DIVISIBILITY & P-INJECTIVITY
ON MONOID (S-ACT) AND FUZZY SET STRUCTURES.

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
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IN THE NAME OF ALLAH

WHO IS

KIND AND THE MOST MERCIFUL

Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

We accept this thesis as confirming to the required standard.

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Dedicated To . . .

My Parents

&

Teachers

for supporting and
encouraging me in my studies.

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ABSTRACT:

The notion of divisibility plays an important role in Algebra. In 1991, Ahsan et al. [2] investigated the concept of divisibility in the more genral case of monoids and their representations, called S-acts where S is a monoid .
An S-act over a monoid S is a non additive generalization of modules over ring and the theory of S-acts has led to the development of a non additive and non commutative Homological Algebra.
Aim of the thesis is to study the divisibility for a monoid S and its representation called S-act, in the context of fuzzy sets. As it has been proved in [2] that the S-act M is divisible if and only if M is P-injective. Now we study this result in fuzzy context and see whether the investigation of divisible S-acts made so far can be extended to the more genral context of fuzzy sets

Chapter 1 contains a brief discussion on Fundamental concepts in semigroup, S -acts, basic definition and results on Fuzzy sets and structures.

In chapter 2 we have studied the concepts of divisibility and P-injectivity for S -act.

Chapter 3 contains the basic definitions and results on fuzzy divisibility and fuzzy P-injectivity. Also we prove the embedding of an arbitrary fuzzy S-act into a fuzzy divisible S-act.

This chapter is basically a review of paper Fuzzy Divisible Semigroups by J.Ahsan and M.Shabir [7].

Contents:

Chapter 1

Fundamental Concepts in Semigroups

Semigroup theory is a thriving field in modern abstract algebra. In this chapter we give a brief introduction to the theory of Algebraic semigroups, S-acts, Fuzzy sets and its structures and Fuzzy S-acts.

1.1 Structure of semigroups

A semigroup is a generalization of the concept of a group only one of the group axiom is retained- associativity; this is the explanation of the term semigroup. Now we give the formal definition of semigroup.

1.1.1 Definition: Let S be a nonempty set. A *binary operation* μ on S is defined as a mapping from $S \times S$ into S, i.e., μ assigns to each pair $(a, b) \in S \times S$ *exactly* one element $\mu(a, b) \in S$. Instead of $\mu(a, b)$ one generally writes $a \mu b$ and moreover, replaces μ by symbols common to denote those operations, say, $a.b$, $a+b$ *or* $a \wedge b$ for instance. Henceforth, we shall write ab instead of $a \mu b$, and usually refer to the binary operation as "." on S.

1.1.2 Definition: Let S be a nonempty set and "." a binary operation on S. Then $(S, .)$ is called a *semigroup* if this operation is associative, that is,

$$a.(b.c) = (a.b).c \quad \text{for all } a, b, c \in S.$$

In particular, a semigroup $(S,.)$ is said to be *commutative* if $a.b = b.a$ for all $a,b \in S$. In what follows, ab will denote $a.b$.

1.1.3 Definition: (a) Let S be a Semigroup. An element $e \in S$ is called a *left identity* (*right identity; identity*) of S if $es = s$ ($se = s$; $se = s = es$) for all $s \in S$.

(b) A semi group S is called a *monoid* if S contains an identity element.

In the sequel, we will usually denote the identity element (if it exists) of a semigroup S by 1. If S has no identity element then it is very easy to adjoin an identity 1 to the set S by defining $1.S = S.1 = S$; for all $s \in S$, and $1.1=1$. Then $S \cup \{1\}$ becomes a semigroup with an identity element 1. We shall use the notation S^1 with the following meaning:

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

and call S^1 the semigroup obtained from S by adjoining an identity element.

1.1.4 Definition: Let S be a semigroup. An element $z \in S$ is called a *left zero* (*right zero*; *zero*) of S if $zs = z$ ($sz = z$; $zs = z = sz$) for all $s \in S$.

A zero is often denoted by 0. If a semigroup S has a left zero and a right zero, then they coincide. In particular S has at most one zero.

If a semigroup S with at least two elements contains a zero element 0 then S is called a *Semigroup with zero*. If S has no zero element then it is easy to adjoin an extra element 0 to the set S, by defining $0.S = S.0 = 0$ and $0.0 = 0$, for all $s \in S$. This makes the set $S \cup \{0\}$ a Semigroup with zero element 0. We shall use the notation S^0 with the following meaning:

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element} \\ S \cup \{0\} & \text{otherwise} \end{cases}$$

and call S^0 the semigroup obtained from S by adjoining a zero (if necessary).

If A and B are two nonempty subsets of a semigroup S, we write

$AB = \{ab : a \in A,\ b \in B\}$.

1.1.5 Definition: An element s of a monoid S is called *left* (*right*) *invertible* if there exists $t \in S$ such that $ts = 1$ ($st = 1$). If there exists $t \in S$ with $ts = st = 1$, then s is called *invertible.*

If x is an element of a semi group S without an identity element, then xS or Sx will not in general contain x. In this situation, we use the notation S^1x for $Sx \cup \{x\}$, xS^1 for $xS \cup \{x\}$ and S^1xS^1 for $SxS \cup Sx \cup xS \cup \{x\}$. Note that S^1x, xS^1 and S^1xS^1 are all sub sets of S (which do not contain 1).

For a semigroup S, if $aS = S$ and $Sa = S$ for all $a \in S$, then it can be shown that S is a group in the usual sense.

1.1.6 Definition: A nonempty subset T of a semigroup S is called a *subsemigroup* of S if $ab \in T$ for all $a, b \in T$. Thus T is a subsemigroup if $T^2 = T.T \subseteq T$.

A subsemigroup T of a semi group S is called a *subgroup* of S if T is a group.

A semigroup S is called a *union of groups* if each element of S is contained in some subgroup G of S. If s is an element of such a semigroup S, then $s \in G$, where G is a subgroup of S.

1.1.7 Definition: An element of a semigroup S which commutes with every element of S is called a *central element* of S. The Set of all central elements of S is either empty or a *sub semigroup* of S, and in the latter case, is called the *center* of S.

1.1.8 Definition: Let A be a subset of a semigroup S. The intersection of all sub semigroups of S containing A is a sub semigroup of S, denoted by $\langle A \rangle$. Clearly, $\langle A \rangle$ contains A and is contained in every other *sub semigroup* of S containing A; it is called the *sub semigroup of S generated by A*. $\langle A \rangle$ may also be described as the set of all elements of S which are expressible as finite products of elements of A. If $\langle A \rangle = S$ then A is called a *set of generating elements* of S, or a *generating set* of S. If A is finite, say, $A = \{a_1, a_2, .., a_n\}$, then $\langle A \rangle = \langle a_1, a_2, ...a_n \rangle$. In particular, if $A = \{a\}$, then $\langle A \rangle = \langle a \rangle = \{a, a^2, a^3, ...\}$. $\langle a \rangle$ is called the *cyclic sub semigroup* of S generated by the element a.

S is called *cyclic* or *monogenic* if $S = \langle a \rangle$ for some $a \in S$, a is then called a *generating element* of S.

1.1.9 Definition: A subset M of generating elements of a semigroup S is called a *basis* of S if every element of S can be uniquely presented as a product of elements of M.

1.1.10 Definition: A semigroup is called *free* if it contains a basis. A monoid T is called *free* if $T = S^1$ for a free semigroup S.

1.1.11 Definition: A nonempty set $K \subseteq S$ is called a *left ideal* of S if $SK \subseteq K$, and *right ideal* of S if $KS \subseteq K$, and an *ideal* or a *two-sided ideal* of S if $KS \subseteq K$ and $SK \subseteq K$.

Clearly, S is an ideal of S and if S has a zero element 0, then $\{0\}$ is an ideal of S. An ideal I of S different from these two ideals is called *proper*.

The definitions of right (left) and two-sided ideals of S generated by a non empty set A of S are given in the usual manner. Note that the right ideal of S generated by A is $A \cup AS = AS^1$ and the two-sided ideal of S generated by A is $A \cup AS \cup SA \cup SAS = S^1AS^1$. If A is a finite subset of S such that $I = S^1AS^1$, then I is a *finitely generated ideal* of S.

1.1.12 Definition: A right (left or two-sided) ideal of S generated by one element set $\{a\}$ is called *principal right (left or two-Sided) ideal* generated by a, and are denoted, respectively, by $R(a)$, $L(a)$ and $J(a)$. Thus

$R(a) = \{a\} \cup aS = aS^1$, $L(a) = \{a\} \cup Sa = S^1a$ and

$J(a) = \{a\} \cup aS \cup Sa \cup SaS = S^1aS^1$.

A semigroup S is called a *principal right (left or two-sided) ideal Semigroup* if every right (left or two-sided) ideal in S is principal.

1.1.13 Definition: Let $(S, .)$ and $(T, *)$ be two semigroups. A function $f: S \to T$ is called a *semigroup homomorphism* of S into T if

$$f(a \,.\, b) = f(a) * f(b) \text{ for all } a, b \in S$$

If such a homomorphism is injective, surjective or bijective, it is called a *monomorphism*, an *epimorphism* or an *isomorphism*, respectively. Finally, a homomorphism of $(S, .)$ into $(S, .)$ is called an *endomorphism* of $(S, .)$, and an *isomorphism* of $(S,.)$ onto $(S,.)$, an *automorphism* of $(S, .)$. A semigroup homomorphism between monoids S and T with $f(1_S) = 1_T$ is called a *monoid homomorphism*.

1.1.14 Definition: A binary relation ρ on a set A is a subset of the Cartesian product $A \times A$. We will write $a\rho b$ and say that a and b are *ρ-related* if $(a, b) \in \rho$ and will call ρ simply a relation.

A relation ρ on A is

Reflexive if $a \rho a$ *for* all $a \in A$

Symmetric if $a \rho b$ implies $b \rho a$,

Antisymmetric if $a \rho b$ and $b \rho a$ implies $a = b$,

Transitive if $a \rho b$ and $b \rho c$ implies $a \rho c$ for all $a, b, c \in A$.

1.1.15 Definition: A reflexive, symmetric, transitive relation ρ is an *equivalence relation*; its classes are ρ-classes and the ρ-class containing an element a will be denoted by $a\rho$. The relation ρ on A for which $a\rho b$ if and only if $a = b$ is the *equality relation* on A and will be denoted by ϵ_A; the relation ρ on A for which $a\rho b$ for all $a, b \in A$ is the *universal relation* on A and will be denoted by ω_A. Both ϵ_A and ω_A are equivalence relations; an equivalence relation on A is *proper* if it is different from ϵ_A and ω_A.

1.1.16 Definition: A relation $\rho \subseteq S \times S$ on a semigroup S is said to be *right* (*left*) *compatible* if for $a, b \in S$, $a\rho b$ implies that $as\rho bs$ ($sa\rho sb$) for all $s \in S$. A *congruence* on S is an equivalence relation that is both right and left compatible.

The *universal congruence* on S denoted by ω_S, is the equivalence relation $(a,b) \in \omega_S$, for all $a,b \in S$.

The *trivial congruence* on S denoted by τ_S is the equivalence relation $(a, b) \in \tau_S \Leftrightarrow a = b$ for all $a, b \in S$. If ρ is a congruence on S, then S/ρ denotes the *set of all equivalence classes* of S determined by ρ.

If $a\rho$ denotes the equivalence class of S containing the element a ($a \in S$), then S/ρ can be made into a semigroup by defining $(a\rho)(b\rho) = (ab)\rho$; S/ρ is called the *factor Semigroup* of S modulo ρ. The function $\rho^{\#} : S \rightarrow S/\rho$ defined by $\rho^{\#}(a) = a\rho$ ($a \in S$) is a (semigroup) *homomorphism*. Let I be an ideal of a semigroup S. Define a relation ρ on S by $a\rho b$ ($a, b \in S$) to mean that either $a = b$ or else both a and b belong to I. Clearly, ρ is congruence on S, called the *Rees congruence* modulo I. The equivalence classes of S modulo ρ are I itself and every one element set $\{a\}$ with $a \in S \setminus I$. We shall write S/I instead of S/ρ, and call S/I the *Rees factor semigroup of S modulo I*. It can be noted that if

$f: S \to T$ is a semigroup homomorphism, then

$Ker\, f = \{(a, b) \in S \times S : f(a) = f(b)\}$ is a congruence on S *and* is called the *kernel congruence of the homomorphism f.*

1.1.17 Definition: A semigroup S is called *left simple* if S has no proper left ideals, *right simple* if S has no proper right ideals, *simple* if S has no proper ideals.

A semigroup with zero is called *0-simple* if $\{0\}$ and S are the only ideals of S, and $S^2 \neq \{0\}$.

A semigroup S is called *semi simple* if $K^2 = K$ for every ideal K of S.

1.1.18 Definition: An element $s \in S$ is called *left cancelable* if $sr = st$ for $r, t \in S$ implies $r = t$; and right cancelable if $rs = ts$ for $r, t \in S$ implies $r = t$; *cancelable* if s is left cancelable and right cancelable.

The semigroup S is called *left cancellative*, *right cancellative* or *cancellative* if all elements of S are left cancelable, right cancelable or cancelable, respectively.

1.1.19 Definition: An element x of a semigroup S is called *idempotent* if $x^2 = x.\, x = x$.

S is called an *idempotent Semigroup* (also called a *band*) if each element of S is idempotent. The set of all idempotents of S is denoted by $E(S)$. Thus S is an *idempotent semigroup* or a *band* if $E(S) = S$.

There is a wide variation in the number of idempotents a semigroup may contain. For example, (**N**, **.**) has the only idempotent 1, the semigroup

(2N, .) does not contain any idempotent, but in (N, *gcd*) all elements are idempotent, where N is the set of natural numbers.

1.1.20 Definition: An element a of a semigroup S is called *regular* if $a \in aSa$, that is, there exists an element b in S such that $a = aba$. A semigroup S is called *regular* if every element of S is regular [10, p.26].

An element $x' \in S$ is said to be an *inverse* of $x \in S$ if and only if $xx'x = x$ and $x'xx' = x'$. A semigroup S is called an *inverse semigroup* if every element of S has a unique inverse. A regular Semigroup S is an inverse semigroup if and only if *its idempotent commute* ([10], Lemma 1.16).

1.1.21 Definition: (i) A reflexive, anti-symmetric and transitive relation on a set A is called a *partial ordering* on A and is usually denoted by $\leq$, one writes $a \leq b$ for $a, b \in A$ if $(a, b) \in \leq$ and calls $(A, \leq)$ a *partially ordered set* or simply a *poset.*

(ii) Let $\leq$ be a partial order on a set A. Then $\leq$ is called a *total order* (*or linear order*) if for each $a, b \in A$, either $a \leq b$ *or* $b \leq a$. If this is the case we say that $(A, \leq)$ is a *totally ordered set.* A totally ordered subset of a partially ordered set is called a *chain.*

In a partially ordered set $(A, \leq)$, an element $c \in A$ is called a *maximal element of A* *if* $c \leq x$ implies $c = x$ for all $x \in A$. Similarly, $d \in A$ is *a minimal element of A if* $x \leq d$ implies $x = d$ for all $x \in A$. Furthermore, an element $a \in A$ is a *greatest element* of A if for all $x \in A$ we have $x \leq a$, and $b \in A$ is a *smallest element* of A if $b \leq x$ for all $x \in A$. Clearly, a partially ordered set has at most one greatest and one smallest element. This however need not be the case with maximal or minimal elements. Now suppose, B is a subset of the partially ordered set $(A, \leq)$, then $a \in A$ is called an upper bound of B if $b \leq a$ for all $b \in B$. similarly, $a \in A$ is a *lower bound* of B if $a \leq b$, for all $b \in B$. The greatest among the lower bounds, whenever it exists, is called the *greatest lower bound* (*g.l.b.)* or the *infimum* (*inf B*). Similarly, the least upper bound (*l.u.b.*) of B, whenever it exists, is called the *Supremum* of B, denoted by *Sup B*. A totally ordered set $(A, \leq)$ is said to be *well-ordered* if every nonempty subset B of A contains a (unique) minimal element i.e. if there exists an element $b \in B$ such that $b \leq x$ for all $x \in B$. In other words, b is the smallest element of B. Thus $(\mathbf{N}, \leq)$ is well ordered, but $(Z, \leq)$ is not (**N** and Z are respectively, the sets of natural numbers and integers, and $\leq$ denotes the usual less than or equal to relation). Recall that the important principle of set theory (known as Zorn's lemma), states that *if $(A, \leq)$ is a partially ordered set such that every chain of elements in A has an upper bound in A, then A has at least one maximal element.*

1.1.22 Definition: A partially ordered set $(L, \leq)$ is called a *lattice* if each subset of, two elements of L has both a *supremum* or "*join*" denoted by $x \vee y$ and an *infimum* or " *meet*" denoted by $x \wedge y$. Thus for all x, y in L, $x \leq y$ if and only if $\sup\{x, y\} = y$ if and only if $\inf\{x, y\} = x$. If $(L, \leq)$ or simply L, has a smallest element with respect to $\leq$, then this element is called *zero element* of $(L, \leq)$ and is denoted by 0. It is easy to show that zero of L, if it exists, necessarily unique. Similarly, the greatest element of L with respect to $\leq$, whenever it exists, is called the *unit element* of L, and it is denoted by 1. The unit element of L, if it exists, is unique. The elements 0 and 1 are called the *universal* *bounds* of L and we have $0 \leq x \leq 1$ for all $x \in L$. It is easy to show that every finite lattice has a 0 and a 1.

A lattice is *complete* if not only every finite but also infinite subset has a supremum and infimum.

If $(E, \leq)$ is a lower semi lattice, then E may be characterized as a commutative idempotent semigroup by defining the product of two elements to be their greatest lower bound. Thus for $e, f \in E$, $e \leq f$ if and only if $ef = fe = e$. Conversely, a commutative band S with the partial order $x \leq y \Leftrightarrow x = xy$ for $x, y \in S$ is a *lower semi lattice.*

1.1.23 Definition: If a lattice L satisfies anyone of the following identities:

$(1)\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

$(2)\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for all $x, y, z \in L$,

then it is called a *distributive lattice*.

Any linearly ordered set or a chain is a distributive lattice, and the lattice of all subsets of a set is distributive. We say that a lattice L is *modular* if for all $a, b, c \in L$, $a \geq b$ implies $a \wedge (b \vee c) = b \vee (a \wedge c)$.

1.2 Fuzzy Sets: Basic Properties

In 1965, Lotfi A. Zadeh first introduced the concept of a fuzzy set. In his classic paper [29], Zadeh defined fuzzy subset of a nonempty set as a collection of objects with grade or degree of membership, each object being assigned a value between 0 and 1. So, a fuzzy set is a generalization of characteristic function wherein the degree of membership of an element is more general than 0 or 1. Each fuzzy set is completely and uniquely determined by a particular membership function. Fuzzy set theory was mathematically formulated by the assumption that classical sets were not appropriate or natural in describing the real-life problems. Fuzzy set theory has greater richness in applications than the ordinary set theory.

This theory has attracted the attention of researchers in a wide variety of fields. The subject is growing enormously and finding applications in such diverse areas as computer science, mathematics, artificial intelligence, pattern recognition, robotics, medical science, social science, engineering, and many other disciplines. In this section, we give some definitions and properties of fuzzy sets which pertain to algebraic operations.

1.2.1 Definition: Let X be a nonempty (usual) set. A *fuzzy set (subset)* μ of the set X is a function $\mu : X \to [0,1]$.

It can be mentioned here that Goguen [13] has generalized the fuzzy subsets of X to L-fuzzy subsets, as a function from X to a complete distributive lattice L. If L is the unit interval $[0,1]$ of real numbers, L-fuzzy subsets are fuzzy subsets in the sense of Zadeh, as above.

1.2.2 Definition: A fuzzy *subset* μ of X is *empty* if and only if μ is identically zero on X. Thus μ is *nonempty (or proper)* if it is not the constant function which always takes the value of 0.

1.2.3 Definition: Two fuzzy subsets μ and λ of a set X are said to be *disjoint* if there exists no $x \in X$ such that $\mu(x) = \lambda(x)$. If $\lambda(x) = \mu(x)$ for all $x \in X$, then we say that λ and μ are equal and write $\lambda = \mu$.

1.2.4 Definition: Let λ and μ be fuzzy subsets of X. Then λ is said to be contained in μ, written as $\lambda \subseteq \mu$, iff $\lambda(x) \leq \mu(x)$ for all $x \in X$, and $\lambda \subset \mu$ iff $\lambda \subset \mu$ and $\lambda \neq \mu$, that is, λ is *properly contained in* μ.

1.2.5 Definition: The *union of two fuzzy subsets* λ and μ of a set X, denoted by $\lambda \cup \mu$, is a fuzzy subset of the set X defined as :

$(\lambda \cup \mu)(x) = max\{\lambda(x), \mu(x)\}$ for every $x \in X$.

The union of any family $\{\mu_i : i \in I\}$ of fuzzy subsets of X is defined by

$(\bigcup_{i \in I} \mu_i)(x) = \sup_{i \in I}\{\mu_i(x)\}$ for all $x \in X$.

It can be noted here that the union of λ and μ is the "*smallest*" fuzzy subset containing both λ and μ. More precisely, if δ is any fuzzy subset of X which contains both λ and μ, then δ also contains the union.

1.2.6 Definition: The *intersection* of two fuzzy subsets λ and μ of a set X, denoted by $\lambda \cap \mu$, is a *fuzzy subset* of X defined as

$(\lambda \cap \mu)(x) = min\{\lambda(x), \mu(x)\}$ for all $x \in X$.

The intersection of any family $\{\lambda_i : i \in I\}$ of fuzzy subsets of X is defined by

$(\bigcap_{i \in I} \lambda_i)(x) = \inf_{i \in I}\{\lambda_i(x)\}$ for all $x \in X$.

It can be shown that the intersection of λ and μ is the "*largest*" fuzzy subset which is contained in both λ and μ.

1.2.7 Definition: The complement of a fuzzy subset μ of a set X is denoted by μ^c and is defined as:

$\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

1.2.8 Proposition: Let λ, μ and υ be any fuzzy subsets of a set X. Then the following properties are immediate:

(a) Commutativity : $\lambda \cup \mu = \mu \cup \lambda$ and $\lambda \cap \mu = \mu \cap \lambda$.

(b) Associativity : $\lambda \cup (\mu \cup \upsilon) = (\lambda \cup \mu) \cup \upsilon$ and $\lambda \cap (\mu \cap \upsilon) = (\lambda \cap \mu) \cap \upsilon$

(c) Idempotent: $\lambda \cup \lambda = \lambda$ and $\lambda \cap \lambda = \lambda$

(d)Distributivity: $\lambda \cup (\mu \cap \upsilon) = (\lambda \cup \mu) \cap (\lambda \cup \upsilon)$ and

$\lambda \cap (\mu \cup \upsilon) = (\lambda \cap \mu) \cup (\lambda \cap \upsilon)$

(e) Absorption: $\mu \cap (\mu \cup \lambda) = \mu$ and $\mu \cup (\mu \cap \lambda) = \mu$

(f) Demorgan's law: $(\mu \cap \lambda)^c = \mu^c \cup \lambda^c$ and $(\mu \cup \lambda)^c = \mu^c \cap \lambda^c$

(g) Involution: $(\mu^c)^c = \mu$.

NOTE: The following properties which are true in ordinary set theory are, in general, no longer valid in fuzzy set theory:

(i) $A \cap A^c = \phi$ but $\mu \cap \mu^c \neq \chi_\phi$, *empty Fuzzy set*

(ii) $A \cup A^c = X$ but $\mu \cup \mu^c \neq \chi_X$, where A is any subset of a set X and μ is any fuzzy subset of X; of course $\chi_X(x) = 1$, for all $x \in X$.

1.2.9 Definition: Let μ be any fuzzy subset of a set X and let $t \in [0,1]$. The set $\mu_t = \{x \in X: \mu(x) \geq t\}$, is called a *level subset* of μ.

Clearly, $\mu_t \subseteq \mu_s$ whenever $t > s$.

1.2.10 Definition: A fuzzy subset μ of X is said to be a *normalized* fuzzy subset if there exists $x \in X$ such that $\mu(x) = 1$.

1.2.11 Definition: Let "." be a binary operation on a set X and λ, μ any two fuzzy sub sets of X. Then the product, $\lambda o \mu$ is defined by

$$(\lambda \circ \mu)(x) = \begin{cases} \sup\limits_{x=y.z} \{\min(\lambda(y), \mu(z))\} \text{ for } x=y.z,\ y,z \in X \\ 0, \text{ if } x \text{ is not expressible as } x=y.z \text{ for all } y,z \in X \end{cases}$$

EXAMPLE

(a) Let X be the set **R** of real numbers and let μ be a fuzzy set of real numbers which are "much greater" than 1. It is possible to give a subjective characterization of μ by defining a function μ on X. Representative values of such function might be

$\lambda(0) = 0,\ \lambda(1) = 0,\ \lambda(5) = 0.1,\ \lambda(10) = 0.2;\ \lambda(100) = 0.95,\ \lambda(500) = 1.$

(b) Let **N** be the set of natural numbers and consider the fuzzy subset of "small" natural numbers:

$\lambda = \{(I, 1), (2,0.8), (3,0.4), (4,0.2), (5,0), (6,0),\ldots\}$

1.2.12 Proposition: Let $S(X)$ be the set of all fuzzy subsets of X. Then $(S(X), \leq, \cap)$ is a complete distributive lattice with least and greatest elements χ_ϕ and χ_X.

Proof: It follows from Definitions 1.2.5 and 1.2.6 and Property 1.2.8 (d)

1.3 Fuzzy Algebraic Structures: A Brief Review

The concept of fuzzy set was applied to generalize different algebraic structures, like other branches of mathematics. In this connection the first attempt was made in 1971 by A. Rosenfeld [27], when he defined the fuzzy subgroupoid and fuzzy subgroups of a group. Several other authors continued the investigation of such concepts (P.S.Das [11], P. Bhattacharya and N.P. Mukharjee [9], J.M. Anthony and H. Sherwood [8]).

1.3.1 Definition: [27]. Let S be a groupoid. A fuzzy set $\mu : S \to [0,1]$ will be called a *fuzzy subgroupoid* of S if, for $x, y \in S$,

$$\mu(xy) \geq min(\mu(x), \mu(y)).$$

If S is a group, a fuzzy subgroupoid μ, of S will be called a *fuzzy subgroup* of S if $\mu(x^{-1}) \geq \mu(x)$ for all $x \in S$.

In [20, 21], Wang-Jin Liu introduced and developed basic results concerning the notions of fuzzy subrings as well as fuzzy ideals of a ring.

1.3.2 Definition: [20, 21]. A nonempty fuzzy subset μ, of a ring R is called a *fuzzy subring* of R, if, for all $x, y \in R$, the following conditions hold:

(i) $\mu\ (x - y) \geq min\ (\mu\ (x),\ \mu\ (y))$, and

(ii) $\mu\ (xy) \geq min(\mu\ (x),\ \mu\ (y))$.

It will be called *fuzzy left ideal* if $\mu\ (xy) \geq \mu\ (y)$; a *fuzzy right ideal* if $\mu\ (xy) \geq \mu\ (x)$; and a *fuzzy ideal* if it is a fuzzy left and right ideal or equivalently, if

$\mu\ (xy) \geq max\ (\mu\ (x),\ \mu\ (y))$.

The properties of fuzzy ideals and fuzzy prime ideals of a ring have heen further studied by many authors, among others (Mukharjee and Sen [25], Zhang [30], Malik and Mordeson [23, 24], Dixit et al. [12]).

The concept of fuzzy module was introduced by Negoita and Ralescu (Applications of Fuzzy Sets to System Analysis [26]), In 1979, N. Kuroki laid the foundation of a theory of fuzzy semi groups in [16].

Subsequently, among others, Kuroki himself (see [16, 17, 18, 19]), Ahsan et al. [3], Ahsan and Saifullah [5], M. Shabir [28] have characterized many classes of semigroups using their various fuzzy ideals.

1.4 S-acts: Essential Definitions and Properties

Let S be a monoid, that is, a semigroup with an identity element 1. In the following S is a monoid with a two-sided zero element and S -acts are representations of S.

1.4.1 Definition: A right unitary S -act M, denoted by M_S is a nonempty set M and a function : $M \times S \to M$ such that if ms denotes the image of (m, s) for

$m \in M$ and $s \in S$, then the following conditions hold:

(i) $(ms)t = m(st)$ for all $m \in M$ and $s, t \in S$

(ii) $m1 = m$ for all $m \in M$.

From the above definition it follows that the monoid S is a right S-act over itself, denoted by S_S. More generally, if I is a right ideal of S, then I is a right S-act through the action $(a, s) \rightarrow as$ $(a \in I, s \in S)$, which is induced by the multiplication in S. One can define left S-acts ${}_SS$ similarly.

1. 4.2 Definition: An element $d \in M_S$ with $ds = d$ for all $s \in S$ is called a *fixed element* of M. Let D denote the set of all fixed elements of M. A right S-act M is called *centered* if S is a semigroup with a two-sided zero element 0 and $|D| = 1$. Thus M is *centered* if and only if there is a fixed element (necessarily unique) denoted by θ such that

(i) $\theta s = \theta$ for all $s \in S$; and

(ii) $m0 = \theta$ for all $m \in M$;

θ will be called the *zero* of M.

1.4.3 Definition: A nonempty subset N of a right S-act M is called an S-subact of M written as $N_S \leq M_S$, if $NS \subseteq N$, that is, $ns \in N$, for *all* $n \in N$ and $s \in S$. We note that $\{\theta\}$ and M are improper S-subacts of M. Thus the subacts of S-act S_S (resp. ${}_SS$) are *right* (resp. *left*) *ideals of S.*

1.4.4 Definition: An equivalence relation ρ on an S-act M is called a (*right*) *congruence* on M if $a\rho b$ $(a, b \in M)$ implies $as\rho bs$ for all $s \in S$, that is, $(a, b) \in \rho$ implies $(as, bs) \in \rho$.

The set of all congruences on M_S form a lattice with universal congruence denoted by ω_M and identity congruence ι_M (as defined in semigroups). Let ρ be a congruence on M_S, then the set of all equivalence classes of M determined by ρ is denoted by M/ρ. Then M/ρ is a right S-act if we define $(m\rho)s = (ms)\rho$ for $m \in M$ and $s \in S$; M/ρ is called the *factor S-act* of M by ρ. If M_S is centered, the zero of M/ρ is $\theta\rho$. If B is an S-subact of an S-act A, then B determines a congruence ρ on A as follows:

For $a, b \in A$, $a\rho b$ if and only if $a = b$ or both a and b belong to B. In this case we write A/B instead of A/ρ and call it *Rees factor S-act* of A by B. If I is an ideal of a semigroup S, then the Rees factor of S modulo I will be denoted by S/I.

The equivalence classes of S/I are I (the zero of S/I) and every single element set $\{a\}$ with $a \in S - I$.

1.4.5 Definition: A right S-act M is called *totally irreducible* if $M_S \neq \theta$ and the only right S-congruences are the universal congruence ω_M and the identity congruence ι_M. Thus if M_S is totally irreducible, then M_S has no proper S-subact.

1.4.6 Definition: An S-act M is called *cyclic* if there exists $x \in M$ such that $M = xS \cup \{x\}$ where $xS=\{xs: s \in S\}$; x is called a *generator* of M_S.

M is called *strictly cyclic* if there exists $x \in M$ such that $M = xS$ and in this case x is called *strict generator* of M_S. If $S = S^1$ then, of course, the difference between the strictly cyclic and cyclic disappears.

1.4.7 Definition: A function f: $M_S \to N_S$ between right S-acts M and N is called an S*-homomorphism* if for each $m \in M$ and $s \in S$,

$$f(ms) = f(m)s.$$

S-monomorphism, *S-epimorphism*, *S-isomorphism* and *S-endomorphism* are defined as usual.

1.5 Fuzzy S-acts: Preliminary Results

Using the basic concepts of fuzzy set occurring in Zadeh [29], we develop some fundamental results regarding fuzzy S-subacts and apply them to generalize some of the results mentioned in [4]. First we recall the definition of fuzzy S-subact of an S-act.

1.5.1 Definition: [1]. Let S be a monoid with a two-sided zero, and M_S a right S-act with a zero element θ_M, A function $\lambda : M \to [0,1]$ is called a fuzzy subact of M if the following condition hold:

$\lambda\ (ms) \geq \lambda\ (m)$ for all $m \in M$, $s \in S$

Similarly, one can define a fuzzy S-subact of a left S-act $_SM$. If $M_S = S_S$, then fuzzy S-subacts are just *fuzzy right ideals* of S. Analogously, the fuzzy S-subacts of $_SS$ are *fuzzy left ideals* of S. A fuzzy subset of the monoid S, which is both a fuzzy right ideal and a fuzzy left ideal of S, is a *fuzzy ideal* of S. We pay special attention to the fuzzy S-subacts $\mathbf{M}$ and $\mathbf{\Phi}$ of M_S defined, respectively, as follows:

$$\mathbf{M}(m) = 1 \text{ for all } m \in M, \text{ and}$$

$$\mathbf{\Phi}(m) = \begin{cases} 0 & \text{if } m \neq \theta_M \\ 1 & \text{if } m = \theta_M \end{cases}$$

1.5.2 Lemma: Let $\{ \alpha_i, i \in \Omega \}$ be a family of fuzzy S-subacts of a right S-act M. Then

(a) $\left(\bigwedge_{i\in\Omega} \alpha_i\right)$ is a *fuzzy* S-subact of M.

(b) $\left(\bigvee_{i\in\Omega} \alpha_i\right)$ is a fuzzy S-subact of M.

Proof: (a) As $\left(\bigwedge_{i\in\Omega} \alpha_i\right)(ms) = \bigwedge_{i\in\Omega} (\alpha_i(ms)) \geq \bigwedge_{i\in\Omega} \alpha_i(m)$ for all $m \in M$ and for all $s \in S$.

Hence $\left(\bigwedge_{i\in\Omega} \alpha_i\right)$ is a *fuzzy S-subact* of M

(b) As $\left(\bigvee_{i\in\Omega} \alpha_i\right)(ms) = \bigvee_{i\in\Omega} (\alpha_i(ms)) \geq \bigvee_{i\in\Omega} (\alpha_i(m))$ for all $m \in M$ and for all $s \in S$.

Thus $\left(\bigvee_{i\in\Omega} \alpha_i\right)$ is a *fuzzy S-subact* of M

1.5.3 Proposition: Let $FA(M)$ be the set of fuzzy right S-subact of M. Then (FA(M), $\leq$, $\wedge$, $\vee$) is distributive complete lattice.

Proof: Let $\alpha, \beta \in FA(M)$. Then by considering Lemma 1.5.2, we see that $(FA(M), \leq, \wedge, \vee)$ is a lattice.

To show $FA(M)$ is complete , let $\{\alpha_i\}_{i\in\Omega} \subseteq FA(M)$. Define α and β in $FA(M)$ as follows:

$$\alpha(x) = \left(\bigwedge_{i\in\Omega} \alpha_i\right)(x) \quad \text{and} \quad \beta(x) = \left(\bigvee_{i\in\Omega} \alpha_i\right)(x) \quad \text{for all } x \in M.$$

By Lemma 1.5.2, $\alpha \in FA(M)$ and it is the greatest lower bound of $\{\alpha_i\}_{i\in\Omega}$ in $FA(M)$.

We note that

(i) $\alpha\wedge(\beta\vee\gamma) = (\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$,

(ii) $(\beta\vee\gamma)\wedge\alpha = (\beta\wedge\alpha)\vee(\gamma\wedge\alpha)$

is true for fuzzy sets α, β and γ.

1.5.4 Definition [1]: Let λ be a fuzzy S-subact of a right S-act M and μ be a fuzzy right ideal of S. Then the product $\lambda \circ \mu$ is the *fuzzy subset* of M defined

1.5.4 Definition [1]: Let λ be a fuzzy S-subact of a right S-act M and μ be a fuzzy right ideal of S. Then the product $\lambda \circ \mu$ is the *fuzzy subset* of M defined by

$$(\lambda \circ \mu)(m) = \bigvee_{m = xs} (\lambda(x) \wedge \mu(s)) \text{ for all } m \in M \, (x \in M, s \in S).$$

The contents of this chapter have been paraphrased from the following sources.

1. N. Kuroki; **On Fuzzy semi groups**, Infor.Scin.53(1991) 203-236.
2. Mati Kilp, Ulrich Knauer, Alexander V. Mikhalev; **Monoids, Acts and Categories**, Walter de Gruyter . Berlin . New York.
3. Khalid Saifullah; **Ph.D Thesis** (supervised by J. Ahsan) August 2005.

Chapter 2

Divisible Monoid

In this chapter we discuss the concept of divisibility in the more general case of monoids and their representations, called S-acts where S is a monoid. Where S-act over monoid S is a non additive generalization of modules over ring. Aim of discussing this chapter is to study the concept of divisibility of S-acts over monoids S in a fuzzy context in next chapter. Most of the contents are taken from [2].

2.1 P-injective and divisible S-acts.

2.1.1 Definition: Let M be a fixed right S-act. Then an S-act Q is called *PM-injective* if each right S-homomorphism from a cyclic S-subact aS $(a \in M)$ of M to Q extends to an S-homomorp ıism from M to Q.

2.1.2 Definition: An S-act Q is called P-injective S-act. If Q is "PS-injective". An S-act all of whose factor S-acts are PM-injective is called completely PM-injective S-act.

2.1.3 Definition: Let Q be an S-act over a monoid S then Q is called divisible or S-divisible if for all $x \in Q$ and $a \in S$ there exists $y \in Q$ such that $x = ya$. The S-act Q is called S-divisible if $Qa = Q$ for all $a \in S$.

2.1.4 Definition: We can define G-act over a groupoid G as follow:

Let G be a groupoid and Let M be a set then a G-act is a mapping

: $M \times G \to G$, such that the image of the pair (m,g) $(m \in M, g \in G)$ is

denoted by mg. Thus every groupoid G is a G-act.

The concept of divisibility is not true in case of groupoid (G-act).

2.1.5 Example: Let $A = \{1, 0, x\}$ be a groupoid with following table.

	1	0	x
1	0	0	x
0	1	0	0
x	x	x	0

Then $1 \in A$ is not divisible by $x \in A$ because there does not exist $y \in A$ such that $1 = yx$

From this example we can also conclude that divisibility cannot be defined in finite structures. Also in case of some infinite Groupoid structure divisibility can not be defined.

2.1.6 Example: **Z** is a groupoid G (and therefore a G-act) then the G-act **Z** does not admit divisibility because $2 \in \mathbf{Z}$ is not divisible by $3 \in \mathbf{Z}$ as there do not exists $y \in \mathbf{Z}$ such that $2 = 3y$.

2.1.7 Proposition: If Q is S-divisible then Q is P-injective.

Proof: Suppose that Q is S-divisible. We show that Q is P-injective. Let

aS $(a \in S)$ be any principal right ideal of S and let $\phi : aS \rightarrow Q$ be an S-homomorphism determined by the element $\phi(a) = x \in Q$ that is $\phi(as) = xs$ for all $s \in S$. Since Q is S-divisible, there exists an element $y \in Q$ such that: $x = ya \rightarrow (*)$.

Define $\psi : S \rightarrow Q$ by $\psi(1) = y$ that is, $\psi(s) = ys$ for all $s \in S$. Then $\psi(as) = \psi(1)as = yas = xs = \phi(as)$ for $s \in S$. This shows that ψ is an extension of ϕ. Thus Q is P-injective.

2.1.8 Proposition: *If A is a retract of an S-divisible S-act Q, then A is S-divisible.*

Proof: Let p be the retraction and q the coretraction such that $poq = i_A$. Let $x \in A$ and $a \in S$, then $q(x) \in Q$. Since Q is S-divisible, there exists $y \in Q$ such that $q(x) = ya$. Then $x = poq(x) = p(q(x)) = p(ya) = p(y)a$ and $p(y) \in A$. This shows that A is S-divisible.

2.1.9 Definition: A right S-act A is said to be right S-cancellative if A has the following property:

$xs = x's$ for all $x, x' \in A$ and $s \in S$ implies that $x = x'$. Thus S is right cancellative if S_S is right S-cancellative. And A is said to be left S-cancellative if $xs = xs'$ for $x \in A$ and $s, s' \in S$ implies that $s = s'$. Thus S is left cancellative if S is left S-cancellative, that is, S is left cancellative as a left S-act.

2.1.10 Proposition: *If A is a retract of a right S-cancellative (left S-cancellative) S-act B, then A is right S-cancellative (left S-cancellative).*

Proof: Let p be the retraction and q the corectraction such that $poq = i_A$. Let $xs=x's$ for $x,x' \in A$ and $s \in S$ then $q(xs)=q(x's)$. This implies that $q(x)s = q(x')s$. Thus $q(x)=q(x')$, since B is S-cancellative. As $poq=i_A$, $p(q(x))=p(q(x'))$. This implies $x = x'$. Hence A is S-cancellative. Similarly, if $xs = xs'$ then $q(xs)=q(xs')$. This implies that $q(x)s=q(x)s'$. But since B is left S-cancellative, $s = s'$. Hence A is left S-cancellative.

2.1.11 Proposition: For a left cancellative monoid S, the following assertions are equivalent:

(1) Q is a (completely) P *-injective right* S-act.

(2) Q is a (completely) S *-divisible right* S-act.

Proof: (2) $\Rightarrow$ (1): This follows from Proposition 2.1.7

(1) $\Rightarrow$ (2): Let $x \in Q$ and $a \in S$. Define a map $\phi : aS \to Q$ by $\phi(as) = xs$ for all $s \in S$. Since S is left cancellative, ϕ is a well defined S-homorphism. Also, since Q is P-injective, there exists an extension ψ from S to Q. Then $x=\phi(a)=\psi(a)=\psi(1.a)=\psi(1)a$ and $\psi(1) \in Q$, this shows that Q is S-divisible.

2.1.12 Proposition: The following assertions are equivalent:

(1) All right S-acts are S *-divisible.*

(2) All right ideals of S are S-divisible.

(3) S is S-divisible.

(4) S is a group.

(5) All right S-acts are P-injective.

(6) S is P-injective.

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4): Let a be an element of S. Then, since S_S is divisible, there exists an element b of S with $1 = ba$. Thus a is left invertible. It, then, follows that a is invertible. This shows that S is a group.

(4) $\Rightarrow$ (1) Let a be an element of S. From (4), there is an element $b \in S$ with $1 = ba$. Thus $x = x\,1 = x(ba) = (xb)a$. Hence $Q = Qa$. That is Q is S-divisible.

Therefore, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4):

Now suppose S is a group and so, in particular, cancellative.

Hence by Proposition 2.1.11, (1) $\Leftarrow$ (5) and (3) $\Leftrightarrow$ (6). This completes the proof of above proposition.

2.2 EMBEDDING Of AN ARBITRARY S–ACT INTO A DIVISIBLE S-ACT.

We construct an S-divisible S-act $Q(A)$. from a right S-act A under some conditions. Consider the set $A \times S = \{ (x,a) : x \in A \text{ and } a \in S \}$, we define S-action on this set as follow:

$(x,a)s = (xs,a)$ for all $s \in S$. Then the set $A \times S$ together with this S-action, is a right S-act and we denote it by $Q(A)$. Now we define a relation $\equiv$ on $Q(A)$ as, $(x,a) \equiv (x',a') \Leftrightarrow xa = x'a$.

2.2.1 Lemma: *If S is a commutative monoid and A is a right S-cancellative S-act then the above relation $\equiv$ is an S-congruence on $Q(A)$.*

Proof: To prove that $\equiv$ defined above is an S-congruence we show that the relation $\equiv$ is an *Equivalence relation* and is *compatible.*

a) By definition, the relation $\equiv$ is *reflexive*

b) Also *symmetric*

c) To show that $\equiv$ is *transitive*, Let $(x,a)\equiv(x',a')$ and $(x',a')\equiv(x'',a'')$ for $x,x',x''\in A$ and $a,a',a''\in S$. Since by assumption $xa'=x'a$ and $x'a''=x''a'$ and S is assumed to be commutative, we have

$xa''a' = xa'a'' = x'aa'' = x'a''a = x''a'a = x''aa'$. Thus $xa''a' = x''aa'$.

Since A is right S-cancellative, we have $xa'' = x''a$ this shows that $(x,a)\equiv(x'',a'')$.

Thus $\equiv$ is an *Equivalence relation.*

Finally, *compatibility* with S follows directly from definition and commutativity of S. Thus the relation $\equiv$ is an S-*congruence* on $Q(A)$.

Note: From this lemma we are able to construct a factor S-act $Q(A)/\equiv$ which is denoted by $\overline{Q(A)}$. For each element $(x,a)\in Q(A)$, we shall denote by $\overline{(x,a)}$ the corresponding element of $\overline{Q(A)}$. Moreover the S-action on $\overline{Q(A)}$ is defined by $\left(\overline{x,a}\right)s = \overline{(x,a)s} = \overline{(xs,a)}$ for $s\in S$.

2.2.2 Proposition: *Let S be a commutative monoid and A a right S-cancellative S-act. Then $\overline{Q(A)}$ has the following properties:*

1) $\overline{Q(A)}$ is right S-divisible with A considered as an S-subact of $\overline{Q(A)}$.

2) $\overline{Q(A)}$ is a right S-cancellative.

3) For every $\overline{(x,a)} \in \overline{Q(A)}$, $\overline{(x,a)}a = \overline{(x,1)}$

Proof: (1) Define $q : A \to \overline{Q(A)}$ by $q(x) = \overline{(x,1)}$. Then q is an injective S-homomorphism, thus we may consider A as an S-subact of $\overline{Q(A)}$. Let $\overline{(x,a)} \in \overline{Q(A)}$ and $s \in S$. Since S is commutative, $xas = xsa$. This shows that $(x,a) \equiv (xs, as)$ and $\overline{(x,a)} = \overline{(x,as)}\,s$ this means that $\overline{Q(A)}$ is S-divisible.

(2) Suppose that $\overline{(x,a)}\,s = \overline{(x',a')}\,s$ then $\overline{(xs,a)} = \overline{(x's,a')}$. Thus $xsa' = x'sa$. Since S is commutative, $xa's = xas$ and since A is right S-cancellative, we have $xa' = xa$ .This means that $\overline{(x,a)} = \overline{(x',a')}$. Hence $\overline{Q(A)}$ is right S-cancellative.

(3) For every $\overline{(x,a)} \in \overline{Q(A)}$, $\overline{(x,a)}\,a = \overline{(xa,a)} = \overline{(x,1)}$.

2.2.3Corollary: Let S be a commutative and cancellative monoid. Then $\overline{Q(S)}$ is S-divisible and $S \subseteq Q(S)$. In this case, $\overline{Q(S)}$ is a commutative group with the following multiplication:

$$\overline{(b,a)} \cdot \overline{(b',a')} = \overline{(bb',aa')}$$

2.2.4 Proposition: Let S be a commutative monoid and A a right S-cancellative S-act. Then the following assertions are equivalent:

(1) A is *S-divisible.*

(2) A is retract of $\overline{Q(A)}$.

Proof: (2) $\Rightarrow$ (1), this follows from Proposition 2.1.8 since $\overline{Q(A)}$ is S-divisible.

(1) $\Rightarrow$ (2): In order to define a retraction $p : \overline{Q(A)} \rightarrow A$, let $\overline{(x,a)} \in \overline{Q(A)}$. Since A is S-divisible, there exists $y \in A$ such that $x = ya$. Since A is right S-cancellative, y is unique. Then we define p by $p\,\overline{(x,a)} = y$ for all $\overline{(x,a)} \in \overline{Q(A)}$. Now suppose that $(x,a) \equiv (x',a')$ with $x = ya$ and $x' = y'a'$. Since $xa' = x'a$ and $yaa' = y'a'a$, $yaa' = y'aa'$, by the commutativity of S. Also, since A is right S-cancellative, $y = y'$. Thus the map p is well defined.To show that p is an S-homomorphism, let $p\,\overline{(x,a)} = y$ with $x = ya$, and $p\,\overline{(xs,a)} = y'$ with $xs = y'a$. Then $yas = y'a$. Since S is commutative, we have $ysa = y'a$. Since A is right S-cancellative, it follows that $y' = ys$.

Hence $p(\overline{(x,a)}) = p\,\overline{(xs,a)} = y' = ys = p\,\overline{(x,a)}s = p\,\overline{(x,a)s}$. This shows that p is an S-homomorphism. Let $q : A \rightarrow \overline{Q(A)}$ be the inclusion defined by $q\,(x) = \overline{(x,1)}$. Then $p \circ q(x) = p\overline{(x,1)} = x$ because $x = x1$. Thus A is retract of $\overline{Q(A)}$.

2.2.5 Corollary: Let S be a commutative and cancellative monoid then the following assertions are equivalent:

(1) S is a commutative group.

(2) S, considered as an S-act, is P-injective.

(3) S, considered as an S-act, is S-divisible.

(4) S is a retract of $\overline{Q(S)}$.

Finally, we prove the following universal property for $Q(A)$:

2.2.6 Theorem: Let S be a commutative monoid and A a right S-cancellative S-act. Then there exist an S-act $\overline{A}$ and an S-homomorphism $f: A \to \overline{A}$ satisfying the following four conditions:

(1) f is injective.

(2) Each element of $f(A)$ is S-divisible in $\overline{A}$.

(3) $\overline{A}$ is right S-cancellative.

(4) For each $y \in \overline{A}$, there exist $a \in S$ and $x \in A$ such that $ya = f(x)$.

If $\overline{A'}$ and f' satisfy the conditions (1) through (4) then there exists a unique S-isomorphism $\overline{\phi}: \overline{A} \to \overline{A'}$ such that $f' = \overline{\phi} \circ f$.

Proof: Since $Q(A)$ satisfy conditions (1) through (4), we need only to prove the last part. To define a map $\overline{\phi}: \overline{A} \to \overline{A'}$, let y be an element of $\overline{A}$. By condition (4), there exist $a \in S$ and $x \in A$ such that $ya = f(x)$. For $f'(x) \in \overline{A'}$ and $a \in S$, there exist $y' \in \overline{A'}$ such that $y'a = f'(x)$, by condition (2). Now let $ya' = f(x')$ and $y''a' = f'(x')$ be another expression. Then $yaa' = f(x)a' = f(xa')$ and $ya'a = f(x')a = f(x'a)$. Hence we have $xa' = x'a$ by the commutativity of S and injectivity of f. Thus it follows that:

$yaa' = f'(x)a' = f'(xa') = f'(x'a) = f'(x')a = y''a'a$. By the commutativity of S and right S-cancellativity of $\overline{A'}$, we have $y' = y''$. This shows that $y' \in \overline{A'}$ is uniquely determined from $y \in \overline{A}$ by the rule: $ya = f(x)$ and $y'a = f'(x)$. Thus we may define a mapping $\phi : \overline{A} \to \overline{A'}$ by $\phi(y)a = f'(x)$, for all $y \in \overline{A}$. To show that $\phi(ys) = \phi(y)s$, let $\phi(y) = y'$ and $\phi(ys) = y''$. Since $ya = f(x)$, $ysa = yas = f(xs)$. Therefore, $y''a = f'(xs) = f'(x)s = y'sa$. Since $\overline{A'}$ is S-cancellative, $y'' = y's$. Thus we have an S- homomorphism $\phi : \overline{A} \to \overline{A'}$. By the definition of ϕ, we may easily check that ϕ is an S-isomorphism such that $f' = \phi \circ f$.

Finally, suppose that $f' = \phi' \circ f$ and $ya = f(x)$. Then $\phi'(y)a = f'(x) = \phi(y)a$. Since $\overline{A'}$ is right S-cancellative, we have $\phi'(y) = \phi(y)$. This establishes the uniqueness of ϕ with the property that $f' = \phi \circ f$.

Remark: If $\overline{A}$ satisfy condition (1) through (4), then $\overline{A}$ is S-divisible (and, therefore, P-injective). To see this, suppose $y \in \overline{A}$ and $a \in S$. By condition (4), there exists $b \in S$ and $x \in A$ such that $yb = f(x)$. By condition (2), for $f(x) \in f(A)$ and $ab \in S$, there exists $z \in A$ such that $f(x) = zab$. Hence $yb = zab$. By condition (3), it follows that $y = za$. This shows that $\overline{A}$ is S-divisible.

2.3 Characterization of Monoids by P-injective S-act.

2.3.1 Definition: A right S-act M is called *regular* if, for each $a \in M$, there exists an S-homomorphism $f \in Hom_S(aS,S)$ such that $a = af(a)$. A monoid S is called *regular* if S_S is regular as an S-act. An S-act M is called *von Neuman regular* if, for each $a \in M$, there exists an S-homomorphism $g \in Hom_S(S,S)$ such that $a = ag(a)$. Thus if S_S is von *Neuman regular*, then for each $a \in S$, there exists $g \in Hom_S(S,S)$ such that $a = ag(a) = ag(1)a$ and $g(1) \in S$. Hence S is von Neuman regular in the familiar sense.

2.3.2 Definition: Let M and Q be the right S-acts. Q is called M-Projective if for each S-epimorphism $g: M \to \overline{M}$ and each S-homomorphism $h: Q \to \overline{M}$ there exists an S-homomorphism $k: Q \to M$ such that $g \circ k = h$. Thus Q is projective if Q is M-Projective for each S-act M. We notice that every monoid S is always Projective.

Dually, Q is M-injective if, for each S-monomorphism $g: N \to M$ and each S-homomorphism $h: N \to Q$ there exists an S-homomorphism $k: M \to Q$ such that $k \circ g = h$. Thus Q is injective if Q is M-injective for each S-act M.

2.3.3 Definition: A right S-act M is called a *right PP S-act* if each cyclic S-subact aS of M with $a \in M$ is projective. S is called a *right PP monoid* if all its principal right ideals are projective as right S-acts.

2.3.4 Proposition: For an S-act M, the following are equivalent:

(1) M is regular S-act.

(2) M is PP S-act.

2.3.5 Corollary: For a monoid S the following are equivalent:

(1) S is *regular*.

(2) S is a *PP-monoid.*

(3) Every projective S-act is *regular.*

For a right S-act M and $a \in M$, we may always define an S-epimorphism $\pi : S \rightarrow aS$ defined by $\pi(s) = as$, for all $s \in S$, and also we have an inclusion $k : as \rightarrow M$.

2.3.6 Proposition: The following conditions for an S- act M are equivalent:

(1) M is regular.

(2) For each $a \in M$, aS is *retract* of S.

(3) For each $a \in M$, $\pi : S \rightarrow aS$ defined by $\pi(s) = as$, for all $s \in S$, is *retraction.*

Also, a monoid S is von *Neumann regular* if and only if the inclusion $k : aS \rightarrow S$ is *coretraction* for each $a \in S$.

2.3.7 Proposition: For an S-act M the following assertions are equivalent:

(1) M is von Neumann regular.

(2) M is regular and S is PM-injective.

Proof: (1) $\Rightarrow$ (2). Suppose that M is von Neumann regular. It easily follows that

M is regular. We show that S is PM-injective. Let $aS\,(a \in M)$ be a cyclic S-sub act of M and let $f : aS \to S$ be an S-homomorphism.

Since M is von Neumann regular and $a \in M$, there exists an S-homomorphism $g : M \to S$ such that $a == ag(a)$. Define $\overline{f} : M \to S$ by $\overline{f}(x) = f(a)g(x)$, for all $x \in M$. Clearly, $\overline{f}$ is an S-homomorphism which extends f. Hence S is PM-injective.

$(2) \Rightarrow (1)$: Suppose that M is regular and S is PM-injective. Then, for every $a \in M$, there exists an S-homomorphism $f : aS \to S$ such that $a == af(a)$. Since S is PM-injective, there exists an S-homomorphism $g : M \to S$, extending f. Hence $a == ag(a)$. Showing that M is von *Neumann regular*.

Chapter 3

Fuzzy Divisibility and Fuzzy P-injectivity

In this chapter we shall define and characterize the fuzzy divisibility and fuzzy P-injectivity for monoid (semigroup) S and its S-acts. We shall also prove the embedding of an arbitrary fuzzy S-act into a fuzzy divisible S-acts for a (Commutative) Cancellative monoid S.

3.1 Fuzzy Divisibility

3.1.1 Definition: Let $m \in M$ and $t \in (0,1]$, then the fuzzy sub-set of M defined as

$$m_t(x) = \begin{cases} t & \text{if } x = m \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } x \in M$$

is called a *fuzzy point* with support m and value t. A fuzzy point m_t is said to belongs to a fuzzy sub-set λ of M written as $m_t \in \lambda$ if

$\lambda(m) \geq t$ (cf. [6]).

3.1.2 Definition: A fuzzy sub-act λ of a right S-act M is called weakly divisible if for each $m_t \in \lambda$ and $s \in S$ there exists $y_p \in \lambda$ such that $\lambda(m) = \lambda(ys)$.

3.1.3 Definition: A fuzzy subact λ of a right S-act M is called *divisible* if for each $m_t \in \lambda$ and $s \in S$ there exists $y_p \in \lambda$ such that $m = ys$.

Every divisible fuzzy sub-act 'λ' of right S-act M is weakly divisible but the converse is not true.

3.1.4 Example: Consider the monoid $S=\{1,o,a,b\}$ *as* a right S-act over it self with calay's table

	1	0	a	b
1	1	0	a	b
0	0	0	0	0
a	a	0	a	b
b	b	0	a	b

Here for $a \in S$ and $b \in S$ there do not exist any $x \in S$ such that $a = xb$.

This shows that 'S' is not *divisible* as a right S-act but we can show that there exists a fuzzy *subact* of S which is *weakly divisible.* A fuzzy sub-set $\lambda : S \to [0,1]$ is fuzzy *subact* of S if and only if

(i) $\lambda(0) \geq \lambda(x)$ for all $x \in S$,

(ii) $\lambda(a) = \lambda(b)$ and

(iii) $\lambda(x) \geq \lambda(1)$ for all $x \in S$.

as $\lambda(1) = \lambda(0) = \lambda(a) = \lambda(b) = 1$

Then 'λ' *is a weakly divisible fuzzy subact of* S *but not fuzzy divisible.*

3.1.5 Lemma: *Let M_S be a right S-act and 'A' be a non-empty sub-set of 'M' then the characteristic function δ_A of A is fuzzy sub-act of M if and only if A is an S-subact of M.*

3.1.6 Lemma: Let M be a right S-act and A be a non-empty sub-set of M. If A is a *divisible* S-subact of M then *the characteristic function* δ_A *of* A *is fuzzy divisible.*

Proof: Suppose that A is divisible S-subact of M. Then by *Lemma 3.1.5,* δ_A is a fuzzy subact of M. Let $m_t \in \delta_A$ for some $t \in (0,1]$ and $s \in S$. Then $\delta_A(m) \geq t > 0$. Thus $\delta_A(m) = 1$ and so $m \in A$. Since A is divisible so there exists $y \in A$, such that

$m = ys$. Since $y \in A$, so $y_p \in \delta_A$ for each $p \in (0,1]$. Thus δ_A is a *divisible fuzzy* subact of M .

3.1.7 Corollary: If A is a divisible S sub-act of M then δ_A is *weakly divisible fuzzy sub-act of M.*

Proof: By above Lemma 3.1.6 δ_A is *divisible* and since every divisible fuzzy sub-act is weakly divisible so δ_A *is weakly divisible but the converse is not true.*

3.1.8 Example: Consider the same example 3.1.4, S is not divisible but δ_S ($\delta_s(1)= \delta_s(0)=\delta_s(a) =\delta_s(b)=1$) *the characteristic function of* S *is weakly divisible.*

3.1.9 Proposition: Let M be a right S-act and A be a non empty sub-set of M then δ_A is a *divisible fuzzy sub-act of M if and only if* A *is divisible.*

Proof: Suppose that δ_A is divisible fuzzy sub-act of M. Then by Lemma 3.1.5 A is an S-subact of M. Let $a \in A$ and $s \in S$ then $a_t \in \delta_A$ for all $t \in (0,1]$, so there exists $x_p \in \delta_A$ such that $a = xs$. Since $x_p \in \delta_A$, so $\delta_A(x) \geq p > 0$ that is $\delta_A(x) = 1$. Hence $x \in A$. This shows that A is divisible.

Conversely, if A is divisible S-subact of M then by Lemma 3.1.6 δ_A is *divisible* fuzzy subact of M.

3.1.10 Note: *Let* M *be a right* S*-act and* λ *a fuzzy sub-act of M, then the pair* (M, λ) *is called a fuzzy* S*-act.*

3.1.11 Definition: Let (M, λ) and (N, μ) be two *fuzzy* S*-act.* An S-homomorphism $f: M \to N$ is called a fuzzy *S-homomorphism* from (M, λ) to (N, μ) if $\mu(f(m)) \geq \lambda(m)$ for all $m \in M$.

3.1.12 Definition: A fuzzy S-act (M, λ) is called a *retract* of a fuzzy S-act (N, μ) if there exist fuzzy S-homomorphisms $p:(N, \mu) \to (M, \lambda)$ and $q:(M, \lambda) \to (N, \mu)$ such that $poq = 1_M$.

3.1.13 Definition: A fuzzy S-act (M, λ) is called *divisible* (*weakly divisible*) fuzzy S-act if M is *divisible* S-act and λ is *divisible* (*weakly divisible*) fuzzy subact of M.

3.1.14 Proposition: *A retract of a divisible fuzzy S-act is divisible.*

Proof: Let (M, λ) be a divisible fuzzy S-act and (N, μ) be a retract of (M, λ). Then there exist fuzzy S-homomorphisms $f:(N, \mu) \to (M, \lambda)$ and $g:(M, \lambda) \to (N, \mu)$ such that $gof = 1_N$. By Proposition *2.1.8* N is a S-divisible S-act. Let $x_t \in \mu$ and $s \in S$. Then $\mu(x) \geq t$. As $f(x) \in M$ and $\lambda(f(x)) \geq \mu(x) \geq t$, so $(f(x))_t \in \lambda$. Since 'λ' is divisible fuzzy sub-act of M so there exists $y_p \in \lambda$ such that $f(x) = ys$. Thus $g(f(x)) = g(ys)$ implies $x = g(y)s$. Also $\mu((g(y))) \geq \lambda(y) \geq p$ implies $(g(y))_p \in \mu$. This shows that 'μ' is a *divisible* fuzzy subact of N. Hence (N, μ) is a *divisible* fuzzy S-act.

3.1.15 Definition: Let f be a mapping from a set X into a set Y and μ, γ be fuzzy sub-sets of X and Y respectively. The fuzzy sub-sets $f(\mu)$ and $f^{-1}(\gamma)$ of X and Y respectively are defined by:

$$f(\mu)(y) = \begin{cases} \bigvee\{\mu(x) : x \in X \text{ and } f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } y \in Y$$

$f^{-1}(\gamma)(x) = \gamma(f(x))$ for all $x \in X$ are called, respectively, the *image* of μ under f and the *pre image* of γ under f.

3.1.16 Proposition: Let M and N be S-acts and f an S-homomorphism from M into N. Then

a) If λ is a fuzzy subact of M, then $f(\lambda)$ is a *fuzzy subact* of N.

b) If μ is a fuzzy subact of N, then $f^{-1}(\mu)$ is a *fuzzy subact* of M.

Proof: Let $n \in N$ then

$$f(\lambda)(n) = \begin{cases} \bigvee \{\mu(m) : m \in M \text{ and } f(m) = n \text{ if } f^{-1}(n) \neq \phi\} \\ 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

If $f(\lambda)(n) = 0$ Then $f(\lambda)(ns) \geq f(\lambda)(n)$.

If $f(\lambda)(n) = \bigvee_{m \in f^{-1}(n)} \mu(m)$, then $f(m) = n$ and so $f(ms) = f(m)s = ns$

Thus $f^{-1}(ns) \neq \phi$ and $ms \in f^{-1}(ns)$ for all $m \in f^{-1}(n)$.

Hence $f(\lambda)(ns) = \bigvee_{x \in f^{-1}(ns)} \mu(m) \geq \bigvee_{m \in f^{-1}(n)} \mu(ms) \geq \bigvee_{m \in f^{-1}(n)} \mu(m) = f(\lambda)(n)$.

Thus $f(\lambda)(ns) \geq f(\lambda)(n)$. This shows that $f(\lambda)$ is a fuzzy sub-act of N.

b) Let μ be a fuzzy subact of N. Then for all $m \in M$ and $s \in S$,

$f^{-1}(\mu)(ms) = \mu(f(ms) = \mu(f(m)s) \geq \mu(f(m)) = f^{-1}(\mu)(m)$. Hence $f^{-1}(\mu)$ is a *fuzzy subact* of M.

3.1.17 Lemma: Let A be a subact of right S-act and $\widetilde{\eta}$ be a fuzzy sub-act of A. Then the fuzzy sub-set of M defined by

$$\widetilde{\eta}(m) = \begin{cases} \eta(m) & \text{if } m \in A \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } m \in M$$

is a fuzzy subact of M.

Proof: Let $m \in M$ if $\widetilde{\eta}(m) = 0$, then $\widetilde{\eta}(ms) \geq \widetilde{\eta}(m)$.

If $\widetilde{\eta}(m) = \eta(m)$. Then $m \in A$ and so $ms \in A$ for all $s \in S$.

Thus $\widetilde{\eta}(ms) = \eta(ms) \geq \eta(m) = \widetilde{\eta}(m)$. Hence $\widetilde{\eta}$, is a fuzzy subact of M.

We know that a right S-act Q is called PM-injective (M is a fixed right S-act) if each S-homomorphism from a cyclic S-subact aS $(a \in M)$ of M to Q extends to an S-homomorphism from M to Q. In particular Q is called P-injective S-act if Q is PS-injective.

3.2 Fuzzy *P*-injectivity

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3.2.1 Definition: Let M be a P-injective S-act. A fuzzy S-act (M, λ) is called *fuzzy P-injective* If each fuzzy S-homomorphism $f : (aS, \mu) \to (M, \lambda)$ can be extended to a fuzzy S-homomorphism $\phi : (S, \widetilde{\mu}) \to (M, \lambda)$ for all $a \in S$.

3.2.2 Theorem: *Every weakly divisible fuzzy S-act is fuzzy P-injective.*

Proof: Let (M, μ) be a weakly divisible fuzzy S-act. Then M is divisible right S-act and μ is a weakly divisible fuzzy subact of M. By Proposition 3.1.14 M is a P-injective S-act. Let $f : (aS, \lambda) \to (M, \mu)$ be a fuzzy S-homomorphism that is $f : aS \to M$ is an S-homomorphism and $\mu(f(x)) \geq \lambda(x)$ for all $x \in aS$. Since M is P-injective so there exist an S-homomorphism $\phi : S \to M$ which extends f. This homomorphism ϕ is defined as, if $f(a) = x \in M$ then there exists $y \in M$ such that $x = ya$. Define $\phi(1) = y$ and $\phi(s) = ys$. We show that ϕ is a fuzzy S-homomorphism that is $\mu(\phi(s)) \geq \widetilde{\lambda}(s)$.

If $\lambda(s) = 0$ then $\mu(\phi(s)) \geq \tilde{\lambda}(s)$.

If $\tilde{\lambda}(s)) = \lambda s$ then $s \in aS$ so $\phi(s) = f(s)$.

Hence $\mu(\phi(s)) = \mu(f(s)) \geq \lambda(s) = \tilde{\lambda}(s)$.

3.2.3 Corollary: *Every fuzzy divisible S-act is fuzzy P-injective.*

Proof: Since every divisible fuzzy S -act is a weakly divisible fuzzy S-act, so by above theorem it is fuzzy P-injective.

3.2.4 Definition: A fuzzy sub-act λ of a right S -act M is called right S-cancellative. If $\lambda(xs) = \lambda(x's) \Rightarrow \lambda(x) = \lambda(x')$ for all $x, x' \in M$ and $s \in S$.

It is not necessary that if M is right S-cancellative then every fuzzy subact of M is right S-cancellative.

3.2.5 Example:

Let N be the set of natural numbers. Then N under usual multiplication of numbers is a cancellative semigroup. Consider N as a right N-cancellative right N-act. Consider the fuzzy sub-act λ of N defined by

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in 4\mathrm{N} \\ 1/2 & \text{if } x \in 2\mathrm{N} - 4\mathrm{N} \\ 0 & \text{otherwise} \end{cases}$$

Then λ is not a right N -cancellative because $\lambda(2.2) = \lambda(4.2) = 1$, but $\lambda(2) = 1/2 \neq 1 = \lambda(4)$.

3.2.6 Example: Consider the semigroup $S = \{0, 1, a, b, c\}$

	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	a	a	a
b	0	b	a	a	a
c	0	c	a	a	a

Then S is a commutative non-cancellative semigroup. Consider S as a right S-act. A *fuzzy sub-set* λ of S is a *fuzzy subact* of S if and only if

i) $\lambda(0) \geq \lambda(x)$ for all $x \in S$

ii) $\lambda(a) \geq \lambda(x)$ for all non zero x in S and

iii) $\lambda(x) \geq \lambda(1)$ for all x in S

Consider the fuzzy subact λ which maps every element of S on 1 .Then λ is a right S-cancellative.

3.2.7 Definition: A fuzzy S-act (M,λ) is called *right S-cancellative* if M is right S-cancellative S-act and λ is a right S-cancellative fuzzy sub-act of M.

3.3 EMBEDDING AN ARBITRARY FUZZY S -ACT INTO A FUZZY DIVISIBLE S -ACT.

Concept used in the following is discussed in Previous Chapter. *Q(A)* denotes the right S-act defined on the set $A \times S$ and a relation defined on *Q(A)* as follows:

$(x,a) \equiv (x',a') \Leftrightarrow xa' = x'a.$

we construct a factor S-act $\overline{Q(A)}.= Q(A)/_{\equiv}$, the S-action on $\overline{Q(A)}$ is defined

as: $(\overline{x,a})s = (\overline{xs,a})$ for all $s \in S$.

Note: Let (A,λ) be a fuzzy S-act we define a fuzzy subact λ_1 of $A \times S$ (a right S-act) by $\lambda_1 : A \times S \to [0,1]$ such that $\lambda_1((a,s)) = \lambda(a)$

3.3.1 Lemma: λ_1 *is a fuzzy sub-act* of $A \times S$

Proof: Let $(a,s) \in A \times S$ and $t \in S$ then, $\lambda_1((a,s)t) = \lambda_1((at,s)) = \lambda(at) \geq \lambda(a) = \lambda_1(a,s)$.

So λ_1 is a fuzzy sub-act of $A \times S$.

3.3.2 Lemma: If S is a Commutative monoid and A is a right S-cancellative S-act and λ be a fuzzy sub-act of A. Then $\lambda_2 : \overline{Q(A)} \to [0,1]$ defined by

$$\lambda_2((\overline{x,a})) = \bigvee_{(y,b) \in (\overline{x,a})} \lambda_1(y,b) = \bigvee_{(y,b) \in (\overline{x,a})} \lambda(y)$$

is a fuzzy sub-act of $\overline{Q(A)}$.

Proof: Let

$$(\overline{x,a}) = (\overline{x_1,a_1})$$

$$\lambda_2((\overline{x,a})) = \bigvee_{(y,b) \in (\overline{x,a})} \lambda(y). \text{ As } (y,b) \in (\overline{x,a}) = (\overline{x_1,a_1}) \Leftrightarrow (y,b) \in (\overline{x_1,a_1})$$

Thus $$\lambda_2((\overline{x,a})) = \bigvee_{(y,b) \in (\overline{x,a})} \lambda(y) = \bigvee_{(y,b) \in (\overline{x_1,a_1})} \lambda(y) = \lambda_2((\overline{x_1,a_1}))$$

Hence λ_2 is well defined.

Furthermore $(\overline{x,a})s = (\overline{xs,a})$ if $(y,b) \in (\overline{x,a})$.

Then $$(y,b) \equiv (x,a)$$

$$\Rightarrow ya = xb \Rightarrow yas = xbs \Rightarrow (ys)a = (xs)b$$

$$\Rightarrow (ys,b) \equiv (xs,a) \Rightarrow (ys,b) \in (\overline{x,a})s$$

Thus $$\bigvee_{(z,c) \in (\overline{x,a})s} \lambda(z) \geq \bigvee_{(y,b) \in (\overline{x,a})} \lambda(ys) \geq \bigvee_{(y,b) \in (\overline{x,a})} \lambda(y)$$

$\Rightarrow \lambda_2((\overline{x,a})s) \geq \lambda_2((\overline{x,a}))$. Thus λ_2 is a fuzzy subact of $\overline{Q(A)}$.

3.3.3 Lemma: λ_2 *is a divisible sub-act* of $\overline{Q(A)}$.

Proof: Let $(\overline{x,a})_t \in \lambda_2$ and $s \in S$ then there exists $(\overline{x,as}) \in \overline{Q(A)}$ such that $(\overline{x,a}) = (\overline{x,as})s$ because $(\overline{x,as})s = (\overline{xs,as})$ and $x(as) = (xs)a$ so $(x,a) \equiv (xs,as)$. Thus $(\overline{x,a}) = (\overline{xs,as}) = (\overline{x,as})s$. Also $\lambda_2((\overline{x,a})) \geq t$, because $(\overline{x,a})_t \in \lambda_2$. Thus

$$\lambda_2((\overline{x,a})) = \bigvee_{(y,b) \in (\overline{x,a})} \lambda(y) \geq t.$$

But

$$\lambda_2((\overline{x,as})) = \bigvee_{(z,c) \in (\overline{x,as})} \lambda(z) \geq \bigvee_{(y,b) \in (\overline{x,a})} \lambda(y) \geq t$$

Since if $(y,b) \in (\overline{x,a})$ then $(y,b) \equiv (x,a) \Rightarrow ya = xb$

$$\Rightarrow yas = xbs \Rightarrow (x,as) \equiv (y,bs)$$

$$\Rightarrow (y,bs) \in (\overline{x,as})$$

$$\Rightarrow ((\overline{x,as}))_t \in \lambda_2$$

Thus λ_2 is *divisible sub-act* of $\overline{Q(A)}$.

3.3.4 Theorem: If S is a Commutative monoid and A is a right S-cancellative S-act. Then the fuzzy S-act (A,λ) can be *embedded* into a divisible fuzzy S-act $(\overline{Q(A)},\lambda_2)$.

Proof: The mapping $q : A \to Q(A)$ defined by $q(x) = (x,1)$ is an S-monomorphism. Also

$$\lambda_2(q(x)) = \lambda_2((\overline{x,1})) = \bigvee_{(y,a)\in(\overline{x,1})} \lambda(y) \geq \lambda(x) \quad \text{because} \quad (x,1) \in (\overline{x,1}).$$

Thus q is a *fuzzsy* S-monomorphism.

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