Roughness in (α, β)-Fuzzy Filters in Ordered Semigroups

By

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***MASTER OF SCIENCE (MS) IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS & STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY ISLAMABAD.

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MS IN MATHEMATICS***

We accept this dissertation as conforming to the required standard.

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2016

DECLARATION

I, hereby declare that this thesis neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor.

No portion of the work presented in this thesis has been submitted in support of an application for any degree or qualification of this or any other Institute of learning.

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DEDICATION

This work is dedicated

To

My Beloved Parents, my Family, Valued Teacher

Dr. Muhammad Irfan Ali & Dr. Tahir Mahmood

for Supporting and Encouraging me

ACKNOWLEDGEMENTS

First and foremost, all praise to '**ALLAH**', Lord of the world, the Almighty, and I find no words to thank '**ALLAH**', who created me and all the universe around us. Who is the sole creator of all the things tinier than an electron to huge galaxies. I am thankful to the Prophet Muhammad (S.A.W.) whose teachings are a blessing for the whole mankind. May Allah guide us and the whole humanity to the right path.

Words of gratitude and appreciation do not always convey the depth of one's feelings, yet I wish to record my thanks to my most respected supervisor **Dr. Muhammad Irfan Ali** and honorable teacher **Dr. Tahir Mahmood** who encouraged me very much in time, made me realize and enjoy the hard work. I am obliged to him for his able guidance, help, immense encouragement and limitless patience, which made massive task accomplished.

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In the end I want to pay my attribute to my parents whose love and guidance always gave me a ray of hope in the darkness of desperation. I am thankful to all of my family members specially my elder brother for their support and well wishing, enormous love, support, encouragements and constant patience. I cannot forget their prayers for me throughout my life, without them this effort would have been nothing.

Azmat Hussain

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CONTENTS

Preface

In daily life, there are many situations where certain order exist among the elements of a set. For example prices of commodities can be descried by terms like very cheap, cheap, affordable, costly and very costly. It is clear that there is an order among these terms and commodities can be arranged with the help of order among their prices. Algebraically ordered semigroups are sets with associative binary operation having a certain partial order. Ordered semigroups have wide ranging applications in computer science, automata theory and coding theory.

L. A. Zadeh [40] put forward the basic concepts of fuzzy set and furnish a model for generalizing many of the concepts and several basic algebraic notions in general mathematics. Fuzzy semigroup concept is innovated by Kuroki [28, 29] for the first time, and fuzzy ordered groupoids and ordered semigroup was presented by Kehayopulu and Tsingelis in [19, 20].

Rough set theory and fuzzy set theory are two distinct concepts, but both of them are very handy to deal with uncertainty. These two can be hybridized in a very fruitful manner. Therefore, concepts of fuzzy rough sets and rough fuzzy sets are introduced in [8].

In fuzzy set the idea of quasi-coincidence which is stated in [3, 4] played a key role in several fields of fuzzy subgroups. Rosenfeld [35] used the generalized form of ($\in$, $\in\vee q$)-fuzzy subgroup. Bhakat and Das [2] introduced the concept of (α, β)-fuzzy subgroups by using "belongs to" relation ($\in$) and "quasi-coincident with" relation (q) between fuzzy point and fuzzy subgroups. An ($\in$, $\in\vee q$)-fuzzy subgroup concept was also introduced by them. In several algebraic structures the generalized notions of fuzzy sets and fuzzy subsystems was initiated by Davvaz, Kazanci et al. [6, 16].

Kehayopulu and Tsingelis for the first time presented fuzzy filter in ordered semigroup. They also studied vestigial attribute of fuzzy filter and prime fuzzy ideal which is mentioned in [20]. A generalized fuzzy filter of R_0 algebra was acquainted by Ma et al. [31] and some properties were provided in terms of this notion.

The concept of a fuzzy ideals in a semigroups was first developed by Kuroki [27, 28, 29]. He studied fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals and semiprime ideals of semigroups. Jun et al. [11] introduce the concept of ($\in$, $\in\vee q$)-fuzzy bi-ideals of ordered semigroup and gave some characterization Theorem. The fuzzy quasi-ideals in a semigroups were studied by Kuroki and Ahsan [29, 1], where the basic properties of semigroups in term of fuzzy quasi-ideals are also given by them.

Jan and Song [10] discuss the general form of fuzzy interior ideals in semigroups and also initiated the study of (α, β)-fuzzy interior ideals of a semigroup. Shabir et al. [36] gave the concept of more generalized forms of (α, β)-fuzzy ideals and defined (∈, ∈∨qk)-fuzzy ideals of semigroups. (Also see [12, 13, 26, 37]).

The properties of fuzzy ideals with thresholds were characterized by Shabir et al. [38] in semigroups. The (∈, ∈∨q)-fuzzy bi-ideal were studied by Kazanci and Yamak [15] and they also initiated fuzzy bi-ideal with thresholds in semigroup. Manzoor et al. [32] bring together the standpoint of fuzzy quasi-ideals with thresholds in ordered semigroup. They also established some result on semiprime fuzzy quasi-ideals with thresholds in ordered semigroup.

Pawlak is the founder of rough set theory [34]. Many applications of this theory have been reported. Actually it is a nice tool to discuss uncertainty among the elements of a set. Equivalence relations play a fundamental role in it. Due to limited knowledge about the elements of a set, it is too complicated to determine the equivalence relation among the objects of a set. So authors studied different models with less restrictions. The study of generalized rough sets was initiated by Davvaz [5]. Instead of equivalence relation, set valued maps are used to define approximations of a set in generalized rough set theory. In algebraic structures roughness has be discussed by many authors. Kuroki studied roughness in semigroups and fuzzy semigroups in [30]. Then this concept is studied for prime ideals in semigroups [39]. In ordered semigroups rough approximations as proposed in [30] cannot be a good idea. As in ordered semigroup there is a partial order associated with the semigroup, therefore non-trivial equivalence relations for such semigroups are difficult to find. Perhaps, this is the major reason that no study of roughness in case of ordered semigroups has been reported till now according to our knowledge. Therefore in this thesis some weaker tool to study roughness for fuzzy filters, fuzzy ideals, fuzzy filters with thresholds and fuzzy ideals with thresholds in ordered semigroups have been introduced. Set valued maps give rise to binary relations in general. These maps with monotone or isotone order help us to study roughness in fuzzy filters of ordered semigroups.

Chapter 1

Preliminaries

In this chapter we will discuss some basic definitions, examples and notions. These definitions will help us in later chapters. For undefined terms and notions , we refer to [7, 9, 13, 25, 26, 29, 32].

1.1 Ordered Semigroup

1.1.1 Definition

Let S be a non-empty set and "$*$" be a binary operation on S. Then $(S, *)$ is called a semigroup if this operation is associative, that is $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$ holds for all $x_1, x_2, x_3 \in S$.

1.1.2 Example

(a) The set $N = \{1, 2, 3, ...\}$ of natural numbers gives the semigroup $(N, \cdot)$, under multiplication.

(b) Similarly, the set $W = \{0, 1, 2, 3, ...\}$ of whole numbers forms a semigroup $(W, \cdot)$, under multiplication.

(c) $\{0, 1\}$ is semigroup under "$\cdot$".

(d) Let S be a non-empty set. Then $(P(S), \cup)$ and $(P(S), \cap)$ are semigroups.

1.1.3 Definition

A binary relation "$\leq$" defined on a non-empty set S is partial order on the set S if the following conditions hold

(o_1) $a \leq a$ (reflexivity)

(o_2) $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry)

(o_3) $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity),

for all $a, b, c \in S$.

A non-empty set S with some partial order "$\leq$" on it is called a partially order set, or more briefly a po-set. Then we write $(S, \leq)$ is a po-set.

1.1.4 Example

(a) Let $S = \{1, 2, 3, 4, 5, 6\}$ and define a relation on S as "for all $x_1, x_2 \in S$, x_1 is related to x_2 if and only if x_1 divides x_2". Then S is a partial ordered set.

(b) Let S be a non-empty set. Then $(P(S), \subseteq)$ is a po-set.

1.1.5 Definition

Let S be a non-empty set. Then an algebraic system $(S, \cdot, \leq)$ is called partially ordered semigroup (po-semigroup) if it satisfies

(os_1) S is a semigroup with respect to "$\cdot$"

(os_2) S is a po-set with respect to "$\leq$"

(os_3) (for all $x, x_1, x_2 \in S$) (If $x_1 \leq x_2 \rightarrow x_1 x \leq x_2 x$ and $x x_1 \leq x x_2$).

Partially ordered semigroup (po-semigroup) is also called as ordered semigroup.

1.1.6 Definition

A non-empty subset A of ordered semigroup S is called ordered subsemigroup of S if $A^2 \subseteq A$.

1.1.7 Example [18]

Let $S = \{a, b, c, d, e\}$ be a set with the following multiplication table and order relation "$\leq$",

·	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

and

$\leq := \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (c,c), (d,d), (e,e)\}$. Then S is called an ordered semigroup.

1.2 Fuzzy Sets

L. A. Zadeh [40] put forward the basic concepts of fuzzy set and provide a model for generalizing many of the concepts and several basic algebraic notions in general mathematics. In this section, we will give a review of some basic concepts of fuzzy sets.

1.2.1 Definition [40]

A function Ψ from S to a unit closed interval $[0,1]$ ($\Psi : S \rightarrow [0,1]$) is known as fuzzy set or fuzzy subset of S.

1.2.2 Definition [33]

A fuzzy subset Ψ of an ordered semigroup S of the form

$$\Psi(x_1) = \begin{cases} t(t \neq 0) & \text{if } x_1 = x \\ 0 & \text{if } x_1 \neq x \end{cases}$$

represent fuzzy point with value t and support by x, where $t \in (0,1]$. It is denoted by x_t. A fuzzy point x_t of ordered semigroup S is said to 'belong to' fuzzy subset Ψ denoted as $x_t \in \Psi$, if $\Psi(x) \geq t$, and is said to 'quasi-coincident with' a fuzzy subset Ψ denoted by $x_t q \Psi$, if $\Psi(x) + t > 1$. To say that $x_t \in \vee q \Psi$ (resp. $x_t \in \wedge q \Psi$) means that $x_t \in \Psi$ or $x_t q \Psi$ ($x_t \in \Psi$ and $x_t q \Psi$).

1.2.3 Remark

(i) Two fuzzy subsets Ψ and μ of ordered semigroup S are said to be disjoint if there is no $x \in S$ such that $\Psi(x) = \mu(x)$. If $\Psi(x) = \mu(x)$ for each $x \in S$, then we say that Ψ and μ are equal and write as $\Psi = \mu$.

(ii) Let Ψ and μ be two fuzzy subset of ordered semigroup S. Then a fuzzy subset Ψ is said to be included in a fuzzy subset μ i.e. $\Psi \subseteq \mu$ if and only if $\Psi(x) \leq \mu(x)$ for all $x \in S$.

(iii) Let Ψ and μ be two fuzzy subsets of ordered semigroup S. Then a fuzzy subset Ψ is said to be properly included in a fuzzy subset μ i.e. $\Psi \subset \mu$ if and only if $\Psi(x) < \mu(x)$ for all $x \in S$.

(iv) The union of any family $\{\Psi_i : i \in \Omega\}$ of fuzzy subsets Ψ_i of ordered semigroup S is denoted by $\left(\bigcup_{i\in\Omega} \Psi_i\right)$ and defined by $\left(\bigcup_{i\in\Omega} \Psi_i\right)(x) = \sup_{i\in\Omega} \Psi_i(x) = \bigvee_{i\in\Omega} \Psi_i(x)$ for all $x \in S$.

(v) The intersection of any family $\{\Psi_i : i \in \Omega\}$ of fuzzy subsets Ψ_i of ordered semigroup S is denoted by $\left(\bigcap_{i\in\Omega} \Psi_i\right)$ and defined by $\left(\bigcap_{i\in\Omega} \Psi_i\right)(x) = \inf_{i\in\Omega} \Psi_i(x) = \bigwedge_{i\in\Omega} \Psi_i(x)$ for all $x \in S$. Moreover $\left(\bigcap_{i\in\Omega} \Psi_i\right)$ is largest fuzzy subset which is contained in Ψ_i.

1.2.4 Definition

In what follows let α, β denote any one of $\in, q, \in \vee q$, or $\in \wedge q$ unless otherwise specified. Consider a fuzzy point x_t of a fuzzy subset Ψ on an ordered semigroup S, and $x_t \alpha \Psi$ means x_t belong to (be quasi-coincident with) a fuzzy subset Ψ. A fuzzy point x_t of ordered semigroup S is said to 'belong to' fuzzy subset Ψ denoted as $x_t \in \Psi$, if

$\Psi(x) \geq t$, and is said to 'quasi-coincident with' a fuzzy subset Ψ denoted by $x_t q\Psi$, if $\Psi(x) + t > 1$. To say that $x_t \in \vee q\Psi$ (resp. $x_t \in \wedge q\Psi$) means that $x_t \in \Psi$ or $x_t q\Psi$ ($x_t \in \Psi$ and $x_t q\Psi$).

1.3 Filters in Ordered Semigroups

1.3.1 Definition [7]

A non-empty subset F of ordered semigroup S is known as a left (resp. right) filter of S if it satisfies the following conditions

(F_1) (for all $x_1 \in S$ *and* for all $x_2 \in F$) ($x_2 \leq x_1$ implies $x_1 \in F$)

(F_2) (for all $x_1, x_2 \in S$) ($x_1, x_2 \in F$ implies $x_1 x_2 \in F$)

(F_3) (for all $x_1, x_2 \in S$) ($x_1 x_2 \in F$ implies $x_1 \in F$(resp. $x_2 \in F$)).

If F is both a left filter and a right filter of ordered semigroup S, then F is known as a filter of S.

1.3.2 Example

Let $S = \{a, b, c, d, e, f\}$ be a set with the following multiplication table and order relation "$\leq$".

·	a	b	c	d	e	f
a	a	b	b	d	e	f
b	b	b	b	b	b	b
c	b	b	b	b	b	b
d	d	b	b	d	e	f
e	e	f	f	e	e	f
f	f	f	f	f	f	f

and

$\leq:=\{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(a,d),(a,e),(d,e),(b,f),(b,e),(c,f),$ $(f,e),(c,e)\}$. Then $(S,\cdot,\leq)$ is an ordered semigroup. Filters of S are $\{a,d,e\}$ and S.

Next we give definitions of bi-filter and quasi filter. Theses are actually generalizations of filter in ordered semigroups.

1.3.3 Definition [7]

A non-empty subset F of ordered semigroup S is known as bi-filter of S if $(F_1),(F_2)$ hold and

(F_4) (for all $x_1, x_2 \in S$) $(x_1x_2x_1 \in F$ implies $x_1 \in F)$.

1.3.4 Definition [9]

A non-empty subset F of ordered semigroup S is known as a qausi filter if it satisfies $(F_1),(F_2)$ and

(F_5) (for all $x_1, x_2, x_3 \in S$) $(x_1x_2 = x_3x_1 \in F$ implies $x_1 \in F)$.

In the following some basic concepts about fuzzy filters and their generalizations in ordered semigroups are given.

1.4 Fuzzy Filters in Ordered Semigroup

1.4.1 Definition

A fuzzy subset Ψ of S is known as a fuzzy ordered subsemigroup of ordered semigroup S if it satisfies

$$(\text{for all } x_1, x_2 \in S)\,(\Psi(x_1x_2) \geq \min\{\Psi(x_1), \Psi(x_2)\}).$$

In the following this inequality will be denoted by (FF_2).

1.4.2 Definition [7]

A fuzzy subset Ψ of S is known as a fuzzy left (resp. right) filter of ordered semigroup S if the following assertions holds,

(FF_1) (for all $x_1, x_2 \in S$) ($x_1 \leq x_2$ implies $\Psi(x_1) \leq \Psi(x_2)$)

(FF_2) (for all $x_1, x_2 \in S$) ($\Psi(x_1x_2) \geq \min\{\Psi(x_1), \Psi(x_2)\}$)

(FF_3) (for all $x_1, x_2 \in S$) ($\Psi(x_1x_2) \geq \Psi(x_1)$ (resp. $\Psi(x_2)$)).

A fuzzy subset Ψ of S is known as a fuzzy filter of ordered semigroup S, if Ψ is both a fuzzy left filter and a fuzzy right filter of S.

1.4.3 Definition [7]

A fuzzy subset Ψ of ordered semigroup S is known as a fuzzy bi-filter of ordered semigroup S if it satisfies $(FF_1), (FF_2)$ and

(FF_4) (for all $x_1, x_2 \in S$) $(\Psi(x_1x_2x_1) \geq \Psi(x_1))$.

1.4.4 Definition [9]

A fuzzy subset Ψ of S is known as a fuzzy quasi-filter of ordered semigroup S if it holds $(FF_1), (FF_2)$ and

(FF_5) (for all $x_1, x_2, x_3 \in S$) $(x_1x_2 = x_3x_1$ implies $\Psi(x_1x_2) \leq \Psi(x_1))$.

1.5 Ideals in Ordered Semigroups

1.5.1 Definition [17]

A non-empty subset I of ordered semigroup S is known as a left (resp. right) ideal of S if it satisfies

(I_1) $SI \subseteq I$(resp. $IS \subseteq I$)

(I_2) if $x_1 \in S$ and $x_2 \in I$ such that $x_1 \leq x_2$, then $x_1 \in I$.

If I is both a left ideal and a right ideal of ordered semigroup S, then I is known as two sided ideal of S or simply an ideal of S.

For $A \subseteq S$, we denote $(A] := \{t \in S/t \leq h$ for some $h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For non-empty subsets A, B of S, we denote, $AB = \{ab/a \in A, b \in B\}$

1.5.2 Example [24]

Let $S = \{a, b, c, d, e, f\}$ be a set with the following multiplication table and the order relation "$\leq$":

$\cdot$	a	b	c	d	e	f
a	a	a	a	d	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

and

$\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (f,e)\}$. Then $(S, \cdot, \leq)$ is an ordered semigroup. Right ideals of S are $\{a, d\}$, $\{a, b, d\}$ and S. Left ideals of S are $\{a\}$, $\{d\}$, $\{a, b\}$, $\{a, d\}$, $\{a, b, d\}$, $\{a, b, c, d\}$, $\{a, b, d, e, f\}$ and S. Ideals of S are $\{a, d\}$, $\{a, b, d\}$ and S.

Next we are going to define bi-ideal, interior ideal and quasi ideal. Theses are actually generalizations of ideals in ordered semigroups.

1.5.3 Definition [21]

A non-empty subset I of ordered semigroup S is called bi-ideal of S if it satisfies (I_2) and

(I_3) $ISI \subseteq I$

(I_4) $I^2 \subseteq I$.

1.5.4 Definition [22]

Let I be a non-empty subset of ordered semigroup S. Then I is known as an interior ideal of S if it satisfies $(I_2), (I_4)$ and

(I_5) $SIS \subseteq I$.

1.5.5 Definition [23]

A non-empty subset Q of ordered semigroup S is known as quasi ideal of S if it hold (I_2) and

(I_6) $(QS] \cap (SQ] \subseteq Q$.

1.5.6 Example [18]

Let $S = \{a, b, c, d, e\}$ be an ordered semigroup with the following multiplication table and order relation " $\leq$ "

·	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

and

$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(c,c),(d,d),(e,e)\}$. Quasi ideals of S are $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b,d\}, \{a,c,d\}, \{a,b,e\}, \{a,c,e\}$ and S.

1.6 Fuzzy Ideals in Ordered Semigroups

In the following some basic concepts about fuzzy ideals and their generalizations in ordered semigroups are given.

1.6.1 Definition [20]

A fuzzy subset Ψ of ordered semigroup S is called a fuzzy ordered subsemigroup of S if it satisfies

(FI_1) $(\forall x_1, x_2 \in S)\left(\Psi(x_1x_2) \geq \min\{\Psi(x_1), \Psi(x_2)\}\right)$.

1.6.2 Definition [20]

A fuzzy subset Ψ of ordered semigroup S is known as a fuzzy left (resp. right) ideal of S if it satisfies

(FI_2) $(\forall x_1, x_2 \in S)(x_1 \leq x_2 \Longrightarrow \Psi(x_1) \geq \Psi(x_2))$

(FI_3) $(\forall x_1, x_2 \in S)(\Psi(x_1x_2) \geq \Psi(x_2)(\text{resp. } \Psi(x_1x_2) \geq \Psi(x_1)))$.

A fuzzy subset Ψ is known as fuzzy ideal of S, if it is both a fuzzy left ideal and a fuzzy right ideal of S.

From this definition we can also conclude the following.

1.6.3 Definition

A fuzzy subset Ψ is called fuzzy ideal of ordered semigroup S if it satisfy (FI_2) and

(FI_4) $(\forall x_1, x_2 \in S)(\Psi(x_1x_2) \geq \max\{\Psi(x_1), \Psi(x_2)\})$.

1.6.4 Example

Consider a set $S = \{a, b, c, d, e, f\}$ with the following multiplication table and order relation "$\leq$".

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

and

$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(f,e)\}$. Then $(S,\cdot,\leq)$ is an ordered semigroup. Right ideals of S are $\{a,d\}$, $\{a,b,d\}$ and S. Left ideals of S are $\{a\}$, $\{a,b\}$, $\{a,d\}$, $\{a,b,d\}$, $\{a,b,c,d\}$, $\{a,b,d,e,f\}$ and S. Define a fuzzy subset $\Psi : S \to [0,1]$ by $\Psi(a) = 0.8$, $\Psi(b) = 0.5$, $\Psi(d) = 0.6$ and $\Psi(c) = \Psi(e) = \Psi(f) = 0.4$. Then Ψ is a fuzzy ideal of S.

1.6.5 Definition [22]

A fuzzy subset Ψ is known as fuzzy interior ideal of ordered semigroup S if it satisfies (FI_1), (FI_2) and

(FI_5) $(\forall x_1, x_2, x_3 \in S)\,(\Psi(x_1x_3x_2) \geq \Psi(x_3))$.

1.6.6 Definition [21]

Let Ψ be a fuzzy subset of ordered semigroup S. Then a fuzzy subset Ψ is said to be a fuzzy bi-ideal of S if it satisfies $(FI_1), (FI_2)$ and

(FI_6) $(\forall x_1, x_2, x_3 \in S)(\Psi(x_1x_2x_3) \geq \min\{\Psi(x_1), \Psi(x_3)\})$.

1.6.7 Definition

Let X be a non-empty subset of ordered semigroup S, then we define a set X_{x_1} by

$$X_{x_1} = \{(x_2, x_3) \in S \times S / x_1 \leq x_2x_3\}.$$

For any two fuzzy subsets Ψ and μ we define

$$(\Psi \circ \mu)(x_1) = \begin{cases} \bigvee\limits_{(x_2,x_3)\in X_{x_1}} \min\{\Psi(x_2), \mu(x_3)\} & \text{if } X_{x_1} \neq \phi \\ 0 & \text{if } X_{x_1} = \phi \end{cases}$$

$\Psi \leq \mu$ means $\Psi(x) \leq \mu(x)$.

1.6.8 Definition [23]

A fuzzy subset Ψ of ordered semigroup S is known as a fuzzy quasi ideal of S if it satisfies (FI_2) and

(FI_7) $\Psi \geq (\Psi \circ 1) \wedge (1 \circ \Psi)$.

1.6.9 Example

Consider quasi ideals as given in example 1.5.6 and define a fuzzy subset $\Psi : S \to [0, 1]$ by $\Psi(a) = 0.8, \Psi(b) = 0.7, \Psi(d) = 0.6, \Psi(c) = \Psi(e) = 0.5$, Then a fuzzy subset Ψ is a fuzzy quasi ideal of S.

1.7 Rough Set

Rough set theory and fuzzy set theory are two distinct concepts, but both of them are very handy to deal with uncertainty. These two can be hybridized in a very fruitful manner. Therefore, concepts of fuzzy rough sets and rough fuzzy sets are introduced in [8].

In the following we recall some basic concepts of rough set theory introduced by Pawlak [34]. The idea of a rough set could be placed in a more general setting, leading to a fruitful further research and applications in classification theory, cluster analysis, measurement theory, taxonomy, etc. The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set.

Recall that an equivalence relation $\Bbbk$ on a universal set U is a reflexive, symmetric and transitive binary relation on U. For any $x \in U$, the set $\{y \in U : (y, x)\}$ is called equivalence class determined by x and it is denoted by $[x]$. Consider an equivalence relation $\Bbbk$ on a universal set U. The pair $(U, \Bbbk)$ is known as an approximation space. Let A be a non-empty subset of a universal set U, then A is said to be definable if we can express it in the form of some equivalence classes of U, else A is said to be undefinable. If A is undefinable, then it may be approximated in the form of definable subsets called lower approximation and upper approximation of A, defined as

$$\underline{app}(A) = \cup\{x_1 \in U : [x_1]_k \subseteq A\}$$

$$\overline{app}(A) = \cup\{x_1 \in U : [x_1]_k \cap A \neq \phi\}$$

The pair $(\underline{app}(A), \overline{app}(A))$ is known as a rough set. If $\underline{app}(A) = \overline{app}(A)$ then A is a definable set.

1.7.1 Example

For the sake of illustration, let $(U, \Bbbk)$ be an approximation space, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and an equivalence relation $\Bbbk$ with the following equivalence classes:

$A_1 = \{x_1, x_4, x_8\}, A_2 = \{x_2, x_5, x_7\}, A_3 = \{x_3\}, A_4 = \{x_6\}$. Let $A = \{x_3, x_5\}$, then $\underline{app}(A) = \{x_3\}$ and $\overline{app}(A) = \{x_2, x_3, x_5, x_7\}$ and so $(\{x_3\}, \{x_2, x_3, x_5, x_7\}) = (\underline{app}(A), \overline{app}(A))$ is a rough set.

This notion of lower and upper approximations can be generalized to fuzzy sets as well.

1.7.2 Definition [14]

Let us consider the approximation space $(U, \mathbf{k})$ and Ψ as a fuzzy subset of U. Define the lower and upper approximation of a fuzzy subset Ψ as the following

$$\underline{app}\Psi(x) = \bigwedge_{x' \in [x]_k} \Psi(x') \text{ and } \overline{app}\Psi(x) = \bigvee_{x' \in [x]_k} \Psi(x'), \text{ for any } x \in U.$$

The pair $(\underline{app}\Psi, \overline{app}\Psi)$ is said to be a rough fuzzy set if $\underline{app}\Psi \neq \overline{app}\Psi$.

1.7.3 Definition

Consider the ordered semigroups S_1 and S_2. A mapping $H : S_1 \longrightarrow P^*(S_2)$ is said to be set-valued homomorphism in short (SVH) if the following hold

$$(h_1) \quad H(x_1)H(x_2) = H(x_1x_2)$$

Where $P^*(S_2)$ denotes the collection of all non-empty subsets of S_2.

1.7.4 Definition

Consider the ordered semigroups S_1 and S_2. A mapping $H : S_1 \longrightarrow P^*(S_2)$ is said to be set-valued monotone homomorphism in short $(SVMH)$ if it hold condition (h_1) of definition 1.7.3 and

(h_2) if $x_1 \leq x_2$ then $H(x_1) \subseteq H(x_2)$ for each $x_1, x_2 \in S_1$.

Where $P^*(S_2)$ denotes the collection of all non-empty subsets of S_2.

1.7.5 Definition

Consider the ordered semigroups S_1 and S_2. A mapping $H : S_1 \longrightarrow P^*(S_2)$ is said to be set-valued isotone homomorphism in short $(SVIH)$ if it hold condition (h_1) of definition 1.7.3 and

(h_3) if $x_1 \leq x_2$ then $H(x_2) \subseteq H(x_1)$ for each $x_1, x_2 \in S_1$.

Where $P^*(S_2)$ denotes the collection of all non-empty subsets of S_2.

In the following concept of roughness in fuzzy sets is being generalized by SVMH and SVIH.

1.7.6 Definition

Let $H : S \longrightarrow P^*(S)$ be a SVMH or SVIH. Then for every $x \in S$, we define the generalized lower and upper approximation of Ψ with respect to mapping H as,

$$\underline{H}(\Psi)(x) = \bigwedge_{x' \in H(x)} \Psi(x') \text{ and } \overline{H}(\Psi)(x) = \bigvee_{x_1' \in H(x)} \Psi(x')$$

The pair $(\underline{H}(\Psi), \overline{H}(\Psi))$ is said to be a rough fuzzy set with respect to H if $\underline{H}(\Psi) \neq \overline{H}(\Psi)$.

Chapter 2

Roughness in Fuzzy Filters in Ordered Semigroups

In this chapter, study of roughness in fuzzy filters of ordered semigroups is being initiated. Thus we start with the following result.

- Approximations of fuzzy left (resp. right) filter
- Approximations of fuzzy bi-filter and fuzzy quasi filter
- Approximations of $(\in, \in \vee q)$-fuzzy left (resp. right) filter
- Approximations of $(\in, \in \vee q)$-fuzzy bi-filter
- Approximations of $(\in, \in \vee qk)$-fuzzy left (resp. right) filter
- Approximations of $(\in, \in \vee qk)$-fuzzy bi-filter

From here onward in discussion below S stands for ordered semigroup and Ψ for fuzzy subset on ordered semigroup S unless stated otherwise.

2.1 Approximations of Fuzzy Filters in Ordered Semigroup

In chapter 1, we have defined fuzzy filters and their generalization. In this section we recall it and then discuss the approximations of fuzzy filters and their generalization.

2.1.1 Definition

A fuzzy subset Ψ is known as a fuzzy ordered subsemigroup of ordered semigroup S if it satisfies

$$(\text{for all } x_1, x_2 \in S)\,(\Psi(x_1x_2) \geq \min\{\Psi(x_1), \Psi(x_2)\}).$$

In the following this inequality will be denoted by (FF_2).

2.1.2 Definition [7]

A fuzzy subset Ψ is known as a fuzzy left (resp. right) filter of ordered semigroup S if the following assertions holds,

(FF_1) (for all $x_1, x_2 \in S$) $(x_1 \leq x_2$ implies $\Psi(x_1) \leq \Psi(x_2))$

(FF_2) (for all $x_1, x_2 \in S$) $(\Psi(x_1x_2) \geq \min\{\Psi(x_1), \Psi(x_2)\})$

(FF_3) (for all $x_1, x_2 \in S$) $(\Psi(x_1x_2) \geq \Psi(x_1)$ (resp. $\Psi(x_2)))$.

A fuzzy subset Ψ of S is known as a fuzzy filter of ordered semigroup S, if Ψ is both a fuzzy left filter and a fuzzy right filter of S.

2.1.3 Definition [7]

A fuzzy subset Ψ of ordered semigroup S is known as a fuzzy bi-filter of ordered semigroup S if it satisfies $(FF_1), (FF_2)$ and

(FF_4) (for all $x_1, x_2 \in S$) $(\Psi(x_1x_2x_1) \geq \Psi(x_1))$.

2.1.4 Definition [9]

A fuzzy subset Ψ is known as a fuzzy quasi-filter of ordered semigroup S if it holds $(FF_1), (FF_2)$ and

(FF_5) (for all $x_1, x_2, x_3 \in S$) $(x_1x_2 = x_3x_1$ implies $\Psi(x_1x_2) \leq \Psi(x_1))$.

2.1.5 Theorem

Let Ψ be a fuzzy ordered subsemigroup of S and $H : S \longrightarrow P^*(S)$ be a SVH. Then $\overline{H}(\Psi)$ is a fuzzy ordered subsemigroup of S.

Proof. For any $x_1, x_2 \in S$. Consider

$$\begin{aligned}
\overline{H}(\Psi)(x_1x_2) &= \bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{ab \in H(x_1)H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab \text{ such that } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\geq \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \min\{\Psi(a), \Psi(b)\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), \bigvee_{b \in H(x_2)} \Psi(b)\right\}
\end{aligned}$$

$$\text{implies } \overline{H}(\Psi)(x_1x_2) \geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2)\right\}$$

Hence $\overline{H}(\Psi)$ is a fuzzy ordered subsemigroup of S. ■

2.1.6 Theorem

Let Ψ be a fuzzy ordered subsemigroup of S and $H : S \longrightarrow P^{*}(S)$ be a SVH. Then $\underline{H}(\Psi)$ is a fuzzy ordered subsemigroup of S.

Proof. Similar to the proof of Theorem 2.1.5. ■

In the following, study of roughness in fuzzy filters of ordered semigroups is being initiated. Certain restrictions are imposed on SVH for this study.

2.1.7 Theorem

Let $H : S \longrightarrow P^{*}(S)$ be a SVMH and Ψ be a fuzzy left (right) filter of S. Then $\overline{H}(\Psi)$ is a fuzzy left (right) filter of S.

Proof. For any $x_1, x_2 \in S$, let $x_1 \leq x_2$, implies $H(x_1) \subseteq H(x_2)$. Now we may consider the following

$$\begin{aligned}\overline{H}(\Psi)(x_1) &= \bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right)\\ &\leq \bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right)\\ \text{That is } \overline{H}(\Psi)(x_1) &\leq \overline{H}(\Psi)(x_2)\end{aligned}$$

Next consider

$$
\begin{aligned}
\overline{H}(\Psi)(x_1x_2) &= \bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{ab \in H(x_1)H(x_2)} \Psi(ab) \ \ (\text{as } x_1' = ab \text{ such that } a \in H(x_1) \text{ and } b \in H(x_2)) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\geq \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \min\{\Psi(a), \Psi(b)\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), \bigvee_{b \in H(x_2)} \Psi(b)\right\} \\
\text{implies } \overline{H}(\Psi)(x_1x_2) &\geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2)\right\}
\end{aligned}
$$

Next consider

$$
\begin{aligned}
\overline{H}(\Psi)(x_1x_2) &= \bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{ab \in H(x_1)H(x_2)} \Psi(ab) \ \ \left(\text{as } x_1' = ab, \text{ such that } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\geq \bigvee_{a \in H(x_1)} \Psi(a) \\
\text{implies } \overline{H}(\Psi)(x_1x_2) &\geq \overline{H}(\Psi)(x_1).
\end{aligned}
$$

Hence $\overline{H}(\Psi)$ is a fuzzy left filter of S. Similarly it can be shown that $\overline{H}(\Psi)$ is a fuzzy right filter of S. ■

The following example shows that if H is a SVMH, then for a fuzzy filter Ψ, its lower approximation $\underline{H}(\Psi)$ may not be a fuzzy filter.

2.1.8 Example

Consider a set $S = \{a, b, c, d, e, f\}$ with multiplication table 1 and order relation.

Multiplication table for S

$\cdot$	**a**	**b**	**c**	**d**	**e**	**f**
a	a	b	b	d	e	f
b	b	b	b	b	b	b
c	b	b	b	b	b	b
d	d	b	b	d	e	f
e	e	f	f	e	e	f
f	f	f	f	f	f	f

Table 1

and

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(a,d),(a,e),(d,e),(b,f),(b,e),(c,f),(c,e),(f,e)\}.$$

Then $(S, \cdot, \leq)$ is an ordered semigroup. Filter of S are $\{a, d, e\}$ and S. Define a fuzzy subset $\Psi : S \to [0,1]$ by $\Psi(e) = \Psi(f) = \Psi(b) = 0.8$, $\Psi(d) = 0.7$, $\Psi(a) = 0.3$, $\Psi(c) = 0.4$. Then Ψ is a fuzzy filter of S.

Now consider $H : S \to P^*(S)$ be a $SVMH$ i.e.

$$(i) \quad H(x_1)H(x_1) = H(x_1x_2)$$

$$(ii) \quad \text{if } x_1 \leq x_2 \to H(x_1) \subseteq H(x_2).$$

Where $P^*(S)$ consist of all non-empty subset of S. Now $H(e) = \{a, d, e, f\}$ $H(d) = \{a, d, f\}$ $H(a) = \{a, d\}$ $H(b) = H(c) = H(f) = \{d, e, f\}$. As $b \leq e \Rightarrow H(b) \subseteq H(e)$ but $\underline{H}(\Psi)(b) \nleq \underline{H}(\Psi)(e)$, also $f \leq e \Rightarrow H(f) \subseteq H(e)$ but $\underline{H}(\Psi)(f) \nleq \underline{H}(\Psi)(e)$. Hence in $SVMH$ it is prove that $\underline{H}(\Psi)$ is not a fuzzy left (resp. right) filter of S.

2.1.9 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVIH and Ψ be a fuzzy left (right) filter of S. Then $\underline{H}(\Psi)$ is a fuzzy left (right) filter of S.

Proof. For any $x_1, x_2 \in S$, let $x_1 \le x_2$ implies $H(x_2) \subseteq H(x_1)$. Now we may consider the following

$$\begin{aligned}
\underline{H}(\Psi)(x_1) &= \bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\
&\le \bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right) \\
\text{That is } \underline{H}(\Psi)(x_1) &\le \underline{H}(\Psi)(x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\underline{H}(\Psi)(x_1 x_2) &= \bigwedge_{x_1' \in H(x_1 x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1' \in H(x_1) H(x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{ab \in H(x_1) H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab \text{ such that } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\ge \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \min\{\Psi(a), \Psi(b)\} \\
&= \min\left\{\bigwedge_{a \in H(x_1)} \Psi(a), \bigwedge_{b \in H(x_2)} \Psi(b)\right\} \\
\text{implies } \underline{H}(\Psi)(x_1 x_2) &\ge \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2)\}
\end{aligned}$$

Next consider

$$\begin{aligned}
\underline{H}(\Psi)(x_1x_2) &= \bigwedge_{x_1'\in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1'\in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{ab\in H(x_1)H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab, \text{ such that } a\in H(x_1) \text{ and } b\in H(x_2)\right) \\
&= \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)}} \Psi(ab) \\
&\geq \bigwedge_{a\in H(x_1)} \Psi(a)
\end{aligned}$$

implies $\underline{H}(\Psi)(x_1x_2) \geq \underline{H}(\Psi)(x_1)$.

Hence $\underline{H}(\Psi)$ is a fuzzy left filter of S. Similarly it can be shown that $\underline{H}(\Psi)$ is a fuzzy right filter of S. ■

The following example shows that if H is a SVIH, then for a fuzzy filter Ψ, its upper approximation $\overline{H}(\Psi)$ may not be a fuzzy filter.

2.1.10 Example

Consider the fuzzy filter Ψ of S from example 2.1.8.

Now let us consider $H : S \to P^*(S)$ be a $SVIH$ i.e.

$$(i) \quad H(x_1)H(x_2) = H(x_1x_2)$$

$$(ii) \quad \text{if } x_1 \leq x_2 \to H(x_2) \subseteq H(x_1).$$

Where $P^*(S)$ consist of all non-empty subset of S. Now if $H(f) = \{a, d, e\}$ and $H(e) = \{a\}$, then as $f \leq e \Rightarrow H(e) \subseteq H(f)$ but $\overline{H}(\Psi)(f) \nleq \overline{H}(\Psi)(e)$. Hence in $SVIH$ it is prove that $\overline{H}(\Psi)$ is not a fuzzy left (resp. right) filter of S.

2.1.11 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVMH and Ψ be a fuzzy bi-filter of S. Then $\overline{H}(\Psi)$ is a fuzzy bi-filter of S.

Proof. From Theorem 2.1.7, we have if $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$. Then $\overline{H}(\Psi)(x_1) \leq \overline{H}(\Psi)(x_2)$ for all $x_1, x_2 \in S$. Also $\overline{H}(\Psi)(x_1x_2) \geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2)\right\}$. Therefore we consider the following for any $x_1, x_2 \in S$.

$$\begin{aligned}
\overline{H}(\Psi)(x_1x_2x_1) &= \bigvee_{x_1' \in H(x_1x_2x_1)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)H(x_1)} \Psi\left(x_1'\right) \\
&= \bigvee_{aba \in H(x_1)H(x_2)H(x_1)} \Psi(aba) \quad \left(\begin{array}{c} \text{as } x_1' = aba \text{ such that } a \in H(x_1) \\ \text{and } b \in H(x_2) \end{array}\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2) \\ a \in H(x_1)}} \Psi(aba) \\
&\geq \bigvee_{a \in H(x_1)} \Psi(a)
\end{aligned}$$

$$\text{implies } \overline{H}(\Psi)(x_1x_2x_1) \geq \overline{H}(\Psi)(x_1)$$

Hence $\overline{H}(\Psi)$ on S is a fuzzy *bi-filter* of S. ■

2.1.12 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVIH and Ψ be a fuzzy bi-filter of S. Then $\underline{H}(\Psi)$ is a fuzzy bi-filter of S.

Proof. From Theorem 2.1.9, we have if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Then $\underline{H}(\Psi)(x_1) \leq \underline{H}(\Psi)(x_2)$ for all $x_1, x_2 \in S$. Also $\underline{H}(\Psi)(x_1x_2) \geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2)\}$.

Therefore we consider the following for any $x_1, x_2 \in S$.

$$
\begin{aligned}
\underline{H}(\Psi)(x_1x_2x_1) &= \bigwedge_{x_1' \in H(x_1x_2x_1)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1' \in H(x_1)H(x_2)H(x_1)} \Psi\left(x_1'\right) \\
&= \bigwedge_{aba \in H(x_1)H(x_2)H(x_1)} \Psi(aba) \quad \left(\begin{array}{c} \text{as } x_1' = aba \text{ such that } a \in H(x_1) \\ \text{and } b \in H(x_2) \end{array}\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2) \\ a \in H(x_1)}} \Psi(aba) \\
&\geq \bigwedge_{a \in H(x_1)} \Psi(a)
\end{aligned}
$$

$$\text{implies } \underline{H}(\Psi)(x_1x_2x_1) \geq \underline{H}(\Psi)(x_1)$$

Hence $\underline{H}(\Psi)$ on S is a fuzzy *bi-filter* of S. ■

2.1.13 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVMH and Ψ be a fuzzy quasi-filter of S. Then $\overline{H}(\Psi)$ is a fuzzy qausi filter of S.

Proof. From Theorem 2.1.7, we have if $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$. Then $\overline{H}(\Psi)(x_1) \leq \overline{H}(\Psi)(x_2)$ for all $x_1, x_2 \in S$. Also $\overline{H}(\Psi)(x_1x_2) \geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2)\right\}$. Therefore we consider the following for each $x_1, x_2, x_3 \in S$ we have $x_1x_2 = x_3x_1$ holds.

Then consider

$$
\begin{aligned}
\overline{H}(\Psi)(x_1x_2) &= \bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{ab \in H(x_1)H(x_2)} \Psi(ab) \;\; (\text{as } x_1' = ab \text{ where } a \in H(x_1) \text{ and } b \in H(x_2)) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\leq \bigvee_{a \in H(x_1)} \Psi(a) \\
\text{implies } \overline{H}(\Psi)(x_1x_2) &\leq \overline{H}(\Psi)(x_1)
\end{aligned}
$$

Hence $\overline{H}(\Psi)$ on S is a fuzzy quasi-filter of S. ■

2.1.14 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVIH and Ψ be a fuzzy quasi-filter of S. Then $\underline{H}(\Psi)$ is a fuzzy qausi filter of S.

Proof. From Theorem 2.1.9, we have if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Then $\underline{H}(\Psi)(x_1) \leq \underline{H}(\Psi)(x_2)$ for all $x_1, x_2 \in S$. Also $\underline{H}(\Psi)(x_1x_2) \geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2)\}$. Therefore we consider the following for each $x_1, x_2, x_3 \in S$ we have $x_1x_2 = x_3x_1$ holds.

Then consider

$$
\begin{aligned}
\underline{H}(\Psi)(x_1x_2) &= \bigwedge_{x_1'\in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1'\in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{ab\in H(x_1)H(x_2)} \Psi(ab) \ \ (\text{as } x_1' = ab \text{ where } a \in H(x_1) \text{ and } b \in H(x_2)) \\
&= \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)}} \Psi(ab) \\
&\leq \bigwedge_{a\in H(x_1)} \Psi(a)
\end{aligned}
$$

implies $\underline{H}(\Psi)(x_1x_2) \leq \underline{H}(\Psi)(x_1)$

Hence $\underline{H}(\Psi)$ on S is a fuzzy quasi-filter of S. ■

2.2 Approximations of $(\in, \in \vee q)$-Fuzzy Filters in Ordered Semigroups

In this section, roughness in $(\in, \in \vee q)$-fuzzy filters is being studied. A fuzzy point x_t of S is said to 'belong to' fuzzy subset Ψ denoted as $x_t \in \Psi$, if $\Psi(x) \geq t$, and is said to 'quasi-coincident with' a fuzzy subset Ψ denoted by $x_t q \Psi$, if $\Psi(x) + t > 1$. To say that $x_t \in \vee q \Psi$ means that $x_t \in \Psi$ or $x_t q \Psi$.

2.2.1 Definition [7]

A fuzzy subset Ψ on S, is known as $(\in, \in \vee q)$-fuzzy left (right) filter of S if

(FF_6) $(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $(x_1 \leq x_2, x_{1t_1} \in \Psi$ implies $x_{2t_1} \in \vee q \Psi)$

(FF_7) $(\forall x_1, x_2 \in S$ and $\forall t_1, t_2 \in (0,1])$ $\left(x_{1t_1} \in \Psi, x_{2t_2} \in \Psi \text{ implies } (x_1x_2)_{\min\{t_1,t_2\}} \in \vee q \Psi\right)$

(FF_8) $(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $(x_{1t_1} \in \Psi$ implies $(x_1x_2)_{t_1} \in \vee q\Psi(resp.(x_2x_1)_{t_1} \in \vee q\Psi))$.

If Ψ is both $(\in, \in \vee q)$-fuzzy left filter and $(\in, \in \vee q)$-fuzzy right filter of S, then Ψ is called $(\in, \in \vee q)$-fuzzy filter of S.Or equivalently, Ψ is known as $(\in, \in \vee q)$-fuzzy filter of S if it holds condition $(FF_6), (FF_7)$, and

$(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $((x_1x_2)_{t_1} \in \Psi$ implies $x_{1t_1} \in \vee q\Psi, x_{2t_1} \in \vee q\Psi)$.

2.2.2 Definition [7]

A fuzzy subset Ψ is called $(\in, \in \vee q)$-fuzzy bi-filter of S if it holds conditions $(FF_6), (FF_7)$ and

(FF_9) $(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $(x_{1t_1} \in \Psi$ implies $(x_1x_2x_1)_{t_1} \in \vee q\Psi)$.

2.2.3 Lemma [7]

Ψ is called $(\in, \in \vee q)$-fuzzy *left* (resp. *right*) filter of S if and only if it satisfies

$(FF_{6'})$ $(\forall x_1, x_2 \in S)$ $(x_1 \leq x_2, \Psi(x_2) \geq \min\{\Psi(x_1), 0.5\})$

$(FF_{7'})$ $(\forall x_1, x_2 \in S)$ $(\Psi(x_1x_2) \geq \min\{\Psi(x_1), \Psi(x_2), 0.5\})$

$(FF_{8'})$ $(\forall x_1, x_2 \in S)$ $(\Psi(x_1x_2) \geq \min\{\Psi(x_1), 0.5\}$ (resp. $\Psi(x_1x_2) \geq \min\{\Psi(x_2), 0.5\}))$.

2.2.4 Lemma [7]

Ψ is known as $(\in, \in \vee q)$-fuzzy bi-filter of S if and only if it holds conditions $(FF_{6'}), (FF_{7'})$ of lemma 2.2.3, and

$(FF_{9'})$ $(\forall x_1, x_2 \in S)$ $(\Psi(x_1x_2x_1) \geq \min\{\Psi(x_1), 0.5\})$.

2.2.5 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVMH and Ψ be $(\in, \in \vee q)$-fuzzy *left* (resp. *right*) filter of S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy *left* (*resp. right*) filter of S.

Proof. To prove this theorem we have to see that $\overline{H}(\Psi)$ satisfies $(FF_{6'}), (FF_{7'})$ and $(FF_{8'})$. If for each $x_1, x_2 \in S$ we have $x_1 \leq x_2$, then $H(x_1) \subseteq H(x_2)$. Now consider

$$\begin{aligned}
\min\left(\overline{H}(\Psi)(x_1), 0.5\right) &= \min\left(\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), 0.5\right) \\
&= \bigvee_{x_1' \in H(x_1)} \min\left(\Psi\left(x_1'\right), 0.5\right) \\
&\leq \bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right) \\
\text{That is } \min\left(\overline{H}(\Psi)(x_1), 0.5\right) &\leq \overline{H}(\Psi)(x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left(\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), 0.5\right) &= \min\left(\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), \bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right), 0.5\right) \\
&= \bigvee_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \min\left(\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right) \\
&= \bigvee_{x_1' x_2' H(x_1) H(x_2)} \min\left(\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right) \\
&= \bigvee_{x_1' x_2' \in H(x_1 x_2)} \min\left(\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right) \\
&\leq \bigvee_{x_1' x_2' \in H(x_1 x_2)} \Psi\left(x_1' x_2'\right) \\
\text{implies } \min\left(\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), 0.5\right) &\leq \overline{H}(\Psi)(x_1 x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left(\overline{H}(\Psi)(x_1), 0.5\right) &= \min\left(\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), 0.5\right) \\
&= \bigvee_{x_1' \in H(x_1)} \min\left(\Psi\left(x_1'\right), 0.5\right) \\
&\leq \bigvee_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \Psi\left(x_1' x_2'\right) \\
&= \bigvee_{x_1' x_2' \in H(x_1) H(x_2)} \Psi\left(x_1' x_2'\right) \\
&= \bigvee_{x_1' x_2' \in H(x_1 x_2)} \Psi\left(x_1' x_2'\right) \\
\text{implies } \min\left(\overline{H}(\Psi)(x_1), 0.5\right) &\leq \overline{H}(\Psi)(x_1 x_2)
\end{aligned}$$

Therefore $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy left filter of S. Similarly it can be shown that $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy right filter of S. ■

2.2.6 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVIH and Ψ be $(\in, \in \vee q)$-fuzzy *left* (resp. *right*) filter of S. Then $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy *left* (resp. *right*) filter of S.

Proof. To prove this theorem we have to see that $\underline{H}(\Psi)$ satisfies $(FF_{6'}), (FF_{7'})$ and $(FF_{8'})$. If for each $x_1, x_2 \in S$ we have $x_1 \leq x_2$, then $H(x_2) \subseteq H(x_1)$. Now consider

$$\begin{aligned}
\min\left(\underline{H}(\Psi)(x_1), 0.5\right) &= \min\left(\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), 0.5\right) \\
&= \bigwedge_{x_1' \in H(x_1)} \min\left(\Psi\left(x_1'\right), 0.5\right) \\
&\leq \bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right) \\
\text{That is } \min\left(\underline{H}(\Psi)(x_1), 0.5\right) &\leq \underline{H}(\Psi)(x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left(\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), 0.5\right) &= \min\left(\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), \bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right), 0.5\right) \\
&= \bigwedge_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \min\left(\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right) \\
&= \bigwedge_{x_1'x_2' H(x_1)H(x_2)} \min\left(\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right) \\
&= \bigwedge_{x_1'x_2' \in H(x_1x_2)} \min\left(\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right) \\
&\leq \bigwedge_{x_1'x_2' \in H(x_1x_2)} \Psi\left(x_1'x_2'\right) \\
\text{implies } \min\left(\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), 0.5\right) &\leq \underline{H}(\Psi)(x_1x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left(\underline{H}(\Psi)(x_1), 0.5\right) &= \min\left(\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), 0.5\right) \\
&= \bigwedge_{x_1' \in H(x_1)} \min\left(\Psi\left(x_1'\right), 0.5\right) \\
&\leq \bigwedge_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \Psi\left(x_1'x_2'\right) \\
&= \bigwedge_{x_1'x_2' \in H(x_1)H(x_2)} \Psi\left(x_1'x_2'\right) \\
&= \bigwedge_{x_1'x_2' \in H(x_1x_2)} \Psi\left(x_1'x_2'\right) \\
\text{implies } \min\left(\underline{H}(\Psi)(x_1), 0.5\right) &\leq \underline{H}(\Psi)(x_1x_2)
\end{aligned}$$

Therefore $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy left filter of S. Similarly it can be shown that $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy right filter of S. ■

2.2.7 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVMH and a fuzzy subset Ψ be an $(\in, \in \vee q)$-fuzzy bi-filter of S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy bi-filter of S.

Proof. From Theorem 2.2.5, we have if $x_1 \leq x_2$, then $H(x_1) \subseteq H(x_2)$, therefore $\overline{H}(\Psi)(x_2) \geq \min\left(\overline{H}(\Psi)(x_1), 0.5\right)$ for all $x_1, x_2 \in S$. Also $\overline{H}(\Psi)(x_1x_2) \geq \min\left(\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), 0.5\right)$. Next to prove this theorem we have to see that $\overline{H}(\Psi)$ satisfies $(FF_{9'})$. Therefore we consider the following for each $x_1, x_2 \in S$.

$$
\begin{aligned}
\min\left(\overline{H}(\Psi)(x_1), 0.5\right) &= \min\left(\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), 0.5\right) \\
&= \bigvee_{x_1' \in H(x_1)} \min\left(\Psi\left(x_1'\right), 0.5\right) \\
&\leq \bigvee_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2) \\ x_1' \in H(x_1)}} \Psi\left(x_1'x_2'x_1'\right) \\
&= \bigvee_{x_1'x_2'x_1' \in H(x_1)H(x_2)H(x_1)} \Psi\left(x_1'x_2'x_1'\right) \\
&= \bigvee_{x_1'x_2'x_1' \in H(x_1x_2x_1)} \Psi\left(x_1'x_2'x_1'\right) \\
\text{implies } \min\left(\overline{H}(\Psi)(x_1), 0.5\right) &\leq \overline{H}(\Psi)(x_1x_2x_1)
\end{aligned}
$$

Hence it is clear that $\overline{H}(\Psi)$ on S is a $(\in, \in \vee q)$-fuzzy *bi-filter*. ■

2.2.8 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVIH and a fuzzy subset Ψ be an $(\in, \in \vee q)$-fuzzy bi-filter of S. Then $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy bi-filter of S.

Proof. From Theorem 2.2.6, we have if $x_1 \leq x_2$, then $H(x_2) \subseteq H(x_1)$, therefore $\underline{H}(\Psi)(x_2) \geq \min(\underline{H}(\Psi)(x_1), 0.5)$ for all $x_1, x_2 \in S$. Also $\underline{H}(\Psi)(x_1x_2) \geq \min(\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), 0.5)$. Next to prove this theorem we have to see that

$\underline{H}(\Psi)$ satisfies $(FF_{g'})$. Therefore we consider the following for each $x_1, x_2 \in S$.

$$\begin{aligned}
\min\left(\underline{H}(\Psi)(x_1), 0.5\right) &= \min\left(\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), 0.5\right) \\
&= \bigwedge_{x_1' \in H(x_1)} \min\left(\Psi\left(x_1'\right), 0.5\right) \\
&\leq \bigwedge_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2) \\ x_1' \in H(x_1)}} \Psi\left(x_1' x_2' x_1'\right) \\
&= \bigwedge_{x_1' x_2' x_1' \in H(x_1) H(x_2) H(x_1)} \Psi\left(x_1' x_2' x_1'\right) \\
&= \bigwedge_{x_1' x_2' x_1' \in H(x_1 x_2 x_1)} \Psi\left(x_1' x_2' x_1'\right) \\
\text{implies } \min\left(\underline{H}(\Psi)(x_1), 0.5\right) &\leq \underline{H}(\Psi)(x_1 x_2 x_1)
\end{aligned}$$

Hence it is clear that $\underline{H}(\Psi)$ on S is a $(\in, \in \vee q)$-fuzzy *bi-filter*. ■

2.3 Approximations of $(\in, \in \vee qk)$-Fuzzy Filters in Ordered Semigroups

In this section, roughness in $(\in, \in \vee qk)$-fuzzy filters is being studied. A fuzzy point x_{1t} of S is known to 'belong to' a fuzzy subset Ψ, denoted as $x_{1t} \in \Psi$ if $\Psi(x_1) \geq t$, and is said to be a 'quasi-coincident with' to Ψ, denoted as $x_{1t} qk \Psi$, if $\Psi(x_1) + t + k > 1$, where $k \in [0, 1)$.

2.3.1 Definition

A fuzzy subset Ψ on S is known as $(\in, \in \vee qk)$-fuzzy left (right) filter of S where $k \in [0, 1)$ if it holds the following conditions

(FF_{10}) $(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $(x_1 \leq x_2, x_{1t_1} \in \Psi$ implies $x_{2t_1} \in \vee qk\Psi)$

(FF_{11}) $(\forall x_1, x_2 \in S$ and $\forall t_1, t_2 \in (0,1])$ $\left(x_{1t_1}, x_{2t_2} \in \Psi \text{ implies } (x_1x_2)_{\min\{t_1,t_2\}} \in \vee qk\Psi\right)$

(FF_{12}) $(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $\left(x_{1t_1} \in \Psi \text{ implies } (x_1x_2)_{t_1} \in \vee qk\Psi \left(\text{resp. } (x_2x_1)_{t_1} \in \vee qk\Psi\right)\right)$

If Ψ is both $(\in, \in \vee qk)$-fuzzy left filter and $(\in, \in \vee qk)$-fuzzy right filter of S, then Ψ is called $(\in, \in \vee qk)$-fuzzy filter of S.Or equivalently Ψ is known as $(\in, \in \vee qk)$-fuzzy filter of S if it holds $(FF_{10}), (FF_{11})$, and

$(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $((x_1x_2)_{t_1} \in \Psi$ implies $x_{1t_1} \in \vee qk\Psi, x_{2t_1} \in \vee qk\Psi)$.

2.3.2 Definition

A fuzzy subset Ψ on S is known as $(\in, \in \vee qk)$-fuzzy bi-filter of S if it holds $(FF_{11}), (FF_{12})$ and

(FF_{13}) $(\forall x_1, x_2 \in S$ and $\forall t_1 \in (0,1])$ $\left(x_{1t_1} \in \Psi \text{ implies } (x_1x_2x_1)_{t_1} \in \vee qk\Psi\right)$.

2.3.3 Lemma

A fuzzy subset Ψ on S is known as $(\in, \in \vee qk)$-fuzzy left (resp. right) filter of S if and only if it satisfies the following assertions

$(FF_{10'})$ $(\forall x_1, x_2 \in S)$ $\left(x_1 \leq x_2, \Psi(x_2) \geq \min\left\{\Psi(x_1), \frac{1-k}{2}\right\}\right)$

$(FF_{11'})$ $(\forall x_1, x_2 \in S)$ $\left(\Psi(x_1x_2) \geq \min\left\{\Psi(x_1), \Psi(x_2), \frac{1-k}{2}\right\}\right)$

$(FF_{12'})$ $(\forall x_1, x_2 \in S)$ $\left(\Psi(x_1x_2) \geq \min\left\{\Psi(x_1), \frac{1-k}{2}\right\} \left(\text{resp. } \Psi(x_1x_2) \geq \min\left\{\Psi(x_2), \frac{1-k}{2}\right\}\right)\right)$.

2.3.4 Lemma

Ψ is known as $(\in, \in \vee qk)$-fuzzy bi-filter of S if and only if it holds $(FF_{10'}), (FF_{11'})$ of lemma 2.3.3, and

$$(FF_{13'})\ (\forall x_1, x_2 \in S)\left(\Psi(x_1x_2x_1) \geq \min\left\{\Psi(x_1), \tfrac{1-k}{2}\right\}\right).$$

2.3.5 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVMH and Ψ be $(\in, \in \vee qk)$-fuzzy left (right) filter of S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee qk)$-fuzzy left (right) filter of S.

Proof. To prove this theorem we have to see that $\overline{H}(\Psi)$ satisfies $(FF_{10'}), (FF_{11'})$ and $(FF_{12'})$. If for each $x_1, x_2 \in S$ we have $x_1 \leq x_2$, then $H(x_1) \subseteq H(x_2)$. Now consider

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_1' \in H(x_1)} \min\left\{\Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&\leq \bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right) \\
\text{that is } \min\left\{\overline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &\leq \overline{H}(\Psi)(x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1),\overline{H}(\Psi)(x_2),\frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_1'\in H(x_1)}\Psi\left(x_1'\right),\bigvee_{x_2'\in H(x_2)}\Psi\left(x_2'\right),\frac{1-k}{2}\right\}\\
&= \bigvee_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)}}\min\left\{\Psi\left(x_1'\right),\Psi\left(x_2'\right),\frac{1-k}{2}\right\}\\
&= \bigvee_{x_1'x_2'\in H(x_1)H(x_2)}\min\left\{\Psi\left(x_1'\right),\Psi\left(x_2'\right),\frac{1-k}{2}\right\}\\
&= \bigvee_{x_1'x_2'\in H(x_1x_2)}\min\left\{\Psi\left(x_1'\right),\Psi\left(x_2'\right),\frac{1-k}{2}\right\}\\
&\leq \bigvee_{x_1'x_2'\in H(x_1x_2)}\underline{H}(\Psi)\left(x_1'x_2'\right)\\
\text{implies } \min\left\{\overline{H}(\Psi)(x_1),\overline{H}(\Psi)(x_2),\frac{1-k}{2}\right\} &\leq \overline{H}(\Psi)(x_1x_2)
\end{aligned}$$

Now consider

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1),\frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_1'\in H(x_1)}\Psi\left(x_1'\right),\frac{1-k}{2}\right\}\\
&= \bigvee_{x_1'\in H(x_1)}\min\left\{\Psi\left(x_1'\right),\frac{1-k}{2}\right\}\\
&\leq \bigvee_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)}}\Psi\left(x_1'x_2'\right)\\
&= \bigvee_{x_1'x_2'\in H(x_1)H(x_2)}\Psi\left(x_1'x_2'\right)\\
&= \bigvee_{x_1'x_2'\in H(x_1x_2)}\Psi\left(x_1'x_2'\right)\\
\text{implies } \min\left\{\overline{H}(\Psi)(x_1),\frac{1-k}{2}\right\} &\leq \overline{H}(\Psi)(x_1x_2)
\end{aligned}$$

Hence $\overline{H}(\Psi)$ is $(\in,\in\vee qk)$-fuzzy left filter of S. Similarly we can show that $\overline{H}(\Psi)$ is $(\in,\in\vee qk)$-fuzzy right filter of S. ■

2.3.6 Theorem

Let $H : S \longrightarrow P^*(S)$ be a SVIH and Ψ be $(\in, \in \vee qk)$-fuzzy left (right) filter of S. Then $\underline{H}(\Psi)$ is $(\in, \in \vee qk)$-fuzzy left (right) filter of S.

Proof. To prove this theorem we have to see that $\underline{H}(\Psi)$ satisfies $(FF_{10'}), (FF_{11'})$ and $(FF_{12'})$. If for each $x_1, x_2 \in S$ we have $x_1 \leq x_2$, then $H(x_2) \subseteq H(x_1)$. Now consider

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1' \in H(x_1)} \min\left\{\Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right) \\
\text{that is } \min\left\{\underline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &\leq \underline{H}(\Psi)(x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), \bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1' x_2' \in H(x_1) H(x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1' x_2' \in H(x_1 x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{x_1' x_2' \in H(x_1 x_2)} \underline{H}(\Psi)\left(x_1' x_2'\right) \\
\text{implies } \min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &\leq \underline{H}(\Psi)(x_1 x_2)
\end{aligned}$$

Now consider

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1'\in H(x_1)} \min\left\{\Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)}} \Psi\left(x_1'x_2'\right) \\
&= \bigwedge_{x_1'x_2'\in H(x_1)H(x_2)} \Psi\left(x_1'x_2'\right) \\
&= \bigwedge_{x_1'x_2'\in H(x_1x_2)} \Psi\left(x_1'x_2'\right) \\
\text{implies } \min\left\{\underline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &\leq \underline{H}(\Psi)(x_1x_2)
\end{aligned}$$

Hence $\underline{H}(\Psi)$ is $(\in, \in \vee qk)$-fuzzy left filter of S. Similarly we can show that $\underline{H}(\Psi)$ is $(\in, \in \vee qk)$-fuzzy right filter of S. ■

2.3.7 Theorem

Suppose $H : S \longrightarrow P^{*}(S)$ is a SVMH and Ψ is $(\in, \in \vee qk)$-fuzzy bi-filter of S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee qk)$-fuzzy bi-filter of S.

Proof. From Theorem 2.3.5, we can see that $(FF_{10'})$ and $(FF_{11'})$ hold for $\overline{H}(\Psi)$.

For $(FF_{13'})$ therefore we consider the following for each $x_1, x_2 \in S$.

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_1'\in H(x_1)} \min\left\{\Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&\leq \bigvee_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)\\ x_1'\in H(x_1)}} \Psi\left(x_1' x_2' x_1'\right) \\
&= \bigvee_{x_1' x_2' x_1' \in H(x_1)H(x_2)H(x_1)} \Psi\left(x_1' x_2' x_1'\right) \\
&= \bigvee_{x_1' x_2' x_1' \in H(x_1 x_2 x_1)} \Psi\left(x_1' x_2' x_1'\right) \\
\text{implies } \min\left\{\overline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &\leq \overline{H}(\Psi)(x_1 x_2 x_1)
\end{aligned}$$

Hence $\overline{H}(\Psi)$ on S is an $(\in, \in \vee qk)$-fuzzy bi-filter of S. ■

2.3.8 Theorem

Suppose $H : S \longrightarrow P^*(S)$ is a SVIH and Ψ is $(\in, \in \vee qk)$-fuzzy bi-filter of S. Then $\underline{H}(\Psi)$ is $(\in, \in \vee qk)$-fuzzy bi-filter of S.

Proof. From Theorem 2.3.6, we can see that $(FF_{10'})$ and $(FF_{11'})$ hold for $\underline{H}(\Psi)$.

For $(FF_{13'})$ therefore we consider the following for each $x_1, x_2 \in S$.

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1'\in H(x_1)} \min\left\{\Psi\left(x_1'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)\\ x_1'\in H(x_1)}} \Psi\left(x_1'x_2'x_1'\right) \\
&= \bigwedge_{x_1'x_2'x_1'\in H(y_1)H(y_2)H(y_1)} \Psi\left(x_1'x_2'x_1'\right) \\
&= \bigwedge_{x_1'x_2'x_1'\in H(x_1x_2x_1)} \Psi\left(x_1'x_2'x_1'\right) \\
\text{implies } \min\left\{\underline{H}(\Psi)(x_1), \frac{1-k}{2}\right\} &\leq \underline{H}(\Psi)(x_1x_2x_1)
\end{aligned}$$

Hence $\underline{H}(\Psi)$ on S is an $(\in, \in \vee qk)$-fuzzy bi-filter of S. ■

Chapter 3

Roughness in Fuzzy Ideals in Ordered Semigroups

In this chapter, study of roughness in fuzzy ideals in ordered semigroups is being presented. Thus we will start with the following result.

- ►Approximations of fuzzy left (resp. right) ideal
- ►Approximations of fuzzy interior ideal
- ►Approximations of fuzzy bi-ideal and fuzzy quasi ideal
- ►Approximations of $(\in, \in \vee q)$-fuzzy left (resp. right) ideal
- ►Approximations of $(\in, \in \vee q)$-fuzzy interior ideal
- ►Approximations of $(\in, \in \vee q)$-fuzzy bi-ideal
- ►Approximations of $(\in, \in \vee q)$-fuzzy quasi ideal
- ►Approximations of $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal
- ►Approximations of $(\in, \in \vee qk)$-fuzzy interior ideal
- ►Approximations of $(\in, \in \vee qk)$-fuzzy bi-ideal and $(\in, \in \vee qk)$-fuzzy quasi ideal

3.1 Approximations of Fuzzy Ideals in Ordered Semigroups

In chapter 1st, we have defined fuzzy ideals and their generalization. In this section we will recall it and then discuss the approximations of fuzzy ideals and approximations of their generalization.

3.1.1 Definition [20]

A fuzzy subset Ψ of ordered semigroup S is called a fuzzy ordered subsemigroup of S if it satisfies

(FI_1) $(\forall x_1, x_2 \in S)\left(\Psi(x_1x_2) \geq \min\left\{\Psi(x_1), \Psi(x_2)\right\}\right)$.

3.1.2 Definition [20]

A fuzzy subset Ψ of ordered semigroup S is known as a fuzzy left (resp. right) ideal of S if it satisfies

(FI_2) $(\forall x_1, x_2 \in S)(x_1 \leq x_2 \Longrightarrow \Psi(x_1) \geq \Psi(x_2))$

(FI_3) $(\forall x_1, x_2 \in S)(\Psi(x_1x_2) \geq \Psi(x_2)(\text{resp. } \Psi(x_1x_2) \geq \Psi(x_1)))$.

A fuzzy subset Ψ is known as fuzzy ideal of S, if it is both a fuzzy left ideal and a fuzzy right ideal of S.

From this definition we can also conclude the following

3.1.3 Definition

A fuzzy subset Ψ is called fuzzy ideal of ordered semigroup S if it satisfy (FI_2) and

(FI_4) $(\forall x_1, x_2 \in S)(\Psi(x_1x_2) \geq \max\{\Psi(x_1), \Psi(x_2)\})$.

3.1.4 Definition [22]

A fuzzy subset Ψ is known as fuzzy interior ideal of ordered semigroup S if it satisfies $(FI_1), (FI_2)$ and

(FI_5) $(\forall x_1, x_2, x_3 \in S)(\Psi(x_1x_3x_2) \geq \Psi(x_3))$.

3.1.5 Definition [21]

Let Ψ be a fuzzy subset of ordered semigroup S. Then a fuzzy subset Ψ is said to be a fuzzy bi-ideal of S if it satisfies $(FI_1), (FI_2)$ and

(FI_6) $(\forall x_1, x_2, x_3 \in S)(\Psi(x_1x_2x_3) \geq \min\{\Psi(x_1), \Psi(x_3)\})$.

3.1.6 Definition [23]

A fuzzy subset Ψ of ordered semigroup S is known as a fuzzy quasi ideal of S if it satisfies (FI_2) and

(FI_7) $\Psi \geq (\Psi \circ 1) \wedge (1 \circ \Psi)$.

3.1.7 Theorem

Let $H : S \to P^*(S)$ be a SVH and Ψ be a fuzzy ordered subsemigroup of S. Then $\overline{H}(\Psi)$ is a fuzzy ordered subsemigroup of S.

Proof. For any $x_1, x_2 \in S$, consider

$$\begin{aligned}
\overline{H}(\Psi)(x_1x_2) &= \bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{ab \in H(x_1)H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab \text{ such that } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\geq \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \min\{\Psi(a), \Psi(b)\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), \bigvee_{b \in H(x_2)} \Psi(b)\right\} \\
\text{implies } \overline{H}(\Psi)(x_1x_2) &\geq \min\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2)\}
\end{aligned}$$

Hence $\overline{H}(\Psi)$ is a fuzzy ordered subsemigroup of S. ∎

3.1.8 Theorem

Let $H : S \to P^*(S)$ be a SVH and Ψ be a fuzzy ordered subsemigroup of S. Then $\underline{H}(\Psi)$ is a fuzzy ordered subsemigroup of S.

Proof. Similarly as above Theorem 3.1.7. ∎

In the following, study of roughness in fuzzy ideals of ordered semigroups is being initiated. Certain restrictions are imposed on SVH for this study.

3.1.9 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and Ψ be a fuzzy left (resp. right) ideal of ordered semigroup S. Then $\underline{H}(\Psi)$ is a fuzzy left (resp. right) ideal of ordered semigroup S.

Proof. For each $x_1, x_2 \in S$ with $x_1 \leq x_2$, then $H(x_1) \subseteq H(x_2)$. Now we may consider the following

$$\begin{aligned}
\underline{H}(\Psi)(x_1) &= \bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\
&\geq \bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right) \\
\text{implies } \underline{H}(\Psi)(x_1) &\geq \underline{H}(\Psi)(x_2)
\end{aligned}$$

Next

$$\begin{aligned}
\underline{H}(\Psi)(x_1x_2) &= \bigwedge_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{ab \in H(x_1)H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab \text{ such that } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\geq \bigwedge_{b \in H(x_2)} \Psi(b) \\
\text{implies } \underline{H}(\Psi)(x_1x_2) &\geq \underline{H}(\Psi)(x_2)
\end{aligned}$$

Hence $\underline{H}(\Psi)$ is a fuzzy left ideal of ordered semigroup S. Similarly it can be shown that $\underline{H}(\Psi)$ is a fuzzy right ideal of S. ■

The following example shows that if H is a SVMH, then for a fuzzy ideal Ψ, its upper approximation $\overline{H}(\Psi)$ may not be a fuzzy ideal.

3.1.10 Example

Consider a set $S = \{a, b, c, d, e, f\}$ with multiplication table 1 and order relation.

Multiplication table for S

·	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

Table 1

and

$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(f,e)\}$. Then $(S, \cdot, \leq)$ is an ordered semigroup. Right ideals of S are $\{a,d\}$, $\{a,b,d\}$ and S. Left ideals of S are $\{a\}$, $\{a,b\}$, $\{a,d\}$, $\{a,b,d\}$, $\{a,b,c,d\}$, $\{a,b,d,e,f\}$ and S. Define a fuzzy subset $\Psi : S \to [0,1]$ by $\Psi(a) = 0.8, \Psi(b) = 0.5, \Psi(d) = 0.6$ and $\Psi(c) = \Psi(e) = \Psi(f) = 0.4$. Then Ψ is a fuzzy ideal of S,

Now consider $H : S \to P^*(S)$ be a $SVMH$ i.e.

$$(i) \quad H(x_1)H(x_1) = H(x_1x_2)$$

$$(ii) \quad \text{if } x_1 \leq x_2 \to H(x_1) \subseteq H(x_2).$$

Where $P^*(S)$ consist of all non-empty subset of S. Now if $H(e) = \{b,c,d,e,f\}$ and $H(f) = \{c,f\}$, as $f \leq e \Rightarrow H(f) \subseteq H(e)$ but $\overline{H}(\Psi)(f) \ngeq \overline{H}(\Psi)(e)$. Hence in $SVMH$ it is prove that $\overline{H}(\Psi)$ is not a fuzzy left (resp. right) ideal of S.

3.1.11 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ be a fuzzy left (resp. right) ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is a fuzzy left (resp. right) ideal of ordered semigroup S.

Proof. For each $x_1, x_2 \in S$ with $x_1 \leq x_2$, then $H(x_2) \subseteq H(x_1)$. Now consider the following

$$\begin{aligned} \overline{H}(\Psi)(x_1) &= \bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\ &\geq \bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right) \\ \text{implies } \overline{H}(\Psi)(x_1) &\geq \overline{H}(\Psi)(x_2) \end{aligned}$$

Next

$$\begin{aligned} \overline{H}(\Psi)(x_1x_2) &= \bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\ &= \bigvee_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\ &= \bigvee_{ab \in H(x_1)H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab \text{ such that } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\ &= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\ &\geq \bigvee_{b \in H(x_2)} \Psi(b) \\ \text{implies } \overline{H}(\Psi)(x_1x_2) &\geq \overline{H}(\Psi)(x_2) \end{aligned}$$

Hence $\overline{H}(\Psi)$ is a fuzzy left ideal of ordered semigroup S. Similarly it can be shown that $\overline{H}(\Psi)$ is a fuzzy right ideal of S. ■

The following example shows that if H is a SVIH, then for a fuzzy ideal Ψ, its lower approximation $\underline{H}(\Psi)$ may not be a fuzzy ideal.

3.1.12 Example

Consider a fuzzy ideal Ψ of S as given in remark 3.1.10. Now consider $H : S \to P^{*}(S)$ be a $SVIH$ i.e.

$$(i) \quad H(y_1) H(y_2) = H(y_1 y_2)$$

$$(ii) \quad \text{if } y_1 \leq y_2 \to H(y_2) \subseteq H(y_1).$$

Where $P^{*}(S)$ consist of all non-empty subset of S. Now if $H(f) = \{a, b, d\}$ and $H(e) = \{a, d\}$, as $f \leq e \Rightarrow H(e) \subsetneq H(f)$ but $\underline{H}(\Psi)(f) \ngeq \underline{H}(\Psi)(e)$. Hence in $SVIH$ it is prove that $\underline{H}(\Psi)$ is not a fuzzy left (resp. right) ideal of S.

3.1.13 Theorem

Let $H : S \to P^{*}(S)$ be $SVMH$ and Ψ be a fuzzy interior ideal of S. Then $\underline{H}(\Psi)$ is a fuzzy interior ideal of ordered semigroup S.

Proof. From Theorem 3.1.9, we have $x_1 \leq x_2$, implies $H(x_1) \subseteq H(x_2)$, for each $x_1, x_2 \in S$, then $\underline{H}(\Psi)(x_1) \geq \underline{H}(\Psi)(x_2)$.Next consider

$$\begin{aligned}
\underline{H}(\Psi)(x_1 x_2) &= \bigwedge_{x_1' \in H(x_1 x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1' \in H(x_1) H(x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{ab \in H(x_1) H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab, \text{ where } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\geq \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \min\{\Psi(a), \Psi(b)\} \\
&= \min\left\{\bigwedge_{a \in H(x_1)} \Psi(a), \bigwedge_{b \in H(x_2)} \Psi(b)\right\}
\end{aligned}$$

$$\text{implies } \underline{H}(\Psi)(x_1 x_2) \geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2)\}$$

Consider

$$\begin{aligned}
\underline{H}(\Psi)(x_1x_3x_2) &= \bigwedge_{x_1' \in H(x_1x_3x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1' \in H(x_1)H(x_3)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigwedge_{acb \in H(x_1)H(x_3)H(x_2)} \Psi(acb) \quad \left(\begin{array}{c} \text{as } x_1' = acb \text{ where } a \in H(x_1), \\ c \in H(x_3) \text{ and } b \in H(x_2) \end{array}\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ c \in H(x_3) \\ b \in H(x_2)}} \Psi(acb) \\
&\geq \bigwedge_{c \in H(x_3)} \Psi(c) \\
\text{implies } \underline{H}(\Psi)(x_1x_3x_2) &\geq \underline{H}(\Psi)(x_3)
\end{aligned}$$

Hence $\underline{H}(\Psi)$ is a fuzzy interior ideal of ordered semigroup S. ∎

3.1.14 Theorem

Let $H : S \to P^*(S)$ be $SVIH$ and Ψ be a fuzzy interior ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is a fuzzy interior ideal of ordered semigroup S.

Proof. From Theorem 3.1.11, if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$ for each $x_1, x_2 \in$

S, then $\overline{H}(\Psi)(x_1) \geq \overline{H}(\Psi)(x_2)$.Next we may consider the following

$$\begin{aligned}
\overline{H}(\Psi)(x_1x_2) &= \bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{ab \in H(x_1)H(x_2)} \Psi(ab) \quad \left(\text{as } x_1' = ab, \text{ where } a \in H(x_1) \text{ and } b \in H(x_2)\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \Psi(ab) \\
&\geq \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \min\{\Psi(a), \Psi(b)\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), \bigvee_{b \in H(x_2)} \Psi(b)\right\} \\
\text{implies } \overline{H}(\Psi)(x_1x_2) &\geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2)\right\}
\end{aligned}$$

Consider

$$\begin{aligned}
\overline{H}(\Psi)(x_1x_3x_2) &= \bigvee_{x_1' \in H(x_1x_3x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_3)H(x_2)} \Psi\left(x_1'\right) \\
&= \bigvee_{acb \in H(x_1)H(x_3)H(x_2)} \Psi(acb) \quad \left(\begin{array}{c} \text{as } x_1' = acb \text{ where } a \in H(x_1), \\ c \in H(x_3) \text{ and } b \in H(x_2) \end{array}\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ c \in H(x_3) \\ b \in H(x_2)}} \Psi(acb) \\
&\geq \bigvee_{c \in H(x_3)} \Psi(c) \\
\text{implies } \overline{H}(\Psi)(x_1x_3x_2) &\geq \overline{H}(\Psi)(x_3)
\end{aligned}$$

Hence $\overline{H}(\Psi)$ is a fuzzy interior ideal of ordered semigroup S. ∎

3.1.15 Theorem

Let $H : S \rightarrow P^{*}(S)$ be a $SVMH$ and Ψ be a fuzzy bi-ideal of ordered semi-group S. Then $\underline{H}(\Psi)$ is a fuzzy bi-ideal of ordered semigroup S.

Proof. From Theorem 3.1.13, we have for each $x_1, x_2 \in S$ such that $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$, then $\underline{H}(\Psi)(x_1) \geq \underline{H}(\Psi)(x_2)$ and also $\underline{H}(\Psi)(x_1x_2) \geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2)\}$. Next for each $x_1, x_2, x_3 \in S$, Consider the following

$$\begin{aligned}
\underline{H}(\Psi)(x_1x_2x_3) &= \bigwedge_{x_1' \in H(x_1x_2x_3)} \Psi\left(x_1'\right) \\
&= \bigwedge_{x_1' \in H(x_1)H(x_2)H(x_3)} \Psi\left(x_1'\right) \\
&= \bigwedge_{abc \in H(x_1)H(x_2)H(x_3)} \Psi(abc) \quad \left(\begin{array}{c} \text{as } x_1' = abc \text{ where } a \in H(x_1), \\ b \in H(x_2) \text{ and } c \in H(x_3) \end{array}\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2) \\ c \in H(x_3)}} \Psi(abc) \\
&\geq \bigwedge_{\substack{a \in H(x_1) \\ c \in H(x_3)}} \min\{\Psi(a), \Psi(c)\} \\
&= \min\left\{\bigwedge_{a \in H(x_1)} \Psi(a), \bigwedge_{c \in H(x_3)} \Psi(c)\right\} \\
\text{implies } \underline{H}(\Psi)(x_1x_2x_3) &\geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_3)\}
\end{aligned}$$

Hence $\underline{H}(\Psi)$ satisfies all the conditions of a fuzzy bi-ideal of S, so $\underline{H}(\Psi)$ is a fuzzy bi-ideal of ordered semigroup S. ■

3.1.16 Theorem

Let $H : S \rightarrow P^{*}(S)$ be a $SVIH$ and Ψ be a fuzzy bi-ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is a fuzzy bi-ideal of ordered semigroup S.

Proof. From Theorem 3.1.14, for each $x_1, x_2 \in S$ such that $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$, then $\overline{H}(\Psi)(x_1) \geq \overline{H}(\Psi)(x_2)$ and also $\overline{H}(\Psi)(x_1x_2) \geq \min\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2)\}$. Next for each $x_1, x_2, x_3 \in S$, we consider

$$
\begin{aligned}
\overline{H}(\Psi)(x_1x_2x_3) &= \bigvee_{x_1' \in H(x_1x_2x_3)} \Psi\left(x_1'\right) \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)H(x_3)} \Psi\left(x_1'\right) \\
&= \bigvee_{abc \in H(x_1)H(x_2)H(x_3)} \Psi(abc) \quad \left(\begin{array}{c}\text{as } x_1' = abc \text{ where } a \in H(x_1), \\ b \in H(x_2) \text{ and } c \in H(x_3)\end{array}\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2) \\ c \in H(x_3)}} \Psi(abc) \\
&\geq \bigvee_{\substack{a \in H(x_1) \\ c \in H(x_3)}} \min\{\Psi(a), \Psi(c)\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), \bigvee_{c \in H(x_3)} \Psi(c)\right\} \\
\text{implies } \overline{H}(\Psi)(x_1x_2x_3) &\geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_3)\right\}
\end{aligned}
$$

Hence $\overline{H}(\Psi)$ satisfies all the conditions of a fuzzy bi-ideal, so $\overline{H}(\Psi)$ is a fuzzy bi-ideal of ordered semigroup S. ■

3.1.17 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVMH$ and Ψ be a fuzzy quasi ideal of ordered semigroup S. Then $\underline{H}(\Psi)$ is a fuzzy quasi ideal of ordered semigroup S.

Proof. From Theorem 3.1.9, for each $x_1, x_2 \in S$ such that $x_1 \leq x_2$ implies

$H(x_1) \subseteq H(x_2)$, then $\underline{H}(\Psi)(x_1) \geq \underline{H}(\Psi)(x_2)$. Next consider

$$\begin{aligned} \underline{H}(\Psi)(x_1) &= \bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\ &\geq \bigwedge_{x_1' \in H(x_1)} ((\Psi \circ 1) \wedge (1 \circ \Psi))\left(x_1'\right) \\ &= \underline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1) \\ \text{implies } \underline{H}(\Psi)(x_1) &\geq \underline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1) \end{aligned}$$

Hence $\underline{H}(\Psi)$ is a fuzzy quasi ideal of ordered semigroup S. ∎

3.1.18 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ be a fuzzy quasi ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is a fuzzy quasi ideal of ordered semigroup S.

Proof. From Theorem 3.1.11, for each $x_1, x_2 \in S$ such that $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$, then $\overline{H}(\Psi)(x_1) \geq \overline{H}(\Psi)(x_2)$. Next consider

$$\begin{aligned} \overline{H}(\Psi)(x_1) &= \bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\ &\geq \bigvee_{x_1' \in H(x_1)} ((\Psi \circ 1) \wedge (1 \circ \Psi))\left(x_1'\right) \\ &= \overline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1) \\ \text{implies } \overline{H}(\Psi)(x_1) &\geq \overline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1) \end{aligned}$$

Hence $\overline{H}(\Psi)$ is a fuzzy quasi ideal of ordered semigroup S. ∎

3.2 Approximations of $(\in, \in \vee q)$-Fuzzy Ideals in Ordered Semigroups

In this section, roughness in $(\in, \in \vee q)$-fuzzy ideals is being studied. A fuzzy point x_t of S is said to 'belong to' fuzzy subset Ψ denoted as $x_t \in \Psi$, if $\Psi(x) \geq t$, and is said to 'quasi-coincident with' a fuzzy subset Ψ denoted by $x_t q \Psi$, if $\Psi(x) + t > 1$. To say that $x_t \in \vee q \Psi$ means that $x_t \in \Psi$ or $x_t q \Psi$.

3.2.1 Definition

A fuzzy subset Ψ of ordered semigroup S is called an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S if

(FI_8) (for each $x_1, x_2 \in S$) (for all $t_1, t_2 \in (0, 1]$) $\left(x_{1t_1}, x_{2t_2} \in \Psi \text{ implies } (x_1x_2)_{\min\{t_1,t_2\}} \in \vee q \Psi\right)$.

3.2.2 Definition

A fuzzy subset Ψ of ordered semigroup S is known as $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S if the following conditions are satisfies

(FI_9) (for all $x_1, x_2 \in S$) (for all $t_1 \in (0, 1]$) $(x_1 \leq x_2,\ x_{2t_1} \in \Psi \text{ implies } x_{1t_1} \in \vee q \Psi)$

(FI_{10}) (for all $x_1, x_2 \in S$) ($\forall t_1 \in (0, 1]$) $\left(x_{2t_1} \in \Psi \text{ implies } (x_1x_2)_{t_1} \in \vee q \Psi \ \left((x_2x_1)_{t_1} \in \vee q \Psi\right)\right)$.

A fuzzy subset Ψ is known as $(\in, \in \vee q)$-fuzzy ideals of S, if it is both $(\in, \in \vee q)$-fuzzy left and $(\in, \in \vee q)$-fuzzy right ideals of S.

3.2.3 Definition

A fuzzy subset Ψ of ordered semigroup S is said to be an $(\in, \in \vee q)$-fuzzy interior ideal of S if it satisfies $(FI_8), (FI_9)$ and

(FI_{11}) (for all $x_1, x_2, x_3 \in S$) (for all $t_1 \in (0,1]$) $\left(x_{3t_1} \in \Psi \text{ implies } (x_1x_3x_2)_{t_1} \in \vee q\Psi\right)$.

3.2.4 Definition

A fuzzy subset Ψ of S is called an $(\in, \in \vee q)$-fuzzy bi-ideal of S if satisfies $(FI_8), (FI_9)$ and

(FI_{12}) (for all $x_1, x_2, x_3 \in S$) $(\forall t_1, t_2 \in (0,1])$ $\left(x_{1t_1}, x_{3t_2} \in \Psi \text{ implies } (x_1x_2x_3)_{\min\{t_1,t_2\}} \in \vee q\Psi\right)$.

3.2.5 Definition

A fuzzy subset Ψ of ordered semigroup S is known as $(\in, \in \vee q)$-fuzzy quasi ideal of S if it satisfies (FI_9) and

(FI_{13}) (for all $x_1 \in S$) (for all $t_1 \in (0,1]$) $(x_{1t_1} \in (\Psi \circ 1) \wedge (1 \circ \Psi) \text{ implies } x_{1t_1} \in \vee q\Psi)$.

3.2.6 Lemma [25]

A fuzzy subset Ψ of S is known as $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of ordered semigroup S if and only if it holds

(FI_{14}) (for all $x_1, x_2 \in S$) $(x_1 \leq x_2, \Psi(x_1) \geq \min\{\Psi(x_2), 0.5\})$

(FI_{15}) (for all $x_1, x_2 \in S$) $(\Psi(x_1x_2) \geq \min\{\Psi(x_2), 0.5\}$ (resp. $\Psi(x_1x_2) \geq \min\{\Psi(x_1), 0.5\}$)).

3.2.7 Lemma [25]

A fuzzy subset Ψ of ordered semigroup S is said to be an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if it holds (FI_{14}) of lemma 3.2.6 and

(FI_{16}) (for all $x_1, x_2 \in S$) $(\Psi(x_1x_2) \geq \min\{\Psi(x_1), \Psi(x_2), 0.5\})$

(FI_{17}) (for all $x_1, x_2, x_3 \in S$) $(\Psi(x_1x_2x_3) \geq \min\{\Psi(x_1), \Psi(x_3), 0.5\})$.

3.2.8 Lemma [25]

A fuzzy subset Ψ of ordered semigroup S is known as $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if it holds $(FI_{14}), (FI_{16})$ of lemmas 3.2.6, 3.2.7 and

(FI_{18}) (for all $x_1, x_2 \in S$) $(\Psi(x_1x_3x_2) \geq \min\{\Psi(x_3), 0.5\})$.

3.2.9 Lemma

A fuzzy subset Ψ of S is said to be an $(\in, \in \vee q)$-fuzzy quasi ideal of S if and only if it holds (FI_{14}) of lemma 3.2.6 and

(FI_{19}) (for all $x_1, x_2 \in S$) $(\Psi(x_1) \geq \min\{((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), 0.5\})$.

3.2.10 Theorem

Let $H : S \to P^*(S)$ be a SVH and Ψ be $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S. Then $\underline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S.

Proof. To prove this theorem we have to see that $\underline{H}(\Psi)$ satisfies (FI_{16}). If for each $x_1, x_2 \in S$, now consider

$$\begin{aligned}
\min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), 0.5\} &= \min\left\{\bigwedge_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \bigwedge_{x_2'\in H(x_2)} \Psi\left(x_2'\right), 0.5\right\} \\
&= \bigwedge_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right\} \\
&= \bigwedge_{x_1'x_2'\in H(x_1)H(x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right\} \\
&= \bigwedge_{x_1'x_2'\in H(x_1x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), 0.5\right\} \\
&\leq \bigwedge_{x_1'x_2'\in H(x_1x_2)} \Psi\left(x_1'x_2'\right)
\end{aligned}$$

$$\text{implies } \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), 0.5\} \leq \underline{H}(\Psi)(x_1x_2)$$

Hence $\underline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S. ■

3.2.11 Theorem

Let $H : S \to P^*(S)$ be a SVH and Ψ be $(\in, \in \vee q)$-fuzzy subsemigroup of S. Then $\overline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S.

Proof. Straightforward as Theorem 3.2.10. ■

3.2.12 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and Ψ be $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of ordered semigroup S. Then $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of ordered semigroup S.

Proof. To prove this theorem we have to see that $\underline{H}(\Psi)$ satisfies (FI_{14}) and

(FI_{15}). If for each $x_1, x_2 \in S$ with $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$. Now consider

$$\begin{aligned} \min\{\underline{H}(\Psi)(x_2), 0.5\} &= \min\left\{\bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right), 0.5\right\} \\ &= \bigwedge_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), 0.5\right\} \\ &\leq \bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\ \text{implies } \min\{\underline{H}(\Psi)(x_2), 0.5\} &\leq \underline{H}(\Psi)(x_1) \end{aligned}$$

Next consider

$$\begin{aligned} \min\{\underline{H}(\Psi)(x_2), 0.5\} &= \min\left\{\bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right), 0.5\right\} \\ &= \bigwedge_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), 0.5\right\} \\ &\leq \bigwedge_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \Psi\left(x_1'x_2'\right) \\ &= \bigwedge_{x_1'x_2' \in H(x_1)H(x_2)} \Psi\left(x_1'x_2'\right) \\ &= \bigwedge_{x_1'x_2' \in H(x_1x_2)} \Psi\left(x_1'x_2'\right) \\ \text{implies } \min\{\underline{H}(\Psi)(x_2), 0.5\} &\leq \underline{H}(\Psi)(x_1x_2) \end{aligned}$$

Therefore $\underline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of ordered semigroup S.

■

3.2.13 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of ordered semigroup S.

Proof. To prove this theorem we have to see that $\overline{H}(\Psi)$ satisfies (FI_{14}) and

(FI_{15}). If for each $x_1, x_2 \in S$ with $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Now consider the following

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_2), 0.5\right\} &= \min\left\{\bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right), 0.5\right\} \\
&= \bigvee_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), 0.5\right\} \\
&\leq \bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\
\text{implies } \min\left\{\overline{H}(\Psi)(x_2), 0.5\right\} &\leq \overline{H}(\Psi)(x_1)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_2), 0.5\right\} &= \min\left\{\bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right), 0.5\right\} \\
&= \bigvee_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), 0.5\right\} \\
&\leq \bigvee_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \Psi\left(x_1' x_2'\right) \\
&= \bigvee_{x_1' x_2' \in H(x_1) H(x_2)} \Psi\left(x_1' x_2'\right) \\
&= \bigvee_{x_1' x_2' \in H(x_1 x_2)} \Psi\left(x_1' x_2'\right) \\
\text{implies } \min\left\{\overline{H}(\Psi)(x_2), 0.5\right\} &\leq \overline{H}(\Psi)(x_1 x_2)
\end{aligned}$$

Therefore $\overline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy left ideal of ordered semigroup S. Similarly it can be shown that $\overline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy right ideal of S. ■

3.2.14 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVMH$ and Ψ be $(\in, \in \vee q)$-fuzzy interior ideal of ordered semigroup S. Then $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy interior ideal of ordered semigroup S.

Proof. From Theorems 3.2.10 and 3.2.12, we see that $\underline{H}(\Psi)$ satisfies (FI_{14}) and

(FI_{16}). Next for (FI_{18}) we consider the following for each $x_1, x_2, x_3 \in S$.

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_3), 0.5\right\} &= \min\left\{\bigwedge_{x_3' \in H(x_3)} \Psi\left(x_3'\right), 0.5\right\} \\
&= \bigwedge_{x_3' \in H(x_3)} \min\left\{\Psi\left(x_3'\right), 0.5\right\} \\
&\leq \bigwedge_{\substack{x_1' \in H(x_1) \\ x_3' \in H(x_3) \\ x_2' \in H(x_2)}} \Psi\left(x_1' x_3' x_2'\right) \\
&= \bigwedge_{x_1' x_3' x_2' \in H(x_1) H(x_3) H(x_2)} \Psi\left(x_1' x_3' x_2'\right) \\
&= \bigwedge_{x_1' x_3' x_2' \in H(x_1 x_3 x_2)} \Psi\left(x_1' x_3' x_2'\right) \\
\text{implies } \min\left\{\underline{H}(\Psi)(x_3), 0.5\right\} &\leq \underline{H}(\Psi)(x_1 x_3 x_2)
\end{aligned}$$

Hence $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy interior ideal of ordered semigroup S. ■

3.2.15 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee q)$-fuzzy interior ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy interior ideal of ordered semigroup S.

Proof. From Theorem 3.2.13, we have for each $x_1, x_2 \in S$, if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Then $\min\left\{\overline{H}(\Psi)(x_2), 0.5\right\} \leq \overline{H}(\Psi)(x_1)$, Next let for each $x_1, x_2 \in$

S, we consider the following

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1),\overline{H}(\Psi)(x_2),0.5\right\} &= \min\left\{\bigvee_{x_1'\in H(x_1)}\Psi\left(x_1'\right),\bigvee_{x_2'\in H(x_2)}\Psi\left(x_2'\right),0.5\right\}\\
&= \bigvee_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)}}\min\left\{\Psi\left(x_1'\right),\Psi\left(x_2'\right),0.5\right\}\\
&= \bigvee_{x_1'x_2'\in H(x_1)H(x_2)}\min\left\{\Psi\left(x_1'\right),\Psi\left(x_2'\right),0.5\right\}\\
&= \bigvee_{x_1'x_2'\in H(x_1x_2)}\min\left\{\Psi\left(x_1'\right),\Psi\left(x_2'\right),0.5\right\}\\
&\leq \bigvee_{x_1'x_2'\in H(x_1x_2)}\Psi\left(x_1'x_2'\right)\\
\text{implies }\min\left\{\overline{H}(\Psi)(x_1),\overline{H}(\Psi)(x_2),0.5\right\} &\leq \overline{H}(\Psi)(x_1x_2)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_3),0.5\right\} &= \min\left\{\bigvee_{x_3'\in H(x_3)}\Psi\left(x_3'\right),0.5\right\}\\
&= \bigvee_{x_3'\in H(x_3)}\min\left\{\Psi\left(x_3'\right),0.5\right\}\\
&\leq \bigvee_{\substack{x_1'\in H(x_1)\\ x_3'\in H(x_3)\\ x_2'\in H(x_2)}}\Psi\left(x_1'x_3'x_2'\right)\\
&= \bigvee_{x_1'x_3'x_2'\in H(x_1)H(x_3)H(x_2)}\Psi\left(x_1'x_3'x_2'\right)\\
&= \bigvee_{x_1'x_3'x_2'\in H(x_1x_3x_2)}\Psi\left(x_1'x_3'x_2'\right)\\
\text{implies }\min\left\{\overline{H}(\Psi)(x_3),0.5\right\} &\leq \overline{H}(\Psi)(x_1x_3x_2)
\end{aligned}$$

Hence $\overline{H}(\Psi)$ is $(\in,\in\vee q)$-fuzzy interior ideal of ordered semigroup S. ∎

3.2.16 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and Ψ be $(\in,\in\vee q)$-fuzzy bi-ideal of ordered semigroup S. Then $\underline{H}(\Psi)$ is $(\in,\in\vee q)$-fuzzy bi-ideal of ordered semigroup S.

Proof. From Theorem 3.2.14, we have for each $x_1, x_2 \in S$, if $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$. Then $\min\{\underline{H}(\Psi)(x_2), 0.5\} \leq \underline{H}(\Psi)(x_1)$ and also $\min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), 0.5\} \leq \underline{H}(\Psi)(x_1x_2)$. Next let for each $x_1, x_2, x_3 \in S$,

$$\begin{aligned}
\min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_3), 0.5\} &= \min\left\{\bigwedge_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \bigwedge_{x_3'\in H(x_3)} \Psi\left(x_3'\right), 0.5\right\} \\
&= \bigwedge_{\substack{x_1'\in H(x_1)\\ x_3'\in H(x_3)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), 0.5\right\} \\
&= \bigwedge_{x_1'x_3'\in H(x_1)H(x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), 0.5\right\} \\
&= \bigwedge_{x_1'x_3'\in H(x_1x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), 0.5\right\} \\
&\leq \bigwedge_{\substack{x_1'x_3'\in H(x_1x_3)\\ x_2'\in H(x_2)}} \Psi\left(x_1'x_2'x_3'\right) \\
&= \bigwedge_{x_1'x_2'x_3'\in H(x_1x_3)H(x_2)} \Psi\left(x_1'x_2'x_3'\right) \\
&= \bigwedge_{x_1'x_2'x_3'\in H(x_1x_2x_3)} \Psi\left(x_1'x_2'x_3'\right)
\end{aligned}$$

$$\text{implies } \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_3), 0.5\} \leq \underline{H}(\Psi)(x_1x_2x_3)$$

Hence $\underline{H}(\Psi)$ satisfies all the conditions of $(\in, \in \vee q)$-fuzzy bi-ideal of S. Therefore $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy bi-ideal of S. ∎

3.2.17 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee q)$-fuzzy bi-ideal of ordered semi-group S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy bi-ideal of ordered semi-group S.

Proof. From Theorem 3.2.15, we have for each $x_1, x_2 \in S$, if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Then $\min\{\overline{H}(\Psi)(x_2), 0.5\} \leq \overline{H}(\Psi)(x_1)$ and also $\min\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), 0.5\} \leq \overline{H}(\Psi)(x_1x_2)$. Next we consider the following for

each $x_1, x_2, x_3 \in S$,

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_3), 0.5\right\} &= \min\left\{\bigvee_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \bigvee_{x_3'\in H(x_3)} \Psi\left(x_3'\right), 0.5\right\} \\
&= \bigvee_{\substack{x_1'\in H(x_1)\\ x_3'\in H(x_3)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), 0.5\right\} \\
&= \bigvee_{x_1'x_3'\in H(x_1)H(x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), 0.5\right\} \\
&= \bigvee_{x_1'x_3'\in H(x_1x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), 0.5\right\} \\
&\leq \bigvee_{\substack{x_1'x_3'\in H(x_1x_3)\\ x_2'\in H(x_2)}} \Psi\left(x_1'x_2'x_3'\right) \\
&= \bigvee_{x_1'x_2'x_3'\in H(x_1x_3)H(x_2)} \Psi\left(x_1'x_2'x_3'\right) \\
&= \bigvee_{x_1'x_2'x_3'\in H(x_1x_2x_3)} \Psi\left(x_1'x_2'x_3'\right)
\end{aligned}$$

$$\text{implies } \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_3), 0.5\right\} \leq \overline{H}(\Psi)(x_1x_2x_3)$$

Hence $\overline{H}(\Psi)$ satisfies all the conditions of $(\in, \in \vee q)$-fuzzy bi-ideal of S. Therefore $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy bi-ideal of S. ■

3.2.18 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and Ψ be $(\in, \in \vee q)$-fuzzy quasi ideal of ordered semigroup S. Then $\underline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy quasi ideal of ordered semigroup S.

Proof. From Theorem 3.2.12, we have for each $x_1, x_2 \in S$, if $x_1 \leq x_2$ implies

$H(x_1) \subseteq H(x_2)$. Then $\min\{\underline{H}(\Psi)(x_2), 0.5\} \leq \underline{H}(\Psi)(x_1)$, Next let for each $x_1 \in S$,

$$\begin{aligned} \min\{\underline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), 0.5\} &= \min\left\{\bigwedge_{x_1' \in H(x_1)} ((\Psi \circ 1) \wedge (1 \circ \Psi))\left(x_1'\right), 0.5\right\} \\ &= \bigwedge_{x_1' \in H(x_1)} \min\left(((\Psi \circ 1) \wedge (1 \circ \Psi))\left(x_1'\right), 0.5\right) \\ &\leq \bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\ \text{implies } \min\{\underline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), 0.5\} &\leq \underline{H}(\Psi)(x_1) \end{aligned}$$

Hence $\underline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy quasi ideal of ordered semigroup S. ∎

3.2.19 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee q)$-fuzzy quasi ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is $(\in, \in \vee q)$-fuzzy quasi-ideal of ordered semigroup S.

Proof. From Theorem 3.2.13, we have for each $x_1, x_2 \in S$, if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Then $\min\left\{\overline{H}(\Psi)(x_2), 0.5\right\} \leq \overline{H}(\Psi)(x_1)$, Next let for each $x_1 \in S$,

$$\begin{aligned} \min\left\{\overline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), 0.5\right\} &= \min\left\{\bigvee_{x_1' \in H(x_1)} ((\Psi \circ 1) \wedge (1 \circ \Psi))\left(x_1'\right), 0.5\right\} \\ &= \bigvee_{x_1' \in H(x_1)} \min\left\{((\Psi \circ 1) \wedge (1 \circ \Psi))\left(x_1'\right), 0.5\right\} \\ &\leq \bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\ \text{implies } \min\left\{\overline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), 0.5\right\} &\leq \overline{H}(\Psi)(x_1) \end{aligned}$$

Hence $\overline{H}(\Psi)$ is an $(\in, \in \vee q)$-fuzzy quasi ideal of ordered semigroup S. ∎

3.3 Approximations of $(\in, \in \vee qk)$-Fuzzy Ideals in Ordered Semigroups

A fuzzy point x_{1t} of ordered semigroups S is said to "belong to" a fuzzy subset Ψ, written as $x_{1t} \in \Psi$, if $\Psi(x_1) \geq t$. A fuzzy point x_{1t} of ordered semigroup S is said to be a "quasi-coincident with" to a fuzzy subset Ψ, written as $x_{1t}qk\Psi$, if $\Psi(x_1)+t+k > 1$, where $k \in [0,1)$.

3.3.1 Definition

A fuzzy subset Ψ of S is known as $(\in, \in \vee qk)$-fuzzy ordered subsemigroup of ordered semigroup S if

(FI_{20}) (for all $x_1, x_2 \in S$) $(\forall t_1, t_2 \in (0,1])$ $\left(x_{1t_1}, x_{2t_2} \in \Psi \text{ implies } (x_1x_2)_{\min\{t_1,t_2\}} \in \vee qk\Psi\right)$.

3.3.2 Definition [26]

A fuzzy subset Ψ of S is called an $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal ordered semigroup S if it satisfies the following conditions

(FI_{21}) $(\forall x_1, x_2 \in S)$ $(\forall t_1 \in (0,1])$ $(x_1 \leq x_2,\ x_{2t_1} \in \Psi \text{ implies } x_{1t_1} \in \vee qk\Psi)$

(FI_{22}) $(\forall x_1, x_2 \in S)$ $(\forall t_1 \in (0,1])$ $\left(x_{2t_1} \in \Psi \text{ implies } (x_1x_2)_{t_1} \in \vee qk\Psi \left(\text{resp. } (x_2x_1)_{t_1} \in \vee qk\Psi\right)\right)$

If Ψ is both an $(\in, \in \vee qk)$-fuzzy left ideal and an $(\in, \in \vee qk)$-fuzzy right ideal ordered semigroup S, then Ψ is called an $(\in, \in \vee qk)$-fuzzy ideal of ordered semigroup S.

3.3.3 Definition

A fuzzy subset Ψ of ordered semigroup S is called an $(\in, \in \vee qk)$-fuzzy interior ideal of ordered semigroup S if it holds $(FI_{20}), (FI_{21})$ and

(FI_{23}) (for all $x_1, x_2, x_3 \in S$) (for all $t_1, t_2 \in (0,1]$) $\left(x_{3t_1} \in \Psi \text{ implies } (x_1x_3x_2)_{\min\{t_1,t_2\}} \in \vee qk\Psi\right)$.

3.3.4 Definition

A fuzzy subset Ψ of S is said to be $(\in, \in \vee qk)$-fuzzy bi-ideal of ordered semigroup S if it satisfies $(FI_{20}), (FI_{21})$ and

(FI_{24}) (for all $x_1, x_2, x_3 \in S$) $(\forall t_1, t_2 \in (0,1])$ $\left(x_{1t_1}, x_{3t_2} \in \Psi \text{ implies } (x_1x_2x_3)_{\min\{t_1,t_2\}} \in \vee qk\Psi\right)$.

3.3.5 Definition

A fuzzy subset Ψ is known as $(\in, \in \vee qk)$-fuzzy quasi ideal ordered semigroup S if it holds (FI_{21}) and

(FI_{25}) (for all $x_1 \in S$) (for all $t_1 \in (0,1]$) $(x_{1t_1} \in (\Psi \circ 1) \wedge (1 \circ \Psi) \text{ implies } x_{1t_1} \in \vee qk\Psi)$.

3.3.6 Lemma [26]

A fuzzy subset Ψ is known as $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal of ordered semigroup S if and only if the following conditions are holds

(FI_{26}) (for all $x_1, x_2 \in S$) $\left(x_1 \leq x_2, \text{ implies } \Psi(x_1) \geq \min\left\{\Psi(x_2), \frac{1-k}{2}\right\}\right)$

(FI_{27}) (for all $x_1, x_2 \in S$) $\left(\Psi(x_1x_2) \geq \min\left\{\Psi(x_2), \frac{1-k}{2}\right\} \left(\text{resp. } \Psi(x_1x_2) \geq \min\left\{\Psi(x_1), \frac{1-k}{2}\right\}\right)\right)$.

3.3.7 Lemma

A fuzzy subset Ψ of ordered semigroup S is said to be $(\in, \in \vee qk)$-fuzzy interior ideal of ordered semigroup S if and only if it holds (FI_{26}) and

(FI_{28}) (for all $x_1, x_2 \in S$) $\left(\Psi(x_1x_2) \geq \min\left\{\Psi(x_1), \Psi(x_2), \frac{1-k}{2}\right\}\right)$

(FI_{29}) (for all $x_1, x_2, x_3 \in S$) $\left(\Psi(x_1x_3x_2) \geq \min\left\{\Psi(x_3), \frac{1-k}{2}\right\}\right)$.

3.3.8 Lemma [26]

A fuzzy subset Ψ of ordered semigroup S is knows as $(\in, \in \vee qk)$-fuzzy bi-ideal of ordered semigroup S if and only if it holds (FI_{26}) and (FI_{28}) and

(FI_{30}) (for all $x_1, x_2, x_3 \in S$) $\left(\Psi(x_1x_2x_3) \geq \min\left\{\Psi(x_1), \Psi(x_3), \frac{1-k}{2}\right\}\right)$.

3.3.9 Lemma

A fuzzy subset Ψ of S is said to be an $(\in, \in \vee qk)$-fuzzy quasi ideal of S if and only if it holds (FI_{26}) and

(FI_{31}) (for all $x_1, x_2 \in S$) $\left(\Psi(x_1) \geq \min\left\{((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), \frac{1-k}{2}\right\}\right)$.

3.3.10 Theorem

Let $H : S \to P^*(S)$ be a SVH and Ψ be an $(\in, \in \vee qk)$-fuzzy ordered subsemigroup of S. Then $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy ordered subsemigroup of S.

Proof. To prove this theorem we have to see that $\underline{H}(\Psi)$ satisfies (FI_{28}). For this if for each $x_1, x_2 \in S$, we consider the following

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), \bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1'x_2' \in H(x_1)H(x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1'x_2' \in H(x_1x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{x_1'x_2' \in H(x_1x_2)} \Psi\left(x_1'x_2'\right)
\end{aligned}$$

implies $\min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1x_2)$

Hence $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy ordered subsemigroup of S. ■

3.3.11 Theorem

Let $H : S \to P^*(S)$ be a SVH and Ψ be an $(\in, \in \vee qk)$-fuzzy ordered subsemigroup of S. Then $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy ordered subsemigroup of S.

Proof. Straightforward as Theorem 3.3.10. ■

3.3.12 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and Ψ be $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal of ordered semigroup S. Then $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal of ordered semigroup S.

Proof. To prove this Theorem we have to show that $\underline{H}(\Psi)$ satisfies (FI_{26}) and (FI_{27}). For this if for each $x_1, x_2 \in S$, with $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$, Now

consider

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_2'\in H(x_2)} \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_2'\in H(x_2)} \min\left\{\Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{x_1'\in H(x_1)} \Psi\left(x_1'\right) \\
\text{implies } \min\left\{\underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &\leq \underline{H}(\Psi)(x_1)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_2'\in H(x_2)} \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_2'\in H(x_2)} \min\left\{\Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{\substack{x_1'\in H(x_1)\\ x_2'\in H(x_2)}} \Psi\left(x_1'x_2'\right) \\
&= \bigwedge_{x_1'x_2'\in H(x_1)H(x_2)} \Psi\left(x_1'x_2'\right) \\
&= \bigwedge_{x_1'x_2'\in H(x_1x_2)} \Psi\left(x_1'x_2'\right) \\
\text{implies } \min\left\{\underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &\leq \underline{H}(\Psi)(x_1x_2)
\end{aligned}$$

Therefore $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal of ordered semigroup S. ■

3.3.13 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal of ordered semigroup S. Then $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy left (resp. right) ideal of ordered semigroup S.

Proof. To prove this Theorem we have to show that $\overline{H}(\Psi)$ satisfies (FI_{26}) and (FI_{27}). If for each $x_1, x_2 \in S$, with $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$, Now consider the following

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&\leq \bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right) \\
\text{implies } \left\{\overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &\leq \overline{H}(\Psi)(x_1)
\end{aligned}$$

Next consider

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&\leq \bigvee_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \Psi\left(x_1' x_2'\right) \\
&= \bigvee_{x_1' x_2' \in H(x_1) H(x_2)} \Psi\left(x_1' x_2'\right) \\
&= \bigvee_{x_1' x_2' \in H(x_1 x_2)} \Psi\left(x_1' x_2'\right) \\
\text{implies } \min\left\{\overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &\leq \overline{H}(\Psi)(x_1 x_2)
\end{aligned}$$

Therefore $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy left ideal of ordered semigroup S. Similarly we can show that $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy right ideal of ordered semigroup S. ■

3.3.14 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and Ψ be $(\in, \in \vee qk)$-fuzzy interior ideal of S. Then $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy interior ideal of S.

Proof. From Theorems 3.3.10 and 3.3.12, we see that $\underline{H}(\Psi)$ satisfies $\min\left\{\underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1)$ and also $\min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1x_2)$ for each $x_1, x_2 \in S$ with $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$. Now for (FI_{29}) consider the following for each $x_1, x_2, x_3 \in S$.

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_3'\in H(x_3)} \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_3'\in H(x_3)} \min\left\{\Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{\substack{x_1'\in H(x_1)\\ x_3'\in H(x_3)\\ x_2'\in H(x_2)}} \Psi\left(x_1'x_3'x_2'\right) \\
&= \bigwedge_{x_1'x_3'x_2'\in H(x_1)H(x_3)H(x_2)} \Psi\left(x_1'x_3'x_2'\right) \\
&= \bigwedge_{x_1'x_3'x_2'\in H(x_1x_3x_3)} \Psi\left(x_1'x_3'x_2'\right) \\
\text{implies } \min\left\{\underline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} &\leq \underline{H}(\Psi)(x_1x_3x_2)
\end{aligned}$$

Hence $\underline{H}(\Psi)$ is an $(\in \vee qk)$ fuzzy interior ideal of S. ■

3.3.15 Theorem

Let $H : S \to P^{*}(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee qk)$-fuzzy interior ideal of S. Then $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy interior ideal of S.

Proof. From Theorem 3.3.13, we have if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$, then $\min\left\{\overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1)$, for each $x_1, x_2 \in S$. Next we consider the following for each $x_1, x_2 \in S$,

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), \bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{\substack{x_1' \in H(x_1) \\ x_2' \in H(x_2)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_1' x_2' \in H(x_1)H(x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_1' x_2' \in H(x_1 x_2)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_2'\right), \frac{1-k}{2}\right\} \\
&\leq \bigvee_{x_1' x_2' \in H(x_1 x_2)} \Psi\left(x_1' x_2'\right)
\end{aligned}$$

implies $\min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1 x_2)$

Next consider

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_3' \in H(x_3)} \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_3' \in H(x_3)} \min\left\{\Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&\leq \bigvee_{\substack{x_1' \in H(x_1) \\ x_3' \in H(x_3) \\ x_2' \in H(x_2)}} \Psi\left(x_1' x_3' x_2'\right) \\
&= \bigvee_{x_1' x_3' x_2' \in H(x_1)H(x_3)H(x_2)} \Psi\left(x_1' x_3' x_2'\right) \\
&= \bigvee_{x_1' x_3' x_2' \in H(x_1 x_3 x_2)} \Psi\left(x_1' x_3' x_2'\right)
\end{aligned}$$

implies $\min\left\{\overline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1 x_3 x_2)$

Hence $\overline{H}(\Psi)$ is $(\in, \in \vee qk)$-fuzzy interior ideal of ordered semigroup S. ■

3.3.16 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVMH$ and Ψ be $(\in, \in \vee qk)$-fuzzy bi-ideal of S. Then $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy bi-ideal of S.

Proof. From Theorems 3.3.10 and 3.3.12, we see that $\underline{H}(\Psi)$ satisfies $\min\left\{\underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1)$ and also $\min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1x_2)$ for each $x_1, x_2 \in S$ with $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Now for (FI_{30}) consider the following for each $x_1, x_2, x_3 \in S$.

$$\begin{aligned}
\min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} &= \min\left\{\bigwedge_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \bigwedge_{x_3'\in H(x_3)} \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{\substack{x_1'\in H(x_1)\\ x_3'\in H(x_3)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1'x_3'\in H(x_1)H(x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigwedge_{x_1'x_3'\in H(x_1x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{\substack{x_1'x_3'\in H(x_1x_3)\\ x_2'\in H(x_2)}} \Psi\left(x_1'x_3'x_2'\right) \\
&= \bigwedge_{x_1'x_3'x_2'\in H(x_1x_3)H(x_2)} \Psi\left(x_1'x_3'x_2'\right) \\
&= \bigwedge_{x_1'x_3'x_2'\in H(x_1x_3x_2)} \Psi\left(x_1'x_3'x_2'\right)
\end{aligned}$$

$$\text{implies } \min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1x_3x_2)$$

Hence $\underline{H}(\Psi)$ satisfies all the conditions of $(\in, \in \vee qk)$-fuzzy bi-ideal of S. Therefore $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy bi-ideal of S. ■

3.3.17 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee qk)$-fuzzy bi-ideal of S. Then $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy bi-ideal of S.

Proof. From Theorems 3.3.15, we see that $\overline{H}(\Psi)$ satisfies $\min\left\{\overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1)$ and also $\min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1x_2)$ for each $x_1, x_2 \in$

S with $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Now for (FI_{30}) consider the following for each $x_1, x_2, x_3 \in S$.

$$\begin{aligned}
\min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} &= \min\left\{\bigvee_{x_1'\in H(x_1)} \Psi\left(x_1'\right), \bigvee_{x_3'\in H(x_3)} \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{\substack{x_1'\in H(x_1)\\ x_3'\in H(x_3)}} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_1'x_3'\in H(x_1)H(x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&= \bigvee_{x_1'x_3'\in H(x_1x_3)} \min\left\{\Psi\left(x_1'\right), \Psi\left(x_3'\right), \frac{1-k}{2}\right\} \\
&\leq \bigvee_{\substack{x_1'x_3'\in H(x_1x_3)\\ x_2'\in H(x_2)}} \Psi\left(x_1'x_3'x_2'\right) \\
&= \bigvee_{x_1'x_3'x_2'\in H(x_1x_3)H(x_2)} \Psi\left(x_1'x_3'x_2'\right) \\
&= \bigvee_{x_1'x_3'x_2'\in H(x_1x_3x_2)} \Psi\left(x_1'x_3'x_2'\right)
\end{aligned}$$

$$\text{implies } \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_3), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1x_3x_2)$$

Hence $\overline{H}(\Psi)$ satisfies all the conditions of $(\in, \in \vee qk)$-fuzzy bi-ideal of S. Therefore $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy bi-ideal of S. ■

3.3.18 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVMH$ and a fuzzy subset Ψ be an $(\in, \in \vee qk)$-fuzzy quasi ideal of S. Then $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy quasi ideal of S.

Proof. From Theorem 3.3.12, we have for each $x_1, x_2 \in S$, if $x_1 \leq x_2$ implies

$H(x_1) \subseteq H(x_2)$. Then $\min\left\{\underline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1)$, Next let for each $x_1 \in S$,

$$\begin{aligned}
\min\left\{\underline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), \frac{1-k}{2}\right\} &= \min \bigwedge_{x_1' \in H(x_1)} \left\{ \begin{array}{c} ((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1'), \\ \frac{1-k}{2} \end{array} \right\} \\
&= \bigwedge_{x_1' \in H(x_1)} \min \left\{ \begin{array}{c} ((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1'), \\ \frac{1-k}{2} \end{array} \right\} \\
&\leq \bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right)
\end{aligned}$$

implies $\min\left\{\underline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), \frac{1-k}{2}\right\} \leq \underline{H}(\Psi)(x_1)$

Hence $\underline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy quasi ideal of ordered semigroup S. ∎

3.3.19 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ be $(\in, \in \vee qk)$-fuzzy quasi ideal of S. Then $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy quasi ideal of S.

Proof. From Theorem 3.3.13, we have for each $x_1, x_2 \in S$, if $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$. Then $\min\left\{\overline{H}(\Psi)(x_2), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1)$, Next let for each $x_1 \in S$,

$$\begin{aligned}
\min\left\{\overline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), \frac{1-k}{2}\right\} &= \min \left\{ \begin{array}{c} \bigvee_{x_1' \in H(x_1)} ((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1'), \\ \frac{1-k}{2} \end{array} \right\} \\
&= \bigvee_{x_1' \in H(x_1)} \min \left\{ \begin{array}{c} ((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1'), \\ \frac{1-k}{2} \end{array} \right\} \\
&\leq \bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right)
\end{aligned}$$

implies $\min\left\{\overline{H}((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), \frac{1-k}{2}\right\} \leq \overline{H}(\Psi)(x_1)$

Hence $\overline{H}(\Psi)$ is an $(\in, \in \vee qk)$-fuzzy quasi ideal of ordered semigroup S. ∎

Chapter 4

Roughness in Fuzzy Filters and Fuzzy Ideals with Thresholds (u_1, u_2) in Ordered Semigroups

In this chapter we recall some basic definitions and theorem related to fuzzy filters with thresholds (u_1, u_2) and fuzzy ideals with thresholds (u_1, u_2) of ordered semigroups. Thus we will start with the following result.

►Approximations of fuzzy left (resp. right) filter with thresholds (u_1, u_2)

►Approximations of fuzzy bi-filter with thresholds (u_1, u_2)

►Approximations of fuzzy ordered subsemigroup with thresholds (u_1, u_2)

►Approximations of fuzzy left (resp. right) ideal with thresholds (u_1, u_2)

►Approximations of fuzzy bi-ideal and fuzzy interior ideal with thresholds (u_1, u_2)

►Approximations of fuzzy quasi ideal with thresholds (u_1, u_2)

4.1 Approximations of Fuzzy Filters with Thresholds (u_1, u_2) in Ordered Semigroups

4.1.1 Definition [7]

Let $u_1, u_2 \in (0,1]$ such that $u_1 < u_2$. Then Ψ is known as a fuzzy left (resp. right) filter with thresholds (u_1, u_2) of S if the following assertions are satisfies

(FF_{14}) $(\forall x_1, x_2 \in S)\,(x_1 \leq x_2, \max\{\Psi(x_2), u_1\} \geq \min\{\Psi(x_1), u_2\})$

(FF_{15}) $(\forall x_1, x_2 \in S)\,(\max\{\Psi(x_1 x_2), u_1\} \geq \min\{\Psi(x_1), \Psi(x_2), u_2\})$

(FF_{16}) $(\forall x_1, x_2 \in S)\,(\max\{\Psi(x_1 x_2), u_1\} \geq \min\{\Psi(x_1), u_2\}\,(\text{resp. } \min\{\Psi(x_2), u_2\}))$.

if Ψ is a fuzzy filter of S with thresholds of S, then we conclude that Ψ is ordinary fuzzy filter when $u_1 = 0$ and $u_2 = 1$ and Ψ is $(\in, \in \vee q)$-fuzzy filter when $u_1 = 0$ and $u_2 = 0.5$.

4.1.2 Definition [7]

Let $u_1, u_2 \in (0,1]$ and $u_1 < u_2$. Ψ is called a fuzzy bi-filter with *thresholds* (u_1, u_2) of S if it holds $(FF_{14}), (FF_{15})$ and

(FF_{17}) $(\forall x_1, x_2 \in S)(\max\{\Psi(x_1 x_2 x_1), u_1\} \geq \min\{\Psi(x_1), u_2\})$.

4.1.3 Theorem

Let $H{:}S \to P^*(S)$ be a $SVMH$ and Ψ be a fuzzy left (right) filter with *thresholds* (u_1, u_2). Then $\overline{H}(\Psi)$ is a fuzzy left (right) filter with *thresholds* (u_1, u_2) of S.

Proof. For each $x_1, x_2 \in S$ with $x_1 \leq x_2$ then $H(x_1) \subseteq H(x_2)$ and $\forall\, u_1, u_2 \in (0,1]$

with $u_1 < u_2$. Now we may Consider the following

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_2), u_1\right\} &= \max\left\{\bigvee_{x_2'\in H(x_2)} \Psi\left(x_2'\right), u_1\right\} \\
&= \bigvee_{x_2'\in H(x_2)} \max\left\{\Psi\left(x_2'\right), u_1\right\} \\
&\geq \bigvee_{x_1'\in H(x_1)} \min\left\{\Psi\left(x_1'\right), u_2\right\} \\
&= \min\left\{\bigvee_{x_1'\in H(x_1)} \Psi\left(x_1'\right), u_2\right\} \\
&= \min\left\{\overline{H}(\Psi)(x_1), u_2\right\} \\
\text{implies } \max\left\{\overline{H}(\Psi)(x_2), u_1\right\} &\geq \min\left\{\overline{H}(\Psi)(x_1), u_2\right\}
\end{aligned}$$

Next consider

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} &= \max\left\{\bigvee_{x_1'\in H(x_1x_2)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1'\in H(x_1x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1'\in H(x_1)H(x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{ab\in H(x_1)H(x_2)} \max\left\{\Psi(ab), u_1\right\} \quad \left(\begin{array}{c}\text{where } x_1' = ab \text{ and} \\ a\in H(x_1), b\in H(x_2)\end{array}\right) \\
&= \bigvee_{\substack{a\in H(x_1)\\ b\in H(x_2)}} \max\left\{\Psi(ab), u_1\right\} \\
&\geq \bigvee_{\substack{a\in H(x_1)\\ b\in H(x_2)}} \min\left\{\Psi(a), \Psi(b), u_2\right\} \\
&= \min\left\{\bigvee_{a\in H(x_1)} \Psi(a), \bigvee_{b\in H(x_2)} \Psi(b), u_2\right\} \\
&= \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_1), u_2\right\} \\
\text{implies } \max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} &\geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_1), u_2\right\}
\end{aligned}$$

consider

$$
\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} &= \max\left\{\bigvee_{x_1'\in H(x_1x_2)}\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1'\in H(x_1x_2)}\max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1'\in H(x_1)H(x_2)}\max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{ab\in H(x_1)H(x_2)}\max\left\{\Psi(ab), u_1\right\} \quad \left(\begin{array}{c}\text{where } x_1' = ab \text{ and } a\in H(x_1) \\ , b\in H(x_2)\end{array}\right) \\
&= \bigvee_{\substack{a\in H(x_1)\\ b\in H(x_2)}}\max\left\{\Psi(ab), u_1\right\} \\
&\geq \bigvee_{a\in H(x_1)}\min\left\{\Psi(a), u_2\right\} \\
&= \min\left\{\bigvee_{a\in H(x_1)}\Psi(a), u_2\right\} \\
&= \min\left\{\overline{H}(\Psi)(x_1), u_2\right\}
\end{aligned}
$$

$$
\text{implies } \max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_1), u_2\right\}
$$

Hence it is clear that $\overline{H}(\Psi)$ is a fuzzy left filter with *thresholds* (u_1, u_2) of S. Similarly we can show that $\overline{H}(\Psi)$ is a fuzzy right filter with *thresholds* (u_1, u_2) of S. ■

4.1.4 Theorem

Let Ψ be a fuzzy left (right) filter with thresholds (u_1, u_2) of S and $H : S \longrightarrow P^*(S)$ be a $SVIH$. Then $\overline{H}(\Psi)$ is a fuzzy left (right) filter with thresholds (u_1, u_2) of S.

Proof. For each $x_1, x_2 \in S$ with $x_1 \leq x_2$ then $H(x_2) \subseteq H(x_1)$ and $\forall\, u_1, u_2 \in (0, 1]$

with $u_1 < u_2$. Now we may Consider the following

$$\begin{aligned}
\max\{\underline{H}(\Psi)(x_2), u_1\} &= \max\left\{\bigwedge_{x_2'\in H(x_2)} \Psi\left(x_2'\right), u_1\right\} \\
&= \bigwedge_{x_2'\in H(x_2)} \max\left\{\Psi\left(x_2'\right), u_1\right\} \\
&\geq \bigwedge_{x_1'\in H(x_1)} \min\left\{\Psi\left(x_1'\right), u_2\right\} \\
&= \min\left\{\bigwedge_{x_1'\in H(x_1)} \Psi\left(x_1'\right), u_2\right\} \\
&= \min\{\underline{H}(\Psi)(x_1), u_2\} \\
\text{implies } \max\{\underline{H}(\Psi)(x_2), u_1\} &\geq \min\{\underline{H}(\Psi)(x_1), u_2\}
\end{aligned}$$

Now consider

$$\begin{aligned}
\max\{\underline{H}(\Psi)(x_1x_2), u_1\} &= \max\left\{\bigwedge_{x_1'\in H(x_1x_2)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1'\in H(x_1x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1'\in H(x_1)H(x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{ab\in H(x_1)H(x_2)} \max\{\Psi(ab), u_1\} \quad \left(\begin{array}{c}\text{where } x_1' = ab \text{ and} \\ a \in H(x_1), b \in H(x_2)\end{array}\right) \\
&= \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)}} \max\{\Psi(ab), u_1\} \\
&\geq \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)}} \min\{\Psi(a), \Psi(b), u_2\} \\
&= \min\left\{\bigwedge_{a\in H(x_1)} \Psi(a), \bigwedge_{b\in H(x_2)} \Psi(b), u_2\right\} \\
&= \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_1), u_2\} \\
\text{implies } \max\{\underline{H}(\Psi)(x_1x_2), u_1\} &\geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_1), u_2\}
\end{aligned}$$

Next

$$
\begin{aligned}
\max\left\{\underline{H}(\Psi)(x_1x_2), u_1\right\} &= \max\left\{\bigwedge_{x_1'\in H(x_1x_2)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1'\in H(x_1x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1'\in H(x_1)H(x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{ab\in H(x_1)H(x_2)} \max\left\{\Psi(ab), u_1\right\} \quad \left(\begin{array}{c}\text{where } x_1' = ab \text{ and } a\in H(x_1) \\ , b\in H(x_2)\end{array}\right) \\
&= \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)}} \max\left\{\Psi(ab), u_1\right\} \\
&\geq \bigwedge_{a\in H(x_1)} \min\left\{\Psi(a), u_2\right\} \\
&= \min\left\{\bigwedge_{a\in H(x_1)} \Psi(a), u_2\right\} \\
&= \min\left\{\underline{H}(\Psi)(x_1), u_2\right\}
\end{aligned}
$$

$$
\text{implies } \max\left\{\underline{H}(\Psi)(x_1x_2), u_1\right\} \geq \min\left\{\underline{H}(\Psi)(x_1), u_2\right\}
$$

Hence it is clear that $\underline{H}(\Psi)$ is a fuzzy left filter with *thresholds* (u_1, u_2) of S. Similarly we can show that $\underline{H}(\Psi)$ is a fuzzy right filter with *thresholds* (u_1, u_2) of S. ∎

4.1.5 Theorem

Let Ψ be a fuzzy bi-filter with *thresholds* (u_1, u_2) of S and $H : S \longrightarrow P^*(S)$ be a $SVMH$. Then $\overline{H}(\Psi)$ is a fuzzy bi-filter *thresholds* (u_1, u_2) of S.

Proof. From Theorem 4.1.3, we see for each $x_1, x_2 \in S$ with $x_1 \leq x_2$ implies $H(x_1) \subseteq H(x_2)$ and $\forall u_1, u_2 \in (0, 1]$ with $u_1 < u_2$. Then $\max\left\{\overline{H}(\Psi)(x_2), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_1), u_2\right\}$ and also $\max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_1'), \overline{H}(\Psi)(x_1), u_2\right\}$.

Therefore we consider the following $\forall x_1, x_2 \in S$

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1x_2x_1), u_1\right\} &= \max\left\{\bigvee_{x_1' \in H(x_1x_2x_1)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1x_2x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)H(x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{aba \in H(x_1)H(x_2)H(x_1)} \max\left\{\Psi(aba), u_1\right\} \quad \begin{pmatrix} x_1' = aba \text{ where} \\ a \in H(x_1), \\ b \in H(x_2) \end{pmatrix} \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2) \\ a \in H(x_1)}} \max\left\{\Psi(aba), u_1\right\} \\
&\geq \bigvee_{a \in H(x_1)} \min\left\{\Psi(a), u_2\right\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), u_2\right\} \\
&= \min\left\{\overline{H}(\Psi)(x_1), u_2\right\}
\end{aligned}$$

implies $\max\left\{\overline{H}(\Psi)(x_1x_2x_1), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_1), u_2\right\}$

Hence it is cleared that $\overline{H}(\Psi)$ is a fuzzy bi-filter with with *thresholds* (u_1, u_2) of S.

■

4.1.6 Theorem

Let Ψ be a fuzzy bi-filter with *thresholds* (u_1, u_2) of S and $H : S \longrightarrow P^*(S)$ be a $SVIH$. Then $\underline{H}(\Psi)$ is a fuzzy bi-filter *thresholds* (u_1, u_2) of S.

Proof. From Theorem 4.1.4, we see for each $x_1, x_2 \in S$ with $x_1 \leq x_2$ implies $H(x_2) \subseteq H(x_1)$ and $\forall u_1, u_2 \in (0, 1]$ with $u_1 < u_2$. Then $\max\left\{\underline{H}(\Psi)(x_2), u_1\right\} \geq \min\left\{\underline{H}(\Psi)(x_1), u_2\right\}$ and also $\max\left\{\underline{H}(\Psi)(x_1x_2), u_1\right\} \geq \min\left\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_1), u_2\right\}$.

Therefore we consider the following $\forall x_1, x_2 \in S$

$$
\begin{aligned}
\max\{\underline{H}(\Psi)(x_1x_2x_1), u_1\} &= \max\left\{\bigwedge_{x_1'\in H(x_1x_2x_1)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1'\in H(x_1x_2x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1'\in H(x_1)H(x_2)H(x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{aba\in H(x_1)H(x_2)H(x_1)} \max\{\Psi(aba), u_1\} \quad \left(\begin{array}{c} x_1' = aba \text{ where} \\ a\in H(x_1), b\in H(x_2) \end{array}\right) \\
&= \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)\\ a\in H(x_1)}} \max\{\Psi(aba), u_1\} \\
&\geq \bigwedge_{a\in H(x_1)} \min\{\Psi(a), u_2\} \\
&= \min\left\{\bigwedge_{a\in H(x_1)} \Psi(a), u_2\right\} \\
&= \min\{\underline{H}(\Psi)(x_1), u_2\}
\end{aligned}
$$

$$
\text{implies } \max\{\underline{H}(\Psi)(x_1x_2x_1), u_1\} \geq \min\{\underline{H}(\Psi)(x_1), u_2\}
$$

Hence it is cleared that $\underline{H}(\Psi)$ is a fuzzy bi-filter with with *thresholds* (u_1, u_2) of S.

■

4.2 Approximations of Fuzzy Ideals with Thresholds (u_1, u_2) in Ordered Semigroups

In this section roughness in fuzzy ideals with thresholds (u_1, u_2) in ordered semigroup is being initiated.

4.2.1 Definition [32]

Let $u_1, u_2 \in (0,1]$ such that $u_1 < u_2$, then Ψ is known as a fuzzy subsemigroup with thresholds (u_1, u_2) of S if the following assertions are satisfied

(FI_{32}) $(\forall x_1, x_2 \in S)\left(x_1 \leq x_2, \text{ implies } \max\{\Psi(x_1), u_1\} \geq \min\{\Psi(x_2), u_2\}\right)$

(FI_{33}) $(\forall x_1, x_2 \in S)\left(\max\{\Psi(x_1x_2), u_1\} \geq \min\{\Psi(x_1), \Psi(x_2), u_2\}\right).$

4.2.2 Definition

Let $u_1, u_2 \in (0,1]$ where $u_1 < u_2$, then Ψ is known as a fuzzy left (right) ideal with thresholds (u_1, u_2) of S if it satisfy condition (FI_{32}) and

(FI_{34}) $(\forall x_1, x_2 \in S)\left(\max\{\Psi(x_1x_2), u_1\} \geq \min\{\Psi(x_2), u_2\} \left(\text{resp. } \min\{\Psi(x_1), u_2\}\right)\right).$

4.2.3 Definition [32]

Let $u_1, u_2 \in (0,1]$ with $u_1 < u_2$, then Ψ is known as fuzzy bi-ideal with thresholds (u_1, u_2) of S if it satisfies $(FI_{32}), (FI_{33})$ and

(FI_{35}) $(\forall x_1, x_2, x_3 \in S)\left(\max\{\Psi(x_1x_2x_3), u_1\} \geq \min\{\Psi(x_1), \Psi(x_3), u_2\}\right).$

4.2.4 Definition

Let $u_1, u_2 \in (0,1]$ where $u_1 < u_2$, then Ψ is known as a fuzzy interior ideal with thresholds (u_1, u_2) of S if it satisfies $(FI_{32}), (FI_{33})$ and

(FI_{36}) $(\forall x_1, x_2, x_3 \in S)\left(\max\{\Psi(x_1x_3x_2), u_1\} \geq \min\{\Psi(x_3), u_2\}\right).$

4.2.5 Definition [32]

Let $u_1, u_2 \in (0,1]$ such that $u_1 < u_2$, then Ψ is called fuzzy quasi ideal with thresholds (u_1, u_2) of S if it satisfies (FI_{32}) and

(FI_{37}) (for each $x_1 \in S$) $\left(\max\{\Psi(x_1), u_1\} \geq \min\{((\Psi \circ 1) \wedge (1 \circ \Psi))(x_1), u_2\}\right)$.

4.2.6 Definition [32]

Let $u_1, u_2 \in (0,1]$ with $u_1 \leq u_2$, and Ψ be fuzzy quasi ideal with thresholds (u_1, u_2) of S. Then Ψ is known as semiprime fuzzy quasi-ideal with thresholds (u_1, u_2) if

(FI_{38}) (for each $x_1 \in S$) $\left(\max\{\Psi(x_1), u_1\} \geq \min\{\Psi(x_1^2), u_2\}\right)$.

4.2.7 Theorem

Let Ψ be a fuzzy ordered subsemigroup with thresholds (u_1, u_2) of S and $H : S \rightarrow P^*(S)$ be a $SVMH$. Then $\underline{H}(\Psi)$ is a fuzzy ordered subsemigroup with thresholds (u_1, u_2) of S.

Proof. For each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_1) \subseteq H(x_2)$ and $\forall u_1, u_2 \in (0,1]$ with $u_1 < u_2$. Consider the following

$$\begin{aligned}
\max\{\underline{H}(\Psi)(x_1), u_1\} &= \max\left\{\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&\geq \bigwedge_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), u_2\right\} \\
&= \min\left\{\bigwedge_{x_2' \in H(x_2)} \Psi\left(x_2'\right), u_2\right\} \\
\text{implies } \max\{\underline{H}(\Psi)(x_1), u_1\} &\geq \min\{\underline{H}(\Psi)(x_2), u_2\}
\end{aligned}$$

Now consider

$$
\begin{aligned}
\max\left\{\underline{H}(\Psi)(x_1x_2),u_1\right\} &= \max\left\{\bigwedge_{x_1'\in H(x_1x_2)}\Psi\left(x_1'\right),u_1\right\}\\
&= \bigwedge_{x_1'\in H(x_1x_2)}\max\left\{\Psi\left(x_1'\right),u_1\right\}\\
&= \bigwedge_{x_1'\in H(x_1)H(x_2)}\max\left\{\Psi\left(x_1'\right),u_1\right\}\\
&= \bigwedge_{ab\in H(x_1)(x_2)}\max\left\{\Psi(ab),u_1\right\} \quad \left(\begin{array}{c}\text{as } x_1'=ab \text{ where}\\ a\in H(x_1) \text{ and } b\in H(x_2)\end{array}\right)\\
&= \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)}}\max\left\{\Psi(ab),u_1\right\}\\
&\geq \bigwedge_{\substack{a\in H(x_1)\\ b\in H(x_2)}}\min\left\{\Psi(a),\Psi(b),u_2\right\}\\
&= \min\left\{\bigwedge_{a\in H(x_1)}\Psi(a),\bigwedge_{b\in H(x_2)}\Psi(b),u_2\right\}
\end{aligned}
$$

$$
\text{implies } \max\left\{\underline{H}(\Psi)(x_1x_2),u_1\right\} \geq \min\left\{\underline{H}(\Psi)(x_1),\underline{H}(\Psi)(x_2),u_2\right\}
$$

Therefore $\underline{H}(\Psi)$ is a fuzzy ordered subsemigroup with thresholds (u_1, u_2) of S. ■

4.2.8 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ be a fuzzy ordered subsemigroup with thresholds (u_1, u_2) of S. Then $\overline{H}(\Psi)$ is a fuzzy ordered subsemigroup of S.

Proof. For each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_2) \subseteq H(x_1)$ and $\forall u_1, u_2 \in$

$(0,1]$ with $u_1 < u_2$. Consider the following

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1), u_1\right\} &= \max\left\{\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&\geq \bigvee_{x_2' \in H(x_2)} \min\left\{\Psi\left(x_2'\right), u_2\right\} \\
&= \min\left\{\bigvee_{x_2' \in H(x_2)} \Psi\left(x_2'\right), u_2\right\} \\
\text{implies } \max\left\{\overline{H}(\Psi)(x_1), u_1\right\} &\geq \min\left\{\overline{H}(\Psi)(x_2), u_2\right\}
\end{aligned}$$

Now consider

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} &= \max\left\{\bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{ab \in H(x_1)(x_2)} \max\left\{\Psi(ab), u_1\right\} \quad \left(\begin{array}{c} \text{as } x_1' = ab \text{ where } a \in H(x_1) \\ \text{and } b \in H(x_2) \end{array}\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \max\left\{\Psi(ab), u_1\right\} \\
&\geq \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \min\left\{\Psi(a), \Psi(b), u_2\right\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), \bigvee_{b \in H(x_2)} \Psi(b), u_2\right\} \\
\text{implies } \max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} &\geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), u_2\right\}
\end{aligned}$$

Therefore $\overline{H}(\Psi)$ is a fuzzy ordered subsemigroup with thresholds (u_1, u_2) of S. ■

4.2.9 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and Ψ be a fuzzy left (right) ideal with thresholds (u_1, u_2) of S. Then $\underline{H}(\Psi)$ is a fuzzy left (right) ideal with threshold (u_1, u_2) of S.

Proof. From Theorem 4.2.7, we see that $\max\{\underline{H}(\Psi)(x_1), u_1\} \geq \min\{\underline{H}(\Psi)(x_2), u_2\}$ is hold for each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_1) \subseteq H(x_2)$. Now consider the following

$$\begin{aligned}
\max\{\underline{H}(\Psi)(x_1x_2), u_1\} &= \max\left\{\bigwedge_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1)H(x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{ab \in H(x_1)H(x_2)} \max\{\Psi(ab), u_1\} \quad \left(\begin{array}{c}\text{as } x_1' = ab \text{ where } a \in \\ H(x_1), \text{and } b \in H(x_2)\end{array}\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \max\{\Psi(ab), u_1\} \\
&\geq \bigwedge_{b \in H(x_2)} \min\{\Psi(b), u_2\} \\
&= \min\left\{\bigwedge_{b \in H(x_2)} \Psi(b), u_2\right\} \\
\text{implies } \max\{\underline{H}(\Psi)(x_1x_2), u_1\} &\geq \min\{\underline{H}(\Psi)(x_2), u_2\}
\end{aligned}$$

Hence $\underline{H}(\Psi)$ is a fuzzy left ideal with threshold (u_1, u_2) of S. Similarly we can show that $\underline{H}(\Psi)$ is a fuzzy right ideal with threshold (u_1, u_2) of S. ■

4.2.10 Theorem

Let Ψ be a fuzzy left (right) ideal with thresholds (u_1, u_2) of S and $H : S \to P^*(S)$ be a $SVIH$. Then $\overline{H}(\Psi)$ is a fuzzy left (right) ideal with threshold (u_1, u_2) of S.

Proof. From Theorem 4.2.8, we see that $\max\left\{\overline{H}(\Psi)(x_1), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_2), u_2\right\}$ is hold for each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_2) \subseteq H(x_1)$. Now consider the following

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} &= \max\left\{\bigvee_{x_1' \in H(x_1x_2)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{ab \in H(x_1)H(x_2)} \max\left\{\Psi(ab), u_1\right\} \quad \left(\begin{array}{c}\text{as } x_1' = ab \text{ where } a \in H(x_1) \\ \text{and } b \in H(x_2)\end{array}\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2)}} \max\left\{\Psi(ab), u_1\right\} \\
&\geq \bigvee_{b \in H(x_2)} \min\left\{\Psi(b), u_2\right\} \\
&= \min\left\{\bigvee_{b \in H(x_2)} \Psi(b), u_2\right\}
\end{aligned}$$

$$\text{implies } \max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_2), u_2\right\}$$

Hence $\overline{H}(\Psi)$ is a fuzzy left ideal with threshold (u_1, u_2) of S. Similarly we can show that $\overline{H}(\Psi)$ is a fuzzy right ideal with threshold (u_1, u_2) of S. ■

4.2.11 Theorem

Let $H : S \rightarrow P^*(S)$ be a $SVMH$ and Ψ be a fuzzy bi-ideal with thresholds (u_1, u_2) of S. Then $\underline{H}(\Psi)$ is a fuzzy bi-ideal with threshold (u_1, u_2) of S.

Proof. From Theorem 4.2.7, we see that $\max\{\underline{H}(\Psi)(x_1), u_1\} \geq \min\{\underline{H}(\Psi)(x_2), u_2\}$, and also $\max\{\underline{H}(\Psi)(x_1x_2), u_1\} \geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), u_2\}$ are hold for each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_1) \subseteq H(x_2)$. Next we consider the following for

each $x_1, x_2, x_3 \in S$,

$$\begin{aligned}
\max\{\underline{H}(\Psi)(x_1x_2x_3), u_1\} &= \max\left\{\bigwedge_{x_1' \in H(x_1x_2x_3)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1x_2x_3)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1)H(x_2)H(x_3)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{abc \in H(x_1)H(x_2)H(x_3)} \max\{\Psi(abc), u_1\} \quad \left(\begin{array}{c}\text{as } x_1' = abc \text{ with} \\ a \in H(x_1), \\ b \in H(x_2) \\ \text{and } c \in H(x_3)\end{array}\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ b \in H(x_2) \\ c \in H(x_3)}} \max\{\Psi(abc), u_1\} \\
&\geq \bigwedge_{\substack{a \in H(x_1) \\ c \in H(x_3)}} \min\{\Psi(a), \Psi(c), u_2\} \\
&= \min\left\{\bigwedge_{a \in H(x_1)} \Psi(a), \bigwedge_{c \in H(x_3)} \Psi(c), u_2\right\}
\end{aligned}$$

implies $\max\{\underline{H}(\Psi)(x_1x_2x_3), u_1\} \geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_3), u_2\}$

Hence $\underline{H}(\Psi)$ satisfies all the conditions of fuzzy bi-ideal with thresholds (u_1, u_2). Therefore $\underline{H}(\Psi)$ is a fuzzy bi-ideal with thresholds (u_1, u_2). ■

4.2.12 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and a fuzzy subset Ψ be a fuzzy bi-ideal with thresholds (u_1, u_2) of S. Then $\overline{H}(\Psi)$ is a fuzzy bi-ideal with thresholds (u_1, u_2) of S.

Proof. From Theorem 4.2.8, we see that $\max\{\overline{H}(\Psi)(x_1), u_1\} \geq \min\{\overline{H}(\Psi)(x_2), u_2\}$, and also $\max\{\overline{H}(\Psi)(x_1x_2), u_1\} \geq \min\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), u_2\}$ are hold for each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_2) \subseteq H(x_1)$. Next we consider the following for each $x_1, x_2, x_3 \in S$,

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1x_2x_3), u_1\right\} &= \max\left\{\bigvee_{x_1' \in H(x_1x_2x_3)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1x_2x_3)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1)H(x_2)H(x_3)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{abc \in H(x_1)H(x_2)H(x_3)} \max\left\{\Psi(abc), u_1\right\} \quad \left(\begin{array}{c} \text{as } x_1' = abc \text{ where} \\ a \in H(x_1), b \in H(x_2), \\ \text{and } c \in H(x_3) \end{array}\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ b \in H(x_2) \\ c \in H(x_3)}} \max\left\{\Psi(abc), u_1\right\} \\
&\geq \bigvee_{\substack{a \in H(x_1) \\ c \in H(x_3)}} \min\left\{\Psi(a), \Psi(c), u_2\right\} \\
&= \min\left\{\bigvee_{a \in H(x_1)} \Psi(a), \bigvee_{c \in H(x_3)} \Psi(c), u_2\right\}
\end{aligned}$$

implies $\max\left\{\overline{H}(\Psi)(x_1x_2x_3), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_3), u_2\right\}$

Hence $\overline{H}(\Psi)$ satisfies all the conditions of fuzzy bi-ideal with thresholds (u_1, u_2). Therefore $\overline{H}(\Psi)$ is a fuzzy bi-ideal with thresholds (u_1, u_2). ■

4.2.13 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and a fuzzy subset Ψ be a fuzzy interior ideal with thresholds (u_1, u_2) of S. Then $\overline{H}(\Psi)$ is a fuzzy interior ideal with thresholds (u_1, u_2) of S.

Proof. From Theorem 4.2.7, we see that $\max\{\underline{H}(\Psi)(x_1), u_1\} \geq \min\{\underline{H}(\Psi)(x_2), u_2\}$, and also $\max\{\underline{H}(\Psi)(x_1x_2), u_1\} \geq \min\{\underline{H}(\Psi)(x_1), \underline{H}(\Psi)(x_2), u_2\}$ are hold for each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_1) \subseteq H(x_2)$. Next we consider the following for

each $x_1, x_2, x_3 \in S$,

$$\begin{aligned}
\max\{\underline{H}(\Psi)(x_1x_3x_2), u_1\} &= \max\left\{\bigwedge_{x_1' \in H(x_1x_3x_2)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1x_3x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1)H(x_3)H(x_2)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{acb \in H(x_1)H(x_3)H(x_2)} \max\{\Psi(acb), u_1\} \quad \left(\begin{array}{c}\text{as } x_1' = acb \text{ with} \\ a \in H(x_1), \\ c \in H(x_3) \\ \text{and } b \in H(x_2)\end{array}\right) \\
&= \bigwedge_{\substack{a \in H(x_1) \\ c \in H(x_3) \\ b \in H(x_2)}} \max\{\Psi(acb), u_1\} \\
&\geq \bigwedge_{c \in H(x_3)} \min\{\Psi(c), u_1\} \\
&= \min\left\{\bigwedge_{c \in H(x_3)} \Psi(c), u_1\right\} \\
&= \min\{\underline{H}(\Psi)(x_3), u_1\}
\end{aligned}$$

implies $\max\{\underline{H}(\Psi)(x_1x_3x_2), u_1\} \geq \min\{\underline{H}(\Psi)(x_3), u_1\}$

Hence $\underline{H}(\Psi)$ is a fuzzy interior ideal with thresholds (u_1, u_2) of S. ■

4.2.14 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and a fuzzy subset Ψ be a fuzzy interior ideal with thresholds (u_1, u_2) of S. Then $\overline{H}(\Psi)$ is a fuzzy interior ideal with thresholds (u_1, u_2) of S.

Proof. From Theorem 4.2.8, we see that $\max\left\{\overline{H}(\Psi)(x_1), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_2), u_2\right\}$, and also $\max\left\{\overline{H}(\Psi)(x_1x_2), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_1), \overline{H}(\Psi)(x_2), u_2\right\}$ are hold for each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_2) \subseteq H(x_1)$. Next we consider the following for each $x_1, x_2, x_3 \in S$,

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1x_3x_2), u_1\right\} &= \max\left\{\bigvee_{x_1^{'} \in H(x_1x_3x_2)} \Psi\left(x_1^{'}\right), u_1\right\} \\
&= \bigvee_{x_1^{'} \in H(x_1x_3x_2)} \max\left\{\Psi\left(x_1^{'}\right), u_1\right\} \\
&= \bigvee_{x_1^{'} \in H(x_1)H(x_3)H(x_2)} \max\left\{\Psi\left(x_1^{'}\right), u_1\right\} \\
&= \bigvee_{acb \in H(x_1)H(x_3)H(x_2)} \max\{\Psi(acb), u_1\} \quad \left(\begin{array}{c}\text{as } x_1^{'} = acb \text{ where} \\ a \in H(x_1), c \in H(x_3) \\ \text{and } b \in H(x_2)\end{array}\right) \\
&= \bigvee_{\substack{a \in H(x_1) \\ c \in H(x_3) \\ b \in H(x_2)}} \max\{\Psi(acb), u_1\} \\
&\geq \bigvee_{c \in H(x_3)} \min\{\Psi(c), u_1\} \\
&= \min\left\{\bigvee_{c \in H(x_3)} \Psi(c), u_1\right\} \\
&= \min\left\{\overline{H}(\Psi)(x_3), u_1\right\}
\end{aligned}$$

implies $\max\left\{\overline{H}(\Psi)(x_1x_3x_2), u_1\right\} \geq \min\left\{\overline{H}(\Psi)(x_3), u_1\right\}$

Hence $\overline{H}(\Psi)$ is a fuzzy interior ideal with thresholds (u_1, u_2) of S. ■

4.2.15 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and a fuzzy subset Ψ be a fuzzy quasi ideal with thresholds (u_1, u_2) of S. Then $\underline{H}(\Psi)$ is a fuzzy quasi ideal with thresholds (u_1, u_2) of S.

Proof. From Theorem 4.2.7, we see that $\max\{\underline{H}(\Psi)(x_1), u_1\} \geq \min\{\underline{H}(\Psi)(x_2), u_2\}$ is hold for each $x_1, x_2 \in S$ with $x_1 \leq x_2$, implies $H(x_1) \subseteq H(x_2)$. Now consider the following

$$\begin{aligned}
\max\left\{\underline{H}(\Psi)(x_1),u_1\right\} &= \max\left\{\bigwedge_{x_1'\in H(x_1)}\Psi\left(x_1'\right),u_1\right\} \\
&= \bigwedge_{x_1'\in H(x_1)}\max\left\{\Psi\left(x_1'\right),u_1\right\} \\
&\geq \bigwedge_{x_1'\in H(x_1)}\min\left\{((\Psi\circ 1)\wedge(1\circ\Psi))\left(x_1'\right),u_2\right\} \\
&= \min\left\{\bigvee_{x_1'\in H(x_1)}((\Psi\circ 1)\wedge(1\circ\Psi))\left(x_1'\right),u_2\right\} \\
\text{implies } \max\left\{\underline{H}(\Psi)(x_1),u_1\right\} &\geq \min\left\{\underline{H}((\Psi\circ 1)\wedge(1\circ\Psi))(x_1),u_2\right\}
\end{aligned}$$

Hence $\underline{H}(\Psi)$ satisfies all the conditions of fuzzy quasi ideal with thresholds (u_1,u_2).

So $\underline{H}(\Psi)$ is a fuzzy quasi ideal with thresholds (u_1,u_2). ■

4.2.16 Theorem

Let Ψ be a fuzzy quasi-ideal with thresholds (u_1,u_2) of S and $H:S\to P^*(S)$ be a $SVIH$. Then $\overline{H}(\Psi)$ is a fuzzy quasi ideal with thresholds (u_1,u_2) of S.

Proof. From Theorem 4.2.8, we see that $\max\left\{\overline{H}(\Psi)(x_1),u_1\right\}\geq\min\left\{\overline{H}(\Psi)(x_2),u_2\right\}$ is hold for each $x_1,x_2\in S$ with $x_1\leq x_2$, implies $H(x_2)\subseteq H(x_1)$. Now consider the following

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1),u_1\right\} &= \max\left\{\bigvee_{x_1'\in H(x_1)}\Psi\left(x_1'\right),u_1\right\} \\
&= \bigvee_{x_1'\in H(x_1)}\max\left\{\Psi\left(x_1'\right),u_1\right\} \\
&\geq \bigvee_{x_1'\in H(x_1)}\min\left\{((\Psi\circ 1)\wedge(1\circ\Psi))\left(x_1'\right),u_2\right\} \\
&= \min\left\{\bigvee_{x_1'\in H(x_1)}((\Psi\circ 1)\wedge(1\circ\Psi))\left(x_1'\right),u_2\right\} \\
\text{implies } \max\left\{\overline{H}(\Psi)(x_1),u_1\right\} &\geq \min\left\{\overline{H}((\Psi\circ 1)\wedge(1\circ\Psi))(x_1),u_2\right\}
\end{aligned}$$

Hence $\overline{H}(\Psi)$ satisfies all the conditions of fuzzy quasi ideal with thresholds (u_1, u_2).

So $\overline{H}(\Psi)$ is a fuzzy quasi ideal with thresholds (u_1, u_2). ∎

4.2.17 Theorem

Let $H : S \to P^*(S)$ be a $SVMH$ and a fuzzy subset Ψ be a semiprime fuzzy quasi ideal with thresholds (u_1, u_2). Then $\underline{H}(\Psi)$ is a semiprime fuzzy quasi ideal with thresholds (u_1, u_2) of S.

Proof. For each $x_1 \in S$, consider the following

$$\begin{aligned}
\max\{\underline{H}(\Psi)(x_1), u_1\} &= \max\left\{\bigwedge_{x_1' \in H(x_1)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigwedge_{x_1' \in H(x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&\geq \bigwedge_{x_1' \in H(x_1)} \min\left\{\Psi\left(x_1'^2\right), u_2\right\} \\
&= \bigwedge_{x_1' x_1' \in H(x_1)H(x_1)} \min\left\{\Psi\left(x_1'^2\right), u_2\right\} \\
&= \bigwedge_{x_1'^2 \in H(x_1^2)} \min\left\{\Psi\left(x_1'^2\right), u_2\right\} \\
&= \min\left\{\bigwedge_{x_1'^2 \in H(x_1^2)} \Psi\left(x_1'^2\right), u_2\right\} \\
&= \min\left\{\underline{H}(\Psi)\left(x_1^2\right), u_2\right\} \\
\text{implies } \max\{\underline{H}(\Psi)(x_1), u_1\} &\geq \min\left\{\underline{H}(\Psi)\left(x_1^2\right), u_2\right\}
\end{aligned}$$

Hence $\underline{H}(\Psi)$ is a semiprime fuzzy quasi ideal with threshold (u_1, u_2). ∎

4.2.18 Theorem

Let $H : S \to P^*(S)$ be a $SVIH$ and Ψ is a semiprime fuzzy quasi ideal. Then $\overline{H}(\Psi)$ is a semiprime fuzzy quasi ideal with thresholds (u_1, u_2) of S.

Proof. For each $x_1 \in S$, consider the following

$$\begin{aligned}
\max\left\{\overline{H}(\Psi)(x_1), u_1\right\} &= \max\left\{\bigvee_{x_1' \in H(x_1)} \Psi\left(x_1'\right), u_1\right\} \\
&= \bigvee_{x_1' \in H(x_1)} \max\left\{\Psi\left(x_1'\right), u_1\right\} \\
&\geq \bigvee_{x_1' \in H(x_1)} \min\left\{\Psi\left(x_1'^2\right), u_2\right\} \\
&= \bigvee_{x_1' x_1' \in H(x_1)H(x_1)} \min\left\{\Psi\left(x_1'^2\right), u_2\right\} \\
&= \bigvee_{x_1'^2 \in H(x_1^2)} \min\left\{\Psi\left(x_1'^2\right), u_2\right\} \\
&= \min\left\{\bigvee_{x_1'^2 \in H(x_1^2)} \Psi\left(x_1'^2\right), u_2\right\} \\
&= \min\left\{\overline{H}(\Psi)\left(x_1^2\right), u_2\right\}
\end{aligned}$$

$$\text{implies } \max\left\{\overline{H}(\Psi)(x_1), u_1\right\} \geq \min\left\{\overline{H}(\Psi)\left(x_1^2\right), u_2\right\}$$

Hence $\overline{H}(\Psi)$ is a semiprime fuzzy quasi ideal with thresholds (u_1, u_2). ■

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