STUDY OF NON-NEWTONIAN FLUID TROUGH AN INFINITE PIPE

By

TARIQ MAHMOOD

Department of Mathematics

International Islamic University, Islamabad PAKISTAN

2010

1. Non-Newtonian fluids
2. Chemical engineering

STUDY OF NON-NEWTONIAN FLUID TROUGH AN INFINITE PIPE

By

TARIQ MAHMOOD

Supervised

By

DR. NASIR ALI

Department of Mathematics

International Islamic University, Islamabad PAKISTAN

2010

STUDY OF NON-NEWTONIAN FLUID TROUGH AN INFINITE PIPE

By

TARIQ MAHMOOD

MASTER OF PHILOSOPHY

IN

MATHEMATICS

Supervised

By

DR. NASIR ALI

Department of Mathematics
International Islamic University, Islamabad PAKISTAN

2010

IN THE NAME OF

ALLAH

THE MOST BENEFICIENT
THE MOST MERCIFUL

Dedicated

To my

<u>Parents</u>

Who are the most precious gems
of my life.
Who've always given me perpetual love,
care, and cheers. Whose prayers have
always been a source of great
inspiration for me and whose
sustained hope in me led me
to where I stand today.

Certificate

Study of non-Newtonian fluid through an infinite pipe

By

Tariq Mahmood

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTRER OF SCIENCE IN MATHEMATICS

We accept this dissertation as conforming to the required standard.

1. Dr. Rahmat Ellahi
(Chairman)

2. Dr. Nasir Ali
(Supervisor)

3. Prof. Dr. Saleem Asghar
(External Examiner)

4. Dr. Ahmer Mehmood
(Internal Examiner)

Department of Mathematics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2010

ACKNOWLEDGEMENT

In the name of Almighty Allah, The Most Merciful, The Most beneficent, Who inculcated in me the strength and spirit to complete such a mandatory task of this dissertation. His peace and blessings be on the Holy Prophet Hazrat Muhammad (PBUH), who has been sent to the whole mankind to guide them the way to their Allah.

I am deeply indebted to my supervisor Dr. Nasir Ali for his advice, encouragement and patience throughout the course of my M.S studies. His unparalleled, unsurpassed and selfless guidance and inspiration showered on me to expose my potentialities in order to complete the work.

I have to pay my special gratitude to Dr. Rehmat Ellahi, Chairman Department of Mathematics for his generous encouragement and providing necessary research facilities.

I also acknowledge my deepest gratitude to all of my teachers for their kindness and moral support during my stay in the University, especially Dr. Tariq Javed.

I am extremely grateful to my parents, my brother and sisters for their patience and moral support who prayed all the time for the completion of my work. Deep thanks are to my wife for her encouragement from time to time and for sparing me to work long hours especially in the last days of my work.

I am very thankful to my research colleagues especially Mr. Irfan Mustafa and Mr. Hussain Ahmed for their useful discussions during the course of my studies. Acknowledgement is also made to my friends, office and library staff for their useful contribution during the course of my studies.

TARIQ MAHMOOD

Preface

Mechanics of non-Newtonian fluids has fascinated the researcher due to its tremendous applications in technology and industry. Flow of such fluids present a major challenge for researcher in fluid dynamics: examples are polymers, slurries, mixtures of oil-water and gas liquid and many biological fluids (such as blood, synovial fluid and food stuffs).

There is no universal relation available in the literature like Newtonian fluids, which can describe the rheology of all Newtonian fluids. Therefore, several constitutive relations of non-Newtonian fluids have been proposed. These constitutive relations are then used with the equations of continuity and momentum to solve flow problems. It is important to mention that consideration of the non-Newtonian constitutive equation in the mathematical modeling give rise two types of difficulties: i) an increase in the order of equation of motion ii) Paucity of boundary conditions. These critical issues regarding some of the non-Newtonian fluids have been discussed by Rajagopal [1] and Rajagopal and kaloni [2].

A simplest constitutive equation that captures some observed non-Newtonian behavior is the generalized Newtonian fluid (GNF) constitutive equation. In GNF constitutive equation the viscosity is a function of share rate. Based on the functional form of viscosity several GNF constitutive model may be found in literature [3]. Widely used GNF models for the study of viscoelastic fluids are power -law model, Carreau model, Bingham model, Ellis and Eyring-Powell models. A detailed discussion on GNF models may be found in Bird et al. [4].

Flows of fluids in pipe/tubes are extensively studied in the literature because of their practical applications and as a starting point of investigation. Fully developed pipe flow gives an exact solution of Navier-stokes equations [5]. However, the possibility of getting exact solutions narrows down for non-Newtonian fluids. For such case one has to relay on approximate methods. Some recent studies on pipes flow of non-Newtonian fluids were performed by Erdogan [6], Hayat et al. [7,8] and Khan et al. [9]. Pressure driven flow of power-law and Ellis fluids can be found in [3].

Motivated by the above mentioned studies and facts we propose to study magnetohydrodynamics (MHD) pipe flow of an Eyring-Powell fluid in this dissertation. The dissertation is organized as follows. Introductory material consisting of some definitions and governing equations is presented in chapter 1. In chapter 2 MHD flow of fourth grade fluid in a pipe is investigated by using Hybrid method proposed by Ariel [10]. The same problem was studied by Hayat et al. [7] using Homotopy Analysis method. The results obtained by both the methods are compared. Effects of emerging parameters on velocity profile are illustrated through graphs. Chapter 3 contains the study of MHD flow of an Eyring-Powell fluid in a pipe. Governing equation is obtained for the fully developed flow. Numerical solution of the problem is presented using hybrid method. Graphs are sketched to analyze the effect of the material parameters of Eyring-Powell fluid on the flow velocity.

Contents

Chapter 1

Preliminaries

This chapter includes some basic definitions and governing laws relevant to the material presented in the subsequent chapters.

1.1 Basic definitions

1.1.1 Fluid

A fluid is a substance that deforms continuously under an applied shear stress regardless of how small the applied stress may be.

1.1.2 Flow

It is a phenomenon in which the material deformation increases continuously without limit when different forces act upon it.

1.1.3 Fluid mechanics

It is a branch of continuum mechanics, which deals with the properties of stationary and moving fluids. Fluid mechanics can be divided into **fluid statics** (*i.e.*the study of fluids at rest) and **fluid dynamics** (*i.e.*the study of fluids in motion).

1.2 Some physical properties of fluids

1.2.1 Density

Density of a fluid is defined as the mass per unit volume. Mathematically, the density ρ at a point p_0 may be defined as

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V},$$

where δV is the volume of the fluid enclosing point p_0.

1.2.2 Viscosity

It is a property of fluid by virtue of which it offers resistance to the flow. Mathematically, it is defined as the ratio of shear stress to the rate of shear strain i.e.,

$$\text{Viscosity} = \mu = \frac{\text{shear stress}}{\text{rate of shear strain}}.$$

In above definition, μ is called the absolute or dynamic viscosity.

1.2.3 Kinematic viscosity

The ratio of dynamic viscosity to the density of the fluid is known as kinematic viscosity and is denoted by ν. Mathematically,

$$\nu = \frac{\mu}{\rho}.$$

1.3 Classification of fluids

1.3.1 Ideal fluids

Ideal fluids are those which have zero viscosity. Gases are usually treated as an ideal fluid for engineering purposes. Ideal fluids are also known as inviscid fluids.

1.3.2 Real fluids

Real fluids are those which have finite viscosity. These fluids may be compressible or incompressible. Depending on the relationship between the shear stress and the rate of shear strain, real fluids are further divided into two main classes, namely **Newtonian** and **non-Newtonian** fluids.

1.3.3 Newtonian fluids

Newtonian fluids are those fluids for which the shear stress is directly and linearly proportional to the rate of deformation. These fluids obey the Newton's law of viscosity, which is defined as

$$\tau_{yx} = \mu \frac{du}{dy},$$

where τ_{yx} is the shear stress acting on a plane normal to $y-$axis and u is the velocity in the $x-$direction. Water is a common example of a Newtonian fluid.

1.3.4 Non-Newtonian fluids

Non-Newtonian fluids are those for which the shear stress is not linearly proportional to the rate of deformation. These fluids obey power law model, which is given by

$$\tau_{yx} = k \left(\frac{du}{d\theta} \right)^n, \quad n \neq 1,$$

where k is the consistency index and n is the flow behavior index.

For $n = 1$ with $k = \mu$, the above equation reduces to the Newton's law of viscosity. Examples of non-Newtonian fluids include ketchup, tooth paste, blood, paints, greases, biological fluids, polymer melts etc.

1.4 Types of flow

1.4.1 Laminar and turbulent flow

A flow in which paths taken by the individual particles do not cross one another and each particle moves along a well defined path is known as a laminar flow.

A turbulent flow is one in which each fluid particle does not has a definite path and it moves randomly.

1.4.2 Steady and unsteady flow

A flow in which fluid properties at each point in the flow field does not depend upon time is called steady flow. For such a flow, we can write

$$\frac{\partial \eta}{\partial t} = 0,$$

where η represents any fluid property and t is the time.

A flow in which fluid properties at each point in flow field depend upon time is called unsteady flow. For such a flow, we can write

$$\frac{\partial \eta}{\partial t} \neq 0.$$

1.4.3 Incompressible and compressible flow

A flow is said to be incompressible if the density remains constant throughout the motion. Therefore, the volume of every portion of fluid remains unchanged over the course of its motion when the flow is incompressible.

Mathematically, incompressibility is expressed by saying that the density ρ of a fluid particle does not change as it moves in the flow field, i.e.,

$$\rho \neq \rho\,(x, y, z, t) \quad \text{or} \quad \rho = \text{constant}.$$

or

$$\frac{d\rho}{dt} = 0,$$

where d/dt is the substantial (total) derivative, which is the sum of local and convective derivatives and has the following form:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla}). \tag{1.1}$$

In the above expression $\mathbf{V}$ is the velocity of the fluid and ∇ is del operator. All liquid flows are generally incompressible.

A flow in which density of the fluid varies during the flow is termed as compressible flow i.e.,

$$\rho = \rho\,(x, y, z, t)\,.$$

1.5 Magnetohydrodynamics

Magnetohydrodynamics (MHD) deals with the mutual interaction of fluid flow and magnetic fields. Magnetohydrodynamics is the academic discipline which studies the dynamics of electrically conducting fluids. Examples of such fluids include plasmas, liquid metals and salt water.

1.5.1 Magnetism

The study of nature, and cause of magnetic force fields and how different substances are effected by them is called magnetism.

1.5.2 Magnetic field

The region or space around a magnet in which it exerts a force on other magnet, is called its magnetic field. It is strong near poles.

1.5.3 Electric field

A region in which a force would be exerted on an electric charge. It is completely defined in magnitude and direction at any point by the force upon a unit positive charge situated at that point. It can be produced by electric charge.

1.6 Maxwell's equations

In electromagnetism, a set of four partial differential equations is given by James Clark Maxwell. These equations show the interrelationship between electricity and magnetism and are as follows:

a. Gauss's law

This law relates electric charge contained in a close surface to the surrounding electric field i.e.,

$$\nabla.\mathbf{E} = \frac{\rho_e}{\epsilon_o}, \tag{1.2}$$

where **E** is the electric field, ρ_e is the charge density and ϵ_o is the permittivity of the free space.

b. Gauss's law for magnetism

It states that the total magnetic flux through a close surface is zero i.e.,

$$\nabla.\mathbf{B} = 0, \tag{1.3}$$

where **B** is the total magnetic field.

c. Faraday's law of induction

This law explains that how an electric field is generated by changing magnetic field. Mathematically,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.4}$$

d. Ampere-Maxwell law

It states that magnetic field can be generated in two ways, firstly by electric current and secondly by changing electric field. The mathematical statement of this law is given as

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t}, \tag{1.5}$$

where $\mathbf{J}$ is the current density and μ_o is the magnetic permeability.

1.7 Generalized Ohm's law

In 1827 George Simon Ohm introduced a law according to which current in a conductor is directly proportional to the voltage drop. The generalized Ohm's law for an electrically conducting fluid has the form

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{V} \times \mathbf{B} \right), \tag{1.6}$$

where σ is the electric conductivity of the fluid.

1.8 Governing laws

1.8.1 Law of conservation of mass

According to the law of conservation of mass (also known as continuity equation), **mass can neither be created nor destroyed in any classical system.** For a moving fluid it can be written in vector form as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.\left(\rho \mathbf{V}\right) = 0. \tag{1.7}$$

For an incompressible flow, the density is constant and thus it becomes

$$\boldsymbol{\nabla}.\mathbf{V} = \mathbf{0}. \tag{1.8}$$

1.8.2 Law of conservation of momentum

Every particle of fluid at rest or in steady or accelerated motion obeys Newton's second law of motion which states that, **the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system.**

In vector form, it can be written as

$$\rho\frac{d\mathbf{V}}{dt} = \nabla.\mathbf{T}+\rho\mathbf{b}, \tag{1.9}$$

where $\mathbf{b}$ the body force per unit mass and $\mathbf{T}$ the Cauchy stress tensor.

1.9 The Hybrid method

Hybrid method is an explicit numerical method proposed by Ariel [10] for solution of ordinary differential equations. It is a combination of the shooting and the finite difference methods. It is useful for the solution of problems:

- with sigularity.
- where the order of differential equation is higher than the number of available boundary conditions.
- where the higher derivative gets vanished for certain values of the parameters involved in the equation.

The detailed procedure of the implementation of this method can be found in the next chapters.

Chapter 2

MHD Pipe Flow of Fourth Grade Fluid

2.1 Introduction

The present chapter deals with the study of steady MHD flow of fourth grade fluid in a pipe. This work was previously undertaken by Sajid et al. [7] by employing an analytical technique known as homotopy analysis method (HAM). Here a hybrid numerical method proposed by Ariel [10] is used to study the same problem. Details of the implementation procedure are provided and a comparison of solution reported in ref. [7] is made with present one. The effects of emerging parameters on the flow velocity are explained graphically at the end of the chapter.

2.2 Governing equations

Consider the steady flow of an electrically conducting, incompressible, fourth grade fluid in a non-conducting circular pipe. The z-axis is taken along the axis of the pipe. A constant magnetic field of strength B_0 is applied transverse to the flow direction. The electric field assumed to be zero. The flow is induced due to a constant applied pressure gradient in the z-direction. For the flow under consideration the governing equations are the continuity and momentum given by Eqs. (1.8) and (1.9). The body force term in Eq. (1.9) for an applied magnetic field shall be derived through Eqs. (1.2) – (1.6).

The Cauchy stress tensor $\mathbf{T}$ appearing in Eq (1.9) for a fourth grade fluid is given by

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2\right) \\ &\quad +\beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1 + \gamma_1\mathbf{A}_4 + \gamma_2\left(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3\right) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 \\ &\quad +\mathbf{A}_1^2\mathbf{A}_2) + \gamma_5\left(tr\mathbf{A}_2\right)\mathbf{A}_2 + \gamma_6\left(tr\mathbf{A}_2\right)\mathbf{A}_1^2 + \left(\gamma_7 tr\mathbf{A}_3 + \gamma_8 tr\left(\mathbf{A}_2\mathbf{A}_1\right)\right)\mathbf{A}_1, \end{aligned} \tag{2.1}$$

where p is the pressure, $\mathbf{I}$ the identity tensor, α/s, β/s and γ/s are the material constants and $\mathbf{A}_{1-4}$ are the Rivlin-Ericksen tensors defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{2.2}$$

and

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{n-1}, (n > 1) \tag{2.3}$$

where

$$\mathbf{L} = \nabla\mathbf{V}. \tag{2.4}$$

For the problem under consideration, the velocity field has the following form

$$\mathbf{V} = (0,\ 0,\ u(r)), \tag{2.5}$$

in which $u(r)$ is the z-component of velocity in the pipe. It is noted that using Eq. (2.5) the continuity equation (1.8) is satisfied identically. Now we proceed to obtain the full form of $\mathbf{T}$ for the velocity field (2.5) and write

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix}, \quad \mathbf{L}^T = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_1 = L + L^T = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix}, \tag{2.6}$$

$$\mathbf{A}_1\mathbf{L} = \begin{bmatrix} \left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L}^T\mathbf{A}_1 = \begin{bmatrix} \left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.7}$$

$$\mathbf{A}_2 = \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.8}$$

$$\mathbf{A}_2\mathbf{L} = \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.9}$$

$$\mathbf{L}^T\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.10}$$

$$\mathbf{A}_3 = \mathbf{A}_2\mathbf{L} + \mathbf{L}^T\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.11}$$

$$\mathbf{A}_4 = \mathbf{A}_3\mathbf{L} + \mathbf{L}^T\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.12}$$

$$\mathbf{A}_2\mathbf{A}_1 = \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2\left(\frac{\partial u}{\partial r}\right)^3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.13}$$

$$\mathbf{A}_1\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix} \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\left(\frac{\partial u}{\partial r}\right)^3 & 0 & 0 \end{bmatrix}, \tag{2.14}$$

$$\mathbf{A}_1\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.15}$$

$$\mathbf{A}_3\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.16}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\partial u}{\partial r}\right)^2 \end{bmatrix}, \tag{2.17}$$

$$trace\mathbf{A}_1^2 = trace \begin{bmatrix} \left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\partial u}{\partial r}\right)^2 \end{bmatrix} = 2(\frac{\partial u}{\partial r})^2, \tag{2.18}$$

$$trace\mathbf{A}_3 = 0, \tag{2.19}$$

$$trace\mathbf{A}_2 = trace \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2(\frac{\partial u}{\partial r})^2, \tag{2.20}$$

$$trace\mathbf{A}_2\mathbf{A}_1 = trace \begin{bmatrix} 0 & 0 & 2\left(\frac{\partial u}{\partial r}\right)^3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.21}$$

$$\mathbf{A}_2\mathbf{A}_1^2 = \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\partial u}{\partial r}\right)^2 \end{bmatrix} = \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.22}$$

$$\mathbf{A}_1^2\mathbf{A}_2 = \begin{bmatrix} \left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\partial u}{\partial r}\right)^2 \end{bmatrix} \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2\left(\frac{\partial u}{\partial r}\right)^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.23}$$

and

$$\mathbf{A}_2^2=\begin{bmatrix}2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}2\left(\frac{\partial u}{\partial r}\right)^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}4\left(\frac{\partial u}{\partial r}\right)^4 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{2.24}$$

Substitution of Eqs. (2.6) – (2.24) into Eq. (2.1) results in

$$\mathbf{T}=\begin{bmatrix}\begin{matrix}-p+2\alpha_1\left(\frac{\partial u}{\partial r}\right)^2+\alpha_2\left(\frac{\partial u}{\partial r}\right)^2+4\gamma_3\left(\frac{\partial u}{\partial r}\right)^4\\ +4\gamma_4\left(\frac{\partial u}{\partial r}\right)^4+4\gamma_5\left(\frac{\partial u}{\partial r}\right)^4+2\gamma_6\left(\frac{\partial u}{\partial r}\right)^4\end{matrix} & 0 & \mu\frac{\partial u}{\partial r}+2\beta_2\left(\frac{\partial u}{\partial r}\right)^3+2\beta_3\left(\frac{\partial u}{\partial r}\right)^3\\ 0 & -p & 0\\ \mu\frac{\partial u}{\partial r}+2\beta_2\left(\frac{\partial u}{\partial r}\right)^3+2\beta_3\left(\frac{\partial u}{\partial r}\right)^3 & 0 & -p+\alpha_2\left(\frac{\partial u}{\partial r}\right)^2+2\gamma_6\left(\frac{\partial u}{\partial r}\right)^4\end{bmatrix}. \tag{2.25}$$

In the presences of body force the law of conservation of momentum is given by Eq. (1.9). Using the definition of substantial derivative it takes the form

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}.\boldsymbol{\nabla})\,\mathbf{V}\right]=\boldsymbol{\nabla}.\mathbf{T}+\rho\mathbf{b}. \tag{2.26}$$

For steady flow $\partial\mathbf{V}/\partial t=0$ and

$$(\mathbf{V}.\boldsymbol{\nabla})\,\mathbf{V}=\left[(0,0,u\,(r))\,.\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\theta},\frac{\partial}{\partial z}\right)\mathbf{V}\right]=0, \tag{2.27}$$

therefore Eq. (2.26) reduce to

$$\boldsymbol{\nabla}.\mathbf{T}+\rho\mathbf{b}=\mathbf{0}. \tag{2.28}$$

Using Maxwell's equation's (1.2) – (1.5) and generalized Ohm's law we get

$$\rho\mathbf{b}=\mathbf{J}\times\mathbf{B}=(0,0,-\sigma B_0^2 u). \tag{2.29}$$

Upon making use of Eq. (2.29) and taking the divergence of stress tensor given by Eq. (2.25), Eq. (2.28) in scalar form becomes

$$0=-\frac{\partial p}{\partial r}+(2\alpha_1+\alpha_2)\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial r}\right)^2\right]+2\left[2\left(\gamma_3+\gamma_4+\gamma_5\right)+\gamma_6\right]\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial r}\right)^4\right], \tag{2.30}$$

$$\frac{\partial p}{\partial \theta}=0, \tag{2.31}$$

$$-\frac{\partial p}{\partial z}+\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+2\frac{(\beta_2+\beta_3)}{r}\frac{d}{dr}\left[r\left(\frac{\partial u}{\partial r}\right)^3\right]-\sigma B_o^2 u=0. \tag{2.32}$$

Note that the induced magnetic field is neglected which is valid assumption for small magnetic Reynold's number. Now first the velocity field is determined from equation (2.32) and then the pressure field can easily be calculated using equation (2.30). It is interesting to mention that the governing equation of velocity field for third and fourth grade fluids is similar. However, the expression for pressure differs. The relevant boundary conditions are:

$$\frac{du}{dr}=0 \text{ at } r=0, \tag{2.33}$$

and

$$u=0 \text{ at } r=R. \tag{2.34}$$

Introducing the non-dimensional variables

$$\zeta=\frac{r}{R},\ f=\frac{4\mu}{\left(\frac{\partial p}{\partial z}\right)R^2}u, \tag{2.35}$$

we find that

$$\frac{du}{dr}=\frac{\partial p}{\partial z}\frac{R}{4\mu}\frac{\partial f}{\partial \zeta}, \tag{2.36}$$

$$\frac{d^2u}{dr^2}=\frac{\partial p}{\partial z}\frac{\partial^2 f}{\partial \zeta^2}\times\frac{1}{4\mu}. \tag{2.37}$$

Thus Eq. (2.32) and boundary conditions (2.33) and (2.34) take the following form

$$\zeta f'' + f' - 4\zeta + \lambda \left[(f')^3 + 3\zeta (f')^2 f'' \right] - M^2 \zeta f = 0, \tag{2.38}$$

$$f'(0) = 0, f(1) = 0, \tag{2.39}$$

in which

$$\lambda = 2 \frac{(\beta_2 + \beta_3)}{16\mu^3} R^2 \left(\frac{dp}{dz} \right)^2,$$

and

$$M^2 = \frac{\sigma B_0^2 R^2}{\mu},$$

is the Hartman number and prime denotes differentiation with respect to ζ.

2.3 Solution of the problem

2.3.1 Case I ($M = 0$)

The governing equation (2.38) in absence of applied magnetic field has the following form:

$$\zeta \frac{d^2 f}{d\zeta^2} + \frac{df}{d\zeta} - 4\zeta + \lambda \left[\left(\frac{df}{d\zeta} \right)^3 + 3\zeta \left(\frac{df}{d\zeta} \right)^2 \frac{d^2 f}{d\zeta^2} \right] = 0.$$

After some re-arrangements this can be written as

$$\frac{d}{d\zeta} \left(\zeta \frac{df}{d\zeta} \right) - 4 \frac{d}{d\zeta} \left(\frac{\zeta^2}{2} \right) + \lambda \frac{d}{d\zeta} \left[\zeta \left(\frac{df}{d\zeta} \right)^3 \right] = 0. \tag{2.40}$$

Integrating above equation once and using the boundary condition

$$f'(0) = 0,$$

we get

$$\zeta f' - 2\zeta^2 + \lambda \zeta f')^3 = 0. \tag{2.41}$$

Following the procedure as describe by Sajid and Hayat [11], we note that equation (2.41) is a cubic equation in f' having one or three real solutions. However, since λ is positive therefore Eq. (2.41) has a unique real solution [12]. Solving this equation for its real root using Mathematica we obtain

$$f'(\zeta) = g(\zeta), \tag{2.42}$$

where

$$g(\zeta) = -\frac{1}{3^{\frac{1}{3}}\left(9\lambda^2\zeta + \sqrt{3}\sqrt{\lambda^3 + 27\lambda^4\zeta^2}\right)^{\frac{1}{3}}} + \frac{\left(9\lambda^2\zeta + \sqrt{3}\sqrt{\lambda^3 + 27\lambda^4\zeta^2}\right)^{\frac{1}{3}}}{3^{\frac{2}{3}}\lambda}, \ \lambda > 0. \tag{2.43}$$

Integration of Eq. (2.43) we get

$$f(\zeta) = \int_0^{\zeta} g\left(\hat{\zeta}\right) d\hat{\zeta} + c_1, \ 0 \leq \zeta \leq 1. \tag{2.44}$$

By using second condition of (2.39) we obtain

$$c_1 = -\int_0^1 g\left(\hat{\zeta}\right) d\hat{\zeta}. \tag{2.45}$$

An exact solution can be obtained by evaluating the integral in Eq. (2.44) numerically.

2.3.2 Case II ($M \neq 0$)

In the presence of applied magnetic field the solution of the problem consisting of equation (2.38) and boundary conditions (2.39) is not possible in a manner as described in section (2.3.1). Further, it is noted that the higher order term gets vanished at

$\zeta = 0$. Therefore, the application of shooting method with Runge-Kutta algorithm is not possible. Therefore we proceed to find the solution of problem using a hybrid method proposed by Ariel [10] .

We rewrite equation (2.38) by substituting

$$y = f, y\prime = f\prime, y\prime\prime = f\prime\prime,$$

as

$$\zeta y'' + y' - 4\zeta + \lambda[(y')^3 + 3\zeta (y')^2 y''] - M^2 y\zeta = 0. \tag{2.46}$$

Introduce a mesh defined by

$$\zeta_j = jh : j = 0, 1, 2, 3, ...N,$$

where N is sufficiently large number and h is the step size.

We now replace the first and second order derivatives in equation (2.46) by the following formulae:

$$y\prime = \frac{y^j - y^{j-1}}{h}, \tag{2.47}$$

$$y\prime\prime = \frac{y^{j+1} - 2y^j + y^{j-1}}{h^2}. \tag{2.48}$$

With the help of Eqs. (2.47) and (2.48) , Eq. (2.46) has the following discritized form:

$$\zeta^j \left(\frac{y^{j+1} - 2y^j + y^{j-1}}{h^2}\right) + \frac{y^j - y^{j-1}}{h} - 4\zeta^j$$
$$+\lambda \left[\left(\frac{y^j - y^{j-1}}{h}\right)^3 + 3\zeta^j \left(\frac{y^j - y^{j-1}}{h}\right)^2 \left(\frac{y^{j+1} - 2y^j + y^{j-1}}{h^2}\right)\right] - M^2 y^j \zeta^j = 0,$$

or

$$y^{j+1} = \left(\frac{\zeta^j}{h} + \frac{3\lambda\zeta^j}{h^2}\left(\frac{y^j - y^{j-1}}{h}\right)^2\right)^{-1} \times \left[\begin{array}{c} \frac{2\zeta^j y^j - \zeta^j y^{j-1}}{h^2} - \frac{y^j - y^{j-1}}{h} + 4\zeta^j - \lambda\left(\frac{y^j - y^{j-1}}{h}\right)^3 \\ -3\lambda\zeta^j\left(\frac{y^j - y^{j-1}}{h}\right)^2\left(\frac{-2y^j + y^{j-1}}{h^2}\right) + M^2 y^j \zeta^j \end{array}\right], \tag{2.49}$$

and the boundary conditions (2.39) become

$$\left(y^{0}\right)' = 0,\ y^{N} = 0. \tag{2.50}$$

Now to start the algorithm we need the values of y^0 and y^1. If we assume

$$y^{0} = f\left(0\right) = s,$$

then by using Taylor series we can write

$$y^{1} = y^{0} + h\left(y^{0}\right)' + \frac{h^{2}}{2!}\left(y^{0}\right)'' + ..., \tag{2.51}$$

$$y^{1} = s + hf'\left(0\right) + \frac{h^{2}}{2!}f''\left(0\right) + ... \tag{2.52}$$

With the help of first boundary condition in (2.50), we have

$$y^{1} = s + \frac{h^{2}}{2}f''\left(0\right). \tag{2.53}$$

The value of $f''(0)$ can be obtained from Eq. (2.46). We differentiate Eq. (2.46) with respect to ζ and set $\zeta = 0$ to get

$$f''\left(0\right) = \frac{4 + M^{2} \times s}{2}. \tag{2.54}$$

Inserting the values of $f''(0)$ in Eq. (2.53), we obtain

$$y^{1} = s + \frac{h^{2}}{4}\left(4 + M^{2}s\right). \tag{2.55}$$

Now we have in hand the values of y^0 and y^1. The integration process will proceed as follows. By giving an approximate value to $f(0)$ we calculate y^2 from Eq. (2.49). Having the values of y^1 and y^2, the values of y^3 can easily be obtained from Eq. (2.49). The cycle is repeated until the values of y have calculated at all mesh point. The value of s is refined by any of zero finding algorithm i.e. Secant or Newton's method until the terminal condition $y^N = 1$ is achieved.

2.4 Results and discussion

The purpose of this section is to analyze the effects of non-Newtonian parameter λ and Hartman number M on $f(\zeta)$. Figure 2.1 presents the effects of λ on non-dimensional velocity f. Figure 2.2 illustrates the effects of M on f. It is clearly observed from Fig. 2.1 that f is an increasing function of λ. Further, as evident from Fig. (2.2), large values of M increases f as a result of decreasing boundary layer thickness. In fact the applied magnetic field provides some mechanism to control the growth of the boundary layer thickness. This provides also a striking difference between the hydrodynamics and hydromagnetic boundary layers. A comparison of the solutions obtained by HAM [7] and Hybrid method is presented in Table 2.1.

Table 2.1 : Values of the velocity field when $\lambda = 0.2$ and $M = 0.2$.

ζ	Anal. Sol. at 15th-order	Numer. Sol.
0.1	-0.797763	-0.800633
0.2	-0.769201	-0.771423
0.3	-0.722225	-0.724016
0.4	-0.658225	-0.659894
0.5	-0.578955	-0.580593
0.6	-0.486029	-0.487528
0.7	-0.380729	-0.381944
0.8	-0.264072	-0.264911
0.9	-0.136915	-0.137336

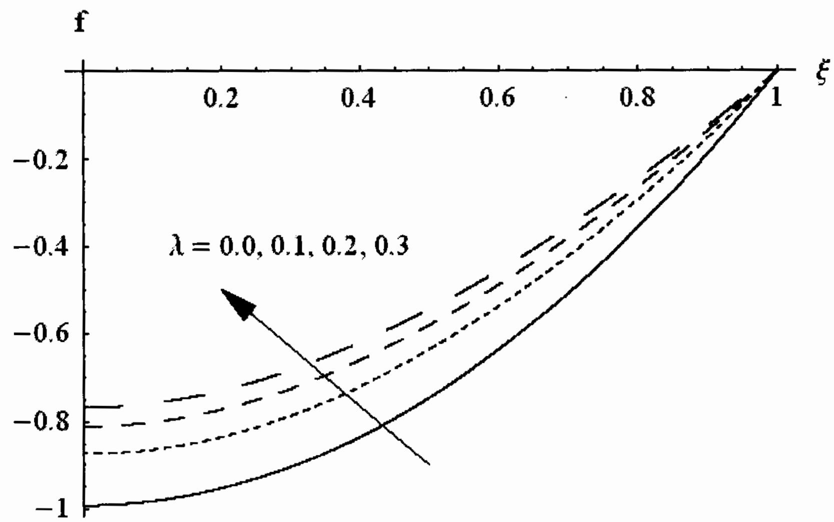

Fig. 2.1 : Variation of the velocity field f for different values of λ $(M = 0.2)$.

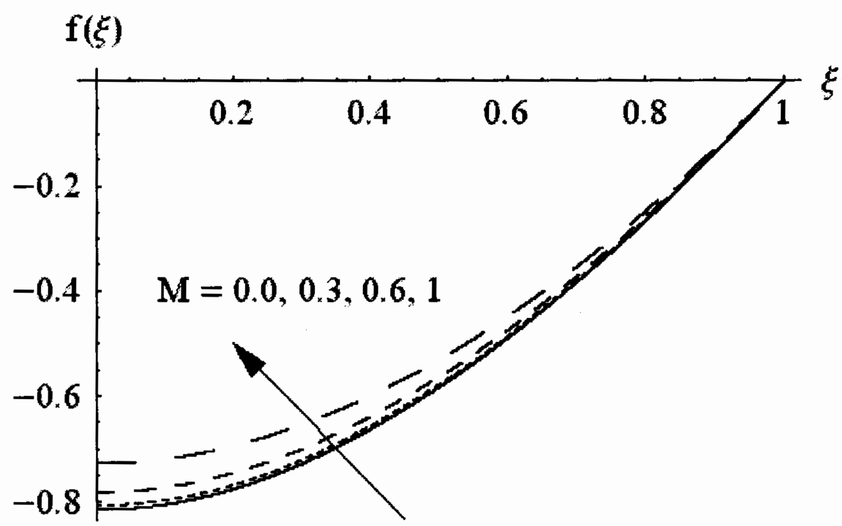

Fig. 2.2 : Variation of the velocity field f for various values of M $(\lambda = 0.2)$.

Chapter 3

Numerical Solution for MHD Pipe Flow of Eyring-Powell Fluid

In this chapter steady flow of an incompressible MHD Eyring-Powell fluid in a circular pipe is studied. The fluid is electrically conducting in the presence of an external uniform magnetic field. Governing equations for velocity and pressure fields are derived. Hybrid numerical method is used to solve the governing equation subject to the appropriate boundary conditions. Graphs are plotted and discussed for various values of the emerging parameters.

3.1 Problem formulation

The geometry and assumptions for the problem under consideration are similar as described in the previous chapter. However, the fluid model considered in this chapter is different. We consider an incompressible Eyring-Powell fluid flowing in a pipe due to a constant applied pressure gradient.

The governing equations for the flow problem are the well known equations of continuity and momentum given by Eqs. (1.8) and (1.9). The assumed form of the velocity field is same as defined in chapter 2. Using this form of velocity field the continuity equcation is satisfied identically and equation of motion in scalar form read as:

$$0 = -\frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial}{\partial r}(r\mathbf{T}_{rr}) - \frac{\mathbf{T}_{\theta\theta}}{r}\right] + (\mathbf{J}\times\mathbf{B})_r, \tag{3.1}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathbf{T}_{r\theta}\right) + (\mathbf{J}\times\mathbf{B})_\theta, \tag{3.2}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\mathbf{T}_{rz}) + (\mathbf{J}\times\mathbf{B})_z. \tag{3.3}$$

Now our aim is to express the components of stress tensor appearing in Eqs. (3.1)–(3.3) in terms of velocity gradients. For this we need the constitutive relationship of an Eyring-Powell fluid, which is given by

$$\mathbf{T} = \mu\nabla\mathbf{V} + \frac{1}{\beta}\sinh^{-1}\left(\frac{1}{c^*}\boldsymbol{\nabla}\mathbf{V}\right). \tag{3.4}$$

In above equcation β and c^* are the material constants. We take the second order approximation of $sinh^{-1}$ function as

$$\sinh^{-1}\left(\frac{1}{c^*}\boldsymbol{\nabla}\mathbf{V}\right) \simeq \frac{1}{c^*}\boldsymbol{\nabla}\mathbf{V} - \frac{1}{6}\left(\frac{1}{c^*}\boldsymbol{\nabla}\mathbf{V}\right)^3, \left[\frac{1}{c^*}\boldsymbol{\nabla}\mathbf{V}\right] \ll \mathbf{1}, \tag{3.5}$$

and re-write (3.4) as:

$$\mathbf{T} = \mu\nabla\mathbf{V} + \frac{1}{\beta c^*}\boldsymbol{\nabla}\mathbf{V} - \frac{1}{6\beta c^*}(\boldsymbol{\nabla}\mathbf{V})^3. \tag{3.6}$$

In index notation the above relation becomes

$$T_{ij} = \mu\frac{\partial u_i}{\partial x_j} + \frac{1}{\beta c^*}\frac{\partial u_i}{\partial x_j} - \frac{1}{6\beta c^{*3}}\left(\frac{\partial u_i}{\partial x_j}\right)^3, \tag{3.7}$$

where u_i $(i = 1, 2, 3)$ denotes the velocity components and x_j $(j = 1, 2, 3)$ are the independent variables. For the velocity field given by Eq. (2.5) the only non-vanishing component of $\mathbf{T}$ is

$$T_{rz} = \mu \frac{\partial u}{\partial r} + \frac{1}{\beta} \sinh^{-1} \left(\frac{1}{c^*} \frac{\partial u}{\partial r} \right), \tag{3.8}$$

or

$$T_{rz} = \mu \frac{\partial u}{\partial r} + \frac{1}{\beta} \left[\frac{1}{c^*} \frac{\partial u}{\partial r} - \frac{1}{6} \frac{1}{c^*} \left(\frac{\partial u}{\partial r} \right)^3 \right], \tag{3.9}$$

or

$$T_{rz} = \left(\mu + \frac{1}{\beta c^*} \right) \frac{\partial u}{\partial r} - \frac{1}{6 \beta c^{*3}} \left(\frac{\partial u}{\partial r} \right)^3. \tag{3.10}$$

With T_{rz} as the only non-vanishing component of $\mathbf{T}$, we can write Eqs. (3.1) – (3.3) as

$$\frac{\partial p}{\partial r} = 0, \tag{3.11}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{3.12}$$

$$-\frac{\partial p}{\partial z} + \frac{1}{r} \left[\begin{array}{c} \left(\frac{du}{dr} + r \frac{d^2 u}{dr^2} \right) \left(\mu + \frac{1}{\beta c^*} \right) - \frac{1}{6 \beta c^{*3}} \left(\frac{du}{dr} \right)^3 - \\ \frac{r}{2 \beta c^{*3}} \left(\frac{du}{dr} \right)^2 \frac{d^2 u}{dr^2} \} - \sigma B_o^2 u r \end{array} \right] = 0. \tag{3.13}$$

The relevant boundary conditions are defined in (2.33) and (2.34). With the help of non-dimensional variables defined by Eq. (2.35) we obtain the following dimensionless equation and boundary conditions for the problem under consideration

$$(1 + N) \left(\frac{\partial f}{\partial \zeta} + \zeta \frac{\partial^2 f}{\partial \zeta^2} \right) - 4\zeta - \frac{\lambda^*}{3} \left(\left(\frac{\partial f}{\partial \zeta} \right)^3 + 3\zeta \frac{\partial^2 f}{\partial \zeta^2} \left(\frac{\partial f}{\partial \zeta} \right)^2 \right) - M^2 \zeta f = 0, \tag{3.14}$$

with

$$N = \frac{1}{\mu \beta c^*},$$

and

$$\lambda^* = \frac{R^2}{32\mu^3\beta c^{*3}}\left(\frac{\partial p}{\partial z}\right)^2,$$

as non-dimensional parameters associated with Eyring-Powell fluid. An exact solution of Eq. (3.14) subject to boundary conditions (2.39) for the special case $M = 0$ can be obtained by a similar procedure as described in section (2.3.1).

3.2 Numerical solution of the problem

Employing the similar procedure as described in section (2.3.2) of chapter 2 for the numerical solution, we obtain the following counterparts of Eqs. (2.49) and (2.55)

$$y^{j+1} = \left[(1+N)\frac{x^j}{h^2} - \lambda^*\frac{x^j}{h^2}\left(\frac{y^j - y^{j-1}}{h}\right)^2\right]^{-1} \times \left[\begin{array}{c} -(1+N)\left(\frac{y^j-y^{j-1}}{h}\right) - \\ (1+N)\,x^j\left(\frac{-2y^j+y^{j-1}}{h^2}\right) + 4x^j + \\ \frac{\lambda^*}{3}\left(\frac{y^j-y^{j-1}}{h}\right)^3 + \\ \lambda^* x^j\left(\frac{-2y^j+y^{j-1}}{h^2}\right)\left(\frac{y^j-y^{j-1}}{h}\right)^2 + \\ M^2 y^j x^j \end{array}\right] = 0, \tag{3.15}$$

and

$$y^1 = s + \frac{h^2}{2}\left(\frac{4 + M^2 s}{2(1+N)}\right). \tag{3.16}$$

After the implementation of the procedure described above in computational software Mathematica, we have obtained the values of f over domain $[0, 1]$ for different values of the parameters. In the next section the results are presented graphically and discussed in detail. A comparison of exact and numerical solutions is provided in Table (3.1).

3.3 Graphical results and discussion

In this section the non-dimensional axial velocity $f(\zeta)$ is plotted against ζ for different values of Hartman number M and non-Newtonian parameters λ^* and N in Figs. 3.1 – 3.3. A comparison of the exact solution (for $M = 0$) is made with the numerical solution in Table 3.1. This table shows that both the solutions are in excellent agreement. Figure 3.1 illustrates the effects of Hartman number M on f. As expected an increase in M decreases the magnitude of f. This is physically admissible because magnetic field causes a resistance to flow. The variation of f with ζ for different values of λ^* is depicted in Fig. 3.2. It is observed from this figure that magnitude of f increases for large values of λ^*. The effects of N on f can be analyzed through Fig. 3.3. It reveals that the magnitude of f decreases rapidly by increasing N.

Table 3.1 : Values of the velocity field

for $\lambda = 0.2$, $M = 0$, $N = 1$.

ζ	Numerical sol.	Exact Sol.
0.0	-0.52558	-0.52558
0.1	-0.52058	-0.52058
0.2	-0.50555	-0.50555
0.3	-0.48041	-0.48041
0.4	-0.44504	-0.44504
0.5	-0.39922	-0.39922
0.6	-0.34270	-0.34270
0.7	-0.27511	-0.27511
0.8	-0.19596	-0.19596
0.9	-0.10457	-0.10457

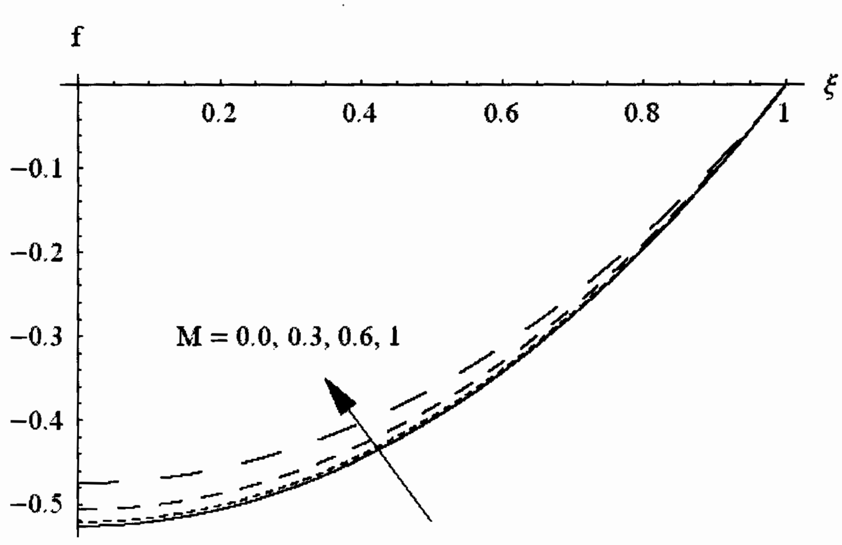

Fig. 3.1 : Variation of the velocity field f with change in parameter M, ($\lambda^* = 0.5$ and $N = 1$).

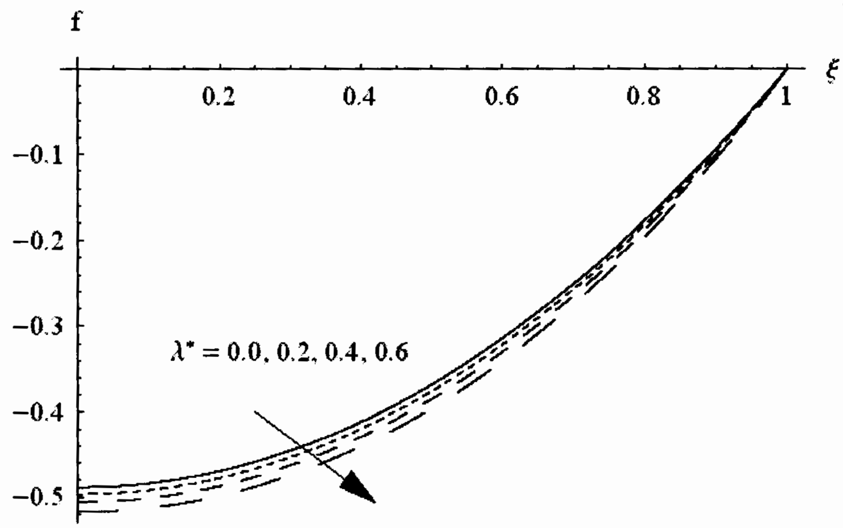

Fig. 3.2 : Variation of the velocity field f with ζ for different values of λ^*, $(M = 0.5$ and $N = 1)$.

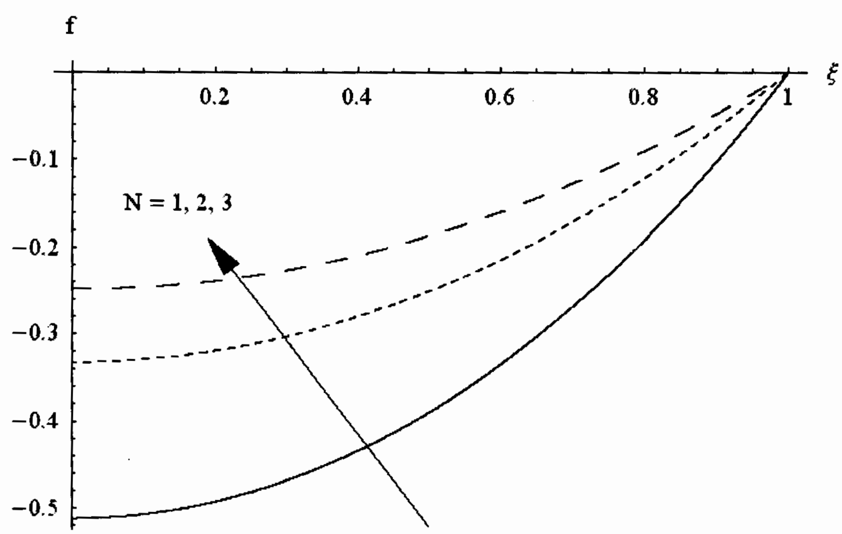

Fig. 3.3 : Variation of the velocity field f with ζ for different values of N, $(\lambda^* = 0.5$ and $M = 0.5)$.

REFERENCES

[1] K.R. Rajagopal, On boundary conditions for the fluids of different type, In; *Sequira* A, editor, Navier-stokes equcation and related non-linear problems. New York; Plenum Press; 273 – 278 (1995).

[2] K.R. Rajagopal, P.N. Kaloni, Some remarks on boundary conditions for fluids of differential type. In: *Graham GAC*, Malik Sk, editors Continuum mechanics and applications. New York: Hemisphere; 935 – 942 (1989).

[3] F.A. Morrison, Understanding rheology. Oxford university press, New York (2001).

[4] R.B. Bird, R.C. Armstrong, O. Hassager, Dynamics of polymeric liquids, Vol.1: *Fluid Mechanics*, Ist ed. JohnWiley & sons, New York, (1977).

[5] F.M. White, Viscous fluid flow, 2nd ed. Mcgraw-Hill, Inc. (1991).

[6] ME. Erdogan, Steady pipe flow of a fluid of fouth grade. Z Angew Math Mech. (*ZAMM*) 61, 466 – 469 (1981).

[7] T. Hayat, M. Sajid, M. Ayub, On explicit analytic solution for MHD pipe flow of a fourth grade fluid. Comm. Non-linear Sci. Num. Simul. 13, 745 – 751 (2008).

[8] T. Hayat, Z. Iqbal, M. Sajid, K. Vajravelu, Heat transfer in pipe flow of a Johnson-Segalman fluid. Int. Comm. Heat Mass Transfer. 35, 1297 – 1301 (2008).

[9] M. Khan, S. Asghar and T. Hayat, Hall effects on the pipe flow of a Burger's fluid: An exact solution. *Nonlinear Anal.*: Real World Appl. 10, 974 – 979 (2009).

[10] P.D. Ariel, A hybrid method for computing the flow of viscoelastic fluids. *Int. J. Num.* Meth. fluids, 14, 757 – 774 (1992).

[11] M. Sajid, T. Hayat, Thin film flow of an Oldroyd 8-constant fluid: an exact solution. Phys. Lett. A 372, 1827 – 1830 (2008).

[12] E.W. Weisstein, Cubic formula. From Wolfram Mathworld, http://mathworld.wolfram.com/cubicformula.html.