Some Contributions in Semigroups in the Framework of Bipolar Soft Sets

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Dedicated
To
My
Loving parents
My family
And respected teachers

DECLARATION

I hereby, declare, that this thesis neither as a whole nor as a part has been copied out from any source. It is further declared that I have prepared this thesis entirely based on my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

Signature: ________________________

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Table of Contents

Introduction:

The study of groups was initiated in the earlier part of 19^{th} century by Evariste Galois, while the theory of semigroups (SGs) was initiated in later on as a generalization of groups. Howie [1] clearly emphasizes the pure semigroup theory. The interpretation theory of SGs was devised by Shain [2] in 1963 by employing a binary relation on a set and for the SG product he employed the composition of relations. Schein and McKenzie [3] achieved the result that every SG is isomorphic to a transitive SG of binary relation. SGs are applied in many areas such as in different fields. In computer science, they are employed in the formal language study and automata theory where they serve as a mathematical framework for the analysis and comprehension of the behavior of finite-state machines. In the study of coding theory, semigroups are used in the development and investigation of the error-correcting codes. Moreover, semigroups are used in the study of algebraic structures like groups and monoids which are algebraic structures constructed on top of semigroups as they are more restrictive algebraic structures. It is much simpler to deal with semigroups than with other algebraic structures, and their study has resulted in numerous theoretical developments and practical applications in many areas of science and technology.

Soft set (SS) theory was devised by a famous mathematician Molodtsov [4]. It has been interpreted as a modification of the crisp set concept for the representation of uncertainties and imprecise data in a parametric manner. Unlike the FS theory that is a tool for assigning membership degrees to elements, the SS theory is a parameterized family of subsets in the universal set. This method is a strong and robust way to handle uncertainties. It can apply in various situations of the data analysis, decision making, and computer science. SS theory is not only universal and capable of dealing with various kinds of uncertainties and imprecisions, but it also does not require any extra quantitative measure as compared to probability theory and FS theory It is a process of presenting and handling data which is indefinite and uncertain, and thus, makes it applicable for different uses like decision making in inexact situations, data mining, and pattern recognition. SST is a theory that has been greatly improved since its initial formulation and has led to many generalizations and extensions. Ali et al. [5] devised certain new operations on SS. The multi-attribute decision-making (MADM) strategy constructed around SST is developed by Zahedi Khameneh and Kilicman [6]. Xiao et al. [7] gave a brief introduction on the concepts of acknowledgment of soft data according to the SS theory. SST are widely used in many different sectors including in real life. The new extension and application of SST in the field of fuzzy mathematics are discussed by Tripathy et al. [8] with the title of latest approach to fuzzy SST and its application in decision making (DM). Cagman [9] also discussed the fuzzy SST and its applications in different ways. Mushrif [10] presents an innovative method for classifying textures, SST based classification algorithm. Moreover, Min [11] developed the similarity in SST. The concept of financial ratio selection for SST-based company failure prediction is covered by Xu et al. [12]. Moreover, Danjuma et al. [13] gave a unique and helpful idea on a different strategy for the SST normal parameter reduction technique. Maji et al. [14] provided the first definition of the term "soft subsets".

A soft semigroup (SSG) structure unites the ideas of SST and SG theory. It was proposed as an extension of SST to study algebraic structures with uncertainties or imperfections. SSGs give the possibility to describe algebraic structures including uncertainties or imprecisions in a parameterized way. They enable the depiction of semi-groups-like structures, with their elements or operations being unclear or vague. The soft ordered SGs were first devised Jun et al. [15]. They studied the basic characteristics of these systems, and subsequently this provided the framework for further advancement in this field. Feng et al. [16] applied the soft relations to SGs. They were concerned with soft bonds and how they are important in SG theory. Khan et al. [17] analyzed the concept of uni-soft structure for the ordered SGs. A new study was conducted and it provided an understanding of uni-soft structure and its usage in ordered SGs. Hamouda [18] studied soft ideals in ordered SGs. It enabled to unravel the role and importance of soft ideals in the realm of ordered SG theory. Shabir et al. [19] studied soft ideals and generalized fuzzy ideals in SGs with a focus on exposing their properties and usages. Muhiuddin and Mahboob [20] devised int-soft notions of SS on ordered SGs with the goal of providing a different viewpoint of the interplay between SS and ordered SGs. Khan et al. [21] devised soft union in ordered SGs via uni-soft quasi-ideals. Yousafzai et al. [22] presented the concept of non-associative ordered SGs based on SSs and expanded the existing knowledge about non-associative structures within the framework of SS theory.

Bipolar statements are actually defining the positive and negative aspects of any objects in real life cases. We know that each and every thing in real-life have up to two aspects and these aspects are very useful for any person who wants to knows about benefits and draw begs of any objects. To understand all these aspects in mathematical ways a lot of frameworks are available in market but first of all the idea of bipolar soft theory or bipolar SSs (BSSs) was given by Shabir and Naz [23] introduce the new theory of on bipolar SSs. They also introduce some theory related operations and properties in this manuscript. One another approach for bipolar SSs in different ways are discussed by Karaaslan and Karatas [24] with the title of novel method for treating bipolar SSs and its applications. Moreover, one another and until last approach for bipolar SSs are discussed by Mahmood [25] with the title of a fresh method for treating bipolar SSs and their applications. This approach was very different from other two approaches and cannot compare able with other two approaches. So, the third approach is veery advance because it covers all the previous concepts and discussed some new concepts and frameworks. Also, application is available in this manuscript. So, using all these concepts many researchers utilized this idea in different fields and discussed some new concepts and applications. Some useful and necessary ideas are discussed here. Al-Shami [26] discuss the idea of BSSs and connections between them, regular points, and their uses. Kamaci and Petchimuthu [27] also gave the new idea of bipolar N-SST with applications. Ali et al. [28] provided some fresh concepts on bipolar neutrosophic SSs and applications in DM. Shabir et al. [29] extended the idea of BSSs and introduce the new concept of approximate BSSs using soft relations and their application in DM. Saleh et al. [30] gives a very interesting idea by extending the concept of BSSs and introduce binary bipolar soft points and topology on binary BSSs with their symmetric Properties. Morover, the concept of multiattribute DM under Fermatean fuzzy bipolar soft framework given by Ali and Ansari [31].

The role of SGs is to operate as the tool for the study and modeling of systems and phenomena in which inverse operation is not relevant or necessary. SGs are encountered in the subject of automata theory, coding theory, language theory, and the study of discrete dynamical systems. They also serve as the basis for the study of formal languages, which are helpful in computer science, programming language theory, and other areas. In addition to this, SGs provide a mathematical framework for algebraic study of structures and their properties. Many classes of SGs, e.g., inverse SGs, regular SGs, and completely regular SGs, are well known and have been researched deeply, leading to important results that are widely used. The motivation for SGs results from the fact that not all types of systems or phenomena can be appropriately modeled using the more restrictive structure of groups. SGs provide a framework which is more general and flexible and, therefore, enable one to study an extended spectrum of algebraic structures and their properties, thus, one may have a deeper insight into various mathematical, scientific, and computational problems. So motivated form this, in this thesis, we study SGs and related results over T-BSSs.

The main contributions and key points of this thesis are discussed underneath in chapter wise study.

Chapter 1

In this chapter, we interpret some fundamental notions of SGs, SSs, SG, BSSs. Additionally, we discuss some of the associated essential operations, such as union, intersection, complement, extended union, extended intersection, restricted union, restricted intersection, AND and OR products. Furthermore, few more fundamental and associated findings are included in this chapter.

Chapter 2

We want to familiarize ourselves with the postulation of T-Bipolar SS (T-BSS) in this Chapter. We are going to show basic properties and related outcomes of T-BSSs and outline their binary operations.

Chapter 3

In this Chapter, we explore the notion and properties of the T-Bipolar Soft Semigroup (T-BSSG), which we constructed as a novel approach to T-BSSs. These properties include the AND and OR product, the Restricted Union (Res-Union) and Restricted Intersection (Res-Intersection), and the Extended Union (Ext-Union) and Extended Intersection (Ext-Intersection) on T-BSSG. Further, we also devise the related algebraic properties of T-BSSG.

Chapter 01

Preliminaries

Here, we'll go over the important ideas such SGs, SSs, SSG, BSSs, and some important results associated with these notions.

1.1 Semigroups:

Here, the basic notion of SG and related result is revised.

1.1.1 Definition [32]:

A set $\ddot{X} \neq \emptyset$ with binary operation $*$ is interpreted as SG, if the underneath property is satisfied,

$$\forall \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3 \in \ddot{X}, (\mathfrak{v}_1 * \mathfrak{v}_2) * \mathfrak{v}_3 = \mathfrak{v}_1 * (\mathfrak{v}_2 * \mathfrak{v}_3)$$

1.1.2 Definition [32]:

Consider $\ddot{X}$ is a SG and $\emptyset \neq \ddot{X}_b \subseteq \ddot{X}$, then $\ddot{X}_b$ is interpreted as subsemigroup of $\ddot{X}$ if $\forall\ \mathfrak{v}_1, \mathfrak{v}_2 \in \ddot{X}_b, \mathfrak{v}_1\mathfrak{v}_2 \in \ddot{X}_b$.

1.1.3 Example:

$(\mathbb{N}, +)$ is a sub-semigroup of $(\mathbb{Z}, +)$.

1.1.4 Proposition [32]:

Consider $\ddot{X}$ is a SG, then any family of subsemigroup of $\ddot{X}$ is again a subsemigroup of $\ddot{X}$.

1.2 Soft Sets:

In this section, we discussed the concept of SS and its operations with examples.

1.2.1 Definition [4]:

Assume, U represent the universal set, with a set of parameter E. For any $ƛ \subseteq E$ such that $ƛ \neq \emptyset$. Then a function $ℸ_1 : ƛ \rightarrow P(U)$ over U is called a soft set.

1.2.2 Example:

Consider $U = \{\check{s}_1, \check{s}_2, \check{s}_3, \check{s}_4, \check{s}_5, \check{s}_6, \check{s}_7\}$, $ƛ_1 = \{\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4\} \subseteq E$. Then $(ℸ_1, ƛ_1)$ is SS over U, where $ℸ_1(\tilde{n}_1) = \{\check{s}_4, \check{s}_5\}$, $ℸ_1(\tilde{n}_2) = \{\check{s}_1, \check{s}_2\}$, $ℸ_1(\tilde{n}_3) = \{\check{s}_6, \check{s}_7\}$, $ℸ_1(\tilde{n}_4) = \{\check{s}_5\}$.

1.2.3 Definition [5]:

Let U represent the universal set, with the set of parameter E. Such that $ƛ \neq \emptyset$, consider $(ℸ_1, ƛ_1)$ and $(ℸ_2, ƛ_2)$ be two SSs over U, then $(ℸ_1, ƛ_1)$ is known as a soft subset of $(ℸ_2, ƛ_2)$, if.

i) $ƛ_1 \subseteq ƛ_2$,

ii) $\forall\, \tilde{n} \in ƛ_1 \Rightarrow ℸ_1(\tilde{n}) \subseteq ℸ_2(\tilde{n})$.

If $(ℸ_1, ƛ_1)$ is a subset of $(ℸ_2, ƛ_2)$ and $(ℸ_2, ƛ_2)$ is a subset of $(ℸ_1, ƛ_1) \Rightarrow (ℸ_1, ƛ_1) = (ℸ_2, ƛ_2)$ i.e. the SS are equal.

If $(ℸ_1, ƛ_1)$ is a soft subset of $(ℸ_2, ƛ_2)$, then $(ℸ_2, ƛ_2)$ is known as a soft super set of $(ℸ_1, ƛ_1)$ is denotad by $(ℸ_1, ƛ_1) \,\tilde{\subseteq}\, (ℸ_2, ƛ_2)$.

1.2.4 Example:

Let $U = \{\mathscr{s}_1, \mathscr{s}_2, \mathscr{s}_3, \mathscr{s}_4, \mathscr{s}_5, \mathscr{s}_6, \mathscr{s}_7\}$ $ƛ_1 = \{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4\}$ and

$ƛ_2 = \{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4, \mathrm{u}_5, \mathrm{u}_6\}$, where $ƛ_1 \subseteq E$ and $ƛ_2 \subseteq E$, then $(ℸ_1, ƛ_1)$ and $(ℸ_2, ƛ_2)$ are SSs as follows,

$$ℸ_1(\mathrm{u}_1) = \{\mathscr{s}_3, \mathscr{s}_4, \mathscr{s}_5\}, ℸ_1(\mathrm{u}_2) = \{\mathscr{s}_1, \mathscr{s}_2\}, ℸ_1(\mathrm{u}_3) = \{\mathscr{s}_6, \mathscr{s}_7\},\ ℸ_1(\mathrm{u}_4) = \{\mathscr{s}_5\},$$

$$ℸ_2(\mathrm{u}_1) = \{\mathscr{s}_2, \mathscr{s}_3, \mathscr{s}_4, \mathscr{s}_5, \mathscr{s}_6\},\ ℸ_2(\mathrm{u}_2) = \{\mathscr{s}_1, \mathscr{s}_2, \mathscr{s}_3\},\ ℸ_2(\mathrm{u}_3) = \{\mathscr{s}_4, \mathscr{s}_5, \mathscr{s}_6, \mathscr{s}_7\},$$

$$ℸ_2(\mathrm{u}_4) = \{\mathscr{s}_1, \mathscr{s}_2, \mathscr{s}_3, \mathscr{s}_4, \mathscr{s}_5, \mathscr{s}_6\},\ ℸ_2(\mathrm{u}_5) = \{\mathscr{s}_1, \mathscr{s}_2, \mathscr{s}_3, \mathscr{s}_4, \mathscr{s}_5\},$$

$$ℸ_2(\mathrm{u}_6) = \{\mathscr{s}_1, \mathscr{s}_2, \mathscr{s}_3, \mathscr{s}_4, \mathscr{s}_5, \mathscr{s}_6, \mathscr{s}_7\}.$$

Then $(ℸ_1, ƛ_1)$ is declared as the soft subset of $(ℸ_2, ƛ_2)$.

1.2.5 Definition [5]:

Let U represents the universal set, with the set of parameter E and $ƛ_1 \subseteq E$ such that $ƛ_1 \neq \emptyset$.

i) $(ℸ_1, ƛ_1)$ is a void SS is denoted by $\emptyset_{ƛ_1}$, if $ℸ_1(ñ_1) = \emptyset \;\; \forall\, ñ_1 \in ƛ_1$.

ii) $(ℸ_1, ƛ_1)$ is a whole SS denoted by $U_{ƛ_1}$, if $ℸ_1(ñ_1) = U \;\; \forall\, ñ_1 \in ƛ_1$.

iii) The compliment of $(ℸ_1, ƛ_1)$ is denoted by $(ℸ_1, ƛ_1)^c = (ℸ_1^c, ƛ_1)$, where

$$ℸ^c(ñ) = U - ℸ(ñ) \; \forall\, ñ \in ƛ_1.$$

We define some other basic notation related to the above 3 points.

i) $U_E =$ The whole SS over parameter E is called Absolute SS.

ii) $\emptyset_{ƛ_1} =$ The relative null SS over the parameter $ƛ_1$.

iii) $\emptyset_\emptyset =$ It is a rare SS with a void parameter over U is known as void SS over U.

The relation being all these soft sets are represented by the form.

$$\emptyset_\emptyset \,\tilde{\subseteq}\, \emptyset_{ƛ_1} \,\tilde{\subseteq}\, (ℸ_1, ƛ_1) \,\tilde{\subseteq}\, U_{ƛ_1} \,\tilde{\subseteq}\, U_E$$

1.2.6 Definition [5]:

Let U represent the universal set, with the set of parameter $\tilde{\alpha}$, and $ƛ_1 \subseteq \tilde{\alpha}$, $ƛ_2 \subseteq \tilde{\alpha}$, such that $ƛ_1, ƛ_2 \neq \emptyset$. Consider a two SSs over the U, $(ℸ_1, ƛ_1)$ and $(ℸ_2, ƛ_2)$ such that $ƛ_1 \cap ƛ_2 \neq \emptyset$. Then

i) "(Res-Union)" of $(ℸ_1, ƛ_1)$ and $(ℸ_2, ƛ_2)$ is denoted and defined by $(ℸ_1, ƛ_1) \cup_R (ℸ_2, ƛ_2) = (ℸ_3, ƛ_1 \cap ƛ_2) = ℸ_1(ñ) \cup ℸ_2(ñ) \;\; \forall\, ñ \in ƛ_1 \cap ƛ_2$.

ii) "(Res-Intersection)" of $(ℸ_1, ƛ_1)$ and $(ℸ_2, ƛ_2)$ is denoted and defined by $(ℸ_1, ƛ_1) \cap_R (ℸ_2, ƛ_2) = (ℸ_3, ƛ_1 \cap ƛ_2) = ℸ_1(ñ) \cap ℸ_2(ñ) \;\; \forall\, ñ \in ƛ_1 \cap ƛ_2$.

iii) If $ƛ_1 \cap ƛ_2 = \emptyset$, then $(ℸ_1, ƛ_1) \cup_R (ℸ_2, ƛ_2) = \emptyset_\emptyset$ and $(ℸ_1, ƛ_1) \cap_R (ℸ_2, ƛ_2) = \emptyset_\emptyset$.

1.2.7 Definition [5]:

Let, $\tilde{\alpha}$ be a set of parameters, U be the universal set, and $ƛ_1 \subseteq \tilde{\alpha}, ƛ_2 \subseteq \tilde{\alpha}$, such that $ƛ_1, ƛ_2 \neq \emptyset$. Consider a two SSs over the U, $(\daleth_1, ƛ_1)$ and $(\daleth_2, ƛ_2)$, then.

i) The Ext-Union of $(\daleth_1, ƛ_1)$ and $(\daleth_2, ƛ_2)$, is symbolized and distinguished by $(\daleth_1, ƛ_1) \cup_E (\daleth_2, ƛ_2) = (\daleth_3, ƛ_3)$, where $ƛ_3 = ƛ_1 \cup ƛ_2$.

$$\daleth_3(\tilde{n}) = \begin{cases} \daleth_1(\tilde{n}) & ; \text{ if } \tilde{n} \in ƛ_1 - ƛ_2 \\ \daleth_2(\tilde{n}) & ; \text{ if } \tilde{n} \in ƛ_2 - ƛ_1 \\ \daleth_1(\tilde{n}) \cup \daleth_2(\tilde{n}) & ; \text{ if } \tilde{n} \in ƛ_1 \cap ƛ_2 \end{cases}$$

ii) The Ext-Intersection of $(\daleth_1, ƛ_1)$ and $(\daleth_2, ƛ_2)$, is symbolized and distinguished by

$(\daleth_1, ƛ_1) \cap_E (\daleth_2, ƛ_2) = (\daleth_3, ƛ_3)$, where $ƛ_3 = ƛ_1 \cup ƛ_2$.

$$\daleth_3(\tilde{n}) = \begin{cases} \daleth_1(\tilde{n}) & ; \text{ if } \tilde{n} \in ƛ_1 - ƛ_2 \\ \daleth_2(\tilde{n}) & ; \text{ if } \tilde{n} \in ƛ_2 - ƛ_1 \\ \daleth_1(\tilde{n}) \cap \daleth_2(\tilde{n}) & ; \text{ if } \tilde{n} \in ƛ_1 \cap ƛ_2 \end{cases}$$

1.2.8 Definition [5]:

Let, $\tilde{\alpha}$ be a set of parameters, U be the universal set, and $ƛ_1 \subseteq \tilde{\alpha}, ƛ_2 \subseteq \tilde{\alpha}$, such that $ƛ_1, ƛ_2 \neq \emptyset$. Consider a two SSs over the U, $(\daleth_1, ƛ_1)$ and $(\daleth_2, ƛ_2)$, then

i) AND Product of $(\daleth_1, ƛ_1)$ and $(\daleth_2, ƛ_2)$, is denoted and defined by

$(\daleth_1, ƛ_1) \wedge (\daleth_2, ƛ_2) = (\daleth_3, ƛ_1 \times ƛ_2)$, Such that

$\daleth_3(\tilde{n}_1, \tilde{n}_2) = \daleth_1(\tilde{n}_1) \cap \daleth_2(\tilde{n}_2) \ \forall (\tilde{n}_1, \tilde{n}_2) \in ƛ_1 \times ƛ_2$.

ii) OR Product of $(\daleth_1, ƛ_1)$ and $(\daleth_2, ƛ_2)$, is denoted and defined by

$(\daleth_1, ƛ_1) \vee (\daleth_2, ƛ_2) = (\daleth_3, ƛ_1 \times ƛ_2)$, Such that

$\daleth_3(\tilde{n}_1, \tilde{n}_2) = \daleth_1(\tilde{n}_1) \cup \daleth_2(\tilde{n}_2) \ \forall (\tilde{n}_1, \tilde{n}_2) \in ƛ_1 \times ƛ_2$.

1.3 Soft Semigroups:

In this portion, we elaborate fundamental notion of SSG and discuss some basic operations on SSGs.

1.3.1 Definition:

Let $(\daleth, \breve{\wp})$ be any SS and let $\ddot{X}$ be any SG. Then $(\daleth, \breve{\wp})$ is called SSG over $\ddot{X}$ iff $\daleth(\lambdabar)$ is subsemigroup of $\ddot{X}$, $(\daleth(\lambdabar) \leq \ddot{X})\ \forall\ \lambdabar \in \breve{\wp}$.

1.3.2 Example:

Consider $\ddot{X} = \{\, ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \,\}$ is a SG, devised as

$\cdot$	$ȩ$	$\boldsymbol{\mathfrak{v}_1}$	$\boldsymbol{\mathfrak{v}_2}$	$\boldsymbol{\mathfrak{v}_3}$	$\boldsymbol{\mathfrak{v}_4}$
$ȩ$	$ȩ$	$ȩ$	$ȩ$	$ȩ$	$ȩ$
$\boldsymbol{\mathfrak{v}_1}$	$ȩ$	$ȩ$	$ȩ$	$ȩ$	$ȩ$
$\boldsymbol{\mathfrak{v}_2}$	$ȩ$	$ȩ$	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$
$\boldsymbol{\mathfrak{v}_3}$	$ȩ$	$ȩ$	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$
$\boldsymbol{\mathfrak{v}_4}$	$ȩ$	$ȩ$	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$

and $\breve{\wp} = \{\lambdabar_1, \lambdabar_2, \lambdabar_3, \lambdabar_4\}$ is a set of parameters, then

$$\daleth(\lambdabar_1) = \{ȩ\};\ \ \daleth(\lambdabar_2) = \{ȩ, \mathfrak{v}_2, \mathfrak{v}_3\};\ \ \daleth(\lambdabar_3) = \{ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\,\};\ \daleth(\lambdabar_4) = \{\mathfrak{v}_3, \mathfrak{v}_4\}.$$

$$(\daleth, \breve{\wp}) = \{< \lambdabar_1, \{ȩ\} >, < \lambdabar_2, \{ȩ, \mathfrak{v}_2, \mathfrak{v}_3\} >,$$

$$< \lambdabar_3, \{ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\,\} >, < \lambdabar_4, \{\mathfrak{v}_3, \mathfrak{v}_4\}, >\}$$

This implies that $(\daleth, \breve{\wp})$ is a SSG over $\ddot{X}$.

1.3.3 Proposition:

The Intersection of two SSG $(\daleth_1, \breve{\wp})$ and $(\daleth_2, \breve{Ɣ})$ over $\ddot{X}$ is SSG over $\ddot{X}$.

1.3.4 Proposition:

The Union of two SSG $(\daleth_1, \breve{\wp})$ and $(\daleth_2, \breve{Ɣ})$ over $\ddot{X}$ is SSG over $\ddot{X}$, if $\breve{\wp} \cap \breve{Ɣ} = \emptyset$.

1.3.5 Definition:

Let $(\daleth_1, \breve{\wp}), (\daleth_2, \breve{Ɣ})$ be two SSGs over $\ddot{X}$. Then $(\daleth_1, \breve{\wp})$ is said to be a SSG subset of $(\daleth_2, \breve{Ɣ})$, if it satisfies:

i) $\breve{\wp} \subseteq \breve{Ɣ}$.

ii) $\forall\, ƛ \in \breve{\wp}, \daleth_1(ƛ) \subseteq \daleth_2(ƛ)$.

1.4 Bipolar Soft Set:

There have been two different attempts to define what bipolar SSs (BSSs). However, the concept of the T-BSS is far closer to bipolarity. Here, are these concepts listed below:

1.4.1 Definition [23]:

Let, us assume U be the universal set and $\breve{\wp} \subseteq E$. Also, $\neg\breve{\wp} = \{\neg\mathfrak{z}, \mathfrak{z} \in \breve{\wp}\}$ represent the NOT set of $\breve{\wp}$. Then, the triplet $(\daleth_{\mathrm{bp}}, \S_{\mathrm{bp}}, \breve{\wp})$ is called a Bipolar SS where $\daleth_{\mathrm{bp}}: \breve{\wp} \to P(U)$ and $\S_{\mathrm{bp}}: \neg\breve{\wp} \to P(U)$ and $\daleth_{\mathrm{bp}}(\mathfrak{z}) \cap \S_{\mathrm{bp}}(\neg\mathfrak{z}) = \phi$ (null set).

1.4.2 Definition [24]:

Let U represent the universal set, with the set of parameter E and $\breve{\wp}_1 \subseteq E$, $\breve{\wp}_2 \subseteq E$ such that $\breve{\wp}_1 \cup \breve{\wp}_2 = E$, and $\breve{\wp}_1 \cap \breve{\wp}_2 = \phi$ (null set). Then the triplet $(\daleth_{\mathrm{bp}}, \S_{\mathrm{bp}}, \breve{\wp})$ is called a Bi-polar SS over U, where $\daleth_{\mathrm{bp}}: \breve{\wp}_1 \to P(U)$ and $\S_{\mathrm{bp}}: \breve{\wp}_2 \to P(U)$ with $\daleth_{\mathrm{bp}}(\mathfrak{z}) \cap \S_{\mathrm{bp}}(\mathrm{g}(\mathfrak{z})) = \phi$, and $\mathrm{g}: \breve{\wp}_1 \to \breve{\wp}_2$ is a bijective mapping.

1.4.3 Example:

Let $U = \{\check{s}_1, \check{s}_2, \check{s}_3, \check{s}_4, \check{s}_5, \check{s}_6, \check{s}_7\}$, $\breve{\wp} = \{ƛ_1, ƛ_2, ƛ_3, ƛ_4\}$, $\breve{\wp} \subseteq E$ s.t. $\breve{\wp} \neq \emptyset$. we define the two functions $\daleth_1: \breve{\wp} \to P(U)$ and $\S_1: \breve{\wp} \to P(U)$.

$$\daleth_1(ƛ_1) = \{\check{s}_3, \check{s}_4, \check{s}_5\};\ \daleth_1(ƛ_2) = \{\check{s}_1, \check{s}_2\};\ \daleth_1(ƛ_3) = \{\check{s}_6, \check{s}_7\};\ \daleth_1(ƛ_4) = \{\check{s}_5\},$$

$$\S_1(ƛ_1) = \{\check{s}_2, \check{s}_6\};\ \S_1(ƛ_2) = \{\check{s}_3\};\ \S_1(ƛ_3) = \{\check{s}_4, \check{s}_5\};\ \S_1(ƛ_4) = \{\check{s}_1, \check{s}_2, \check{s}_3, \check{s}_4\}.$$

So, the triplet $(\daleth_1, \S_1, \breve{\wp})$ is called a Bipolar SS.

Chapter 02

T-Bipolar Soft Sets and their Related Operations and Properties

The perception of the T-Bipolar soft set (T-BSS) will be discussed in this chapter which is devised by Mahmood [25]. For T-BSSs, we shall state binary operations and compute certain fundamental features and outcomes related to these ideas.

2.1. T-Bipolar Soft Set

We want to familiarize ourselves with the concept of T-BSS in this section. We will go over the binary operations connected to outline the binary operations associated with T-BSSs and discuss fundamental properties and related outcomes.

2.1.1 Definition:

Let U represent the universal set, with the set of parameter E $\breve{\wp} \subseteq E$, $\ddot{X} \subset U$ and $Y = U - \ddot{X}$. A triplet $(\bar{\daleth}, \bar{\S}, \breve{\wp})$ is then considered the T-BSSs over U, where $\bar{\daleth}$ and $\bar{\S}$ are set valued mappngs given by $\bar{\daleth}: \breve{\wp} \rightarrow P\ (\ddot{X})$ and $\bar{\S}: \breve{\wp} \rightarrow P\ (Y)$.

Here, we can write $\left(\bar{\daleth}, \bar{\S}, \breve{\wp}\right) = \{< \lambda, \bar{\daleth}(\lambda), \bar{\S}(\lambda): \bar{\daleth}(\lambda) \in P(\ddot{X})$ and $\bar{\S}(\lambda) \in P(Y) >\}$. For the sake of simplicity, we write $\left(\bar{\daleth}, \bar{\S}, \breve{\wp}\right) = \left\{< \lambda, \bar{\daleth}(\lambda), \bar{\S}(\lambda) >\right\}$.

The symbol $(T - BSS)_{(U)}$ represents the collection of all T-BSSs over U.

2.1.2 Remarks:

Assume that $\breve{\wp} = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_l\} \subseteq E$, $\ddot{X} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \dots, \mathfrak{v}_m\}$, $Y = \{y_1, y_2, y_3, \dots, y_n\}$ and $(\bar{\daleth}, \bar{\S}, \breve{\wp})$ represent T-BSS. We express the $(\bar{\daleth}, \bar{\S}, \breve{\wp})$ as under.

$(\bar{\mathcal{F}},\bar{\mathcal{G}},\breve{\mathcal{F}}$	$(x_1,y_1$	$(x_1,y_2$	…	$(x_1,y_n$	$(x_2,y_1$	$(x_2,y_2$	…	$(x_2,y_n$	…	$(x_m,y_1$	$(x_m,y_2$	…	$(x_m,y_n$
$\gimel_1$	$\breve{\xi}_{111}$	$\breve{\xi}_{112}$	…	$\breve{\xi}_{11n}$	$\breve{\xi}_{121}$	$\breve{\xi}_{122}$	…	$\breve{\xi}_{12n}$	…	$\breve{\xi}_{1m1}$	$\breve{\xi}_{1m2}$	…	$\breve{\xi}_{1mn}$
$\gimel_2$	$\breve{\xi}_{211}$	$\breve{\xi}_{212}$	…	$\breve{\xi}_{21n}$	$\breve{\xi}_{221}$	$\breve{\xi}_{222}$	…	$\breve{\xi}_{22n}$	…	$\breve{\xi}_{2m1}$	$\breve{\xi}_{2m2}$	…	$\breve{\xi}_{2mn}$
$\gimel_3$	$\breve{\xi}_{311}$	$\breve{\xi}_{312}$	…	$\breve{\xi}_{31n}$	$\breve{\xi}_{321}$	$\breve{\xi}_{322}$	…	$\breve{\xi}_{32n}$	…	$\breve{\xi}_{3m1}$	$\breve{\xi}_{3m2}$	…	$\breve{\xi}_{3mn}$
…	…	…	…	…	…	…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…	…	…	…	…	…	…
$\gimel_l$	$\breve{\xi}_{l11}$	$\breve{\xi}_{l12}$	…	$\breve{\xi}_{l1n}$	$\breve{\xi}_{l21}$	$\breve{\xi}_{l22}$	…	$\breve{\xi}_{l2n}$	…	$\breve{\xi}_{lm1}$	$\breve{\xi}_{lm2}$	…	$\breve{\xi}_{lmn}$

(Tabular form of a T-BSS)

where,

$$\breve{\xi}_{ijk} = (\check{s}_j, \tilde{n}_k) = \begin{cases} (0,0) & if\ \mathfrak{v}_j \notin \bar{\daleth}(\gimel_i)\ \check{\acute{a}}nd\ y_k \notin \bar{\daleth}(\gimel_i) \\ (1,0) & if\ \mathfrak{v}_j \in \bar{\daleth}(\gimel_i)\ \check{\acute{a}}nd\ y_k \notin \bar{\daleth}(\gimel_i) \\ (0,1) & if\ \mathfrak{v}_j \notin \bar{\daleth}(\gimel_i)\ \check{\acute{a}}nd\ y_k \in \bar{\daleth}(\gimel_i) \\ (1,1) & if\ \mathfrak{v}_j \in \bar{\daleth}(\gimel_i)\ \check{\acute{a}}nd\ y_k \in \bar{\daleth}(\gimel_i) \end{cases}$$

And $\breve{\xi}^*_{ijk} = \check{s}_j, \breve{\xi}^\circ_{ijk} = \tilde{n}_k$.

2.1.3 Definition:

Let $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \in (T - BSS)_{(U)}$. Then $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ is considered the T-Bipolar Soft Subset of $(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})$. If,

i) $\breve{\wp} \subseteq \breve{Ɣ}$,

ii) $\forall\ \gimel \in \breve{\wp},\ \bar{\daleth}_1(\gimel) \subseteq \bar{\daleth}_2(\gimel)$ and $\bar{\S}_2(\gimel) \subseteq \bar{\S}_1(\gimel)$.

Then we can write $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \subseteq (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})$. If $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \subseteq (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ}) \subseteq (\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$, then they are considered to be equal i.e. $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) = (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})$.

2.1.4 Definition:

Let, $(\bar{\daleth}, \bar{\S}, \breve{\wp}) \in (T - BSS)_{(U)}$. then,

i) The Compliment of $(\bar{\daleth},\bar{\S},\breve{\wp})$ is indicated and symbolled by $(\bar{\daleth},\bar{\S},\breve{\wp})^c = (\bar{\daleth}^c,\bar{\S}^c,\breve{\wp}) = \{<\text{ƛ},\ \bar{\daleth}^c(\text{ƛ}) = \ddot{X} - \bar{\daleth}(\text{ƛ}),\ \bar{\S}^c(\text{ƛ}) = Y - \bar{\S}(\text{ƛ}) >\}$.

ii) $(\bar{\daleth},\bar{\S},\breve{\wp})$ is said to be null iff $\forall\, \text{ƛ} \in \breve{\wp},\ \bar{\daleth}(\text{ƛ}) = \emptyset$ and $\bar{\S}(\text{ƛ}) = Y$. It is also be categorized in this study according to Φ, i.e. $\Phi = \{<\text{ƛ},\emptyset,Y>\}$.

iii) $(\bar{\daleth},\bar{\S},\breve{\wp})$ is called absolute iff $\forall\, \text{ƛ} \in \breve{\wp},\ \bar{\daleth}(\text{ƛ}) = \ddot{X}$ and $\bar{\S}(\text{ƛ}) = \emptyset$. In this case it will also be identify by A, that is $\mathrm{A} = \{<\text{ƛ},\ddot{X},\emptyset>\}$.

2.1.5 Definition:

Let $(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp}),(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}}) \in (T-BSS)_{(U)}$. then,

i) "AND" product of $(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})$ and $(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}})$ is denoted and given by $(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})\wedge(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}}) = \{<(\lambda,\check{s}),\bar{\daleth}_1(\lambda)\cap\bar{\daleth}_2(\check{s}),\bar{\S}_1(\lambda)\cup\bar{\S}_2(\check{s})>:(\lambda,\check{s}) \in \breve{\wp}\times\breve{\text{Ɣ}}\}$.

ii) "OR" product is denoted and given by, $(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})\vee(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}}) = \{<(\lambda,\check{s}),\bar{\daleth}_1(\lambda)\cup\bar{\daleth}_2(\check{s}),\ \bar{\S}_1(\lambda)\cap\bar{\S}_2(\check{s})>;(\lambda,\check{s}) \in \breve{\wp}\times\breve{\text{Ɣ}}\}$.

2.1.6 Proposition:

Let $(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp}),(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}}) \in (T-BSS)_{(U)}$. Then,

i) $[(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})\wedge(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}})]^c = [(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})]^c \vee [(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}})]^c$,

ii) $[(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})\vee(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}})]^c = [(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})]^c \wedge [(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}})]^c$.

Proof:

i) $$[(\bar{\daleth}_1,\bar{\S}_1,\breve{\wp})\wedge(\bar{\daleth}_2,\bar{\S}_2,\breve{\text{Ɣ}})]^c = \begin{Bmatrix} <(\lambda,\check{s}),\ \bar{\daleth}_1(\lambda)\cap\bar{\daleth}_2(\check{s}), \\ \bar{\S}_1(\lambda)\cup\bar{\S}_2(\check{s})> \end{Bmatrix}^c$$

$$= \begin{Bmatrix} <(\lambda,\check{s}), \\ \ddot{X} - (\bar{\daleth}_1(\lambda)\cap\bar{\daleth}_2(\check{s})), \\ Y - (\bar{\S}_1(\lambda)\cup\bar{\S}_2(\check{s}))> \end{Bmatrix}$$

$$= \{< (\lambda, š), (Ẍ - \overline{7}_1(\lambda)) \cup (Ẍ - \overline{7}_2(š)), \left(Y - \overline{\S}_1(\lambda)\right) \cap \left(Y - \overline{\S}_2(š)\right) >\}$$

$$= \left\{< (\lambda, š), \overline{7}_1^c(\lambda) \cup \overline{7}_2^c(š), \overline{\S}_1^c(\lambda) \cap \overline{\S}_2^c(š) >\right\} = \left[(\overline{7}_1, \overline{\S}_1, \breve{\wp})\right]^c \vee \left[(\overline{7}_2, \overline{\S}_2, \breve{Ɣ})\right]^c$$

ii) The proof of part (i) and part (ii) are comparable.

2.1.7 Definition:

Let $(\overline{7}_1, \overline{\S}_1, \breve{\wp})$, $(\overline{7}_2, \overline{\S}_2, \breve{Ɣ}) \in (T - BSS)_{(U)}$. Than the "Ext−Union" of $(\overline{7}_1, \overline{\S}_1, \breve{\wp})$ and $(\overline{7}_2, \overline{\S}_2, \breve{Ɣ})$ is designated and given by $(\overline{7}_1, \overline{\S}_1, \breve{\wp}) \cup_E (\overline{7}_2, \overline{\S}_2, \breve{Ɣ}) = (\mathcal{H}, \mathcal{K}, \breve{\wp} \cup \breve{Ɣ})$.

Where,

$$\mathcal{H}\ (ể) = \begin{cases} \overline{7}_1(ể) & \text{IF } ể \in \breve{\wp} - \breve{Ɣ} \\ \overline{7}_2(ể) & \text{IF } ể \in \breve{Ɣ} - \breve{\wp} \\ \overline{7}_1(ể) \cup \overline{7}_2(ể) & \text{IF } ể \in \breve{\wp} \cap \breve{Ɣ} \end{cases}$$

And

$$\mathcal{K}\ (ể) = \begin{cases} \overline{\S}_1\ (ể) & if\ ể \in \breve{\wp} - \breve{Ɣ} \\ \overline{\S}_2\ (ể) & if\ ể \in \breve{Ɣ} - \breve{\wp} \\ \overline{\S}_1\ (ể) \cap\ \overline{\S}_2(ể) & if\ ể \in \breve{\wp} \cap \breve{Ɣ} \end{cases}$$

2.1.8 Definition:

Let $(\overline{7}_1, \overline{\S}_1, \breve{\wp})$, $(\overline{7}_2, \overline{\S}_2, \breve{Ɣ}) \in (T - BSS)_{(U)}$. Then the "Ext−Intersection" of $(\overline{7}_1, \overline{\S}_1, \breve{\wp})$ and $(\overline{7}_2, \overline{\S}_2, \breve{Ɣ})$ is signify and described by $(\overline{7}_1, \overline{\S}_1, \breve{\wp}) \cap_E (\overline{7}_2, \overline{\S}_2, \breve{Ɣ}) = (\mathcal{H}, \mathcal{K}, \breve{\wp} \cup \breve{Ɣ})$.

Where,

$$\mathcal{H}\ (ể) = \begin{cases} \overline{7}_1(ể) & \text{IF } ể \in \breve{\wp} - \breve{Ɣ} \\ \overline{7}_2(ể) & \text{IF } ể \in \breve{Ɣ} - \breve{\wp} \\ \overline{7}_1(ể) \cap \overline{7}_2(ể) & \text{IF } ể \in \breve{\wp} \cap \breve{Ɣ} \end{cases}$$

And

$$\mathcal{K}\ (ể) = \begin{cases} \bar{§}_1\ (ể) & if\ ể \in \breve{\wp} - \breve{Ɣ} \\ \bar{§}_2\ (ể) & if\ ể \in \breve{Ɣ} - \breve{\wp} \\ \bar{§}_1\ (ể) \cup \bar{§}_2(ể) & if\ ể \in \breve{\wp} \cap \breve{Ɣ} \end{cases}$$

2.1.9 Proposition:

For a few T-BSSs $(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}), (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})$ and $(\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})$ we have,

i) $(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E \Phi = \Phi,\ (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E \Phi = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}),$

$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E \mathrm{A} = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}),\ (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E \mathrm{A} = \mathrm{A},$

ii) $(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}), (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) =$

$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp})$

iii) $(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) = (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}),$

$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) = (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cup_E (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}).$

iv) $(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})] =$

$[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})] \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}}),$

$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cup_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})] =$

$[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})] \cup_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}}).$

v) $(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cup_E (\bar{ℸ}_1, \bar{§}_1, \breve{\wp})] = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}),$

$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_1, \bar{§}_1, \breve{\wp})] = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}).$

vi) $[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp})^c]^c = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}),\ (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E [(\bar{ℸ}_1, \bar{§}_1, \breve{\wp})]^c = \Phi,$

$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E [(\bar{ℸ}_1, \bar{§}_1, \breve{\wp})]^c = \mathrm{A}.$

vii) $[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})]^c = [(\bar{ℸ}_1, \bar{§}_1, \breve{\wp})]^c \cup_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})]^c,$

$[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})]^c = [(\bar{ℸ}_1, \bar{§}_1, \breve{\wp})]^c \cap_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})]^c.$

Proof:

We illustrate the part (iv) and (vii), the remaining is simple.

(iv) When $ƛ \in \breve{\wp}, ƛ \notin \breve{Ɣ}$ and $ƛ \in \breve{\mathcal{R}}$, then

$$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})] = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})$$

$$= \{< ƛ \in \breve{\wp} \cap \breve{\mathcal{R}} : \bar{ℸ}_1(ƛ) \cap \bar{ℸ}_3(ƛ), \bar{§}_1(ƛ) \cup \bar{§}_3(ƛ) >\}$$

and

$$[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})] \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}}) = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})$$

$$= \{< ƛ \in \breve{\wp} \cap \breve{\mathcal{R}} : \bar{ℸ}_1(ƛ) \cap \bar{ℸ}_3(ƛ), \bar{§}_1(ƛ) \cup \bar{§}_3(ƛ) >\}$$

when $ƛ \in \breve{\wp}, ƛ \in \breve{Ɣ}$ and $ƛ \notin \breve{\mathcal{R}}$, then

$$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})] = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})$$

$$= \{< ƛ \in \breve{\wp} \cap \breve{Ɣ} : \bar{ℸ}_1(ƛ) \cap \bar{ℸ}_2(ƛ), \bar{§}_1(ƛ) \cup \bar{§}_2(ƛ) >\}$$

and

$$[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})] \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}}) = (\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})$$

$$= \{< ƛ \in \breve{\wp} \cap \breve{Ɣ} : \bar{ℸ}_1(ƛ) \cap \bar{ℸ}_2(ƛ), \bar{§}_1(ƛ) \cup \bar{§}_2(ƛ) >\}$$

when $ƛ \notin \breve{\wp}, ƛ \in \breve{Ɣ}$ and $ƛ \in \breve{\mathcal{R}}$, then

$$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})] = (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})$$

$$= \{< ƛ \in \breve{Ɣ} \cap \breve{\mathcal{R}} : \bar{ℸ}_2(ƛ) \cap \bar{ℸ}_3(ƛ), \bar{§}_2(ƛ) \cup \bar{§}_3(ƛ) >\}$$

and

$$[(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})] \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}}) = (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})$$

$$= \{< ƛ \in \breve{Ɣ} \cap \breve{\mathcal{R}} : \bar{ℸ}_2(ƛ) \cap \bar{ℸ}_3(ƛ), \bar{§}_2(ƛ) \cup \bar{§}_3(ƛ) >\}$$

In the case, where $ƛ \in \breve{\wp}, ƛ \in \breve{Ɣ}$ and $ƛ \in \breve{\mathcal{R}}$, then outcome is obvious. Thus, we conclude that.

$$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})]$$
$$= [(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cap_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})] \cap_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}}).$$

Similarly,

$$(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E [(\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ}) \cup_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})]$$
$$= [(\bar{ℸ}_1, \bar{§}_1, \breve{\wp}) \cup_E (\bar{ℸ}_2, \bar{§}_2, \breve{Ɣ})] \cup_E (\bar{ℸ}_3, \bar{§}_3, \breve{\mathcal{R}})$$

(vii) When $ƛ \in \breve{\wp}$ and $ƛ \notin \breve{Ɣ}$, then

$$\left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c = \left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})\right]^c$$

And

$$\left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})\right]^c \cup_E \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c = \left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})\right]^c$$

When $ƛ \notin \breve{\wp}$ and $ƛ \in \breve{Ɣ}$, then

$$\left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c = \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c$$

$$\text{and } \left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})\right]^c \cup_E \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c = \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c$$

In the case, where $ƛ \in \breve{\wp}, \mathring{a}nd\ ƛ \in \breve{Ɣ}$ then outcome is obvious. Thus, we conclude that.

$$\left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c = \left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})\right]^c \cup_E \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c$$

Similarly,

$$\left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cup_E (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c = \left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})\right]^c \cap_E \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right]^c.$$

2.1.10 Remarks:

It is not essential for any arbitrary $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}), (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ}), (\bar{\daleth}_3, \bar{\S}_3, \breve{\mathcal{R}}) \in (T - BSS)_{(U)}$, that.

i) $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ}) \cup_E (\bar{\daleth}_3, \bar{\S}_3, \breve{\mathcal{R}})\right] =$

$\left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right] \cup_E \left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E (\bar{\daleth}_3, \bar{\S}_3, \breve{\mathcal{R}})\right]$,

ii) $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cup_E \left[(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ}) \cap_E (\bar{\daleth}_3, \bar{\S}_3, \breve{\mathcal{R}})\right] =$

$\left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cup_E (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})\right] \cap_E \left[(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cup_E (\bar{\daleth}_3, \bar{\S}_3, \breve{\mathcal{R}})\right]$.

2.1.11 Example:

Let us assume that the

$E = \{ƛ_1, ƛ_2, ƛ_3, ƛ_4, ƛ_5\}$, $\breve{\wp} = \{ƛ_1, ƛ_2, ƛ_3\}$, $\breve{Ɣ} = \{ƛ_3, ƛ_4\}$, $\breve{\mathcal{R}} = \{ƛ_4, ƛ_5\}$,

$U = \{ữ_1, ữ_2, ữ_3, ữ_4, ữ_5\}$, $\ddot{X} = \{ữ_1, ữ_2, ữ_3\}$ And $Y = \{ữ_4, ữ_5\}$.

Further, we assume that.

$$\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) = \{< ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, < ƛ_3, \{\tilde{ư}_1, \tilde{ư}_3\}, \{\tilde{ư}_4\} >\},$$

$$\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) = \{< ƛ_3, \{\tilde{ư}_2, \tilde{ư}_3\}, \{\tilde{ư}_5\} >, < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4, \tilde{ư}_5\} >\},$$

$$\left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right) = \{< ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} >\},$$

Further,

$$\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_E \left[\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) \cup_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right]$$

$$= \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_E \{< ƛ_3, \{\tilde{ư}_2, \tilde{ư}_3\}, \{\tilde{ư}_5\} >, < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} >\}$$

$$= \left\{\begin{matrix} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, \\ < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_3\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, \\ < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} > \end{matrix}\right\}$$

Next $\left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_E \left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right)\right] \cup_E \left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right]$

$$= \left\{\begin{matrix} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, \\ < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_3\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4, \tilde{ư}_5\} > \end{matrix}\right\} \cup_E \left\{\begin{matrix} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, \\ < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_1, \tilde{ư}_3\}, \{\tilde{ư}_4\} >, \\ < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, \\ < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} > \end{matrix}\right\}$$

$$= \left\{\begin{matrix} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, \\ < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_1, \tilde{ư}_3\}, \{\tilde{ư}_4\} >, \\ < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, \\ < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} > \end{matrix}\right\}$$

$$\Rightarrow \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_E \left[\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) \cup_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right]$$

$$\neq \left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_E \left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right)\right] \cup_E \left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right].$$

Now,

$$\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_E \left[\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) \cap_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right]$$

$$= \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_E \{< ƛ_3, \{\tilde{ư}_2, \tilde{ư}_3\}, \{\tilde{ư}_5\} >, < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4, \tilde{ư}_5\} >,$$

$$< ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} >\}$$

$$= \left\{ \begin{array}{c} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} > \end{array} \right\}$$

Further,

$$\left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_E \left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right)\right] \cap_E \left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right]$$

$$= \left\{ \begin{array}{c} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, \\ < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, \\ < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4, \tilde{ư}_5\} > \end{array} \right\} \cap_E \left\{ \begin{array}{c} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, \\ < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_1, \tilde{ư}_3\}, \{\tilde{ư}_4\} >, \\ < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2, \tilde{ư}_3\}, \emptyset >, \\ < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} > \end{array} \right\}$$

$$= \left\{ \begin{array}{c} < ƛ_1, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4\} >, \\ < ƛ_2, \{\tilde{ư}_1\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_3, \{\tilde{ư}_1, \tilde{ư}_3\}, \{\tilde{ư}_4\} >, \\ < ƛ_4, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_4, \tilde{ư}_5\} >, \\ < ƛ_5, \{\tilde{ư}_1, \tilde{ư}_2\}, \{\tilde{ư}_5\} > \end{array} \right\}$$

$$\Rightarrow \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_E \left[\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) \cap_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right]$$

$$\neq \left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_E \left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right)\right] \cap_E \left[\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_E \left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)\right].$$

2.1.12 Definition:

Let, we assume that $\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right)$, $\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) \in (T - BSS)_{(U)}$ with $\breve{\wp} \cap \breve{Ɣ} \neq \emptyset$. Then,

i) "Res-Union" of $\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right)$ and $\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right)$ is given and denoted by,

$$\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_R \left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) = \left\{< ƛ, \bar{ℸ}_1(ƛ) \cup \bar{ℸ}_2(ƛ), \bar{\S}_1(ƛ) \cap \bar{\S}_2(ƛ) >: ƛ \in \breve{\wp} \cap \breve{Ɣ}\right\},$$

ii) "Res-Intersection" of $\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right)$ and $\left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right)$ is given and denoted by,

$$\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_R \left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right) = \left\{< ƛ, \bar{ℸ}_1(ƛ) \cap \bar{ℸ}_2(ƛ), \bar{\S}_1(ƛ) \cup \bar{\S}_2(ƛ) >: ƛ \in \breve{\wp} \cap \breve{Ɣ}\right\}.$$

2.1.13 Proposition:

For, a few T-BSSs $\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right), \left(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}\right)$ and $\left(\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}}\right)$ then the following holds

i) $\left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_R \Phi = \Phi, \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_R \Phi = \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right), \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_R \mathrm{A} = \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right), \left(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}\right) \cup_R \mathrm{A} = \mathrm{A}$

ii) $(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) = (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}), (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) = (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})$

iii) $(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) = (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cap_R (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})$,

$(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) = (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cup_R (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})$

iv) $(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cap_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})] = [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})] \cap_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})$,

$(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cup_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})] = [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})] \cup_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})$

v) $(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cup_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})] = [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})] \cup_R [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})]$,

$(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cap_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})] = [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})] \cap_R [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\urcorner}_3, \bar{\S}_3, \breve{\mathcal{R}})]$

vi) $(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cup_R (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})] = (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})$,

$(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma}) \cap_R (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})] = (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})$

vii) $[(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})^c]^c = (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}), (\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})]^c = \Phi$,

$(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})]^c = \mathrm{A}$

viii) $[(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})]^c = [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})]^c \cup_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})]^c$,

$[(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})]^c = [(\bar{\urcorner}_1, \bar{\S}_1, \breve{\wp})]^c \cap_R [(\bar{\urcorner}_2, \bar{\S}_2, \breve{\gamma})]^c$.

Proof: Straightforward.

2.2. Algebraic Structures Associated with T-BSSs.

We will talk about few algebraic structures related to T-BSSs in this segment. Remember that the collection of all T-BSSs over U is represented by $(T - BSS)_{(U)}$.

And, the collection of all T-BSSs over U with domain A is represented by $(T-BSS)_{(U)}^{A}$.

2.2.1 Proposition:

For each $\Delta \in \{\cap_E, \cup_E, \cap_R, \cup_R\}$, $\left((T-BSS)_{(U)}, \Delta\right)$, whose every element is idempotent is a commutative SG.

Proof:

Using Proposition 2.1.9 and Proposition 2.1.13, the proof of the above theorem is simple and straightforward.

2.2.2 Proposition:

$\left((T-BSS)_{(U)}, \cap_R, \cup_R\right)$ is a commutative SG.

Proof:

By using the definitions of restricted intersection of T-BSSs, restricted union of T-BSSs and parts (iv) and (v) of Proposition 2.1.13, the proof of above theorem is simple and straightforward.

2.2.3 Remarks:

Note that, $\left((T-BSS)_{(U)}, \cap_E, \cup_E\right)$ is not a semiring (SR) as suggested by Proposition 2.1.9 and Remark 2.1.10.

2.2.4 Proposition:

$\left((T-BSS)_{(U)}^{A}, \cap_E, \cup_E\right)$ is a commutative SR.

Proof: Straightforward.

2.2.5 Proposition:

$\left((T-BSS)_{(U)}, \cap_E, \cup_E, {}^{c}, \Phi, \mathrm{A}\right)$ is a bounded lattice.

Proof: By using conditions (i) $-$ (v) of Proposition 2.1.9, the result follows above statement.

2.2.6 Proposition:

$\left((T - BSS)_{(U)}, \cap_R, \cup_R, {}^c, \Phi, \mathrm{A}\right)$ is a bounded distributive lattice.

Proof: By using Proposition 2.1.13, the result follows the above statement.

Chapter 03

T-Bipolar Soft Semigroups and Related Results

In the following chapter, we talk about the notion of the T-bipolar soft semigroup (T-BSSG) and explore the properties of the T-BSSG such as AND and OR product, Res-Union and Res-Intersection, Ext-Union and Ext- Intersection. In this chapter, the following SGs are used $G = \{ ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \}$, $H = \{ ắ, ɓ, c, \mathfrak{v}, y, z \}$ and $I = \{ ắ, ɓ, c, d, ȩ \}$ which are devised as follows respectively;

$\cdot$	ȩ	$\mathfrak{v}_1$	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$
ȩ	ȩ	ȩ	ȩ	ȩ	ȩ
$\mathfrak{v}_1$	ȩ	ȩ	ȩ	ȩ	ȩ
$\mathfrak{v}_2$	ȩ	ȩ	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$
$\mathfrak{v}_3$	ȩ	ȩ	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$
$\mathfrak{v}_4$	ȩ	ȩ	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$

$\cdot$	ắ	ɓ	c	$\mathfrak{v}$	y	z
ắ	ắ	ắ	ắ	ắ	ắ	ắ
ɓ	ắ	ắ	ắ	ắ	ắ	ắ
c	ắ	ắ	ắ	ắ	ắ	ắ
$\mathfrak{v}$	ắ	ắ	ắ	ắ	ắ	ɓ
y	ắ	ắ	ắ	ắ	ɓ	c
z	ắ	ắ	ɓ	ắ	$\mathfrak{v}$	ắ

and,

.	ắ	ɓ	$\boldsymbol{c}$	$\boldsymbol{d}$	ȩ
ắ	ắ	ắ	ắ	ắ	ắ
ɓ	ắ	ắ	ắ	ɓ	c
$\boldsymbol{c}$	ắ	ɓ	c	ắ	ắ
$\boldsymbol{d}$	ắ	ắ	ắ	d	ȩ
ȩ	ắ	d	ȩ	ắ	ắ

3.1. T-Bipolar Soft Semigroups

The fundamental definition of the T-BSSG will be introduced in this section, along with several fundamental operational laws, including extended union and intersection, restricted union, and intersection, AND and OR product.

3.1.1 Definition:

Let G and H be two distinct SGs, $U = G \cup H$. Then for any set $\breve{\wp}$, a T-BSS $< \bar{7}, \bar{\S}, \breve{\wp} >$ is said to be a T-BSSG iff $\bar{7}(\lambda) \leq G$ and $\bar{\S}(\lambda) \leq H \ \ \forall \ \lambda \in \ \breve{\wp}$.

3.1.2 Example:

Let us assume,

$G = \{$ ȩ$, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \}$, $H = \{$ ắ, ɓ$, c, \mathfrak{v}, y, z \}$, $U = G \cup H$, $\breve{\wp} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$.

$\bar{7}(\lambda_1) = \{$ȩ$\}$; $\bar{7}(\lambda_2) = \{$ȩ$, \mathfrak{v}_2, \mathfrak{v}_3\}$; $\bar{7}(\lambda_3) = \{$ȩ$, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \}$; $\bar{7}(\lambda_4) = \{\mathfrak{v}_3, \mathfrak{v}_4\}$.

$\bar{\S}(\lambda_1) = \{$ắ$, z\}$; $\bar{\S}(\lambda_2) = \{$ắ, ɓ$, c, \mathfrak{v}\}$; $\bar{\S}(\lambda_3) = \{$ắ, ɓ$, c, \mathfrak{v}, y, z\}$;

$\bar{\S}(\lambda_4) = \{$ắ, ɓ$, c, \mathfrak{v}, y\}$.

$(\bar{7}, \bar{\S}, \breve{\wp}) = \{< \lambda_1, \{$ȩ$\}, \{$ắ$, z\} >, < \lambda_2, \{$ȩ$, \mathfrak{v}_2, \mathfrak{v}_3\}, \{$ắ, ɓ$, c, \mathfrak{v}\} >,$

$< \lambda_3, \{$ȩ$, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \}, \{$ắ, ɓ$, c, \mathfrak{v}, y, z\} >, < \lambda_4, \{\mathfrak{v}_3, \mathfrak{v}_4\}, \{$ắ, ɓ$, c, \mathfrak{v}, y\} >\}$.

3.1.3 Definition:

Let, $(\bar{\text{٦}}_1, \bar{\S}_1, \breve{\wp}), (\bar{\text{٦}}_2, \bar{\S}_2, \breve{\text{Ɣ}}) \in$ T-BSSG over U. Then, $(\bar{\text{٦}}_1, \bar{\S}_1, \breve{\wp})$ is known to be a T-BSSG subset of $(\bar{\text{٦}}_2, \bar{\S}_2, \breve{\text{Ɣ}})$. If,

i) $\breve{\wp} \subseteq \breve{\text{Ɣ}}$.

ii) $\forall\, \text{ƛ} \in \breve{\wp},\ \bar{\text{٦}}_1(\text{ƛ}) \subseteq \bar{\text{٦}}_2(\text{ƛ})\,, \bar{\S}_1(\text{ƛ}) \supseteq \bar{\S}_2(\text{ƛ})$.

3.1.4 Example:

We suppose that the

$$G = \{\, \text{ȩ}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \,\}, \quad H = \{\, \text{ǻ}, \text{ɓ}, c, \mathfrak{v}, y, z \,\}, \quad U = G \cup H\,, \breve{\wp} = \{\text{ƛ}_1, \text{ƛ}_2\}$$

And

$$\breve{\text{Ɣ}} = \{\text{ƛ}_1, \text{ƛ}_2, \text{ƛ}_3\}, \qquad \bar{\text{٦}}_1(\text{ƛ}_1) = \{\text{ȩ}\}, \qquad \bar{\text{٦}}_1(\text{ƛ}_2) = \{\text{ȩ}, \mathfrak{v}_2, \mathfrak{v}_3\}\,;$$

$$\bar{\text{٦}}_2(\text{ƛ}_1) = \{\text{ȩ}, \mathfrak{v}_2, \mathfrak{v}_3\}\,;\ \bar{\text{٦}}_2(\text{ƛ}_2) = \{\text{ȩ}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\,\}\,;\ \bar{\text{٦}}_2(\text{ƛ}_3) = \{\mathfrak{v}_3, \mathfrak{v}_4\}.$$

$$\bar{\S}_1(\text{ƛ}_1) = \{\text{ǻ}, \text{ɓ}, c, \mathfrak{v}, y\}\,;\ \bar{\S}_1(\text{ƛ}_2) = \{\text{ǻ}, \text{ɓ}, c, \mathfrak{v}, y, z\},$$

$$\bar{\S}_2(\text{ƛ}_1) = \{\text{ǻ}, \text{ɓ}, c, \mathfrak{v}\}\,;\ \bar{\S}_2(\text{ƛ}_2) = \{\text{ǻ}, z\}\,;\ \bar{\S}_2(\text{ƛ}_3) = \{\text{ǻ}, \text{ɓ}\}.$$

Then,

i) $\breve{\wp} \subseteq \breve{\text{Ɣ}}$.

ii) $\forall\, \text{ƛ} \in \breve{\wp},\ \bar{\text{٦}}_1(\text{ƛ}) \subseteq \bar{\text{٦}}_2(\text{ƛ})\,, \bar{\S}_1(\text{ƛ}) \supseteq \bar{\S}_2(\text{ƛ})$.

3.1.5 Remarks:

Generally, if $\breve{\wp} \subseteq \breve{\text{Ɣ}}$, it is not necessarily that $(\bar{\text{٦}}_1, \bar{\S}_1, \breve{\wp})$ is known to be a T-BSSG subset of $(\bar{\text{٦}}_2, \bar{\S}_2, \breve{\text{Ɣ}})$. It holds only $\bar{\text{٦}}_1(\text{ƛ}_n) \subseteq \bar{\text{٦}}_2(\text{ƛ}_n)$ and $\bar{\S}_2(\text{ƛ}_n) \subseteq \bar{\S}_1(\text{ƛ}_n)\ \forall\ \text{ƛ}_n \in \breve{\wp}$.

3.1.6 Theorem:

If $\bar{\S}_1(\text{ƛ})$ is a S-SG of $\bar{\S}_2(\text{ƛ})$ or $\bar{\S}_2(\text{ƛ})$ is a S-SG of $\bar{\S}_1(\text{ƛ})\ \forall\ (\text{ƛ}, \text{ƛ}) \in \breve{\wp} \times \breve{\text{Ɣ}}$. Then the AND product of two T-BSSG $(\bar{\text{٦}}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\text{٦}}_2, \bar{\S}_2, \breve{\text{Ɣ}})$ over U is a T-BSSG over U.

Proof:

Assume that $\bar{\S}_1(\lambda)$ is a sub semigroup of $\bar{\S}_2(\lambda)$ or $\bar{\S}_2(\lambda)$ is a S-SG of $\bar{\S}_1(\lambda)$, then in both cases $\bar{\S}_1(\lambda) \cup \bar{\S}_2(\lambda)$ is a sub semigroup $\forall\ (\lambda, \lambda) \in \breve{\wp} \times \breve{\gamma}$.

Now, consider the sub-SG $\bar{\daleth}_1(\lambda) \cap \bar{\daleth}_2(\lambda)\ \forall\ (\lambda, \lambda) \in \breve{\wp} \times \breve{\gamma}$ trivially. It is because, the intersection of any number of the sub-SG is again sub-SG. Hence, in either case the AND product of two T-BSSG over U is a T-BSSG over U.

3.1.7 Example:

Let us assume that the,

$$G = \{ ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \}, I = \{ ǎ́, ɓ, c, d, ȩ \}, \qquad U = G \cup I,$$

$$\bar{\daleth}_1(\lambda_1) = \{ȩ\};\ \bar{\daleth}_1(\lambda_2) = \{ȩ, \mathfrak{v}_2, \mathfrak{v}_3\};$$

$$\bar{\daleth}_2(\lambda_1) = \{ȩ, \mathfrak{v}_2, \mathfrak{v}_3\};\ \bar{\daleth}_2(\lambda_2) = \{ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\};\ \bar{\daleth}_2(\lambda_3) = \{ȩ, \mathfrak{v}_3, \mathfrak{v}_4\}.$$

$$\bar{\S}_1(\lambda_1) = \{ǎ́, d, ȩ\};\ \bar{\S}_1(\lambda_2) = \{ǎ́, ɓ, c\};$$

$$\bar{\S}_2(\lambda_1) = \{ǎ́, ɓ, c, d\};\ \bar{\S}_2(\lambda_2) = \{ǎ́, ɓ, c, d, ȩ\};\ \bar{\S}_2(\lambda_3) = \{ǎ́, ɓ\}.$$

$$\breve{\wp} \times \breve{\gamma} = \{ (\lambda_1, \lambda_1), (\lambda_1, \lambda_2), (\lambda_1, \lambda_3), (\lambda_2, \lambda_1), (\lambda_2, \lambda_2), (\lambda_2, \lambda_3)\},$$

AND the product of

$$(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \wedge (\bar{\daleth}_2, \bar{\S}_2, \breve{\gamma}) = \{< (\lambda, \lambda); \bar{\daleth}_1(\lambda) \cap \bar{\daleth}_2(\lambda)\ ;\ \bar{\S}_1(\lambda) \cup \bar{\S}_2(\lambda) > (\lambda, \lambda) \in \breve{\wp} \times \breve{\gamma}\}$$

then the

$$(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \wedge (\bar{\daleth}_2, \bar{\S}_2, \breve{\gamma}) =$$

$$
= \begin{cases} (\lambda_1, \lambda_1) \Rightarrow \\ \bar{7}_1(\lambda_1) \cap \bar{7}_2(\lambda_1) = \{ȩ\}\ ;\ \bar{§}_1(\lambda_1) \cup \bar{§}_2(\lambda_1) = \{ǎ, ɓ, c, d, ȩ\} \\ (\lambda_1, \lambda_2) \Rightarrow \\ \bar{7}_1(\lambda_1) \cap \bar{7}_2(\lambda_2) = \{ȩ\}\ ;\ \bar{§}_1(\lambda_1) \cup \bar{§}_2(\lambda_2) = \{ǎ, ɓ, c, d, ȩ\} \\ (\lambda_1, \lambda_3) \Rightarrow \\ \bar{7}_1(\lambda_1) \cap \bar{7}_2(\lambda_3) = \{ȩ\}\ ;\ \bar{§}_1(\lambda_1) \cup \bar{§}_2(\lambda_3) = \{ǎ, ɓ, d, ȩ\} \\ (\lambda_2, \lambda_1) \Rightarrow \\ \bar{7}_1(\lambda_2) \cap \bar{7}_2(\lambda_1) = \{ȩ, \mathfrak{v}_2, \mathfrak{v}_3\}\ ;\ \bar{§}_1(\lambda_2) \cup \bar{§}_2(\lambda_1) = \{ǎ, ɓ, c, d\} \\ (\lambda_2, \lambda_2) \Rightarrow \\ \bar{7}_1(\lambda_2) \cap \bar{7}_2(\lambda_2) = \{ȩ, \mathfrak{v}_2, \mathfrak{v}_3\}\ ;\ \bar{§}_1(\lambda_2) \cup \bar{§}_2(\lambda_2) = \{ǎ, ɓ, c, d, ȩ\} \\ (\lambda_2, \lambda_3) \Rightarrow \\ \bar{7}_1(\lambda_2) \cap \bar{7}_2(\lambda_3) = \{ȩ, \mathfrak{v}_3\}\ ;\ \bar{§}_1(\lambda_2) \cup \bar{§}_2(\lambda_3) = \{ǎ, ɓ, c\} \end{cases}
$$

3.1.8 Remarks:

In general, the product of two T-BSSG needs not to be T-BSSG.

3.1.9 Example:

Let $H = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}$ and $I = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{d}, \bar{ȩ}\}$ be two semigroups, then for any sets $\breve{\wp}$ and $\breve{Ɣ}$, and $(\bar{7}_1, \bar{§}_1, \breve{\wp})$ and $(\bar{7}_2, \bar{§}_2, \breve{Ɣ})$ be two T-BSSG over $U = H \cup I$.

Assume $\breve{\wp} = \{\bar{\mathfrak{v}}_1, \bar{y}_1, \bar{z}_1\}$ and $\breve{Ɣ} = \{\bar{ǎ}_1, \bar{ɓ}_1, \bar{c}_1\}$,

$$\bar{7}_1(\bar{\mathfrak{v}}_1) = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{y}\}\ ;\ \bar{7}_1(\bar{y}_1) = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\}\ ;\ \bar{7}_1(\bar{z}_1) = \{\bar{ǎ}, \bar{ɓ}\} ,$$

$$\bar{7}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}\}\ ;\ \bar{7}_2(\bar{ɓ}_1) = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}\ ;\ \bar{7}_2(\bar{c}_1) = \{\bar{ǎ}, \bar{z}\} ,$$

$$\bar{§}_1(\bar{\mathfrak{v}}_1) = \{\bar{ǎ}, \bar{ɓ}, \bar{c}\}\ ;\ \bar{§}_1(\bar{y}_1) = \{\bar{ǎ}, \bar{ɓ}\}\ ;\ \bar{§}_1(\bar{z}_1) = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{d}\} ,$$

$$\bar{§}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{d}, \bar{ȩ}\}\ ;\ \bar{§}_2(\bar{ɓ}_1) = \{\bar{ǎ}, \bar{ɓ}, \bar{c}, \bar{d}\}\ ;\ \bar{§}_2(\bar{c}_1) = \{\bar{ǎ}, \bar{c}, \bar{d}, \bar{ȩ}\}.$$

$$\breve{\wp} \times \breve{Ɣ}$$
$$= \{(\bar{\mathfrak{v}}_1, \bar{ǎ}_1), (\bar{\mathfrak{v}}_1, \bar{ɓ}_1), (\bar{\mathfrak{v}}_1, \bar{c}_1), (\bar{y}_1, \bar{ǎ}_1), (\bar{y}_1, \bar{ɓ}_1), (\bar{y}_1, \bar{c}_1), (\bar{z}_1, \bar{ǎ}_1), (\bar{z}_1, \bar{ɓ}_1), (\bar{z}_1, \bar{c}_1)\}.$$

AND product of the

$$(\bar{7}_1, \bar{§}_1, \breve{\wp})\ \wedge\ (\bar{7}_2, \bar{§}_2, \breve{Ɣ}) = \{< (\lambda, \lambda)\ ;\ \bar{7}_1(\lambda) \cap \bar{7}_2(\lambda)\ ;\ \bar{§}_1(\lambda) \cup \bar{§}_2(\lambda) > (\lambda, \lambda) \in \breve{\wp} \times \breve{Ɣ}\}.$$

$(\bar{ꞁ}_1, \bar{§}_1, \breve{\wp}) \wedge (\bar{ꞁ}_2, \bar{§}_2, \breve{Ɣ}) =$

$$
\begin{cases}
(\bar{ʋ}_1, \bar{ǎ}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{ʋ}_1) \cap \bar{ꞁ}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}\} \quad ; \quad \bar{§}_1(\bar{ʋ}_1) \cup \bar{§}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}, \bar{ȩ}\}, \\
(\bar{ʋ}_1, \bar{б}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{ʋ}_1) \cap \bar{ꞁ}_2(\bar{б}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{y}\} \quad ; \quad \bar{§}_1(\bar{ʋ}_1) \cup \bar{§}_2(\bar{б}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}\}, \\
(\bar{ʋ}_1, \bar{c}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{ʋ}_1) \cap \bar{ꞁ}_2(\bar{c}_1) = \{\bar{ǎ}\} \quad ; \quad \bar{§}_1(\bar{ʋ}_1) \cup \bar{§}_2(\bar{c}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}, \bar{ȩ}\}, \\
(\bar{y}_1, \bar{ǎ}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{y}_1) \cap \bar{ꞁ}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{ʋ}\} \quad ; \quad \bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{б}, \bar{d}, \bar{ȩ}\}, \\
(\bar{y}_1, \bar{б}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{y}_1) \cap \bar{ꞁ}_2(\bar{б}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{ʋ}, \bar{y}\} \quad ; \quad \bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{б}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}\}, \\
(\bar{y}_1, \bar{c}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{y}_1) \cap \bar{ꞁ}_2(\bar{c}_1) = \{\bar{ǎ}\} \quad ; \quad \bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{c}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}, \bar{ȩ}\}, \\
(\bar{z}_1, \bar{ǎ}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{z}_1) \cap \bar{ꞁ}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{б}\} \quad ; \quad \bar{§}_1(\bar{z}_1) \cup \bar{§}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}, \bar{ȩ}\}, \\
(\bar{z}_1, \bar{б}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{z}_1) \cap \bar{ꞁ}_2(\bar{б}_1) = \{\bar{ǎ}, \bar{б}\} \quad ; \quad \bar{§}_1(\bar{z}_1) \cup \bar{§}_2(\bar{б}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}\}, \\
(\bar{z}_1, \bar{c}_1) \Rightarrow \\
\bar{ꞁ}_1(\bar{z}_1) \cap \bar{ꞁ}_2(\bar{c}_1) = \{\bar{ǎ}\} \quad ; \quad \bar{§}_1(\bar{z}_1) \cup \bar{§}_2(\bar{c}_1) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}, \bar{ȩ}\}.
\end{cases}
$$

It is not a T-BSSG because $\bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{ǎ}_1) = \{\bar{ǎ}, \bar{б}, \bar{d}, \bar{ȩ}\}$ is not a sub-semigroup of I.

3.1.10 Theorem:

If $\bar{ꞁ}_1(ƛ)$ is a sub-semigroup of $\bar{ꞁ}_2(ƛ)$ or $\bar{ꞁ}_2(ƛ)$ is a sub-semigroup of $\bar{ꞁ}_1(ƛ)$ $\forall\, (ƛ, ƛ) \in \breve{\wp} \times \breve{Ɣ}$. Then the OR product of two T-BSSG $(\bar{ꞁ}_1, \bar{§}_1, \breve{\wp})$ and $(\bar{ꞁ}_2, \bar{§}_2, \breve{Ɣ})$ over U is a T-BSSG over U.

Proof:

Assume that $\bar{ꞁ}_1(ƛ)$ is a sub-semigroup of $\bar{ꞁ}_2(ƛ)$ or $\bar{ꞁ}_2(ƛ)$ is a sub-SG of $\bar{ꞁ}_1(ƛ)$, then in both cases $\bar{ꞁ}_1(ƛ) \cup \bar{ꞁ}_2(ƛ)$ is a sub-semigroup $(ƛ, ƛ) \in \breve{\wp} \times \breve{Ɣ}$.

Consider, $\bar{§}_1(ƛ) \cap \bar{§}_2(ƛ)$ $\forall\, (ƛ, ƛ) \in \breve{\wp} \times \breve{Ɣ}$ is sub-SG trivially. It is because a sub-SG is formed by the intersection of any number of the sub-SG. Hence, in either case the OR product of two T-BSSG $(\bar{ꞁ}_1, \bar{§}_1, \breve{\wp})$ and $(\bar{ꞁ}_2, \bar{§}_2, \breve{Ɣ})$ over U is a T-BSSG over U.

3.1.11 Example:

$$\text{Let } G = \{\,\text{ȩ}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}\,, \qquad I = \{\text{ắ}, \text{ɓ}, c, d, \text{ȩ}\}\,, \qquad U = G \cup I,$$

$$\breve{\wp} = \{\text{ƛ}_1, \text{ƛ}_2\} \quad \text{and} \quad \breve{\gamma} = \{\text{ƛ}_1, \text{ƛ}_2, \text{ƛ}_3\}$$

$$\bar{\daleth}_1(\text{ƛ}_1) = \{\text{ȩ}, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\daleth}_1(\text{ƛ}_2) = \{\text{ȩ}\}\,;$$

$$\bar{\daleth}_2(\text{ƛ}_1) = \{\mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\daleth}_2(\text{ƛ}_2) = \{\text{ȩ}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\daleth}_2(\text{ƛ}_3) = \{\text{ȩ}, \mathfrak{v}_2, \mathfrak{v}_3\}.$$

$$\bar{\S}_1(\text{ƛ}_1) = \{\text{ắ}, d, \text{ȩ}\}\,;\ \bar{\S}_1(\text{ƛ}_2) = \{\text{ắ}, \text{ɓ}, c, d, \text{ȩ}\}\,;$$

$$\bar{\S}_2(\text{ƛ}_1) = \{\text{ắ}, \text{ɓ}, c, d\}\,;\ \bar{\S}_2(\text{ƛ}_2) = \{\text{ắ}, \text{ɓ}, c\}\,;\ \bar{\S}_2(\ \text{ƛ}_3) = \{\text{ắ}, \text{ɓ}\}.$$

$$\breve{\wp} \times \breve{\gamma} = \{(\text{ƛ}_1, \text{ƛ}_1), (\text{ƛ}_1, \text{ƛ}_2), (\text{ƛ}_1, \text{ƛ}_3), (\text{ƛ}_2, \text{ƛ}_1), (\text{ƛ}_2, \text{ƛ}_2), (\text{ƛ}_2, \text{ƛ}_3)\}.$$

OR product of $\left(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}\right) \vee \left(\bar{\daleth}_2, \bar{\S}_2, \breve{\gamma}\right) = \left\{< (\text{ƛ}, \text{ƛ})\,;\ \bar{\daleth}_1(\text{ƛ}) \cup \bar{\daleth}_2(\text{ƛ})\,;\ \bar{\S}_1(\text{ƛ}) \cap \bar{\S}_2(\text{ƛ}) > (\text{ƛ}, \text{ƛ}) \in \breve{\wp} \times \breve{\gamma}\right\}.$

$$\left(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}\right) \vee \left(\bar{\daleth}_2, \bar{\S}_2, \breve{\gamma}\right)$$

$$= \left\{ \begin{array}{c} (\text{ƛ}_1, \text{ƛ}_1) \Rightarrow \\ \bar{\daleth}_1(\text{ƛ}_1) \cup \bar{\daleth}_2(\text{ƛ}_1) = \{\text{ȩ}, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\S}_1(\text{ƛ}_1) \cap \bar{\S}_2(\text{ƛ}_1) = \{\text{ắ}, d\}, \\ (\text{ƛ}_1, \text{ƛ}_2) \Rightarrow \\ \bar{\daleth}_1(\text{ƛ}_1) \cup \bar{\daleth}_2(\text{ƛ}_2) = \{\text{ȩ}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\S}_1(\text{ƛ}_1) \cap \bar{\S}_2(\text{ƛ}_2) = \{\text{ắ}\}, \\ (\text{ƛ}_1, \text{ƛ}_3) \Rightarrow \\ \bar{\daleth}_1(\text{ƛ}_1) \cup \bar{\daleth}_2(\text{ƛ}_3) = \{\text{ȩ}, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\S}_1(\text{ƛ}_1) \cap \bar{\S}_2(\text{ƛ}_3) = \{\text{ắ}\}, \\ (\text{ƛ}_2, \text{ƛ}_1) \Rightarrow \\ \bar{\daleth}_1(\text{ƛ}_2) \cup \bar{\daleth}_2(\text{ƛ}_1) = \{\text{ȩ}, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\S}_1(\text{ƛ}_2) \cap \bar{\S}_2(\text{ƛ}_1) = \{\text{ắ}, \text{ɓ}, c, \}, \\ (\text{ƛ}_2, \text{ƛ}_2) \Rightarrow \\ \bar{\daleth}_1(\text{ƛ}_2) \cup \bar{\daleth}_2(\text{ƛ}_2) = \{\text{ȩ}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\S}_1(\text{ƛ}_2) \cap \bar{\S}_2(\text{ƛ}_2) = \{\text{ắ}, \text{ɓ}, c\}, \\ (\text{ƛ}_2, \text{ƛ}_3) \Rightarrow \\ \bar{\daleth}_1(\text{ƛ}_2) \cup \bar{\daleth}_2(\text{ƛ}_3) = \{\text{ȩ}, \mathfrak{v}_2, \mathfrak{v}_3\}\,;\ \bar{\S}_1(\text{ƛ}_2) \cap \bar{\S}_2(\text{ƛ}_3) = \{\text{ắ}, \text{ɓ}\}. \end{array} \right.$$

3.1.12 Remarks:

Typically, the OR product of two T-BSSG does not have to be T-BSSG.

3.1.13 Example:

Let $H = \left\{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\right\}$ and $I = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}$ be two semigroups, then for any sets $\breve{\wp}$ and $\breve{\gamma}$, and $\left(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}\right)$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\gamma})$ be two T-BSSG over $U = H \cup I$.

Let us assume,

$$\breve{\wp} = \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}\ \text{ắ}nd\ \breve{\text{Ɣ}} = \{\bar{\text{ắ}}_1, \bar{\text{б}}_1, \bar{c}_1\},$$

$$\bar{\daleth}_1(\bar{\alpha}) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{y}\}\,;\ \bar{\daleth}_1(\bar{\beta}) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\}\,;\ \bar{\daleth}_1(\bar{\gamma}) = \{\bar{\text{ắ}}, \bar{\text{б}}\},$$

$$\bar{\daleth}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}\}\,;\ \bar{\daleth}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}\,;\ \bar{\daleth}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{z}\},$$

$$\bar{\S}_1(\bar{\alpha}) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}\}\,;\ \bar{\S}_1(\bar{\beta}) = \{\bar{\text{ắ}}, \bar{\text{б}}\}\,;\ \bar{\S}_1(\bar{\gamma}) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{d}\},$$

$$\bar{\S}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}, \bar{d}, \bar{\text{ẹ}}\}\,;\ \bar{\S}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{d}\}\,;\ \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{c}, \bar{d}, \bar{\text{ẹ}}\}.$$

$$\breve{\wp} \times \breve{\text{Ɣ}} = \{(\bar{\alpha}, \bar{\text{ắ}}_1), (\bar{\alpha}, \bar{\text{б}}_1), (\bar{\alpha}, \bar{c}_1), (\bar{\beta}, \bar{\text{ắ}}_1), (\bar{\beta}, \bar{\text{б}}_1), (\bar{\beta}, \bar{c}_1), (\bar{\gamma}, \bar{\text{ắ}}_1), (\bar{\gamma}, \bar{\text{б}}_1), (\bar{\gamma}, \bar{c}_1)\}.$$

OR product of $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \vee (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}}) = \{< (\text{ƛ}, \text{ƛ})\,;\ \bar{\daleth}_1(\text{ƛ}) \cup \bar{\daleth}_2(\text{ƛ})\ ;\ \bar{\S}_1(\text{ƛ}) \cap \bar{\S}_2(\text{ƛ}) > (\text{ƛ}, \text{ƛ}) \in \breve{\wp} \times \breve{\text{Ɣ}}\}$.

$$(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \vee (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}}) =$$

$$\left\{\begin{array}{c}
(\bar{\alpha}, \bar{\text{ắ}}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\alpha}) \cup \bar{\daleth}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\} \quad ; \quad \bar{\S}_1(\bar{\alpha}) \cap \bar{\S}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}\}, \\
(\bar{\alpha}, \bar{\text{б}}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\alpha}) \cup \bar{\daleth}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\} \quad ; \quad \bar{\S}_1(\bar{\alpha}) \cap \bar{\S}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}\}, \\
(\bar{\alpha}, \bar{c}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\alpha}) \cup \bar{\daleth}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{y}, \bar{z}\} \quad ; \quad \bar{\S}_1(\bar{\alpha}) \cap \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{c}\}, \\
(\bar{\beta}, \bar{\text{ắ}}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\beta}) \cup \bar{\daleth}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\} \quad ; \quad \bar{\S}_1(\bar{\beta}) \cap \bar{\S}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}\}, \\
(\bar{\beta}, \bar{\text{б}}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\beta}) \cup \bar{\daleth}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\} \quad ; \quad \bar{\S}_1(\bar{\beta}) \cap \bar{\S}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}\}, \\
(\bar{\beta}, \bar{c}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\beta}) \cup \bar{\daleth}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\} \quad ; \quad \bar{\S}_1(\bar{\beta}) \cap \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ắ}}\}, \\
(\bar{\gamma}, \bar{\text{ắ}}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\gamma}) \cup \bar{\daleth}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}\} \quad ; \quad \bar{\S}_1(\bar{\gamma}) \cap \bar{\S}_2(\bar{\text{ắ}}_1) = \{\bar{\text{ắ}}, \bar{d}\}, \\
(\bar{\gamma}, \bar{\text{б}}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\gamma}) \cup \bar{\daleth}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\} \quad ; \quad \bar{\S}_1(\bar{\gamma}) \cap \bar{\S}_2(\bar{\text{б}}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{d}\}, \\
(\bar{\gamma}, \bar{c}_1) \Rightarrow \\
\bar{\daleth}_1(\bar{\gamma}) \cup \bar{\daleth}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{z}\} \quad ; \quad \bar{\S}_1(\bar{\gamma}) \cap \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{c}, \bar{d}\}.
\end{array}\right.$$

It is not a T-BSSG because $\bar{\daleth}_1(\bar{\alpha}) \cup \bar{\daleth}_2(\bar{c}_1) = \{\bar{\text{ắ}}, \bar{\text{б}}, \bar{c}, \bar{y}, \bar{z}\}$ is not a sub-semigroup of H.

3.1.14 Theorem:

If $\bar{ℸ}_1(ƛ)$ is a sub-semigroup of $\bar{ℸ}_2(ƛ)$ or $\bar{ℸ}_2(ƛ)$ is a sub-semigroup of $\bar{ℸ}_1(ƛ)$ $\forall\, ƛ \in \breve{\wp} \cup \breve{Ɣ}$. Then, the Extended union of two T-BSSG $(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ})$ over U is a T-BSSG over U.

Proof:

Assume that $\bar{ℸ}_1(ƛ)$ is a sub-semigroup of $\bar{ℸ}_2(ƛ)$ or $\bar{ℸ}_2(ƛ)$ is a sub-semigroup of $\bar{ℸ}_1(ƛ)$, then in both cases $\bar{ℸ}_1(ƛ) \cup \bar{ℸ}_2(ƛ)$ is a sub-semigroup $\forall\, ƛ \in \breve{\wp} \cup \breve{Ɣ}$.

If $ƛ \in \breve{\wp} - \breve{Ɣ}$ or $ƛ \in \breve{Ɣ} - \breve{\wp}$, then it is a trivial case.

Now, consider $\bar{\S}_1(ƛ) \cap \bar{\S}_2(ƛ)$ $\forall\, ƛ \in \breve{\wp} \cup \breve{Ɣ}$ is a sub-SG trivially. It is because a sub-SG is formed by the intersection of any number of the sub-SG. Hence, in either case the Ext-U of two T-BSSG $(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ})$ over U is a T-BSSG over U.

3.1.15 Example:

$$\text{Say, } G = \{ ȩ, ɒ_1, ɒ_2, ɒ_3, ɒ_4 \}, I = \{ ǻ, ɓ, c, d, ȩ \},\ U = G \cup I,$$

$$\breve{\wp} = \{ƛ_1, ƛ_2\} \quad \text{and} \quad \breve{Ɣ} = \{ƛ_1, ƛ_2, ƛ_3\},$$

$$\bar{ℸ}_1(ƛ_1) = \{ȩ, ɒ_2, ɒ_3\};\ \bar{ℸ}_1(ƛ_2) = \{ȩ\};$$

$$\bar{ℸ}_2(ƛ_1) = \{ȩ, ɒ_1, ɒ_2, ɒ_3, ɒ_4\};\ \bar{ℸ}_2(ƛ_2) = \{ɒ_3, ɒ_4\};\ \bar{ℸ}_2(ƛ_3) = \{ȩ, ɒ_3, ɒ_4\};$$

$$\bar{\S}_1(ƛ_1) = \{ǻ, c, d, ȩ\};\ \bar{\S}_1(ƛ_2) = \{ǻ, ɓ, c, d, ȩ\};$$

$$\bar{\S}_2(ƛ_1) = \{ǻ, d, ȩ\};\ \bar{\S}_2(ƛ_2) = \{ǻ, c, d\};\ \bar{\S}_2(ƛ_3) = \{ǻ, ɓ, c\}.$$

Then the Ext-Union of $(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ})$ is denoted and defined by $(\bar{ℸ}_1, \bar{\S}_1, \breve{\wp}) \cup_E (\bar{ℸ}_2, \bar{\S}_2, \breve{Ɣ}) = (\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}} = \breve{\wp} \cup \breve{Ɣ}$.

$$\bar{ℸ}_3(ƛ) = \begin{cases} \bar{ℸ}_1(ƛ) & ;\text{if } ƛ \in \breve{\wp} - \breve{Ɣ}, \\ \bar{ℸ}_2(ƛ) & ;\text{if } ƛ \in \breve{Ɣ} - \breve{\wp}, \\ \bar{ℸ}_1(ƛ) \cup \bar{ℸ}_2(ƛ) & ;\text{if } ƛ \in \breve{\wp} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{\S}_3(ƛ) = \begin{cases} \bar{\S}_1(ƛ) & ;\text{if } ƛ \in \breve{\wp} - \breve{Ɣ}, \\ \bar{\S}_2(ƛ) & ;\text{if } ƛ \in \breve{Ɣ} - \breve{\wp}, \\ \bar{\S}_1(ƛ) \cap \bar{\S}_2(ƛ) & ;\text{if } ƛ \in \breve{\wp} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{⁊}_3(ƛ_1) = \bar{⁊}_1(ƛ_1) \cup \bar{⁊}_2(ƛ_1) = \{ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ ƛ_1 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{⁊}_3(ƛ_2) = \bar{⁊}_1(ƛ_2) \cup \bar{⁊}_2(ƛ_2) = \{ȩ, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ ƛ_2 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{⁊}_3(ƛ_3) = \bar{⁊}_2(ƛ_3) = \{ȩ, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ ƛ_3 \in \breve{Ɣ} - \breve{\wp}\,,$$

$$\bar{\S}_3(ƛ_1) = \bar{\S}_1(ƛ_1) \cap \bar{\S}_2(ƛ_1) = \{å, d, ȩ\}\,;\ ƛ_1 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{\S}_3(ƛ_2) = \bar{\S}_1(ƛ_2) \cap \bar{\S}_2(ƛ_2) = \{å, c, d\}\,;\ ƛ_2 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{\S}_3(ƛ_3) = \bar{\S}_2(ƛ_3) = \{å, ɓ, c\}\,;\ ƛ_3 \in \breve{Ɣ} - \breve{\wp}.$$

3.1.16 Remarks:

It is noted that the Ext-Union of two T-BSSG is not a T-BSSG.

3.1.17 Example:

Consider, $H = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}$ and $I = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{d}, \bar{ȩ}\}$ be two semigroups, then for any sets $\breve{\wp}$ and $\breve{Ɣ}$, and $(\bar{⁊}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{⁊}_2, \bar{\S}_2, \breve{Ɣ})$ be two T-BSSG over $U = H \cup I$.

Now assume,

$\breve{\wp} = \{\alpha, \beta, \gamma\}$ and $\breve{Ɣ} = \{\alpha, \beta, \eta\}$

$$\bar{⁊}_1(\alpha) = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{y}\}\,;\ \bar{⁊}_1(\beta) = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\}\,;\ \bar{⁊}_1(\gamma) = \{\bar{å}, \bar{ɓ}\}\,,$$

$$\bar{⁊}_2(\alpha) = \{\bar{å}, \bar{z}\}\,;\ \bar{⁊}_2(\beta) = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}\,;\ \bar{⁊}_2(\eta) = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}\}\,,$$

$$\bar{\S}_1(\alpha) = \{\bar{å}, \bar{ɓ}, \bar{c}\}\,;\ \bar{\S}_1(\beta) = \{\bar{å}, \bar{ɓ}\}\,;\ \bar{\S}_1(\gamma) = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{d}\}\,,$$

$$\bar{\S}_2(\alpha) = \{\bar{å}, \bar{d}, \bar{ȩ}\}\,;\ \bar{\S}_2(\beta) = \{\bar{å}, \bar{ɓ}, \bar{c}, \bar{d}\}\,;\ \bar{\S}_2(\eta) = \{\bar{å}, \bar{c}, \bar{d}, \bar{ȩ}\}.$$

Then the Ext-Union of $(\bar{⁊}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{⁊}_2, \bar{\S}_2, \breve{Ɣ})$ is denoted and given by $(\bar{⁊}_1, \bar{\S}_1, \breve{\wp}) \cup_E (\bar{⁊}_2, \bar{\S}_2, \breve{Ɣ}) = (\bar{⁊}_3, \bar{\S}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}} = \breve{\wp} \cup \breve{Ɣ}$.

$$\bar{⁊}_3(ƛ) = \begin{cases} \bar{⁊}_1(ƛ) & ;\text{if } ƛ \in \breve{\wp} - \breve{Ɣ}, \\ \bar{⁊}_2(ƛ) & ;\text{if } ƛ \in \breve{Ɣ} - \breve{\wp}, \\ \bar{⁊}_1(ƛ) \cup \bar{⁊}_2(ƛ) & ;\text{if } ƛ \in \breve{\wp} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{\S}_3(ƛ) = \begin{cases} \bar{\S}_1(ƛ) & ;\text{if } ƛ \in \breve{\wp} - \breve{Ɣ}, \\ \bar{\S}_2(ƛ) & ;\text{if } ƛ \in \breve{Ɣ} - \breve{\wp}, \\ \bar{\S}_1(ƛ) \cap \bar{\S}_2(ƛ) & ;\text{if } ƛ \in \breve{\wp} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{ך}_3(\alpha) = \bar{ך}_1(\alpha) \cup \bar{ך}_2(\alpha) = \{\bar{\check{å}}, \bar{ɓ}, \bar{c}, \bar{y}, \bar{z}\};\ \alpha \in \breve{\wp} \cap \breve{Ɣ},$$

$$\bar{ך}_3(\beta) = \bar{ך}_1(\beta) \cup \bar{ך}_2(\beta) = \{\bar{\check{å}}, \bar{ɓ}, \bar{c}, \bar{ɒ}, \bar{y}, \bar{z}\};\ \beta \in \breve{\wp} \cap \breve{Ɣ},$$

$$\bar{ך}_3(\gamma) = \bar{ך}_1(\gamma) = \{\bar{\check{å}}, \bar{ɓ}\};\ \gamma \in \breve{\wp} - \breve{Ɣ},$$

$$\bar{ך}_3(\eta) = \bar{ך}_2(\eta) = \{\bar{\check{å}}, \bar{ɓ}, \bar{c}, \bar{ɒ}\};\ \eta \in \breve{Ɣ} - \breve{\wp},$$

$$\bar{§}_3(\alpha) = \bar{§}_1(\alpha) \cap \bar{§}_2(\alpha) = \{\bar{\check{å}}\};\ \alpha \in \breve{\wp} \cap \breve{Ɣ},$$

$$\bar{§}_3(\beta) = \bar{§}_1(\beta) \cap \bar{§}_2(\beta) = \{\bar{\check{å}}, \bar{ɓ}\};\ \beta \in \breve{\wp} \cap \breve{Ɣ},$$

$$\bar{§}_3(\gamma) = \bar{§}_1(\gamma) = \{\bar{\check{å}}, \bar{ɓ}, \bar{c}, \bar{d}\};\ \gamma \in \breve{\wp} - \breve{Ɣ},$$

$$\bar{§}_3(\eta) = \bar{§}_2(\eta) = \{\bar{\check{å}}, \bar{c}, \bar{d}, \bar{ȩ}\};\ \eta \in \breve{Ɣ} - \breve{\wp}.$$

It is not a T-BSSG because,

$\bar{ך}_3(\alpha) = \bar{ך}_1(\alpha) \cup \bar{ך}_2(\alpha) = \{\bar{\check{å}}, \bar{ɓ}, \bar{c}, \bar{y}, \bar{z}\};\ \alpha \in \breve{\wp} \cap \breve{Ɣ}$, is not a sub-semigroup of H.

3.1.18 Theorem:

If $\bar{§}_1(ƛ)$ is a sub-semigroup of $\bar{§}_2(ƛ)$ or $\bar{§}_2(ƛ)$ is a sub-semigroup of $\bar{§}_1(ƛ)$ $\forall\, ƛ \in \breve{\wp} \cup \breve{Ɣ}$. Then the Ext-Intersection of two T-BSSG $(\bar{ך}_1, \bar{§}_1, \breve{\wp})$ and $(\bar{ך}_2, \bar{§}_2, \breve{Ɣ})$ over U is a T-BSSG over U.

Proof:

Suppose that, if $\bar{§}_1(ƛ)$ is a sub-semigroup of $\bar{§}_2(ƛ)$ or $\bar{§}_2(ƛ)$ is a sub-semigroup of $\bar{§}_1(ƛ)$, then in both cases $\bar{§}_1(ƛ) \cup \bar{§}_2(ƛ)$ is a sub-semigroup $\forall\, ƛ \in \breve{\wp} \cup \breve{Ɣ}$. If $ƛ \in \breve{\wp} - \breve{Ɣ}$ or $ƛ \in \breve{Ɣ} - \breve{\wp}$, then it became a trivial one. Next, we consider $\bar{ך}_1(ƛ) \cap \bar{ך}_2(ƛ)$ $\forall\, ƛ \in \breve{\wp} \cup \breve{Ɣ}$ is a sub-SG trivially. It is because a sub-SG is formed by the intersection of any number of the sub-SG. Thus, in either case the Ext-Intersection of two T-BSSG $(\bar{ך}_1, \bar{§}_1, \breve{\wp})$ and $(\bar{ך}_2, \bar{§}_2, \breve{Ɣ})$ over U is a T-BSSG over U.

3.1.19 Example:

Let, $G = \{ȩ, ɒ_1, ɒ_2, ɒ_3, ɒ_4\}$, $I = \{\check{å}, ɓ, c, d, ȩ\}$, $U = G \cup I$,

$\breve{\wp} = \{ƛ_1, ƛ_2\}$ and $\breve{Ɣ} = \{ƛ_1, ƛ_2, ƛ_3\}$,

$$\bar{\daleth}_1(ƛ_1) = \{\mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\daleth}_1(ƛ_2) = \{ȩ\}\,;$$

$$\bar{\daleth}_2(ƛ_1) = \{ȩ, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ \bar{\daleth}_2(ƛ_2) = \{ȩ, \mathfrak{v}_2, \mathfrak{v}_3\}\,;\ \bar{\daleth}_2(ƛ_3) = \{ȩ, \mathfrak{v}_3, \mathfrak{v}_4\}\,;$$

$$\bar{\S}_1(ƛ_1) = \{ắ, c, d, ȩ\}\,;\ \bar{\S}_1(ƛ_2) = \{ắ, ɓ, c, d, ȩ\}\,;$$

$$\bar{\S}_2(ƛ_1) = \{ắ, d, ȩ\}\,;\ \bar{\S}_2(ƛ_2) = \{ắ, c, d\}\,;\ \bar{\S}_2(ƛ_3) = \{ắ, ɓ, c\}.$$

Then, the Ext-Intersection of $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})$ is denoted and given by $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E (\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ}) = (\bar{\daleth}_3, \bar{\S}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}} = \breve{\wp} \cup \breve{Ɣ}$.

$$\bar{\daleth}_3(ƛ) = \begin{cases} \bar{\daleth}_1(ƛ) & ;\text{if } ƛ \in \breve{\wp} - \breve{Ɣ}, \\ \bar{\daleth}_2(ƛ) & ;\text{if } ƛ \in \breve{Ɣ} - \breve{\wp}, \\ \bar{\daleth}_1(ƛ) \cap \bar{\daleth}_2(ƛ) & ;\text{if } ƛ \in \breve{\wp} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{\S}_3(ƛ) = \begin{cases} \bar{\S}_1(ƛ) & ;\text{if } ƛ \in \breve{\wp} - \breve{Ɣ}, \\ \bar{\S}_2(ƛ) & ;\text{if } ƛ \in \breve{Ɣ} - \breve{\wp}, \\ \bar{\S}_1(ƛ) \cup \bar{\S}_2(ƛ) & ;\text{if } ƛ \in \breve{\wp} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{\daleth}_3(ƛ_1) = \bar{\daleth}_1(ƛ_1) \cap \bar{\daleth}_2(ƛ_1) = \{\mathfrak{v}_3, \mathfrak{v}_4\}\,;\ ƛ_1 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{\daleth}_3(ƛ_2) = \bar{\daleth}_1(ƛ_2) \cap \bar{\daleth}_2(ƛ_2) = \{ȩ\}\,;\ ƛ_2 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{\daleth}_3(ƛ_3) = \bar{\daleth}_2(ƛ_3) = \{ȩ, \mathfrak{v}_3, \mathfrak{v}_4\}\,;\ ƛ_3 \in \breve{Ɣ} - \breve{\wp}\,,$$

$$\bar{\S}_3(ƛ_1) = \bar{\S}_1(ƛ_1) \cup \bar{\S}_2(ƛ_1) = \{ắ, c, d, ȩ\}\,;\ ƛ_1 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{\S}_3(ƛ_2) = \bar{\S}_1(ƛ_2) \cup \bar{\S}_2(ƛ_2) = \{ắ, ɓ, c, d, ȩ\}\,;\ ƛ_2 \in \breve{\wp} \cap \breve{Ɣ}\,,$$

$$\bar{\S}_3(ƛ_3) = \bar{\S}_2(ƛ_3) = \{ắ, ɓ, c\}\,;\ ƛ_3 \in \breve{Ɣ} - \breve{\wp}.$$

3.1.20 Remarks:

It is noted that the Ext-Intersection of two T-BSSG is not a T-BSSG.

3.1.21 Example:

Consider, $H = \{\bar{ắ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}$ and $I = \{\bar{ắ}, \bar{ɓ}, \bar{c}, \bar{d}, \bar{ȩ}\}$ be two semigroups, then for any sets $\breve{\wp}$ and $\breve{Ɣ}$, and $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{Ɣ})$ be two T-BSSG over $U = H \cup I$.

Assume $\breve{\wp} = \{\alpha, \beta, \gamma\}$ and $\breve{Ɣ} = \{\alpha, \beta, \eta\}$.

$$\bar{\daleth}_1(\alpha) = \{\bar{ắ}, \bar{ɓ}, \bar{c}, \bar{y}\}\,;\ \bar{\daleth}_1(\beta) = \{\bar{ắ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\}\,;\ \bar{\daleth}_1(\gamma) = \{\bar{ắ}, \bar{ɓ}\}\,,$$

$$\bar{\daleth}_2(\alpha) = \{\bar{ắ}, \bar{z}\}\,;\ \bar{\daleth}_2(\beta) = \{\bar{ắ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}\,;\ \bar{\daleth}_2(\eta) = \{\bar{ắ}, \bar{ɓ}, \bar{c}, \bar{\mathfrak{v}}\}\,,$$

$$\bar{\S}_1(\alpha) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}\}\,;\ \bar{\S}_1(\beta) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{c}\}\,;\ \bar{\S}_1(\gamma) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}\,,$$

$$\bar{\S}_2(\alpha) = \{\bar{\mathring{a}}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \bar{\S}_2(\beta) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}\,;\ \bar{\S}_2(\eta) = \{\bar{\mathring{a}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}.$$

Then, the Ext-Intersection of $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}})$ is denoted and given by $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cap_E (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}}) = (\bar{\daleth}_3, \bar{\S}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}} = \breve{\wp} \cup \breve{\text{Ɣ}}$.

$$\bar{\daleth}_3(\lambdabar) = \begin{cases} \bar{\daleth}_1(\lambdabar) & ;\text{if } \lambdabar \in \breve{\wp} - \breve{\text{Ɣ}}, \\ \bar{\daleth}_2(\lambdabar) & ;\text{if } \lambdabar \in \breve{\text{Ɣ}} - \breve{\wp}, \\ \bar{\daleth}_1(\lambdabar) \cap \bar{\daleth}_2(\lambdabar) & ;\text{if } \lambdabar \in \breve{\wp} \cap \breve{\text{Ɣ}}. \end{cases}$$

$$\bar{\S}_3(\lambdabar) = \begin{cases} \bar{\S}_1(\lambdabar) & ;\text{if } \lambdabar \in \breve{\wp} - \breve{\text{Ɣ}}, \\ \bar{\S}_2(\lambdabar) & ;\text{if } \lambdabar \in \breve{\text{Ɣ}} - \breve{\wp}, \\ \bar{\S}_1(\lambdabar) \cup \bar{\S}_2(\lambdabar) & ;\text{if } \lambdabar \in \breve{\wp} \cap \breve{\text{Ɣ}}. \end{cases}$$

$$\bar{\daleth}_3(\alpha) = \bar{\daleth}_1(\alpha) \cap \bar{\daleth}_2(\alpha) = \{\bar{\mathring{a}}\}\,;\ \alpha \in \breve{\wp} \cap \breve{\text{Ɣ}}\,,$$

$$\bar{\daleth}_3(\beta) = \bar{\daleth}_1(\beta) \cap \bar{\daleth}_2(\beta) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{c}, \bar{\upsilon}, \bar{y}\}\,;\ \beta \in \breve{\wp} \cap \breve{\text{Ɣ}}\,,$$

$$\bar{\daleth}_3(\gamma) = \bar{\daleth}_1(\gamma) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}\}\,; \gamma \in \breve{\wp} - \breve{\text{Ɣ}}\,,$$

$$\bar{\daleth}_3(\eta) = \bar{\daleth}_2(\eta) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{c}, \bar{\upsilon}\}\,;\ \eta \in \breve{\text{Ɣ}} - \breve{\wp}\,,$$

$$\bar{\S}_3(\alpha) = \bar{\S}_1(\alpha) \cup \bar{\S}_2(\alpha) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \alpha \in \breve{\wp} \cap \breve{\text{Ɣ}}\,,$$

$$\bar{\S}_3(\beta) = \bar{\S}_1(\beta) \cup \bar{\S}_2(\beta) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}\,;\ \beta \in \breve{\wp} \cap \breve{\text{Ɣ}}\,,$$

$$\bar{\S}_3(\gamma) = \bar{\S}_1(\gamma) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}\,;\ \gamma \in \breve{\wp} - \breve{\text{Ɣ}}\,,$$

$$\bar{\S}_3(\eta) = \bar{\S}_2(\eta) = \{\bar{\mathring{a}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \eta \in \breve{\text{Ɣ}} - \breve{\wp}.$$

It is not a T-BSSG because,

$\bar{\S}_3(\alpha) = \bar{\S}_1(\alpha) \cup \bar{\S}_2(\alpha) = \{\bar{\mathring{a}}, \bar{\text{ƃ}}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \alpha \in \breve{\wp} \cap \breve{\text{Ɣ}}$, is not a sub-semigroup of I.

3.1.22 Theorem:

If $\bar{\daleth}_1(\lambdabar)$ is a sub-semigroup of $\bar{\daleth}_2(\lambdabar)$ or $\bar{\daleth}_2(\lambdabar)$ is a sub-semigroup of $\bar{\daleth}_1(\lambdabar)$ $\forall\ \lambdabar \in \breve{\wp} \cap \breve{\text{Ɣ}}$. Then, the Res-Union of two T-BSSG $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}})$ over U is a T-BSSG over U.

Proof:

Assume that $\bar{\daleth}_1(\lambda)$ is a sub-semigroup of $\bar{\daleth}_2(\lambda)$ or $\bar{\daleth}_2(\lambda)$ is a sub-semigroup of $\bar{\daleth}_1(\lambda), \forall\, \lambda \in \breve{\wp} \cap \breve{\text{Ɣ}}$, then in both cases $\bar{\daleth}_1(\lambda) \cup \bar{\daleth}_2(\lambda)$ is a sub-semigroup.

Now, consider $\bar{\S}_1(\lambda) \cap \bar{\S}_2(\lambda)\ \forall\, \lambda \in \breve{\wp} \cap \breve{\text{Ɣ}}$ is a sub-SG trivially. It is because a sub-SG is formed by the intersection of any number of the sub-SG. Thus, in either case the Res-Union of two T-BSSG $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}})$ over U is a T-BSSG over U.

3.1.23 Theorem:

If $\bar{\S}_1(\lambda)$ is a sub-SG of $\bar{\S}_2(\lambda)$ or $\bar{\S}_2(\lambda)$ is a sub-SG of $\bar{\S}_1(\lambda)\ \forall\, \lambda \in \breve{\wp} \cap \breve{\text{Ɣ}}$. Then the Res-Intersection of two T-BSSG $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}})$ over U is a T-BSSG over U.

Proof: Same as the above theorem.

3.1.24 Example:

Assume that,

$$G = \{\, \text{ȩ}, \text{ʋ}_1, \text{ʋ}_2, \text{ʋ}_3, \text{ʋ}_4 \}\,, I = \{\, \text{ắ}, \text{ɓ}, c, d, \text{ȩ}\,\}\,, U = G \cup I\,,$$

$$\breve{\wp} = \{\lambda_1, \lambda_2\} \quad \text{and} \quad \breve{\text{Ɣ}} = \{\lambda_1, \lambda_2, \lambda_3\},$$

$$\bar{\daleth}_1(\lambda_1) = \{\text{ʋ}_3, \text{ʋ}_4\}\,;\ \bar{\daleth}_1(\lambda_2) = \{\text{ȩ}\};$$

$$\bar{\daleth}_2(\lambda_1) = \{\text{ȩ}, \text{ʋ}_1, \text{ʋ}_2, \text{ʋ}_3, \text{ʋ}_4\}\,;\ \bar{\daleth}_2(\lambda_2) = \{\text{ȩ}, \text{ʋ}_2, \text{ʋ}_3\}\,;\ \bar{\daleth}_2(\lambda_3) = \{\text{ȩ}, \text{ʋ}_3, \text{ʋ}_4\};$$

$$\bar{\S}_1(\lambda_1) = \{\text{ắ}, c, d, \text{ȩ}\}\,;\ \bar{\S}_1(\lambda_2) = \{\text{ắ}, \text{ɓ}, c, d, \text{ȩ}\};$$

$$\bar{\S}_2(\lambda_1) = \{\text{ắ}, d, \text{ȩ}\}\,;\ \bar{\S}_2(\lambda_2) = \{\text{ắ}, c, d\}\,;\ \bar{\S}_2(\lambda_3) = \{\text{ắ}, \text{ɓ}, c\}.$$

Then the Res-Union of $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}})$, is denoted and given by $(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}}) = \{< \lambda\,, \bar{\daleth}_1(\lambda) \cup \bar{\daleth}_2(\lambda),\ \bar{\S}_1(\lambda) \cap \bar{\S}_2(\lambda) >\,; \forall\, \lambda \in \breve{\wp} \cap \breve{\text{Ɣ}}\}$.

$$(\bar{\daleth}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Ɣ}}) = \left\{ \begin{array}{c} < \lambda_1\,,\ \bar{\daleth}_1(\lambda_1) \cup \bar{\daleth}_2(\lambda_1) = \{\text{ȩ}, \text{ʋ}_1, \text{ʋ}_2, \text{ʋ}_3, \text{ʋ}_4\}\,; \\ \bar{\S}_1(\lambda_1) \cap \bar{\S}_2(\lambda_1) = \{\text{ắ}, d, \text{ȩ}\} >\,;\ \forall\, \lambda_1 \in \breve{\wp} \cap \breve{\text{Ɣ}}. \\ < \lambda_2\,,\ \bar{\daleth}_1(\lambda_2) \cup \bar{\daleth}_2(\lambda_2) = \{\text{ȩ}, \text{ʋ}_2, \text{ʋ}_3\}\,; \\ \bar{\S}_1(\lambda_2) \cap \bar{\S}_2(\lambda_2) = \{\text{ắ}, c, d\} >\,;\ \forall\, \lambda_2 \in \breve{\wp} \cap \breve{\text{Ɣ}}. \end{array} \right.$$

"Res-Intersection" of $(\bar{7}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{7}_2, \bar{\S}_2, \breve{\mathrm{Y}})$, is denoted and defined by $(\bar{7}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{7}_2, \bar{\S}_2, \breve{\mathrm{Y}}) = \{< \lambda, \bar{7}_1(\lambda) \cap \bar{7}_2(\lambda), \bar{\S}_1(\lambda) \cup \bar{\S}_2(\lambda) >; \forall \lambda \in \breve{\wp} \cap \breve{\mathrm{Y}}\}$.

$$(\bar{7}_1, \bar{\S}_1, \breve{\wp}) \cap_R (\bar{7}_2, \bar{\S}_2, \breve{\mathrm{Y}}) = \begin{cases} < \lambda_1, \bar{7}_1(\lambda_1) \cap \bar{7}_2(\lambda_1) = \{\mathfrak{v}_3, \mathfrak{v}_4\}; \\ \bar{\S}_1(\lambda_1) \cup \bar{\S}_2(\lambda_1) = \{\mathring{\text{ǎ}}, c, d, \text{ę}\} >; \forall \lambda_1 \in \breve{\wp} \cap \breve{\mathrm{Y}}. \\ < \lambda_2, \bar{7}_1(\lambda_2) \cap \bar{7}_2(\lambda_2) = \{\text{ę}\}; \\ \bar{\S}_1(\lambda_2) \cup \bar{\S}_2(\lambda_2) = \{\mathring{\text{ǎ}}, \text{ɓ}, c, d, \text{ę}\} >; \forall \lambda_2 \in \breve{\wp} \cap \breve{\mathrm{Y}}. \end{cases}$$

3.1.25 Remarks:

It is noted that Res-Intersection and Res-Union of two T-BSSG over U, is not a T-BSSG.

3.1.26 Example:

Consider, $H = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}$ and $I = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}, \bar{\text{ę}}\}$ be two semigroups, then for any sets $\breve{\wp}$ and $\breve{\mathrm{Y}}$, and $(\bar{7}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{7}_2, \bar{\S}_2, \breve{\mathrm{Y}})$ be two T-BSSG over $U = H \cup I$.

Assume $\breve{\wp} = \{\alpha, \beta, \gamma\}$ $\mathring{\text{ǎ}}nd$ $\breve{\mathrm{Y}} = \{\alpha, \beta\}$.

$$\bar{7}_1(\alpha) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{y}\}; \bar{7}_1(\beta) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\}; \bar{7}_1(\gamma) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}\},$$

$$\bar{7}_2(\alpha) = \{\bar{\text{ǎ}}, \bar{z}\}; \bar{7}_2(\beta) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}.$$

$$\bar{\S}_1(\alpha) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}\}; \bar{\S}_1(\beta) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}\}; \bar{\S}_1(\gamma) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}\},$$

$$\bar{\S}_2(\alpha) = \{\bar{\text{ǎ}}, \bar{d}, \bar{\text{ę}}\}; \bar{\S}_2(\beta) = \{\bar{\text{ǎ}}, \bar{c}, \bar{d}\}.$$

Then the "Res-Union" of $(\bar{7}_1, \bar{\S}_1, \breve{\wp})$ and $(\bar{7}_2, \bar{\S}_2, \breve{\mathrm{Y}})$, is denoted and defined by $(\bar{7}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{7}_2, \bar{\S}_2, \breve{\mathrm{Y}}) = \{< \lambda, \bar{7}_1(\lambda) \cup \bar{7}_2(\lambda), \bar{\S}_1(\lambda) \cap \bar{\S}_2(\lambda) >; \forall \lambda \in \breve{\wp} \cap \breve{\mathrm{Y}}\}$.

$$(\bar{7}_1, \bar{\S}_1, \breve{\wp}) \cup_R (\bar{7}_2, \bar{\S}_2, \breve{\mathrm{Y}}) = \begin{cases} < \alpha, \bar{7}_1(\alpha) \cup \bar{7}_2(\alpha) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{y}, \bar{z}\}; \\ \bar{\S}_1(\alpha) \cap \bar{\S}_2(\alpha) = \{\bar{\text{ǎ}}\} >; \forall \alpha \in \breve{\wp} \cap \breve{\mathrm{Y}}. \\ < \beta, \bar{7}_1(\beta) \cup \bar{7}_2(\beta) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}, \bar{z}\}; \\ \bar{\S}_1(\beta) \cap \bar{\S}_2(\beta) = \{\bar{\text{ǎ}}, \bar{c}\} >; \forall \beta \in \breve{\wp} \cap \breve{\mathrm{Y}}. \end{cases}$$

It is not a T-BSSG because $\bar{7}_1(\alpha) \cup \bar{7}_2(\alpha) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{y}, \bar{z}\}$; $\forall \alpha \in \breve{\wp} \cap \breve{\mathrm{Y}}$, is not a sub-semigroup of H.

"Res-Intersection" of $\left(\bar{ٵ}_1, \bar{\S}_1, \breve{\wp}\right)$ and $\left(\bar{ٵ}_2, \bar{\S}_2, \breve{\gamma}\right)$, is denoted and given by $\left(\bar{ٵ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_R \left(\bar{ٵ}_2, \bar{\S}_2, \breve{\gamma}\right) = \left\{< \lambda, \bar{ٵ}_1(\lambda) \cap \bar{ٵ}_2(\lambda), \bar{\S}_1(\lambda) \cup \bar{\S}_2(\lambda) > ; \forall \lambda \in \breve{\wp} \cap \breve{\gamma}\right\}$.

$$\left(\bar{ٵ}_1, \bar{\S}_1, \breve{\wp}\right) \cap_R \left(\bar{ٵ}_2, \bar{\S}_2, \breve{\gamma}\right) = \left\{\begin{array}{c} < \alpha, \bar{ٵ}_1(\alpha) \cap \bar{ٵ}_2(\alpha) = \{\bar{\check{a}}\} ; \\ \bar{\S}_1(\alpha) \cup \bar{\S}_2(\alpha) = \{\bar{\check{a}}, \bar{b}, \bar{d}, \bar{e}\} > ; \forall \alpha \in \breve{\wp} \cap \breve{\gamma}. \\ < \beta, \bar{ٵ}_1(\beta) \cap \bar{ٵ}_2(\beta) = \{\bar{\check{a}}, \bar{b}, \bar{c}, \bar{\mathfrak{v}}, \bar{y}\} ; \\ \bar{\S}_1(\beta) \cup \bar{\S}_2(\beta) = \{\bar{\check{a}}, \bar{b}, \bar{c}, \bar{d}\} > ; \forall \beta \in \breve{\wp} \cap \breve{\gamma}. \end{array}\right.$$

As, $\bar{\S}_1(\alpha) \cup \bar{\S}_2(\alpha) = \{\bar{\check{a}}, \bar{b}, \bar{d}, \bar{e}\} ; \forall \alpha \in \breve{\wp} \cap \breve{\gamma}$, is not a sub-SG of I. So, due to this reason "Res-Intersection" of $\left(\bar{ٵ}_1, \bar{\S}_1, \breve{\wp}\right)$ and $\left(\bar{ٵ}_2, \bar{\S}_2, \breve{\gamma}\right)$ is not a T-BSSG.

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