definition

Let $B = (\lambda^{+}, \lambda^{-})$ be a BVF subset of R. Then B is called BVF h-subhemiring of R if it satisfies

(1) $x_{t'} \in B,\ y_{r'} \in B \Longrightarrow (x+y)_{\min(t',r')} \in B$,

(2) $x_{t'} \in B,\ y_{r'} \in B \Longrightarrow (xy)_{\min(t',r')} \in B$,

(3) $\quad x + a_1 + z = a_2 + z,\ (a_1)_{t'} \in B,\ (a_2)_{r'} \in B \implies (x)_{\min(t',r')} \in B,$

$\forall\, x, y, z, a_1, a_2 \in R \ \&\ t' = (t^+, t^-)\,,\ r' = (r^+, r^-) \in (0,1] \times [-1, 0).$

2.1.2 Definition

Let $B = (\lambda^+, \lambda^-)$ be a BVF subset of R. Then B is called BVF left (resp. right) h-ideal of R if it satisfies (1), (3) and

(4) $\quad x_{t'} \in B \implies (yx)_{t'} \in B$ (resp. (5) $(xy)_{t'} \in B$) $\forall\, x,\, y \in R \ \&\ t' = (t^+, t^-) \in (0,1] \times [-1, 0).$

B is called a BVF h-ideal if it is both left and right BVF h-ideal of R.

2.1.3 Example

Consider hemiring $R = \{0, 1, p, p^*\}$ defined by

+	0	1	p	p^*
0	0	1	p	p^*
1	1	1	p	p^*
p	p	p	p	p^*
p^*	p^*	p^*	p^*	p^*

,

.	0	1	p	p^*
0	0	0	0	0
1	0	1	1	1
p	0	1	1	1
p^*	0	1	1	1

.

We define BVF set B as follows

	0	1	p	p^*
μ^+	0.52	0.52	0.32	0.32
μ^-	-0.73	-0.73	-0.23	-0.23

.

Clearly, B is a BVF h-ideal of R.

2.1.4 Definition

Let $B = (\lambda^+, \lambda^-)$ be a BVF subset of R. Then B is called BVF interior h-ideal of R if it satisfies (1), (2), (3) and

(6) $\quad y_{t'} \in B \Longrightarrow (xyz)_{t'} \in B \; \forall x, y, z \in R \;\&\; t' \in (0,1] \times [-1,0)$.

2.1.5 Definition

Let $B = (\lambda^+, \lambda^-)$ be a BVF subset of R. Then B is called BVF h-bi-ideal of R if it satisfies (1), (2), (3) and

(7) $\; x_{t'} \in B,\; y_{r'} \in B \Longrightarrow (xzy)_{\min(t',r')} \in B \; \forall x,\; y,\; z \in R \;\&\; t', r' \in (0,1] \times [-1,0)$.

2.1.6 Example

Consider hemiring $\mathbb{N}_0$ with respect to the usual "+" and ".". Let $t_1', t_2' \in [0,1)$ be such that $t_1' \le t_2'$ i.e, $(t_1, -t_1) \le (t_2, -t_2)$. Define BVF set $B = (\lambda^+, \lambda^-)$ by

$$\lambda^+(x) = \begin{cases} t_1 & \text{if } x \in \langle 3 \rangle \\ t_2 & \text{if } x \notin \langle 3 \rangle \end{cases},$$

and

$$\lambda^-(x) = \begin{cases} -t_1 & \text{if } x \in \langle 3 \rangle \\ -t_2 & \text{if } x \notin \langle 3 \rangle \end{cases},$$

$\forall x \in \mathbb{N}_0$. Then $B = (\lambda^+, \lambda^-)$ is BVF h-bi-ideal of $\mathbb{N}_0$.

2.1.7 Definition

Let $B = (\lambda^+, \lambda^-)$ be a BVF subset of a commutative hemirig R with unity. Then B is called BVF prime h-ideal of R if it satisfies (1), (3), (4), (5) and

(8) $(xy)_{t'} \in B \Longrightarrow (x)_{t'} \in B$ or $y_{t'} \in B$ $\forall x, y \in R$ & $t' \in (0,1] \times [-1,0)$.

2.1.8 Definition

Let $B = (\lambda^+, \lambda^-)$ be a BVF subset of a commutative hemiring R with unity. Then B is called BVF semiprime h-ideal of R if it satisfies $(1), (3), (4), (5)$ and

(9) $(x^2)_{t'} \in B \Longrightarrow x_{t'} \in B$ $\forall x \in R$ & $t' \in (0,1] \times [-1,0)$.

2.1.9 Remark

In rest of the thesis, we denote set of all BVF left h-ideals of R (resp. BVF right h-ideals of R, BVF h-ideals of R, BVF interior h-ideals of R, BVF h-bi-ideals of R, BVF prime h-ideals of R, BVF semiprime h-ideals of R) by $BVFLhI(R)$ (resp. $BVFRhI(R), BVFhI(R), BVFIhI(R), BVFhbI(R), BVFPhI(R), BVFShI(R)$).

2.1.10 Theorem

Let B be a BVF subset of R. Then (1) to (9) are equivelent to $(1)'$ to $(9)'$ respectively, $\forall x, y, z, r_1, r_2$ where:

$(1)'$ $\lambda^+(x+y) \geq \min\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(x+y) \leq \max\{\lambda^-(x), \lambda^-(y)\}$.

$(2)'$ $\lambda^+(xy) \geq \min\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xy) \leq \max\{\lambda^-(x), \lambda^-(y)\}$.

$(3)'$ $x + r_1 + z = r_2 + z \Longrightarrow \lambda^+(x) \geq \min\{\lambda^+(r_1), \lambda^+(r_2)\}$, $\lambda^-(x) \leq \max\{\lambda^-(r_1), \lambda^-(r_2)\}$.

$(4)'$ $\lambda^+(xy) \geq \lambda^+(y)$, $\lambda^-(xy) \leq \lambda^-(y)$.

$(5)'$ $\lambda^+(xy) \geq \lambda^+(x)$, $\lambda^-(xy) \leq \lambda^-(x)$.

$(6)'$ $\lambda^+(xyz) \geq \lambda^+(y)$, $\lambda^-(xyz) \leq \lambda^-(y)$.

$(7)'$ $\lambda^+(xzy) \geq \min\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xzy) \leq \max\{\lambda^-(x), \lambda^-(y)\}$.

$(8)'$ $\lambda^+(xy) \geq \max\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xy) \leq \min\{\lambda^-(x), \lambda^-(y)\}$.

$(9)'$ $\lambda^+(x^2) \geq \lambda^+(x)$, $\lambda^-(x^2) \leq \lambda^-(x)$.

Proof. First we prove (1) is equivallent to $(1)'$.

$(1) \Longrightarrow (1)'$. Suppose $(1)'$ fasle. Then $\forall x, y \in R$, so that $\lambda^+(x+y) < \min\{\lambda^+(x), \lambda^+(y)\}$ and $\lambda^-(x+y) > \max\{\lambda^-(x), \lambda^-(y)\}$. Then $\exists\ t' = (t^+, t^-) \in (0,1] \times [-1, 0)$ such that $\lambda^+(x+y) < t^+ < \min\{\lambda^+(x), \lambda^+(y)\}$ & $\lambda^-(x+y) > t^- > \max\{\lambda^-(x), \lambda^-(y)\}$. Here $\lambda^+(x) > t^+$, $\lambda^+(y) > t^+$ & $\lambda^-(x) < t^-$, $\lambda^-(y) < t^-$. This implies, $x_{t'} \in B$, $y_{t'} \in B$ but $(x+y)_{t'} \overline{\in} B$. Which is contradiction. Hence $(1)'$ holds.

$(1)' \Longrightarrow (1)$. Let x, $y \in R$ for all $t' = (t^+, t^-)$ and $r' = (r^+, r^-) \in (0,1] \times [-1, 0)$. Such that $x_{t'} \in B$, $y_{r'} \in B$. Here $\lambda^+(x) \geq t^+$, $\lambda^-(x) \leq t^-$ and $\lambda^+(y) \geq r^+$, $\lambda^-(y) \leq r^-$. Then by $(1)'$ $\lambda^+(x+y) \geq \min\{\lambda^+(x), \lambda^+(y)\}$ and $\lambda^-(x+y) \leq \max\{\lambda^-(x), \lambda^-(y)\}$. So that $\lambda^+(x+y) \geq \min\{\lambda^+(x), \lambda^+(y)\} \geq \min\{t^+, r^+\}$ and $\lambda^-(x+y) \leq \max\{\lambda^-(x), \lambda^-(y)\} \leq \max\{t^-, r^-\}$. So $(\lambda^+(x+y), \lambda^-(x+y)) \geq (\min\{t^+, r^+\}, \max\{t^-$ This implies $(x+y)_{\min\{t', r'\}} \in B$. This proves (1).

Similarly, we can prove other conditions of theorem. ■

2.1.11 Theorem

A BVF set $B = (\lambda^+, \lambda^-) \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$, $BVFhbI(R)$, $BVFPhI(R)$, $BVFShI(R)$) iff it holds following sets of conditions $\{(1)', (3)', (4)'\}$(resp. $\{(1)', (3)', (5)'\}$, $\{(1)', (3)', (4)', (5)'\}$, $\{(1)', (2)',$

$(3)', (6)'\}, \{(1)', (2)', (3)', (7)'\}, \{(1)', (3)', (4)', (5)', (8)'\}, \{(1)', (3)', (4)', (5)', (9)'\})$.

2.1.12 Remark

If $B = (\lambda^+, \lambda^-) \in BVFLhI(R)$ $(BVFRhI(R), BVFhI(R), BVFIhI(R), BVFhbI(R),$ $BVFPhI(R)$, $BVFShI(R)) \Longrightarrow \lambda^+(0) \geq \lambda^+(x)$ and $\lambda^-(0) \leq \lambda^-(x)$ $\forall x \in R$.

2.1.13 Theorem

Let $\emptyset \neq I \subseteq R$. Then $C_I \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$, $BVFhbI(R)$) iff I is a left h-ideal (resp. right h-ideal, h-ideal, interior h-ideal, h-bi-ideal) of R.

Proof. Straightforward. ■

2.1.14 Theorem

Let $\emptyset \neq I \subseteq R$, where R is a commutative hemiring with unity. Then $C_I = (C_I^+, C_I^-) \in BVFPhI(R)$ (resp. $BVFShI(R)$) iff I is a prime h-ideal (resp. semiprime h-ideals) of R.

Proof. Straightforward. ■

2.1.15 Theorem

If a BVF set $B = (\lambda^+, \lambda^-) \in BVFhI(R)$ then $B = (\lambda^+, \lambda^-) \in BVFIhI(R)$.

2.1.16 Remark

Generally, converse of Theorem 2.1.15, is not true.

2.1.17 Example

Consider hemiring $R = \{0, p, q, r\}$ defined by the following operations

$+$	0	p	q	r
0	0	p	q	r
p	p	0	r	q
q	q	r	0	p
r	r	q	p	0

,

$\cdot$	0	p	q	r
0	0	0	0	0
p	0	q	0	q
q	0	0	0	0
r	0	q	0	q

.

Define B as follows

	0	p	q	r
μ^+	0.41	0.42	0.11	0.10
μ^-	-0.72	-0.71	-0.31	-0.33

.

Then $B = (\mu^+, \mu^-) \in BVFIhI(R)$ but it is not a $B = (\mu^+, \mu^-) \overline{\in} BVFhI(R)$.

As $B(pqr) = B(0) = (0.4, -0.7)$ and $B(q) = (0.1, -0.3)$, this shows $\mu^+(pqr) > \mu^+(q)$ and $\mu^-(pqr) < \mu^-(q)$. On the other hand $B(pp) = (0.1, -0.3)$ and $B(p) = (0.4, -0.7)$, this shows $\mu^+(pp) \ngeq \mu^+(p)$ and $\mu^-(pp) \nleq \mu^-(p)$.

2.1.18 Theorem

If $B = (\mu^+, \mu^-) \in BVFPhI(R)$ then $B = (\mu^+, \mu^-) \in BVFShI(R)$.

Proof. Suppose $B = (\lambda^+, \lambda^-) \in BVFPhI(R)$. Then by definition $\lambda^+(xy) \leq \max\{\lambda^+(x), \lambda^+(y)\}$ and $\lambda^-(xy) \geq \min\{\lambda^-(x), \lambda^-(y)\}$. For $y = x$ $\lambda^+(x) \geq \lambda^+(x^2)$ and $\lambda^-(x) \leq \lambda^-(x^2)$. Hence $B \in BVFShI(R)$. This complete the proof. ∎

2.1.19 Remark

Generally, converse of Theorem 2.1.18, is not true.

2.1.20 Example

Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and p_1, p_2, p_3, ... be the distinct prime numbers in $\mathbb{N}_0$. If $J^0 = \mathbb{N}_0$ and $J^l = p_1 p_2 p_3 ... p_l \mathbb{N}_0$, where $l = 1, 2, 3, ...$ then $J^0 \supset J^1 \supset J^2 \supset .... J^n \supset J^{n+1} \supset$ ...... As every non-zero elememt of $\mathbb{N}_0$ has unique prime factorization, for $l = 2$, 3, ... J^l is a semiprime h-ideal but not a prime h-ideal. Then by Theorem 2.1.14, $C_{J^l} = (C^+_{J^l}, C^-_{J^l}) \in BVFShI(R)$, but $C_{J^l} = (C^+_{J^l}, C^-_{J^l}) \overline{\in} BVFPhI(R)$.

2.1.21 Theorem

A family of BVF set $B_i = \{(\lambda_i^+, \lambda_i^-) : i \in \Omega\} \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$). Then $B = \bigwedge_{i\in\Omega} B_i \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$) where $B = (\lambda^+, \lambda^-)$ that is $\lambda^+ = \bigwedge_{i\in\Omega} \lambda_i^+$ and $\lambda^- = \bigvee_{i\in\Omega} \lambda_i^-$ $(\lambda^+ \leq \lambda_i^+, \lambda^- \geq \lambda_i^-$ $\forall\, i \in \Omega)$.

2.2 Bipolar-valued fuzzy h-intrinsic product

2.2.1 Definition

Let $B_1 = (\lambda^+, \lambda^-)$ and $B_2 = (\mu^+, \mu^-)$ be two BVF set of R. Then the h-intrinsic product of B_1 and B_2 is denoted and described as $(B_1 \odot_h B_2)(x) = ((\lambda^+ \odot_h \mu^+)(x)$ $, (\lambda^- \odot_h \mu^-)(x))$ $\forall x \in R$, if x can be signified as $x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z$, so that

$$(\lambda^+ \odot_h \mu^+)(x) = \bigvee_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \left[\left(\bigwedge_{i=1}^m \lambda^+(a_i) \right) \wedge \left(\bigwedge_{i=1}^m \mu^+(b_i) \right) \wedge \left(\bigwedge_{j=1}^n \lambda^+(c_j) \right) \wedge \left(\bigwedge_{j=1}^n \mu^+(d_j) \right) \right]$$

$$(\lambda^- \odot_h \mu^-)(x) = \bigwedge_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \left[\left(\bigvee_{i=1}^m \lambda^-(a_i) \right) \vee \left(\bigvee_{i=1}^m \mu^-(b_i) \right) \vee \left(\bigvee_{j=1}^n \lambda^-(c_j) \right) \vee \left(\bigvee_{j=1}^n \mu^-(d_j) \right) \right].$$

$(B_1 \odot_h B_2)(x) = (0, 0)$ $\forall x \in R$, if x cannot be signified as $x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z$.

2.2.2 Theorem

Let M_1, M_2 be h-ideals of R. Then we have

(i) $M_1 \subseteq M_2$, iff $C_{M_1}^+ \leq C_{M_2}^+$, $C_{M_1}^- \geq C_{M_2}^-$.

(ii) $C_{M_1}^+ \wedge C_{M_2}^+ = C_{M_1 \cap M_2}^+$, $C_{M_1}^- \vee C_{M_2}^- = C_{M_1 \cup M_2}^-$.

(iii) $C_{M_1}^+ \odot_h C_{M_2}^+ = C_{\overline{M_1 M_2}}^+$, $C_{M_1}^- \odot_h C_{M_2}^- = C_{\overline{M_1 M_2}}^-$.

Proof. (i), (ii) Straight forwad.

(iii) Let $C_{\overline{M_1 M_2}} = \left(C_{\overline{M_1 M_2}}^+, C_{\overline{M_1 M_2}}^- \right)$. Suppose $x \in R$ and $x \in \overline{M_1 M_2}$ so $C_{\overline{M_1 M_2}}^+ = 1$, $C_{\overline{M_1 M_2}}^- = -1$. Now, let $x + \Sigma_{i=1}^m p_i q_i + z = \Sigma_{j=1}^n r_j s_j + z$ for some p, $r \in A_1$ and q, $s \in M_2$

$$(C^+_{M_1} \odot_h C^+_{M_2})(x) =$$

$$\bigvee_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \left[(\bigwedge_{i=1}^m C^+_{M_1}(a_i)) \wedge (\bigwedge_{i=1}^m C^+_{M_2}(b_i)) \wedge (\bigwedge_{j=1}^n C^+_{M_1}(c_j)) \wedge (\bigwedge_{j=1}^n C^+_{M_2}(d_j)) \right]$$

$$\geq \left[(\bigwedge_{i=1}^m C^+_{M_1}(p_i)) \wedge (\bigwedge_{i=1}^m C^+_{M_2}(q_i)) \wedge (\bigwedge_{j=1}^n C^+_{M_1}(r_j)) \wedge (\bigwedge_{j=1}^n C^+_{M_2}(s_j)) \right] = 1$$

$$(C^-_{M_1} \odot_h C^-_{M_2})(x) =$$

$$\bigwedge_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \left[(\bigvee_{i=1}^m C^-_{M_1}(a_i)) \vee (\bigvee_{i=1}^m C^-_{M_2}(b_i)) \vee (\bigvee_{j=1}^n C^-_{M_1}(c_j)) \vee (\bigvee_{j=1}^n C^+_{M_2}(d_j)) \right]$$

$$\leq \left[(\bigvee_{i=1}^m C^-_{M_1}(p_i)) \vee (\bigvee_{i=1}^m C^-_{M_2}(q_i)) \vee (\bigvee_{j=1}^n C^-_{M_1}(r_j)) \vee (\bigvee_{j=1}^n C^-_{M_2}(s_j)) \right] = -1$$

Hence (iii) is proved. ■

2.2.3 Theorem

A BVF subset $B = (\lambda^+, \lambda^-) \in BVFLhI(R)$ (resp. $BVFRhI(R)$) iff it holds $(1)'$, $(3)'$ and $(C^+_R \odot_h \lambda^+)(x) \leq \lambda^+(x)$, $(C^-_R \odot_h \lambda^-)(x) \geq \lambda^-(x)$ (resp., $(\lambda^+ \odot_h C^+_R)(x) \leq \lambda^+(x)$, $(\lambda^- \odot_h C^-_R)(x) \geq \lambda^-(x)$).

2.2.4 Lemma

Let $B_1 = (\lambda^+, \lambda^-) \in BVFRhI(R)$ and $B_2 = (\mu^+, \mu^-) \in BVFLhI(R)$. Then $\lambda^+ \odot_h \mu^+ \leq \lambda^+ \wedge \mu^+$ and $\lambda^- \odot_h \mu^- \geq \lambda^- \vee \mu^-$.

2.3 Bipolar-valued fuzzy h-quasi-ideals of hemirings

2.3.1 Definition

Let $B = (\lambda^+, \lambda^-)$ be a BVF subset of R. Then B is called BVF h-quasi-ideal of R if and only if it satisafied for $x,\ y, z, r_1, r_2 \in R$,

$(1)'$ $\lambda^+(x+y) \geq \min\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(x+y) \leq \max\{\lambda^-(x), \lambda^-(y)\}$

$(3)'$ $x + r_1 + z = r_2 + z \implies \lambda^+(x) \geq \min\{\lambda^+(r_1), \lambda^+(r_2)\}$, $\lambda^-(x) \leq \max\{\lambda^-(r_1), \lambda^-(r_2)\}$.

$(10)'$ $(\lambda^+ \odot_h C_R^+) \cap (C_R^+ \odot_h \lambda^+) \leq \lambda^+$, $(\lambda^- \odot_h C_R^-) \cup (C_R^- \odot_h \lambda^-) \geq \lambda^-$.

2.3.2 Remark

In rest of thesis, set of all BVF h-quasi-ideal of R is denoted by $BVFhqI(R)$.

2.3.3 Example

In Example 2.1.6, $B = (\lambda^+, \lambda^-) \in BVFhqI(\mathbb{N}_0)$.

2.3.4 Theorem

Let $B = (\lambda^+, \lambda^-)$ is a BVF subset of R. If $B = (\lambda^+, \lambda^-) \in BVFhqI(R)$ iff all level subsets $U(B, t') \neq \emptyset$ are h-quasi-ideal of R.

2.3.5 Theorem

Let $\emptyset \neq I \subseteq R$. Then $C_I \in BVFhqI(R)$ iff I is a h-quasi-ideal of R.

2.3.6 Lemma

Let $B = (\lambda^+, \lambda^-) \in BVFRhI(R)$ and $B' = (\mu^+, \mu^-) \in BVFLhI(R)$. Then $B \cap B' \in BVFhqI(R)$.

2.3.7 Lemma

If $B = (\lambda^+, \lambda^-) \in BVFhqI(R)$ then $B = (\lambda^+, \lambda^-) \in BVFhbI(R)$.

Proof. Let $B = (\lambda^+, \lambda^-) \in BVFhqI(R)$. It is sufficient to prove $\lambda^+(xyz) \geq \min\{\lambda^+(x), \lambda^+(z)\}$, $\lambda^-(xyz) \leq \max\{\lambda^-(x), \lambda^-(z)\}$ and $\lambda^+(xy) \geq \min\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xy) \leq \max\{\lambda^-(x), \lambda^-(y)\}$ $\forall x, y, z \in R$. Now, we have

$$\begin{aligned}
\lambda^+(xyz) &\geq ((\lambda^+ \odot_h C_R^+) \cap (C_R^+ \odot_h \lambda^+))(xyz) \\
&= \min\{(\lambda^+ \odot_h C_R^+)(xyz), (C_R^+ \odot_h \lambda^+)(xyz)\} \\
&= \min\left\{\begin{array}{l} \bigvee\limits_{xyz+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{(\bigwedge\limits_{i=1}^{m}(\lambda^+)(a_i)) \wedge (\bigwedge\limits_{j=1}^{n}(\lambda^+)(c_j))\right\}, \\ \bigvee\limits_{xyz+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{(\bigwedge\limits_{i=1}^{m}(\lambda^+)(b_i)) \wedge (\bigwedge\limits_{j=1}^{n}(\lambda^+)(d_j))\right\} \end{array}\right\} \\
&\geq \min\{\min\{\lambda^+(0), \lambda^+(x)\}, \min\{\lambda^+(0), \lambda^+(z)\}\} \\
&= \min\{\lambda^+(x), \lambda^+(z)\}.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
\lambda^-(xyz) &\leq ((\lambda^- \odot_h C_R^-) \cap (C_R^- \odot_h \lambda^-))(xyz) \\
&= \max\{(\lambda^- \odot_h C_R^-)(xyz), (C_R^- \odot_h \lambda^-)(xyz)\}
\end{aligned}$$

$$= \max \left\{ \begin{array}{c} \bigwedge_{xyz+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ \left(\bigvee_{i=1}^{m} (\lambda^-)(a_i) \right) \vee \left(\bigvee_{j=1}^{n} (\lambda^-)(c_j) \right) \right\}, \\ \bigwedge_{xyz+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ \left(\bigvee_{i=1}^{m} (\lambda^-)(b_i) \right) \vee \left(\bigvee_{j=1}^{n} (\lambda^-)(d_j) \right) \right\} \end{array} \right\}$$

$$\leq \max\{\max\{\lambda^-(0), \lambda^-(x)\}, \max\{\lambda^-(0), \lambda^-(z)\}\}$$

$$= \max\{\lambda^-(x), \lambda^-(z)\}.$$

Similarly, we can prove $\lambda^+(xy) \geq \min\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xy) \leq \max\{\lambda^-(x), \lambda^-(y)\}$. Therefore $B \in BVFhbI(R)$. ■

Converse of Lemma 2.3.7, is not true.

2.4 Characterization of hemirings by their bipolar-valued fuzzy h-ideals

First we recall the definition of h-hemiregular and h-semisimple hemiring. In this section we characterize h-hemiregular and h-semisimple hemirings by using their BVF h-ideal (interior h-ideal).

2.4.1 Definition

Let $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-)$ be two BVF subsets of R. Then we say $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$, if $x_{t'} \in B_1 \Longrightarrow x_{t'} \in B_2$ i.e, $\lambda^+(x) \geq t^+ \Longrightarrow \mu^+(x) \geq t^+$ and $\lambda^-(x) \leq t^- \Longrightarrow \mu^-(x) \leq t^-$ for all $x \in R$ and $t' = (t^+, t^-)$.

2.4.2 Theorem

Let $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-)$ be two BVF subsets of R. Then we say$\lambda^+ \sim \mu^+$, $\mu^- \sim \lambda^-$ iff $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$ and $\mu^+[\in]\lambda^+$, $\lambda^-[\in]\mu^-$.

2.4.3 Lemma

The relation " $\backsim$ " is called equivalance relation on BVF subsets of R.

2.4.4 Theorem

Let $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-) \in BVFhI(R)$. Then $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$ iff $\lambda^+ \leq \mu^+$, $\mu^- \leq \lambda^-$, $\forall x \in R$.

Proof. Let us assume $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$. To prove $\lambda^+ \leq \mu^+$, $\mu^- \leq \lambda^-$, for all $x \in R$. Suppose on contrary for $x \in R$, $\lambda^+(x) > \mu^+(x)$, $\lambda^-(x) < \mu^-(x)$. Then $\exists$ $t' = (t^+, t^-) \in F(B)$ such that $\lambda^+(x) > t^+ > \mu^+(x)$, and $\lambda^-(x) < t^- < \mu^-(x)$. Which implies $\lambda^+(x) > t^+ \nRightarrow \mu^+(x) > t^+$ and $\lambda^-(x) < t^- \nRightarrow \mu^-(x) < t^-$. Which is contradiction. Hence $\lambda^+ \leq \mu^+$, $\mu^- \leq \lambda^-$.

Conversely, assume $\lambda^+ \leq \mu^+$, $\mu^- \leq \lambda^-$, for all $x \in R$. To prove $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$. Suppose $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$ does not hold. Then there exists $x \in R$ and $t' = (t^+, t^-) \in F(B)$ such that $\lambda^+(x) > t^+ \nRightarrow \mu^+(x) > t^+$ and $\lambda^-(x) < t^- \nRightarrow$ $\mu^-(x) < t^-$. Which is a contradiction. Hence $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$. ∎

2.4.5 Theorem

If $B_1 = (\lambda^+, \lambda^-) \in BVFRhI(R)$ and $B_2 = (\mu^+, \mu^-) \in BVFLhI(R)$, then $(\lambda^+ \odot_h \mu^+) \sim (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \sim (\lambda^- \vee \mu^-)$ iff R is h-hemiregular.

Proof. Suppose R is h-hemiregular. Let $B_1 \in BVFRhI(R)$ and $B_2 \in BVFLhI(R)$. Then $\forall s \in R$ by Lemma 2.2.4, $(\lambda^+ \odot_h \mu^+)(s) \leq (\lambda^+ \wedge \mu^+)(s)$, $(\lambda^- \odot_h \mu^-)(s) \geq (\lambda^- \vee \mu^-)(s)$ and so by the Theorem 2.4.4, $(\lambda^+ \odot_h \mu^+)(s) [\in] (\lambda^+ \wedge \mu^+)(s)$, $(\lambda^- \vee \mu^-)(s) [\in] (\lambda^- \odot_h \mu^-)(s)$. Now, since R is h-hemiregular, so $\forall s \in R$, $\exists\, p, q, z \in R$ such that $s+sps+z = sqs+z$. Thus $(\lambda^+ \odot_h \mu^+)(s) = \bigvee_{s+\Sigma_{i=1}^{n} p_i q_i + z = \Sigma_{j=1}^{m} r_j t_j + z} \left[(\bigwedge_{i=1}^{n} \lambda^+(p_i)) \wedge (\bigwedge_{i=1}^{n} \mu^+(q_i)) \wedge (_j \right.$

$$\geq \min\{\lambda^+(sp), \lambda^+(sq), \mu^+(s)\}$$
$$= (\lambda^+ \wedge \mu^+)(s)$$

$$(\lambda^+ \odot_h \mu^+)(s) \geq (\lambda^+ \wedge \mu^+)(s)$$

and

$$(\lambda^- \odot_h \mu^-)(s) =$$
$$\bigwedge_{s+\Sigma_{i=1}^{n} p_i q_i + z = \Sigma_{j=1}^{m} r_j t_j + z} \left[(\bigvee_{i=1}^{n} \lambda^-(p_i)) \vee (\bigvee_{i=1}^{n} \mu^-(q_i)) \vee (\bigvee_{j=1}^{m} \lambda^-(r_j)) \vee (\bigvee_{j=1}^{m} \mu^-(t_j)) \right]$$
$$\leq \max\{\lambda^-(sp), \lambda^-(sq), \mu^-(s)\}$$
$$= (\lambda^- \vee \mu^-)(s)$$

$$(\lambda^+ \odot_h \mu^+)(s) \leq (\lambda^- \vee \mu^-)(s).$$

Hence $(\lambda^+ \odot_h \mu^+) \sim (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \sim (\lambda^- \vee \mu^-)$.

Conversely, let I be a right and J be a left h-ideal of R. Then $C_I = (C_I^+, C_I^-) \in BVFRhI(R)$ and $C_j = (C_j^+, C_j^-) \in BVFLhI(R)$. Then, we know that from Theorem 2.2.2, $C_{\overline{IJ}}^+ = C_I^+ \odot_h C_J^+$ and from (ii) $C_{\overline{IJ}}^+ = C_I^+ \odot_h C_J^+ \sim C_I^+ \wedge C_J^+$ and from Theorem 2.2.2, $C_{\overline{IJ}}^+ = C_I^+ \odot_h C_J^+ \sim C_I^+ \wedge C_J^+ = C_{I \cap J}^+ \Longrightarrow \overline{IJ} = I \cap J$ so R is

h-hemiregular. ■

2.4.6 Theorem

If $B \in BVFhbI(R)$,then $\lambda^+ \leq (\lambda^+ \odot_h C_R^+ \odot_h \lambda^+)$, $\lambda^- \geq (\lambda^- \odot_h C_R^- \odot_h \lambda^-)$ iff R is h-hemiregular.

Proof. $(i) \Longrightarrow (ii)$ Suppose (i) hold.

$$(\lambda^+ \odot_h C_R^+ \odot_h \lambda^+)(x)$$

$$= \bigvee_{xyz+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ \begin{array}{c} (\bigwedge_{i=1}^{m} (\lambda^+ \odot_h C_R^+)(a_i)) \wedge \\ (\bigwedge_{j=1}^{n} (\lambda^+ \odot_h C_R^+)(c_j)) \\ \wedge (\bigwedge_{i=1}^{m} (\lambda^+)(b_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(d_j)) \end{array} \right\}$$

$$\geq \min \left\{ \begin{array}{c} ((\lambda^+ \odot_h C_R^+)(xa)), \\ ((\lambda^+ \odot_h C_R^+)(xc)), \lambda^+(x) \end{array} \right\}$$

$$= \min \left\{ \begin{array}{c} \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigwedge_{i=1}^{m} (\lambda^+)(a_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(c_j)) \right\}, \\ \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigwedge_{i=1}^{m} (\lambda^+)(b_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(d_j)) \right\}, \\ \lambda^+(x) \end{array} \right\}$$

$$\geq \min\{\min\{\lambda^+(xax), \lambda^+(xcx)\}, \min\{\lambda^+(xax), \lambda^+(xcx)\}, \lambda^+(x)\}$$

(since $xa + xaxa + za = xcxa + za$ and $xc + xaxc + zc = xcxc + zc$)

$$= \min\{\lambda^+(x), \lambda^+(x), \lambda^+(x)\} = \lambda^+(x).$$

Similarly, we have

$$(\lambda^- \odot_h C_R^- \odot_h \lambda^-)(x)$$

$$= \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \begin{array}{c} (\bigvee_{i=1}^{m} (\lambda^- \odot_h C_R^-)(a_i)) \vee \\ (\bigvee_{j=1}^{n} (\lambda^- \odot_h C_R^-)(c_j)) \\ \vee (\bigvee_{i=1}^{m} (\lambda^-)(b_i)) \vee (\bigvee_{j=1}^{n} (\lambda^-)(d_j)) \end{array} \right\}$$

$$\leq \max \left\{ \begin{array}{c} ((\lambda^- \odot_h C_R^-)(xa)), \\ ((\lambda^- \odot_h C_R^-)(xc)), \lambda^-(x) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ (\bigvee_{i=1}^{m} (\lambda^-)(a_i)) \vee (\bigvee_{j=1}^{n} (\lambda^-)(c_j)) \right\}, \\ \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ (\bigvee_{i=1}^{m} (\lambda^-)(b_i)) \vee (\bigvee_{j=1}^{n} (\lambda^-)(d_j)) \right\}, \\ \lambda^-(x) \end{array} \right\}$$

$$\leq \max\{\max\{\lambda^-(xax), \lambda^-(xcx)\}, \max\{\lambda^-(xax), \lambda^-(xcx)\}, \lambda^-(x)\}$$

(since $xa + xaxa + za = xcxa + za$ and $xc + xaxc + zc = xcxc + zc$)

$= \max\{\lambda^-(x), \lambda^-(x), \lambda^-(x)\} = \lambda^-(x)$. This prove (ii).

$(ii) \Longrightarrow (i)$ Assume that (ii) holds. Let M be any h-bi-ideal of R. Then by the Theorem 2.1.13, $C_M \in BVFhbI(R)$. Now, from (ii) $C_M^+ \subseteq C_M^+ \odot_h C_R^+ \odot_h C_M^+$, from Theorem 2.2.2, $C_M^+ \subseteq C_M^+ \odot_h C_R^+ \odot_h C_M^+ = C_{\overline{MRM}}^+$ and $M \subseteq \overline{MRM}$. On the other hand, since M is h-bi-ideal of R so that $MRM \subseteq M$. This imlies $\overline{MRM} \subseteq \overline{M}$, and from 1.3.8, $\overline{MRM} \subseteq \overline{M} = M$. Therefore $\overline{MRM} = M$. Hence from 1.4.7, R is h-hemiregular. ∎

2.4.7 Theorem

The following conditions for R are equivalent:

(i) R is h-hemiregular hemiring.

(ii) $\min\{\lambda^+,\mu^+\} \leq \lambda^+ \odot_h \mu^+ \odot_h \lambda^+$, $\max\{\lambda^-,\mu^-\} \geq \lambda^- \odot_h \mu^- \odot_h \lambda^-$ for every $B = (\lambda^+,\lambda^-) \in BVFhbI(R)$ and for every $B' = (\mu^+,\mu^-) \in BVFhI(R)$.

(iii) $\min\{\lambda^+,\mu^+\} \leq \lambda^+ \odot_h \mu^+ \odot_h \lambda^+$, $\max\{\lambda^-,\mu^-\} \geq \lambda^- \odot_h \mu^- \odot_h \lambda^-$ for every $B = (\lambda^+,\lambda^-) \in BVFhqI(R)$ and for every $B' = (\mu^+,\mu^-) \in BVFhI(R)$.

Proof. $(i) \Longrightarrow (ii)$ Suppose (i) hold.

$$
\begin{aligned}
&(\lambda^+ \odot_h \mu^+ \odot_h \lambda^+)(x) \\
&= \bigvee_{xyz+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ \begin{array}{c} (\bigwedge_{i=1}^{m} (\lambda^+ \odot_h \mu^+)(a_i)) \wedge \\ (\bigwedge_{j=1}^{n} (\lambda^+ \odot_h \mu^+)(c_j)) \\ \wedge (\bigwedge_{i=1}^{m} (\lambda^+)(b_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(d_j)) \end{array} \right\} \\
&\geq \min \left\{ \begin{array}{c} ((\lambda^+ \odot_h \mu^+)(xa)), \\ ((\lambda^+ \odot_h \mu^+)(xc)), \lambda^+(x) \end{array} \right\} \\
&= \min \left\{ \begin{array}{c} \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ \begin{array}{c} (\bigwedge_{i=1}^{m} (\lambda^+)(a_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(c_j)) \\ \wedge (\bigwedge_{i=1}^{m} (\mu^+)(b_i)) \wedge (\bigwedge_{j=1}^{n} (\mu^+)(d_j)) \end{array} \right\}, \\ \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ \begin{array}{c} (\bigwedge_{i=1}^{m} (\lambda^+)(a_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(c_j)) \\ \wedge (\bigwedge_{i=1}^{m} (\mu^+)(b_i)) \wedge (\bigwedge_{j=1}^{n} (\mu^+)(d_j)) \end{array} \right\}, \\ \lambda^+(x) \end{array} \right\} \\
&\geq \min\{\min\{\lambda^+(x), \mu^+(axa), \mu^+(cxa)\}, \\
&\quad \min\{\lambda^+(x), \mu^+(axc), \lambda^+(cxc)\}, \lambda^+(x)\} \\
&\quad \left(\begin{array}{c} \text{since } xa + xaxa + za = xcxa + za \text{ and} \\ xc + xaxc + zc = xcxc + zc \end{array} \right) \\
&= \min\{\min\{\lambda^+(x), \mu^+(x)\}. \min\{\lambda^+(x), \mu^+(x)\}\} \\
&= \min\{\lambda^+(x), \mu^+(x)\}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&(\lambda^- \odot_h \mu^- \odot_h \lambda^-)(x) \\
&= \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \begin{array}{c} (\bigvee_{i=1}^{m} (\lambda^- \odot_h \mu^-)(a_i)) \vee \\ (\bigvee_{j=1}^{n} (\lambda^- \odot_h \mu^-)(c_j)) \\ \vee (\bigvee_{i=1}^{m} (\lambda^-)(b_i)) \vee (\bigvee_{j=1}^{n} (\lambda^-)(d_j)) \end{array} \right\} \\
&\leq \max \left\{ \begin{array}{c} ((\lambda^- \odot_h \mu^-)(xa)), \\ ((\lambda^- \odot_h \mu^-)(xc)), \lambda^-(x) \end{array} \right\} \\
&= \max \left\{ \begin{array}{c} \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \begin{array}{c} (\bigvee_{i=1}^{m} (\lambda^-)(a_i)) \vee (\bigvee_{j=1}^{n} (\lambda^-)(c_j)) \\ \vee (\bigvee_{i=1}^{m} (\mu^-)(b_i)) \vee (\bigvee_{j=1}^{n} (\mu^-)(d_j)) \end{array} \right\}, \\ \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \begin{array}{c} (\bigvee_{i=1}^{m} (\lambda^-)(a_i)) \vee (\bigvee_{j=1}^{n} (\lambda^-)(c_j)) \\ \vee (\bigvee_{i=1}^{m} (\mu^-)(b_i)) \vee (\bigvee_{j=1}^{n} (\mu^-)(d_j)) \end{array} \right\}, \\ \lambda^-(x) \end{array} \right\} \\
&\leq \max\{\max\{\lambda^-(x), \mu^-(axa), \mu^-(cxa)\}, \\
&\qquad \max\{\lambda^-(x), \lambda^-(axc), \mu^-(cxc)\}, \lambda^-(x)\} \\
&\qquad \left(\begin{array}{c} \text{since } xa + xaxa + za = xcxa + za \text{ and} \\ xc + xaxc + zc = xcxc + zc \end{array} \right) \\
&= \max\{\max\{\lambda^-(x), \mu^-(x)\}, \max\{\lambda^-(x), \mu^-(x)\}\} \\
&= \max\{\lambda^-(x), \mu^-(x)\}. \text{ This prove } (ii).
\end{aligned}
$$

$(ii) \Longrightarrow (iii)$. This is straightforward by Lemma 2.3.7.

$(iii) \Longrightarrow (i)$. Assume that (iii) holds. Let M be any h-quasi-ideal of R. Then by the Theorem 2.1.13, $C_M \in BVFhqI(R)$. Since $C_R \in BVFhI(R)$. Now, from (iii) $C_M^+ \leq C_M^+ \odot_h C_R^+ \odot_h C_M^+$, from 2.2.2, $C_M^+ \subseteq C_M^+ \odot_h C_R^+ \odot_h C_M^+ = C_{\overline{MRM}}^+$ and $M \subseteq \overline{MRM}$. On the other hand, since M is h-bi-ideal of R so that $MRM \subseteq M$. This imlies $\overline{MRM} \subseteq \overline{M}$, and from 1.3.8, $\overline{MRM} \subseteq \overline{M} = M$. Therefore $\overline{MRM} = M$. Hence from 1.4.7, R is h-hemiregular hemiring. ■

2.4.8 Theorem

Let R be a h-hemisimple and B be a BVF subset of R. Then $B \in BVFhI(R)$ iff $B \in BVFIhI(R)$.

Proof. By Theorem 2.1.15, every $B = (\lambda^+, \lambda^-) \in BVFhI(R)$ is BVF interior h-ideal of R.

Conversely assume $B = (\lambda^+, \lambda^-) \in BVFhI(R)$. Let p, $q \in R$. By 1.4.2, $\exists$ a_i, b_i, c_i, d_i, e_j, f_j, g_j, $h_j \in R$ such that $p + \Sigma_{i=1}^{m} a_i p b_i c_i p d_i + z = \Sigma_{j=1}^{n} e_j p f_j g_j p h_j + z$. Which implies $pq + \Sigma_{i=1}^{m} a_i p b_i c_i p d_i q + zq = \Sigma_{j=1}^{n} e_j p f_j g_j p h_j q + zq$.

Thus $\lambda^+(pq) \geq \min\left\{\lambda^+(\Sigma_{i=1}^{m} a_i p b_i c_i p d_i q), \lambda^+\left(\Sigma_{j=1}^{n} e_j p f_j g_j p h_j q\right), 0.5\right\}$

$\lambda^+(pq) \geq \lambda^+(p)$.

And $\lambda^-(pq) \leq \max\left\{\lambda^-(\Sigma_{i=1}^{m} a_i p b_i c_i p d_i q), \lambda^-\left(\Sigma_{j=1}^{n} e_j p f_j g_j p h_j q\right), -0.5\right\}$

$\lambda^-(pq) \leq \lambda^-(p)$.

Thus $B \in BVFRhI(R)$. Similarly, we can show $B \in BVFLhI(R)$. Hence proved the theorem. ■

2.4.9 Theorem

If $B_1 = (\lambda^+, \lambda^-) \in BVFIhI(R)$ and $B_2 = (\mu^+, \mu^-) \in BVFIhI(R)$, then $(\lambda^+ \odot_h \mu^+) \sim (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \sim (\lambda^- \vee \mu^-)$ iff R is h-semisimple.

Proof. $(i) \Longrightarrow (ii)$ Suppose for any $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-) \in BVFLhI(R)$ of R. Then by the Lemma 2.2.4 $(\lambda^+ \odot_h \mu^+) \leq (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \geq (\lambda^- \vee \mu^-)$.. And by the Theorem 2.4.4 $(\lambda^+ \odot_h \mu^+)\,[\in]\,(\lambda^+ \wedge \mu^+)$, $(\lambda^- \vee \mu^-)\,[\in]\,(\lambda^- \odot_h \mu^-)$. Since, R is h-hemisimple, so $\forall s \in R$, $\exists\, c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j \in R$ such that $s+\Sigma_{i=1}^{m} c_i s d_i e_i s f_i + z = \Sigma_{j=1}^{n} c'_j s d'_j e'_j s f'_j + z$. Thus

$$(\lambda^+ \odot_h \mu^+)(s) = \bigvee_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left[(\bigwedge_{i=1}^{m} \lambda^+(a_i)) \wedge (\bigwedge_{i=1}^{m} \mu^+(b_i)) \wedge (\bigwedge_{j=1}^{n} \lambda^+(a'_j)) \wedge (\bigwedge_{j=1}^{n} \mu^+(b'_j)) \right]$$

$$\geq \min\{\lambda^+(c_i s d_i), \lambda^+(c'_j s d'_j), \mu^+(e_i s f_i), \mu^+(e'_j s f'_j)\}$$

$$\geq \min\{\lambda^+(s), \mu^+(s)\}$$

$$= (\lambda^+ \wedge \mu^+)(s)$$

$$(\lambda^+ \odot_h \mu^+)(s) \geq (\lambda^+ \wedge \mu^+)(s)$$

and

$$(\lambda^- \odot_h \mu^-)(s) = \bigwedge_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left[(\bigvee_{i=1}^{m} \lambda^-(a_i)) \vee (\bigvee_{i=1}^{m} \mu^-(b_i)) \vee (\bigvee_{j=1}^{n} \lambda^-(a'_j)) \vee (\bigvee_{j=1}^{n} \mu^-(b'_j)) \right]$$

$$\leq \max\{\lambda^-(c_i s d_i), \lambda^-(c'_j s d'_j), \mu^+(e_i s f_i), \mu^+(e'_j s f'_j)\}$$

$$\leq \max\{\lambda^-(s), \mu^-(s)\}$$

$$= (\lambda^- \vee \mu^-)(s)$$

$$(\lambda^- \odot_h \mu^-)(s) \leq (\lambda^- \vee \mu^-)(s).$$

Hence (ii) is proved.

$(ii) \Longrightarrow (i)$ Let I be a h-ideal of R. Then from Theorem 2.1.15, I is an interior h-ideal of R. Then from Theorem 2.1.13, $C_I = (C_I^+, C_I^-) \in BVFIhI(R)$. Then we have $C_I^+ = C_I^+ \wedge C_I^+$ and from $(ii)\, C_I^+ = C_I^+ \odot_h C_I^+ \sim C_I^+ \wedge C_I^+$ and from 2.2.2, $C_I^+ = C_{\overline{I^2}}^+ \Longrightarrow I = \overline{I^2}$.Therefore R is h-hemisimple. ■

Chapter 3

Bipolar-valued anti fuzzy h-ideals of hemirings

In this chapter, we define bipolar-valued anti fuzzy h-subhemiring and bipolar-valued anti fuzzy h-ideals. We analysed some basic definitions of bipolar-valued anti fuzzy h-ideals and some basic properties of bipolar-valued anti fuzzy h-ideals. We characterize h-hemiregular and h-semisimple hemirings by using their bipolar-valued anti fuzzy h-ideals (h-bi-ideals, h-quasi-ideals and interior h-ideal). Also we use notion BVAF h-ideals in place of bipolar-valued anti fuzzy h-ideals.

3.1 Bipolar-valued anti fuzzy h-ideals of hemirings

In this chapter, we define BVAF h-subhemiring . We also popularized BVAF h-ideals. We analysed some basic definitions of BVAF h-ideals.

3.1.1 Definition.

Let M be a non-empty subset of R. A bipolar-valued fuzzy subset $C_{M^C} = \left(C^+_{M^C}, C^-_{M^C}\right)$ is described by

$$C^+_{M^C}(x) = \begin{cases} 0 & \text{if } x \in M \\ 1 & \text{otherwise} \end{cases}$$

and

$$C^-_{M^C}(x) = \begin{cases} 0 & \text{if } x \in M \\ -1 & \text{otherwise} \end{cases}$$

is called bipolar-valued anti characteristic function.

3.1.2 Definition

A BVF subset $B = \left(\lambda^+, \lambda^-\right)$ of R is called BVAF h-subhemiring of R if it satisfies

$(W1)$ $x_{t'} \overline{\in} B$, $y_{r'} \overline{\in} B \Longrightarrow (x+y)_{\max\{t',r'\}} \overline{\in} B$

$(W2)$ $x_{t'} \overline{\in} B$, $y_{r'} \overline{\in} B \Longrightarrow (xy)_{\max\{t',r'\}} \overline{\in} B$

$(W3)$ $x + s_1 + z = s_2 + z$, $(s_1)_{t'} \overline{\in} B$, $(s_2)_{r'} \overline{\in} B \Longrightarrow (x)_{\max\{t',r'\}} \overline{\in} B \forall x, z, s_1, s_2 \in R$

and $t' = (t^+, t^-)$, $r' = (r^+, r^-) \in (0,1] \times [-1,0)$.

3.1.3 Definition

A BVF subset $B = \left(\lambda^+, \lambda^-\right)$ of a hemirig R is called BVAF left (resp., right) h-ideal of R if it hold $(W1)$, $(W3)$ and

$(W4)$ $x_{t'} \overline{\in} B \Longrightarrow (yx)_{t'} \overline{\in} B$ (resp., $(W5)$ $(xy)_{t'} \overline{\in} B$) $\forall x, y \in R$ and $t' = (t^+, t^-) \in (0,1] \times [-1,0)$.

B is called a BVAF h-ideal if it is both left and right BVAF h-ideal of R.

3.1.4 Example.

Let us consider hemiring $R = \{0, 1, q_1, q_2, q_3\}$ with zero multiplication and addition defined by,

$+$	0	1	q_1	q_2	q_3
0	0	1	q_1	q_2	q_3
1	1	1	q_3	q_3	q_3
q_1	q_1	q_3	q_3	q_3	q_3
q_2	q_2	q_3	q_3	q_3	q_3
q_3	q_3	q_3	q_3	q_3	q_3

We define BVF $B = (\lambda^+, \lambda^-)$ as follows

	0	1	q_1	q_2	q_3
λ^+	0.13	0.63	0.65	0.64	0.66
λ^-	-0.44	-0.24	-0.25	-0.25	-0.26

Then B is a BVAF h-ideal of R.

3.1.5 Proposition

Let $\emptyset \neq M \subseteq R$. A BVF set $B = (\mu_M^+, \mu_M^-)$ is defined by

$$\mu_M^+(x) = \begin{cases} \epsilon_1 & \text{if } x \in M \\ \epsilon_2 & \text{otherwise} \end{cases}$$

and

$$\mu_M^-(x) = \begin{cases} \delta_1 & \text{if } x \in M \\ \delta_2 & \text{otherwise} \end{cases}$$

where $0 \leq \epsilon_1 \leq \epsilon_2 \leq 1$ and $0 \geq \delta_1 \geq \delta_2 \geq -1$, is a BVAF h-ideal of R iff M is h-ideal of R.

3.1.6 Definition

A BVF subset $B = (\lambda^+, \lambda^-)$ of R is called BVAF interior h-ideal of R if it holds $(W1), (W2), (W3)$ and

$(W6)$ $\quad y_{t'} \overline{\in} B \Longrightarrow (xyz)_{t'} \overline{\in} B\ \forall x,\ y,\ z \in R$ and $t' = (t^+, t^-) \in (0,1] \times [-1,0)$.

3.1.7 Definition

A BVF subset $B = (\lambda^+, \lambda^-)$ of R is called BVAF h-bi-ideal of R if it holds $(W1), (W2), (W3)$ and

$(W7)$ $\quad x_{t'} \overline{\in} B,\ y_{r'} \overline{\in} B \Longrightarrow (xzy)_{\max(t',r')} \overline{\in} B\ \forall x,\ y,\ z \in R$ and $t' = (t^+, t^-)$, $r' = (r^+, r^-) \in (0,1] \times [-1,0)$.

3.1.8 Theorem

Let B be a BVF subset of R. Then $(W1)$ to $(W7)$ are equivelent to $(W1)'$ to $(W7)'$ respectively, $\forall x, y, z, r_1, r_2$ where:

$(W1)'$ $\lambda^+(x+y) \leq \max\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(x+y) \geq \min\{\lambda^-(x), \lambda^-(y)\}$.

$(W2)'$ $\lambda^+(xy) \leq \max\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xy) \geq \min\{\lambda^-(x), \lambda^-(y)\}$.

$(W3)'$ $\quad y + r_1 + z = r_2 + z \quad \Longrightarrow \quad \lambda^+(y) \leq \max\{\lambda^+(r_1), \lambda^+(r_2)\}$, $\lambda^-(y) \geq \min\{\lambda^-(r_1), \lambda^-(r_2)\}$.

$(W4)'$ $\quad \lambda^+(xy) \leq \lambda^+(y)$, $\lambda^-(xy) \geq \lambda^-(y)$.

$(W5)'$ $\quad \lambda^+(xy) \leq \lambda^+(x)$, $\lambda^-(xy) \geq \lambda^-(x)$

$(W6)'$ $\lambda^+(xyz) \leq \lambda^+(y)$, $\lambda^-(xyz) \geq \lambda^-(y)$

$(W7)'$ $\lambda^+(xzy) \leq \max\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xzy) \geq \min\{\lambda^-(x), \lambda^-(y)\}$.

Proof. $(W1) \iff (W1)'$

Suppose $(W1)'$ is false. Then there exist x, $y \in R$, such that $\lambda^+(x+y) > \max\{\lambda^+(x), \lambda^+(y)\}$ and $\lambda^-(x+y) < \min\{\lambda^-(x),\lambda^-(y)\}$. Then for some $t' = (t^+, t^-) \in F(B)$. So that $\lambda^+(x+y) > t^+ \geq \max\{\lambda^+(x), \lambda^+(y)\}$ and $\lambda^-(x+y) < t^- \leq \min\{\lambda^-(x),\lambda^-(y)\}$. Here $\lambda^+(x) \leq t^+$, $\lambda^+(y) \leq t^+$ and $\lambda^-(x) \geq t^-$, $\lambda^-(y) \geq t^-$. This implies, $x_{t'}\overline{\in}B$, $y_{t'}\overline{\in}B$ but $(x+y)_{t'} \in B$. Which is contradiction. Hence $(W1)'$ hold.

Conversely, let x, $y \in R$ for $t' = (t^+, t^-)$ and $r' = (r^+, r^-)$. Such that $x_{t'}\overline{\in}B$, $y_{r'}\overline{\in}B$ means $B(x) \leq t'$ and $B(y) \leq r'$. Here $\lambda^+(x) \leq t^+$, $\lambda^-(x) \geq t^-$ and $\lambda^+(y) \leq r^+$, $\lambda^-(y) \geq r^-$. Then by $(W1)'$ $\lambda^+(x+y) \leq \max\{\lambda^+(x), \lambda^+(y)\}$ and $\lambda^-(x+y) \geq \min\{\lambda^-(x),\lambda^-(y)\}$. So that $\lambda^+(x+y) \leq \max\{\lambda^+(x), \lambda^+(y)\} \leq \max\{t^+, r^+\}$ and $\lambda^-(x+y) \geq \min\{\lambda^-(x),\lambda^-(y)\} \geq \min\{t^-, r^-\}$. So $\left(\lambda^+(x+y), \lambda^-(x+y)\right) \leq (\max\{t^+, r^+\}, \min\{t^-, r^-\})$ *i.e,* $B(x+y) \leq \max\{t', r'\}$. This implies $(x+y)_{\max\{t',r'\}}\overline{\in}B$. This proves $(W1)$.

In addition, we can prove other conditions similarly. ■

3.1.9 Remark

In rest of the thesis, we denote set of all BVAF left h-ideals (resp. BVAF right h-ideals, BVAF h-ideals, BVAF interior h-ideals, BVAF h-bi-ideals) of R by $BVAFLhI(R)$ (resp. $BVAFRhI(R)$, $BVAFhI(R)$, $BVAFIhI(R)$, $BVAFhbI(R)$).

3.1.10 Theorem

A BVF subset $B = (\lambda^+, \lambda^-) \in BVAFLhI(R)$ (resp. $BVAFRhI(R)$, $BVAFhI(R)$, $BVAFIhI(R)$, $BVAFhbI(R)$) iff it holds following sets of conditions $\{(W1)', (W3)', (W4)'\}$ (resp. $\{(W1)', (W3)', (W5)'\}$, $\{(W1)', (W3)', (W4)', (W5)'\}$, $\{(W1)', (W2)', (W3)', (W6)'\}$, and $\{(W1)', (W2)', (W3)', (W7)'\}$).

3.1.11 Remark

If $B = (\lambda^+, \lambda^-) \in BVAFLhI(R)$ ($BVAFRhI(R)$, $BVAFhI(R)$, $BVAFIhI(R)$, $BVAFhbI(R)$) then $\lambda^+(0) \leq \lambda^+(x)$ and $\lambda^-(0) \geq \lambda^-(x)$ $\forall x \in R$.

3.1.12 Theorem

Let $\emptyset \neq I \subseteq R$. Then $C_{I^c} \in BVAFLhI(R)$ ($BVAFRhI(R)$, $BVAFhI(R)$, $BVAFIhI(R)$, $BVAFhbI(R)$) iff I is a left h-ideal (resp. right h-ideal, h-ideal, interior h-ideal, h-bi-ideal) of R.

Proof. Straightforward ■

3.1.13 Theorem

If $B = (\lambda^+, \lambda^-) \in BVAFhI(R)$ then $B = (\lambda^+, \lambda^-) \in BVAFIhI(R)$.

Proof. Straightforward ■

3.1.14 Theorem

A family of BVF set $B_i = \{(\lambda_i^+, \lambda_i^-) : i \in \Omega\} \in BVAFLhI(R)$ (resp. $BVAFRhI(R)$, $BVAFhI(R)$, $BVAFIhI(R)$). Then $B = \bigvee_{i\in\Omega} B_i \in BVAFLhI(R)$ (resp. $BVAFRhI(R)$, $BVAFhI(R)$ and $BVAFIhI(R)$) where $B = (\lambda^+, \lambda^-)$ that is $\lambda^+ = \bigvee_{i\in\Omega} \lambda_i^+$ and $\lambda^- = \bigwedge_{i\in\Omega} \lambda_i^-$ ($\lambda^+ \geq \lambda_i^+, \lambda^- \leq \lambda_i^- \quad \forall\, i \in \Omega$).

3.2 Realation between BVF h-ideals and BVAF h-ideals

In this section, we analysed some realation between BVF h-ideals and BVAF h-ideals of R.

3.2.1 Definition

Let $B = (\lambda^+, \lambda^-)$ be a BVF subset of R. Then complement of B is define by $B^c = ((\lambda^+)^c, (\lambda^-)^c) = (1 - \lambda^+, -1 - \lambda^-)$.

3.2.2 Proposition

Let $B = (\lambda^+, \lambda^-)$ be a BVF set in R. Then $B \in BVAFLhI(R)$ (resp. $BVAFRhI(R)$) iff $B^c \in BVFLhI(R)$ (resp. $BVFRhI(R)$).

Proof. Let $B = (\lambda^+, \lambda^-) \in BVAFLhI(R)$ (resp. $BVAFRhI(R)$). For s_1, $s_2 \in R$, we have

$(\lambda^+)^c(s_1 + s_2) = 1 - \lambda^+(s_1 + s_2)$

$$\geq 1 - \max\{\lambda^+(s_1), \lambda^+(s_2)\}$$

$$= \min\{(1 - \lambda^+(s_1)), (1 - \lambda^+(s_2))\}$$

$$= \min\{(\lambda^+)^c(s_1), (\lambda^+)^c(s_2)\}$$

and

$$(\lambda^-)^c(s_1 + s_2) = -1 - \lambda^-(s_1 + s_2)$$

$$\leq -1 - \min\{\lambda^-(s_1), \lambda^-(s_2)\}$$

$$= \max\{(-1 - \lambda^-(s_1)), (-1 - \lambda^-(s_2))\}$$

$$= \max\{(\lambda^-)^c(s_1), (\lambda^-)^c(s_2)\}.$$

Now, $(\lambda^+)^c(s_2 s_1) = 1 - \lambda^+(s_2 s_1)$

$$\geq 1 - \lambda^+(s_1)$$

$$= (\lambda^+)^c(s_1) \quad (\text{resp., } (\lambda^+)^c(s_1 s_2) \geq (\lambda^+)^c(s_1))$$

$$(\lambda^-)^c(s_2 s_1) = -1 - \lambda^-(s_2 s_1)$$

$$\leq -1 - \lambda^-(s_1)$$

$$= (\lambda^-)^c(s_1) \quad (\text{resp., } (\lambda^-)^c(s_1 s_2) \leq (\lambda^-)^c(s_1)).$$

Let us suppose, for y, z, s_1, $s_2 \in R$, so that $y + s_1 + z = s_2 + z$ this implies

$$(\lambda^+)^c(y) = 1 - \lambda^+(y)$$

$$\geq 1 - \max\{\lambda^+(s_1), \lambda^+(s_2)\}$$

$$= \min\{(1 - \lambda^+(s_1)), (1 - \lambda^+(s_2))\}$$

$$= \min\{(\lambda^+)^c(s_1), (\lambda^+)^c(s_2)\}$$

Suppose, for y, z, s_1, $s_2 \in R$, so that $y + s_1 + z = s_2 + z$ this implies $(\lambda^-)^c(y) = -1 - \lambda^-(y)$

$$\leq -1 - \min\{\lambda^-(s_1), \lambda^-(s_2)\}$$

$$= \max\{(-1 - \lambda^-(s_1)), (-1 - \lambda^-(s_2))\}$$

$$= \max\{(\lambda^-)^c(s_1), (\lambda^-)^c(s_2)\}$$

Hence $B^c \in BVFLhI(R)$ (resp. $BVFRhI(R)$).

Conversely, let $B^c \in BVFLhI(R)$ (resp. $BVFRhI(R)$). Let for x, $y \in R$, we have

$$\begin{aligned}\lambda^+(s_1+s_2) &= 1-(\lambda^+)^c(s_1+s_2)\\ &\leq 1-\min\{(\lambda^+)^c(s_1), (\lambda^+)^c(s_2)\}\\ &= \max\{(1-(\lambda^+)^c(s_1)), (1-(\lambda^+)^c(s_2))\}\\ &= \max\{\lambda^+(s_1), \lambda^+(s_2)\}\end{aligned}$$

and $$\begin{aligned}\lambda^-(s_1+s_2) &= -1-(\lambda^-)^c(s_1+s_2)\\ &\geq -1-\max\{(\lambda^-)^c(s_1), (\lambda^-)^c(s_2)\}\\ &= \min\{(-1-(\lambda^-)^c(s_1)), (-1-(\lambda^-)^c(s_2))\}\\ &= \min\{\lambda^-(s_1), \lambda^-(s_2)\}\end{aligned}$$

$$\begin{aligned}\lambda^+(s_2s_1) &= 1-(\lambda^+)^c(s_2s_1)\\ &\leq 1-(\lambda^+)^c(s_1)\\ &= \lambda^+(s_1) \quad (\text{resp., } \lambda^+(s_1s_2) \leq \lambda^+(s_1))\end{aligned}$$

and $$\begin{aligned}\lambda^-(s_2s_1) &= 1-(\lambda^-)^c(s_2s_1)\\ &\geq 1-(\lambda^-)^c(s_1)\\ &= \lambda^-(s_1) \quad (\text{resp., } \lambda^-(s_1s_2) \geq \lambda^-(s_1))\end{aligned}$$

Let us suppose, for y, z, s_1, $s_2 \in R$, so that $y+s_1+z = s_2+z$ this implies

$$\begin{aligned}(\lambda^+)(y) &= 1-(\lambda^+)^c(y)\\ &\leq 1-\min\{(\lambda^+)^c(s_1), (\lambda^+)^c(s_2)\}\\ &= \max\{(1-(\lambda^+)^c(s_1)), (1-(\lambda^+)^c(s_2))\}\\ &= \max\{(\lambda^+)(s_1), (\lambda^+)(s_2)\}\end{aligned}$$

Suppose, for y, z, s_1, $s_2 \in R$, so that $y+s_1+z = s_2+z$ this implies $(\lambda^-)^c(y) = -1 - \lambda^-(y)$

$$\geq -1 - \max\{(\lambda^-)^c(s_1), (\lambda^-)^c(s_2)\}$$

$$= \min\{(-1 - (\lambda^-)^c(s_1)), (-1 - (\lambda^-)^c(s_2))\}$$

$$= \min\{(\lambda^-)(s_1), (\lambda^-)(s_2)\}$$

Hence $B \in BVAFLhI(R)$ (resp. $BVAFRhI(R)$). ■

3.3 Bipolar-valued anti fuzzy h-intrinsic product

3.3.1 Definition

The anti h-intrinsic product of BVF subset $B_1 = (\lambda^+, \lambda^-)$ and $B_2 = (\mu^+, \mu^-)$ is denoted and defined as $(B_1 \circledast_h B_2)(x) = ((\lambda^+ \circledast_h \mu^+)(x), (\lambda^- \circledast_h \mu^-)(x))\ \forall x \in R$, if it is possible to express as $x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} c_j d_j + z$, so that

$$(\lambda^+ \circledast_h \mu^+)(x) = \bigwedge_{x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}c_jd_j+z} \left[\left(\bigvee_{i=1}^{m}\lambda^+(a_i)\right) \vee \left(\bigvee_{i=1}^{m}\mu^+(b_i)\right) \vee \left(\bigvee_{j=1}^{n}\lambda^+(c_j)\right) \vee \left(\bigvee_{j=1}^{n}\mu^+(d_j)\right)\right]$$

$$(\lambda^- \circledast_h \mu^-)(x) = \bigvee_{x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}c_jd_j+z} \left[\left(\bigwedge_{i=1}^{m}\lambda^-(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m}\mu^-(b_i)\right) \wedge \left(\bigwedge_{j=1}^{n}\lambda^-(c_j)\right) \wedge \left(\bigwedge_{j=1}^{n}\mu^-(d_j)\right)\right].$$

$(B_1 \circledast_h B_2)(x) = (1, 1)\ \forall x \in R$, if it is not possible to express x as $x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} c_j d_j + z$.

3.3.2 Theorem

Let A, B be the subset of R and C_{A^c}, C_{B^c} be the anti characteristic functions of A and B. Then we have

$(i)\, C_A^+ \vee C_B^+ = C_{A\cup B}^+,\ C_A^- \vee C_B^- = C_{A\cup B}^-.$

$(ii)\, C_{A^c}^+ \circledast_h C_{B^c}^+ = C_{\overline{AB}^c}^+,\ C_{A^c}^- \circledast_h C_{B^c}^- = C_{\overline{AB}^c}^-.$

Proof. Proof of (i) is straghtforwad.

(ii) Let $C_{\overline{AB}^c} = \left(C_{\overline{AB}^c}^+, C_{\overline{AB}^c}^-\right)$. Suppose $x \in R$ and $x \in \overline{AB}^c$ so $C_{\overline{AB}^c}^+ = 1, C_{\overline{AB}^c}^- = -1$, then there $\nexists x$ so that $x + \Sigma_{i=1}^m p_i q_i + z = \Sigma_{j=1}^n r_j s_j + z$ for some p_i, $r_j \in A$ and q_i, $s_j \in B$. Therefore, $C_{A^c}^+ \circledast_h C_{B^c}^+ = 1$, $C_{A^c}^- \circledast_h C_{B^c}^- = -1$.

Now, suppose $x \in R$ and $x \notin \overline{AB}^c$ so $x \in \overline{AB}$ and $C_{\overline{AB}^c}^+ = 0$, $C_{\overline{AB}^c}^- = 0$, then there exist $x + \Sigma_{i=1}^m p_i q_i + z = \Sigma_{j=1}^n r_j s_j + z$ for some p_i, $r_j \in A$ and q_i, $s_j \in B$.

$$(C_{A^c}^+ \circledast_h C_{B^c}^+)(x) = \bigwedge_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \begin{bmatrix} (\bigvee_{i=1}^m C_{A^c}^+(a_i)) \vee (\bigvee_{i=1}^m C_{B^c}^+(b_i)) \\ \vee(\bigvee_{j=1}^n C_{A^c}^+(c_j)) \vee (\bigvee_{j=1}^n C_{B^c}^+(d_j)) \end{bmatrix}$$

$$\leq \begin{bmatrix} (\bigvee_{i=1}^m C_{A^c}^+(p_i)) \vee (\bigvee_{i=1}^m C_{B^c}^+(q_i)) \\ \vee(\bigvee_{j=1}^n C_{A^c}^+(r_j)) \vee (\bigvee_{j=1}^n C_{B^c}^+(s_j)) \end{bmatrix} = 0$$

$$(C_{A^c}^- \circledast_h C_{B^c}^-)(x) = \bigvee_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \begin{bmatrix} (\bigwedge_{i=1}^m C_{A^c}^-(a_i)) \wedge (\bigwedge_{i=1}^m C_{B^c}^-(b_i)) \\ \wedge(\bigwedge_{j=1}^n C_{A^c}^-(c_j)) \wedge (\bigwedge_{j=1}^n C_{B^c}^+(d_j)) \end{bmatrix}$$

$$\geq \begin{bmatrix} (\bigwedge_{i=1}^m C_{A^c}^-(p_i)) \wedge (\bigwedge_{i=1}^m C_{B^c}^-(q_i)) \\ \wedge(\bigwedge_{j=1}^n C_{A^c}^-(r_j)) \wedge (\bigwedge_{j=1}^n C_{B^c}^-(s_j)) \end{bmatrix} = 0$$

Therefore (ii) is proved.

Hence $C_{A^c}^+ \circledast_h C_{B^c}^+ = C_{\overline{AB}^c}^+$, $C_{A^c}^- \circledast_h C_{B^c}^- = C_{\overline{AB}^c}^-$. ■

3.3.3 Theorem

A BVF subset $B = \left(\lambda^{+}, \lambda^{-}\right) \in BVAFLhI(R)$ (resp. $BVAFRhI(R)$) iff it holds $(W1)', (W3)'$ and $C^{+}_{R^{c}} \circledast_{h} \lambda^{+} \geq \lambda^{+}$, $C^{-}_{R} \circledast_{h} \lambda^{-} \leq \lambda^{-}$(resp., $\lambda^{+} \circledast_{h} C^{+}_{R^{c}} \leq \lambda^{+}$, $\lambda^{-} \circledast_{h} C^{-}_{R} \leq \lambda^{-}$).

3.3.4 Lemma

Let $B_1 = \left(\lambda^{+}, \lambda^{-}\right) \in BVAFRhI(R)$ and $B_2 = (\mu^{+}, \mu^{-}) \in BVAFLhI(R)$. Then $\lambda^{+} \circledast_{h} \mu^{+} \geq \lambda^{+} \vee \mu^{+}$ and $\lambda^{-} \circledast_{h} \mu^{-} \leq \lambda^{-} \wedge \mu^{-}$.

Proof. Let $B_1 = \left(\lambda^{+}, \lambda^{-}\right) \in BVAFRhI(R)$ and $B_2 = (\mu^{+}, \mu^{-}) \in BVAFLhI(R)$. Then, we have from Theorem 3.3.3, $\lambda^{+} \circledast_{h} C^{+}_{S^{c}} \geq \lambda^{+}$ and $\lambda^{-} \circledast_{h} C^{-}_{S^{c}} \leq \lambda^{-}$ we have $\lambda^{+} \circledast_{h} \mu^{+} \geq \lambda^{+} \circledast_{h} C^{+}_{S^{c}} \geq \lambda^{+}$ and $\lambda^{-} \circledast_{h} \mu^{-} \leq \lambda^{-} \circledast_{h} C^{-}_{S^{c}} \leq \lambda^{-}$. Moreover, $\lambda^{+} \circledast_{h} \mu^{+} \geq C^{+}_{S^{c}} \circledast_{h} \mu^{+} \geq \mu^{+}$ and $\lambda^{-} \circledast_{h} \mu^{-} \leq C^{-}_{S^{c}} \circledast_{h} \mu^{-} \leq \mu^{-}$. Hence $\lambda^{+} \circledast_{h} \mu^{+} \geq \lambda^{+} \vee \mu^{+}$ and $\lambda^{-} \circledast_{h} \mu^{-} \leq \lambda^{-} \wedge \mu^{-}$. ■

3.4 Bipolar-valued anti fuzzy h-quasi-ideals of hemirings

3.4.1 Definition

Let $B = \left(\lambda^{+}, \lambda^{-}\right)$ be a BVF subset of R. Then B is called BVAF h-quasi-ideal of R if and only if it satisafied for $p,\ q,\ z,\ s_1,\ s_2 \in R$,

$(W1)'$ $\lambda^{+}(p+q) \leq \max\left\{\lambda^{+}(p), \lambda^{+}(q)\right\}$, $\lambda^{-}(p+q) \geq \min\left\{\lambda^{-}(p), \lambda^{-}(q)\right\}$.

$(W3)'$ $p + s_1 + z = s_2 + z \implies \lambda^+(p) \leq \max\{\lambda^+(s_1), \lambda^+(s_2)\}$, $\lambda^-(p) \geq \min\{\lambda^-(s_1), \lambda^-(s_2)\}$

$(W10)'$ $(\lambda^+ \circledast_h C^+_{R^c}) \cup (C^+_{R^c} \circledast_h \lambda^+) \geq \lambda^+$, $(C^-_{R^c} \circledast_h \lambda^-) \cap (\lambda^- \circledast_h C^-_{R^c}) \leq \lambda^-$.

3.4.2 Remark

In rest of the thesis, we denote set of BVAF h-qusai-ideal of R by $BVAFhqI(R)$.

3.4.3 Theorem

Let $\emptyset \neq I \subseteq R$. Then $C_{I^c} \in BVAFhqI(R)$ iff I is a h-quasi-ideal of R.

3.4.4 Lemma.

Let $B = (\lambda^+, \lambda^-) \in BVAFRhI(R)$ and $B' = (\mu^+, \mu^-) \in BVAFLhI(R)$. Then $B \cup B' \in BVAFhqI(R)$.

3.4.5 Lemma.

If $B = (\lambda^+, \lambda^-) \in BVAFhqI(R)$ then $B = (\lambda^+, \lambda^-) \in BVAFhbI(R)$.

Proof. Let $B = (\lambda^+, \lambda^-) \in BVAFhqI(R)$. It is sufficient to prove $\lambda^+(xyz) \leq \max\{\lambda^+(x), \lambda^+(z)\}$, $\lambda^-(xyz) \geq \min\{\lambda^-(x), \lambda^-(z)\}$ and $\lambda^+(xy) \leq \max\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xy) \geq \min\{\lambda^-(x), \lambda^-(y)\}$ $\forall x, y, z \in R$. Now, we have

$$\lambda^+(xyz) \leq ((\lambda^+ \circledast_h C^+_R) \cup (C^+_R \circledast_h \lambda^+))(xyz)$$

$$= \max\{(\lambda^+ \circledast_h C^+_R)(xyz), (C^+_R \circledast_h \lambda^+)(xyz)\}$$

$$= \max \left\{ \begin{array}{l} \bigwedge\limits_{xyz+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigvee\limits_{i=1}^{m} (\lambda^+)(a_i)) \vee (\bigvee\limits_{j=1}^{n} (\lambda^+)(c_j)) \right\}, \\ \bigwedge\limits_{xyz+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigvee\limits_{i=1}^{m} (\lambda^+)(b_i)) \vee (\bigvee\limits_{j=1}^{n} (\lambda^+)(d_j)) \right\} \end{array} \right\}$$

$$\leq \max\{\max\{\lambda^+(0), \lambda^+(x)\}, \max\{\lambda^+(0), \lambda^+(z)\}\}$$

$$= \max\{\lambda^+(x), \lambda^+(z)\}.$$

Similarly, we have

$$\lambda^-(xyz) \geq ((\lambda^- \circledast_h C_R^-) \cap (C_R^- \circledast_h \lambda^-))(xyz)$$

$$= \min\{(\lambda^- \circledast_h C_R^-)(xyz), (C_R^- \circledast_h \lambda^-)(xyz)\}$$

$$= \min \left\{ \begin{array}{l} \bigvee\limits_{xyz+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigwedge\limits_{i=1}^{m} (\lambda^-)(a_i)) \wedge (\bigwedge\limits_{j=1}^{n} (\lambda^-)(c_j)) \right\}, \\ \bigvee\limits_{xyz+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigwedge\limits_{i=1}^{m} (\lambda^-)(b_i)) \wedge (\bigwedge\limits_{j=1}^{n} (\lambda^-)(d_j)) \right\} \end{array} \right\}$$

$$\geq \min\{\min\{\lambda^-(0), \lambda^-(x)\}, \min\{\lambda^-(0), \lambda^-(z)\}\}$$

$$= \min\{\lambda^-(x), \lambda^-(z)\}.$$

Hence $\lambda^+(xyz) \leq \max\{\lambda^+(x), \lambda^+(z)\}$, $\lambda^-(xyz) \geq \min\{\lambda^-(x), \lambda^-(z)\}$. Similarly, we can prove $\lambda^+(xy) \leq \max\{\lambda^+(x), \lambda^+(y)\}$, $\lambda^-(xy) \geq \min\{\lambda^-(x), \lambda^-(y)\}$. Therefore $B \in BVAFhqI(R)$. ∎

Converse of Lemma 3.4.5, is not true.

3.5 Characterization of hemirings by their bipolar-valued anti fuzzy h-ideals

In this section, we provide the concept of h-hemiregular and h-semisimple hemirings and provides their characterizations in term of BVAF h-ideals.

3.5.1 Theorem.

Let $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-)$ be two BVF subsets of R. Then we say $B_1 \backsim B_2$ if and only if $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$ and $\mu^+[\in]\lambda^+$, $\lambda^-[\in]\mu^-$.

The relation " $\backsim$ " is called equivalance relation on BVF subsets of R.

3.5.2 Theorem.

If $B_1 = (\lambda^+, \lambda^-) \in BVAFRhI(R)$ and $B_2 = (\mu^+, \mu^-) \in BVAFLhI(R)$, then $\lambda^+ \circledast_h \mu^+ \sim \lambda^+ \vee \mu^+$, $\lambda^- \circledast_h \mu^- \sim \lambda^- \wedge \mu^-$ iff R is h-hemiregular.

Proof. Let R be a h-hemiregular. Let $B_1 \in BVAFRhI(R)$ and $B_2 \in BVAFLhI(R)$. By Lemma 3.3.4, $(\lambda^+ \circledast_h \mu^+) \geq (\lambda^+ \vee \mu^+)$, $(\lambda^- \circledast_h \mu^-) \leq (\lambda^- \wedge \mu^-)$.and so by Theorem 2.4.4, $(\lambda^+ \vee \mu^+)[\in](\lambda^+ \circledast_h \mu^+)$, $(\lambda^- \circledast_h \mu^-)[\in](\lambda^- \wedge \mu^-)$. Now, since R is h-hemiregular, so $\forall s \in R$, $\exists\ p, q, z \in R$ such that $s + sps + z = sqs + z$. Thus $(\lambda^+ \circledast_h \mu^+)(s) =$

$$\bigwedge_{s+\sum_{i=1}^{n} p_i q_i + z = \sum_{j=1}^{m} r_j t_j + z} \left[(\bigvee_{i=1}^{n} \lambda^+(p_i)) \vee (\bigvee_{i=1}^{n} \mu^+(q_i)) \vee (\bigvee_{j=1}^{m} \lambda^+(r_j)) \vee (\bigvee_{j=1}^{m} \mu^+(t_j)) \right]$$

$$\leq \max\{\lambda^+(sp), \lambda^+(sq), \mu^+(s)\}$$

$$\leq \max\{\lambda^+(s), \mu^+(s)\}$$

$$= (\lambda^+ \vee \mu^+)(s)$$

$$(\lambda^+ \circledast_h \mu^+)(s) \leq (\lambda^+ \vee \mu^+)(s)$$

and

$$(\lambda^- \circledast_h \mu^-)(s) =$$

$$\bigvee_{s+\sum_{i=1}^{n} p_i q_i + z = \sum_{j=1}^{m} r_j t_j + z} \left[(\bigwedge_{i=1}^{n} \lambda^-(p_i)) \wedge (\bigwedge_{i=1}^{n} \mu^-(q_i)) \wedge (\bigwedge_{j=1}^{m} \lambda^-(r_j)) \wedge (\bigwedge_{j=1}^{m} \mu^-(t_j)) \right]$$

$$\geq \min\{\lambda^-(sp), \lambda^-(sq), \mu^-(s)\}$$

$$\geq \min\{\lambda^-(s), \mu^-(s)\}$$

$= (\lambda^- \wedge \mu^-)(s)$

$(\lambda^- \circledast_h \mu^-)(s) \geq (\lambda^- \vee \mu^-)(s)$.

Hence $(\lambda^+ \circledast_h \mu^+) \sim (\lambda^+ \vee \mu^+)$, $(\lambda^- \circledast_h \mu^-) \sim (\lambda^- \wedge \mu^-)$.

Conversely, let L be a right and M be a left h-ideal of R. Then $C_{L^c} = (C^+_{L^c}, C^-_{L^c}) \in BVAFRhI(R)$ and $C_{M^c} = (C^+_{M^c}, C^-_{M^c}) \in BVAFRhI(R)$. Then by Theorem 3.3.2, $C^+_{\overline{LM}^c} = C^+_{L^c} \circledast_h C^+_{M^c}$ and $C^+_{\overline{LM}^c} = C^+_{L^c} \circledast_h C^+_{M^c} \sim C^+_{L^c} \vee C^+_{M^c}$ and from 3.3.2, $C^+_{\overline{LM}^c} = C^+_{L^c} \circledast_h C^+_{M^c} \sim C^+_{L^c} \vee C^+_{M^c} = C^+_{L^c \cup M^c} \Longrightarrow \overline{LM}^c = L^c \cup M^c \Longrightarrow \overline{LM}^c = (L \cap M)^c \Longrightarrow \overline{LM} = (L \cap M)$ so R is h-hemiregular.

This complete the proof. ∎

3.5.3 Theorem.

If $B \in BVAFhbI(R)$, then $(\lambda^+ \circledast_h C^+_{R^c} \circledast_h \lambda^+) \leq \lambda^+$, $(\lambda^- \circledast_h C^-_{R^c} \circledast_h \lambda^-) \geq \lambda^-$ iff R is h-hemiregular.

Proof. Straightforward. ∎

3.5.4 Theorem.

If $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-) \in BVAFIhIs(R)$, then $(\lambda^+ \circledast_h \mu^+) \sim (\lambda^+ \vee \mu^+)$, $(\lambda^- \circledast_h \mu^-) \sim (\lambda^- \wedge \mu^-)$ iff R is h-semisimple.

Proof. Let $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-) \in BVAFIhIs(R)$. By Lemma 3.3.4, $(\lambda^+ \circledast_h \mu^+) \geq (\lambda^+ \vee \mu^+)$, $(\lambda^- \circledast_h \mu^-) \leq (\lambda^- \wedge \mu^-)$. And by Theorem 2.4.4, $(\lambda^+ \vee \mu^+) [\in] (\lambda^+ \circledast_h \mu^+)$, $(\lambda^- \circledast_h \mu^-) [\in] (\lambda^- \wedge \mu^-)$. Now since R is h-hemisimple, so $\forall x \in R$, $\exists$ c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, $f'_j \in R$ such that $x + \Sigma_{i=1}^{m} c_i x d_i e_i x f_i + z =$

$\Sigma_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z$. Thus

$$(\lambda^+ \circledast_h \mu^+)(x) =$$
$$\bigwedge_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left[(\bigvee_{i=1}^{m} \lambda^+(a_i)) \vee (\bigvee_{i=1}^{m} \mu^+(b_i)) \vee (\bigvee_{j=1}^{n} \lambda^+(a'_j)) \vee (\bigvee_{j=1}^{n} \mu^+(b'_j)) \right]$$
$$\leq \max\{\lambda^+(c_i x d_i), \lambda^+(c'_j x d'_j), \mu^+(e_i x f_i), \mu^+(e'_j x f'_j)\}$$
$$\leq (\lambda^+ \vee \mu^+)(x)$$

and

$$(\lambda^- \circledast_h \mu^-)(x) =$$
$$\bigvee_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left[(\bigwedge_{i=1}^{m} \lambda^-(a_i)) \wedge (\bigwedge_{i=1}^{m} \mu^-(b_i)) \wedge (\bigwedge_{j=1}^{n} \lambda^-(a'_j)) \wedge (\bigwedge_{j=1}^{n} \mu^-(b'_j)) \right]$$
$$\geq \min\{\lambda^-(c_i x d_i), \lambda^-(c'_j x d'_j), \mu^+(e_i x f_i), \mu^+(e'_j x f'_j)\}$$
$$\geq (\lambda^- \wedge \mu^-)(x)$$

Hence $(\lambda^+ \circledast_h \mu^+) \sim (\lambda^+ \vee \mu^+)$, $(\lambda^- \circledast_h \mu^-) \sim (\lambda^- \wedge \mu^-)$.

Convesely, let M be a h-ideal of R. Then from Theorem 3.1.12 & 3.1.13, $C_{M^c} = (C^+_{M^c}, C^-_{M^c}) \in BVAFIhI(R)$. Then, we have $C^+_{M^c} = C^+_{M^c} \vee C^+_{M^c}$ and from (ii) $C^+_{M^c} = C^+_{M^c} \circledast_h C^+_{M^c} \sim C^+_{M^c} \vee C^+_{M^c}$ and from Theorem 3.3.2, $C^+_{M^c} = C^+_{M^c} \vee C^+_{M^c} \sim C^+_{M^c} \circledast_h C^+_{M^c}$ $= C^+_{\overline{M^2}^c} \Longrightarrow M^c = \overline{M^2}^c \Longrightarrow M = \overline{M^2}$. Hence R is h-hemisimple. ■

3.6 Cheracterization of bipolar-valued anti fuzzy h-ideals in term of positive anti β-cuts and negative anti α-cuts

In this section, characterizations of BVAF h-ideals are investigated by means of positive anti-cut and negative anti-cut.

3.6.1 Definition.

Let $B = (\mu^+, \mu^-)$ be BVF subset of a hemiring R and $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, then

(1) The set $\widetilde{B}^+_\beta = \{x \in S : \mu^+(x) \leq \beta\}$ is called positive anti β-cut of B.

(2) The set $\widetilde{B}^-_\alpha = \{x \in S : \mu^-(x) \geq \alpha\}$ is called negative anti α-cut of B.

(3)The set $\widetilde{B}_{(\alpha,\beta)} = \{x \in S : \mu^-(x) \geq \alpha$ and $\mu^+(x) \leq \beta\}$ is called anti (α, β)-cut of B.

For every $\gamma \in (0, 1]$ and $\widetilde{B}^+_\gamma \cap \widetilde{B}^-_{-\gamma}$ is called anti γ-cut of B.

3.6.2 Theorem.

A BVF subset $B = (\mu^+, \mu^-) \in BVAFhI(R)$ iff the followings hold:

(i) $\widetilde{B}^+_\beta$ is non-empty this implies $\widetilde{B}^+_\beta$ is an h-ideal of R, $\forall\beta \in [0, 1]$.

(ii) B^-_α is non-empty this implies $\widetilde{B}^-_\alpha$ is an h-ideal of R, $\forall\alpha \in [-1, 0]$.

Proof. Let $B = (\mu^+, \mu^-) \in BVAFhI(R)$. For x, $y \in R$ and x, $y \in \widetilde{B}^+_\beta$ so $\mu^+(x) \leq \beta$, $\mu^+(y) \leq \beta$ where $\beta \in [0, 1]$. Now as $\mu^+(x + y) \leq \max\{\mu^+(x), \mu^+(y)\}$

$$\leq \max\{\beta, \beta\}$$

$$= \beta.$$

So that $x + y \in \widetilde{B}^+_\beta$.

Also for, $r \in R$, $x \in \widetilde{B}^+_\beta$, we have

$$\mu^+(rx) \leq \mu^+(x)$$

$$\leq \beta$$

So that $rx \in \widetilde{B}^+_\beta$, similarly, $rx \in \widetilde{B}^+_\beta$

Now, let y, $z \in R$ and s_1, $s_2 \in \widetilde{B}^+_\beta$ with $y + s_1 + z = s_2 + z \Longrightarrow \mu^+(y) \leq$

$\max\{\mu^+(s_1), \mu^+(s_2)\} \leq \beta$, this means that $y \in \widetilde{B}^+_\beta$. Hence $\widetilde{B}^+_\beta$ is an h-ideal of R. Likewise, we can prove (ii).

Conversely, suppose (i) and (ii) are hold. Suppose for any $x \in R$, if $\mu^+(x) = \beta$, $\mu^-(x) = \alpha$, $x \in \widetilde{B}^+_\beta \cap \widetilde{B}^-_\alpha$. Thus $\widetilde{B}^+_\beta \neq \emptyset$ and $\widetilde{B}^-_\alpha \neq \emptyset$. Suppose on contrary $B = (\mu^+, \mu^-) \overline{\in} BVAFhI(R)$, for y, z, s_1, $s_2 \in R$, such that $y + s_1 + z = s_2 + z$, $\mu^+(y) > \beta > \max\{\mu^+(s_1), \mu^+(s_2)\}$ and $\mu^-(y) < \alpha < \min\{\mu^-(s_1), \mu^-(s_2)\}$. This implies, s_1, $s_2 \in \widetilde{B}^+_\beta$, but $y \notin \widetilde{B}^+_\beta$ and s_1, $s_2 \in \widetilde{B}^-_\alpha$, but $y \notin \widetilde{B}^-_\alpha$. Which is a contradation. Hence $B = (\mu^+, \mu^-) \in BVAFhI(R)$. ∎

As critical importance of Theorem 3.6.2, we have to show more results regarding positive anti-cut and negative anti-cut. Which are discussed as follow:

3.6.3 Corollary.

If $B = (\mu^+, \mu^-) \in BVAFhI(R)$ then, $\forall \gamma \in [0, 1]$ the anti γ-cut of B is an h-ideal of R.

Proof. Suppose $B = (\mu^+, \mu^-) \in BVAFhI(R)$. Then we have to prove that the set

$\widetilde{B}_{(\gamma,-\gamma)} = \{s_1 \in R : \mu^-(s_1) \geq -\gamma$ and $\mu^+(s_1) \leq \gamma\}$ is h-ideal of R. Let s_1, $s_1 \in R$ and s_1, $s_2 \in \widetilde{B}_{(\gamma,-\gamma)}$. As $\mu^+(s_1 + s_2) \leq \max\{\mu^+(s_1), \mu^+(s_2)\} \leq \max\{\gamma, \gamma\} = \gamma$,

and $\mu^-(s_1 + s_2) \geq \min\{\mu^-(s_1), \mu^-(s_2)\} \geq \min\{-\gamma, -\gamma\} = -\gamma$. Therefore $s_1 + s_2 \in \widetilde{B}_{(\gamma,-\gamma)}$.

Now, let $s_1 \in \widetilde{B}_{(\gamma,-\gamma)}$, $r \in S$ so that $\mu^+(rs_1) \leq \mu^+(s_1) \leq \gamma$ and $\mu^-(rs_1) \geq \mu^-(s_1) \geq -\gamma$, $\mu^+(s_1 r) \leq \mu^+(s_1) \leq \gamma$ and $\mu^-(s_1 r) \geq \mu^-(s_1) \geq -\gamma$.

Now, let x, $z \in R$ and s_1, $s_2 \in \widetilde{B}_{(\gamma,-\gamma)}$, $x + s_1 + z = s_2 + z$ this implies $\mu^+(x) \leq \max\{\mu^+(s_1), \mu^+(s_2)\} \leq \gamma$

and $\mu^-(x) \geq \min\{\mu^-(s_1), \mu^-(s_2)\} \geq -\gamma$. Therefore $x \in \widetilde{B}_{(\gamma,-\gamma)}$. Hence $\widetilde{B}_{(\gamma,-\gamma)}$ is a h-ideal of R.

Conversely, suppose $\widetilde{B}_{(\gamma,-\gamma)}$ is an h-ideal of R. Assume that $B = (\mu^+, \mu^-) \overline{\in} BVAFhI(R)$, for y, z, s_1, $s_2 \in R$, so that $y + s_1 + z = s_2 + z$, $\mu^+(y) > \gamma > \mu^+(s_1) \vee \mu^+(s_2)$ and $\mu^-(y) < -\gamma < \mu^-(s_1) \wedge \mu^-(s_2)$. This implies $s_1, s_2 \in \widetilde{B}_{(\gamma,-\gamma)}$, but $y \notin \widetilde{B}_{(\gamma,-\gamma)}$. Which is a contradation. Hence $B = (\mu^+, \mu^-) \in BVAFhI(R)$. ■

3.6.4 Corollary.

If $B = (\mu^+, \mu^-) \in BVAFhI(R)$, then $\widetilde{B}_{(\alpha,\beta)}$ is an h-ideal of R $\forall (\alpha, \beta) \in [-1, 0] \times [0, 1]$.

Proof. Straightforward by using Corollary 3.6.3. ■

3.6.5 Theorem.

If $B = (\mu^+, \mu^-) \in BVAFhI(R)$ and $\mu^+(z) + \mu^-(z) \leq 0$ $\forall z \in R$, then $\widetilde{B}^+_\gamma \cup \widetilde{B}^-_{-\gamma}$ is an h-ideal of R , $\forall$ $\gamma \in [0, 1]$.

Proof. Suppose $B = (\mu^+, \mu^-) \in BVAFhI(R)$ and $\mu^+(z) + \mu^-(z) \leq 0$. Assume $\widetilde{B}^+_\gamma$ and $\widetilde{B}^-_{-\gamma}$ are non-empty $\forall \gamma \in [0, 1]$ and from Theorem 3.6.2, both are h-ideal of R. Let s_1, $s_2 \in \widetilde{B}^+_\gamma \cup \widetilde{B}^-_{-\gamma}$ and y, $z \in R$ with $z + s_1 + y = s_2 + y$. Here we have following cases to prove the theorem:

(i) $s_1 \in \widetilde{B}^+_\gamma$, $s_2 \in \widetilde{B}^-_{-\gamma}$, (ii) $s_2 \in \widetilde{B}^+_\gamma$, $s_1 \in \widetilde{B}^-_{-\gamma}$, (iii) $s_1 \in \widetilde{B}^+_\gamma$, $s_2 \in \widetilde{B}^+_\gamma$, (iv)

$s_1 \in \widetilde{B}^-_{-\gamma}$, $s_2 \in \widetilde{B}^-_{-\gamma}$.

Case (i) $s_1 \in \widetilde{B}^+_\gamma$, $s_2 \in \widetilde{B}^-_{-\gamma}$, *i.e*, $\mu^+(s_1) \leq \gamma$ and $\mu^-(s_2) \geq -\gamma$. Since $\mu^+(s_2) + \mu^-(s_2) \leq 0$, $\mu^+(s_2) \leq -\mu^-(s_2) \leq \gamma$, so that we have $\mu^+(s_1 + s_2) \leq \max\{\mu^+(s_1), \mu^+(s_2)\} \leq \max\{\mu^+(s_1), -\mu^-(s_2)\} \leq \gamma$, $\mu^+(zs_1) \leq \mu^+(s_1) \leq \gamma$ (resp., $\mu^+(s_1z) \leq \mu^+(s_1) \leq \gamma$) and $\mu^+(z) \leq \max\{\mu^+(s_1), \mu^+(s_2)\} \leq \max\{\mu^+(s_1), -\mu^-(s_2)\} \leq \gamma$. Therefore s_1+s_2, zs_1, s_1z, $z \in \widetilde{B}^+_\gamma \subseteq \widetilde{B}^+_\gamma \cup \widetilde{B}^-_{-\gamma}$ this prove the (i).

Similarly, we can prove (ii).

Now, we can prove (iii) $s_1 \in \widetilde{B}^+_\gamma$, $s_2 \in \widetilde{B}^+_\gamma$, *i.e*, $\mu^+(s_1) \leq \gamma$ and $\mu^+(s_2) \leq \gamma$. As $B = (\mu^+, \mu^-) \in BVAFhI(R)$, $\mu^+(s_1 + s_2) \leq \max\{\mu^+(s_1), \mu^+(s_2)\} \leq \gamma$, $\mu^+(zs_1) \leq \mu^+(s_1) \leq \gamma$ (resp., $\mu^+(s_1z) \leq \mu^+(s_1) \leq \gamma$) and $\mu^+(z) \leq \max\{\mu^+(s_1), \mu^+(s_2)\} \leq \gamma$. Therefore $s_1 + s_2$, zs_1, s_1z, $z \in \widetilde{B}^+_\gamma \subseteq \widetilde{B}^+_\gamma \cup \widetilde{B}^-_{-\gamma}$ this prove the (iii). Similarly we can prove (iv). Therefore $\widetilde{B}^+_\gamma \cup \widetilde{B}^-_{-\gamma}$ is an h-ideal of R. ■

3.7 The anti image and anti pre-images of bipolar-valued anti fuzzy h-ideals

In this section, we introduce the notion of anti image and anti pre-image of BVAF h-ideal, and discuss its properties.

3.7.1 Definition.

Let $\psi : R_1 \to R_2$ be a maping of hemirings. If $B = (\mu^+, \mu^-)$ and $V = (\upsilon^+, \upsilon^-)$ are BVF subset of R_1 and R_2 respectively. Then the anti image of B under ψ is a BVF subset $\psi_a(B) = (\psi_a(\mu^+), \psi_a(\mu^-))$ of R_2, given by for each $x \in R_2$

$$\psi_a(\mu^+)(x) = \begin{cases} \bigwedge_{y\in\psi^{-1}(x)} \mu^+(y) & \text{if } \psi^{-1}(x) \neq \emptyset \\ 1 & \text{Otherwise} \end{cases},$$

and

$$\psi_a(\mu^-)(x) = \begin{cases} \bigvee_{y\in\psi^{-1}(x)} \mu^-(y) & \text{if } \psi^{-1}(x) \neq \emptyset \\ -1 & \text{Otherwise} \end{cases}$$

for all $x \in R_2$, $\psi^{-1}(x) = \{z \in R_1 \mid \psi(z) = x\}$. Also the anti pre-image $\psi^{-1}(V)$ of V under ψ is a BVF subset of R_1 defined by for $y \in R_1$, $\psi^{-1}((\upsilon^+)(y)) = \upsilon^+(\psi(y))$, $\psi^{-1}(\upsilon^-(y))) = \upsilon^-(\psi(y))$.

3.7.2 Theorem.

Let $\psi : R_1 \to R_2$ be a homomorphism of hemirings. If $B = (\mu^+, \mu^-)$ and $V = (\upsilon^+, \upsilon^-)$ are BVF subset of R_1 and R_2 respectively, then

(i) $[\psi^{-1}(V)]^c = \psi^{-1}(V^c)$

(ii) $[\psi_a(B)]^c = \psi_a(B^c)$.

3.7.3 Theorem.

Let ψ be a homomorphism from a hemiring R_1 into a hemiring R_2, and $V = (\upsilon^+, \upsilon^-) \in BVAFhI(R_2)$, then the anti pre-image $\psi^{-1}(V) = \left(\psi^{-1}(\upsilon^+), \psi^{-1}(\upsilon^-)\right) \in BVAFhI(R_1)$.

Proof. Suppose that $V = (\upsilon^+, \upsilon^-) \in BVAFhI(R_2)$. Then $\forall p, q \in R_1$, we have

$$\begin{aligned} \psi^{-1}(\upsilon^+)(p+q) = \upsilon^+(\psi(p+q)) = \\ &= \upsilon^+(\psi(p) + \psi(q)) \\ &\leq \max\{\upsilon^+\psi(p)), \upsilon^+(\psi(q))\} \end{aligned}$$

$$= \max\{\psi^{-1}(\upsilon^{+})(p), \psi^{-1}(\upsilon^{+})(q)\}$$

$$\psi^{-1}(\upsilon^{+})(p+q) \leq \max\{\psi^{-1}(\upsilon^{+})(p), \psi^{-1}(\upsilon^{+})(q)\},$$

and

$$\psi^{-1}(\upsilon^{-})(p+q) = \upsilon^{-}(\psi(p+q))$$

$$= \upsilon^{-}(\psi(p) + \psi(q))$$

$$\geq \min\{\upsilon^{-}(\psi(p)), \upsilon^{-}(\psi(q))\}$$

$$= \min\{\psi^{-1}(\upsilon^{-})(p), \psi^{-1}(\upsilon^{-})(q)\}$$

$$\psi^{-1}(\upsilon^{-})(p+q) \geq \min\{\psi^{-1}(\upsilon^{-})(p), \psi^{-1}(\upsilon^{-})(q)\}.$$

Now, let s_1, s_2, y, $z \in R_1$ with $y+s_1+z = s_2+z$, therefore $\psi(y)+\psi(s_1)+\psi(z) = \psi(s_2) + \psi(z)$ and

$$\psi^{-1}(\upsilon^{+})(y) = \upsilon^{+}(\psi(y))$$

$$\leq \max\{\upsilon^{+}\psi(s_1)), \upsilon^{+}(\psi(s_2))\}$$

$$= \max\{\psi^{-1}(\upsilon^{+})(s_1), \psi^{-1}(\upsilon^{+})(s_2)\}$$

$$\psi^{-1}(\upsilon^{+})(y) \leq \max\{\psi^{-1}(\upsilon^{+})(s_1), \psi^{-1}(\upsilon^{+})(s_2)\},$$

and similarly, $\psi^{-1}(\upsilon^{-})(y) \geq \min\{\psi^{-1}(\upsilon^{-})(s_1), \psi^{-1}(\upsilon^{-})(s_2)\}$.

Moreover, for y, $s \in R_1$, $\psi^{-1}(\upsilon^{+})(sy) = \upsilon^{+}(\psi(sy)) = \upsilon^{+}((\psi(s))(\psi(y))) \leq \upsilon^{+}(\psi(y)) = \psi^{-1}(\upsilon^{+})(y)$, and $\psi^{-1}(\upsilon^{+})(ys) \leq \psi^{-1}(\upsilon^{+})(y)$. Similarly, $\psi^{-1}(\upsilon^{-})(sy) = \upsilon^{-}(\psi(sy)) = \upsilon^{-}((\psi(s))(\psi(y))) \geq \upsilon^{-}(\psi(y)) = \psi^{-1}(\upsilon^{-})(y)$, and $\psi^{-1}(\upsilon^{-})(ys) \geq \psi^{-1}(\upsilon^{-})(y)$.

Hence the anti pre-image $\psi^{-1}(V) = (\psi^{-1}(\upsilon^{+}), \psi^{-1}(\upsilon^{-})) \in BVAFhI(R_1)$. ■

3.7.4 Theorem.

Let ψ be an anti homomorphism from a hemiring R_1 into a hemiring R_2. If $V = (\upsilon^+, \upsilon^-) \in BVAFRhI(R_2)$ then the anti pre-image $\psi^{-1}(V) = (\psi^{-1}(\upsilon^+), \psi^{-1}(\upsilon^-)) \in BVAFLhI(R_1)$.

Proof. Let $V = (\upsilon^+, \upsilon^-) \in BVAFRhI(R_2)$. Then $\forall p, q \in R_1$, we have

$$\psi^{-1}(\upsilon^+)(p+q) = \upsilon^+(\psi(p+q))$$

$$= \upsilon^+(\psi(q) + \psi(p))$$

$$\leq \max\{\upsilon^+\psi(p)), \upsilon^+(\psi(q))\}$$

$$= \max\{\psi^{-1}(\upsilon^+)(p), \psi^{-1}(\upsilon^+)(q)\}$$

$$\psi^{-1}(\upsilon^+)(p+q) \leq \max\{\psi^{-1}(\upsilon^+)(p), \psi^{-1}(\upsilon^+)(q)\}$$

and

$$\psi^{-1}(\upsilon^-)(p+q) = \upsilon^-(\psi(p+q))$$

$$= \upsilon^-(\psi(q) + \psi(p))$$

$$\geq \min\{\upsilon^-\psi(p)), \upsilon^-(\psi(q))\}$$

$$= \min\{\psi^{-1}(\upsilon^-)(p), \psi^{-1}(\upsilon^-)(q)$$

$$\psi^{-1}(\upsilon^-)(p+q) \geq \min\{\psi^{-1}(\upsilon^-)(p), \psi^{-1}(\upsilon^-)(q).$$

Now, let $s_1, s_2, x, y \in R_1$ with $x+s_1+y = s_2+y$, therefore $\psi(x)+\psi(s_1)+\psi(y) = \psi(s_2)+\psi(y)$ and

$$\psi^{-1}(\upsilon^+)(x) = \upsilon^+(\psi(x))$$

$$\leq \max\{\upsilon^+(\psi(s_1)), \upsilon^+(\psi(s_2))\}$$

$$= \max\{\psi^{-1}(\upsilon^+)(s_1), \psi^{-1}(\upsilon^+)(s_2)\}$$

$$\psi^{-1}(\upsilon^+)(x) \leq \max\{\psi^{-1}(\upsilon^+)(s_1), \psi^{-1}(\upsilon^+)(s_2)\}$$

and similarly, $\psi^{-1}(\upsilon^{-})(x) \geq \max\{\psi^{-1}(\upsilon^{-})(s_1), \psi^{-1}(\upsilon^{-})(s_2)\}$.

Moreover, let for x, $r \in R_1$ and

$$\begin{aligned}\psi^{-1}(\upsilon^{+})(rx) &= \upsilon^{+}(\psi(rx)) \\ &= \upsilon^{+}((\psi(x))(\psi(r))) \\ &\leq \upsilon^{+}(\psi(x)) \\ &= \psi^{-1}(\upsilon^{+})(x).\end{aligned}$$

Similarly
$$\begin{aligned}\psi^{-1}(\upsilon^{-})(rx) &= \upsilon^{-}(\psi(rx)) \\ &= \upsilon^{-}((\psi(x))(\psi(r))) \\ &\geq \upsilon^{-}(\psi(x)) \\ &= \psi^{-1}(\upsilon^{-})(x).\end{aligned}$$

Hence the anti pre-image $\psi^{-1}(V) = (\psi^{-1}(V), \psi^{-1}(V)) \in BVAFLhI(R_1)$. ■

3.7.5 Remark.

From Theorem 3.7.4, if $V = (\upsilon^{+}, \upsilon^{-}) \in BVAFLhI(R_2)$ then the anti pre-image $\psi^{-1}(V) = (\psi^{-1}(\upsilon^{+}), \psi^{-1}(\upsilon^{-})) \in BVAFRhI(R_1)$.

3.7.6 Theorem.

Let ψ be an anti epimorphism from a hemiring R_1 to a hemiring R_2, $A = (\mu^{+}, \mu^{-}) \in BVAFhI(R_1)$, then the anti image of A, $\psi_a(A) = (\psi_a(\mu^{+}), \psi_a(\mu^{-})) \in BVAFhI(R_2)$.

Proof. Let ψ be an anti epimorphism from a hemiring R_1 to a hemiring R_2, $A = (\mu^{+}, \mu^{-}) \in BVAFhI(R_1)$. Let $\alpha \in [-1, 0]$ and $\beta \in [0, 1]$. Let $\emptyset \neq \psi_a(\widetilde{A}^{+}_{\beta})$, $\emptyset \neq \psi_a(\widetilde{A}^{-}_{\alpha}) \subseteq R_2$. Let x_1, $x_2 \in \psi_a(\widetilde{A}^{+}_{\beta})$ so $\psi_a(\mu^{+}(x_1)) = \min\{\mu^{+}(y) : y \in \psi^{-1}(x_1)\} \leq \beta$

and $\psi_a(\mu^+(x_2)) = \min\{\mu^+(y) : y \in \psi^{-1}(x_2)\} \leq \beta$. Therefore, $y_1 \in \psi^{-1}(x_1)$ and $y_2 \in \psi^{-1}(x_2)$. Then

$$\begin{aligned}\psi_a(\mu^+(x_1+x_2)) &= \min\{\mu^+(y) : y \in \psi^{-1}(x_1+x_2)\}\\ &\leq \mu^+(y_1+y_2)\\ &\leq \max\{\mu^+(y_1), \mu^+(y_2)\} \leq \beta.\end{aligned}$$

This implies $x_1 + x_2 \in \psi_a(\widetilde{A}^+_\beta)$. Now let $x_0 \in \psi_a(\widetilde{A}^+_\beta)$, then $\psi_a(\mu^+(x_0)) = \min\{\mu^+(y) : y \in \psi^{-1}(x_0)\} \leq \beta$, which implies there exists $y_0 \in \psi^{-1}(x_0)$ such that $\mu^+(y_0) \leq \beta$. As $A = (\mu^+, \mu^-) \in BVAFhI(R_1)$. For all $x \in R_2$, since ψ be an anti epimorphism from a hemiring R_1 to a hemiring R_2, then $\exists\ y \in R_1$ such that $\psi_a(y) = x$. Let $\mu^+(yy_0) \leq \mu^+(y_0)$ and $\mu^+(y_0y) \leq \mu^+(y_0)$). Therefore $\psi(\mu^+(xx_0)) = \min\{\mu^+(y) : y \in \psi^{-1}(x_0x)\} \leq \mu^+(yy_0) \leq \mu^+(y_0) \leq \beta$. Therefore, $xx_0 \in \psi(\widetilde{A}^+_\beta)$ (resp., $x_0x \in \psi(\widetilde{A}^+_\beta)$). Now let any x, $z \in R_2$ and x'_1, $x'_2 \in \psi(\widetilde{A}^+_\beta)$, then $x + x'_1 + z = x'_2 + z$. This implies, $\psi_a(\mu^+(x'_1)) = \min\{\mu^+(y) : y \in \psi^{-1}(x'_1)\} \leq \beta$ and $\psi_a(\mu^+(x'_2)) = \min\{\mu^+(y) : y \in \psi^{-1}(x'_2)\} \leq \beta$. Let $\psi^{-1}(x + x'_1 + z) = \psi^{-1}(x'_2 + z)$, i.e, $\psi^{-1}(x) + \psi^{-1}(x'_1) + \psi^{-1}(z) = \psi^{-1}(x'_2) + \psi^{-1}(z)$. So $y_0 \in \psi^{-1}(x)$, $y_1 \in \psi^{-1}(x'_1)$, $y_2 \in \psi^{-1}(x'_2)$, $y_3 \in \psi^{-1}(z)$, such that $y_0 + y_1 + y_3 = y_2 + y_3$ and $\mu^+(y_1) \leq \beta$, $\mu^+(y_2) \leq \beta$.

Therefore,

$$\begin{aligned}\psi_a(\mu^+(x)) &= \min\{\mu^+(y) : y \in \psi^{-1}(x)\}\\ &\leq \mu^+(y_0)\\ &\leq \max\{(\mu^+)(y_1), (\mu^+)(y_2)\} \leq \beta.\end{aligned}$$

Thus $y \in \psi_a(\widetilde{A}^+_\beta)$. Therefore, $\psi_a(\widetilde{A}^+_\beta)$ is an h-ideal of R_2. Similarly, we can show that $\psi_a(\widetilde{A}^-_\alpha)$ is an h-ideal of R_2. Hence from Theorem 3.6.2, $\psi_a(A) = (\psi_a(\mu^+), \psi_a(\mu^-)) \in$

$BVAFhI(R_2)$. ∎

3.7.7 Theorem.

Let ψ be an anti epimorphism from a hemiring R_1 to a hemiring R_2, $A = (\mu^+, \mu^-) \in BVAFhI(R_1)$. Then anti image of A^c, $[\psi_a(A)]^c \in BVAFhI(R_2)$.

Proof. Let ψ be an anti epimorphism from a hemiring R_1 to a hemiring R_2. Let $A = (\mu^+, \mu^-) \in BVAFhI(R_1)$. Then by Proposition 3.2.2, $A^c \in BVAFhI(R_1)$. Now by Theorem 3.7.6, $\psi_a(A^c) = (\psi_a((\mu^c)^+), \psi_a((\mu^c)^-)) \in BVAFhI(R_2)$. Hence by Theorem 3.7.2,

$[\psi_a(A)]^c \in BVAFhI(R_2)$. ∎

3.8 Equivalence relations on bipolar-valued anti fuzzy h-ideals.

In this section, we defined some realation on BVAF h-ideals by using positive anti-cut and negative anti-cut.

3.8.1 Definition.

For any $(\alpha,\beta) \in [0,1] \times [-1,0]$. Define two binary relations P^β and N^α on BVAF h-ideals of R as follows:

$(A,B) \in P^\beta \Leftrightarrow \widetilde{A}^+_\beta = \widetilde{B}^+_\beta$ and $(A,B) \in N^\alpha \Leftrightarrow \widetilde{A}^-_\alpha = \widetilde{B}^-_\alpha$ $\forall$ $A = (\lambda^+, \lambda^-)$ and $B = (\mu^+, \mu^-) \in BVAFhI(R)$. It is easy to check P^β and N^α are equivelance relations on

BVAF h-ideals of R. We have $[A]_{P^\beta}$ express the class of modular P^β and $[A]_{N^\alpha}$ express the class of modular N^α denoted by $BVAFhI(R)/P^\beta$ and $BVAFhI(R)/N^\alpha$. Let $I(R)$ be the family of all h-ideals of R. Define maps

$$g_\beta : BVAFhI(R) \longrightarrow I(R) \cup \{\emptyset\},\ A \longrightarrow \widetilde{A}^+_\beta$$

$$h_\alpha : BVAFhI(R) \longrightarrow I(R) \cup \{\emptyset\},\ A \longrightarrow \widetilde{A}^-_\alpha$$

$\forall A = (\lambda^+, \lambda^-) \in BVAFhI(R)$. Then g_β and h_α are well-defined.

3.8.2 Theorem.

The maps g_β and h_α are surjective for any $(\alpha,\beta) \in [0,1] \times [-1,0]$.

Proof. Let $C_{p^c} = (C^+_{p^c}, C^-_{p^c})$ be a BVF set in R, for $P \neq \emptyset$ in $I(R)$.

$$C^+_{p^c} = \begin{cases} 0 & \text{if } x \in P \\ 1 & \text{if otherwise} \end{cases}$$

$$C^-_{p^c} = \begin{cases} 0 & \text{if } x \in P \\ -1 & \text{otherwise} \end{cases}$$

from Theorem 3.1.12, $C_{p^c} = (C^+_{p^c}, C^-_{p^c}) \in BVAFhI(R)$. We have

$$g_\beta(C_{p^c}) = \widetilde{(C_{p^c})}^+_\beta = \{x \in P \mid C^+_{p^c}(x) \leq \beta\} = \{x \in P \mid C^+_{p^c}(x) = 0\} = P \text{ and}$$

$$h_\alpha(C_{p^c}) = \widetilde{(C_{p^c})}^-_\alpha = \{x \in P \mid C^-_{p^c}(x) \geq \alpha\} = \{x \in P \mid C^-_{p^c}(x) = 0\} = P.$$

Now, as $1 = (1^+, 1^-) \in BVAFhI(R)$, where $1^+(x) = 1$ and $1^-(x) = -1$. Then $g_\beta(1) = \widetilde{(1)}^+_\beta = \{x \in R \mid 1^+(x) \leq \beta\} = \emptyset$, and $h_\alpha(1) = \widetilde{(1)}^-_\alpha = \{x \in R \mid 1^-(x) \geq \alpha\} = \emptyset$. Therefore, g_β and h_α are surjective. ■

3.8.3 Theorem.

The sets $BVAFhI(R)/P^{\beta}$ and $BVAFhI(R)/N^{\alpha}$ are equipotent to $I(R) \cup \{\emptyset\}$ for all $(\alpha,\beta) \in [0,1] \times [-1,0]$.

Proof. For all $(\alpha,\beta) \in [0,1] \times [-1,0]$ and let $A = (\mu^+, \mu^-) \in BVAFhI(R)$. Let us suppose $g'_\beta : BVAFhI(R)/P^{\beta} \longrightarrow I(R) \cup \{\emptyset\}, [A]_{P^\beta} \longrightarrow g_\beta(A)$,

$h'_\alpha : BVAFhI(R)/N^{\alpha} \longrightarrow I(R) \cup \{\emptyset\}, [A]_{P^\alpha} \longrightarrow h_\alpha(A)$.

respectively. For every $A = (\mu^+, \mu^-)$,and $B = (\lambda^+, \lambda^-) \in BVAFhI(R)$, if $\mu^+ = \lambda^+$ and $\mu^- = \lambda^-$, then $(A,B) \in P^{\beta}$ and $(A,B) \in P^{\alpha}$. Therefore, $[A]_{P^\beta} = [B]_{P^\beta}$ and $[A]_{P^\alpha} = [B]_{P^\alpha}$, so g_β and h_α is injective. By Theorem 3.8.2, g'_β and h'_α are surjective. This completes the proof. ■

3.8.4 Theorem.

Let $0 < \gamma < 1$, then the map $g_\gamma : BVAFhI(R) \longrightarrow I(R) \cup \{\emptyset\}$, define by $g_\gamma(A) \longrightarrow A_\gamma$, is surjective.

Proof. Let $0 < \gamma < 1$, we have $g_\gamma(1) = 1^+_\beta \cap 1^-_\alpha = \emptyset$. Let $C_{p^c} = (C^+_{p^c}, C^-_{p^c})$ be a BVF set in R, for $P \neq \emptyset$ in $I(R)$. By Theorem 3.1.12, $C_{p^c} = (C^+_{p^c}, C^-_{p^c}) \in BVAFhI(R)$. We have

$g_\gamma(C_{p^c}) = \widetilde{(C_{p^c})}^+_\gamma \cap \widetilde{(C_{p^c})}^-_{-\gamma}$

$= \{x \in P \mid C^+_{p^c}(x) \leq \gamma\} \cap \{x \in P \mid C^-_{p^c}(x) \geq -\gamma\}$

$= P$. Therefore, g_γ is surjective. This completes the proof. ■

3.8.5 Theorem.

Let $0 < \gamma < 1$, then the quotion set $BVAFhI(R)/R^{\gamma}$ is equipotent to $I(R) \cup \{\emptyset\}$.

Proof. Let $0 < \gamma < 1$, and $g'_{\gamma} : BVAFhI(R)/R^{\gamma} \longrightarrow I(R) \cup \{\emptyset\}$ is a map defined by $g'_{\gamma}([A_{R^{\gamma}}]) = g_{\gamma}(A)$ for all $[A_{R^{\gamma}}] \in BVAFhI(R)/R^{\gamma}$. Let $g'_{\gamma}([A_{R^{\gamma}}]) = g'_{\gamma}([B_{R^{\gamma}}])$ for every $[A_{R^{\gamma}}]$, $[B_{R^{\gamma}}] \in BVAFhI(R)/R^{\gamma}$, then $g_{\gamma}(A) = g_{\gamma}(B)$, *i.e.*, $A_{\gamma} = B_{\gamma}$. Therefore, $(A, B) \in R^{\gamma}$. Hence $[A_{R^{\gamma}}] = [B_{R^{\gamma}}]$ so g'_{γ} is injective.

Now, we have $g'_{\gamma}([1]_{R^{\gamma}}) = g_{\gamma}(1) = \widetilde{1}^{+}_{\beta} \cap \widetilde{1}^{-}_{\alpha} = \emptyset$. Now, suppose for any non-empty P in $I(R)$, consider a BVF set $C_{p^c} = (C^{+}_{p^c}, C^{-}_{p^c})$. By Teorem 3.1.12, $C_{p^c} = (C^{+}_{p^c}, C^{-}_{p^c}) \in BVAFhI(R)$. We have

$g'_{\gamma}([C_{p^c}]_{R^{\gamma}}) = g_{\gamma}(C_{p^c}) = \widetilde{(C_{p^c})}^{+}_{\gamma} \cap \widetilde{(C_{p^c})}^{-}_{-\gamma}$

$= \{x \in P \mid C^{+}_{p^c}(x) \leq \gamma\} \cap \{x \in P \mid C^{-}_{p^c}(x) \geq -\gamma\}$

$= P$. Therefore, g'_{γ} is surjective. This completes the proof. ■

3.9 Normal bipolar anti fuzzy h-ideal

In this section, we introduced and characterized normal BVAF h-ideals in R. By Definition 3.1.1, we have $C_{M^C} = (C^{+}_{M^C}, C^{-}_{M^C})$ is a normal BVAF h-ideals of R giving that $C^{+}_{M^C}(x) = 1$ and $C^{-}_{M^C} = -1$ for all $x \notin M$. However, as a general rule, $C^{+}_{M^C}(x) = 1$ and $C^{+}_{M^C} = -1$ may not always true. Therefore it is necessary for us to define following definition.

3.9.1 Definition.

Let $B \in BVAFhI(R)$. Then B is called normal if $\exists$ an element $x \in R$ such that $B(x) = (1, -1)$.

Set of all normal BVAF h-ideals in R is denoted by $NBVAFhI(R)$.

3.9.2 Proposition.

Let $B \in BVAFhI(R)$. Then B is called normal iff $B(0) = (1, -1)$. *i.e.*, $\mu^+(0) = 1$ and $\mu^-(0) = -1$.

3.9.3 Theorem.

Let $B = (\mu^+, \mu^-) \in BVAFhI(R)$. Let $\widehat{B} = (\widehat{\mu}^+, \widehat{\mu}^-)$ be a BVF set in R defined by $\widehat{\mu}^+(x) = \mu^+(x) + 1 - \mu^+(0)$ and $\widehat{\mu}^-(x) = \mu^-(x) - 1 - \mu^-(0)$ $\forall x \in R$. Then $\widehat{B} = (\widehat{\mu}^+, \widehat{\mu}^-) \in NBVAFhI(R)$ which contains B.

Proof. Let $B = (\mu^+, \mu^-) \in BVAFhI(R)$. For 0, x, $y \in R$, we have $\widehat{\mu}^+(0) = \mu^+(0) + 1 - \mu^+(0) = 1$ and $\widehat{\mu}^-(0) = \mu^-(0) - 1 - \mu^-(0) = -1$.

$$\begin{aligned}
\text{Now, } \widehat{\mu}^+(x+y) &= \mu^+(x+y) + 1 - \mu^+(0) \\
&\leq \max\{\mu^+(x), \mu^+(y)\} + 1 - \mu^+(0) \\
&= \max\{\mu^+(x) + 1 - \mu^+(0), \mu^+(y) + 1 - \mu^+(0)\} \\
&= \max\{\widehat{\mu}^+(x), \widehat{\mu}^+(y)\},
\end{aligned}$$

$$\begin{aligned}
\text{and } \widehat{\mu}^-(x+y) &= \mu^-(x+y) - 1 - \mu^-(0) \\
&\geq \min\{\mu^-(x), \mu^-(y)\} - 1 - \mu^-(0) \\
&= \min\{\mu^-(x) - 1 - \mu^-(0), \mu^-(y) - 1 - \mu^-(0)\}
\end{aligned}$$

$= \min\{\widehat{\mu}^-(x), \widehat{\mu}^-(y)\}.$

Now, let r, $x \in R$,

$\widehat{\mu}^+(rx) = \mu^+(rx) + 1 - \mu^+(0)$

$\leq \mu^+(x) + 1 - \mu^+(0)$

$= \widehat{\mu}^+(x).$

Similarly, $\widehat{\mu}^+(xr) \leq \widehat{\mu}^+(x)$.

And $\widehat{\mu}^-(rx) = \mu^-(rx) - 1 - \mu^-(0)$

$\leq \mu^-(x) - 1 - \mu^-(0)$

$= \widehat{\mu}^-(x).$

Similarly, $\widehat{\mu}^-(xr) \leq \widehat{\mu}^-(x)$.

Now, let y, z, s_1, $s_2 \in R$ such that $y + s_1 + z = s_2 + z$. Then $\widehat{\mu}^+(y) = \mu^+(y) + 1 - \mu^+(0)$

$\leq \max\{\mu^+(s_1), \mu^+(s_2)\} + 1 - \mu^+(0)$

$= \max\{\mu^+(s_1) + 1 - \mu^+(0), \mu^+(s_2) + 1 - \mu^+(0)\}$

$= \max\{\widehat{\mu}^+(s_1), \widehat{\mu}^+(s_2)\}$

and $\widehat{\mu}^-(y) = \mu^-(y) - 1 - \mu^-(0)$

$\geq \min\{\mu^-(s_1), \mu^-(s_2)\} - 1 - \mu^-(0)$

$= \min\{\mu^-(s_1) - 1 - \mu^-(0), \mu^-(s_2) - 1 - \mu^-(0)\}$

$= \min\{\widehat{\mu}^-(s_1), \widehat{\mu}^-(s_2)\}.$

Hence $\widehat{B} = (\widehat{\mu}^+, \widehat{\mu}^-) \in NBVAFhI(R)$ which contains B. ■

3.9.4 Corollary.

From the Theorem 3.9.3, if $B \in NBVAFhI(R)$, then $B = \widehat{B}$. Note that $NBVAFhI(R)$ is a poset under the set inclusion.

3.9.5 Definition.

A BVF set B in R is called a maximal BVAF h-ideal of R if it is non-constant and $\widehat{B}$ is a maximal element in $(NBVAFhI(R), \subseteq)$.

3.9.6 Theorem.

Let $B \in NBVAFhI(R)$ be non-constant such that it is a maximal element of $(NBVAFhI(R), \subseteq)$. Then it takes only values among $(0,0)$, $(0,-1)$, $(1,-1)$.

Proof. Suppose $B = (\mu^+, \mu^-) \in NBVAFhI(R)$ be non-constant and it is a maximal element of $(NBVAFhI(R), \subseteq)$. This implies $B \in NBVAFhI(R)$. Then $\mu^+(0) = 1$ and $\mu^-(0) = -1$. We claim that for $x \in R$, $\mu^+(x) \neq 1$ and $\mu^-(x) \neq -1$. Then $\mu^+(x) = 0$ and $\mu^-(x) = 0$. Contrarily, there exist $z \in R$ such that $0 < \mu^+(z) < 1$ and $0 > \mu^-(z) > -1$. Now, we define BVF set $V = (\upsilon^+, \upsilon^-)$ of R as $\upsilon^+(x) = \frac{1}{2}(\mu^+(x) + \mu^+(z))$, $\upsilon^-(x) = \frac{1}{2}(\mu^-(x) + \mu^-(z))$ for all $x \in R$. Then, we have surely V is well-defined. Therefore, for x, $y \in R$, have

$$
\begin{aligned}
\upsilon^+(x+y) &= \tfrac{1}{2}(\mu^+(x+y) + \mu^+(z)) \\
&\leq \tfrac{1}{2}(\{\max\{(\mu^+(x), \mu^+(y)\} + \mu^+(z)) \\
&= \max\{\tfrac{1}{2}((\mu^+(x) + \mu^+(z)), \tfrac{1}{2}(\mu^+(y) + \mu^+(z))\} \\
&= \max\{\upsilon^+(x), \upsilon^+(y)\}
\end{aligned}
$$

and $\upsilon^-(x+y) = \frac{1}{2}(\mu^-(x+y) + \mu^-(z))$

$$\geq \tfrac{1}{2}(\{\min\{(\mu^-(x), \mu^-(y)\} + \mu^-(z))$$

$$= \min\{\tfrac{1}{2}((\mu^-(x) + \mu^-(z)), \tfrac{1}{2}(\mu^-(y) + \mu^-(z))\}$$

$$= \min\{\upsilon^-(x), \upsilon^-(y)\}.$$

Now, $\upsilon^+(xy) = \frac{1}{2}(\mu^+(xy) + \mu^+(z))$

$$\leq \tfrac{1}{2}(\mu^+(y) + \mu^+(z)) = \upsilon^+(y) \text{ (resp., } \upsilon^+(xy) \leq \upsilon^+(x))$$

and similarly, $\upsilon^-(xy) \geq \upsilon^-(x)$ (resp., $\upsilon^-(xy) \geq \upsilon^-(x)$).

Futhermore, let x, z, a, $b \in R$ such that $x + a + z = b + z$, then

$$\upsilon^+(x) = \tfrac{1}{2}(\mu^+(x) + \mu^+(z))$$

$$\leq \tfrac{1}{2}(\{\max\{(\mu^+(a), \mu^+(b)\} + \mu^+(z))$$

$$= \max\{\tfrac{1}{2}((\mu^+(a) + \mu^+(z)), \tfrac{1}{2}(\mu^+(b) + \mu^+(z))\}$$

$$= \max\{\upsilon^+(a), \upsilon^+(b)\}$$

and $\upsilon^-(x) \geq \min\{\upsilon^-(a), \upsilon^-(b)\}$.

Now, $\upsilon^+(0) = \frac{1}{2}(\mu^+(0) + \mu^+(z))$

$$= \tfrac{1}{2}(1 + \mu^+(z))$$

$$\geq \tfrac{1}{2}(\mu^+(x) + \mu^+(z)) = \upsilon^+(x)$$

and

$$\upsilon^-(0) = \tfrac{1}{2}(\mu^-(0) + \mu^-(z))$$

$$= \tfrac{1}{2}(-1 + \mu^-(z))$$

$$\leq \tfrac{1}{2}(\mu^-(x) + \mu^-(z)) = \upsilon^-(x).$$

Hence this shows $V = (\upsilon^+, \upsilon^-) \in NBVAFhI(R)$. By the Theorem 3.9.3, $\widehat{V} = (\widehat{\upsilon}^+, \widehat{\upsilon}^-)$ is a maximal. Therefore $\widehat{\upsilon}^+(x) = \upsilon^+(x) + 1 - \upsilon^+(0) = \frac{1}{2}(1 + \mu^+(x))$ and $\widehat{\upsilon}^-(x) = \upsilon^-(x) - 1 - \upsilon^-(0) = \frac{1}{2}(-1 + \mu^-(x))$.

Since $\widehat{v}^{+}(x) = v^{+}(x) + 1 - v^{+}(0) = \frac{1}{2}(1 + \mu^{+}(x)) > \mu^{+}(x)$ and $\widehat{v}^{-}(x) = v^{-}(x) - 1 - v^{-}(0) = \frac{1}{2}(-1 + \mu^{-}(x)) < \mu^{-}(x)$.

Hence $B = (\mu^{+}, \mu^{-})$ is proper subset of $\widehat{V} = (\widehat{v}^{+}, \widehat{v}^{-})$. Thus $\widehat{v}^{+}(z) = v^{+}(z) + 1 - v^{+}(0) = \frac{1}{2}(1 + \mu^{+}(z)) < 1 = \widehat{v}^{+}(0)$ and $\widehat{v}^{-}(z) = v^{-}(z) - 1 - v^{-}(0) = \frac{1}{2}(-1 + \mu^{-}(z)) > -1 = \widehat{v}^{-}(0)$. Therefore $\widehat{V} = (\widehat{v}^{+}, \widehat{v}^{-})$ is non-constant and $B = (\mu^{+}, \mu^{-})$ is not a maximal element of $(NBVAFhI(R), \subseteq)$. Which is a contradiction. Thus μ^{+} takes only two positive values, 0 and 1, also μ^{-} takes only two positive values, 0 and -1. This implies B has four possible values, *i.e.*, $(0,0)$, $(0,-1)$, $(1,-1)$, $(1,0)$ then

$\widetilde{B}_{(0,0)} = \{x \in R : \mu^{+}(x) \leq 0\} \cap \{x \in R : \mu^{-}(x) \geq 0\}$

$= \{x \in R : \mu^{+}(x) = 0,\ \mu^{-}(x) = 0\}$

$\widetilde{B}_{(0,-1)} = \{x \in R : \mu^{+}(x) \leq 0\} \cap \{x \in R : \mu^{-}(x) \geq -1\}$

$= \{x \in R : \mu^{+}(x) = 0\}$

$\widetilde{B}_{(1,-1)} = \{x \in R : \mu^{+}(x) \leq 1\} \cap \{x \in R : \mu^{-}(x) \geq -1\} = R$

$\widetilde{B}_{(1,0)} = \{x \in R : \mu^{+}(x) \leq 1\} \cap \{x \in R : \mu^{-}(x) \geq 0\}$

$= \{x \in R :\ \mu^{-}(x) = 0\}$

By Corollary 3.6.4, we have two cases:

(i) $\widetilde{B}_{(0,0)} \subseteq \widetilde{B}_{(0,-1)} \subseteq \widetilde{B}_{(1,-1)}$ $\qquad$ (ii) $\widetilde{B}_{(0,0)} \subseteq \widetilde{B}_{(1,0)} \subseteq \widetilde{B}_{(1,-1)}$

From (i) according to the Proposition 3.1.5, a bipolar subset $B' = (\nu^{+}, \nu^{-})$

$$\nu^{+}(x) = \begin{cases} 0 & \text{if } x \in \widetilde{B}_{(0,-1)} \\ 1 & \text{if } x \notin \widetilde{B}_{(0,-1)} \end{cases} \qquad \nu^{-}(x) = \begin{cases} 0 & \text{if } x \in \widetilde{B}_{(0,-1)} \\ -1 & \text{if } x \notin \widetilde{B}_{(0,-1)} \end{cases}.$$

Suppose $x \in \widetilde{B}_{(0,-1)}$ so $\nu^{+}(x) = 0$, $\nu^{-}(x) = 0$ and $\mu^{+}(x) = 0$, $\mu^{-}(x) \leq 0$, therefore $B' \subseteq B$. Now, for all $x \in \widetilde{B}_{(1,-1)} - \widetilde{B}_{(0,-1)}$ so $\nu^{+}(x) = 1$, $\nu^{-}(x) = -1$ and

$\mu^{+}(x)=1, \mu^{-}(x)\geq -1$, therefore $B\subseteq B'$. If B take the value $(1,-1)$ *i.e.*, $\mu^{+}(x)=1$, $\mu^{-}(x)=-1$, then $B=B'$, and on the other hand $\mu^{+}(x)=1$, $\mu^{-}(x)=0$, then $B\subset B'$ which contradicts the fact that B is maximal element of $(NBVAFhI(R),\subseteq)$. Therefore $B\neq(1,0)$. In addition let $x\in \widetilde{B}_{(0,-1)}-\widetilde{B}_{(0,0)}$ so $\nu^{+}(x)=0$, $\nu^{-}(x)=0$ and $\mu^{+}(x)=0$, $\mu^{-}(x)=-1$, therefore $B'\subseteq B$. Similarly, we have $B\neq(1,0)$ from (ii). Hence B takes the values $(0,0)$, $(0,-1)$ and $(1,-1)$. ■

3.9.7 Remark.

A non-constant $B\in BVAFhI(R)$ is called a maximal element of R when $\widehat{B}$ defined in Theorem 3.9.3, is a maximal element of $(NBVAFhI(R),\subseteq)$.

3.9.8 Proposition.

A maximal $B\in BVAFhI(R)$ is normal and takes a value amoung $(0,0)$, $(1,-1)$, and $(0,-1)$.

Proof. Suppose $B=(\mu^{+},\mu^{-})\in BVAFhI(R)$ is a maximal. Then $\widehat{B}$ is a maximal element of $(NBVAFhI(R),\subseteq)$. By Theorem 3.9.6, $\widehat{B}$ takes three possible values $(0,0)$, $(1,-1)$, and $(0,-1)$. Since $B\subseteq\widehat{B}$ so B also takes a value amoung $(0,0)$, $(1,-1)$, and $(0,-1)$. Now, we have to show that B is normal. By Theorem 3.9.6, $\widehat{\mu}^{+}$ takes only two positive values, 0 and 1, also $\widehat{\mu}^{-}$ takes only two positive values, 0 and -1. Since $\widehat{\mu}^{+}(x)=\mu^{+}(x)+1-\mu^{+}(0)$ and $\widehat{\mu}^{-}(x)=\mu^{-}(x)-1-\mu^{-}(0)$, therefore $\widehat{\mu}^{+}(x)=1$ if and only if $\mu^{+}(x)=\mu^{+}(0)$ and $\widehat{\mu}^{+}(x)=0$ if and only if $\mu^{+}(x)=\mu^{+}(0)-1$. Now, since $B\subseteq\widehat{B}$ we have $\mu^{+}(x)\leq\widehat{\mu}^{+}(x)$ for $x\in R$. Thus,

$\widehat{\mu}^+(x) = 0$ implies $\mu^+(x) = 0$. Consequently, $\mu^+(0) = 1$. Similarly, we can prove $\mu^-(0) = -1$. Hence by Proposition 3.9.2, B is normal. ∎

3.9.9 Proposition.

Let $B \in BVAFhI(R)$ be a maximal, then $\widetilde{B}_{(1,-1)}$ is a maximal h-ideal of R.

3.9.10 Corollary.

Let ψ be an anti epimorphism from a hemiring R_1 to a hemiring R_2, $A = (\mu^+, \mu^-) \in BVAFhI(R_1)$. Then anti image of A, $\psi_a(A) = (\psi_a(\mu^+), \psi_a(\mu^-)) \in BVAFhI(R_2)$ if and only if $\psi^{-1}(0) = \emptyset$.

Chapter 4

Decision making by using bipolar-valued fuzzy soft h-ideals of hemirings

Decision making is observed as a psychological method emerging in the choice of a conviction or a way of deal among various alternative prospects. A concluding choice can be yeild in each decision making process, it may or may not an efficient deal. Classification of alternatives in terms of principles and decision maker's desires is titled as decision making. Decision making has been applied in severval problems of management, economics, engineering and environment etc. Decision making problems on various criteria are called Multi-Criteria Decision Analysis ($MCDA$) problems. In conventional decision of our daily lives, we my be convenient with the results of specific accords of perception and contemplate multiple criteria unquestioningly.

However in case of high wagers, it is essential to make appropriate form of problem and estimate multiple criteria explicitly. For example decision of make sure to construct a nuclear power plant, and where to constract it, there are not only very complicated concerns about multiple criteria, but there are also many parties who are seriously stirred from the outgrowths. To gain preferable and knowledgeable decisions, it is important that we consider multiple criteria precisely and organize compound problem completely. In 1981 a multi-criteria decision analysis method known as "technique for order preference by similarity to ideal solution (TOPSIS)" was established by Hwang and Yoon.

In this chapter, we define TOPSIS method on BVF soft sets. Further, we apply TOPSIS method on normal BVF and normal BVAF soft h-ideals to calculate maximal BVF and BVAF soft h-ideals in R. Moreover, we denote set of BVF soft h-ideals and set of all set of BVAF soft h-ideals by $BVFSfhIs(R)$ and $BVAFSfhIs(R)$ respectively.

4.1 Bipolar-valued fuzzy soft h-ideals

In this section, we review BVF soft h-ideals of R which are defined in [30, 37].

4.1.1 Definition[30]

A BVFS set $L\ (\mu_L^+, \mu_L^-) \in BVFSfLhI(R)$ (resp. $BVFSfRhI(R)$) if it satisafied for $r_1, r_2, s, t \in R$

(1) $\mu_L^+(r_1 + r_2) \geq \min\{\mu_L^+(r_1), \mu_L^+(r_2)\}$, $\mu_L^-(r_1 + r_2) \leq \max\{\mu_L^-(r_1), \mu_L^-(r_2)\}$

(2) $r_1+s+r_2 = t+r_2 \Longrightarrow \mu_L^+(r_1) \geq \min\{\mu_L^+(s), \mu_L^+(t)\}, \mu_L^-(r_1) \leq \max\{\mu_L^-(s), \mu_L^-(t)\}$

(3) $\mu_L^+(r_1r_2) \geq \mu_L^+(r_2)$, $\mu_L^-(r_1r_2) \leq \mu_L^-(r_2)$ (resp., $\mu_L^+(r_1r_2) \geq \mu_L^+(r_1), \mu_L^-(r_1r_2) \leq \mu_L^-(r_1)$).

4.1.2 Example

Consider $R = \{0, 1, p, p^*\}$ which is described in example 2.1.3. Let L is a BVF set defined by

	0	1	p	p^*
μ_L^+	0.72	0.72	0.72	0.22
μ_L^-	-0.82	-0.82	-0.82	-0.12

Clearly, $L \in BVFSfhI(R)$.

4.2 Bipolar-valued anti fuzzy soft h-ideals

In this section, we specify normal BVF soft h-ideals in R.

4.2.1 Definition

A BVFS set $L\ (\mu_L^+, \mu_L^-) \in BVAFSfLhI(R)$ (resp. $BVAFSfRhI(R)$) if it satisafied for $r_1, r_2, s, t, r \in R$

(1) $\mu_L^+(r_1 + r_2) \leq \max\{\mu_L^+(r_1), \mu_L^+(r_2)\}, \mu_L^-(r_1 + r_2) \geq \min\{\mu_L^-(r_1), \mu_L^-(r_2)\}$

(2) $r_1+s+r = t+r \Longrightarrow \mu_L^+(r_1) \leq \max\{\mu_L^+(s), \mu_L^+(t)\}, \mu_L^-(r_1) \geq \min\{\mu_L^-(s), \mu_L^-(t)\}$

(3) $\mu_L^+(r_1r_2) \leq \mu_L^+(r_2)$, $\mu_L^-(r_1r_2) \geq \mu_L^-(r_2)$ (resp., $\mu_L^+(r_1r_2) \leq \mu_L^+(r_1), \mu_L^-(r_1r_2) \geq$

$\mu_L^-(r_1))$.

4.2.2 Example

Consider $R = \{0, 1, p_1, p_2, p_3\}$ which is described in example 3.1.4. Let L be a BVF soft set defined by

	0	1	p_1	p_2	p_3
μ_L^+	0.21	0.41	0.41	0.41	0.41
μ_L^-	-0.61	-0.31	-0.31	-0.31	-0.31

Clearly, $L \in BVAFSfhI(R)$.

4.3 BVF and BVAF normal soft h-ideals

4.3.1 Definition[37]

A BVF soft h-ideal or BVAF soft h-ideal L of R is said to be normal if there exits an element $x \in S$ such that $L(x) = (1, -1)$.

4.3.2 Proposition.

A BVAF soft h-ideal L of a hemiring R is called normal if and only if $L(0) = (1, -1)$. *i.e.*, $\lambda_L^+(0) = 1$ and $\lambda_L^-(0) = -1$.

4.4 TOPSIS on bipolar-valued fuzzy soft sets

In [15, 16, 37, 43], TOPSIS method was introduced. The main concept of TOPSIS technique is that the chosen alternative should have the shortest distance from the positive ideal solution and the longest distance from the negative ideal solution

In uncertainties from our daily life problems, BVFS set has several application. Here we discuss such an application for solving a TOPSIS method. The TOPSIS technique based on BVFS sets carried out following steps:

Step-1

Create a decision matrix based on BVFS set. Suppose that there exists a set of alternatives A =$\{A_1, A_2, ..., A_m\}$. Each alternative is determined on n criteria, which are denoted by $C = \{C_1, C_2, ..., C_n\}$.

$$D = \begin{bmatrix} & C_1 & C_2 & .... & C_n \\ A_1 & L_{11} & L_{12} & .... & L_{1n} \\ A_2 & L_{21} & L_{22} & .... & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & L_{m1} & L_{m2} & .... & L_{mn} \end{bmatrix}$$

Where L_{ij} are BVFS sets, which are defined by $L_{ij} = \{(\mu_{ij}^+, \mu_{ij}^-) : \mu_{ij}^+ \in (0, 1]$ and $\mu_{ij}^- \in [-1, 0)\}$.

Therefore, matrix D express as,

$$D = \begin{bmatrix} & C_1 & C_2 & .... & C_n \\ A_1 & (\mu_{11}^+, \mu_{11}^-) & (\mu_{12}^+, \mu_{12}^-) & .... & (\mu_{1n}^+, \mu_{1n}^-) \\ A_2 & (\mu_{21}^+, \mu_{21}^-) & (\mu_{22}^+, \mu_{22}^-) & .... & (\mu_{2n}^+, \mu_{2n}^-) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & (\mu_{m1}^+, \mu_{m1}^-) & (\mu_{m2}^+, \mu_{m2}^-) & .... & (\mu_{mn}^+, \mu_{mn}^-) \end{bmatrix}.$$

We can write matrix D in form of two matrices D^+ and D^-,

$$D^+ = \begin{bmatrix} & C_1 & C_2 & .... & C_n \\ A_1 & (\mu_{11}^+) & (\mu_{12}^+) & .... & (\mu_{1n}^+) \\ A_2 & (\mu_{21}^+) & (\mu_{22}^+) & .... & (\mu_{2n}^+) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & (\mu_{m1}^+) & (\mu_{m2}^+) & .... & (\mu_{mn}^+) \end{bmatrix} \text{ and } D^- = \begin{bmatrix} & C_1 & C_2 & .... & C_n \\ A_1 & (\mu_{11}^-) & (\mu_{12}^-) & .... & (\mu_{1n}^-) \\ A_2 & (\mu_{21}^-) & (\mu_{22}^-) & .... & (\mu_{2n}^-) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & (\mu_{m1}^-) & (\mu_{m2}^-) & .... & (\mu_{mn}^-) \end{bmatrix}$$

Step-2

Matrix D^+ is normalized in form of matrix $\dot{N}^+ = (t_{ij}^+)_{m\times n}$ where $t_{ij}^+ = \mu_{ij}^+ / \left(\sum_{i=1}^{m} \mu_{ij}^+\right)^{1/2}$ for $i = 1, 2, ..m$ and $j = 1, 2, ..n$. Matrix D^- is normalized in form of matrix $\dot{N}^- = (t_{ij}^-)_{m\times n}$ where $t_{ij}^- = \mu_{ij}^- / \left(\sum_{i=1}^{m} \mu_{ij}^-\right)^{1/2}$ for $i = 1, 2, ..m$ and $j = 1, 2, ..n$.

Step-3

Calculation of weighted normilized decision matrix

$L^+ = (l_{ij}^+)_{m\times n} = (w_j^+ t_{ij}^+)_{m\times n}$, where $w_j^+ = W_j^+ / \sum_{j=1}^{n} W_j^+$, so that $\sum_{j=1}^{n} w_j^+ = 1$, and W_j^+ is origional weight given to the indictors v_j^+, $\forall\ i = 1, 2, ..m$, $j = 1, 2, ..n$.

$L^- = (l_{ij}^-)_{m\times n} = (w_j^- t_{ij}^-)_{m\times n}$, where $w_j^- = W_j^- / \sum_{j=1}^{n} W_j^-$, so that $\sum_{j=1}^{n} w_j^- = 1$, and W_j^- is origional weight given to the indictors v_j^-, $\forall\ i = 1, 2, ..m$, $j = 1, 2, ..n$.

Step-4

In this step we obtain the positive ideal best and negative ideal worst solutions for L^+ and L^-. The positive ideal and negative ideal solutions can be expressed as:

For L^+ we heve

$V_j^{++} = \{\max\{l_{ij}^+/i = 1,2,..m\} \mid j \in J_+, \min\{l_{ij}^+/i = 1,2,..m\} \mid j \in J_-\}$

$V_j^{+-} = \{\min\{l_{ij}^+/i = 1,2,..m\} \mid j \in J_+, \max\{l_{ij}^+/i = 1,2,..m\} \mid j \in J_-\}.$

For L^- we heve

$V_j^{-+} = \{\min\{l_{ij}^-/i = 1,2,..m\} \mid j \in J_+, \max\{l_{ij}^-/i = 1,2,..m\} \mid j \in J_-\}$

$V_j^{--} = \{\max\{l_{ij}^-/i = 1,2,..m\} \mid j \in J_+, \min\{l_{ij}^-/i = 1,2,..m\} \mid j \in J_-\}$

$j \in J_+$ is associated with beneficial attributes and

$j \in J_-$ is associated with non-beneficial attributes.

Step-5

The separation measures, $d_{i\alpha}^{++}$ and $d_{i\alpha}^{+-}$, of each alternative from V_j^{++} and V_j^{+-},

respectively, are gain as

$$d_{i\alpha}^{++} = \left(\sum_{i=1}^{m}(V_j^{++} - l_{\alpha j}^+)\right)^{1/2}$$

$$d_{i\alpha}^{+-} = \left(\sum_{i=1}^{m}(V_j^{+-} - l_{\alpha j}^+)\right)^{1/2}.$$

The separation measures, $d_{i\alpha'}^{-+}$ and $d_{i\alpha'}^{--}$, of each alternative from V_j^{-+} and V_j^{--},

respectively, are gain as

$$d_{i\alpha'}^{-+} = \left(\sum_{i=1}^{m}(V_j^{-+} - l_{\alpha' j}^-)\right)^{1/2}$$

$$d_{i\alpha'}^{--} = \left(\sum_{i=1}^{m}(V_j^{--} - l_{\alpha' j}^-)\right)^{1/2}.$$

Step-6

The relative nearness of an alternative A_i with respect to the positive ideal solution

is defined as the following general formula for each i,

$s_i^+ = d_{i\alpha}^{+-} / \left(d_{i\alpha}^{+-} + d_{i\alpha}^{++}\right)$ and $s_i^- = d_{i\alpha'}^{-+} / \left(d_{i\alpha'}^{-+} + d_{i\alpha'}^{--}\right).$

Where $0 \leq s_i^+ \leq 1$ and $0 \leq s_i^- \leq 1$ and $i = 1, 2, ...m$.

Step-7

Rank of alternatives according to (s_i^+, s_i^-) for each $i = 1, 2, ...m$.

4.5 TOPSIS on normal BVFS and normal BVAFS $h-$ideals

In this section, we introduced TOPSIS on normal BVF soft h-ideals. In [30], we calculate maximal BVF soft h-ideals. Here we calculate maximal BVF soft h-ideals and maximal BVAF soft h-ideals by using TOPISIS method.

4.5.1 TOPSIS on normal BVFS $h-$ideals

Consider normal BVF soft h-ideals as follows. Suppose that a car dealer has a set of cars $U = \{u_1, u_2, u_3, u_4, u_5\}$ which may be associated with $E = \{z_1, z_2, z_3, z_4\}$ for $j = 1, 2, 3, 4$ the parameters z_j stand for in "beautiful", "costly", "modern technology", "fuel efficient", respectively. Suppose that the pair , Mr. Y and Mrs. Y, come to the car dealer to buy a car. If each partner has to consider their own set of parameters, then we select a car on the basis of the sets of partners' parameters by using BVFS sets as follows.

Suppose that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universe and $E = \{z_1, z_2, z_3, z_4\}$ set of all parameters. Our plan is to find the appealing car for Mr. Y. Assume that the hoping parameters of Mr. Y be $L = \{e_1, e_2, e_5\}$ is subset of E.

$$G(e_1) = \{(A_1, 0, 0), (A_2, 1, 0), (A_3, 1, -1), (A_4, 1, -1)\}$$

$$G(e_2) = \{(A_1, 1, 0), (A_2, 0, 0), (A_3, 0, -1), (A_4, 1, 0)\}$$

$$G(e_5) = \{(A_1, 1, -1), (A_2, 0, -1), (A_3, 0, 0), (A_4, 0, -1)\}$$

Now, we can define TOPSIS method for Mr. Y to choose attractive car or beneficial car

Step-1

We define decision matrix as follows

$$D = \begin{bmatrix} & e_1 & e_2 & e_5 \\ A_1 & (0,0) & (1,0) & (1,-1) \\ A_2 & (1,0) & (0,0) & (0,-1) \\ A_3 & (1,-1) & (0,-1) & (0,0) \\ A_4 & (1,-1) & (1,0) & (0,-1) \end{bmatrix}.$$

In form of two matrices

$$D^+ = \begin{bmatrix} & C_1 & C_2 & C_n \\ A_1 & 0 & 1 & 1 \\ A_2 & 1 & 0 & 0 \\ A_3 & 1 & 0 & 0 \\ A_m & 1 & 1 & 0 \end{bmatrix}, \quad D^- = \begin{bmatrix} & C_1 & C_2 & C_n \\ A_1 & 0 & 0 & -1 \\ A_2 & 0 & 0 & -1 \\ A_3 & -1 & -1 & 0 \\ A_m & -1 & 0 & -1 \end{bmatrix}.$$

Step-2

Normalized matrices of D^+ and D^- are

$$\check{N}^{+}=\begin{bmatrix} & C_1 & C_2 & C_5 \\ A_1 & 0 & 0.707 & 1 \\ A_2 & 0.557 & 0 & 0 \\ A_3 & 0.557 & 0 & 0 \\ A_4 & 0.557 & 0.707 & 0 \end{bmatrix}, \check{N}^{+}=\begin{bmatrix} & C_1 & C_2 & C_5 \\ A_1 & 0 & 0 & -0.557 \\ A_2 & 0 & 0 & -0.557 \\ A_3 & -0.707 & -1 & 0 \\ A_4 & -0.707 & 0 & -0.557 \end{bmatrix}$$

respectively.

Step-3

As there exist 1 and -1 in D^+ and D^- respectively, therefore there is no change in case of weighted matrices.

Step-4

The positive ideal solution and negative ideal solution for $\check{N}^+$ are $V_j^{++} = [0.557, 0.707, 1]$ and $V_j^{+-} = [0, 0, 0]$ respectively for all $j = 1, 2, 3$.

The positive ideal solution and negative ideal solution for $\check{N}^-$ are $V_j^{-+} = [-0.707, -1, -0.557]$ and $V_j^{--} = [0,0,0]$ respectively for all $j = 1, 2, 3$.

Step-5

The separation measures, $d_{i\alpha}^{++}$ and $d_{i\alpha}^{+-}$, of each A_i from V_j^{++} and V_j^{+-}, respectively, are gain as

$$d_i^{++}=\begin{bmatrix} ((0.557-0)^2+(0.707-0.707)^2+(1-1)^2)^{1/2} \\ ((0.557-0.557)^2+(0.707-0)^2+(1-0)^2)^{1/2} \\ ((0.557-0.557)^2+(0.707-0)^2+(1-0)^2)^{1/2} \\ ((0.557-0.557)^2+(0.707-0.707)^2+(1-0)^2)^{1/2} \end{bmatrix}=\begin{bmatrix} 0.557 \\ 1.225 \\ 1.225 \\ 1 \end{bmatrix}$$

$$\text{and } d_i^{+-} = \begin{bmatrix} ((0-0)^2+(0-0.707)^2+(0-1)^2)^{1/2} \\ ((0-0.557)^2+(0-0)^2+(0-0)^2)^{1/2} \\ ((0-0.557)^2+(0-0)^2+(0-0)^2)^{1/2} \\ ((0-0.557)^2+(0-0.707)^2+(0-0)^2)^{1/2} \end{bmatrix} = \begin{bmatrix} 1.2247 \\ 0.557 \\ 0.557 \\ 0.9126 \end{bmatrix},$$

The separation measures, $d_{i\alpha'}^{-+}$ and $d_{i\alpha'}^{--}$, of each A_i from V_j^{-+} and V_j^{--}, respectively, are gain as

$$d_i^{-+} = \begin{bmatrix} ((-0.707-0)^2+(-1-0)^2+(-0.557+0.557)^2)^{1/2} \\ ((-0.707-0)^2+(-1-0)^2+(-0.557+0.557)^2)^{1/2} \\ ((-0.707+0.707)^2+(-1+1)^2+(-0.557-0)^2)^{1/2} \\ ((-0.707+0.707)^2+(-1-0)^2+(-0.557+0.557)^2)^{1/2} \end{bmatrix} = \begin{bmatrix} 1.2247 \\ 1.2247 \\ 0.557 \\ 1 \end{bmatrix}$$

$$\text{and } d_i^{--} = \begin{bmatrix} ((0-0)^2+(0-0)^2+(0+0.557)^2)^{1/2} \\ ((0-0)^2+(0-0)^2+(0+0.557)^2)^{1/2} \\ ((0+0.707)^2+(0+1)^2+(0-0)^2)^{1/2} \\ ((0+0.707)^2+(0-0)^2+(0+0.557)^2)^{1/2} \end{bmatrix} = \begin{bmatrix} 0.557 \\ 0.557 \\ 1.2247 \\ 0.9126 \end{bmatrix}.$$

Step-6

The relative nearness of A_i with respect to the positive

ideal solution

$$s_i^+ = \begin{bmatrix} 0.6797 \\ 0.3202 \\ 0.3202 \\ 0.4772 \end{bmatrix} \quad and \quad s_i^- = \begin{bmatrix} 0.6797 \\ 0.6797 \\ 0.3202 \\ 0.5228 \end{bmatrix}.$$

Step-7

We note that s_i^+ has maximal value 0.6797 and s_i^- has maximal vaue 0.6795. Hence A_1 is a maximal result so that Mr. Y buy A_1. Hence maximal BVF soft h-ideals are

$(0,0),(1,0),(1,-1)$.

4.5.2 TOPSIS of normal BVAFS h-ideals

In case of BVAF soft h-ideals, Mr. Y need to choose non-beneficial car in 4.5.1. For non-beneficial car there is no change in step-1, step-2 and step-3 of 4.5.1.

Step-4

The positive ideal solution and negative ideal solution of $\tilde{N}^+$ are

$V_j^{++} = [0,0,0]$ and $V_j^{+-} = [0.557, 0.707, 1]$ respectively.

The positive ideal solution and negative ideal solution of $\tilde{N}^-$ are $V_j^{-+} = [0,0,0]$ and $V_j^{--} = [-0.707, -1, -0.557]$ respectively.

Step-5

The separation measures, $d_{i\alpha}^{++}$ and $d_{i\alpha}^{+-}$, of each A_i from V_j^{++} and V_j^{+-}, respectively, are gain as

$$d_i^{++} = \begin{bmatrix} ((0-0)^2+(0-0.707)^2+(0-1)^2)^{1/2} \\ ((0-0.557)^2+(0-0)^2+(0-0)^2)^{1/2} \\ ((0-0.557)^2+(0-0)^2+(0-0)^2)^{1/2} \\ ((0-0.557)^2+(0-0.707)^2+(0-0)^2)^{1/2} \end{bmatrix} = \begin{bmatrix} 1.225 \\ 0.557 \\ 0.557 \\ 0.913 \end{bmatrix}$$

$$\text{and } d_i^{+-} = \begin{bmatrix} ((0.557-0)^2+(0.707-0.707)^2+(1-1)^2)^{1/2} \\ ((0.557-0.557)^2+(0.707-0)^2+(1-0)^2)^{1/2} \\ ((0.557-0.557)^2+(0.707-0)^2+(1-0)^2)^{1/2} \\ ((0.557-0.557)^2+(0.707-0.707)^2+(1-0)^2)^{1/2} \end{bmatrix} = \begin{bmatrix} 0.557 \\ 1.225 \\ 1.225 \\ 1 \end{bmatrix},$$

The separation measures, $d_{i\alpha}^{++}$ and $d_{i\alpha}^{+-}$, of each A_i from V_j^{++} and V_j^{+-}, respectively, are gain as

$$d_i^{-+} = \begin{bmatrix} ((0-0)^2+(0-0)^2+(0+0.557)^2)^{1/2} \\ ((0-0)^2+(0-0)^2+(0+0.557)^2)^{1/2} \\ ((0+0.707)^2+(0+1)^2+(0-0)^2)^{1/2} \\ ((0+0.707)^2+(0-0)^2+(0+0.557)^2)^{1/2} \end{bmatrix} = \begin{bmatrix} 0.557 \\ 0.557 \\ 1.2247 \\ 0.9126 \end{bmatrix}$$

$$\text{and } d_i^{--} = \begin{bmatrix} ((-0.707-0)^2+(-1-0)^2+(-0.557+0.557)^2)^{1/2} \\ ((-0.707-0)^2+(-1-0)^2+(-0.557+0.557)^2)^{1/2} \\ ((-0.707+0.707)^2+(-1+1)^2+(-0.557-0)^2)^{1/2} \\ ((-0.707+0.707)^2+(-1-0)^2+(-0.557+0.557)^2)^{1/2} \end{bmatrix} = \begin{bmatrix} 1.2247 \\ 1.2247 \\ 0.557 \\ 1 \end{bmatrix}.$$

Step-6

The relative nearness of A_i with respect to the positive

ideal solution

$$s_i^+ = \begin{bmatrix} 0.3202 \\ 0.6797 \\ 0.6797 \\ 0.5228 \end{bmatrix} \quad and \quad s_i^- = \begin{bmatrix} 0.3202 \\ 0.3202 \\ 0.6797 \\ 0.4772 \end{bmatrix}.$$

Step-7

Here s_3^+ has maximal value 0.6797 and s_3^- has maximal value 0.6797. We note that A_3 is maximal result, so that A_3 is non-beneficial car for Mr. Y. Hence maximal BVAF soft h-ideals are $(0,0),(0,-1),(1,-1)$.

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